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Manipulating an ordering*

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Abstract

It is well known that many social decision procedures are manipulable through strategic behaviour. Typically, the decision procedures considered in the literature are social choice correspondences. In this paper we investigate the problem of constructing a social welfare function that is non-manipulable. In this context, individuals attempt to manipulate a social ordering as opposed to a social choice.

Using techniques from an ordinal version of fuzzy set theory, we introduce a class of ordinally fuzzy binary relations of which exact binary relations are a special case. Operating within this family enables us to prove an impossibility theorem. This theorem states that all non-manipulable social welfare functions are dictatorial, provided that they are not constant. This theorem generalizes the one in Perote-Peña and Piggins [Perote-Peña, J., Piggins, A., 2007. Strategy-proof fuzzy aggregation rules. J. Math. Econ. vol 43, p.564-p.580]. We conclude by considering several ways of circumventing this impossibility theorem.

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1 Introduction

The fact that many social decision procedures are manipulable through strategic behaviour is now well understood in the literature. Typically, the social decision procedures considered in the literature have been social choice correspondences. A social choice correspondence selects a nonempty subset of the feasible set of alternatives at each profile of individual preferences. If these chosen subsets contain exactly one element, then we obtain the special case of a social choice function.

In this paper, we investigate the problem of constructing a social welfare function that is non-manipulable. In this context, individuals attempt to manipulate a social ordering as opposed to a social choice.[1]

In order to illustrate the importance of strategic behaviour in this setting, consider the following example. Imagine that an academic department wishes to appoint a new member, in order to fill a vacant position. Five candidates have been interviewed for the position and each member of the department has been asked to rank them. In accordance with an established procedure, the chair of the department aggregates these individual rankings using the Borda rule and then uses this aggregate ranking to determine who is to receive the offer of appointment. Imagine that the chair makes the initial offer to the candidate who emerges at the top of this aggregate ranking (ties are broken by employing a device that gives each candidate an equal probability of being selected). If the top-ranked candidate rejects the offer then the chair either offers the position to the other tied candidate(s), or moves down the list to the second-placed candidate. This process continues until someone accepts the position.

When placing the candidates in order, each member of the department has no idea as to which of the candidates would accept the position in the event of it being offered to them. For instance, a candidate might decline the position if he or she has already received a better offer elsewhere. This means that it is not just the candidate(s) at the top of the aggregate ordering that matters to the individual department member, but the entire ordering itself becomes relevant.

Naturally, an individual member of the department could behave strategically in such circumstances and submit an insincere ranking as opposed to a sincere one. This is done in the hope that the aggregate ranking which emerges from such strategic behaviour is "closer" to the member's truthful ranking than would otherwise be the case. Therefore, studying the manipu-

[1] There has been hardly any treatment of manipulation in the context of social welfare functions. Exceptions are Pattanaik (1973) and Bossert and Storcken (1992).

lability of social welfare functions is important in its own right. This is the problem we analyze in this paper.

1.1 Preferences

An important feature of the paper is the assumption we make about preferences. Using techniques from an ordinal version of fuzzy set theory, we introduce a class of ordinally fuzzy binary relations (OFBRs) of which exact binary relations are a special case. We use these OFBRs to represent preferences, both individual and social. The assumption that preferences are represented by OFBRs gives us some mathematical generality. However, it can also be given an independent philosophical motivation. In order to illustrate this, let us return to our original example.

Suppose that a member of our hypothetical department is comparing two possible candidates, and what she cares about is how they fare with respect to teaching and research. Imagine that one of the candidates (candidate A) is better at research than the other (candidate B). To complicate matters, imagine that candidate A is worse at teaching than candidate B. How would our department member place these two candidates in order? Often it is hard to say, but not always.

For instance, imagine that candidate A is much better at research than candidate B and only slightly worse at teaching. In such cases, it seems reasonable to suppose that our department member would place A above B in her ranking. The reason for this is that most members of an academic department would be willing to trade-off slightly inferior teaching quality in order to acquire a colleague who is significantly better at research. In cases like this we say that preferences are "exact".

Unfortunately, things are not always this straightforward. For instance, what if candidate A is much worse at teaching than candidate B? In cases like this, it might be extremely difficult for our department member to place the two candidates in a clear order. She might feel that to some extent candidate A is better than candidate B. At the same time however, she might also feel that to some extent candidate B is better than candidate A. These conflicting feelings may be difficult to integrate into a clear expression of preference or indifference.

In cases like this we could perhaps describe our department member's preferences as "non-exact" or "vague". Fuzzy binary relations are a natural mathematical device for representing preferences that are either exact or non-exact.[2] To see how non-exactness can be expressed, recall our second

[2] An exact preference is an element of the set of fuzzy preferences.

example. In that example the department member feels that to some extent candidate A is better than candidate B, and yet at the same time she also feels that to some extent candidate B is better than candidate A. In the version of fuzzy set theory that we use in this paper, which originates in work by Goguen (1967), these "extents" are elements of a set L of which there are at least two members. Importantly, in Goguen's theory the elements of L are ordered (possibly incompletely) by a binary relation $\succeq$.

A special case of this framework is the standard version of fuzzy set theory. In the standard version L is taken to be $[0, 1]$ and the elements of L are ordered by $\geq$. This is sometimes referred to as the "cardinal" approach to fuzziness. Goguen pioneered the "ordinal" approach to fuzziness which is formally more general. This is the approach we adopt in this paper.[3]

An OFBR defined on a set of alternatives X is a function $f : X \times X \to L$. If the semantic concept the fuzzy relation f represents is (weak) preference then $f(x, y)$ can be interpreted as the degree of confidence that "x is at least as good as y". This is not the only possible interpretation of $f(x, y)$. It can be interpreted as the degree of truth of the sentence "x is at least as good as y". Others refer to it as the extent to which (or the degree to which) x is at least as good as y.

1.2 Outline

This paper is a contribution to the literature on social choice with fuzzy preferences.[4] This literature has been motivated by the idea that fuzziness can have a "smoothing" effect on preference aggregation and so perhaps the famous impossibility results of Arrow (1951) and others can be avoided.[5] Unfortunately, this is not always the case.[6] In fact, in this paper a very strong concept of dictatorship emerges.[7]

[3] Some arguments in favour of the ordinal approach can be found in Basu, Deb and Pattanaik (1992).

[4] A comprehensive survey of the literature is Salles (1998). The papers closest to this one are Barrett, Pattanaik and Salles (1992) and Basu, Deb and Pattanaik (1992). Both of these papers consider ordinal approaches to fuzziness. The former paper deals with social choice theory explicitly. A discussion of the underlying philosophical issues is contained in Piggins and Salles (2007).

[5] Our approach, like others in the literature, allows for social preference to be vague even if the underlying profile of individual preferences is exact. This is what we mean by smoothing. A similar suggestion is made by Sen (1970a).

[6] Barrett, Pattanaik and Salles (1986), Dutta (1987), Ovchinnikov (1991), Banerjee (1994), Billot (1995), Richardson (1998), Dasgupta and Deb (1999), and Fono and Andjiga (2005). See also Leclerc (1984, 1991) and Leclerc and Monjardet (1995).

[7] We do, however, establish several possibility results at the end of the paper. These involve weakening our transitivity assumption on social preferences. For one of these

We investigate the structure of social welfare functions which, for every permissible profile of fuzzy individual preferences, specify a fuzzy social preference. We show that all social welfare functions that are non-manipulable and not constant must be dictatorial. This means that there is an individual whose fuzzy preferences determine the entire fuzzy social ranking at every profile in the domain of the social welfare function. To prove this theorem, we show that all social welfare functions that are non-manipulable and not constant must satisfy counterparts of independence of irrelevant alternatives and a condition that is like the Pareto principle.

Of course, this result is a variant of the Gibbard-Satterthwaite theorem but in the context of social welfare functions with fuzzy preferences.[8] A proof of this theorem first appeared in Perote-Peña and Piggins (2007). However, in that paper L is taken to be $[0, 1]$ and the elements of L are ordered by $\geq$. The theorem in this paper generalises this earlier theorem in that L is any set with at least two elements and the elements of L are ordered by a binary relation $\succeq$.[9] Moreover, the earlier proof in the cardinal framework involved an unnecessarily complicated argument involving vectors. The proof of the more general theorem contained in this paper is considerably simpler.

2 Preliminaries

Environment

Let A be a set of social alternatives with $\#A \geq 3$.

Individuals

Let $N = \{1, ..., n\}$, $n \geq 2$, be a finite set of individuals.

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results we permit intransitive exact preference, which is also considered in the cardinal setting by Perote-Peña and Piggins (2008a).

[8]Gibbard (1973) and Satterthwaite (1975). An important precursor to the present study in the case of exact preferences is Pattanaik (1973). To the best of our knowledge, the only other papers that consider the manipulability problem in a fuzzy framework are Tang (1994), Côrte-Real (2007), Perote-Peña and Piggins (2007, 2008a, 2008b) and Abdelaziz, Figueira and Meddeb (2008). Barberà (2001) and Taylor (2005) are good introductions to the conventional social choice literature.

[9]The precise mathematical structure is described in the next section.

Degree structure

Let L be a set of degrees with $\#L \geq 2$. Let $\succeq$ be a reflexive, transitive and complete binary relation on L.[10] The asymmetric part of $\succeq$ is denoted by $\succ$ and the symmetric part is denoted by $\sim$. We assume that the sets $\{d^* \in L \mid d^* \succeq d \text{ for all } d \in L\}$ and $\{d_* \in L \mid d \succeq d_* \text{ for all } d \in L\}$ are non-empty. Moreover, these sets are assumed to be singletons with $d^* \succ d_*$.

d^* and d_* are the counterparts, respectively, of 1 and 0 in the cardinal theory.

Preferences

An ordinally fuzzy binary relation (OFBR) is a function $f : A \times A \to L$. An exact binary relation is an OFBR g such that $g(A \times A) \subseteq \{d_*, d^*\}$.

Let T denote the set of all OFBRs.

Let H be the set of all $r \in T$ which satisfy the following three conditions.

(i) For all $a \in A$, $r(a, a) = d^*$.
(ii) For all distinct $a, b \in A$, $r(a, b) = d_*$ implies that $r(b, a) = d^*$.
(iii) For all $a, b, c \in A$, $r(a, c) \succeq r(a, b)$ or $r(a, c) \succeq r(b, c)$.

The OFBRs in H will be interpreted as fuzzy weak preference relations. Thus if $r_i \in H$ is interpreted as the fuzzy weak preference relation of individual i, then $r_i(a, b)$ is to be interpreted as the degree to which individual i is confident that "a is at least as good as b".

Property (i) is the fuzzy counterpart of the traditional reflexivity condition, property (ii) is a completeness condition, and property (iii) is the ordinal version of the familiar max-min transitivity condition. The cardinal version of this condition was used in Perote-Peña and Piggins (2007). As is noted in that paper, this condition is somewhat controversial.[11] Despite this, max-min transitivity remains the most commonly used transitivity condition in the literature on fuzzy relations. Since our objective in this paper is to generalise the theorem in Perote-Peña and Piggins (2007), we leave to future research the task of systematically exploring weakenings of this condition. That said, we do demonstrate at the end of this paper that weakening transitivity can lead to possibility results.[12]

[10] Sen (1970b, p.8).

[11] Barrett and Pattanaik (1989) and Dasgupta and Deb (1996, 2001).

[12] It is perhaps worth comparing the conditions in this paper to the corresponding ones in Perote-Peña and Piggins (2007). In that paper a fuzzy binary relation α is a function from $\bar{A}$ to $[0, 1]$ where $\bar{A} = \{(a, b) \in A \times A \mid a \neq b\}$. For this reason there is no counterpart of condition (i) in that paper. The counterpart of condition (ii) is: for all distinct $a, b \in A, \alpha(a, b) + \alpha(b, a) \geq 1$. The counterpart of condition (iii) is: for all $a, b, c \in A, \alpha(a, c) \geq$

Note that within H are all (exact) reflexive, transitive and complete weak preference relations. The standard approach to preferences is, therefore, a special case.

Social welfare function

A social welfare function (SWF) is a function $\Psi : H^n \rightarrow H$.

Intuitively, an SWF specifies a fuzzy social weak preference relation given an n-tuple of fuzzy individual weak preference relations (one for each individual). The elements of H^n are indicated by $(r_1, ..., r_n)$, $(r_1^{'}, ..., r_n^{'})$, etc. We write $r = \Psi(r_1, ..., r_n)$, $r^{'} = \Psi(r_1^{'}, ..., r_n^{'})$ and so on (where Ψ is the SWF). We write $r(a, b)$ to denote the restriction of r to (a, b), and $r^{'}(a, b)$ to denote the restriction of $r^{'}$ to (a, b) and so on.

Manipulation

We need some way of prohibiting the profitable misrepresentation of a social welfare function. For expositional purposes assume for the moment that we are operating in the cardinal framework.

Take any pair of alternatives (a, b) and any preference profile at which you truthfully report your preferences. Imagine that at this profile the social welfare function assigns a larger social degree of confidence to (a, b) than the one you happen to hold. Then, if the social welfare function is non-manipulable, whenever you misrepresent your preferences the social degree assigned to (a, b) will either rise or remain constant. Conversely, if the social degree assigned to (a, b) is smaller than your individual (a, b) value, whenever you misrepresent your preferences the social degree assigned to (a, b) will either fall or remain constant. Loosely speaking, what this means is as follows. Whenever someone unilaterally switches from telling the truth to lying, the fuzzy social ranking moves "at least as far away" from their truthful ranking as was initially the case. In other words, whenever someone misrepresents their preferences, the "distance" between their truthful ranking and the social ranking (weakly) increases. In such circumstances, individuals do not gain by misrepresenting their preferences. We say that a social welfare function is non-manipulable if and only if it satisfies this property.[13]

Here are the relevant formal definitions, stated in the ordinal framework.

$\min\{\alpha(a, b), \alpha(b, c)\}$. The theorem in this paper shows that a weakening of condition (ii) is possible in the cardinal framework.

[13]Obviously there are other ways of formulating a non-manipulation condition for social welfare functions, but this one strikes us as a natural place to start. Weaker conditions are possible but inevitably they would be more controversial as conditions of non-manipulation. Our condition should be viewed as establishing a benchmark case.

We denote by $(r_1, .., r_i^{'}, .., r_n) \in H^n$ the profile obtained from $(r_1, .., r_i, .., r_n)$ when individual i replaces $r_i \in H$ with $r_i^{'} \in H$.

We write $r_{-i} \otimes r_i^{'} = \Psi(r_1, .., r_i^{'}, .., r_n)$ and $r_{-i} \otimes r_i^{'}\{a, b\}$ denotes the restriction of $r_{-i} \otimes r_i^{'}$ to (a, b). Similarly, $r_{-i-j} \otimes r_i^{'} \otimes r_j^{'} = \Psi(r_1, .., r_i^{'}, .., r_j^{'}, ..r_n)$ and $r_{-i-j} \otimes r_i^{'} \otimes r_j^{'}\{a, b\}$ denotes the restriction of $r_{-i-j} \otimes r_i^{'} \otimes r_j^{'}$ to (a, b).

An SWF Ψ is *non-manipulable* if and only if it satisfies the following property.
(NM) For all $(a, b) \in A \times A$, all $(r_1, ..., r_n) \in H^n$, all $i \in N$ and all $r_i' \in H$, both (i) and (ii) hold.
(i) $r(a, b) \prec r_i(a, b) \rightarrow r_{-i} \otimes r_i^{'}\{a, b\} \preceq r(a, b)$.
(ii) $r(a, b) \succ r_i(a, b) \rightarrow r_{-i} \otimes r_i^{'}\{a, b\} \succeq r(a, b)$.

To clarify the nature of this condition, consider Figure 1. Again, for expositional purposes, assume that we are operating in the cardinal framework.

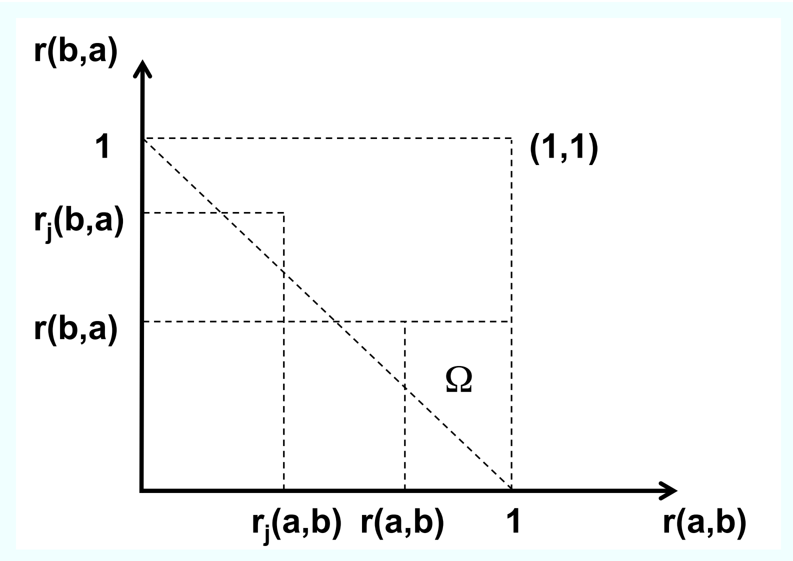

Figure 1: An illustration of the NM condition.

In Figure 1, we restrict attention to the ordered pair (a, b) and the ordered pair (b, a). Individual j's fuzzy preferences are denoted by the vector $(r_j(a, b)$, $r_j(b, a))$. Social preferences are denoted by the vector $(r(a, b),\ r(b, a))$. If individual j misrepresents her preferences, then the new vector of social values $(r^*(a, b),\ r^*(b, a))$ is constrained to lie in Ω.

Finally, we need a condition that eliminates trivially non-manipulable SWFs. Let $\bar{A} = \{(a, b) \in A \times A \mid a \neq b\}$.

SWFs that are not constant

An SWF Ψ is *not constant* if and only if it satisfies the following property.
(NC) For all $(a,b) \in \bar{A}$, there exists $(r_1,...,r_n) \in H^n$ such that $r(a,b) = d_*$.

This condition is mild and is reminiscent of the familiar non-imposition axiom in social choice theory. It implies, by virtue of the connectedness condition, that for each pair of social alternatives two profiles exist in the domain of the social welfare function that produce different social values in $\{d_*, d^*\}$ for this pair. This condition rules out social welfare functions that assign constant values to pairs of alternatives, irrespective of individual preferences.

Coalitions

Let P denote the set of all subsets of N. A non-empty subset of N is called a coalition. Given a coalition $C = \{i_1,...,i_m\}$ in which $i_1 < i_2 < ... < i_m$, and given $(r_1,...,r_n) \in H^n$, $r_C(a,b)$ denotes $(r_{i_1}(a,b),...,r_{i_m}(a,b)) \in L^m$.

Note that $r_N(a,b)$ denotes $(r_{i_1}(a,b),...,r_{i_n}(a,b)) \in L^n$. So given $N = \{1,...,n\}$, $r_{i_1}(a,b)$ in the vector $r_N(a,b)$ denotes individual 1's fuzzy preference over (a,b), $r_{i_n}(a,b)$ in the vector $r_N(a,b)$ denotes individual n's fuzzy preference over (a,b) and so on.

We write $r_C(a,b) \succeq r'_C(a,b)$ if $r_i(a,b) \succeq r'_i(a,b)$ for all $i \in C$. We write $r_C(a,b) \sim r'_C(a,b)$ if $r_i(a,b) \sim r'_i(a,b)$ for all $i \in C$. We write $r_C(a,b) \nsim r'_C(a,b)$ if there exists an $i \in C$ such that $r_i(a,b) \nsim r'_i(a,b)$.

Arrow-like properties

We now introduce some other properties that SWFs might satisfy.

Let $Max\,(r_N(a,b))$ denote the set $\{\overline{d} \in L \mid \exists i \in N$ with $r_i(a,b) \sim \overline{d}$ and $r_i(a,b) \succeq r_j(a,b)$ for all $j \in N - \{i\}\}$.

Let $Min\,(r_N(a,b))$ denote the set $\{\underline{d} \in L \mid \exists i \in N$ with $r_i(a,b) \sim \underline{d}$ and $r_j(a,b) \succeq r_i(a,b)$ for all $j \in N - \{i\}\}$.

An SWF Ψ is *Arrow-like* if and only if it satisfies the following two properties.
(IIA) For all $(r_1,...,r_n)$, $(r'_1,...,r'_n) \in H^n$, and all $(a,b) \in A \times A$,

$$r_N(a,b) \sim r'_N(a,b) \text{ implies } r(a,b) \sim r'(a,b).$$

(PC) For all $(r_1,...,r_n) \in H^n$, all $(a,b) \in A \times A$, all $\overline{d} \in Max\,(r_N(a,b))$, and all $\underline{d} \in Min\,(r_N(a,b))$,

$$\overline{d} \succeq r(a,b) \succeq \underline{d}.$$

Of course, IIA is a version of Arrow's (1951) independence of irrelevant alternatives condition. Similarly, PC is a Pareto-like condition.

Neutrality

An SWF Ψ is *neutral* if and only if it satisfies the following property. For all $(r_1, ..., r_n)$, $(r_1^{'}, ..., r_n^{'}) \in H^n$, and all $(a, b), (c, d) \in A \times A$,

$$r_N(a, b) \sim r_N^{'}(c, d) \text{ implies } r(a, b) \sim r^{'}(c, d).$$

Neutrality is a strengthening of independence. Loosely speaking, neutrality says that the names of the alternatives do not matter.

Dictatorship

An SWF Ψ is *dictatorial* if and only if there exists an individual $i \in N$ such that for all $(a, b) \in A \times A$, and for every $(r_1, ..., r_n) \in H^n$, $r_i(a, b) \sim r(a, b)$.

In order to explain this condition, let Ψ be a dictatorial SWF. Then there is an individual (the dictator) who can can ensure that at every profile in the domain of Ψ, the social degree of confidence for every pair of alternatives is in the same equivalence class (induced by $\sim$) as his or her own.

3 Theorem

We now state and prove our central result.

Theorem 1. *Any non-manipulable SWF that is not constant is dictatorial.*

The proof of this theorem involves a number of steps.[14]

Lemma 1. *Let Ψ be a non-manipulable SWF that is not constant. Then Ψ is Arrow-like.*

Proof. Let Ψ be a non-manipulable SWF that is NC. We start by proving that Ψ must satisfy IIA. Assume, by way of contradiction, that Ψ does not satisfy IIA. Therefore, $\exists (a, b) \in A \times A$ and $\exists (r_1, ..., r_n), (r_1^{'}, ..., r_n^{'}) \in H^n$ with

[14] Our proof uses an argument in Perote-Peña and Piggins (2008a) that simplifies the original proof in Perote-Peña and Piggins (2007). Both of these papers consider cardinal fuzziness and so the theorem presented here is more general.

$r_j(a,b) \sim r_j^{'}(a,b)\ \forall j \in N$ such that $r(a,b) \nsim r^{'}(a,b)$. Consider the following sequence of fuzzy preference profiles:

$$
\begin{aligned}
\mathbf{R}^{(0)} &= (r_1, ..., r_n), \\
\mathbf{R}^{(1)} &= (r_1', r_2, .., r_n), \\
\mathbf{R}^{(2)} &= (r_1', r_2', r_3, .., r_n), \\
&\ldots \\
\mathbf{R}^{(n)} &= (r_1', ..., r_n').
\end{aligned}
$$

Assume, without loss of generality, that $r(a,b) \sim k$ and $r^{'}(a,b) \sim k'$ with $k' \succ k$. First of all, compare $r_{-1} \otimes r_1^{'}\{a,b\}$ with $r(a,b)$. There are two possibilities.

Case 1. $r_{-1} \otimes r_1^{'}\{a,b\} \nsim k$. If $k \prec r_{-1} \otimes r_1^{'}\{a,b\} \preceq r_1(a,b)$ or $k \succ r_{-1} \otimes r_1^{'}\{a,b\} \succeq r_1(a,b)$ then NM is violated in the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$. Similarly, if $k \preceq r_1(a,b) \prec r_{-1} \otimes r_1^{'}\{a,b\}$ then NM is violated either in the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$ or in the move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(0)}$. If $k \succeq r_1(a,b) \succ r_{-1} \otimes r_1^{'}\{a,b\}$ then NM is violated either in the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$ or in the move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(0)}$. Finally, if $r_1(a,b) \succ k \succ r_{-1} \otimes r_1^{'}\{a,b\}$ or $r_1(a,b) \prec k \prec r_{-1} \otimes r_1^{'}\{a,b\}$ then NM is violated in the move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(0)}$. The only case remaining is Case 2.

Case 2. $k \sim r_{-1} \otimes r_1^{'}\{a,b\}$.

We now proceed to move from $\mathbf{R}^{(1)}$ to $\mathbf{R}^{(2)}$ by changing the fuzzy preferences of individual 2. However, we can treat this case in exactly the same manner as the move from $\mathbf{R}^{(0)}$ to $\mathbf{R}^{(1)}$ and so $r_{-1-2} \otimes r_1^{'} \otimes r_2^{'}\{a,b\} \sim k$. Repeating this argument for each individual ensures that when we reach $\mathbf{R}^{(n)}$ we have $r^{'}(a,b) \sim k$ which contradicts the assumption that $r^{'}(a,b) \sim k' \succ k$.

Therefore, Ψ satisfies IIA.

We now prove that Ψ satisfies PC.

First of all, we prove that Ψ satisfies the following property.

(*) For all $(r_1, ..., r_n), (r_1^{'}, ..., r_n^{'}) \in H^n$ and every $(a,b) \in A \times A$, (i) if $r_N(a,b) = (d_*, ..., d_*)$ then $r(a,b) = d_*$ and (ii) if $r_N^{'}(a,b) = (d^*, ..., d^*)$ then $r^{'}(a,b) = d^*$.[15]

To see that (*) holds note that NC implies that there exists $(r_1, ..., r_n) \in H^n$ such that $r(a,b) = d_*$. Let $(r_1^*, ..., r_n^*) \in H^n$ denote a profile such that $r_N^*(a,b) = (d_*, ..., d_*)$. If $r_N(a,b) = (d_*, ..., d_*)$ then (i) of (*) holds immediately by IIA. Assume that $r_N^*(a,b) \nsim r_N(a,b)$. Therefore, $\exists Q \subseteq N$ such that $r_j(a,b) \succ d_*$ for all $j \in Q$. Let $q \in Q$ and note that $r_{-q} \otimes r_q^*\{a,b\} = d_*$. If not, then NM is violated in the move from $(r_1, .., r_q, .., r_n) \in H^n$ to

[15]We write $r_N(a,b) = (d, ..., d)$ to denote that $r_N(a,b)$ is a n-vector of d's. Writing $r_N(a,b) \neq (d, ..., d)$ means that $r_N(a,b)$ is not a n-vector of d's.

$(r_1, .., r_q^*, .., r_n) \in H^n$. If $Q - \{q\}$ is non-empty then let $z \in Q - \{q\}$ and note that $r_{-q-z} \otimes r_q^* \otimes r_z^*\{a, b\} = d_*$. If not, then NM is violated in the move from $(r_1, .., r_q^*, r_z, .., r_n) \in H^n$ to $(r_1, .., r_q^*, r_z^*, .., r_n) \in H^n$. Simply repeating this argument for the remaining members of Q ensures that $r^*(a, b) = d_*$. Since $(r_1^*, ..., r_n^*) \in H^n$ is arbitrary, part (i) of (*) is proved.

The proof of part (ii) of (*) is similar and therefore is omitted. We now prove that Ψ satisfies PC.

We prove by contradiction. Assume that $\exists(a, b) \in A \times A$, $\exists(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$, and $\exists d \in Min\,(\widehat{r}_N(a, b))$ such that $\widehat{r}(a, b) \prec d$. Note that if $\widehat{r}_N(a, b) = (d^*, ..., d^*)$ then (*) implies that $\widehat{r}(a, b) = d^*$ and so $d^* \prec d$, a contradiction. So $\widehat{r}_N(a, b) \neq (d^*, ..., d^*)$.

Consider any fuzzy preference profile $(r_1^*, ..., r_n^*) \in H^n$ such that $r_i^*(a, b) = d^*$ for all $i \in N$.

Consider the following sequence of fuzzy preference profiles:

$$\begin{aligned}
\mathbf{G}^{(0)} &= (\widehat{r}_1, ..., \widehat{r}_n), \\
\mathbf{G}^{(1)} &= (r_1^*, \widehat{r}_2, .., \widehat{r}_n), \\
\mathbf{G}^{(2)} &= (r_1^*, r_2^*, \widehat{r}_3, .., \widehat{r}_n), \\
&... \\
\mathbf{G}^{(n)} &= (r_1^*, ..., r_n^*).
\end{aligned}$$

Consider $\mathbf{G}^{(1)}$. If $\widehat{r}_{-1} \otimes r_1^*\{a, b\} \succ \widehat{r}(a, b)$ then NM is violated in the move from $\mathbf{G}^{(0)}$ to $\mathbf{G}^{(1)}$. If $\widehat{r}_{-1} \otimes r_1^*\{a, b\} \prec \widehat{r}(a, b)$ then NM is violated in the move from $\mathbf{G}^{(1)}$ to $\mathbf{G}^{(0)}$. Therefore, $\widehat{r}_{-1} \otimes r_1^*\{a, b\} \sim \widehat{r}(a, b)$.

We can repeat this argument as we move from $\mathbf{G}^{(1)}$ to $\mathbf{G}^{(2)}$ and so $\widehat{r}_{-1-2} \otimes r_1^* \otimes r_2^*\{a, b\} \sim \widehat{r}(a, b)$. Again, repeating this argument for each individual ensures that when we reach $\mathbf{G}^{(n)}$ we have $r^*(a, b) \sim \widehat{r}(a, b)$. However, this contradicts (*) and so $\widehat{r}(a, b) \succeq d$ for all $d \in Min\,(\widehat{r}_N(a, b))$.

In order to complete the proof that Ψ satisfies PC, assume that $\exists(a, b) \in A \times A$, $\exists(\overline{r}_1, ..., \overline{r}_n) \in H^n$, and $\exists d \in Max\,(\overline{r}_N(a, b))$ such that $\overline{r}(a, b) \succ d$. Note that if $\overline{r}_N(a, b) = (d_*, ..., d_*)$ then (*) implies that $\overline{r}(a, b) = d_*$ and so $d_* \succ d$, a contradiction. So $\overline{r}_N(a, b) \neq (d_*, ..., d_*)$.

Consider any fuzzy preference profile $(r_1^{**}, ..., r_n^{**}) \in H^n$ such that $r_i^{**}(a, b) = d_*$ for all $i \in N$.

Consider the following sequence of fuzzy preference profiles:

$$\begin{aligned}
\mathbf{H}^{(0)} &= (\overline{r}_1, ..., \overline{r}_n),\\
\mathbf{H}^{(1)} &= (r_1^{**}, \overline{r}_2, .., \overline{r}_n),\\
\mathbf{H}^{(2)} &= (r_1^{**}, r_2^{**}, \overline{r}_3, .., \overline{r}_n),\\
&\;\;...\\
\mathbf{H}^{(n)} &= (r_1^{**}, ..., r_n^{**}).
\end{aligned}$$

Consider $\mathbf{H}^{(1)}$. If $\overline{r}_{-1} \otimes r_1^{**}\{a,b\} \prec \overline{r}(a,b)$ then NM is violated in the move from $\mathbf{H}^{(0)}$ to $\mathbf{H}^{(1)}$. If $\overline{r}_{-1} \otimes r_1^{**}\{a,b\} \succ \overline{r}(a,b)$ then NM is violated in the move from $\mathbf{H}^{(1)}$ to $\mathbf{H}^{(0)}$. Therefore, $\overline{r}_{-1} \otimes r_1^{**}\{a,b\} \sim \overline{r}(a,b)$.

We can repeat this argument as we move from $\mathbf{H}^{(1)}$ to $\mathbf{H}^{(2)}$ and so $\overline{r}_{-1-2} \otimes r_1^{**} \otimes r_2^{**}\{a,b\} \sim \overline{r}(a,b)$. Again, repeating this argument for each individual ensures that when we reach $\mathbf{H}^{(n)}$ we have $r^{**}(a,b) \sim \overline{r}(a,b)$. However, this contradicts (*) and so $\overline{r}(a,b) \preceq d$.

Therefore, Ψ satisfies PC.

We have proved that Ψ is Arrow-like. $\square$

Lemma 2. *An Arrow-like SWF Ψ is neutral.*

Proof. Case 1: If $(a,b) = (c,d)$ then the result follows immediately from the fact that Ψ is Arrovian.

Case 2: $(a,b),(a,d) \in A \times A$. Take $(r_1, ..., r_n) \in H^n$ such that $r_N(b,d) = (d^*, ..., d^*)$. Then PC implies that $r(b,d) = d^*$. Since r is max-min transitive, we have $r(a,d) \succeq r(a,b)$.

In addition, since $r_N(b,d) = (d^*, ..., d^*)$ and individual preferences are max-min transitive, it follows that $r_N(a,d) \succeq r_N(a,b)$.

Select a profile $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ such that $\overline{r}_N(b,d) = (d^*, ..., d^*)$ and $\overline{r}_N(d,b) = (d^*, ..., d^*)$. From the argument above we know that $\overline{r}(a,d) \succeq \overline{r}(a,b)$ and $\overline{r}_N(a,d) \succeq \overline{r}_N(a,b)$. However, an identical argument shows that $\overline{r}(a,b) \succeq \overline{r}(a,d)$ and $\overline{r}_N(a,b) \succeq \overline{r}_N(a,d)$. Therefore, it must be the case that $\overline{r}(a,b) \sim \overline{r}(a,d)$ and $\overline{r}_N(a,b) \sim \overline{r}_N(a,d)$.

Since $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1, ..., r_n) \in H^n$ such that $r_N(b,d) = (d^*, ..., d^*)$ and $r_N(d,b) = (d^*, ..., d^*)$. Let F^n denote the set of such profiles. Take any profile $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ such that $\widehat{r}_N(a,b) \sim \widehat{r}_N(a,d)$. Then there exists a profile $(r_1^{'}, ..., r_n^{'}) \in F^n$ such that $\widehat{r}_N(a,b) \sim \widehat{r}_N(a,d) \sim r_N^{'}(a,b) \sim r_N^{'}(a,d)$. IIA implies that $\widehat{r}(a,b) \sim \widehat{r}(a,d) \sim r^{'}(a,b) \sim r^{'}(a,d)$.

Take any pair of distinct profiles $(r_1^{''}, ..., r_n^{''}), (r_1^{*}, ..., r_n^{*}) \in H^n$ such that $r_N^{''}(a,b) \sim r_N^{*}(a,d)$. Then there exists a profile $(r_1^{**}, ..., r_n^{**}) \in F^n$ such that

$r''_N(a,b) \sim r^*_N(a,d) \sim r^{**}_N(a,b) \sim r^{**}_N(a,d)$. IIA implies that $r''(a,b) \sim r^*(a,d) \sim r^{**}(a,b) \sim r^{**}(a,d)$.

Case 3: $(a,b),(c,b) \in A \times A$. Take $(r_1,...,r_n) \in H^n$ such that $r_N(a,c) = (d^*,...,d^*)$. Then PC implies that $r(a,c) = d^*$. Since r is max-min transitive, we have $r(a,b) \succeq r(c,b)$.

In addition, since $r_N(a,c) = (d^*,...,d^*)$ and individual preferences are max-min transitive, it follows that $r_N(a,b) \succeq r_N(c,b)$.

Select a profile $(\overline{r}_1,...,\overline{r}_n) \in H^n$ such that $\overline{r}_N(a,c) = (d^*,...,d^*)$ and $\overline{r}_N(c,a) = (d^*,...,d^*)$. From the argument above we know that $\overline{r}(a,b) \succeq \overline{r}(c,b)$ and $\overline{r}_N(a,b) \succeq \overline{r}_N(c,b)$. However, an identical argument shows that $\overline{r}(c,b) \succeq \overline{r}(a,b)$ and $\overline{r}_N(c,b) \succeq \overline{r}_N(a,b)$. Therefore, it must be the case that $\overline{r}(a,b) \sim \overline{r}(c,b)$ and $\overline{r}_N(a,b) \sim \overline{r}_N(c,b)$.

Since $(\overline{r}_1,...,\overline{r}_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1,...,r_n) \in H^n$ such that $r_N(a,c) = (d^*,...,d^*)$ and $r_N(c,a) = (d^*,...,d^*)$. Let G^n denote the set of such profiles. Take any profile $(\widehat{r}_1,...,\widehat{r}_n) \in H^n$ such that $\widehat{r}_N(a,b) \sim \widehat{r}_N(c,b)$. Then there exists a profile $(r'_1,...,r'_n) \in G^n$ such that $\widehat{r}_N(a,b) \sim \widehat{r}_N(c,b) \sim r'_N(a,b) \sim r'_N(c,b)$. IIA implies that $\widehat{r}(a,b) \sim \widehat{r}(c,b) \sim r'(a,b) \sim r'(c,b)$.

Take any pair of distinct profiles $(r''_1,...,r''_n),(r^*_1,...,r^*_n) \in H^n$ such that $r''_N(a,b) \sim r^*_N(c,b)$. Then there exists a profile $(r^{**}_1,...,r^{**}_n) \in G^n$ such that $r''_N(a,b) \sim r^*_N(c,b) \sim r^{**}_N(a,b) \sim r^{**}_N(c,b)$. IIA implies that $r''(a,b) \sim r^*(c,b) \sim r^{**}(a,b) \sim r^{**}(c,b)$.

Case 4: $(a,b),(c,d) \in A \times A$ with a,b,c,d distinct. Take $(r_1,...,r_n) \in H^n$ such that $r_N(b,d) = r_N(d,b) = r_N(a,c) = r_N(c,a) = (d^*,...,d^*)$. Then PC implies that $r(d,b) = d^*$. Since r is max-min transitive, we have $r(a,b) \succeq r(a,d)$. However, an identical argument shows that $r(a,d) \succeq r(a,b)$ and so $r(a,b) \sim r(a,d)$.

In addition, since $r_N(d,b) = r_N(b,d) = (d^*,...,d^*)$ and individual preferences are max-min transitive, it follows that $r_N(a,b) \sim r_N(a,d)$.

We can repeat this argument to show that $r(a,d) \sim r(c,d)$ and $r_N(a,d) \sim r_N(c,d)$. Since $(r_1,...,r_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1,...,r_n) \in H^n$ such that $r_N(b,d) = r_N(d,b) = r_N(a,c) = r_N(c,a) = (d^*,...,d^*)$. Let J^n denote the set of such profiles.

Take any profile $(\widehat{r}_1,...,\widehat{r}_n) \in H^n$ such that $\widehat{r}_N(a,b) \sim \widehat{r}_N(c,d)$. Then there exists a profile $(r'_1,...,r'_n) \in J^n$ such that $\widehat{r}_N(a,b) \sim \widehat{r}_N(c,d) \sim r'_N(a,b) \sim r'_N(c,d)$. IIA implies that $\widehat{r}(a,b) \sim \widehat{r}(c,d) \sim r'(a,b) \sim r'(c,d)$.

Take any pair of distinct profiles $(r''_1,...,r''_n),(r^*_1,...,r^*_n) \in H^n$ such that $r''_N(a,b) \sim r^*_N(c,d)$. Then there exists a profile $(r^{**}_1,...,r^{**}_n) \in J^n$ such that $r''_N(a,b) \sim r^*_N(c,d) \sim r^{**}_N(a,b) \sim r^{**}_N(c,d)$. IIA implies that $r''(a,b) \sim r^*(c,d) \sim r^{**}(a,b) \sim r^{**}(c,d)$.

Case 5: $(a,b),(b,a) \in A \times A$. Take any profile $(r_1,...,r_n) \in H^n$ such that $r_N(a,b) \sim r_N(a,c) \sim r_N(b,c) \sim r_N(b,a)$. Cases (2) and (3) imply that $r(a,b) \sim r(a,c) \sim r(b,c) \sim r(b,a)$. Let W^n denote the set of such profiles. Take any profile $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ such that $\overline{r}_N(a,b) \sim \overline{r}_N(b,a)$. Then there exists a profile $(r_1^{'}, ..., r_n^{'}) \in W^n$ such that $\overline{r}_N(a,b) \sim \overline{r}_N(b,a) \sim r_N^{'}(a,b) \sim r_N^{'}(b,a)$. IIA implies that $\overline{r}(a,b) \sim \overline{r}(b,a) \sim r^{'}(a,b) \sim r^{'}(b,a)$.

Take any pair of distinct profiles $(r_1^{''}, ..., r_n^{''}), (r_1^*, ..., r_n^*) \in H^n$ such that $r_N^{''}(a,b) \sim r_N^*(b,a)$. Then there exists a profile $(r_1^{**}, ..., r_n^{**}) \in W^n$ such that $r_N^{''}(a,b) \sim r_N^*(b,a) \sim r_N^{**}(a,b) \sim r_N^{**}(b,a)$. IIA implies that $r^{''}(a,b) \sim r^*(b,a) \sim r^{**}(a,b) \sim r^{**}(b,a)$. $\square$

Given that Ψ is neutral we can complete the proof in the following way. Take any $(a,b) \in A \times A$ and any profile $(r_1, ..., r_n) \in H^n$ such that $r_N(a,b) = (d_*, ..., d_*)$. By PC it must be the case that $r(a,b) = d_*$. Take some other profile $(r_1^*, ..., r_n^*) \in H^n$ such that $r_N^*(a,b) = (d^*, ..., d^*)$. By PC it must be the case that $r^*(a,b) = d^*$.

Consider the following sequence of fuzzy preference profiles:

$$
\begin{aligned}
\mathbf{W}^{(0)} &= (r_1, ..., r_n), \\
\mathbf{W}^{(1)} &= (r_1^*, r_2, .., r_n), \\
\mathbf{W}^{(2)} &= (r_1^*, r_2^*, r_3, .., r_n), \\
&\cdots \\
\mathbf{W}^{(n)} &= (r_1^*, ..., r_n^*).
\end{aligned}
$$

At some profile in this sequence, the social value assigned to (a,b) must rise from d_* to a degree d such that $d \succ d_*$. By PC, the latest this can happen is when we reach $\mathbf{W}^{(n)}$. We shall assume, without loss of generality, that this happens at $\mathbf{W}^{(2)}$ when individual 2 raises her (a,b) value from d_* to d^*.

Now consider the profile $\mathbf{W}^{(\alpha)} = (r_1^*, r_2, r_3^*, .., r_n^*)$. We claim that the social value of (a,b) at this profile is d_*. To see this note that, by assumption, the social value of (a,b) at $\mathbf{W}^{(1)}$ is d_*. We can construct a profile $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ in which individuals have the following preferences over three alternatives a, b and c. Individual preferences over (a,b) at this profile are the same as they are over (a,b) at $\mathbf{W}^{(\alpha)}$. Individual preferences over (a,c) at this profile are the same as they are over (a,b) at $\mathbf{W}^{(1)}$. Finally, individual preferences over (b,c) at this profile are the same as they are over (a,b) at $\mathbf{W}^{(2)}$. We write $aRb \longleftrightarrow \widehat{r}(a,b) = d^*$ and $aPb \longleftrightarrow aRb \wedge \widehat{r}(b,a) = d_*$. Therefore at $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ individuals hold the following preferences:

Individual 1: $aRbRc$

Individual 2: $bRcPa$

Everyone else: $cPaRb$.

Neutrality implies that $\widehat{r}(a,b)$ is identical to the value (a,b) takes at $\mathbf{W}^{(\alpha)}$. Similarly, it implies that $\widehat{r}(b,c) = d \succ d_*$ and that $\widehat{r}(a,c) = d_*$. Note that by max-min transitivity $\widehat{r}(a,c) \succeq \widehat{r}(a,b)$ or $\widehat{r}(a,c) \succeq \widehat{r}(b,c)$ and so $\widehat{r}(a,b) = d_*$. Therefore, the social value (a,b) takes at $\mathbf{W}^{(\alpha)}$ is d_*. At $\mathbf{W}^{(\alpha)}$ individual 2 assigns the value d_* to (a,b) but everyone else assigns the value d^*. Despite this, the social value of (a,b) is d_*. Neutrality implies that this will remain the case whenever these preferences are replicated over any other pair of distinct social alternatives at any profile.

Now consider any profile $\mathbf{W}^{(\alpha\alpha)} = (\overline{r}_1, ..., \overline{r}_n) \in H^n$ in which individual 2 assigns the value d_* to some pair of distinct social alternatives. Furthermore, at this profile, individual 1 assigns the value d_1 to this pair, individual 3 assigns the value d_3 to this pair, and so on with $d_1, d_3, ..., d_n \in L$. The NM condition implies that the social value assigned to this pair must remain d_* for all $d_1, d_3, ..., d_n \in L$.

Let us now return to $\mathbf{W}^{(\alpha)}$. To recall, individual 2 assigns the value d_* to (a,b) at this profile but everyone else assigns the value d^*. Despite this, the social value of (a,b) is d_*. Completeness implies that, at this profile, individual 2 must assign the value d^* to (b,a) and so must society. This is true irrespective of everyone else's (b,a) value. Neutrality implies that any profile $\mathbf{W}^{(\beta\beta)} = (\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ in which individual 2 assigns the value d^* to some pair of distinct social alternatives, and in which individual 1 assigns the value d_1^* to this pair, individual 3 assigns the value d_3^* to this pair, and so on with $d_1^*, d_3^*, ..., d_n^* \in L$, must be consistent with the social welfare function assigning a social value of d^* to this pair.

To see that individual 2 is a dictator, fix some ordered pair (a,b). By the above argument, whenever individual 2 assigns a value of d^* to this pair then so must society, irrespective of everyone else's (a,b) value. Imagine now that individual 2 changes his or her (a,b) value to some value in $L - \{d^*\}$. If this value is d_* then the social value of (a,b) must be d_* due to the argument above about $\mathbf{W}^{(\alpha\alpha)}$. Imagine that individual 2 selects a value v where $d^* \succ v \succ d_*$. Let d denote the social value of (a,b) at this profile. If $d \succ v$ then individual 2 can profitably misrepresent by lowering his or her value to d_*. If $v \succ d$ then individual 2 can profitably misrepresent by changing his or her value to d^*. Neither of these things can happen and so $d \sim v$.

We have demonstrated at every profile in the domain of the social welfare function Ψ, individual 2 can ensure that the social degree of confidence that "a is at least as good as b" is always in the same equivalence class as his or her own. Since Ψ is neutral, individual 2 is a dictator.

This completes the proof of the theorem.

4 Discussion

It is worth pointing out that Arrow's theorem can be viewed as a special case of the theorem above. It corresponds to the case where the cardinality of L is 2.

One way of circumventing the impossibility theorem is to relax the assumption that exact social preference is transitive.[16] This enables us to state the following, rather trivial, theorem.

Theorem 2. *There exists a function $\Phi : H^n \to T$ that is non-manipulable, not constant and not dictatorial.*

Proof. Define the function $\Phi : H^n \to T$ as follows. For all $(a, b) \in A \times A$ and all $(r_1, ..., r_n) \in H^n$, $r(a, b) = Max\,(r_N(a, b))$. This function is non-manipulable, not constant and not dictatorial. □

Are there functions that satisfy our normative properties without resorting to social intransitivity? Such functions would be much more attractive than the one proposed in the theorem above. In order to answer this question, we need to introduce a new set of preferences and use this set to expand the set of social welfare functions.

Let D be the set of all $r \in T$ which satisfy the following three conditions.

(i) For all $a \in A$, $r(a, a) = d^*$.
(ii) For all distinct $a, b \in A$, $r(a, b) = d_*$ implies that $r(b, a) = d^*$.
(iii*) For all $a, b, c \in A$, $r(a, b) = d^*$ and $r(b, c) = d^*$ implies that $r(a, c) = d^*$.

Note that $H \subseteq D \subseteq T$. These conditions on preferences are identical to our earlier ones with the exception of (iii*). Condition (iii*) is the weakest possible transitivity condition that respects transitive exact preference. D^n is, therefore, the largest possible domain of fuzzy preferences.

However, for our next theorem, we only require that the co-domain of the social welfare function is D. For consistency, we shall keep as our domain H^n. This is because enlarging the co-domain is all that is required to generate a possibility result.

Theorem 3. *Assume that the cardinality of L is 3. Then there exists a function $\Xi : H^n \to D$ that is non-manipulable, not constant and not dictatorial.*

[16]This is actually Peter Fishburn's position expressed in Fishburn (1970). Fishburn argues that the idea of a social welfare function is untenable since it assumes social transitivity. Fishburn suggests that transitivity is a much less appealing assumption than Arrow's independence condition. Of course, if we accept this line of reasoning then much of traditional social choice theory loses its paradoxical character. Despite this, an interpretation of the classic impossibility theorems in terms of social choice functions would still be possible and this may be, in fact, what Fishburn is implicitly arguing for.

Proof. Define the function $\Xi : H^n \to D$ as follows. For all $(a, b) \in A \times A$ and all $(r_1, ..., r_n) \in H^n$, if $Max\,(r_N(a, b)) = Min\,(r_N(a, b)) = \alpha \in \{d^*, d_*\}$ then $r(a, b) = \alpha$, otherwise $r(a, b) = L - \{d^*, d_*\}$. This function is non-manipulable, not constant and not dictatorial. □

This social welfare function is very simple to describe. It respects, for each pair of alternatives, unanimous exact preference whenever it exists. In the event that it does not exist, then society assigns the value $L - \{d^*, d_*\}$ to this pair.

To give an interpretation to this social welfare function, suppose that the elements of L correspond to degrees of truth in a three-valued logic.[17] Then $r(a, b) = d^*$ means that the proposition "a is socially at least as good as b" is true, $r(a, b) = d_*$ means that the proposition is false, and $r(a, b) = L - \{d^*, d_*\}$ means that the proposition is neither true nor false. Put simply, social preference is vague whenever unanimous exact preference is absent.

This social welfare function shares some conceptual similarities with Sen's "Pareto-extension" rule.[18] However, there are some important differences. For one thing, vagueness replaces indifference whenever people in society hold conflicting preferences.[19] More significantly, this social welfare function is transitive. As is well-known, Sen's rule is quasi-transitive but not fully transitive, and the above rule does not suffer from this defect.[20]

That said, one weakness with the social welfare function above is that it requires the cardinality of L to be 3. If the cardinality of L is 4 or larger, then the function above is manipulable (provided that the degree society assigns in the absence of unanimity is always the same).[21]

However, the following social welfare function remedies this deficiency. In fact, it *coincides* with the function above whenever the cardinality of L is 3 and so it can be viewed as a generalisation. In order to describe this new function, we need to formally define the concept of a median. In the following definition $\mathbb{N}$ denotes the set of natural numbers.

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[17] Williamson (1994) contains a detailed discussion of these kinds of logics.

[18] Sen (1970b), Gaertner (2006).

[19] Note that this is not the same thing as social incomparability which would require both $r(a, b) = d_*$ and $r(b, a) = d_*$. The difference between vagueness and incomparability is discussed in Broome (1997) and Piggins and Salles (2007).

[20] In the language of our theory, Sen's rule is a function $\Lambda : H^n \to T$ defined as: for all $(a, b) \in A \times A$ and all $(r_1, ..., r_n) \in H^n$, if $r_N(a, b) = (d_*, ..., d_*)$ then $r(a, b) = d_*$, otherwise $r(a, b) = d^*$.

[21] This degree can be interpreted as a social "disagreement" point.

Median

For all $m \in \mathbb{N} - \{0\}$ and all $(d_1, \ldots, d_k) \in L^k$ where $k = 2m - 1$, let $Med(d_1, \ldots, d_k)$ denote the set $\{d \in L \mid \exists\{x_1, \ldots, x_k\} = \{1, \ldots, k\}$ with $d_{x_1} \succeq d_{x_2} \succeq \ldots \succeq d_{x_k}$ and $d \sim d_{x_m}\}$.

It might help to translate this formal definition into English. Imagine that m is 2 and so k is 3. L^3 is the set of all logically possible combinations of 3 degrees. Since $\succeq$ is complete, we can index these 3 degrees by x_1, x_2 and x_3 according to the relationship $d_{x_1} \succeq d_{x_2} \succeq d_{x_3}$. Since m is 2, the median degree is the set containing d_{x_2} and all degrees in the same equivalence class as d_{x_2}. The reader will be able to see that this procedure can be applied to numbers larger than 2.

We now state our final possibility theorem, which can be viewed as a generalisation of theorem 3.

Theorem 4. *Assume that the cardinality of L is 3 or more. Then there exists a function $\Gamma : H^n \to D$ that is non-manipulable, not constant and not dictatorial.*

Proof. Define the function $\Gamma : H^n \to D$ as follows. For all $(a, b) \in A \times A$ and all $(r_1, ..., r_n) \in H^n$, $r(a, b) \in Med(\overline{d}, d, \underline{d})$ where $\overline{d} \in Max\,(r_N(a, b))$, $\underline{d} \in Min\,(r_N(a, b))$ and $d \in L - \{d^*, d_*\}$. This function is non-manipulable, not constant and not dictatorial. $\square$

It should be clear that this function coincides with our earlier one whenever the cardinality of L is 3. In such a case, the elements of L must be d^*, d_* and $d = L - \{d^*, d_*\}$. As before the social welfare function respects, for each pair of alternatives, unanimous exact preference whenever it exists. It assigns the disagreement value $L - \{d^*, d_*\}$ in all other cases.

Now let us consider how this social welfare function performs whenever the cardinality of L exceeds 3. First of all, note that this function respects both unanimous exact preference and unanimous inexact preference. To demonstrate the latter, note that if $d_\alpha = Max\,(r_N(a, b)) = Min\,(r_N(a, b))$ and $d_\alpha \notin \{d^*, d_*\}$ then $d_\alpha = Med(d_\alpha, d, d_\alpha)$ provided that $d_\alpha \nsim d$. Secondly, the disagreement value itself varies; it is not fixed at $L - \{d^*, d_*\}$. Consider a profile that is consistent with $\overline{d} \succ \underline{d} \succ d$. At this profile the social value of (a, b) is $\underline{d}$, not d. In other words, the disagreement point is specific to a profile and not fixed to a particular degree. To put the matter somewhat loosely, in the absence of unanimity society adopts the value d unless there is a consensus that this value is too low or too high. In this sense, unlike our earlier formulation, the disagreement point responds to individual preferences.

It should be emphasised that all of the social welfare functions discussed in this section satisfy IIA and PC. In fact any function from H^n to T that satisfies NC and NM *must* satisfy IIA and PC. The reader will notice from lemma 1 that the co-domain of the social welfare function is irrelevant for establishing this result.

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