Genus Two Zhu Theory
for Vertex Operator Algebras

Thomas Patrick Gilroy
06549977
Ph.D Thesis
Supervised by Prof. Michael P. Tuite
National University of Ireland, Galway
Final submission June 2014
Abstract

In this thesis we consider the recursive properties of correlation functions for a vertex operator algebra on a genus two Riemann surface formed by sewing two tori together. We derive a system of formal recursive identities, which allow us to express an arbitrary genus two $n$-point correlation function in terms of $(n - 1)$-point functions. This generalises Zhu reduction for genus one correlation functions. We apply these recursive identities to compute the genus two Heisenberg $n$-point correlation functions, the genus two Virasoro $n$-point functions and the genus two Ward Identity. We define a formal differential operator with respect to the sewing parameters and derive differential equations for holomorphic 1-forms, the normalised 2-form, the Heisenberg partition function and the partition function for the Virasoro $(2,5)$-minimal model. We prove that this formal differential operator is defined on an open subset of the sewing domain.
Contents

1 Introduction 5
  1.1 Summary ...................................... 5
  1.2 Acknowledgements .............................. 8
  1.3 Basic Notational Conventions .................. 9
  1.4 Vertex Operator Algebras ....................... 10
    1.4.1 Local Fields ............................ 10
    1.4.2 Vertex Algebras ......................... 11
    1.4.3 Vertex Operator Algebras ................. 12
    1.4.4 The Square Bracket Formalism ............ 13
    1.4.5 Modules ................................ 14
    1.4.6 Adjoint Operators and the Li-Zamolodchikov Metric 15
    1.4.7 Heisenberg VOAs .......................... 17
  1.5 Genus One .................................. 17
    1.5.1 Elliptic Functions and Modular Forms .... 17
    1.5.2 Genus One Correlation Functions and Zhu Recursion 18
  1.6 Genus Two .................................. 20
    1.6.1 Genus Two Surfaces Formed from Sewn Tori 20
    1.6.2 Genus Two Correlation Functions ........... 24

2 1-point Correlation Functions 27
  2.1 The General Quasiprimary Case .................. 27
  2.2 Quasiprimary Descendants ........................ 37

3 Applications of the 1-point Formula 39
  3.1 Genus Two Heisenberg 1-point Functions ........ 39
  3.2 Genus Two Virasoro 1-point Functions .......... 40

4 n-point Correlation Functions 44
  4.1 The General Quasiprimary Case .................. 44
  4.2 Genus Two Generalised Weierstrass Functions .... 61
  4.3 Quasiprimary Descendants ........................ 64

5 Applications of the n-point Recursion Formula 65
  5.1 Genus Two Ward Identities ...................... 65
  5.2 Genus Two Virasoro n-point Functions .......... 68
5.3 Genus Two Heisenberg n-point Functions 73

6 Geometric Results 76

6.1 A differential equation for the genus two Heisenberg partition function 76
6.2 The convergence of $D_x$ 77
6.3 A differential equation for holomorphic 1-forms 80
6.4 A differential equation for the normalised 2-form 82
6.5 The $(2, 5)$ minimal model 83
6.6 Conjectures 86
1 Introduction

1.1 Summary

First described by R.E. Borcherds [B] and I. Frenkel, J. Lepowsky and A. Meurmann [FLM], a vertex operator algebra (VOA) is an algebraic structure closely related to conformal field theory in physics [BPZ].

Due to Y. Zhu [Z], we can express the genus one $n$-point correlation functions for a VOA in terms of $(n-1)$-point functions using formal recursive identities. These formal recursive identities give rise to differential equations which can be used to prove the convergence of the genus one partition functions of a class of VOAs known as the $C_2$-cofinite VOAs.

In a program of research by G. Mason and M.P. Tuite in conjunction with A. Zuevsky, the correlation functions for VOAs and super-VOAs on a genus two surface formed by sewing procedures (due to A. Yamada [Y]) have been defined. This thesis considers the recursive properties of correlation functions for vertex operator algebras on a genus two surface formed by sewing two tori together.

This thesis is divided into sections, each with a particular theme.

The remainder of Section 1 contains the basic definitions and results regarding vertex operator algebras, elliptic functions and modular forms, genus one correlation functions, genus two Riemann surfaces formed by sewing two tori and genus two correlation functions which will be needed for this thesis.

In Section 2 we derive a formal identity for a 1-point function of a VOA on a genus two surface formed by the sewing of two tori. By first considering a general quasiprimary vector, we use Zhu’s recursion formula on the genus one factors of the genus two 1-point function to derive recursive relationships between a certain pair of infinite vectors. These recursive relationships give rise to the formal identity for the genus two 1-point correlation function for the quasiprimary vector. The quasiprimary case is extended to the case of a general quasi-primary descendant. Since the quasiprimary descendants span the VOA, we may write the genus two 1-point correlation function for any vector using the quasiprimary and quasi-primary descendant identities. An interesting counting of parameters for the 1-point function of a quasiprimary vector of weight $N$ highlights a conjectured
connection to the Riemann-Roch theorem concerning $N$-forms (differentials) on a genus two Riemann surface.

In Section 3 we apply the formal identity for the 1-point function to specific examples. We calculate the genus two Heisenberg vector 1-point function for a pair of Heisenberg modules. The result agrees with a known result of [MT3] obtained by a different method. We also calculate the Virasoro 1-point function for an arbitrary VOA. This calculation introduces a formal differential operator with respect to the sewing parameters, which is of particular importance in later sections.

In Section 4 we derive the main result of this thesis; a formal recursive identity which allows us to express an arbitrary genus two $n$-point correlation function in terms of $(n - 1)$-point functions. The approach is similar to that of Section 2. Beginning again with a quasiprimary vector, we use Zhu’s recursion formula on the genus one factors of the genus two $n$-point correlation function to derive recursive relationships between a pair of infinite vectors. These recursive relationships give rise to the recursive identity for the genus two $n$-point correlation function in terms of $(n - 1)$-point functions. This recursive identity is expressed in terms of new generalised genus two Weierstrass functions, which again depend on the weight of the quasiprimary. These play the role that the usual Weierstrass functions have in the elliptic case. The quasiprimary case is extended to the cases of a quasiprimary descendant, and by means of these identities all genus two $n$-point correlation functions may be expressed recursively.

In Section 5 we apply the formal recursive identity for $n$-point correlation functions. We calculate the genus two Ward Identity for an arbitrary number of primary vectors. We also calculate the Virasoro $n$-point function for an arbitrary VOA. These calculations involve the formal differential operator introduced in Section 3. We also calculate the Heisenberg $n$-point function for a pair of Heisenberg modules. This result agrees with a known result of [MT3].

In Section 6 we use our $n$-point recursion formula to derive formal differential equations. In particular, differential equations are derived for the genus two Heisenberg partition function, the holomorphic 1-forms, the normalised 2-form and the genus two partition function for the Virasoro $(2,5)$-minimal model. By comparison with known results for the Heisenberg VOA, we show that the formal differential
operator introduced in Section 3 is defined on an open subset of the sewing domain. We close with a discussion of the results and by stating several important conjectures motivated by the work of this thesis.
1.2 Acknowledgements

I wish to thank my supervisor Prof. Michael P. Tuite for his continued guidance, encouragement and support since I was an undergraduate. I have been privileged to have had a supervisor as enthusiastic and dedicated as he is.

I would also like to acknowledge the Irish Research Council for funding my postgraduate studies.

Finally, I wish to thank my family, my girlfriend Lorraine Kiely and all of my friends for their support throughout the years.
1.3 Basic Notational Conventions

We begin with some basic notations and definitions that will be used throughout the thesis. \( \mathbb{Z} \) is the set of integers, \( \mathbb{R} \) is the set of real numbers, \( \mathbb{C} \) is the set of complex numbers and \( \mathbb{H} \) is the complex upper-half plane, that is

\[ \mathbb{H} = \{ \tau \in \mathbb{C} | \Im(\tau) > 0 \}. \]

We make uses of the following \( q \)-conventions

\[ q^x = e^x, \]

for formal \( x \) where \( e^x \) is sensible and

\[ q = q_{2\pi i\tau} = e^{2\pi i\tau}, \]

for \( \tau \in \mathbb{H} \). In most instances we will have two parameters \( \tau_1, \tau_2 \in \mathbb{H} \) and we will use the convention that

\[ q_1 = q_{2\pi i\tau_1} = e^{2\pi i\tau_1}, \]

\[ q_2 = q_{2\pi i\tau_2} = e^{2\pi i\tau_2}. \]

Where it appears, \( q_0 = e^0 = 1 \). Finally, we use the convention that

\[ \partial_x = \frac{\partial}{\partial x}, \]

\[ \partial_x^k = \frac{\partial^k}{\partial x^k}. \]
1.4 Vertex Operator Algebras

In this section we review the necessary results and definitions regarding vertex algebras (first described in [B]) and vertex operator algebras (introduced in [FLM]). In this thesis, we follow the treatment of the review article [MT4]. Further details may be found there. For alternative approaches, see [B], [FHL], [FLM], [K], [LL], [MN].

1.4.1 Local Fields

Let $V$ be a vector space and let $\text{End}(V)$ denote the space of endomorphisms of $V$. We consider formal Laurent series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1},$$

where $a(n) \in \text{End}(V)$ and $z$ is a formal parameter. We then have

$$a(n) = \text{Res}_z a(z)z^n.$$

We let $\text{End}(V)[[z, z^{-1}]]$ denote the space of all such formal Laurent series. The series $a(z)$ defines a linear map

$$a(z) : V \to V[[z, z^{-1}]]$$
$$v \mapsto \sum_{n \in \mathbb{Z}} (a(n)v)z^{-n-1}.$$

The endomorphisms $a(n)$ are called the modes of $a(z)$. The element of $V$ are referred to as states, and we call $V$ the state-space or the Fock space.

**Definition 1.1.** The series $a(z) \in \text{End}(V)[[z, z^{-1}]]$ is a field if $\forall v \in V$, $\exists N \in \mathbb{Z}$ such that $a(n)v = 0 \ \forall n > N$, or equivalently, if $\forall v \in V$ we have $a(z)v \in V[[z]][z^{-1}]$.

We let

$$\mathcal{F}(V) = \{ a(z) \in \text{End}(V)[[z, z^{-1}]] \mid a(z) \text{ a field} \}.$$

It is clear that $\mathcal{F}(V)$ is a subspace of $\text{End}(V)[[z, z^{-1}]]$. 

10
We distinguish between indeterminates to define the commutator of two fields \( a(z), b(z) \). Set
\[
[a(z_1), b(z_2)] = \left[ \sum_{m \in \mathbb{Z}} a(m) z_1^{-m-1}, \sum_{n \in \mathbb{Z}} b(n) z_2^{-n-1} \right] = \sum_{m,n \in \mathbb{Z}} [a(m), b(n)] z_1^{-m-1} z_2^{-n-1},
\]
where \([a(m), b(n)]\) is the usual Lie bracket.

**Definition 1.2.** \( a(z), b(z) \in \mathfrak{F}(V) \) are **mutually local** if \( \exists k \in \mathbb{Z}, \ (k \geq 0) \) such that
\[
(z_1 - z_2)^k a(z_1) b(z_2) = (z_1 - z_2)^k b(z_1) a(z_2).
\]
We write \( a(z) \sim b(z) \) if \( a(z), b(z) \) are mutually local fields.

For negative \( k \), we use the convention that \((z_1 - z_2)^k\) is expanded in the formal domain \(|z_1| > |z_2|\) that is
\[
(z_1 - z_2)^k = \sum_{l \geq 0} \binom{k}{l} z_1^{k-l} (-z_2)^l.
\]

If we fix a nonzero state \( 1 \in V \), we say that \( a(z) \in \mathfrak{F}(V) \) is **creative** (with respect to \( 1 \)) and **creates** the state \( a \) if
\[
a(z)1 = a + O(z) \in V[[z]].
\]

### 1.4.2 Vertex Algebras

**Definition 1.3.** A **vertex algebra** (VA) is a quadruple \((V, Y, 1, D)\)
where
\[
Y : V \to \mathfrak{F}(V), \quad v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} \text{ is a linear map},
\]
\[
1 \in V, \quad 1 \neq 0,
\]
\[
D \in \text{End}(V), \quad D1 = 0,
\]
and \( \forall u, v \in V \), we have:
1. \( Y(u, z) \sim Y(v, z) \) (locality)
2. \( Y(u, z) 1 = u + O(z) \) (creativity)

3. \([D, Y(u, z)] = \partial_z Y(u, z)\) (translation-covariance).

The state 1 is called the vacuum vector, while the fields \( Y(v, z) \) are called vertex operators.

There are a number of equivalent formulations of these axioms known. An important consequence of these axioms is the associativity formula

\[
(z_1 + z_2)^k Y(u, z_1 + z_2)Y(v, z_2)w = (z_1 + z_2)^k Y(Y(u, z_1)v, z_2)w.
\]

for \( k \gg 0 \). The following theorem allows the construction of vertex algebras [FKRW]

**Theorem 1.1.** Let \( V \) be a vector space with \( 0 \neq 1 \in V \) and \( D \in \text{End}(V) \). Suppose \( S \subseteq \mathcal{F}(V) \) is a set of mutually local, creative, translation-covariant fields which generates \( V \) in the sense that

\[
V = \text{span}\{a^i(-n_1) \ldots a^i(-n_k)1 | a^i(z) \in S, n_1, \ldots, n_k \geq 1, k \geq 0\}.
\]

Then, there is a unique vertex algebra \((V, Y, 1, D)\) such that \( Y(a^i(-1)1, z) = a^i(z) \).

**1.4.3 Vertex Operator Algebras**

**Definition 1.4.** The Virasoro algebra is the Lie algebra with underlying vector space

\[
\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L(n) \oplus \mathbb{C}K,
\]

where \( K \) is a central element and Lie Bracket

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m,-n}K.
\]

**Definition 1.5.** A vertex operator algebra (VOA) is a quadruple \((V, Y, 1, \omega)\) where \( V = \bigoplus_{n \in \mathbb{Z}} V(n) \) is a \( \mathbb{Z} \)-graded vector space and

\[
Y : V \rightarrow \mathcal{F}(V), \quad v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1}
\]

\( 1, \omega \in V, \quad 1 \neq 0 \).
The fields $Y(v, z)$ are mutually local and creative, and the following hold:

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \text{ with constant } c \text{ such that }$$

$$[L(m), L(n)] = (m - n) L(m + n) + \frac{m^3 - m}{12} \delta_{m,-n} \text{Id}_V$$

$$V_{(n)} = \{ v \in V \mid L(0)v = nv \}$$

$$\dim V_{(n)} < \infty, V_{(n)} = 0, \text{ for } n \ll 0$$

$$Y(L(-1)u, z) = \partial_z Y(u, z).$$

The state $\omega$ is referred to as the conformal vector or the Virasoro vector, while $c$ is called the central charge. For a vector $v \in V_{(k)}$ we find

$$v(n) : V_{(m)} \to V_{(m+k-n-1)}.$$  

**Definition 1.6.** The zero mode of a homogeneous vector $v$ is given by

$$o(v) = v(k - 1).$$

Notice that

$$o(v) : V_{(m)} \to V_{(m)},$$

for each homogeneous space of $V$.

**Definition 1.7.** A state $v \in V$ is quasiprimary if $L(1)v = 0$.

**Definition 1.8.** A state $v \in V$ is primary if $v$ is quasiprimary and additionally $L(2)v = 0$.

The Virasoro commutation relationships imply that the conformal vector $\omega$ is quasiprimary.

### 1.4.4 The Square Bracket Formalism

Given a VOA $(V, Y, 1, \omega)$, we can find a second isomorphic VOA introduced by Zhu [Z], called the square bracket VOA $(V, Y[,], 1, \tilde{\omega})$. Both VOAs have the same underlying Fock space, the same central
charge and the same vacuum vector $1$. The operator $Y[v, ]$ is defined by a change of coordinates involving an exponential map where

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)}v, q_z - 1),$$

where $q_z^{L(0)}$ is the operator

$$q_z^{L(0)} : V \rightarrow V[[z]], \quad v \mapsto q_k z v \quad (v \in V(k)).$$

The new conformal vector is

$$\tilde{\omega} = \omega - \frac{c}{24},$$

with vertex operator

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-1}.$$

States homogenous with respect to $L(0)$ are not necessarily homogenous with respect to $L[0]$, however, both Virasoro vectors have the same set of primary states. We write $\text{wt}(v) = k$ if $L(0)v = kv$ and we write $\text{wt}[v] = k$ if $L[0]v = kv$.

The main purpose of this process is to construct vertex operators which are automatically periodic in $z$ with period $2\pi i$.

### 1.4.5 Modules

**Definition 1.9.** A **weak $V$-module** is a pair $(M, Y_M)$ where $Y_M : M \rightarrow \mathfrak{F}(M), \quad v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_M(n) z^{-n-1}$ is a linear map, and the following hold $\forall u, v \in V, w \in M$:

1. **locality:** $Y_M(u, z) \sim Y_M(v, z)$
2. **vacuum:** $Y(1, z) = \text{Id}_M$
3. **associativity:** $(z_1 + z_2)^k Y_M(u, z_1 + z_2) Y_M(v, z_2) w = (z_1 + z_2)^k Y_M(Y(u, z_1)v, z_2) w$ for $k \gg 0$. 

14
Definition 1.10. A V-module is a weak V-module \((M, Y_M)\) equipped with a grading \(M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}\) such that

\[
\begin{align*}
\dim M_{\lambda} &< \infty, \\
\forall \lambda, \ M_{\lambda+n} &= 0 \text{ for } n \ll 0, \\
L(0)m &= \lambda m, \ m \in M_{\lambda}.
\end{align*}
\]

A V-module is irreducible if no proper, nonzero subspace of \(M\) is invariant under all modes \(v_{M}(n)\). A simple module is a module in which 0 and \(M\) are the only submodules. For V-modules, the terms irreducible and simple are synonymous.

1.4.6 Adjoint Operators and the Li-Zamolodchikov Metric

The subalgebra \(\{L(-1), L(0), L(1)\}\) generates a natural action on vertex operators associated with \(SL(2, \mathbb{C})\) Möbius transformations on the parameter \(z\). In particular, we note the mapping \(z \mapsto A/z, \ A \in \mathbb{C}\) for which

\[
Y(v, z) \mapsto Y_{A}^{\dagger}(v, z) = Y\left(\exp\left(\frac{z}{A}L(1)\right)\left(-\frac{A}{z^2}\right)^{L(0)}v, \frac{A}{z}\right).
\]

Definition 1.11. \(Y_{A}^{\dagger}(v, z)\) is called the adjoint vertex operator (with respect to \(A\)). \([FHL]\)

For \(u\) quasiprimary of weight \(\text{wt}(u)\), it follows that \(Y_{A}^{\dagger}(u, z) = \sum_{n} u_{\dagger}(n)z^{-n-1}\) has modes

\[
u_{\dagger}(n) = (-1)^{\text{wt}(u)}A^{m+1-\text{wt}(u)}u(2\text{wt}(u) - n - 2).
\]

A bilinear form \((,): V \times V \to \mathbb{C}\) is called invariant if the following identity holds for all \(a, b, c \in V\):

\[
(Y(a, z)b, c) = (b, Y_{A}^{\dagger}(a, z)c),
\]

where \(Y_{A}^{\dagger}(a, z)\) is the adjoint operator as before \([FHL]\). By a theorem of \([FHL]\), any invariant bilinear form on \(V\) is necessarily symmetric. In terms of modes, we have
\[ \langle u(n)a, b \rangle = \langle a, u^\dagger(n)b \rangle. \]

Choosing \( u = \omega \) and \( n = 1 \) gives

\[ \langle L(0)a, b \rangle = \langle a, L(0)b \rangle, \]

and thus for homogeneous states \( a, b \) we have that \( \langle a, b \rangle = 0 \) if \( \text{wt}(a) \neq \text{wt}(b) \).

[Li] guarantees that if \( V(0) = \mathbb{C}1 \) and \( V \) is simple and self-dual in the sense that \( V \) is isomorphic to the dual module \( V' \) as a \( V \)-module, then \( V \) has an invariant non-degenerate bilinear form and this bilinear form is unique up to scalars. All VOAs considered in this thesis satisfy these conditions. If we normalise so that \( \langle 1, 1 \rangle = 1 \), the bilinear form is unique. These results motivate the following definition

**Definition 1.12.** The **Li-Zamolodchikov metric** on \( V \) is the unique bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) such that

\[ \langle Y(a, z)b, c \rangle = \langle b, Y^\dagger_A(a, z)c \rangle, \]

for all \( a, b, c \in V \) and

\[ \langle 1, 1 \rangle = 1. \]

We refer to this as the Li-Z metric for short. We can similarly define a Li-Z metric for the square bracket VOA corresponding to \( V \) by

**Definition 1.13.** The **square bracket Li-Z metric** on \( V \) is the unique bilinear form \( \langle \cdot, \cdot \rangle_{\text{sq}} : V \times V \to \mathbb{C} \) such that

\[ \langle Y[a, z]b, c \rangle_{\text{sq}} = \langle b, Y^\dagger_A[a, z]c \rangle_{\text{sq}}, \]

for all \( a, b, c \in V \) where

\[ Y^\dagger_A[a, z] = Y \left[ \exp \left( \frac{z}{A} L[1] \right) \left( -\frac{A}{z^2} \right)^{L[0]} \frac{A}{z} \right], \]

and

\[ \langle 1, 1 \rangle_{\text{sq}} = 1. \]

The subscript \( \text{sq} \) may be omitted where there is no ambiguity. The subscript \( A \) may be omitted from the adjoints \( Y^\dagger_A(a, z), Y^\dagger_A[a, z] \) where the choice of \( A \in \mathbb{C} \) is unambiguous.
1.4.7 Heisenberg VOAs

Definition 1.14. The Heisenberg Lie algebra is the Lie algebra with bracket

\[ [a(m), a(n)] = m\delta_{m+n,0}. \]

Theorem 1.1 implies there is a unique vertex algebra \((M, Y, 1, D)\) with one generator

\[ Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \]

and \(D(a(n)1) = -na(n - 1)1\).

Definition 1.15. \((M, Y, 1, D)\) is the (rank 1) Heisenberg vertex algebra.

To find a Virasoro structure on \(M\), we may use the Segal-Sugawara construction detailed in [LL]. We find

Definition 1.16. \((M, Y, 1, \omega)\) is the Heisenberg vertex operator algebra, where the Virasoro vector is given by

\[ \omega = \frac{1}{2}a(-1)^21. \]

The state \(a = a(-1)1\) is primary of weight 1 with respect to this Virasoro structure.

In several calculations we will be concerned with simple Heisenberg modules \(M_\alpha = M \otimes e^\alpha\) where \(\alpha \in \mathbb{C}\) is the eigenvalue of \(a(0)\) [MT4].

1.5 Genus One

1.5.1 Elliptic Functions and Modular Forms

In this subsection we define the various elliptic functions and modular forms used throughout the thesis.

Definition 1.17. The Weierstrass function is given by

\[ \wp(z, \tau) = \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}} \left( \frac{1}{(z - \omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right), \]

where \(\omega_{m,n} = 2\pi i(m\tau + n), \tau \in \mathbb{H}\) and the prime indicates that the \((m, n) = (0, 0)\) term is omitted. The sum is absolutely convergent and independent of the order of summation when \(|q| < |q_z| < 1\).
\( \wp(z, \tau) \) is an elliptic function with periods \( 2\pi i \) and \( 2\pi i \tau \) [La].

**Definition 1.18.** The **Eisenstein series** for an integer \( k \geq 2 \) is given by

\[
E_k(\tau) = E_k(q) = \begin{cases} 
0 & \text{if } k \text{ is odd,} \\
-\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n & \text{if } k \text{ is even.}
\end{cases}
\]

where \( \sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \) and \( B_k \) is the \( k \)th Bernoulli number.

If \( k \geq 4 \) then \( E_k(\tau) \) is a holomorphic modular form of weight \( k \) on \( SL(2, \mathbb{Z}) \), while \( E_2(\tau) \) is a quasi-modular form.

**Definition 1.19.**

\[
P_1(z, \tau) = \frac{1}{z} - \sum_{k \geq 2} E_k(\tau) z^{k-1},
\]

\[
P_k(z, \tau) = \frac{(-1)^{k-1}}{(k-1)!} \partial_z^{k-1} P_1(z, \tau).
\]

In particular, we have

\[
P_2(z, \tau) = \wp(z, \tau) + E_2(\tau)
\]

\[
= \frac{1}{z^2} + \sum_{k=2}^{\infty} (k-1) E_k(\tau) z^{k-2}.
\]

Finally, several specific examples will involve the following function.

**Definition 1.20.** The **Dedekind \( \eta \)-function** is defined by

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]

### 1.5.2 Genus One Correlation Functions and Zhu Recursion

**Definition 1.21.** The **genus one partition function** for \( V \) is given by

\[
Z_V^{(1)}(\tau) = \text{Tr}_V \left( q^{L(0) - c/24} \right).
\]

18
Definition 1.22. A genus one $n$-point correlation function for $V$ is of the form

\[ Z^{(1)}_V(v_1, z_1; \ldots; v_n, z_n; \tau) = \text{Tr}_V \left( Y(q^{L(0)}_{z_1}) v_1, q_{z_1} \ldots Y(q^{L(0)}_{z_n}) v_n, q_{z_n} \right) q^{L(0) - c/24}. \]

Any $n$-point function can be expressed in terms of 1-point functions [MT1] as

\[ Z^{(1)}_V(v_1, z_1; \ldots; v_n, z_n; \tau) = Z^{(1)}_V(Y[v_1, z_1] \ldots Y[v_n, z_n] 1; \tau) \]

where $z_{in} = z_i - z_n$. Throughout the thesis we make repeated use of the following theorem of Zhu [Z], which gives a recursive identity for $n$-point correlation functions in terms of $(n - 1)$-point functions.

**Theorem 1.2.** [Zhu’s Recursion Formula] The genus one $n$-point correlation functions obey

\[ Z^{(1)}_V(v_1, z_1; \ldots; v_n, z_n; \tau) = \text{Tr}_V \left( o(v_1)Y(q^{L(0)}_{z_2}) v_2, q_{z_2} \ldots Y(q^{L(0)}_{z_n}) v_n, q_{z_n} \right) q^{L(0) - c/24} \]

\[ + \sum_{k=2}^{n} \sum_{j \geq 0} P_{1+j}(z_1 - z_k, \tau) Z^{(1)}_V(v_2, z_2; \ldots; v_1[j] v_k, z_k; \ldots; v_n, z_n; \tau). \]

This powerful result has many important implications. Among these, we have that the genus one 1-point functions are given by

\[ Z^{(1)}_V(v; \tau) = \text{Tr}_V \left( o(v)q^{L(0) - c/24} \right), \]

and so we have that

\[ Z^{(1)}_V(\tilde{\omega}; \tau) = \text{Tr}_V \left( o(\tilde{\omega})q^{L(0) - c/24} \right) \]

\[ = \text{Tr}_V \left( (L(0) - \frac{c}{24}) q^{L(0) - c/24} \right) \]

\[ = q \partial q Z^{(1)}_V(\tau). \]
We find that for primary vectors $v_1, \ldots, v_n \in V$, we have the genus one Ward Identity

$$Z^{(1)}_{\bar{V}}(\tilde{\omega}, x; v_1, x_1; \ldots; v_n, x_n; \tau) = \left(q \partial q + \sum_{k=1}^{n} (P_1(x - x_k, \tau) \partial x_k + \mathrm{wt}[v_k]P_2(x - x_k, \tau)) \right) Z^{(1)}_{\bar{V}}(v_1, z_1; \ldots; v_n, z_n; \tau).$$

We have also that the Virasoro $n$-point function is

$$Z^{(1)}_{\bar{V}}(\tilde{\omega}, x_1; \ldots; \tilde{\omega}, x_n; \tau) = \left(q \partial q + \sum_{k=2}^{n} (P_1(x_1 - x_k, \tau) \partial x_k + 2P_2(x_1 - x_k, \tau)) \right) Z^{(1)}_{\bar{V}}(\tilde{\omega}, x_2; \ldots; \tilde{\omega}, x_n; \tau)$$

$$+ \frac{c}{2} \sum_{k=2}^{n} P_4(x_1 - x_k, \tau) Z^{(1)}_{\bar{V}}(\tilde{\omega}, x_2; \ldots; \tilde{\omega}, \hat{x}_k; \ldots; \tilde{\omega}, x_n; \tau),$$

where the $\hat{x}_k$ indicates that the insertion of $\tilde{\omega}$ at $x_k$ is omitted.

Theorem 1.2 gives identities between formal series, while the differential equations from the Ward Identities allow us to prove convergence. The primary aim of this thesis is to find result analogous to Theorem 1.2 for genus two correlation functions.

1.6 Genus Two

1.6.1 Genus Two Surfaces Formed from Sewn Tori

We consider a compact Riemann surface $S$ of genus two with canonical homology basis $a_1, a_2, b_1, b_2$. There exists two holomorphic 1-forms $\nu_i$, $i = 1, 2$ which we may normalize by [FK]

$$\oint_{a_i} \nu_j = 2\pi i \delta_{ij}, \tag{1}$$

These forms can also be defined via the unique singular bilinear two form $\omega^{(2)}$, known as the normalized differential of the second kind. It is defined by the following properties [FK],[Y]

$$\omega^{(2)}(x, y) = \left(\frac{1}{(x - y)^2} + \text{regular terms}\right) dx dy, \tag{2}$$

20
for any local coordinates \( x, y \), with normalization

\[
\int_{a_i} \omega^{(2)}(x, \cdot) = 0,
\]

for \( i = 1, 2 \). An explicit construction of \( \omega^{(2)} \) can be found in [F]. Using the Riemann bilinear relations, one finds that

\[
\nu_i(x) = \oint_{a_i} \omega^{(2)}(x, \cdot),
\]

with \( \nu_i \) normalized as in (1).

**Definition 1.23.** The **projective connection** \( s^{(2)}(x) \) is defined by

\[
s^{(2)}(x) = 6 \lim_{x \to y} \left( \omega^{(2)}(x, y) - \frac{dx dy}{(x - y)^2} \right).
\]

**Definition 1.24.** The **genus two period matrix** \( \Omega \) is defined by

\[
\Omega_{ij} = \frac{1}{2\pi i} \oint_{b_i} \nu_j,
\]

for \( i, j = 1, 2 \).

We have that \( \Omega \in \mathbb{H}_2 \), the Siegel upper half plane.

We now review a general method due to Yamada [Y] and discussed in detail in [MT2] for calculating \( \omega^{(2)}(x, y) \), \( \nu(x) \) and \( \Omega_{ij} \) on the genus two Riemann surface formed by sewing together two tori \( S_a \) for \( a = 1, 2 \). We may sometimes refer to \( S_1 \) and \( S_2 \) as the left and right torus respectively.

We consider an oriented torus \( S_a = \mathbb{C}/\Lambda_a \) with lattice \( \Lambda_a = 2\pi i (\mathbb{Z} \tau_a \oplus \mathbb{Z}) \) for \( \tau_a \in \mathbb{H}_1 \). For local coordinate \( z_a \in \mathbb{C}/\Lambda_a \) consider the closed disk \( |z_a| \leq r_a \) which is contained in \( S_a \) provided \( r_a < \frac{1}{2} D(q_a) \) where

\[
D(q_a) = \min_{\lambda \in \Lambda_a, \lambda \neq 0} |\lambda|,
\]

is the minimal lattice distance. We introduce a complex sewing parameter \( \epsilon \) where \( |\epsilon| \leq r_1 r_2 < \frac{1}{2} D(q_1) D(q_2) \) and excise the disk \( \{z_a, |z_a| \leq |\epsilon|/r_a\} \) centered at \( z_a = 0 \) to form a punctured torus

\[
\hat{S}_a = S_a \setminus \{z_a, |z_a| \leq |\epsilon|/r_a\},
\]

(6)
where we use the convention
\[ \mathcal{I} = 2, \quad \mathcal{J} = 1. \] (7)

Defining the annulus
\[ \mathcal{A}_a = \{ z_a, |\epsilon|/r_a \leq |z_a| \leq r_a \} \subset \mathcal{S}_a, \] (8)
we identify \( \mathcal{A}_1 \) with \( \mathcal{A}_2 \) via the sewing relation
\[ z_1 z_2 = \epsilon. \] (9)

We call this procedure the \( \epsilon \)-formalism. The genus two Riemann surface is parameterized by the domain
\[ \mathcal{D}^\epsilon = \{ (\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} | |\epsilon| < \frac{1}{4} \mathcal{D}(q_1)\mathcal{D}(q_2) \}. \] (10)

We next introduce the infinite dimensional matrices

**Definition 1.25.**
\[ A_a(\tau_a, \epsilon) = A_a(k, l, \tau_a, \epsilon) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} C(k, l, \tau_a), \]
for \( k, l \geq 1 \), where
\[ C(k, l, \tau_a) = (-1)^{k+1} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau_a). \]

The matrices \( A_1, A_2 \) play an important role both here and in our later calculations of the free bosonic VOA and its modules on a genus two Riemann surface. In particular, the matrix \( \mathbb{1} - A_1 A_2 \) and \( \det(\mathbb{1} - A_1 A_2) \) (where \( \mathbb{1} \) denotes the infinite identity matrix) play an important role where...
**Definition 1.26.** $\det(\mathbb{1} - A_1 A_2)$ is defined by

$$\log \det(\mathbb{1} - A_1 A_2) = \text{Tr} \log(\mathbb{1} - A_1 A_2) = -\sum_{n \geq 1} \frac{1}{n} \text{Tr}((A_1 A_2)^n).$$

We have

**Theorem 1.3.**

1. ([MT2], Proposition 1) The infinite matrix

$$(\mathbb{1} - A_1 A_2)^{-1} = \sum_{n \geq 0} (A_1 A_2)^n,$$

is convergent for $(\tau_1, \tau_2, \epsilon) \in D^\epsilon$.

2. ([MT2], Theorem 2 & Proposition 3) $\det(\mathbb{1} - A_1 A_2)$ is non-vanishing and holomorphic for $(\tau_1, \tau_2, \epsilon) \in D^\epsilon$.

The bilinear two form $\omega^{(2)}(x, y)$, the holomorphic one forms $\nu_i(x)$ and the period matrix $\Omega_{ij}$ are given in terms of the matrices $A_a$ and holomorphic one forms on the punctured torus $\hat{S}_a$ given by

**Definition 1.27.**

$$a_a(k, x) = \sqrt{k} \epsilon^{k/2} P_{k+1}(x, \tau_a) dx.$$  

Letting $a_a(x)$, $a_a^T(x)$ denote the infinite row and column vector (respectively) with elements $a_a(k, x)$ we have

**Theorem 1.4.** ([MT2], Lemma 2, Proposition 1, Theorem 4)

$$\omega^{(2)}(x, y) = \begin{cases} P_2(x - y, \tau_a) dxdy + a_a(x) A_a(\mathbb{1} - A_a A_a)^{-1} a_a^T(y), & x, y \in \hat{S}_a, \\ -a_a(x)(\mathbb{1} - A_a A_a)^{-1} a_a^T(y), & x \in \hat{S}_a, y \in \hat{S}_\bar{a}. \end{cases}$$

Applying (4) we have

**Theorem 1.5.** ([MT2], Theorem 4)

$$\nu_a(x) = \begin{cases} dx + \epsilon^{1/2}(a_a(x) A_a(\mathbb{1} - A_a A_a)^{-1})(1), & x \in \hat{S}_a, \\ -\epsilon^{1/2}(a_a(x)(\mathbb{1} - A_a A_a)^{-1})(1), & x \in \hat{S}_a, \end{cases}$$

where (1) refers to the (1)-entry of a vector.
Furthermore, by applying (5) we have

**Theorem 1.6.** ([MT2], Theorem 4) The $\epsilon$-formalism determines a holomorphic map

$$F^\epsilon : D^\epsilon \to \mathbb{H}_2,$$

$$((\tau_1, \tau_2, \epsilon) \mapsto \Omega(\tau_1, \tau_2, \epsilon),$$

where $\Omega = \Omega(\tau_1, \tau_2, \epsilon)$ is given by

$$2\pi i \Omega_{11} = 2\pi i \tau_1 + \epsilon(A_2(1 - A_1 A_2)^{-1})(1, 1),$$

$$2\pi i \Omega_{22} = 2\pi i \tau_2 + \epsilon(A_1(1 - A_2 A_1)^{-1})(1, 1),$$

$$2\pi i \Omega_{12} = -\epsilon(1 - A_1 A_2)^{-1}(1, 1),$$

where $(1, 1)$ refers to the $(1, 1)$-entry of a matrix.

**Remark 1.1.** $D^\epsilon$ is preserved under the action of $G \cong (SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) \rtimes \mathbb{Z}_2$, the direct product of two copies of $SL(2, \mathbb{Z})$ (the left and right torus modular groups) which are interchanged upon conjugation by an involution.

We let $D^\epsilon_0$ be the subset of $D^\epsilon$ for which $\epsilon = 0$. We then have

**Theorem 1.7.** ([MT2], Proposition 5) Let $x \in D^\epsilon_0$. Then there exists a $G$-invariant neighbourhood $N_x^\epsilon \subseteq D^\epsilon$ of $x$ throughout which $F^\epsilon$ is invertible.

### 1.6.2 Genus Two Correlation Functions

In this section we define the genus two partition function and $n$-point correlation functions for a VOA [MT3]. We assume that $V$ has a non-degenerate Li-Z metric throughout. A consequence of this assumption is that for any $V$ basis $\{u(a)\}$ we may define the dual basis $\{\pi(a)\}$ with respect to the Li-Z metric where

$$\langle u(a), \pi(b) \rangle = \delta_{a,b}.$$

Recall that we sew together a pair of punctured tori $\hat{S}_a$ of (6) with modular parameters $\tau_a$ for $a = 1, 2$ by the sewing relation (9).
Remark 1.2. In [MT3], the authors write

\[ Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(u; \tau_1)Z_V^{(1)}(\overline{u}; \tau_2), \]

where the sum is taken over any basis of \( V \) and \( \overline{u} \) is the dual of \( u \) with respect to \( \langle , \rangle_{\text{sq}} \) and the adjoint is with \( A = \epsilon \) (see Definition 1.13), i.e.,

\[ Y^r[v, \overline{z}] = Y \left[ \exp \left( \frac{\tau}{\epsilon} L[1] \right) \left( -\frac{\epsilon}{\tau^2} \right)^{L[0]} v, \frac{\epsilon}{\tau^2} \right]. \]

Definition 1.28. The genus two partition function for \( V \) is

\[ Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(u; \tau_1)Z_V^{(1)}(\overline{u}; \tau_2), \]

where the sum is taken over any basis of \( V \) and \( \overline{u} \) is the dual of \( u \) with respect to \( \langle , \rangle_{\text{sq}} \) and the adjoint is with \( A = \epsilon \) (see Definition 1.13), i.e.,

\[ Y^r[v, \overline{z}] = Y \left[ \exp \left( \frac{\tau}{\epsilon} L[1] \right) \left( -\frac{\epsilon}{\tau^2} \right)^{L[0]} v, \frac{\epsilon}{\tau^2} \right]. \]

Definition 1.29. The genus two \( n \)-point correlation function

for \( a_1, \ldots, a_L \) inserted at \( x_1, \ldots, x_L \in \mathcal{S}_1 \) and \( b_1, \ldots, b_R \) inserted at \( y_1, \ldots, y_R \in \mathcal{S}_2 \) with \( L + R = n \) is

\[ Z_V^{(2)}(a_1, x_1; \ldots; a_L, x_L, b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(a_1, x_1; \ldots; a_L, x_L; u; 0; \tau_1)Z_V^{(1)}(b_R, y_R; \ldots; b_1, y_1; \overline{u}; 0; \tau_2) \]

\[ = \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]u; \tau_1)Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]\overline{u}; \tau_2), \]

where the sum is taken over any basis of \( V \) and \( \overline{u} \) is the dual of \( u \) with respect to \( \langle , \rangle_{\text{sq}} \) and the adjoint as in Definition 1.28.

The primary aim of this thesis is to find an analogue of Theorem 1.2 for these genus two \( n \)-point correlation functions. As such, many expressions will involve these \( n \)-point functions, and some shorthand notations will be introduced as needed.

Remark 1.2. In [MT3], the authors write

\[ Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{r \geq 0} \epsilon^r \sum_{u \in V[r]} Z_V^{(1)}(u; \tau_1)Z_V^{(1)}(\overline{u}; \tau_2), \]

for the partition function and

\[ Z_V^{(2)}(a_1, x_1; \ldots; a_L, x_L, b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) = \sum_{r \geq 0} \epsilon^r \sum_{u \in V[r]} Z_V^{(1)}(a_1, x_1; \ldots; a_L, x_L; u; 0; \tau_1)Z_V^{(1)}(b_R, y_R; \ldots; b_1, y_1; \overline{u}; 0; \tau_2) \]

\[ = \sum_{r \geq 0} \epsilon^r \sum_{u \in V[r]} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]u; \tau_1)Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]\overline{u}; \tau_2), \]
for the $n$-point function, where the internal sum is taken over any basis of $V_{[n]}$ and $\overline{u}$ is the dual of $u$ with respect to the Li-Z metric and adjoint operators defined by the mapping $z \mapsto 1/z$, rather than $z \mapsto \epsilon/z$. These definitions are equivalent to Definitions 1.28 and 1.29. The definitions presented here have the benefit of streamlining notation as much as possible.

Notice that both definitions can be naturally extended for any pair of $V$ modules $M_1, M_2$, where the left (right) 1-point function in the definitions is considered for $M_1$ (respectively $M_2$).
2 1-point Correlation Functions

2.1 The General Quasiprimary Case

In this section we will derive a formal expression for the genus two 1-point correlation functions of a VOA $V$.

We let $v \in V$ be a quasiprimary vector of weight $N$. By the square bracket Li-Z metric with (Definition 1.13) $A = \epsilon$, we have that

$$v^\dagger[m] = (-1)^N \epsilon^{m+1-N} v[2N-m-2].$$

We consider the genus two 1-point function for $v$ (left insertion),

$$Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(v, x; u, 0; \tau_1) Z_V^{(1)}(\pi; \tau_2)$$

The $Z_V^{(1)}(Y[v, x]u; \tau_1)$ factors are genus one correlation functions and we may apply Zhu’s recursion formula (Theorem 1.2) to these factors. By Zhu recursion,

$$Z_V^{(1)}(Y[v, x]u; \tau_1) = \text{Tr}_V \left( o(v) o(u) q_1^{L(0)-c/24} \right) + \sum_{m \geq 1} \frac{(-1)^m}{m!} P_1^{(m)}(x, \tau_1) Z_V^{(1)}(v[m]u; \tau_1)$$

and so
\[ Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon) \]
\[ = \sum_{u \in V} \text{Tr}_V \left( o(v) o(u) q_1^{L(0)-c/24} \right) Z_V^{(1)}(\overline{v}; \overline{\tau}_2) \]
\[ + \sum_{u \in V} \sum_{m \geq 1} P_{m+1}(x, \tau_1) Z_V^{(1)}(v[m]u; \tau_1) Z_V^{(1)}(\overline{v}; \overline{\tau}_2) \]
\[ = \sum_{u \in V} \text{Tr}_V \left( o(v) o(u) q_1^{L(0)-c/24} \right) Z_V^{(1)}(\overline{v}; \overline{\tau}_2) \]
\[ + \sum_{m \geq 1} P_{m+1}(x, \tau_1) \sum_{u \in V} Z_V^{(1)}(v[m]u; \tau_1) Z_V^{(1)}(\overline{v}; \overline{\tau}_2). \]

We define the infinite vectors \( X_1(v) \), \( X_2(v) \) by

\[ X_1(v; m) = X_1(v; m; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(v[m]u; \tau_1) Z_V^{(1)}(\overline{v}; \overline{\tau}_2), \]
\[ X_2(v; m) = X_2(v; m; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(u; \tau_1) Z_V^{(1)}(v[m]u; \tau_2), \]

where \( m \geq 1 \). Thus, we have that

\[ Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} \text{Tr}_V \left( o(v) o(u) q_1^{L(0)-c/24} \right) Z_V^{(1)}(\overline{v}; \overline{\tau}_2) \]
\[ + \sum_{m \geq 1} P_{m+1}(x, \tau_1) X_1(v; m). \]

Our aim is to develop recursive formulae between vectors proportional to \( X_1 \) and \( X_2 \). By combining these recursive formulae, we will be able to write a closed formal expression for the 1-point function above.
We notice that since $v$ is quasiprimary, we may move the modes of $v$ from left to right. We have

$$v[m]u = \sum_{w \in V} \langle w, v[m]u \rangle w$$

$$= \sum_{w \in V} \langle v^+[m]w, u \rangle w,$$

where here and further $\langle \cdot, \cdot \rangle$ denotes the square bracket Li-Z metric with $A = \epsilon$. Thus we have

$$X_1(v; m) = \sum_{w \in V} Z^{(1)}_{V'}(v[m]u; \tau_1) Z^{(1)}_{V'}(\bar{w}; \tau_2)$$

$$= \sum_{w \in V} \sum_{w' \in V} \langle w, v[m]u \rangle Z^{(1)}_{V'}(w; \tau_1) Z^{(1)}_{V'}(\bar{w}; \tau_2)$$

$$= \sum_{w \in V} \sum_{w' \in V} \langle v^+[m]w, u \rangle Z^{(1)}_{V'}(w; \tau_1) Z^{(1)}_{V'}(\bar{w}; \tau_2)$$

$$= \sum_{w \in V} \sum_{w' \in V} Z^{(1)}_{V'}(w; \tau_1) Z^{(1)}_{V'}((v^+[m]w, u) \bar{w}; \tau_2)$$

$$= \sum_{w \in V} Z^{(1)}_{V'}(w; \tau_1) Z^{(1)}_{V'}(v^+[m]w; \tau_2)$$

$$= (-1)^N \epsilon^{m+1-N} \sum_{w \in V} Z^{(1)}_{V'}(w; \tau_1) Z^{(1)}_{V'}(v[2N - m - 2]\bar{w}; \tau_2)$$

$$= (-1)^N \epsilon^{m+1-N} \sum_{w \in V} Z^{(1)}_{V'}(w; \tau_1) Z^{(1)}_{V'}(v[2N - m - 2]\bar{w}; \tau_2).$$

The $Z^{(1)}_{V'}(v[N - m - 2]\bar{w}; \tau_2)$ factors are genus 1 correlation functions, and again we may apply Zhu’s recursion formula to these factors, provided

$$m + 2 - 2N \geq 1,$$

that is,

$$m \geq 2N - 1.$$
We let 

$$s = -2N + m + 2$$

and so we have by Zhu recursion that

$$Z_{V}^{(1)}(v[N - m - 2]\pi; \tau_2) = Z_{V}^{(1)}(v[-s]\pi; \tau_2)$$

$$= \delta_{s,1} \text{Tr}_V \left( o(v)o(\pi)q_2^{L(0)-c/24} \right)$$

$$+ \sum_{j \geq 1} (-1)^{j+1} \binom{j + s - 1}{j} E_{s+j}(\tau_2) Z_{V}^{(1)}(v[j]|\pi; \tau_2),$$

where $$s = 1 \iff m = 2N - 1$$. Substituting this back into our previous expression for $$X_1(v; m)$$, we find that for $$m \geq 2N - 1$$, we have

$$X_1(v; m) = (-1)^N \epsilon^{m+1-N} \delta_{m,2N-1} \sum_{u \in V} Z_{V}^{(1)}(u; \tau_1) \text{Tr}_V \left( o(v)o(\pi)q_2^{L(0)-c/24} \right)$$

$$+ (-1)^N \epsilon^{m+1-N} \sum_{j \geq 1} \sqrt{\frac{\pi}{j}} \epsilon^{-\frac{(s+j)}{2}} A_2(s, j) X_2(v; j).$$

It is convenient to define the following sums of formal traces

**Definition 2.1.**

$$O_1(v) = O_1(v; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} \text{Tr}_V \left( o(v)o(u)q_1^{L(0)-c/24} \right) Z_{V}^{(1)}(\pi; \tau_2),$$

$$O_2(v) = O_2(v; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_{V}^{(1)}(u; \tau_1) \text{Tr}_V \left( o(v)o(\pi)q_2^{L(0)-c/24} \right),$$

which may occasionally be referred to as the left and right traces (respectively). It is also convenient to rescale the $$X_1(v), X_2(v)$$ vectors as

$$X_1(v; m) = X_1(m; v; \tau_1, \tau_2, \epsilon) = \frac{\epsilon^{-m/2}}{\sqrt{m}} X_1(v; m),$$

$$X_2(v; m) = X_2(m; v; \tau_1, \tau_2, \epsilon) = \frac{\epsilon^{-m/2}}{\sqrt{m}} X_2(v; m).$$
Thus we have that for $m \geq 2N - 1$,

$$X_1(v; m) = \frac{\epsilon^{-m/2}}{\sqrt{m}}(-1)^N \epsilon^{m+1-N} \delta_{m,2N-1} O_2(v)$$

$$+ (-1)^N \sqrt{\frac{s}{m}} (A_2 X_2(v))(s).$$

For convenience, we let

$$K = 2N - 2$$

so that

$$s = m - K.$$ 

This gives that

$$X_1(v; m) = \frac{s - K}{\sqrt{m}}(-1)^N \delta_{m,K+1} O_2(v)$$

$$+ (-1)^N \sqrt{\frac{s - K}{m}} (A_2 X_2(v))(m - K).$$

We define the infinite matrices

**Definition 2.2.** For $K = 2N - 2$,

$$\Gamma(m,n) = \sqrt{\frac{n}{m}} \delta_{m+n,K},$$

$$\Delta(m,n) = \sqrt{\frac{n}{m}} \delta_{m-n,K},$$

$$\Theta(m,n) = \sqrt{\frac{n}{m}} \delta_{n-m,K},$$

where $m, n \geq 1$.

Notice that for any $K$, $\Delta$ and $\Theta$ have infinitely many non-zero values while $\Gamma$ has only finitely many non-zero values. Further, $\Theta$ is the left-inverse of $\Delta$, that is $\Theta \Delta = 1$, and we have that $\Theta \Gamma = 0$ and that $\Delta \Theta$ is a projection operator. In the degenerate case where $N = 1$, we have
$K = 0$, and so $\Delta = 1 = \Theta$ and $\Gamma = 0$. For $N \geq 2$ where $K \geq 2$, these matrices have the following shapes:

$$
\Gamma = \begin{bmatrix}
0 & \cdots & 0 & \sqrt{\frac{K-1}{K-1}} & 0 & \cdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \vdots \\
0 & \ddots & \ddots & \vdots & \ddots & \vdots \\
\sqrt{\frac{1}{K-1}} & 0 & \cdots & 0 & 0 & \cdots \\
0 & \cdots \\
\vdots & \ddots
\end{bmatrix},
$$

$$
\Delta = \begin{bmatrix}
\sqrt{\frac{1}{K+1}} & 0 & \cdots \\
0 & \sqrt{\frac{2}{K+2}} & 0 & \cdots \\
\vdots & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
0 & \cdots \\
\vdots & \ddots
\end{bmatrix},
$$

$$
\Theta = \begin{bmatrix}
0 & \cdots & 0 & \sqrt{\frac{K+1}{K-1}} & 0 & \cdots \\
\vdots & \ddots & \ddots & 0 & \sqrt{\frac{K+2}{K+2}} & 0 & \cdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots \\
\vdots & \ddots
\end{bmatrix},
$$

that is, $\Gamma$ is cross-diagonal in the the upper-left $(K-1) \times (K-1)$ block and zero elsewhere, while $\Delta$ and $\Theta$ have a skewed-diagonal structure, where the first $K$ rows of $\Delta$ are zero and the first $K$ columns of $\Theta$ are zero. Using the $\Delta$ matrix, for $m > K$ we may write

$$
\mathcal{X}_1(v;m) = \frac{\epsilon^{1/2}}{\sqrt{m}}(-1)^N \delta_{m,K+1} O_2(v) + (-1)^N(DA_2 \mathcal{X}_2(v))(m).
$$
Now for $1 \leq m \leq K$, we first notice that

$$X_1(v; K) = \frac{\epsilon^{-K/2}}{\sqrt{K}} \sum_{u \in V} Z^{(1)}_V(v|K)u; \tau_1)Z^{(1)}_V(\pi; \tau_2)$$

$$= (-1)^N \epsilon^{K/2+1-N} \sqrt{K} \sum_{u \in V} Z^{(1)}_V(u; \tau_1)Z^{(1)}_V(v|2N-K-2)\pi; \tau_2)$$

$$= (-1)^N \epsilon^{K/2+1-N} \sqrt{K} \sum_{u \in V} Z^{(1)}_V(u; \tau_1)Z^{(1)}_V(v|0)\pi; \tau_2) = 0,$$

since $Z^{(1)}_V(v|0)\pi; \tau_2) = 0$ by a theorem of Zhu [Z]. Now for $1 \leq m \leq K$, we have

$$X_1(v; m) = (-1)^N \sum_{n=1}^{K} \Gamma(m, n)X_2(v; n).$$

The values $X_1(v; 1), \ldots, X_1(v; K-1)$ are independent and hence we will refer to these as the seed values. Alternatively, we may define the vectors

**Definition 2.3.**

$$X_1^0(v; m) = X_1^0(v; m; \tau_1, \tau_2, \epsilon) = \begin{cases} X_1(v; m) & \text{for } 1 \leq m < K \\ 0 & \text{for } m \geq K \end{cases},$$

$$X_2^0(v; m) = X_2^0(v; m; \tau_1, \tau_2, \epsilon) = \begin{cases} X_2(v; m) & \text{for } 1 \leq m < K \\ 0 & \text{for } m \geq K \end{cases}. $$

We notice that

$$X_1^0(v) = (1 - \Delta\Theta)X_1(v) = (-1)^N \Gamma X_2(v) = (-1)^N \Gamma X_2^0(v),$$

$$X_2^0(v) = (1 - \Delta\Theta)X_2(v) = (-1)^N \Gamma X_1(v) = (-1)^N \Gamma X_2^0(v).$$
Combining our expression for $1 \leq m \leq K$ with our expression for $m > K$, we find

$$X_1(v; m) = \frac{\epsilon^{1/2}}{\sqrt{m}} (-1)^N \delta_{m,K+1} O_2(v)$$

$$+ (-1)^N ((\Gamma + \Delta A_2)X_2(v)) (m).$$

We define the vectors $\Theta_1, \Theta_2$ by

$$\Theta_1(m) = \frac{\epsilon^{1/2}}{\sqrt{m}} \delta_{m,K+1} O_1(v),$$

$$\Theta_2(m) = \frac{\epsilon^{1/2}}{\sqrt{m}} \delta_{m,K+1} O_2(v),$$

where $m \geq 1$. Thus, the previous equation in vector form is

$$X_1(v) = (-1)^N \Theta_2 + (-1)^N (\Gamma + \Delta A_2)X_2(v).$$

To derive an expression for the $X_1$ in terms of itself, it is helpful to project out the effects of the $\Gamma$ matrix at this stage. To achieve this, we define

$$\mathcal{X}_1 = (-1)^N \Theta X_1(v),$$

$$\mathcal{X}_2 = (-1)^N \Theta X_2(v).$$

Now,

$$\mathcal{X}_1 = (-1)^N \Theta X_1$$

$$= \Theta O_2 + \Theta (\Gamma + \Delta A_2)X_2$$

$$= \Theta O_2 + A_2((-1)^N \Delta \mathcal{X}_2 + \mathcal{X}_2(v))$$

$$= \Theta O_2 + A_2 \mathcal{X}_2(v) + (-1)^N (A_2 \Delta) \mathcal{X}_2$$

$$= \Theta O_2 + A_2 \mathcal{X}_2(v) + (-1)^N \tilde{A}_2 \mathcal{X}_2$$

where $\tilde{A}_2 = A_2 \Delta$. 

34
Notice that we have

\[ X_2 = \Theta \Theta_1 + A_1 X_1^0(v) + (-1)^N \tilde{A}_1 X_1 \]

by left-right symmetry, where \( \tilde{A}_1 = A_1 \Delta \). Combining these formulae we have

\[ X_1 = \Theta \Theta_2 + A_2 X_2^0(v) + (-1)^N \tilde{A}_2 X_2 \]

\[ = \Theta \Theta_2 + A_2 X_2^0(v) + (-1)^N \tilde{A}_2 \left( \Theta \Theta_1 + A_1 X_1^0(v) + (-1)^N \tilde{A}_1 X_1 \right) \]

\[ = \Theta \Theta_2 + A_2 X_2^0(v) + (-1)^N \tilde{A}_2 \Theta \Theta_1 + (-1)^N \tilde{A}_2 A_1 X_1^0(v) + \tilde{A}_2 \tilde{A}_1 X_1, \]

and so we have

\[ X_1 \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \Theta \Theta_2 + A_2 X_2^0(v) + (-1)^N \tilde{A}_2 \Theta \Theta_1 + (-1)^N \tilde{A}_2 A_1 X_1^0(v) \right), \]

where \( \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \) is the formal inverse

\[ \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} = \sum_{n \geq 0} \left( \tilde{A}_2 \tilde{A}_1 \right)^n. \]

Now, we recall that

\[ Z^{(2)}_V(v, x; \tau_1, \tau_2, \epsilon) = O_1(v) + \sum_{m \geq 1} P_{m+1}(x, \tau_1) X_1(v; m). \]

**Definition 2.4.** For \( x \in \hat{S}_a \), \( \mathbb{P}_2(x) = \mathbb{P}_2(x, \tau_a) \) is the infinite vector given by

\[ (\mathbb{P}_2(x))(m) = \epsilon^{m/2} \sqrt{m} P_{m+1}(x, \tau_a). \]
Then we have that

\[ Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon) = O_1(v) + P_2(x)X_1(v) \]
\[ = O_1(v) + P_2(x)X_1^0(v) + (-1)^N \Delta X_1 \]
\[ = O_1(v) + P_2(x)X_1^0(v) \]
\[ + (-1)^N P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \Theta \mathcal{O}_2 \]
\[ + (-1)^N P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} A_2 X_0^0(v) \]
\[ + P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \Theta \mathcal{O}_1 \]
\[ + P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 A_1 X_0^0(v). \]

We notice that

\[
(\Theta \mathcal{O}_1)(m) = \delta_{m,1} \epsilon^{1/2} O_1(v),
\]
\[
(\Theta \mathcal{O}_2)(m) = \delta_{m,1} \epsilon^{1/2} O_2(v).
\]

That is, only the first components of $\Theta \mathcal{O}_1, \Theta \mathcal{O}_2$ are non-zero. Thus we have proved the following theorem;

**Theorem 2.1.** The genus two 1-point correlation function for a quasiprimary vector $v \in V_N$ inserted at $x \in \hat{S}_1$ is formally given by

\[
Z_V^{(2)}(v, x; \tau_1, \tau_2, \epsilon)
= \left( 1 + \epsilon^{1/2} \left( P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \right) O_1(v; \tau_1, \tau_2, \epsilon)
+ \left( (-1)^N \epsilon^{1/2} \left( P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \right) \right) O_2(v; \tau_1, \tau_2, \epsilon)
+ \left( P_2(x) + P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \tilde{A}_2 A_1 + A_2 \Gamma \right) \right) X_1^0(v; \tau_1, \tau_2, \epsilon).
\]

with $O_1(v; \tau_1, \tau_2, \epsilon), O_2(v; \tau_1, \tau_2, \epsilon)$ from Definition 2.1, $\Delta$ and $\Gamma$ from Definition 2.2, $X_1^0(v; \tau_1, \tau_2, \epsilon)$ from Definition 2.3 and $P_2(x)$ from Definition 2.4 and $A_a = A_a \Delta$ with $A_a$ from Definition 1.25.
Remark 2.1. Notice that the coefficients of the terms in Theorem 2.1 depend on $N = \text{wt}[v]$ and the insertion point $x \in \mathcal{S}_1$. This differs from genus one Zhu recursion, where the 1-point function is independent of the insertion point and the coefficient does not depend on weight.

Remark 2.2. The Riemann-Roch theorem implies that on a genus two Riemann surface there exist 2 independent holomorphic 1-forms and $2N - 1$ independent holomorphic $N$-forms for $N \geq 2$ [FK]. Since only $K - 1$ components of $\mathcal{X}_0^0(v; \tau_1, \tau_2, \epsilon)$ are independent, the final quasiprimary 1-point function formula depends only on the left and right traces and these $K - 1$ seed values, for a total of $K + 1 = 2N - 1$ parameters when $N \geq 2$. When $N = 1$, all components of $\mathcal{X}_0^0(v; \tau_1, \tau_2, \epsilon)$ are zero, giving only 2 parameters. It is interesting to note that these numbers are equal to the counting in the Riemann-Roch theorem. See Conjecture 6.2 for further remarks.

2.2 Quasiprimary Descendants

We now consider a general quasiprimary descendant $L[-1]^k v$ and its genus two 1-point correlation function. Since we have that

$$Z_{V'}^{(2)}(L[-1]^k v, x; \tau_1, \tau_2, \epsilon)$$

$$= \sum_{u \in V} Z_{V'}^{(1)}(L[-1]^k v, x; u, 0; \tau_1) Z_{V'}^{(1)}(u; \tau_2)$$

$$= \sum_{u \in V} Z_{V'}^{(1)}(Y[L[-1]^k v, x]u; \tau_1) Z_{V'}^{(1)}(u; \tau_2)$$

$$= \sum_{u \in V} Z_{V'}^{(1)}(\partial_x^k Y[v, x]u; \tau_1) Z_{V'}^{(1)}(u; \tau_2)$$

$$= \partial_x^k Z_{V'}^{(2)}(v, x; \tau_1, \tau_2, \epsilon),$$

we have the following corollary to Theorem 2.1.
Corollary 2.1. For \( k \geq 1 \), the genus two 1-point correlation function for a quasiprimary descendant \( L[-1]^kv \) is given by

\[
Z^{(2)}_V(L[-1]^kv, x; \tau_1, \tau_2, \epsilon)
= \left( \epsilon^{1/2} \partial_x^k \left( P_2(x) \Delta \left( \mathbb{1} - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) (1) \right) O_1(v; \tau_1, \tau_2, \epsilon)
+ \left( (-1)^N \epsilon^{1/2} \partial_x^k \left( P_2(x) \Delta \left( \mathbb{1} - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \right) (1) \right) O_2(v; \tau_1, \tau_2, \epsilon)
+ \partial_x^k \left( P_2(x) + P_2(x) \Delta \left( \mathbb{1} - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \tilde{A}_2 A_1 + A_2 \Gamma \right) \right) X^0_1(v; \tau_1, \tau_2, \epsilon).
\]

This result differs from genus one Zhu recursion where all quasiprimary descendant vectors have 1-point function equal to zero. Finally, we recall that the quasiprimary descendants span \( V \), and so by linearity, all genus two 1-point correlation functions can be determined in terms of the seed values and the left and right traces.
3 Applications of the 1-point Formula

In this section, we will specialise Theorem 2.1 from the previous section to derive formal expressions for specific examples of genus two 1-point correlation functions.

3.1 Genus Two Heisenberg 1-point Functions

We first consider a pair of Heisenberg modules \( M_{\alpha_1}, M_{\alpha_2} \) with the partition function defined as

\[
Z^{(2)}_{M_{\alpha_1}, \alpha_2}(\tau_1, \tau_2, \epsilon) = \sum_{u \in M} Z^{(1)}_{M_{\alpha_1}}(u; \tau_1) Z^{(1)}_{M_{\alpha_2}}(u; \tau_2),
\]

and the Heisenberg 1-point function defined as

\[
Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x; \tau_1, \tau_2, \epsilon) = \sum_{u \in M} Z^{(1)}_{M_{\alpha_1}}(Y[a, x]u; \tau_1) Z^{(1)}_{M_{\alpha_2}}(u; \tau_2).
\]

We find that

\[
O_1(a; \tau_1, \tau_2, \epsilon) = \sum_{u \in M} \text{Tr}_{M_{\alpha_1}} \left( o(a) o(u) q_1^{L(0) - c/24} \right) Z^{(1)}_{M_{\alpha_2}}(u; \tau_2)
= \alpha_1 \sum_{u \in M} Z^{(1)}_{M_{\alpha_1}}(u; \tau_1) Z^{(1)}_{M_{\alpha_2}}(u; \tau_2)
= \alpha_1 Z^{(2)}_{M_{\alpha_1}, \alpha_2}(\tau_1, \tau_2, \epsilon).
\]

Similarly, we have that \( O_2(a; \tau_1, \tau_2, \epsilon) = \alpha_2 Z^{(2)}_{M_{\alpha_1}, \alpha_2}(\tau_1, \tau_2, \epsilon) \). Since

\[
K = 2 \text{wt}[a] - 2 = 2 - 2 = 0,
\]

we have that \( \Gamma = 0 \) and thus \( \mathcal{X}^{0}_{1}(a; \tau_1, \tau_2, \epsilon) = 0 = \mathcal{X}^{0}_{2}(a; \tau_1, \tau_2, \epsilon) \).

Further, we have that \( \Delta = \mathbb{1} = \Theta \) and so we have that the Heisenberg 1-point function is

\[
Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x; \tau_1, \tau_2, \epsilon) = \alpha \left( 1 + \epsilon^{1/2} \left( P_2(x)(1 - A_2 A_1)^{-1} A_2 \right) (1) \right) Z^{(2)}_{M_{\alpha_1}, \alpha_2}(\tau_1, \tau_2, \epsilon)
- \beta \left( \epsilon^{1/2} \left( P_2(x)(1 - A_2 A_1)^{-1} \right) (1) \right) Z^{(2)}_{M_{\alpha_1}, \alpha_2}(\tau_1, \tau_2, \epsilon).
\]
By Theorem 1.3 [MT2], the matrix \((1 - A_2 A_1)^{-1}\) converges on \(D^\epsilon\).

Notice that

\[
P_2(x)dx = a_1(x),
\]

where \(a_1(x)\) is the holomorphic 1-form from Definition 1.27, since

\[
a_1(m, x) = \frac{\epsilon^{m/2}}{2\pi \sqrt{m}} \oint \frac{z^{-m} \omega^{(1)}(x, z_1)}{c_1(z_1)} dz_1 = \epsilon^{m/2} \sqrt{m} P_{m+1}(x, \tau_1) dx = (P_2(x))(m) dx.
\]

Thus, we have that

**Proposition 3.1.** The genus two 1-point correlation function for a pair of Heisenberg modules \(M_{\alpha_1}, M_{\alpha_2}\) is given by

\[
Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x; \tau_1, \tau_2, \epsilon) dx = \alpha_1 \left(1 + \epsilon^{1/2} \left(a_1(x)(1 - A_2 A_1)^{-1} A_2\right)(1)\right) Z^{(2)}_{M_{\alpha_1}, \alpha_2} (\tau_1, \tau_2, \epsilon)
- \alpha_2 \left(\epsilon^{1/2} \left(a_1(x)(1 - A_2 A_1)^{-1}\right)(1)\right) Z^{(2)}_{M_{\alpha_1}, \alpha_2} (\tau_1, \tau_2, \epsilon),
\]

where \(\nu_1, \nu_2\) are holomorphic 1-forms of Theorem 1.5.

This agrees with Theorem 12 of [MT3].

**Remark 3.1.** Notice that the coefficients in Theorem 2.1 depend on the weight of the quasiprimary vector, and not on the VOA. In particular, for any weight 1 quasiprimary vector, the coefficient functions give rise to the holomorphic 1-forms \(\nu_1, \nu_2\), as in the Proposition 3.1 above. See Conjecture 6.5 for further remarks.

### 3.2 Genus Two Virasoro 1-point Functions

We consider a VOA \(V\) with Virasoro vector \(\tilde{\omega} \in V_{[2]}\), the partition function defined as

\[
Z^{(2)}_V(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z^{(1)}_V(u; \tau_1) Z^{(1)}_V(u; \tau_2),
\]
and the 1-point function defined as

$$Z^{(2)}_V(\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z^{(1)}_V(Y[\tilde{\omega}, x]u; \tau_1)Z^{(1)}_V(\overline{\omega}; \tau_2).$$

We find that

$$O_1(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} \text{Tr}_V \left( o(\tilde{\omega})o(u)q_1^{L(0)-c/24} \right) Z^{(1)}_V(\overline{\omega}; \tau_2) = q_1 \partial_{q_1} \sum_{u \in V} Z^{(1)}_V(u; \tau_1)Z^{(1)}_V(\overline{\omega}; \tau_2) = q_1 \partial_{q_1} Z^{(2)}_V(\tau_1, \tau_2, \epsilon).$$

Similarly, we have that $O_2(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = q_2 \partial_{q_2} Z^{(2)}_V(\tau_1, \tau_2, \epsilon)$. Since $K = 2\text{wt}[\tilde{\omega}] - 2 = 4 - 2 = 2$, we have that

$$\Gamma(m, n) = \delta_{m, 1}\delta_{n, 1}$$

and so we have that $X^0_1(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = X^0_2(\tilde{\omega}; \tau_1, \tau_2, \epsilon)$, where

$$X^0_1(m; \tilde{\omega}; \tau_1, \tau_2, \epsilon) = \delta_{m, 1}X_1(m; \tilde{\omega}; \tau_1, \tau_2, \epsilon).$$

Now,

$$X^0_1(1) = \epsilon^{-1/2} \sum_{u \in V} Z^{(1)}_V(\tilde{\omega}[1]u; \tau_1)Z^{(1)}_V(\overline{\omega}; \tau_2) = \epsilon^{-1/2} \sum_{u \in V} Z^{(1)}_V(L[0]u; \tau_1)Z^{(1)}_V(\overline{\omega}; \tau_2),$$

and by choosing a basis $\{u\}$ of homogeneous vectors,

$$X^0_1(1) = \epsilon^{-1/2} \sum_{n \geq 0} \sum_{u \in V[n]} n Z^{(1)}_V(u; \tau_1)Z^{(1)}_V(\overline{\omega}; \tau_2).$$
Now

\[ 1 = \langle u, \bar{u} \rangle = \langle u[-1], \bar{u} \rangle, \]

implies that

\[ u[2N - 1]\bar{u} = (-1)^N \epsilon^N 1, \]

and so \( \bar{u} \) has an \( \epsilon^N \) multiplier, which implies that

\[ Z^0_1(1) = e^{-1/\epsilon} \partial_\epsilon \sum_{n \geq 0} \sum_{u \in V^{[n]}} Z^{(1)}_V (u; \tau_1) Z^{(1)}_V (\bar{u}; \tau_2) \]

\[ = e^{1/2} \partial_\epsilon Z^{(2)}_V (\tau_1, \tau_2, \epsilon). \]

We define the formal differential operator

**Definition 3.1.**

\[ \mathbb{D}_x = A(x)q_1 \partial q_1 + B(x)q_2 \partial q_2 + C(x) \epsilon \partial_\epsilon, \]

for

\[ A(x) = 1 + \epsilon^{1/2} \left( \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) (1), \]

\[ B(x) = \epsilon^{1/2} \left( \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \right) \]

\[ C(x) = \epsilon^{-1/2} \left( \mathbb{P}_2(x) + \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \tilde{A}_2 A_1 + A_2 \right) \right) (1). \]

Thus, we have

**Proposition 3.2.** The genus two Virasoro 1-point correlation function for a VOA \( V \) is given by

\[ Z^{(2)}_V (\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = \mathbb{D}_x Z^{(2)}_V (\tau_1, \tau_2, \epsilon). \]
Notice that the genus two Virasoro 1-point function is similar in form to the genus one Virasoro 1-point function, given by

\[ Z_V^{(1)}(\tilde{\omega}; \tau) = q \partial_q Z_V^{(1)}(\tau). \]

The operator \( \mathcal{D}_x \) is of deep geometric significance, being a derivative with respect to the moduli \( \tau_1, \tau_2, \epsilon \). This geometric significance will be discussed later in Section 6.
4 \textit{n-point Correlation Functions} \\
In this section we will derive a formal expression for the genus two \textit{n-point} correlation functions of a VOA \( V \).

4.1 The General Quasiprimary Case \\
We let \( v \in V \) be a quasiprimary vector of weight \( N \), and we consider the genus two \( n \)-point function for \( v \) and vectors \( a_1, \ldots, a_L \in V \), and \( b_1, \ldots, b_R \in V \), with \( a_1, \ldots, a_L \) inserted on the left torus at \( x_1, \ldots, x_L \) respectively and \( b_1, \ldots, b_R \) inserted on the right torus at \( y_1, \ldots, y_R \) respectively. With \( L + R + 1 = n \), this \( n \)-point function is defined by

\[
Z_V^{(2)}(v, x_1, x_2; a_1, y_1; \ldots, a_L, y_L | b_1, \tau_1; \ldots; b_R, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(v, x_1, x_2; a_1, y_1; \ldots, a_L, y_L | u, \tau_1) Z_V^{(1)}(b_1, \tau_2; \ldots; b_R, \tau_2) 
\]

The \( Z_V^{(1)}(Y[v, x_1] \ldots Y[a_L, x_L]u | \tau_1) \) factors are genus one correlation functions and we may apply Zhu’s recursion formulae (Theorem 1.2) to these factors as follows,

\[
Z_V^{(1)}(Y[v, x_1] \ldots Y[a_L, x_L]u | \tau_1) = Z_V^{(1)}(v, x_1; a_1, x_2; \ldots; a_L, x_L; u, 0; \tau_1) = \text{Tr}_V \left( o(v)Y(q^{L(0)}_{x_1} a_1, q_{x_1}) \ldots Y(q^{L(0)}_{x_L} a_L, q_{x_L})Y(q^{L(0)}_0 u, q_0)q_1^{L(0) - \epsilon/24} \right) 
\]

\[
+ \sum_{l=1}^{L} \sum_{j \geq 0} P_{j+1}(x - x_1, \tau_1) Z_V^{(1)}(a_1, x_1; \ldots; v[j]a_i, x_i; \ldots; a_L, x_L; u, 0; \tau_1) 
\]

\[
+ \sum_{m \geq 0} P_{m+1}(x, \tau_1) Z_V^{(1)}(a_1, x_1; \ldots; a_L, x_L; v[m]u, 0; \tau_1) 
\]
\[ \begin{align*}
&= \text{Tr}_V \left( o(v)Y(q_{x_1}^{L(0)}a_1, q_{x_1}) \ldots Y(q_{x_L}^{L(0)}a_L, q_{x_L})Y(q_0^{L(0)}u, q_0)q_1^{L(0) - \epsilon/24} \right) \\
&\quad + \sum_{l=1}^{L} \sum_{j \geq 0} P_{j+1}(x - x_l, \tau_l)Z_{V}^{(1)}(Y[a_1, x_1] \ldots Y[v[j]a_l, x_l] \ldots Y[a_L, x_L]u; \tau_1) \\
&\quad + P_1(x, \tau_1)Z_{V}^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]v[0]u; \tau_1) \\
&\quad + \sum_{m \geq 1} P_{m+1}(x, \tau_1)Z_{V}^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]v[m]u; \tau_1).
\end{align*} \]

Thus, we have

\[ Z_{V}^{(2)}(v, x; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]

\[ = O_1(v; a_l, x_l|b_r, y_r) \]

\[ + \sum_{l=1}^{L} \sum_{j \geq 0} P_{j+1}(x - x_l, \tau_l)Z_{V}^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \\
+ P_1(x, \tau_1) \sum_{u \in V} Z_{V}^{(1)}(a_1, x_1; \ldots; a_L, x_L; v[0]u, 0; \tau_1)Z_{V}^{(1)}(b_R, y_R; \ldots; b_1, y_1; \pi, 0; \tau_2) \\
+ \sum_{m \geq 1} P_{m+1}(x, \tau_1)X_1(m), \]

where

**Definition 4.1.**

\[ O_1(v; a_l, x_l|b_r, y_r) \]

\[ = O_1(v; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]

\[ = \sum_{u \in V} \text{Tr}_V \left( o(v)Y(q_{x_1}^{L(0)}a_1, q_{x_1}) \ldots Y(q_{x_L}^{L(0)}a_L, q_{x_L})Y(q_0^{L(0)}u, q_0)q_1^{L(0) - \epsilon/24} \right) \\
\quad \cdot Z_{V}^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]; \pi; \tau_2), \]

**Definition 4.2.**

\[ Z_{V}^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]

\[ = Z_{V}^{(2)}(a_1, x_1; \ldots; v[j]a_l, x_l; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]

\[ = \sum_{u \in V} Z_{V}^{(1)}(a_1, x_1; \ldots; v[j]a_l, x_l; \ldots; a_L, x_L; u, 0; \tau_1)Z_{V}^{(1)}(b_R, y_R; \ldots; b_1, y_1; \pi, 0; \tau_2), \]

45
and the vector \( X_1(v) \) is defined by

\[
X_1(v; m) = X_1(v; m; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \\
= \sum_{u \in V} Z_V^{(1)}(a_1, x_1; \ldots; a_L, x_L; v|m|u; 0; \tau_1) Z_V^{(1)}(b_R, y_R; \ldots; b_1, y_1; v|m|\overline{u}; 0; \tau_2) \\
= \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]v|m|u; \tau_1) Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]v|m|\overline{u}; \tau_2),
\]

where \( m \geq 1 \). We also define the vector \( X_2(v) \) by

\[
X_2(v; m) = X_1(m; v; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \\
= \sum_{u \in V} Z_V^{(1)}(a_1, x_1; \ldots; a_L, x_L; u; 0; \tau_1) Z_V^{(1)}(b_R, y_R; \ldots; b_1, y_1; v|m|\overline{u}; 0; \tau_2) \\
= \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]u; \tau_1) Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]v|m|\overline{u}; \tau_2).
\]

We notice that since \( v \) is quasiprimary, we may move the modes of \( v \) from left to right as before,

\[
X_1(v; m) = \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]v|m|u; \tau_1) Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]v|m|\overline{u}; \tau_2) \\
= \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]u; \tau_1) Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]|v|m|\overline{u}; \tau_2) \\
= (-1)^{N} \epsilon^{m+1-N} \left( \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \ldots Y[a_L, x_L]u; \tau_1) \right) \\
\quad \cdot Z_V^{(1)}(Y[b_R, y_R] \ldots Y[b_1, y_1]|v|2N - m - 2|\overline{u}; \tau_2). \]

We introduce the general shorthand notations

\[
Y[a_1, x_L] = Y[a_1, x_1] \ldots Y[a_L, x_L], \\
Y[b_R, y_R] = Y[b_R, y_1] \ldots Y[b_1, y_1].
\]

46
The $Z_V^{(1)}(Y[b_r, y_r]v[2N - m - 2]|\tau_2)$ factors are genus one correlation functions, and again we may apply Zhu’s recursion formula (Theorem 1.2) to these factors, provided

$$m + 2 - 2N \geq 1,$$

that is,

$$s \geq 1,$$

where

$$s = -2N + m + 2.$$

Zhu’s recursion formulae are typically written with the recursion performed in the first vector. We wish to perform Zhu recursion in $v$, the second to last vector. To achieve this, we make use of locality. By locality, there exists $M_1, \ldots, M_R \in \mathbb{N}$ such that

$$\left(\prod_{r=1}^{R} (y_r - x)^{M_r}\right) Z_V^{(1)}(Y[b_r, y_r] Y[v, x] Y[\mu, z]; \tau_2)$$

$$= \left(\prod_{r=1}^{R} (y_r - x)^{M_r}\right) Z_V^{(1)}(Y[v, x] Y[b_r, y_r] Y[\mu, z]; \tau_2).$$

Now,

$$Z_V^{(1)}(Y[v, x] Y[b_r, y_r] Y[\mu, z]; \tau_2)$$

$$= \text{Tr}_V\left(o(v)Y(q_{y_R}^{L(0)} b_R, q_{y_R}) \cdots Y(q_{y_1}^{L(0)} b_1, q_{y_1}) Y(q_{\mu}^{L(0)} \bar{\mu}, q_{\bar{\mu}}^{L(0)} - c/24)\right)$$

$$+ \sum_{r=1}^{R} \sum_{j \geq 0} P_{j+1}(x - y_r, \tau_2) Z_V^{(1)}(Y[b_R, y_R] \cdots Y[v[j] b_r, y_r] \cdots Y[b_1, y_1] Y[\mu, z]; \tau_2)$$

$$+ \sum_{j \geq 0} P_{j+1}(x - z, \tau_2) Z_V^{(1)}(Y[b_r, y_r] Y[v[j] \mu, z]; \tau_2).$$

From locality, we now have that
\[ Z_{V}^{(1)}(Y[b_r, y_r]Y[v, x]Y[\mu, z]; \tau_2) \]
\[ = \text{Tr}_V \left( o(v)Y(q_{y_R}^{L(0)}b_R) \ldots Y(q_{y_1}^{L(0)}b_1, q_{y_1})Y(q_0^{L(0)}\mu, q_2^{L(0)}-c/24) \right) \]
\[ + \sum_{r=1}^{R} \sum_{j \geq 0} (-1)^{j+1} P_{j+1}(y_r - x, \tau_2) Z_{V}^{(1)}(Y[b_R, y_R] \ldots Y[v[j]b_r, y_r] \ldots Y[b_1, y_1]Y[\mu, z]; \tau_2) \]
\[ + \sum_{j \geq 0} P_{j+1}(x - z, \tau_2) Z_{V}^{(1)}(Y[b_r, y_r]Y[v[j]\mu, z]; \tau_2). \]

Thus, setting \( z = 0 \) gives

\[ Z_{V}^{(1)}(Y[b_r, y_r]Y[v, x]\mu; \tau_2) \]
\[ = \text{Tr}_V \left( o(v)Y(q_{y_R}^{L(0)}b_R) \ldots Y(q_{y_1}^{L(0)}b_1, q_{y_1})Y(q_0^{L(0)}\mu, q_0^{L(0)}-c/24) \right) \]
\[ + \sum_{r=1}^{R} \sum_{j \geq 0} (-1)^{j+1} P_{j+1}(y_r - x, \tau_2) Z_{V}^{(1)}(Y[b_R, y_R] \ldots Y[v[j]b_k, y_k] \ldots Y[b_1, y_1]\mu; \tau_2) \]
\[ + \sum_{j \geq 0} P_{j+1}(x, \tau_2) Z_{V}^{(1)}(Y[b_r, y_r]v[j]\mu; \tau_2). \]

To restrict to the \( v[-s] \) mode, we extract the coefficient of \( x^{s-1} \). Where \( s \geq 1 \).

\[ P_{j+1}(y_r - x, \tau_2) = \sum_{i \geq 0} \binom{j + i}{j} P_{j+1+i}(y_r, \tau_2)x^i, \]

gives the coefficient

\[ \binom{j + s - 1}{j} P_{j+s}(y_r, \tau_2), \]

and

\[ P_{j+1}(x, \tau_2) = \frac{1}{x^{j+1}} + (-1)^{j+1} \sum_{i \geq j+1} \binom{i - 1}{j} E_{i}(\tau_2)x^{i-j-1} , \]

48
where $i - j + 1 = s - 1$, gives the coefficient

$$(-1)^{j+1} \binom{j+s-1}{j} E_{j+s}(\tau_2).$$

Thus, by Zhu recursion we have that

$$Z^{(1)}_V(Y[b_r, y_r]v[2N - m - 2]\pi; \tau_2) = Z^{(1)}_V(Y[b_r, y_r]v[-s]\pi; \tau_2) = \delta_{s,1} \text{Tr}_V \left(o(v)Y(q_{y_R}^{L(0)}b_R, q_{y_R}) \cdots Y(q_{y_2}^{L(0)}b_1, q_{y_1})Y(q_0^{L(0)}\pi, q_0^{L(0)} - \epsilon/24)\right) + \sum_{j = 1}^{R} \sum_{j \geq 0} (-1)^{j+1} \binom{j+s-1}{j} P_{j+s}(y_r, \tau_2) Z^{(1)}_V(Y[b_R, y_R] \cdots Y[v[j]b_r, y_r] \cdots Y[b_1, y_1]\pi; \tau_2) - E_s(\tau_2) Z^{(1)}_V(Y[b_r, y_r]v[0]\pi; \tau_2) + \sum_{j \geq 1} (-1)^{j+1} \binom{j+s-1}{j} E_{j+s}(\tau_2) Z^{(1)}_V(Y[b_r, y_r]v[j]\pi; \tau_2).$$

where $s = 1 \iff m = 2N - 1$. Substituting this back into our previous expression for $X_1(m)$, we find that for $m \geq 2N - 1$,

$$X_1(v; m) = (-1)^{N+1} \epsilon^{m+1-N} \delta_{m,2N-1} O_2(v; \mathbf{a}_t, \mathbf{x}_t|b_r, y_r) + (-1)^{N+1} \epsilon^{m+1-N} \sum_{r=1}^{R} \sum_{j \geq 0} (-1)^{j+1} \binom{j+s-1}{j} P_{j+s}(y_r, \tau_2) Z^{(2)}_V(\cdots; v[j]b_r, y_r; \cdots) - (-1)^{N+1} \epsilon^{m+1-N} E_s(\tau_2) \sum_{u \in V} Z^{(1)}_V(Y[a_t, x_t]u; \tau_1) Z^{(1)}_V(Y[b_r, y_r]v[0]\pi; \tau_2) + (-1)^{N+1} \epsilon^{m+1-N} \sum_{j \geq 1} \sqrt{\frac{s}{j}} \epsilon^{-(j+1)} A_2(s, j) X_2(v; j).$$

49
where

**Definition 4.3.**

\[ O_2(v; a_1, x_1|b_r, y_r) \]
\[ = O_2(v; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]
\[ = \sum_{u \in V} \text{Tr}_V \left( o(v)Y(q_{y_R}^{L(0)}b_R, q_{y_R}) \ldots Y(q_{y_L}^{L(0)}b_1, q_{y_1})Y(q_0^{L(0)}\pi, q_0)q_2^{L(0)}c/24 \right) \]
\[ \cdot Z_1^{(1)}(Y[a_1, x_1]u; \tau_1), \]

and

**Definition 4.4.**

\[ Z_2^{(2)}(\ldots; v[j]b_r, y_r; \ldots) \]
\[ = Z_2^{(2)}(a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; v[j]b_r, y_r; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]
\[ = \sum_{u \in V} Z_2^{(1)}(a_1, x_1; \ldots; a_L, x_L|u, 0; \tau_1)Z_1^{(1)}(b_R, y_R; \ldots; v[j]b_r, y_r; \ldots; b_1, y_1; \pi, 0; \tau_2). \]

It is convenient to rescale the \( X_1(v), X_2(v) \) vectors as before to find

\[ X_1(v; m) = X_1(v; m; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]
\[ = \frac{\epsilon^{-m/2}}{\sqrt{m}} X_1(v; m), \]
\[ X_2(v; m) = X_2(v; m; a_1, x_1; \ldots; a_L, x_l|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]
\[ = \frac{\epsilon^{-m/2}}{\sqrt{m}} X_2(v; m). \]
Thus we have that for \( m \geq 2N - 1 \),

\[
\mathbb{X}_1(v; m) = \frac{\epsilon^{-m/2}}{\sqrt{m}} (-1)^N \epsilon^{m+1-N} \delta_{m,2N-1} O_2(v; a_l, x_l | b_r, y_r) \\
+ \frac{\epsilon^{-m/2}}{\sqrt{m}} (-1)^N \epsilon^{m+1-N} \sum_{r=1}^{R} \sum_{j \geq 0} (-1)^{j+1} \binom{j + s - 1}{j} P_{j+s}(y_r, \tau_2) Z_V^{(2)}(\ldots; v[j]b_r, y_r; \ldots) \\
- \frac{\epsilon^{-m/2}}{\sqrt{m}} (-1)^N \epsilon^{m+1-N} E_s(\tau_2) \sum_{u \in V} Z_V^{(1)}(Y[a_l, x_l]u; \tau_1) Z_V^{(1)}(Y[a_l, x_l]v[0]u; \tau_2) \\
+ (-1)^N \sqrt{\frac{s}{m}} (A_2 \mathbb{X}_2(v))(s).
\]

For convenience, we let

\[
K = 2N - 2,
\]

as before, so that

\[
s = m - K.
\]

This gives that for \( m > K \),

\[
\mathbb{X}_1(v; m) = \frac{\epsilon^{-m/2}}{\sqrt{m}} (-1)^N \epsilon^{m+1-N} \delta_{m,K+1} O_2(v; a_l, x_l | b_r, y_r) \\
+ \frac{\epsilon^{-m/2}}{\sqrt{m}} (-1)^N \epsilon^{m+1-N} \sum_{r=1}^{R} \sum_{j \geq 0} (-1)^{j+1} \binom{j + s - 1}{j} P_{j+s-K}(y_r, \tau_2) \\
\cdot Z_V^{(2)}(\ldots; v[j]b_r, y_r; \ldots) \\
- \frac{\epsilon^{-m/2}}{\sqrt{m}} (-1)^N \epsilon^{m+1-N} E_{m-K}(\tau_2) \sum_{u \in V} Z_V^{(1)}(Y[a_l, x_l]u; \tau_1) Z_V^{(1)}(Y[a_l, x_l]v[0]u; \tau_2) \\
+ (-1)^N \sqrt{\frac{s}{m}} (A_2 \mathbb{X}_2(v))(m - K).
\]
We recall the infinite matrices (Definition 2.2)

\[
\begin{align*}
\Gamma(m, n) &= \sqrt{\frac{n}{m}} \delta_{m+n,K}, \\
\Delta(m, n) &= \sqrt{\frac{n}{m}} \delta_{m-n,K}, \\
\Theta(m, n) &= \sqrt{\frac{n}{m}} \delta_{n-m,K},
\end{align*}
\]

where \( m, n \geq 1 \). Using the \( \Delta \) matrix as before, for \( m > K \) we may write

\[
\mathbb{X}_1(v; m) = \frac{1/2}{\sqrt{m}} (-1)^N \delta_{m,K+1} O_2(v; a_1, x_1|b_r, y_r)
\]

\[
+ \frac{\varepsilon^{-m/2}}{\sqrt{m}} (-1)^N e^{m+1-N} \sum_{r=1}^R \sum_{j \geq 0} (-1)^{j+1} \left( j + m - K - 1 \right) P_{j+m-K}(y_r, \tau_2)
\]

\[
\cdot Z_V^{(2)}(\ldots v[j] b_r, y_r; \ldots)
\]

\[
- \frac{\varepsilon^{-m/2}}{\sqrt{m}} (-1)^N e^{m+1-N} E_{m-K}(\tau_2) \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] u; \tau_1) Z_V^{(1)}(Y[a_1, x_1] v[0] u; \tau_2)
\]

\[
+ (-1)^N (\Delta A_2 \mathbb{X}_2(v))(m).
\]

Now for \( 1 \leq m \leq K \), we first notice that

\[
\mathbb{X}_1(v; K) = \frac{\varepsilon^{-K/2}}{\sqrt{K}} \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] v[K] u; \tau_1) Z_V^{(1)}(Y[b_r, y_r] u; \tau_2)
\]

\[
= (-1)^N \frac{e^{-K/2+1-N}}{\sqrt{K}} \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] u; \tau_1) Z_V^{(1)}(Y[b_r, y_r] v[2N-K-2] u; \tau_2)
\]

\[
= (-1)^N \frac{1}{\sqrt{K}} \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] u; \tau_1) Z_V^{(1)}(Y[b_r, y_r] v[0] \tau; \tau_2),
\]

is not necessarily zero. However, by a theorem of Zhu [Z], we have

\[
Z_V^{(1)}(Y[b_r, y_r] v[0] \tau; \tau_2) = - \sum_{r=1}^R Z_V^{(1)}(Y[b_r, y_r] \ldots Y[v[0] b_r, y_r] \ldots Y[b_1, y_1] \tau; \tau_2).
\]

52
This gives that
\[
\sum_{u \in V} Z_{V}^{(1)} (Y[a_l, x_l]u; \tau_1) Z_{V}^{(1)} (Y[b_r, y_r]v[0]w; \tau_2) = \sum_{r=1}^{R} Z_{V}^{(2)} (\ldots; v[0]b_r, y_r; \ldots).
\]

Now for \(1 \leq m \leq K\), we have
\[
X_1(v; m) = \frac{(-1)^N}{\sqrt{K}} \delta_{m,K} \sum_{r=1}^{R} Z_{V}^{(2)} (\ldots; v[0]b_r, y_r; \ldots)
\]
\[
+ (-1)^N \sum_{n=1}^{K} \Gamma(m,n) X_2(v; n).
\]
The values \(X_1(v; 1), \ldots, X_1(v; K-1)\) are again independent and we will refer to these as the seed values. Alternatively, we may define the vectors

**Definition 4.5.**
\[
X_1^0(v; m; a_l, x_l|b_r, y_r) = X_1(m; v; a_1, x_1; \ldots; a_l, x_l|b_1, y_1; \ldots; b_r, y_r; \tau_1, \tau_2, \epsilon)
\]
\[
= \begin{cases} X_1(v; m) & \text{for } 1 \leq m < K \\ 0 & \text{for } m \geq K \end{cases},
\]
and
\[
X_2^0(v; m; a_l, x_l|b_r, y_r) = X_2(m; v; a_1, x_1; \ldots; a_l, x_l|b_1, y_1; \ldots; b_r, y_r; \tau_1, \tau_2, \epsilon)
\]
\[
= \begin{cases} X_2(v; m) & \text{for } 1 \leq m < K \\ 0 & \text{for } m \geq K \end{cases}.
\]

We notice that
\[
X_1^0(v; a_l, x_l|b_r, y_r) = (-1)^N \Gamma X_2(v) = (-1)^N \Gamma X_2^0(v; a_l, x_l|b_r, y_r),
\]
\[
X_2^0(v; a_l, x_l|b_r, y_r) = (-1)^N \Gamma X_1(v) = (-1)^N \Gamma X_2^0(v; a_l, x_l|b_r, y_r).
\]
Combining our expression for $1 \leq m \leq K$ with our expression for $m > K$, we find

$$X_1(v; m) = (-1)^N \frac{e^{1/2}}{\sqrt{m}} \delta_{m,K+1} O_2(v; a_1, x_l | b_r, y_r) + \delta_{m,K} X_1(v; K)$$

$$+ (-1)^N \sum_{r=1}^{R} G_r^2(m) + (-1)^N ((\Gamma + \Delta A_2)X_2(v)) (m),$$

where $G_r^2(m) = 0$ for $1 \leq m \leq K$ and

$$G_r^2(m) = \epsilon_{m-K}^2 \sqrt{m} \sum_{j=0}^{m-K} (-1)^{j+1} \left( \frac{m-K-1}{j} \right) P_{j+m-K}(y_r, \tau_2) Z_\nu^{(2)}(\ldots; v[j]b_r, y_r; \ldots)$$

$$+ \epsilon_{m-K}^2 \sqrt{m} E_{m-K}(\tau_2) Z_\nu^{(2)}(\ldots; v[0]b_r, y_r; \ldots),$$

for $m > K$. We define the vectors $\Theta_1, \Theta_2$ by

$$\Theta_1(m) = \frac{e^{1/2}}{\sqrt{m}} \delta_{m,K+1} O_1(v; a_1, x_l | b_r, y_r),$$

$$\Theta_2(m) = \frac{e^{1/2}}{\sqrt{m}} \delta_{m,K+1} O_2(v; a_1, x_l | b_r, y_r),$$

where $m \geq 1$. To derive an expression for the $X_1$ in terms of itself, we project out the effects of the $\Gamma$ matrix at this stage. To achieve this, we define

$$X_1 = (-1)^N \Theta X_1(v),$$

$$X_2 = (-1)^N \Theta X_2(v),$$

and

$$G_r^2 = \Theta G_r^2.$$
Now,

\[ x_1 = (-1)^N \Theta x_1(v) \]

\[ = \Theta O_2 + \sum_{r=1}^{R} \Theta G_r^2 + \Theta (\Gamma + \Delta A_2) x_2(v) \]

\[ = \Theta O_2 + \sum_{r=1}^{R} G_r^2 + A_2 \left((-1)^N \Delta x_2 + x_2^0(v; a_l, x_l|b_r, y_r)\right) \]

\[ = \Theta O_2 + A_2 x_2^K(v) + A_2 x_2^0(v; a_l, x_l|b_r, y_r) + \sum_{r=1}^{R} G_r^2 + (-1)^N (A_2 \Delta) x_2 \]

\[ = \Theta O_2 + A_2 x_2^K(v) + A_2 x_2^0(v; a_l, x_l|b_r, y_r) + \sum_{r=1}^{R} G_r^2 + (-1)^N \tilde{A}_2 x_2 , \]

where \( \tilde{A}_2 = A_2 \Delta \) and \( x_2^K(v) \) is defined by

\[ x_2^K(v; m) = \delta_{m,K} x_2(v; m) , \]

for \( m \geq 1 \). By left-right symmetry, we have

\[ x_2 = \Theta O_1 + A_1 x_1^K(v) + A_1 x_1^0(v; a_l, x_l|b_r, y_r) + \sum_{l=1}^{L} G_l^1 + (-1)^N \tilde{A}_1 x_1 , \]

where \( \tilde{A}_1 = A_1 \Delta \), \( x_1^K(v) \) is defined by

\[ x_1^K(v; m) = \delta_{m,K} x_1(v; m) , \]

for \( m \geq 1 \) and \( G_l^1 = \Theta G_l^1 \), with \( G_l^1(m) = 0 \) for \( 1 \leq m \leq K \) and

\[ G_l^1(m) = \frac{\epsilon^{m-K}}{\sqrt{m}} \sum_{j=0}^{m-K} (-1)^{j+1} \binom{j + m - K - 1}{j} P_{j+m-K}(x_l, \tau_1) Z^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]

\[ + \frac{\epsilon^{m-K}}{\sqrt{m}} E_{m-K}(\tau_1) Z^{(2)}(\ldots; v[0]a_l, x_l; \ldots) , \]

for \( m > K \).

55
Combining these formulae, we have

\[ X_1 = \Theta \Theta_2 + A_2 X_2^1(v) + A_2 X_2^0(v; a_l, x_l|b_r, y_r) + \sum_{r=1}^{R} G_r^2 + (-1)^N \tilde{A}_2 X_2 \]

\[ = \Theta \Theta_2 + A_2 X_2^1(v) + A_2 X_2^0(v; a_l, x_l|b_r, y_r) + \sum_{r=1}^{R} G_r^2 \]

\[ + (-1)^N \tilde{A}_2 \left( \Theta \Theta_1 + A_1 X_1^1(v) + A_1 X_1^0(v; a_l, x_l|b_r, y_r) + \sum_{l=1}^{L} G_l^1 + (-1)^N \tilde{A}_1 X_1 \right) \]

\[ = \Theta \Theta_2 + A_2 X_2^1(v) + A_2 X_2^0(v; a_l, x_l|b_r, y_r) + \sum_{r=1}^{R} G_r^2 + (-1)^N \tilde{A}_2 \Theta \Theta_1 \]

\[ + (-1)^N \tilde{A}_2 A_1 X_1^1(v) + (-1)^N \tilde{A}_2 A_1 X_1^0(v; a_l, x_l|b_r, y_r) + (-1)^N \sum_{l=1}^{L} \tilde{A}_2 G_l^1 + \tilde{A}_2 A_1 X_1 \]

and so

\[ X_1 = \left( I - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \Theta \Theta_2 + A_2 X_2^1(v) + A_2 X_2^0(v; a_l, x_l|b_r, y_r) \right) \]

\[ + \sum_{r=1}^{R} G_r^2 + (-1)^N \tilde{A}_2 \Theta \Theta_1 + (-1)^N \tilde{A}_2 A_1 X_1^1(v) \]

\[ + (-1)^N \tilde{A}_2 A_1 X_1^0(v; a_l, x_l|b_r, y_r) + (-1)^N \sum_{l=1}^{L} \tilde{A}_2 G_l^1 \right) \]

where \( X_1 = \left( I - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \) is the formal inverse.
Now, we recall that

\[ Z_V^{(2)}(v, x; a_1, x_1; \ldots; a_L, x_L| b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]

\[ = O_1(v; a_l, x_l| b_r, y_r) + \sum_{m \geq 1} P_{m+1}(x, \tau_1)X_1(v; m) \]

\[ + \sum_{l=1}^{L} \sum_{j \geq 0} P_{j+1}(x - x_l, \tau_1)Z_V^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]

\[ - P_1(x, \tau_1) \sum_{l=1}^{L} Z_V^{(2)}(\ldots; v[0]a_l, x_l; \ldots). \]

We recall \( P_2(x) \), the infinite vector defined by (Definition 2.4)

\[(P_2(x))(m) = \epsilon^{m/2} \sqrt{m} P_{m+1}(x, \tau_1), \]

where \( m \geq 1 \). We then have that

\[ Z_V^{(2)}(v, x; a_1, x_1; \ldots; a_L, x_L| b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon) \]

\[ = O_1(v; a_l, x_l| b_r, y_r) + P_2(x)X_1(v) \]

\[ + \sum_{l=1}^{L} \sum_{j \geq 0} P_{j+1}(x - x_l, \tau_1)Z_V^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]

\[ - P_1(x, \tau_1) \sum_{l=1}^{L} Z_V^{(2)}(\ldots; v[0]a_l, x_l; \ldots) \]

\[ = O_1(v; a_l, x_l| b_r, y_r) + P_2(x) \left( X_0^0(v; a_l, x_l| b_r, y_r) + X_1^K(v) + (-1)^N \Delta X_1 \right) \]

\[ + \sum_{l=1}^{L} \sum_{j \geq 0} P_{j+1}(x - x_l, \tau_1)Z_V^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]

\[ - P_1(x, \tau_1) \sum_{l=1}^{L} Z_V^{(2)}(\ldots; v[0]a_l, x_l; \ldots) \]
\[ = O_1(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r) + P_2(x)X_1^0(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r) + P_2(x)X_1^K(v) + (-1)^N P_2(x) \Delta X_1 \]
\[ + \sum_{l=1}^L \sum_{j \geq 0} P_{l+1}(x - x_l, \tau_l)Z_{V}{^2}^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]
\[ - P_1(x, \tau_l) \sum_{l=1}^L Z_{V}{^2}^{(2)}(\ldots; v[0]a_l, x_l; \ldots) \]
\[ = O_1(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r) + P_2(x)X_1^0(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r) + P_2(x)X_1^K(v) \]
\[ + (-1)^N P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \Theta \Omega_2 \]
\[ + (-1)^N P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 X^K(v) \]
\[ + (-1)^N P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 X_2^0(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r) \]
\[ + (-1)^N \sum_{r=1}^R P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} G_2^r \]
\[ + P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \Theta \Omega_1 \]
\[ + P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 A_1 X^K(v) \]
\[ + P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 A_1 X_1^0(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r) \]
\[ + \sum_{l=1}^L P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 G_1^l \]
\[ + \sum_{l=1}^L \sum_{j \geq 0} P_{l+1}(x - x_l, \tau_l)Z_{V}{^2}^{(2)}(\ldots; v[j]a_l, x_l; \ldots) \]
\[ - P_1(x, \tau_l) \sum_{l=1}^L Z_{V}{^2}^{(2)}(\ldots; v[0]a_l, x_l; \ldots) \].

We notice that
\[ (\Theta \Omega_1)(m) = \delta_{m,1} \epsilon^{1/2} O_1(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r), \]
\[ (\Theta \Omega_2)(m) = \delta_{m,1} \epsilon^{1/2} O_2(v; \mathbf{a}_l, x_l|\mathbf{b}_r, y_r). \]

That is, only the first components of \( \Theta \Omega_1, \Theta \Omega_2 \) are non-zero.
We have that

\[
\mathbb{P}_2(x)X^K_1(v) = -(-1)^N \epsilon^{K/2} P_{K+1}(x, \tau_1) \sum_{r=1}^R Z^{(2)}_V (\ldots; v[0]b_r, y_r; \ldots).
\]

We also notice that

\[
G_1^l(m) = (\Theta G_1^l)(m) = \epsilon m^2 \frac{\sqrt{m}}{m} \sum_{j \geq 0} (-1)^{j+1} \binom{j + m - 1}{j} P_{j+m}(x_l, \tau_1) Z^{(2)}_V (\ldots; v[j]a_l, x_l; \ldots)
\]

\[
+ \epsilon m^2 \frac{\sqrt{m}}{m} E_m(\tau_1) Z^{(2)}_V (\ldots; v[0]a_l, x_l; \ldots),
\]

and

\[
G_2^r(m) = (\Theta G_2^r)(m) = \epsilon m^2 \frac{\sqrt{m}}{m} \sum_{j \geq 0} (-1)^{j+1} \binom{j + m - 1}{j} P_{j+m}(y_r, \tau_2) Z^{(2)}_V (\ldots; v[j]b_r, y_r; \ldots)
\]

\[
+ \epsilon m^2 \frac{\sqrt{m}}{m} E_m(\tau_2) Z^{(2)}_V (\ldots; v[0]b_r, y_r; \ldots).
\]

Where \( m \geq 1 \), we define the vectors \( G^l_{1,j}, G^r_{2,j} \) by

\[
G^l_{1,0}(m) = \epsilon m^2 \frac{\sqrt{m}}{m} (E_m(\tau_1) - P_m(x_l, \tau_1)) Z^{(2)}_V (\ldots; v[0]a_l, x_l; \ldots),
\]

\[
G^r_{2,0}(m) = \epsilon m^2 \frac{\sqrt{m}}{m} (E_m(\tau_2) - P_m(y_r, \tau_2)) Z^{(2)}_V (\ldots; v[0]b_r, y_r; \ldots),
\]

and

\[
G^l_{1,j}(m) = \epsilon m^2 \frac{\sqrt{m}}{m} (-1)^{j+1} \binom{j + m - 1}{j} P_{j+m}(x_l, \tau_1) Z^{(2)}_V (\ldots; v[j]a_l, x_l; \ldots),
\]

\[
G^r_{2,j}(m) = \epsilon m^2 \frac{\sqrt{m}}{m} (-1)^{j+1} \binom{j + m - 1}{j} P_{j+m}(y_r, \tau_2) Z^{(2)}_V (\ldots; v[j]b_r, y_r; \ldots),
\]

59
for $j \geq 1$. This gives that

\[
G_l^j = \sum_{j \geq 0} G_{1,j}^l ,
\]

\[
G_r^j = \sum_{j \geq 0} G_{2,j}^r .
\]

Now, we have that

\[
\mathbb{P}_2(x) \Delta \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} \tilde{A}_2 G_i^r = \sum_{j \geq 0} \tilde{\mathbb{P}}(x) \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} \tilde{A}_2 G_{i,j}^r
\]

and

\[
\mathbb{P}_2(x) \Delta \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} G_j^l = \sum_{j \geq 0} \tilde{\mathbb{P}}(x) \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} G_{j}^l
\]

Notice that

\[
\mathbb{P}_2(x) \Delta \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} \tilde{A}_2 G_{1,0}^l
\]

\[
= \sum_{m \geq 1} \left(\mathbb{P}_2(x) \Delta \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} \tilde{A}_2\right)(m) \frac{\epsilon m}{\sqrt{m}} (E_m(\tau_l) - P_m(x_l, \tau_l)) \cdot Z^{(2)}(\ldots; v[0]a_l, x_l; \ldots),
\]

\[
\mathbb{P}_2(x) \Delta \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1} G_{2,0}^r
\]

\[
= \sum_{m \geq 1} \left(\mathbb{P}_2(x) \Delta \left(1 - \tilde{A}_2 \tilde{A}_1^\top\right)^{-1}\right)(m) \frac{\epsilon m}{\sqrt{m}} (E_m(\tau_r) - P_m(y_r, \tau_r)) \cdot Z^{(2)}(\ldots; v[0]b_r, y_r; \ldots),
\]

and that

60
\[ \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \mathcal{G}_{1,j}^l \]

\[ = \sum_{m \geq 1} \left( \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right)(m) \frac{\epsilon^m}{\sqrt{m}} (-1)^{j+1} \binom{j + m - 1}{j} P_{j+m}(x_l, \tau_1) \]

\[ \cdot Z^{(2)}_{l} (\ldots; v[j]a_l, x_l; \ldots), \]

\[ \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \mathcal{G}_{2,j}^r \]

\[ = \sum_{m \geq 1} \left( \mathbb{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \right)(m) \frac{\epsilon^m}{\sqrt{m}} (-1)^{j+1} \binom{j + m - 1}{j} P_{j+m}(y_r, \tau_2) \]

\[ \cdot Z^{(2)}_{r} (\ldots; v[j]b_r, y_r; \ldots). \]

for \( j \geq 1. \)

### 4.2 Genus Two Generalised Weierstrass Functions

For \( x \in \hat{S}_a \), we have defined \( \mathbb{P}_2(x) = \mathbb{P}_2(x, \tau_a) \) by

\[ (\mathbb{P}_2(x))(m) = \epsilon^{m/2} \sqrt{m} P_{m+1}(x, \tau_a). \]

where \( m \geq 1 \). If we let \( \mathbb{P}_1(x) = \mathbb{P}_1(x, \tau_a) \) be defined by

\[ (\mathbb{P}_1(x))(m) = \frac{\epsilon^{m/2}}{\sqrt{m}} \left( P_m(x, \tau_a) - E_m(\tau_a) \right), \]

where \( m \geq 1 \) and \( \mathbb{P}_{1+j}(x) = \mathbb{P}_{1+j}(x, \tau_a) \) be defined by

\[ (\mathbb{P}_{1+j}(x))(m) = \frac{(-1)^j}{j!} \partial_x^j (\mathbb{P}_1(x))(m), \]

where \( j \geq 0 \) and \( m \geq 1 \), we then notice that

\[ \mathbb{P}_2(x) = -\partial_x \mathbb{P}_1(x). \]

We have that

61
\[
\frac{\epsilon P}{\sqrt{m}} \left( j + m - 1 \right) P_{j+m}(x, \tau_a) = \left( \frac{-1}{j!} \partial^j_2 \left( \frac{e^{m/2}}{\sqrt{m}} (P_m(x, \tau_a) - E_m(\tau_a)) \right) \right)
= \left( \frac{-1}{j!} \partial^j_2 (P_1(x)) (m) \right)
= (P_{1+j}(x)) (m).
\]

We consider the previous expressions for \( P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right) \) and \( P_2(x) \Delta \left( 1 - \bar{A}_2 \bar{A}_1 \right) \), and we find

\[
P_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right) \tilde{A}_2 \mathcal{G}^j_{1,j}
= (-1)^{1+j}P_2(x) \left( \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right) \right) P^T_1(x_1) Z_{V}^{(2)}(\ldots ; v[j]a_l, x_l; \ldots)
\]

and

\[
P_2(x) \Delta \left( 1 - \bar{A}_2 \bar{A}_1 \right) \mathcal{G}^j_{2,j}
= (-1)^{1+j}P_2(x) \left( \Delta \left( 1 - \bar{A}_2 \bar{A}_1 \right) \right) P^T_1(y_r) Z_{V}^{(2)}(\ldots ; v[j]b_r, y_r; \ldots).
\]

We recall the \( \tilde{a} \) convention from (7) and define genus two generalised Weierstrass functions

**Definition 4.6.**

\[
N \mathcal{P}_{2,1}(x, y) = \begin{cases}
P_1(x - y, \tau_a) - P_1(x, \tau_a) \\
\frac{-1}{\sqrt{2N - 2}} \left( P_2(x) \Delta \left( 1 - \bar{A}_a \bar{A}_a \right) \tilde{A}_a \right) (2N - 2) \\
-\left( P_2(x) \Delta \left( 1 - \bar{A}_a \bar{A}_a \right) \tilde{A}_a \right) P^T(y) \text{ for } x, y \in \tilde{S}_a, \\
-\left( -1 \right)^N \left( P_2(x) \Delta \left( 1 - \tilde{A}_a \tilde{A}_a \right) \tilde{A}_a \right) P^T(y) \text{ for } x \in \tilde{S}_a, y \in \tilde{S}_a.
\end{cases}
\]
Definition 4.7. For \(j \geq 1\), we define
\[
N \mathcal{P}_{2,1+j}(x, y) := \frac{1}{j!} \partial_y^j \left( N \mathcal{P}_{2,1}(x, y) \right)
\]
\[
= \begin{cases} 
  P_{1+j}(x - y, \tau_a) 
  & \text{for } x, y \in \hat{S}_a, \\
  +(-1)^{1+j} \mathcal{P}_2(x) \left( \Delta \left( 1 - A_a \tilde{A}_a \right)^{-1} \tilde{A}_a \right) \mathcal{P}_1^{T}(y) 
  & \text{for } x \in \hat{S}_a, y \in \hat{S}_a.
\end{cases}
\]

Remark 4.1. The generalised Weierstrass functions depend on \(N\) through the matrices \(\Delta\) of Definition 2.2 and \(\tilde{A}_a = A_a \Delta\) with \(A_a\) from Definition 1.25.

Thus, we have proved the following recursive theorem,

Theorem 4.1. [Genus Two Zhu Recursion] The genus two \(n\)-point correlation function for a quasiprimary vector \(v \in V_{[N]}\) inserted at \(x \in \hat{S}_1\) and arbitrary vectors \(a_1, \ldots, a_L \in V\) and \(b_1, \ldots, b_R \in V\) inserted at \(x_1, \ldots, x_L \in \hat{S}_1\) and \(y_1, \ldots, y_R \in \hat{S}_2\) respectively obeys
\[
Z_V^{(2)}(v, x_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; x_1, x_1, \tau_1, \tau_2, \epsilon)
= \left( 1 + \epsilon^{1/2} \left( \mathcal{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) (1) \right) O_1(v; a_1, x_l|b_r, y_r)
+ \left( (-1)^N \epsilon^{1/2} \left( \mathcal{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) (1) \right) O_2(v; a_1, x_l|b_r, y_r)
+ \left( \mathcal{P}_2(x) + \mathcal{P}_2(x) \Delta \left( 1 - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \tilde{A}_2 A_1 + A_2 \Gamma \right) \right) \mathcal{P}_1^T(v; a_1, x_l|b_r, y_r)
+ \sum_{l=1}^{L} \sum_{j \geq 0} N \mathcal{P}_{2,1+j}(x, x_l) Z_V^{(2)}(\ldots; v[j]a_1, x_l; \ldots)
+ \sum_{r=1}^{R} \sum_{j \geq 0} N \mathcal{P}_{2,1+j}(x, y_r) Z_V^{(2)}(\ldots; v[j]b_r, y_r; \ldots),
\]

with \(O_1(v; a_1, x_l|b_r, y_r)\) and \(O_2(v; a_1, x_l|b_r, y_r)\) from Definitions 4.1 and 4.3 respectively, \(\Gamma\) and \(\Delta\) from Definition 2.2, \(\mathcal{P}_1^T(v; a_1, x_l|b_r, y_r)\) from Definition 4.5, \(\mathcal{P}_2(x)\) from Definition 2.4, \(\tilde{A}_a = A_a \Delta\) with \(A_a\) from Definition 1.25, \(Z_V^{(2)}(\ldots; v[j]a_1, x_l; \ldots)\) and \(Z_V^{(2)}(\ldots; v[j]b_r, y_r; \ldots)\) from Definitions 4.2 and 4.4 respectively and the functions \(N \mathcal{P}_{2,1+j}(x, x_l)\) and \(N \mathcal{P}_{2,1+j}(x, y_r)\) from Definitions 4.6 and 4.7.
Notice that the recursive identity in the above theorem is very similar in form to the recursive identity in genus one Zhu Recursion (Theorem 1.2). Notice also that this theorem reduces to Theorem 2.1 for a genus two quasiprimary 1-point function.

### 4.3 Quasiprimary Descendants

We now generalise to the case of a general quasiprimary descendant $L[-1]^k v$. As before,

$$Z^{(2)}_V(L[-1]^k v, x; a_1, x_1; \ldots; a_L, x_L| b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon)$$

$$= \partial_x^k Z^{(2)}_V(v, x; a_1, x_1; \ldots; a_L, x_L| b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon).$$

Thus, as an immediate corollary to Theorem 4.1 we have

**Corollary 4.1.** For $k \geq 1$, the genus two $n$-point correlation function for a quasiprimary descendant $L[-1]^k v$ inserted at $x \in \hat{S}_1$ and arbitrary vectors $a_1, \ldots, a_L \in V$ and $b_1, \ldots, b_R \in V$ inserted at $x_1, \ldots, x_L \in \hat{S}_1$ and $y_1, \ldots, y_R \in \hat{S}_2$ respectively is given by

$$Z^{(2)}_V(L[-1]^k v, x; a_1, x_1; \ldots; a_L, x_L| b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon)$$

$$= \Big( \epsilon^{1/2} \partial_x^k \left( \mathcal{P}_2(x) \Delta \left( \mathbb{1} - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \Big) (1) \ O_1(v; a_l, x_l| b_r, y_r)$$

$$+ \Big( (-1)^N \epsilon^{1/2} \partial_x^k \left( \mathcal{P}_2(x) \Delta \left( \mathbb{1} - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \right) \Big) (1) \ O_2(v; a_l, x_l| b_r, y_r)$$

$$+ \partial_x^k \left( \mathcal{P}_2(x) + \mathcal{P}_2(x) \Delta \left( \mathbb{1} - \tilde{A}_2 \tilde{A}_1 \right)^{-1} \left( \tilde{A}_2 A_1 + A_2 \Gamma \right) \right) \chi_0^0(v; a_l, x_l| b_r, y_r)$$

$$+ \sum_{l=1}^L \sum_{j=0}^\infty \partial_x^k \mathcal{P}_{2,1+j}(x, x_l) Z^{(2)}_V(\ldots; v[j] a_l, x_l; \ldots)$$

$$+ \sum_{r=1}^R \sum_{j=0}^\infty \partial_x^k \mathcal{P}_{2,1+j}(x, y_r) Z^{(2)}_V(\ldots; v[j] b_r, y_r; \ldots).$$

**Remark 4.2.** Since the quasiprimary descendants span $V$, we have demonstrated that genus two Zhu Recursion holds for all vectors.
5 Applications of the \( n \)-point
Recursion Formula

In this section, we will specialise the final \( n \)-point recursion formula of
the previous section to derive formal expressions for specific examples
of genus two \( n \)-point correlation functions.

5.1 Genus Two Ward Identities

We consider a VOA \( V \) with Virasoro vector \( \tilde{\omega} \in V[2] \), the partition
function defined as

\[
Z^{(2)}_V(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z^{(1)}_V(u; \tau_1) Z^{(1)}_V(\bar{u}; \tau_2),
\]

and we consider the \( n \)-point function

\[
Z^{(2)}_V(\tilde{\omega}, x; a_1, x_1; \ldots; a_L, x_L; b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon)
= \sum_{u \in V} Z^{(1)}_V(Y[a_1, x_1]u; \tau_1) Z^{(1)}_V(Y[b_R, y_R]\bar{u}; \tau_2),
\]

where \( a_1, \ldots, a_L \) and \( b_1, \ldots, b_R \) are primary vectors. We find that

\[
O_1(\tilde{\omega}; a_L, x_L|b_R, y_R)
= \sum_{u \in V} \text{Tr}_V \left( o(\tilde{\omega}) Y(q^{L(0)} a_1, q x_1) \ldots Y(q^{L(0)} a_L, q x_L) Y(q^{L(0)} u, q_0) q^{L(0) - c/24}_1 \right)
\cdot Z^{(1)}_V(Y[b_R, y_R]\bar{u}; \tau_2)
= q_1 \partial q_1 \sum_{u \in V} Z^{(2)}_V(Y[a_1, x_1]u; \tau_1) Z^{(1)}_V(Y[b_R, y_R]\bar{u}; \tau_2)
= q_1 \partial q_1 Z^{(2)}_V(a_L, x_L|b_R, y_R),
\]

where

\[
Z^{(2)}_V(a_L, x_L|b_R, y_R) = Z^{(2)}_V(a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon).
\]
Similarly, we find that

\[ O_2(\tilde{\omega}; a_l, x_l|b_r, y_r) = q_2 \partial_{q_2} Z^{(2)}_V (a_l, x_l|b_r, y_r). \]

Since

\[ K = 2 \text{wt}[\tilde{\omega}] - 2 = 4 - 2 = 2, \]

we have that

\[ \Gamma(m, n) = \delta_{m,1}\delta_{n,1}, \]

and so we have that \( X^0_1(\tilde{\omega}; a_l, x_l|b_r, y_r) = X^0_1(\tilde{\omega}; a_l, x_l|b_r, y_r), \) where

\[ X^0_1(\tilde{\omega}; m; a_l, x_l|b_r, y_r) = \delta_{m,1} X^1_1(\tilde{\omega}; m). \]

Now,

\[
X^0_1(\tilde{\omega}; 1; a_l, x_l|b_r, y_r) = \epsilon^{-1/2} \sum_{u \in V} Z^{(1)}_V (Y[a_l, x_l]|\tilde{\omega}[1]u; \tau_1) Z^{(1)}_V (Y[b_r, y_r]|\tilde{\omega}; \tau_2) \\
= \epsilon^{-1/2} \sum_{u \in V} Z^{(1)}_V (Y[a_l, x_l]|L[0]u; \tau_1) Z^{(1)}_V (Y[b_r, y_r]|\tilde{\omega}; \tau_2) \\
= \epsilon^{-1/2} \epsilon \partial_\epsilon \sum_{u \in V} Z^{(1)}_V (Y[a_l, x_l]|u; \tau_1) Z^{(1)}_V (Y[b_r, y_r]|\tilde{\omega}; \tau_2) \\
= \epsilon^{1/2} \partial_\epsilon Z^{(2)}_V (a_l, x_l|b_r, y_r).
\]

We now consider the contraction terms, \( Z^{(2)}_V (\ldots; \tilde{\omega}[j]a_l, x_l; \ldots) \) and \( Z^{(2)}_V (\ldots; \tilde{\omega}[j]b_r, y_r; \ldots) \) for \( j \geq 0 \). Since \( \tilde{\omega}[j] = L[j - 1] \) for all \( j \in \mathbb{Z}, \) we have that \( \tilde{\omega}[0] = L[-1] \tilde{\omega} \) and so we find that

\[
Z^{(2)}_V (\ldots; \tilde{\omega}[0]a_l, x_l; \ldots) = \partial_{x_l} Z^{(2)}_V (a_l, x_l|b_r, y_r), \\
Z^{(2)}_V (\ldots; \tilde{\omega}[0]b_r, y_r; \ldots) = \partial_{y_r} Z^{(2)}_V (a_l, x_l|b_r, y_r).
\]

We have that \( \tilde{\omega}[1]a_l = L[0]a_l = \text{wt}[a_l]a_l, \) and so we have that
\[ Z^{(2)}_V(\ldots; \tilde{\omega}[1]a_l, x_l; \ldots) = \text{wt}[a_l]Z^{(2)}_V(a_l, x_l|b_r, y_r). \]

Similarly, we have

\[ Z^{(2)}_V(\ldots; \tilde{\omega}[1]b_r, y_r; \ldots) = \text{wt}[b_r]Z^{(2)}_V(a_l, x_l|b_r, y_r). \]

Now, since \( a_1, \ldots, a_L \) and \( b_1, \ldots, b_R \) are primary vectors, we have that for all \( j \geq 2 \),

\[ Z^{(2)}_V(\ldots; \tilde{\omega}[j]a_l, x_l; \ldots) = 0 = Z^{(2)}_V(\ldots; \tilde{\omega}[j]b_r, y_r; \ldots). \]

Thus, we obtain

**Proposition 5.1.** The genus two Ward Identity is given by

\[
Z^{(2)}_V(\omega, x; a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \tau_1, \tau_2, \epsilon)
= \left( D_x + \sum_{l=1}^{L} \left( 2P_{2,1}(x, x_l) \partial_{x_l} + \text{wt}[a_l] \cdot 2P_{2,2}(x, x_l) \right) 
+ \sum_{r=1}^{R} \left( 2P_{2,1}(x, y_r) \partial_{y_r} + \text{wt}[b_r] \cdot 2P_{2,2}(x, y_r) \right) \right) Z^{(2)}_V(a_l, x_l|b_r, y_r).
\]

where \( a_1, \ldots, a_L \in V \) and \( b_1, \ldots, b_R \in V \) are primary vectors.

Notice that the genus two Ward Identity is very similar in form to the genus one Ward Identity, which is given by

\[
Z^{(1)}_V(\omega, x; v_1, x_1; \ldots; v_n, x_n; \tau)
= \left( q\partial_q + \sum_{k=1}^{n} \left( P_1(x - x_k, \tau) \partial_{x_k} + \text{wt}[v_k]P_2(x - x_k, \tau) \right) \right) Z^{(1)}_V(v_1, z_1; \ldots; v_n, z_n; \tau).
\]

where \( v_1, \ldots, v_n \) are primary vectors.
5.2 Genus Two Virasoro $n$-point Functions

We consider a VOA $V$ with Virasoro vector $\tilde{\omega} \in V[2]$, the partition function defined as

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(u; \tau_1) Z_V^{(1)}(\bar{u}; \tau_2),$$

and the Virasoro $n$-point function defined as

$$Z_V^{(2)}(\tilde{\omega}, x; \tilde{\omega}, x_1; \ldots; \tilde{\omega}, x_L; y_1; \ldots; \tilde{\omega}, y_R; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(Y[\tilde{\omega}, x]Y[\bar{\omega}, x_1]u; \tau_1) Z_V^{(1)}(Y[\tilde{\omega}, y]u; \tau_2).$$

We find that

$$O_1(\tilde{\omega}; \tilde{\omega}, x_l|\tilde{\omega}, y_r) = q_1 \partial_{q_1} Z_V^{(2)}(\tilde{\omega}, x_l|\tilde{\omega}, y_r).$$

where

$$Z_V^{(2)}(\tilde{\omega}, x_l|\tilde{\omega}, y_r) = Z_V^{(2)}(\tilde{\omega}, x_1; \ldots; \tilde{\omega}, x_L|\tilde{\omega}, y_1; \ldots; \tilde{\omega}, y_R; \tau_1, \tau_2, \epsilon).$$

Similarly, we find that

$$O_2(\tilde{\omega}; \tilde{\omega}, x_l|\tilde{\omega}, y_r) = q_2 \partial_{q_2} Z_V^{(2)}(\tilde{\omega}, x_l|\tilde{\omega}, y_r).$$
Since
\[ K = 2 \text{wt}[\tilde{\omega}] - 2 = 4 - 2 = 2, \]
we have that
\[ \Gamma(m, n) = \delta_{m, 1} \delta_{n, 1}, \]
and so we have that \( X^0_1(\tilde{\omega}; \tilde{\omega}, x_l|\tilde{\omega}, y_r) = X^0_2(\tilde{\omega}; \tilde{\omega}, x_l|\tilde{\omega}, y_r) \), where
\[ X^0_1(\tilde{\omega}; m; \tilde{\omega}, x_l|\tilde{\omega}, y_r) = \delta_{m, 1} X^1_1(\tilde{\omega}; \tilde{\omega}) \].

Now,
\[
X^0_1(\tilde{\omega}; 1; \tilde{\omega}, x_l|\tilde{\omega}, y_r)
= \epsilon^{-1/2} \sum_{u \in V} Z^{(1)}_V(\hat{Y}[\tilde{\omega}, x_l] \hat{\omega}[1] u; \tau_1) Z^{(1)}_V(\hat{Y}[\tilde{\omega}, y_r] \hat{\pi}; \tau_2)
= \epsilon^{-1/2} \sum_{u \in V} Z^{(1)}_V(\hat{Y}[\tilde{\omega}, x_l] \hat{L}[0] u; \tau_1) Z^{(1)}_V(\hat{Y}[\tilde{\omega}, y_r] \hat{\pi}; \tau_2)
= \epsilon^{-1/2} \epsilon \partial_x \sum_{u \in V} Z^{(1)}_V(\hat{Y}[\tilde{\omega}, x_l] u; \tau_1) Z^{(1)}_V(\hat{Y}[\tilde{\omega}, y_r] \hat{\pi}; \tau_2)
= \epsilon^{1/2} \partial_x Z^{(2)}_V(\tilde{\omega}, x_l|\tilde{\omega}, y_r).
\]

We now consider the contraction terms, \( Z^{(2)}_V(\ldots; \tilde{\omega}[j]|\tilde{\omega}, x_l; \ldots) \) and \( Z^{(2)}_V(\ldots; \tilde{\omega}[j]|\tilde{\omega}, y_r; \ldots) \) for \( j \geq 0 \). Since \( \tilde{\omega}[j] = \tilde{L}[j - 1] \) for all \( j \in \mathbb{Z} \), we have that \( \tilde{\omega}[0]|\omega = \tilde{L}[1]|\omega \) and so we find
\[
Z^{(2)}_V(\ldots; \tilde{\omega}[0]|\tilde{\omega}, x_l; \ldots) = Z^{(2)}_V(\ldots; \tilde{L}[1]|\tilde{\omega}, x_l; \ldots)
= \partial_{x_l} Z^{(2)}_V(\tilde{\omega}, x_l|\tilde{\omega}, y_r).
\]
Similarly,
\[
Z^{(2)}_V(\ldots; \tilde{\omega}[0]|\tilde{\omega}, y_r; \ldots) = \partial_{y_r} Z^{(2)}_V(\tilde{\omega}, x_l|\tilde{\omega}, y_r).
\]
We have that $\tilde{\omega}[1] \tilde{\omega} = L[0] \tilde{\omega} = \text{wt}[\tilde{\omega}] \tilde{\omega} = 2 \tilde{\omega}$, and so we have that

$$Z_V^{(2)}(\ldots; \tilde{\omega}[1] \tilde{\omega}, x_l; \ldots) = 2Z_V^{(2)}(\tilde{\omega}, x_l | \tilde{\omega}, y_r).$$

Similarly, we have

$$Z_V^{(2)}(\ldots; \tilde{\omega}[1] \tilde{\omega}, y_r; \ldots) = 2Z_V^{(2)}(\tilde{\omega}, x_l | \tilde{\omega}, y_r).$$

We have that $\tilde{\omega}[2] = L[1]$ and by creativity, we have that

$$\tilde{\omega} = \tilde{\omega}[-1]1 = L[-2]1.$$

By the Virasoro commutation relationships, we now have that

$$\tilde{\omega}[2] \tilde{\omega} = L[1]L[-2]1$$

$$= (L[-2]L[1] + [L[1], L[-2]]) 1$$

$$= (L[-2]L[1] + (1 - (-2))L[1 - 2]) 1$$

$$= 3L[-1]1$$

$$= 0,$$

since $L[1]1 = 0 = L[-1]1$. This implies that

$$Z_V^{(2)}(\ldots; \tilde{\omega}[2] \tilde{\omega}, x_l; \ldots) = 0 = Z_V^{(2)}(\ldots; \tilde{\omega}[2] \tilde{\omega}, y_r; \ldots).$$

We have that $\tilde{\omega}[3] = L[2]$ and so

$$\tilde{\omega}[3] \tilde{\omega} = L[2]L[-2]1$$

$$= (L[-2]L[2] + [L[2], L[-2]]) 1$$

$$= \left(L[-2]L[2] + (2 - (-2))L[2 - 2] + \frac{23 - 2}{12} c\right) 1$$

$$= \left(4L[0] + \frac{c}{2}\right) 1$$

$$= \frac{c}{2} 1,$$

since $L[2]1 = 0 = L[0]1$. Thus we have that
\[ Z_V^{(2)}(\ldots; \bar{\omega}[3] \bar{\omega}, x_l; \ldots) = Z_V^{(2)}(\ldots; \frac{c}{2}, x_l; \ldots) \]
\[ = \frac{c}{2} Z_V^{(2)}(\ldots; \bar{\omega}, \bar{x}_l; \ldots), \]

where the \( \bar{x}_l \) indicates that the insertion of \( \bar{\omega} \) at \( x_l \) is now omitted.

Similarly,
\[ Z_V^{(2)}(\ldots; \bar{\omega}[3] \bar{\omega}, y_r; \ldots) = \frac{c}{2} Z_V^{(2)}(\ldots; \bar{\omega}, \hat{y}_r; \ldots). \]

Finally, for \( j \geq 4 \), we have \( \bar{\omega}[j] = L[j - 1] \) and so
\[ \bar{\omega}[j] \bar{\omega} = L[j - 1] L[-2] \mathbf{1} \]
\[ = (L[-2] L[j - 1] + [L[j - 1], L[-2]]) \mathbf{1} \]
\[ = (L[-2] L[j - 1] + [L[j - 1], L[-2]]) \mathbf{1} \]
\[ = (j - 1 - (-2)) L[j - 1 - 2] \mathbf{1} \]
\[ = 0, \]

since \( L[j - 1] \mathbf{1} = 0 = L[j - 1 - 2] \mathbf{1} \). This implies that for \( j \geq 4 \), we have
\[ Z_V^{(2)}(\ldots; \bar{\omega}[j] \bar{\omega}, x_l; \ldots) = 0 = Z_V^{(2)}(\ldots; \bar{\omega}[j] \bar{\omega}, y_r; \ldots). \]
Thus we have

**Proposition 5.2.** The genus two Virasoro $n$-point correlation function is given by

$$ Z_V^{(2)}(\tilde{\omega}, x; \tilde{\omega}, x_1; \ldots; \tilde{\omega}, x_L|\tilde{\omega}, y_1; \ldots; \tilde{\omega}, y_R; \tau_1, \tau_2, \epsilon) $$

$$ = \left( D_x + \sum_{l=1}^L \left( 2P_{2,1}(x, x_l)\partial_{x_l} + 2 \cdot 2P_{2,2}(x, x_l) \right) \right) + \sum_{r=1}^R \left( 2P_{2,1}(x, y_r)\partial_{y_r} + 2 \cdot 2P_{2,2}(x, y_r) \right) Z_V^{(2)}(\tilde{\omega}, x_l|\tilde{\omega}, y_r) $$

$$ + \frac{c}{2} \sum_{l=1}^L 2P_{2,4}(x, x_l)Z_V^{(2)}(\ldots; \tilde{\omega}, \hat{x}_l; \ldots) $$

$$ + \frac{c}{2} \sum_{r=1}^R 2P_{2,4}(x, y_r)Z_V^{(2)}(\ldots; \tilde{\omega}, \hat{y}_r; \ldots), $$

where $D_x$ is the differential operator given by Definition 3.1.

Notice that the genus two Virasoro $n$-point correlation function is very similar in form to the genus one Virasoro $n$-point correlation function, which is given by

$$ Z_V^{(1)}(\tilde{\omega}, x_1; \ldots; \tilde{\omega}, x_n; \tau) $$

$$ = \left( q\partial_q + \sum_{k=2}^n \left( P_1(x_1 - x_k, \tau)\partial_{x_k} + 2P_2(x_1 - x_k, \tau) \right) \right) Z_V^{(1)}(\tilde{\omega}, x_2; \ldots; \tilde{\omega}, x_n; \tau) $$

$$ + \frac{c}{2} \sum_{k=2}^n P_4(x_1 - x_k, \tau) Z_V^{(1)}(\tilde{\omega}, x_2; \ldots; \tilde{\omega}, \hat{x}_k; \ldots; \tilde{\omega}, x_n; \tau).$$
5.3 Genus Two Heisenberg $n$-point Functions

We consider a pair of Heisenberg modules $M_{\alpha_1}, M_{\alpha_2}$ with the partition function defined as

$$Z^{(2)}_{M_{\alpha_1}, \alpha_2}(\tau_1, \tau_2, \epsilon) = \sum_{u \in M} Z^{(1)}_{M_{\alpha_1}}(u; \tau_1) Z^{(1)}_{M_{\alpha_2}}(\bar{u}; \tau_2),$$

and the Heisenberg $n$-point function defined as

$$Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x_1| a, y_1; \ldots; a, x_L| a, y_L; \tau_1, \tau_2, \epsilon) = \sum_{u \in M} Z^{(1)}_{M_{\alpha_1}}(Y[a, x_1]u; \tau_1) Z^{(1)}_{M_{\alpha_2}}(Y[a, y_1]u; \tau_2).$$

We find that

$$O_1(a; a, x_l| a, y_r) = \alpha_1 \cdot \sum_{u \in M} \text{Tr}_{M_{\alpha_1}} \left( \alpha(a) Y(q^{L(0)} a_{x_1}) \ldots Y(q^{L(0)} a_{x_L}) Y(q^{L(0)} u_{0}) Y(q^{L(0)} u_{0})^{1/24} \right) Z^{(1)}_{M_{\alpha_2}}(Y[a, y_r]\bar{u}; \tau_2)$$

$$= \alpha_1 \sum_{u \in M} Z^{(1)}_{M_{\alpha_1}}(Y[a, x_l]u; \tau_1) Z^{(1)}_{M_{\alpha_2}}(Y[a, y_r]\bar{u}; \tau_2) = \alpha_1 Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x_l| a, y_r),$$

where

$$Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x_l| a, y_r) = Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x_1; \ldots; a, x_L| a, y_1; \ldots; a, y_R; \tau_1, \tau_2, \epsilon).$$

Similarly, we have that

$$O_2(a; a, x_l| a, y_r) = \alpha_2 Z^{(2)}_{M_{\alpha_1}, \alpha_2}(a, x_l| a, y_r).$$
Since

\[ K = 2 \text{wt}[a] - 2 = 2 - 2 = 0, \]

we have that \( \Gamma = 0 \) and thus \( \Phi_1^0(a; x_l|a, y_r) = 0 = \Phi_2^0(a; x_l|a, y_r) \).

Further, we have that \( \Delta = 1 = \Theta \). We now consider the contraction terms \( Z_{N_{\alpha,\beta}}^{(2)}(\ldots; a[j]a, x_l; \ldots) \) and \( Z_{N_{\alpha,\beta}}^{(2)}(\ldots; a[j]a, y_r; \ldots) \) for \( j \geq 0 \).

By creativity we have that \( a = a[-1]1 \), and thus by the Heisenberg commutation relationships we have that

\[ a[j]a = \delta_{j,1}1. \]

Thus we have that

\[ Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a[j]a, x_l; \ldots) = 0 = Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a[j]a, y_r; \ldots), \]

for all \( j \neq 1 \). We also find that

\[ Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a[1]a, x_l; \ldots) = Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a, \hat{x}_l; \ldots), \]
\[ Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a[1]a, y_r; \ldots) = Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a, \hat{y}_r; \ldots), \]

where \( \hat{x}_l, \hat{y}_r \) indicates that the insertions of \( a \) at \( x_l, y_r \) are omitted.

Thus we have that the Heisenberg \( n \)-point function is given by

\[
\begin{align*}
Z_{M_{\alpha_1,\alpha_2}}^{(2)}(a, x; a_1; \ldots; a, x_L|a_y; \ldots; a, y_R|\tau_1, \tau_2, \epsilon) & = \alpha_1 \left( 1 + \epsilon^{1/2} \left( P_2(x) \left( 1 - A_2A_1 \right)^{-1} A_2 \right) \right) \left( 1 \right) Z_{M_{\alpha_1,\alpha_2}}^{(2)}(a, x_l|a, y_r) \\
& - \alpha_2 \left( \epsilon^{1/2} \left( P_2(x) \left( 1 - A_2A_1 \right)^{-1} A_2 \right) \right) \left( 1 \right) Z_{M_{\alpha_1,\alpha_2}}^{(2)}(a, x_l|a, y_r) \\
& + \sum_{l=1}^{L} 1_{P_2}(x, x_l) Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a, \hat{x}_l; \ldots) \\
& + \sum_{r=1}^{R} 1_{P_2}(x, y_l) Z_{M_{\alpha_1,\alpha_2}}^{(2)}(\ldots; a, \hat{y}_r; \ldots).
\end{align*}
\]
We notice that

$$1 \mathcal{P}_{2,2}(x, x_l) = P_2(x - x_1, \tau_1) + \mathbb{F}_2(x) \left( (1 - A_2 A_1)^{-1} A_2 \right) \mathbb{F}_2^T(x_l),$$

$$1 \mathcal{P}_{2,2}(x, y_r) = -\mathbb{F}_2(x) \left( (1 - A_2 A_1)^{-1} \right) \mathbb{F}_2^T(y_r),$$

and since

$$\mathbb{F}_2(x) dx = a_1(x),$$

we see that (Theorem 1.4)

$$1 \mathcal{P}_{2,2}(x, x_l) dx dx_l = \omega^{(2)}(x, x_l),$$

$$1 \mathcal{P}_{2,2}(x, y_r) dx dy_r = \omega^{(2)}(x, y_r),$$

and that (Theorem 1.5)

$$\nu_1(x) = \left(1 + \epsilon^{1/2} \left( \mathbb{F}_2(x) (1 - A_2 A_1)^{-1} A_2 \right)(1) \right) dx,$$

$$\nu_2(x) = -\left( \epsilon^{1/2} \left( \mathbb{F}_2(x) (1 - A_2 A_1)^{-1} \right)(1) \right) dx,$$

and so we have

**Proposition 5.3.** The genus two $n$-point correlation function for a pair of Heisenberg modules $M_{\alpha_1}, M_{\alpha_2}$ is given by

$$Z^{(2)}_{M_{\alpha_1, \alpha_2}}(a, x_l|a, x_1; \ldots; a, x_L|a, y_1; \ldots; a, y_R; \tau_1, \tau_2, \epsilon) dx \prod_{j=1}^L dx_j \prod_{k=1}^R dy_k$$

$$= (\alpha_1 \nu_1(x) + \alpha_2 \nu_2(x)) Z^{(2)}_{M_{\alpha_1, \alpha_2}}(a, x_l|a, y_r) \prod_{j=1}^L dx_j \prod_{k=1}^R dy_k$$

$$+ \sum_{l=1}^L \omega^{(2)}(x, x_l) Z^{(2)}_{M_{\alpha_1, \alpha_2}}(\ldots; a, \hat{x}_l; \ldots) \prod_{j \neq l}^L dx_j \prod_{k=1}^R dy_k$$

$$+ \sum_{r=1}^R \omega^{(2)}(x, y_r) Z^{(2)}_{M_{\alpha_1, \alpha_2}}(\ldots; a, \hat{y}_r; \ldots) \prod_{j=1}^L dx_j \prod_{k \neq r}^R dy_k.$$
6 Geometric Results

In this section, we discuss the geometric significance of the $D_x$ operator (Definition 3.1) which was employed in previous sections. In particular, we derive differential equations involving $D_x$ and use these differential equations to show that $D_x$ is defined on a given domain, i.e., the coefficient functions $A(x), B(x), C(x)$ are convergent. This is achieved by comparing calculations from genus two Zhu Recursion (Theorem 4.1) to results for the Heisenberg VOA. We discuss the significance of these results and we end this section by stating several conjectures motivated by these results.

6.1 A differential equation for the genus two Heisenberg partition function

We consider the genus two Virasoro 1-point function for the Heisenberg VOA. We have seen previously (Proposition 3.2) that this Virasoro 1-point function is

$$Z_M^{(2)}(\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = D_x Z_M^{(2)}(\tau_1, \tau_2, \epsilon).$$

In [MT3], the authors prove that the genus two partition function and the Virasoro 1-point function for the Heisenberg VOA converge on $D^\epsilon$. In particular,

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1) Z_M^{(1)}(\tau_2) (\det (I - A_2 A_1))^{-\frac{1}{2}},$$

where

$$Z_M^{(1)}(\tau) = \frac{1}{\eta(\tau)},$$

and $(\det (I - A_2 A_1))^{-\frac{1}{2}}$ is holomorphic and convergent. Furthermore,

$$dx^2 Z_M^{(2)}(\tilde{\omega}, x; \tau_1, \tau_2, \epsilon)$$

$$= \lim_{x \to y} \frac{1}{2} \left( \frac{dxdy Z_M^{(2)}(a, x; a, y; \tau_1, \tau_2, \epsilon)}{(x-y)^2} - \frac{dxdy}{(x-y)^2} Z_M^{(2)}(\tau_1, \tau_2, \epsilon) \right)$$

$$= \frac{1}{12} s(2)(x) Z_M^{(2)}(\tau_1, \tau_2, \epsilon),$$

76
where \( s^{(2)}(x) \) is the projective connection (Definition 1.23)

\[
  s^{(2)}(x) = 6 \lim_{x \to y} \left( \omega^{(2)}(x, y) - \frac{dx dy}{(x - y)^2} \right).
\]

Thus, we have

**Proposition 6.1.**

\[
dx^2 \mathbb{D}_x Z^{(2)}_{M \alpha_1, \alpha_2}(\bar{\omega}, x; \tau_1, \tau_2, \epsilon) = \frac{1}{12} s^{(2)}(x) Z^{(2)}_{M \alpha_1, \alpha_2}(\tau_1, \tau_2, \epsilon).
\]

(11)

Alternatively, we may write

**Corollary 6.1.**

\[
dx^2 \mathbb{D}_x \left( \log Z^{(2)}_{M \alpha_1, \alpha_2}(\bar{\omega}, x; \tau_1, \tau_2, \epsilon) \right) = \frac{1}{12} s^{(2)}(x).
\]

(12)

Notice that this result is very similar in form to the genus one case, where

\[
q \partial_q Z'^{(1)}_{M \alpha_1, \alpha_2}(\tau) = \frac{1}{2} E_2(\tau) Z^{(1)}_{M \alpha_1, \alpha_2}(\tau),
\]

where

\[
Z^{(1)}_{M}(\tau) = \frac{1}{\eta(\tau)},
\]

and

\[
\frac{s^{(1)}}{6} = E_2(\tau) dx \, dy = \lim_{x \to y} \left( P_2(x - y, \tau) - \frac{1}{(x - y)^2} \right) dx \, dy.
\]

**6.2 The convergence of \( \mathbb{D}_x \)**

We now consider the genus two Virasoro 1-point function for a pair of Heisenberg modules \( M_{\alpha_1}, M_{\alpha_2} \). We have seen previously (Proposition 3.2) that this Virasoro 1-point function is

\[
Z^{(2)}_{M_{\alpha_1, \alpha_2}}(\bar{\omega}, x; \tau_1, \tau_2, \epsilon) = \mathbb{D}_x Z^{(2)}_{M_{\alpha_1, \alpha_2}}(\tau_1, \tau_2, \epsilon).
\]
In [MT3], the authors prove that

$$Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tau_1, \tau_2, \epsilon) = e^{i\pi \alpha \cdot \Omega \cdot \alpha} Z_M^{(2)}(\tau_1, \tau_2, \epsilon),$$

where

$$\alpha \cdot \Omega \cdot \alpha = \sum_{i,j=1,2} \alpha_i \Omega_{ij} \alpha_j.$$

where $\Omega$ is the period matrix (Definition 1.24). Thus we have that

$$dx^2 Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = \lim_{x \to y} \frac{1}{2} \left( (\nu_\alpha(x) \nu_\alpha(y) + \omega(x, y)) - \frac{dxdy}{(x-y)^2} \right) Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tau_1, \tau_2, \epsilon).$$

From [MT3] or by Proposition 5.3, we have that

$$dx dy Z_{M_{\alpha_1, \alpha_2}}^{(2)}(a, x; a, y; \tau_1, \tau_2, \epsilon) = (\nu_\alpha(x) \nu_\alpha(y) + \omega(x, y)) Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tau_1, \tau_2, \epsilon),$$

where

$$\nu_\alpha(x) = \alpha_1 \nu_1(x) + \alpha_2 \nu_2(x).$$

By the associativity formula, we have [MT3]

$$Z_M^{(1)}(Y[a, x] Y[a, y] v; \tau) = Z_M^{(1)}(Y[Y[a, x - y] a, y] v; \tau) = \frac{Z_M^{(1)}(v; \tau)}{(x - y)^2} + 2 Z_M^{(1)}(Y[\tilde{\omega}, y] v; \tau) + \cdots,$$

and so we have that [MT3]

$$dx^2 Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = \lim_{x \to y} \frac{1}{2} \left( (\nu_\alpha(x) \nu_\alpha(y) + \omega(x, y)) - \frac{dxdy}{(x-y)^2} \right) Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tau_1, \tau_2, \epsilon) = \left( \frac{1}{12} s^{(2)}(x) + \frac{1}{2} \nu_\alpha(x)^2 \right) Z_{M_{\alpha_1, \alpha_2}}^{(2)}(\tau_1, \tau_2, \epsilon).$$
By comparing these expressions, we see that

$$2\pi i \, dx^2 \, \mathbb{D}_x \, \Omega_{ij} = \nu_i(x) \nu_j(x),$$

which gives

$$2\pi i \, dx^2 \, \Omega_{ij} = \nu_i(x) \nu_j(x).$$

We define the differential operator

**Definition 6.1.**

$$\nabla_x = \frac{1}{2\pi i} \sum_{1 \leq i, j \leq 2} \nu_i(x) \nu_j(x) \frac{\partial}{\partial \Omega_{ij}}.$$  \hspace{1cm} (13)

**Remark 6.1.** Notice that $\nabla_x$ is defined on all differentiable functions $f(\Omega)$ on $\mathbb{H}_2$.

Thus, by the chain rule, we have

**Proposition 6.2.** We have the formal equality of differential operators on $\mathcal{D}^\epsilon$,

$$dx^2 \mathbb{D}_x = \nabla_x.$$  \hspace{1cm} (14)

By Theorem 1.7 [MT2], there exists a $G$-invariant neighbourhood $\mathcal{N}^\epsilon \subseteq \mathcal{D}^\epsilon$ where the mapping $F^\epsilon : \mathcal{D}^\epsilon \to \mathbb{H}_2$ is invertible. In particular

$$(\partial \Omega_{11}, \partial \Omega_{12}, \partial \Omega_{21}) = \frac{\partial(\tau_1, \tau_2, \epsilon)}{\partial(\Omega_{11}, \Omega_{12}, \Omega_{21})}(\partial \tau_1, \partial \tau_2, \partial \epsilon).$$

Further, since $\nabla_x$ is defined on differentiable functions of $\Omega_{ij}$ with invertible Jacobian, we have

**Proposition 6.3.** The differential operator $\mathbb{D}_x$ is defined on differentiable functions of $\Omega(\tau_1, \tau_2, \epsilon)$ for $(\tau_1, \tau_2, \epsilon) \in \mathcal{N}^\epsilon$. In particular, if $f$ is a differentiable function of $\Omega(\tau_1, \tau_2, \epsilon)$, then on $\mathcal{N}^\epsilon$

$$dx^2 \mathbb{D}_x f(\Omega) = \nabla_x f(\Omega).$$
We have that the projective connection obeys the modular transformation property \[F],[U]\]
s\(^{(2)}\)(x) \overset{\gamma}{\mapsto} s\(^{(2)}\)(x) - 6 \frac{\nabla_x \det(C \Omega + D)}{\det(C \Omega + D)},

where \(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G\). Thus, as an immediate consequence of Proposition 6.3 we have

**Proposition 6.4.** The differential equation

\[\nabla_x Z\(^{(2)}\)\_M(\tau_1, \tau_2, \epsilon) = \frac{1}{12} s\(^{(2)}\)(x)Z\(^{(2)}\)\_M(\tau_1, \tau_2, \epsilon),\]

is invariant under the transformation

\[Z\(^{(2)}\)\_M(\tau_1, \tau_2, \epsilon) \overset{\gamma}{\mapsto} \det(C \Omega + D)^{-1/2} \chi(\gamma) Z\(^{(2)}\)\_M(\tau_1, \tau_2, \epsilon),\]

for \(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G\) and multiplier system \(\chi\).

**Remark 6.2.** Proposition 6.4 agrees with Theorem 8 of [MT3], where it is shown by a different method that the genus two Heisenberg partition function \(Z\(^{(2)}\)\_M(\tau_1, \tau_2, \epsilon)\) transforms up to a multiplier as a Siegel form of weight \(-\frac{1}{2}\) under \(G\).

### 6.3 A differential equation for holomorphic 1-forms

The genus two 3-point function for a pair of Heisenberg modules \(M_{\alpha_1}, M_{\alpha_2}\) is given by [MT3] or Proposition 5.3

\[
dx_1 \dx_2 \dy Z\(^{(2)}\)\_{M_{\alpha_1},\alpha_2}(a, x_1; a, x_2; a, y; \tau_1, \tau_2, \epsilon)
= \left( \nu_{\alpha}(x_1) \nu_{\alpha}(x_2) \nu_{\alpha}(y) + \nu_{\alpha}(x_1) \omega(x_2, y) + \nu_{\alpha}(x_2) \omega(x_1, y) + \nu_{\alpha}(y) \omega(x_1, x_2) \right) Z\(^{(2)}\)\_{M_{\alpha_1},\alpha_2}(\tau_1, \tau_2, \epsilon).
\]

We then have that
\[ dx^2 dy Z_{M_{01},02}^{(2)}(\tilde{\omega}, x; a, y; \tau_1, \tau_2, \epsilon) \]
\[ = \lim_{x_1 \to x_2} \frac{1}{2} \left( dx_1 dx_2 dy Z_{M_{01},02}^{(2)}(a, x_1; a, x_2; a, y; \tau_1, \tau_2, \epsilon) - \frac{dx_1 dx_2}{(x_1 - x_2)^2} \nu_\alpha(y) Z_{M_{01},02}^{(2)}(\tau_1, \tau_2, \epsilon) \right) \]
\[ = \left( \frac{1}{2} \nu_\alpha(x)^2 \nu_\alpha(y) + \nu_\alpha(x) \omega(x, y) + \frac{1}{12} \nu_\alpha(y) s^{(2)}(x) \right) Z_{M_{01},02}^{(2)}(\tau_1, \tau_2, \epsilon). \]

By Proposition 5.1, we also have that

\[ dx^2 dy Z_{M_{01},02}^{(2)}(\tilde{\omega}, x; a, y; \tau_1, \tau_2, \epsilon) \]
\[ = dx^2 \left( \mathbb{D}_x + \left( 2 P_{2,1}(x, y) \partial_y + 2 P_{2,2}(x, y) \right) \right) Z_{M_{01},02}^{(2)}(a, y; \tau_1, \tau_2, \epsilon) dy \]
\[ = dx^2 \left( \mathbb{D}_x + \left( 2 P_{2,1}(x, y) \partial_y + 2 P_{2,2}(x, y) \right) \right) \nu_\alpha(y) Z_{M_{01},02}^{(2)}(\tau_1, \tau_2, \epsilon). \]

We recall that

\[ dx^2 \mathbb{D}_x Z_{M_{01},02}^{(2)}(\tau_1, \tau_2, \epsilon) = \left( \frac{1}{12} s^{(2)}(x) + \frac{1}{2} \nu_\alpha(x)^2 \right) Z_{M_{01},02}^{(2)}(\tau_1, \tau_2, \epsilon), \]

and so we find the differential equation

\[ \left( \frac{1}{2} \nu_\alpha(x)^2 \nu_\alpha(y) + \nu_\alpha(x) \omega(x, y) + \frac{1}{12} \nu_\alpha(y) s^{(2)}(x) \right) \]
\[ = \left( \frac{1}{12} s^{(2)}(x) + \frac{1}{2} \nu_\alpha(x)^2 \right) \nu_\alpha(y) + dx^2 \mathbb{D}_x \nu_\alpha(y) \]
\[ + dx^2 \left( 2 P_{2,1}(x, y) \partial_y + 2 P_{2,2}(x, y) \right) \nu_\alpha(y), \]

and after simplifying this expression, we have

**Proposition 6.5.**

\[ \nu_\alpha(x) \omega(x, y) = dx^2 \left( \mathbb{D}_x + 2 P_{2,1}(x, y) \partial_y + 2 P_{2,2}(x, y) \right) \nu_\alpha(y). \]
6.4 A differential equation for the normalised 2-form

The genus two 4-point function for the Heisenberg VOA is given by [MT3] or Proposition 5.3

\[ dx_1 dx_2 dy_1 dy_2 Z_M^{(2)}(a, x_1; a, x_2; a, y_1; a, y_2; \tau_1, \tau_2, \epsilon) \]

\[ = \left( \omega(x_1, x_2)\omega(y_1, y_2) + \omega(x_1, y_1)\omega(x_2, y_2) + \omega(x_1, y_2)\omega(x_2, y_1) \right) Z_M^{(2)}(\tau_1, \tau_2, \epsilon). \]

We then have that

\[ dx_1 dx_2 dy_1 dy_2 Z_M^{(2)}(\tilde{\omega}, x; a, y_1; a, y_2; \tau_1, \tau_2, \epsilon) \]

\[ = \lim_{x_1 \to x_2} \frac{1}{2} \left( dx_1 dx_2 dy_1 dy_2 Z_M^{(2)}(a, x_1; a, x_2; a, y_1; a, y_2; \tau_1, \tau_2, \epsilon) \right. \]

\[ \left. - \frac{dx_1 dx_2}{(x_1 - x_2)^2} \omega(y_1, y_2) Z_M^{(2)}(\tau_1, \tau_2, \epsilon) \right) \]

\[ = \left( \frac{1}{12} s^{(2)}(x)\omega(y_1, y_2) + \omega(x, y_1)\omega(x, y_2) \right) Z_M^{(2)}(\tau_1, \tau_2, \epsilon). \]

By Proposition 5.1, we also have that

\[ dx_1 dx_2 dy_1 dy_2 Z_M^{(2)}(\tilde{\omega}, x; a, y_1; a, y_2; \tau_1, \tau_2, \epsilon) \]

\[ = dx^2 \left( \mathbb{D}_x + \sum_{r=1}^2 \left( 2P_{2,1}(x, y_r)\partial_{y_r} + 2P_{2,2}(x, y_r) \right) \right) Z_M^{(2)}(a, y_1; a, y_2; \tau_1, \tau_2, \epsilon) dy_1 dy_2 \]

\[ = dx^2 \left( \mathbb{D}_x + \sum_{r=1}^2 \left( 2P_{2,1}(x, y_r)\partial_{y_r} + 2P_{2,2}(x, y_r) \right) \right) \omega(y_1, y_2) Z_M^{(2)}(\tau_1, \tau_2, \epsilon). \]

We recall that

\[ dx^2 \mathbb{D}_x Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = \frac{1}{12} s^{(2)}(x) Z_M^{(2)}(\tau_1, \tau_2, \epsilon), \]

and so we find the differential equation

\[ \left( \frac{1}{12} s^{(2)}(x)\omega(y_1, y_2) + \omega(x, y_1)\omega(x, y_2) \right) \]

\[ = \left( \frac{1}{12} s^{(2)}(x)\omega(y_1, y_2) + dx^2 \left( \mathbb{D}_x + \sum_{r=1}^2 \left( 2P_{2,1}(x, y_r)\partial_{y_r} + 2P_{2,2}(x, y_r) \right) \right) \right) \omega(y_1, y_2), \]

82
and after simplifying the above expression we find

**Proposition 6.6.**

\[
\omega(x, y_1) \omega(x, y_2) = dx^2 \left( \mathbb{D}_x + \sum_{r=1}^{2} \left( 2p_{2,1}(x, y_r) \partial_{y_r} + 2p_{2,2}(x, y_r) \right) \right) \omega(y_1, y_2).
\]

Notice that this differential equation is similar in form to the genus one case ([HT], Theorem 3.3)

\[
P_2(x - y_1, \tau) P_2(x - y_2, \tau) = \left( q \partial_q - P_1(x - y_1, \tau) \partial_x - P_1(x - y_2, \tau) \partial_y + P_2(x - y_1, \tau) + P_2(x - y_2, \tau) \right) P_2(y_2 - y_1, \tau).
\]

Notice also that Proposition 6.6 implies Proposition 6.5 by integration.

### 6.5 The (2, 5) minimal model

We now derive a linear partial differential equation for the partition function of the Virasoro (2, 5)-minimal model. This example illustrates the potential for genus two Zhu theory, as this partition function cannot be computed by graphical methods, as are used for the Heisenberg VOA (or fermionic SVOAs). It is known that the (2, 5)-minimal model has a singular vector \( v = L[-2]^2 \mathbf{1} + \alpha L[-4] \mathbf{1} \) of weight 4. Therefore, there exists \( v \in V_4 \) such that

\[
0 = \langle L[-2]^2 \mathbf{1}, v \rangle = \langle \mathbf{1}, L[2]^2 v \rangle.
\]

Thus, by the properties of the Li-Z metric, we have that \( L[2]^2 v = 0 \). Now,

\[
L[2]^2 L[-2]^2 \mathbf{1} = L[2] \left( L[-2] L[2] + 4L[0] \frac{c}{2} \right) L[-2] \mathbf{1}
\]

\[
= \left( L[2] L[-2] \right)^2 \mathbf{1} + \left( 8 + \frac{c}{2} \right) L[2] L[-2] \mathbf{1}
\]

\[
= c \left( 4 + \frac{c}{2} \right) \mathbf{1},
\]

and
\[ = 6L[2]L[-2]1 \]
\[ = 3c1. \]

Thus, we have that

\[ c \left( 4 + \frac{c}{2} \right) = -\alpha 3c. \]

Since for the (2,5)-minimal model we have that \( c = -\frac{22}{3} \), we conclude that \( \alpha = -\frac{3}{5} \). Further, we have that

\[ L[-4]1 = \frac{1}{2} L[-1]^2L[-2]1. \]

Since \( L[-2]1 = \tilde{\omega} \) by creativity, we now have

\[ L[-2]\tilde{\omega} = \frac{3}{10} L[-1]^3\tilde{\omega}. \]

By Theorem 2.1, Corollary 2.1 and Proposition 3.2, we have that

\[ Z_{V}^{(2)}(L[-1]^2\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = \partial_x^2 D_x Z_{V}^{(2)}(\tau_1, \tau_2, \epsilon). \]

Now, by associativity

\[ Z_{V}^{(2)}(\tilde{\omega}, x; \tilde{\omega}, y; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_{V}^{(1)}(Y[\tilde{\omega}, x]Y[\tilde{\omega}, y]u; \tau_1)Z_{V}^{(1)}(\bar{\tau}; \tau_2) \]
\[ = \sum_{u \in V} Z_{V}^{(1)}(Y[Y[\tilde{\omega}, x - y]\tilde{\omega}, y]u; \tau_1)Z_{V}^{(1)}(\bar{\tau}; \tau_2), \]

where

\[ Y[\tilde{\omega}, x - y] = \sum_{n \in \mathbb{Z}} L[n](x - y)^{-n-2}. \]

From Proposition 5.2, we have

84
\[ Z_V^{(2)}(\tilde{\omega}, x; \tilde{\omega}, y; \tau_1, \tau_2, \epsilon) = \left( \left( \mathbb{D}_x + 2P_{2,1}(x, y)\partial_y + 2P_{2,2}(x, y) \right) \mathbb{D}_y - \frac{11}{5} \cdot 2P_{2,4}(x, y) \right) Z_{V}^{(2)}(\tau_1, \tau_2, \epsilon). \]

To find \( Z_V^{(2)}(L[-2]\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) \), we extract the 0th power of \( \xi = (x-y) \) from \( Z_V^{(2)}(\tilde{\omega}, x; \tilde{\omega}, y; \tau_1, \tau_2, \epsilon) \). We have

\[ 2P_{2,1}(x, y) = P_1(\xi, \tau_1) - P_1(x, \tau_1) - \mathbb{P}_2(x) \left( \Delta \left( 1 - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_{1}^T(x - \xi) \]

so the coefficient of the 0th power of \( \xi \) is

\[ 2P_{2,1}|_{x=y} = -P_1(x, \tau_1) - \mathbb{P}_2(x) \left( \Delta \left( 1 - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_{1}^T(x). \]

Similarly, we have

\[ 2P_{2,2}(x, y) = P_2(\xi, \tau_1) + \mathbb{P}_2(x) \left( \Delta \left( 1 - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_{2}^T(x - \xi) \]

so the coefficient of the 0th power of \( \xi \) is

\[ 2P_{2,2}|_{x=y} = E_2(\tau_1) + \mathbb{P}_2(x) \left( \Delta \left( 1 - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_{2}^T(x). \]

Finally, we have

\[ 2P_{2,4}(x, y) = P_4(\xi, \tau_1) + \mathbb{P}_2(x) \left( \Delta \left( 1 - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_{4}^T(x - \xi) \]

so the coefficient of the 0th power of \( \xi \) is

\[ 2P_{2,4}|_{x=y} = E_4(\tau_1) + \mathbb{P}_2(x) \left( \Delta \left( 1 - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_{4}^T(x). \]
so the coefficient of the 0th power of $\xi$ is

$$2^2\mathcal{P}_{2,4}|_{x=y} = E_4(\tau_1) + \mathbb{P}_2(x) \left( \Delta \left( \mathbbm{1} - \tilde{A}_2\tilde{A}_1 \right)^{-1} \tilde{A}_2 \right) \mathbb{P}_4^T(x).$$

Now, we have

$$Z_V^{(2)}(L[-2]|\bar{\omega}, x; \tau_1, \tau_2, \epsilon) = \left( [\mathbb{D}_x + 2\mathcal{P}_{2,1}|_{x=y}\partial_x + 2 \cdot 2\mathcal{P}_{2,2}|_{x=y}]\mathbb{D}_x - \frac{11}{5} \cdot 2\mathcal{P}_{2,4}|_{x=y} \right) Z_V^{(2)}(\tau_1, \tau_2, \epsilon).$$

Thus by the properties of the singular vector, we have

**Proposition 6.7.** The partition function for the (2, 5)-minimal model obeys the differential equation

$$\left( \frac{3}{10} \partial^2_x \mathbb{D}_x + \left( \mathbb{D}_x + 2\mathcal{P}_{2,1}|_{x=y}\partial_x + 2 \cdot 2\mathcal{P}_{2,2}|_{x=y} \right)\mathbb{D}_x - \frac{11}{5} \cdot 2\mathcal{P}_{2,4}|_{x=y} \right) Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = 0.$$

**Remark 6.3.** The partition function $Z_{V_{M_1,M_2}}^{(2)}(\tau_1, \tau_2, \epsilon)$ where $M_1, M_2$ are either of the modules for the (2, 5)-minimal model must obey the same differential equation.

**Remark 6.4.** An alternative approach to higher genus correlation functions with results specific to the Virasoro (2,5)-minimal model has been explored in recent work by M. Leitner [Le].

### 6.6 Conjectures

We now state a series of conjectures motivated by the work of this thesis.

**Conjecture 6.1.** The formal differential operator $\mathbb{D}_x$ is defined everywhere on $\mathcal{D}^\epsilon$ with

$$dx^2 \mathbb{D}_x = \nabla_x,$$

or equivalently, the map $F^\epsilon : \mathcal{D}^\epsilon \rightarrow \mathbb{H}_2$ is invertible everywhere on $\mathcal{D}^\epsilon$.

**Conjecture 6.2.** For all $N \geq 1$, the inverse matrix

$$\left( \mathbbm{1} - \tilde{A}_a\tilde{A}_a \right)^{-1} = \sum_{n \geq 0} \left( A_a\tilde{A}_a \right)^n$$

is convergent on $\mathcal{D}^\epsilon$.  

86
Conjecture 6.3. For all $N \geq 1$ and $j \geq 0$, the generalised Weierstrass functions $N P_{2,1+j}(x,y)$ are holomorphic for all $x \in \mathring{S}_a, y \in \mathring{S}_b$ where $x \neq y$.

Conjecture 6.4. The generalised Weierstrass functions $N P_{2,1+j}(x,y)$ and the differential equations of this section are geometrical in origin.

Conjecture 6.5. For $N > 1$, the $2N - 1$ terms described by Remark 2.2 form a basis of holomorphic $N$-forms on $\mathcal{D}^\nu$.

Conjecture 6.6. The genus two partition function for a $C_2$-cofinite VOA is convergent. In particular, a differential equation derived from Theorem 4.1 may be used to prove this.
References


