<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Waves, wrinkles and creases in deformed soft solids</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Mangan, Robert</td>
</tr>
<tr>
<td><strong>Publication Date</strong></td>
<td>2018-07-10</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>NUI Galway</td>
</tr>
<tr>
<td><strong>Item record</strong></td>
<td><a href="http://hdl.handle.net/10379/7435">http://hdl.handle.net/10379/7435</a></td>
</tr>
</tbody>
</table>
WAVES, WRINKLES AND CREASES
IN DEFORMED SOFT SOLIDS

PhD Thesis

by

Robert Mangan

Supervisor: Professor Michel Destrade

School of Mathematics, Statistics and Applied Mathematics
Discipline of Applied Mathematics
National University of Ireland, Galway

July 2018
Abstract

This article-based thesis comprises a collection of four articles, each of which constitutes a chapter written and formatted in pre-print manuscript form. The general aim underlying these articles is to model the large deformations of soft solids including soft biological tissues, with particular interest in wrinkle and crease formation, and in wave propagation. These wrinkles and creases occur frequently in nature, often as a result of an instability of a finite deformation. Waves can be used to characterise the mechanical properties of a soft solid, including biological soft tissues. These soft tissues may undergo large deformations in service or in testing, so we study the propagation of waves in a soft solid subject to an underlying finite deformation.
Contents

Declaration vii

Acknowledgement ix

Introduction 1

References ................................................. 7

1 Strain energy function for isotropic non-linear elastic incompressible solids with linear finite strain response in shear and torsion 11

1.1 Introduction ........................................ 12

1.2 Results ........................................... 13

1.3 Conclusion ........................................ 14

References ................................................. 15

Corrigendum .............................................. 16

Supplementary material .................................. 16

2 Guided waves in pre-stressed hyperelastic plates and tubes: Application to the ultrasound elastography of thin-walled soft materials 19

2.1 Introduction ........................................ 20

2.2 Theoretical analysis ............................... 22

2.2.1 Governing equations for waves in the pre-stressed hyperelastic plate ............................. 22

2.2.2 Dispersion analysis of the guided waves ........ 24

2.3 Finite element simulations ......................... 27

2.3.1 Comparison with the theoretical solutions ... 27

2.3.2 Guided circumferential waves (GCWs) in pre-stressed tubes ................................. 28

2.4 Experiments on phantom gels ....................... 30

2.5 Discussion ......................................... 34

2.6 Conclusion ......................................... 35

References ................................................. 36

3 Wrinkles in the opening angle method 43

3.1 Introduction ......................................... 44

3.2 The opening angle method .......................... 45

3.3 Wrinkling of a coated sector ....................... 48

3.4 Experimental & numerical results ................. 50
## CONTENTS

3.4.1 Results for polymers ........................................ 50
3.4.2 Results for soft tissues ................................. 51
3.5 Discussion .................................................. 53
Appendix: Derivation of the target condition .............. 55
References ....................................................... 56

4 Wrinkles and creases in the bending, unbending and eversion of soft sectors 59
  4.1 Introduction .................................................. 60
  4.2 Large bending, unbending and eversion ................. 62
  4.3 Existence, uniqueness and thin-wall expansion ......... 65
  4.4 Wrinkles ..................................................... 66
  4.5 Numerical results for wrinkles ............................ 67
  4.6 Numerical results for creases ............................. 74
Appendix A: Proofs of existence and uniqueness; Thin-wall expansions .................................................. 78
  4.6.1 Existence and uniqueness ............................... 78
  4.6.2 Thin-walled sectors ....................................... 81
Appendix B: Algorithms for the analysis of the Stroh problem 82
References ....................................................... 84

Conclusion ........................................................ 89
References ....................................................... 90
Declaration

I declare that the work in this thesis is my own, or, in the case of Chapters 1-4, joint work with my co-authors. I have not obtained a degree in this University or elsewhere on the basis of this work. My contributions to each chapter are as follows:

Chapter 1: I wrote the first draft of the article, proposed we extend the article to include torsion as well as simple shear, did the curve fitting, created all of the figures, came up with examples of the generalized Mooney-Rivlin model and found the connections of these models with fourth-order elasticity.

Chapter 2: I did the non-linear implicit curve fitting for the experiments on phantom gels and I (jointly) contributed to the Theoretical Analysis section. I also checked the results of our Chinese collaborators.

Chapter 3: I performed the stability analysis for polymers and soft tissues and wrote most of the corresponding section and I (jointly) contributed to the sections on the opening angle method and wrinkling of a coated sector.

Chapter 4: I performed the finite element simulations (using ABAQUS on a supercomputer) of bending, unbending and eversion in a cylindrical sector, and wrote much of the corresponding section.
Acknowledgment

I thank my supervisor, Michel Destrade, for his support and guidance throughout my PhD. I also thank all of my collaborators who helped with the articles presented in this thesis, and Valentina Balbi for helpful discussions. Finally, I thank the Irish Research Council for their financial support.
Introduction

About this thesis

This article-based thesis consists of four articles, each of which constitutes its own chapter. The general aim of the thesis is to study large deformations of soft solids, with particular interest in wrinkle and crease formation, and in wave propagation.

The four articles have been peer-reviewed and published in international scientific journals [2-5]. I have published one other article which is not included in this thesis [1]. Here is a list of my peer-reviewed publications so far:


Background

Much progress was made in the topic of rubber elasticity in the 1940s and 1950s by Rivlin [1] [2], who studied and modelled deformations of rubber-like materials. He modelled the solid as hyperelastic (its constitutive relation can be derived from a strain energy function) and incompressible (its volume remains unchanged at all times). While no solid behaves in exactly this way, experiments show that it is a reasonable assumption in modelling many rubber-like materials and soft biological tissues. This theory was successfully used to accurately predict a number of physical phenomena [3], and eventually the theory was extended to a general theory of non-linear elasticity [4]. In recent decades much work has been put into applying this theory to biomechanics [5]. For example, it has been used to model biological tissues such as arteries [6] and the brain [7].

In non-linear elasticity the equations governing the motion of a continuum include the equation of mass conservation,

\[ \dot{\rho} + \rho \text{div} \mathbf{v} = 0, \]  

(1)

the equation of motion

\[ \text{div} \mathbf{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \]  

(2)

and the equation of angular momentum balance

\[ \mathbf{\sigma}^T = \mathbf{\sigma}, \]  

(3)

where \( \mathbf{\sigma} \) is the Cauchy stress tensor, \( \mathbf{b} \) is the body force, \( \rho \) is the mass density, \( \mathbf{v} \) is the velocity, and the dot denotes the material time derivative. In addition, constitutive equations are needed to solve the system in full. For incompressible isotropic hyperelastic materials which are of interest to us, the stress is derived from the strain-energy density function \( W = W(I_1, I_2) \) where \( I_1, I_2 \) are principal invariants of the left Cauchy-Green deformation tensor \( \mathbf{B} = \mathbf{F}^T \mathbf{F} \), where \( \mathbf{F} \) is the deformation gradient. The relation is given by [8]

\[ \mathbf{\sigma} = -p \mathbf{I} + 2W_1 \mathbf{B} - 2W_2 \mathbf{B}^{-1}, \]  

(4)

where \( p \) is a Lagrange multiplier associated with the constraint of incompressibility, i.e., \( I_3 = 1 \) where \( I_3 = (\text{det} \mathbf{F})^2 = \text{det} \mathbf{B} \) is the third principal invariant of \( \mathbf{B} \).

For the class of materials defined by (4) there exist six families of universal solutions [9], i.e. a solution that is independent of the form of the constitutive law. Examples of universal solutions include all homogeneous deformations (e.g. extension of a rectangular block, simple shear of a rectangular block) and inhomogeneous deformations such as bending of a rectangular block, and torsion of a cylinder, and inflation of a cylindrical tube. Experiments can be performed based on these deformations, leading to a stress-strain curve. This curve can be fitted with the theoretical stress relation for a given constitutive model, which allows for the evaluation of the best-fit material parameters.
The theory of nonlinear elasticity has been extended to include small “incremental” motions and deformations [10]. In the incremental theory of elasticity, or "small-on-large" theory, the small disturbance, such as a wave or static wrinkle, is modelled as an infinitesimal perturbation imposed onto a large static deformation of a hyperelastic solid, see Fig. 3. The resulting equations of motion can be linearised in the neighbourhood of a finite static deformation, to obtain

$$\text{div}\Sigma = \rho \ddot{u},$$

where \(u\) is the displacement, \(\Sigma = \mathcal{A}_0 - \dot{p}I + p\Gamma\), \(\Gamma\) is the incremental displacement gradient, \(\dot{p}\) is an arbitrary scalar, and \(\mathcal{A}_0\) is a fourth tensor whose components are the instantaneous elastic moduli

$$\mathcal{A}_{ijkl} = \frac{\partial^2 W}{\partial F_{ia} \partial F_{j\beta}} F_{j\alpha} F_{k\beta}.$$  

For inhomogeneous sinusoidal disturbances the equation of motion can be reformulated as a first-order system of coupled linear differential equations known as the Stroh formulation [12]. This formulation allows for the implementation of robust numerical methods (surface impedance matrix method, compound matrix method) to overcome numerical stiffness.
In recent decades great advancements have been made in elastography, i.e., the mapping of the elastic and stiffness properties of soft tissues. Various methods have been developed including static elastography [13], magnetic resonance elastography [14] and shear wave elastography [15]. Shear wave elastography is a particularly promising method as it avoids the slow acquisition times and large, expensive equipment of magnetic resonance elastography. In this method, transient pulses are used to generate shear waves in tissue. The wave can be seen and its speed determined using ultrafast ultrasound imaging. Then the speed of the wave is linked to the stiffness of the material. Thus, by measuring variations in the speed, one can infer the mechanical properties of the tissue by applying the theory of wave propagation in solids. The wave can also be combined with a deformation (e.g. compression) to obtain the non-linear parameters of the material. This method has been used to characterise the mechanical properties of soft biological tissues such as the breast [16] and brain [17] in a non-destructive manner, see Fig. 4.

If the wave propagates in thin-walled structures such as plates or tubes, then the wave is called a guided wave. The presence of boundaries has an effect on the properties of the wave and, unlike bulk waves, guided waves are dispersive. Guided waves have been used extensively in engineering, for example in non-destructive testing of a pipe. They have also been used to characterise biological tissues such as arteries [18].

When a soft solid is subject to certain large deformations it may lead to wrinkle or crease formation. Theoretical work has been done on wrinkle formation in, for example, tubes subject to torsion [19] and blocks subject to bending [20]. Wrinkles and creases are very common in biology, for example in the skin of a human elbow when it is unbent or in the valve leaflets of the heart [21]. Aside from mechanical deformations, wrinkles and creases can also be caused by growth in biological tissues.
Introduction

Figure 4: Left: Porcine brain being tested with a probe which generates a shear and generates an ultrasound image. Right: speed map of the brain in an undeformed state (upper) and in a compressed state (lower). The speed can be connected with material properties of the tissue. Image retrieved from [17].

Understanding the formation of wrinkles during growth can help explain the morphology of tissues, for example intestinal villi [22] and brain sulci [23]. Creases have been observed experimentally in, for example, bending of a homogeneous block [24] and also studied analytically [25] and numerically using finite element analysis [26].

Overview of the articles

Many soft incompressible solids such as rubbers and soft tissues have a linear response in simple shear and torsion, deformations often used in destructive testing of materials. In Chapter 1 we find the strain energy function for isotropic incompressible solids exhibiting a linear relationship between shear stress and amount of shear. The model is inclusive of the neo-Hookean and Mooney-Rivlin models but may also include extra terms which can be used to capture the Poynting effect and strain-stiffening effects which occur in many soft solids. This article was published in Extreme Mechanics Letters. This chapter serves as an introduction to universal solutions of non-linear elasticity and constitutive relations.

In Chapter 2 we study the propagation of guided waves in deformed hyperelastic plates and tubes. We carry out both a theoretical analysis using incremental elasticity, and a numerical analysis using finite element simulations. We compare the results of these with experimental results on a stretched phantom gel surrounded by fluid. We characterise the mechanical properties (including non-linear parameters) of the material
using guided wave theory and validate the results using a standard tensile test. This article was published in the Journal of the Mechanics and Physics of Solids.

Many biological structures, such as arteries, experience residual stresses (stresses which exist in the absence of any external forces). This stress can be demonstrated by isolating a cylindrical shape and cutting it axially. The structure will open up, revealing that the cylinder was under a large circumferential stress. In Chapter 3, we investigate the so-called opening angle method, a method used to model residual stress, in a cylindrical structure. In the opening angle method, one begins with an opened up sector and then deforms it into an intact cylinder. In this way, one can model the residual stresses which would be present in an intact cylindrical structure. Since many biological tissues are comprised of several layers, here we look at a two-layered structure. However, the large deformation solution in the opening angle method may not exist, as it can become unstable for certain combinations of dimensions and material parameters, leading to wrinkle formation. We thus study the formation of wrinkles during the opening angle method. This article was published in the International Journal of Solids and Structures, using the theory of incremental elasticity.

In Chapter 4 we again look at instabilities of a cylindrical sector, this time in bending, unbending and eversion of a homogeneous solid. This deformation is common in nature, e.g. aortic valve leaflets or artery rings. We again investigate wrinkle formation using incremental elasticity, and also crease formation using finite element simulations. We also prove existence and uniqueness of the solution. This article was published by
Proceedings of the Royal Society A.

References


REFERENCES


Chapter 1

Strain energy function for isotropic non-linear elastic incompressible solids with linear finite strain response in shear and torsion

Robert Mangan†, Michel Destrade†‡, Giuseppe Sacco-mandi §

Abstract

We find the strain energy function for isotropic incompressible solids exhibiting a linear relationship between shear stress and amount of shear, and between torque and amount of twist, when subject to large simple shear or torsion deformations. It is inclusive of the well-known neo-Hookean and the Mooney-Rivlin models, but also can accommodate other terms, as certain arbitrary functions of the principal strain invariants. Effectively, the extra terms can be used to account for several non-linear effects observed experimentally but not captured by the neo-Hookean and Mooney-Rivlin models, such as strain stiffening effects due to limiting chain extensibility.

†School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, University Road, Galway, Ireland
‡School of Mechanical & Materials Engineering, University College Dublin, Belfield, Dublin 4, Ireland
§Dipartimento di Ingegneria, Università degli Studi di Perugia, Via G. Duranti, Perugia 06125, Italy
CHAPTER 1. SOLIDS WITH LINEAR RESPONSE IN SHEAR

1.1 Introduction

Many soft incompressible materials have a linear response in shear and in torsion, including rubbers and soft tissues (FIG.1.1). But how should that property be modeled? The strain energy functions that come to mind are those of the neo-Hookean and the Mooney-Rivlin [1] materials,

\[
W_{\text{nh}} = \frac{1}{2} C_1 (I_1 - 3), \\
W_{\text{MR}} = \frac{1}{2} C_1 (I_1 - 3) + \frac{1}{2} C_2 (I_2 - 3),
\]

respectively, where \( C_1 > 0, C_2 > 0 \) are constants, and \( I_1 = \text{tr} \, C, \, I_2 = \text{tr}(C^{-1}) \) are the first two principal invariants of the right Cauchy-Green deformation tensor \( C \). These models provide indeed an exact linear relationship between the Cauchy shear stress component \( T_{12} \) and the amount of shear \( K \), and between the torque \( M \) and the twist \( \psi \). This can be checked directly by recalling the general relationships

\[
T_{12} = 2 \left( \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right) K, \\
M = 4\pi \psi \int_0^a r^3 \left( \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right) dr,
\]

(1.2)

(where \( r \) is the radial distance and \( a \) is the radius of the twisted cylinder [2]) because the term in the parentheses is a constant for these two models.

![Graphs](image.png)

Figure 1.1: Left: Shear stress response in the simple shear of porcine brain matter; experiments conducted at University College Dublin, see Ref. [3] for details. Right: Torsion of a right cylinder of rubber with radius \( a = 1.27 \) cm; digitized data from Ref. [2]. The straight lines represent linear fittings, indicating that the shear modulus is \( \mu = 163 \) Pa for brain and 38.2 kPa for rubber.

However popular, these models present some significant limitations when it comes to capturing certain non-linear effects: (1) Poynting effect: experiments show that a normal stress develops for soft solids in simple shear [3], but this cannot be captured by models that depend on \( I_1 \) only, like the neo-Hookean model, because their normal stress component \( T_{22} = -2(\partial W/\partial I_2)K^2 \) is zero; (2) Strain-stiffening effect: for large
1.2 Results

We arrive at the desired linear relationships by enforcing that the strain energy function \( W \) satisfy \( \partial W / \partial I_1 + \partial W / \partial I_2 = \text{constant} \). Furthermore, compatibility with the linear theory imposes that \( \partial W / \partial I_1 + \partial W / \partial I_2 = \mu / 2 \). We note that the Mooney-Rivlin material (1.1) is a particular solution of that inhomogeneous partial differential equation, with \( C_1 + C_2 = \mu \). Thus the general solution may be written as

\[
W = W_{\text{MR}} + H(I_1, I_2),
\]

where \( H \) is an arbitrary function of the two variables \( I_1, I_2 \). Then, after substitution, we obtain a homogeneous partial differential equation for \( H \),

\[
\frac{\partial H}{\partial I_1} + \frac{\partial H}{\partial I_2} = 0.
\]

The general solution of this equation is simply \( H = H(I_1 - I_2) \) where \( H \) remains an arbitrary function, but now of the single variable \( I_1 - I_2 \). We call the corresponding class of solids, the generalized Mooney-Rivlin materials

\[
W = \frac{1}{2} C_1 (I_1 - 3) + \frac{1}{2} C_2 (I_2 - 3) + H(I_1 - I_2).
\]

As an illustration we consider the following example of a generalized Mooney-Rivlin material,

\[
W_{\text{gMR}} = W_{\text{MR}} - \frac{1}{2} C_3 J_m \ln \left( 1 - \frac{I_1 - I_2}{J_m} \right).
\]

This model is chosen in an attempt to capture the strain-hardening effects which occur for moderate to large extensions of rubber, and which cannot be captured by the Mooney-Rivlin model alone [4]. The final term of \( W_{\text{gMR}} \) is obtained from Gent’s model [6] after substituting \( I_1 \) by \( I_1 - I_2 \).
and we expect that it will be able to capture limiting chain extensibility by tuning the parameter $J_m$.

For $W_{MR}$ and $W_{gMR}$ we perform curve fitting to the uni-axial extension data of Treloar [7], by minimizing the relative error. The *engineering tensile stress* $\sigma$ is given by

\[
\sigma(\lambda) = \frac{\partial W}{\partial \lambda} = 2(\lambda - \lambda^{-2}) \left( \frac{\partial W}{\partial I_1} + \lambda^{-1} \frac{\partial W}{\partial I_2} \right), \tag{1.8}
\]

where $\lambda$ is the stretch along the direction of extension, and the *Mooney-plot* scales these variables as $g(z) := \sigma/(\lambda - \lambda^{-2})$ against $z := \lambda^{-1}$. For the Mooney-Rivlin material $W_{MR}$ the fit is made over the first seven data points only, which correspond to the linear regime in the Mooney-plot, see [4] for details and the lower (green) curves of FIG.1.2. Over that limited range ($1 \leq \lambda \lesssim 2$), it gives a maximal relative error of 1.70% by adjusting $C_1$ and $C_2$ appropriately (explicitly, $C_1 = 1.7725$, $C_2 = 2.7042$.) Over the entire range ($1 \leq \lambda \lesssim 8$) it gives a terrible fit because it cannot accommodate the upturn in the Mooney-plot, only its early, linear part. For the model $W_{gMR}$ we perform the fitting over the entire range of stretches: we keep the same $C_1$ and $C_2$ throughout, and adjust the parameters $C_3$ and $J_m$. The fitted curves for $W_{gMR}$ are plotted as the upper (red) graphs of FIG.1.2; the maximum relative error over the full range is 4.89%, which is well within the experimental error of Treloar.

Further, in the fourth-order expansion [8] of these models, we find the following connections for $W_{gMR}$,

\[
\mu = C_1 + C_2, \quad A = -4(C_1 + 2C_2 + 2C_3), \quad D = C_1 + 3C_2 + 4C_3. \tag{1.9}
\]

### 1.3 Conclusion

We note that the generalized Mooney-Rivlin models still exhibit some *special* mechanical behavior. Indeed, when we calculate the coefficient...
of non-linearity of non-linear acoustics [9] \( \beta = (\mu + A/2 + D)/(2\mu) \) we obtain \( \beta = 0 \), not only for the specific example (1.7) but for the entire class of generalized Mooney-Rivlin materials. As a result, these materials cannot be used to model non-linear shear wave propagation. Moreover, because \( \beta = 0 \), they will not predict unbounded growth for the bending moment of a rectangular block with increasing values of the product of the block aspect ratio by the bending angle [10, 8]. To overcome these problems associated with the linearity of the models in shear and in torsion, we have to recognize that the linearity property exists only over a limited range of stretches, and we then have to undertake a completely different approach to the modeling, as explained in a recent contribution on mathematical models of rubber-like materials [11].

Nonetheless, the class of generalized Mooney-Rivlin materials achieves Mooney’s aspiration [1] of a model obeying Hooke’s law in shear over a wide range of deformation and for which neither the force-elongation nor the stress-elongation relationship agrees with Hooke’s law in simple extension. The models proposed improve on the predictions of the Mooney-Rivlin model in simple extension over the whole range of admissible deformations and hence provide a rich alternative to the model first proposed by Mooney [1] and later re-elaborated by Rivlin and co-authors [2].

Acknowledgements

We are thankful to Jerry Murphy for helpful discussions and to Badar Rashid for the simple shear experiment of FIG.1.1. RM gratefully acknowledges the funding of his PhD by a scholarship from the Irish Research Council. The research of GS is partially funded by GNFM of Istituto Nazionale di Alta Matematica.

References


Corrigendum

The curve fitting in FIG.1.2 was actually done using all four parameters. The best-fit parameters were $C_1 = -2.22$, $C_2 = 6.59$, $C_3 = 4.29$, and $J_m = 70.54$, with a maximum relative error of 4.89%. Fixing $C_1 = 1.7725$, $C_2 = 2.7042$ and fitting for the other two parameters, we find $C_3 = 0.99$, $J_m = 48.1$, with a maximum relative error of 16.85%

Supplementary material

We note that for simple shear and pure torsion we have $I_1 = I_2$, so that we may generalize the solution (1.6) further by introducing additional terms which tend to zero in the limit $I_1 - I_2 \rightarrow 0$. Therefore the most general solution is

$$ W = \frac{1}{2}C_1(I_1 - 3) + \frac{1}{2}C_2(I_2 - 3) + H(I_1 - I_2) + \sum_{k \geq 1} (I_1 - I_2)^k F_k(I_1, I_2), $$

where the $F_k$ are arbitrary non-constant functions of the two principal invariants and the exponents $k$ are real numbers. This information was removed from the final version of the article following the suggestion of a referee.

In the figure below we provide the experimental data used for FIG.1.1.
Chapter 2

Guided waves in pre-stressed hyperelastic plates and tubes: Application to the ultrasound elastography of thin-walled soft materials

Guo-Yang Li†, Qiong He‡, Robert Mangan§, Guoqiang Xu†, Chi Mo†, Jianwen Luo‡, Michel Destrade§, and Yanping Cao†

Abstract

In vivo measurement of the mechanical properties of thin-walled soft tissues (e.g., mitral valve, artery and bladder) and in situ mechanical characterization of thin-walled artificial soft biomaterials in service are of great challenge and difficult to address via commonly used testing methods. Here we investigate the properties of guided waves generated by focused acoustic radiation force in immersed pre-stressed plates and tubes, and show that they can address this challenge. To this end, we carry out both (i) a theoretical analysis based on incremental wave motion in finite deformation theory and (ii) finite element simulations. Our analysis leads to a novel method based on the ultrasound elastography to image the elastic properties of pre-stressed thin-walled soft tissues and artificial soft materials in a non-destructive and non-invasive manner. To validate the theoretical and numerical solutions and demonstrate

†Institute of Biomechanics and Medical Engineering, AML, Department of Engineering Mechanics, Tsinghua University, Beijing 100084, PR China
‡Department of Biomedical Engineering, School of Medicine, Tsinghua University, Beijing 100084, PR China
§School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, Galway, Ireland
the usefulness of the corresponding method in practical measurements, we perform (iii) experiments on polyvinyl alcohol cryogel phantoms immersed in water, using the Verasonics V1 System equipped with a L10-5 transducer. Finally, potential clinical applications of the method have been discussed.

2.1 Introduction

Guided waves in thin-walled structures are widely used in non-destructive testing (NDT) (Achenbach, 2000; Chimenti, 1997; Kim et al., 2006; Raghavan and Cesnik, 2007; Rose, 2002; Su et al., 2006). Understanding the dispersion relations (i.e., the variation of the phase velocities with frequency) of guided waves in the tested material is essential in the NDT and this topic has received considerable attention over the years (Achenbach, 1973; Lowe, 1995; Rose, 2014). Because dispersion relations are sensitive to physical and geometrical parameters, the dispersion features of guided waves can be harnessed for the characterization of materials and structures, e.g., by yielding the elastic moduli, thicknesses and curvatures of elastic shells (Cès et al., 2012; Chimenti, 1997; Moilanen et al., 2007; Yeh and Yang, 2011). To achieve this goal, dispersion curves measured in experiments are fitted with theoretical dispersion curves in order to infer the mechanical and geometrical parameters.

In many circumstances, the tested materials and structures are surrounded by fluid media, e.g., underwater pipelines. It is well known that the dispersion properties of guided waves in fluid-loaded media are also sensitive to the physical properties of the surrounding fluid media. Hence, if the phase velocities of the guided waves are larger than the bulk wave velocities of the surrounding fluid media, the guided waves are the so-called “leaky guided waves” (LGWs), due to energy leakage into the surroundings. Otherwise, the waves are “trapped” within the waveguides (Mazzotti et al., 2014; Rose, 2014). Due to the rich dispersion properties and important engineering applications of fluid-loaded waveguides, the study of guided waves in these structures spans several disciplines (Rose, 2014). For instance, the analytical dispersion relations for waves in a fluid-loaded elastic plate were originally obtained by Osborne and Hart (1945) and have been widely used in the scientific and engineering literature ever since (Aristégui et al., 2001; Chimenti, 1997; Chimenti and Nayfeh, 1985). For waveguides with complicated geometries, numerical methods have been established and validated to calculate the dispersion relations (Hayashi and Inoue, 2014; Mazzotti et al., 2014; Pavlakovic, 1998).

Beyond their use in the characterization of stiff engineering materials, guided waves have also recently been adopted in the ultrasound elastography of soft thin-walled biological tissues (Bernal et al., 2011; Couade et al., 2010; Li et al., 2017a; Li et al., 2017b; Nenadic et al., 2016; Nenadic et al., 2011; Urban et al., 2015). The key idea behind ultrasound-based shear wave elastography is to generate elastic waves inside soft biologi-
2.1. INTRODUCTION

cal tissues and then track their propagation with an ultrafast ultrasound imaging method. The elastic properties of the biological soft tissues can then be quantitatively inferred from the measured elastic wave velocities (Bercoff et al., 2004; Jiang et al., 2015; Sarvazyan et al., 1998). In this testing method, the frequencies of the shear waves are usually limited to less than 2 kHz, because of the rapid dissipation of higher frequency shear waves in soft biological tissues (Gennisson et al., 2013; Sarvazyan et al., 2013), and the wavelength is thus of the order of millimeter (recall that bulk shear wave velocities in soft biological tissues are typically 1-10 m/s). For soft biological tissues such as mitral valve, bladder, cornea and artery, the wall thicknesses are smaller than, or comparable to the wavelength, indicating that the elastic waves in these soft tissues are guided by the thin walls. Therefore, guided wave theory (instead of bulk wave theory) should be used in this case to analyze the experimental data and infer their material parameters (Bernal et al., 2011; Couade et al., 2010; Li et al., 2017b).

It is important to note that thin-walled soft tissues and thin-walled artificial soft biomaterials in their working state are usually subject to pre-stress. Although the effects of pre-stresses on the dispersion relations of guided waves in elastic materials have been systematically investigated in the literature (Ogden and Roxburgh, 1993; Rogerson and Fu, 1995; Bagno and Guz’, 1997; Kaplunov et al., 2000, 2002; Wijeyewickrema and Leungvichcharoen, 2009; Kayestha et al., 2010; Akbarov et al., 2011), guided waves in fluid-loaded pre-stressed thin-walled soft materials have been less studied (see for example, Bagno and Guz’ (1997); and references therein). Compared with hard engineering materials, soft materials can undergo large deformations in response to external loads, which influences greatly the characteristics of the elastic waves (Destrade and Ogden, 2010; Gennisson et al., 2007; Jiang et al., 2015). In this sense, to address the effects of pre-stresses on the dispersion properties of guided waves in thin-walled soft materials, it is necessary and important to conduct the analysis within the framework of finite deformation theory.

Based on the premise above, this paper investigates theoretically, numerically, and experimentally, the propagation of guided waves in pre-stressed soft plates and tubes surrounded by fluid. The results may serve as fundamental solutions to characterize the mechanical properties of thin-walled soft materials and soft tissues including mitral valve, artery, cornea and bladder, using the ultrasound elastography method.

The paper is organized as follows. A theoretical analysis is conducted in Section 2 based on the incremental theory of nonlinear elasticity (Ogden, 1984; Ogden, 2007). The analytical dispersion relations for both the anti-symmetric and symmetric modes of the guided waves in a pre-stressed hyperelastic plate immersed in inviscid fluid are derived. In Section 3, a finite element model is built to compare with the analytical solutions and demonstrate their applicability in describing the dispersion relations of the guided circumferential waves in a tube. In Section 4, phantom experiments are carried out on pre-stretched soft plates immersed in fluid to verify the theoretical solutions and to demonstrate
the usefulness of the corresponding method in practical measurements. Section 5 discusses the potential applications of the method in medical image analysis. Section 6 concludes the paper.

2.2 Theoretical analysis

We consider that thin-walled soft tissues in their in vivo state, and thin-walled artificial soft biomaterials in their working state, are all subject to pre-stress. In this section, we first derive analytically the dispersion relations of guided waves in pre-stressed hyperelastic plates immersed in an inviscid fluid (Fig. 2.1). Then we extend our analysis to curved plates to include tubular geometries. To address the effect of pre-stresses on the dispersion relations, we rely on the incremental theory of nonlinear elasticity (Ogden, 1984; Ogden, 2007). In the following section, we recall the basic equations for wave motion in pre-stressed hyperelastic plates and inviscid fluids.

Figure 2.1: Sketch of the model. The elastic plate immersed in fluid is pre-stretched homogeneously, which causes a finite deformation of the plate from (a) the reference configuration to (b) the current (or deformed) configuration. Then, small amplitude elastic wave propagation is studied in the current configuration.

2.2.1 Governing equations for waves in the pre-stressed hyperelastic plate

The plate

The soft plate is modelled as an incompressible isotropic hyperelastic solid, characterized by its mass density $\rho$ and strain energy $W$. It is subject to a pre-stress $\sigma$ and maintained in a state of homogeneous deformation characterized by the principal stretch ratios $\lambda_1$, $\lambda_2$, $\lambda_3$, along the principal Eulerian axes $x_1$, $x_2$, $x_3$. Then $W = W(\lambda_1, \lambda_2, \lambda_3)$ subject to $\lambda_1\lambda_2\lambda_3 = 1$. The dimension of the plate in the $x_1$, $x_3$ plane is much greater than its thickness. Initially, it is of thickness $2h_0$, and in its deformed state, of thickness $2h = 2\lambda_2h_0$. This assumption on the
2.2. THEORETICAL ANALYSIS

deformation state is acceptable in the region of interest (ROI) with finite
dimension in ultrasound elastography (Jiang et al., 2015).

Then we can use the equations of incremental elasticity to describe the
propagation of a small-amplitude wave in $x_1, x_2$ plane with displacement
$\mathbf{u} = \mathbf{u}(x_1, x_2, t)$. Because of incompressibility, we have $u_{1,1} + u_{2,2} = 0$, so
that we may introduce a stream function $\psi$ such that

$$u_1 = \psi, \quad u_2 = -\psi,$$

(2.1)

It can be shown that the propagation of the plane wave in the pre-
stressed solid is governed by the following equation of motion (Ogden,
2007)

$$\alpha \psi_{,1111} + 2\beta \psi_{,1122} + \gamma \psi_{,2222} = \rho (\psi_{,11tt} + \psi_{,22tt}),$$

(2.2)

and that the incremental nominal traction components are

$$\dot{S}_{021} = (\sigma_{22} - \gamma) \psi_{,11} + \gamma \psi_{,22},$$

(2.3)

$$\dot{S}_{022,1} = \rho \psi_{,22t} - (2\beta + \gamma - \sigma_{22}) \psi_{,112} - \gamma \psi_{,222},$$

(2.4)

Here the acousto-elastic coefficients $\alpha, \beta, \gamma$ are given by

$$\alpha = \frac{\lambda_1^2 (\lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2})}{\lambda_1^2 - \lambda_2^2}, \quad \gamma = \frac{\lambda_2^2}{\lambda_1^2} \alpha,$$

$$2\beta = \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} - 2\lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} - 2 \frac{\lambda_1 \lambda_2 (\lambda_2 \frac{\partial W}{\partial \lambda_1} - \lambda_1 \frac{\partial W}{\partial \lambda_2})}{\lambda_1^2 - \lambda_2^2}. \quad (2.5)$$

To model the hyperelastic deformation behavior of the plate we will
use in turn the Fung-Demiray strain energy density (Demiray, 1972; Fung
et al., 1979) for isotropic soft tissues,

$$W = \frac{\mu_0}{2b} (e^{b(I_1-3)} - 1),$$

(2.6)

where $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $\mu_0 > 0$ is the initial shear modulus and $b > 0$ is
a hardening parameter (Demiray, 1972); the neo-Hookean model,

$$W = \frac{\mu_0}{2} (I_1 - 3),$$

(2.7)

(which corresponds to $b = 0$ in the Fung-Demiray model), and the general
fourth-order model of weakly non-linear elasticity

$$W = \mu_0 i_2 + \frac{A}{3} i_3 + D (i_2)^2,$$

(2.8)

where $i_k = [\sum_{i=1}^3 (\lambda_i^k - 1)]/2^k$, $k = 2, 3$, and $A, D$ are the Landau coeffi-
cients of non-linearity (Destrade and Ogden, 2010).

For those three models, the acousto-elastic coefficients are easily com-
puted from the formulas (2.5). For instance, in the case of the neo-
Hookean solid (2.7), they are

$$\alpha = \mu_0 \lambda_1^2, \quad \beta = \frac{\mu_0}{2} (\lambda_1^2 + \lambda_2^2), \quad \gamma = \mu_0 \lambda_2^2.$$
CHAPTER 2. GUIDED WAVES IN PLATES AND TUBES

The fluid

The surrounding fluid is modelled as compressible and inviscid, and we
assume that the motion is irrotational. In the static state, the fluid is
subject only to a hydrostatic pressure $\sigma^* = -P I$. However, because
continuity of stress is required across the fluid-solid boundaries in the
reference configuration, we must have $P = -\sigma_{22}$.

The governing equation of motion for the small displacement of the
fluid reads (Jensen et al, 2011)

$$\nabla (\kappa \nabla \cdot u^F) = \rho^F \ddot{u}^F, \quad (2.10)$$

where $\kappa$ and $\rho^F$ denote the bulk modulus and mass density of the fluid,
respectively, and $u^F$ is the mechanical displacement of the fluid, measured
in the current configuration. Because the motion is irrotational, we have
$\nabla \times u^F = 0$, and hence the displacement can be written in the form $u^F = \nabla \chi$, where $\chi = \chi(x_1, x_2, t)$ is a scalar potential. Using this expression,
Eq.(2.10) can be rewritten as

$$\nabla^2 \chi = \frac{1}{c_p^2} \ddot{\chi}, \quad (2.11)$$

where $c_p = \sqrt{\kappa/\rho^F}$ is the speed of sound in the fluid. The incremental
pressure in the fluid induced by the deformation is

$$p = -\kappa \nabla \cdot u^F. \quad (2.12)$$

The interface

Let $n$ denote the unit outward normal of the plate. The interfacial con-
nitions between the pre-stressed plate and the fluid are (Otténio et al.,
2007)

$$\dot{S}_T^0 n = -p n - \sigma_{22}(\nabla u)^T n, \quad (2.13)$$

and

$$u \cdot n = u^F \cdot n, \quad (2.14)$$

from which we find the following boundary conditions in component form

$$u_2 = u_2^F, \quad \dot{S}_{021} = -\sigma_{22} u_{2,1}, \quad \dot{S}_{022,1} = -p_{,1} - \sigma_{22} u_{2,2,1}, \quad (2.15)$$

at $x_2 = \pm h$.

2.2.2 Dispersion analysis of the guided waves

We now consider a guided wave, travelling in the $x_1$-direction with angular
frequency $\omega$, wave-number $k$ and speed $c = \omega/k$, which is attenuated
with distance away from the plate. Hence we seek a wave solution of the form

$$\chi = A e^{r x_2} e^{i k (x_1 - ct)} \quad \psi = B e^{s x_2} e^{i k (x_1 - ct)}, \quad (2.16)$$

where $r$ and $s$ are attenuation factors, and $A, B$ are constants.
2.2. THEORETICAL ANALYSIS

In the fluid we find from Eq. (2.11) that

\[ r^2 - 1 = -\frac{c^2}{c_p^2}. \] (2.17)

The solutions to Eq. (2.17) are \( r = \pm \xi \), where \( \xi = \sqrt{1 - \frac{c^2}{c_p^2}} \). It follows that the solution for the fluid in the region \( x_2 \geq h \) \((x_2 \leq -h\), respectively\) is

\[ \chi^{\pm} = A^{\pm} e^{(\mp \xi k x_2)} e^{ik(x_1 - ct)}. \] (2.18)

In the plate we find from Eq. (2.2) that

\[ \gamma s^4 - (2\beta - \rho c^2)s^2 + \alpha - \rho c^2 = 0. \] (2.19)

We call \( s_1^2, s_2^2 \) the roots of this quadratic in \( s^2 \), and conclude that

\[ \psi = \phi(x_2) e^{ik(x_1 - ct)}, \] (2.20)

with

\[ \phi(x_2) = B_1 \cosh(s_1 k x_2) + B_2 \sinh(s_1 k x_2) + B_3 \cosh(s_2 k x_2) + B_4 \sinh(s_2 k x_2), \] (2.21)

where \( B_1, B_2, B_3, B_4 \) are constants.

Similar to the analysis for Lamb waves in plates surrounded by vacuum (Ogden and Roxburgh, 1993), we find that anti-symmetric modes \((B_2 = B_4 = 0)\) decouple from symmetric modes \((B_1 = B_3 = 0)\). In contrast to that analysis, we find that \( \sigma_{22} \) disappears from the equations once the boundary conditions are applied, a hallmark of waves at the interface between two media (Otténio et al., 2007). For instance, for the anti-symmetric mode, the boundary conditions (2.15) written at \( x_2 = \pm h \) first show that \( A^+ = A^- = A \) (say) and then read

\[ \cosh(s_1 k h)B_1 + \cosh(s_2 k h)B_3 + i \xi e^{-\xi k h} A = 0, \] (2.22)

\[ (1 + s_1^2) \cosh(s_1 k h)B_1 + (1 + s_2^2) \cosh(s_2 k h)B_3 = 0, \] (2.23)

\[ \gamma s_1 (1 + s_2^2) \sinh(s_1 k h)B_1 + \gamma s_2 (1 + s_2^2) \sinh(s_2 k h)B_3 + i \rho F c^2 e^{-\xi k h} A = 0, \] (2.24)

where we used the identity \( 2\beta - \rho c^2 = \gamma(s_1^2 + s_2^2) \).

This homogeneous system has non-trivial solutions for the amplitudes \( B_1, B_3, A \) when its determinant is zero, which is the dispersion equation:

\[ \gamma s_1 (1 + s_2^2) \tanh(s_1 k h) - \gamma s_2 (1 + s_2^2) \tanh(s_2 k h) + \frac{\rho F c^2}{\xi} (s_1^2 - s_2^2) = 0. \] (2.25)

When there is no fluid around the plate, \( \rho F = 0 \) and we recover the dispersion equation of (Ogden and Roxburgh, 1993) for a pre-deformed plate in vacuum (with \( \sigma_{22} = 0 \) in their equation). When the plate is not stretched \( \alpha = \beta = \gamma = \mu_0 \), so that \( s_1^2 = 1, s_2^2 = 1 - \frac{c^2}{\mu_0} \) and the dispersion equation simplifies to

\[ \left( 2 - \frac{\rho c^2}{\mu} \right)^2 \tanh(k h_0) - 4 \sqrt{1 - \frac{\rho c^2}{\mu_0}} \tanh \left( \sqrt{1 - \frac{\rho c^2}{\mu_0}} k h_0 \right) + \frac{\rho p F c^4}{\mu_0^2 \sqrt{1 - \frac{c^2}{\mu_0}}^3} = 0, \] (2.26)
in agreement with the linear elasticity result of (Osborne and Hart, 1945).

Finally, for symmetric modes, the roles of Cosh and Sinh are interchanged in the boundary conditions, and the dispersion equation is (2.25) where Tanh is replaced with Cotanh.

To plot the dispersion curves, we fix the circular frequency \( f = \omega / 2\pi \) (noticing \( k = \omega / c \)) and solve numerically the dispersion equation for the speed \( c \). There are infinitely many solutions to the dispersion equation, which correspond to the different branches of the anti-symmetric and symmetric modes (denoted by \( A_n, S_n, n = 0, 1, 2, \ldots \)). For illustration, we solve the dispersion equation for the neo-Hookean plate, and show the numerical results in Fig. 2.2: solid lines show the results without pre-stress \( (\lambda_1 = \lambda_2 = \lambda_3 = 1) \) and dashed lines show the dispersion curves with a homogeneous pre-stress leading to \( \lambda_1 = 1.1, \lambda_2 = 1/1.1, \lambda_3 = 1 \). Curves in different colors represent different modes, but in the present study we concentrate on the lowest anti-symmetric mode \( A_0 \) and symmetric mode \( S_0 \) (the fundamental modes) which are predominant in the low frequency range.

![Figure 2.2: Dispersion curves for the neo-Hookean plate without pre-stress (solid lines) and with pre-stress (dashed lines) when \( \lambda_1 = 1.1, \lambda_2 = 1/1.1, \lambda_3 = 1 \). The initial shear modulus, mass density and thickness of the plate are 100 kPa, 1000 kg/m\(^3\) and 1 mm, respectively, and the bulk modulus and mass density of the fluid are \( \kappa = 2.2 \) GPa and \( \rho_F = 1000 \) kg/m\(^3\), respectively.

Our analytical solution reveals the effect of the state of pre-deformation on the dispersion relation. The state of homogeneous deformation of the plate is entirely determined by the principal stretch ratios \( \lambda_i \) \( (i = 1, 2, 3) \). Defining \( \lambda_1 = \lambda \), and \( \lambda_2 = \lambda^{-\zeta} \), then we obtain \( \lambda_3 = \lambda^{1-\zeta} \) from the constraint \( \lambda_1 \lambda_2 \lambda_3 = 1 \) for incompressible materials. The parameter \( \zeta \) is in the range of 0.5 to 1 and determined by the deformation state of the plate: for example \( \zeta = 1 \) describes plane strain and \( \zeta = 0.5 \) describes uni-axial tension. For the neo-Hookean model (2.7) and Fung-Demiray model (2.6) with \( b = 5 \) (corresponding to the value found experimentally for arteries of older humans, see Horgan and Saccomandi (2003)), we study the variation of the \( A_0 \) and \( S_0 \) modes with \( \zeta \). From Fig. 2.3, it
is interesting to find that $\zeta$ basically has no effect on the $A_0$ mode, as the curves are almost indistinguishable from one another, indicating that only the principal stretch ratio $\lambda$ along the wave propagation direction needs to be measured to predict the effect of pre-stress on the $A_0$ mode using our analytical solution. This finding brings great ease for the practical application of our analytical solution because the $A_0$ mode is used in the ultrasound elastography of thin-walled soft biological tissues (Bernal et al., 2011; Li et al., 2017a) and it is not easy to accurately evaluate the parameter $\zeta$ in practical measurements.

![Figure 2.3: Effect of the parameter $\zeta$ on the $A_0$ and $S_0$ modes for plates with a 20% extension ($\lambda = 1.2$). Here $\zeta = 1$ (solid line), $\zeta = 0.75$ (dashed line), $\zeta = 0.5$ (dash-dot line) for: (a) neo-Hookean model; (b) Fung-Demiray model with $b = 5$. The initial shear modulus and wall thickness of the plate are 100 kPa and 1 mm, respectively, and the bulk modulus and mass density of the fluid are $\kappa = 2.2$ GPa and $\rho^F = 1000$ kg/m$^3$, respectively.](image)

2.3 Finite element simulations

To compare with the analytical solutions obtained in Section 2 and investigate the guided waves in a curved plate or tubular structures, we establish Finite Element (FE) models to calculate the dispersion relations from numerical simulations.

### 2.3.1 Comparison with the theoretical solutions

A plane strain model as shown in Fig. 2.4(a) is firstly established to validate the analytical solutions. The model consists of three parts: fluid/plate/fluid. The interfacial conditions between the fluid and the plate given by Eqs. (2.13) and (2.14) are realized with the ‘tie’ constraint in ABAQUS (2010).

In the simulation, the model is firstly stretched from its initial length $L_0$ with $L_0 = 30h_0$ to $L = \lambda L_0$ in order to pre-stress the plate, and during this process, the arbitrary Lagrangian-Eulerian (ALE) mesh technique is adopted to avoid mesh distortion in the fluid region, which is modeled
as the acoustic medium (AC2D8) (ABAQUS, 2010). Hybrid elements (CPE8RH) are adopted to model the incompressible plate and the number of the elements along thickness direction is 12.

Then the dispersion curve is calculated on the deformed configuration with the frequency domain FE method. In this step, the following periodic boundary conditions (PBCs) are invoked in the simulations

\[
\mathbf{u}^{BC} = \mathbf{u}^{C'}B', \quad p^{AB} = p^{A'B'}, \quad p^{CD} = p^{C'D'},
\]  

(2.27)

where the superscript indicates the boundary region as shown in Fig. 2.4(b). Those constraint equations can be realized by multi-point constrains (MPCs). The natural frequency analysis is conducted. Supposing that for a specified modal shape, the frequency is \( f_n \), where \( n = 1, 2, 3, \ldots \) is the number of waves between two sides of the model. The corresponding wavelength is \( l_n = L/n \) and the phase velocity can be determined as

\[
c_n = l_n f_n.
\]  

(2.28)

For illustration, Figs. 4(c)-(d) show the modal shape of the plate and the corresponding pressure field in the fluid for the anti-symmetric mode and the symmetric mode when \( n = 3 \). Taking the stretch ratio \( \lambda \) as 1.0 (no pre-stress) and then 1.1 (10% extension), the dispersion curves for the fundamental anti-symmetric and symmetric modes are plotted in Fig. 2.5(a) for the neo-Hookean model (2.7) and in Fig. 2.5(b) for the Fung-Demiray model (2.6) with \( b = 1 \). The solid and the dashed lines are the theoretical solutions and the discrete points are the FE results, and they match each other very well. Clearly, the pre-stretch increases the phase velocities of both the anti-symmetric and symmetric modes, and the higher the frequency, the more significant the increase. Similarly, the greater the hardening parameter \( b \) is (going from \( b = 0 \) to \( b = 1 \)), the larger the increase in phase velocities will be, especially for the symmetric mode. When the frequency tends to infinity, the phase velocities of the anti-symmetric and the symmetric modes both tend to the same phase velocities of the (half-spaces) interfacial waves, consistent with previous studies (Otténio et al., 2007).

### 2.3.2 Guided circumferential waves (GCWs) in pre-stressed tubes

For guided waves in curved structures, such as circular cylindrical tubes, explicit dispersion equations cannot be obtained and a numerical integration of the equations of motion through the wall thickness is required, for instance based on the Stroh formulation and the surface impedance method (Shuvalov, 2003). Here we do not need that treatment, because it is found from our numerical simulations, that when the ratio between the wall thickness and the radius of curvature is smaller than a critical ratio, the dispersion equation of the curved structure can be accurately approximated by that of flat plate established in the previous section,
2.3. FINITE ELEMENT SIMULATIONS

Figure 2.4: The FE model used to calculate the dispersion relations. (a)-(b) The plate is pre-stretched. The dispersion relation is calculated based on the deformed configuration by using the PBCs. The resulting modal shape of the plate and corresponding pressure field in the fluid for: (c) anti-symmetric mode; and (d) symmetric mode. Here the number of the waves between the two sides of the model is \( n = 3 \).

similar to the result for the stress-free tube reported by Fong (2005) and our previous study (Li et al., 2017b).

We consider that the tube is pre-stressed by the internal pressure, which is \( P \) mmHg higher than that of the external pressure. As shown in Fig. 2.6(b), the inner radius and wall thickness of the tube are \( R_0 \) and \( 2h_0 \) in the stress-free configuration, and \( R \) and \( 2h \) in the deformed configuration, respectively. The dispersion graphs for this case are found with the FE method and results are shown in Fig. 2.7 (points). We see that the phase velocities significantly increase with an increase in the pressure \( P \).

To compare the dispersion curves obtained from the FE analysis for tubes to those obtained with a plane geometry, we consider a plate as shown in Fig. 2.6(c), which is totally constrained along \( x_3 \)-direction and pre-stretched along the \( x_1 \)-direction with the prescribed stretch ratio \( \lambda_{\theta\theta} \). Here \( \lambda_{\theta\theta} \) is the average circumferential stretch ratio in the tube shown in Fig. 2.6(b), determined by

\[
\lambda_{\theta\theta} = \frac{R + H}{R_0 + h_0}. \tag{2.29}
\]
Figure 2.5: Dispersion curves for the fundamental anti-symmetric (lower curves) and symmetric (upper curves) modes: Comparison of the theoretical solutions (solid and dashed lines) with FE results (discrete points) when (a) \( b = 0 \) (neo-Hookean model); (b) \( b = 1 \) (Fung-Demiray model), and when the plate is unstretched (\( \lambda = 1.0 \)) and stretched by 10\% (\( \lambda = 1.1 \)). The initial thickness, the initial shear modulus and the mass density of the plate are \( 2h_0 = 1 \text{ mm}, \) \( \mu_0 = 100 \text{ kPa}, \) and \( \rho = 1000 \text{ kg/m}^3 \), respectively, and the bulk modulus and mass density of the fluid are \( \kappa = 2.2 \text{ GPa} \) and \( \rho^F = 1000 \text{ kg/m}^3 \), respectively.

Different pressure levels \( P \) lead to different curvature radii \( R \) and wall thicknesses \( 2h, \) and the corresponding stretch ratio \( \lambda_{\theta\theta} \) can be calculated from Eq.(2.29). Then the anti-symmetric mode of the dispersion curve for an immersed plate can be calculated with Eq.(2.25).

The theoretical predictions of the dispersion curves are shown in Fig. 2.7 (lines). We see that the theoretical solution given by Eq.(2.25) derived for the pre-stressed flat plate with the stretch ratio given by Eq.(2.29) can describe well the dispersion properties of the GCWs in a specified frequency range, e.g., 200 Hz-3000 Hz. The curvature effects of the tube are dictated by the ratio \( R_0/(2h_0) \), and the larger it is, the less it affects the dispersion curves. In the present example, \( R_0/(2h_0) = 2 \), which is a common value for some arterial walls, e.g., carotid artery (for the bladder it is a much larger number). The results in Fig. 2.7 indicate that, at least for \( R_0/(2h_0) \geq 2 \), the dispersion curves of the GCWs in a pre-stressed tube can be very well approximated with those of a flat plate. This finding indicates that our analytical treatment of Section 2 can be readily used in the ultrasound elastography of arteries. For example, consider an in vivo arterial wall as shown in Fig. 2.6(a), subject to a variable blood pressure (typically 80-120 mmHg). Our present analytical solution enables quantitative understanding of the effect of blood pressure on the dispersion curves which are used in elastography of the arteries in order to infer the elastic properties of arterial wall.

2.4 Experiments on phantom gels

Our above theoretical and numerical analyses lead the way to an ultrasound-based elastography method to infer the mechanical properties of pre-
2.4. EXPERIMENTS ON PHANTOM GELS

Figure 2.6: (a) The arterial wall (image credit: Boston Children’s Hospital’s Science and Clinical innovation blog, website: https://vector.childrenshospital.org) and the variation of the blood pressure; (b) Guided circumferential wave in a pre-stressed tube subjected to internal fluid pressure $P$; and (c) the corresponding approximation model with plane geometry, when the wall thickness is small compared to the radius of curvature.

stressed thin-walled soft tissues and artificial thin-walled soft biomaterials in their working state. To validate the method and demonstrate its usefulness in practical measurements, we performed experiments on phantom gels, as reported in this section.

We prepared a polyvinyl alcohol (PVA) cryogel phantom as follows. The PVA solution was made of 10% (by weight) PVA (Sigma-Aldrich, Shanghai, China), 87% distilled water, and 3% Sigmacell cellulose (20 µm, Sigma-Aldrich, Shanghai, China). The latter provided ultrasound scattering particles. The mixture was kept at a temperature of 85°C and stirred until the powder was fully dissolved. Then the mixture was cooled and underwent four freeze/thaw (F/T) cycles with 12h of freezing (-20 °C) and 12h of thawing (20 °C). After that, the phantom was mounted on a tensile machine to introduce pre-stress by prescribing the stretch ratio. The local deformation in the phantom along the stretch direction, as quantified by $\lambda$, was determined from the displacements of the grids marked on the surface of the phantom as shown in Fig. 2.8(a). According to the finding in Fig. 2.4, we may simply take $\zeta = 1$ in our data analysis, because the actual value of $\zeta$ has negligible effect on the $A_0$ mode used in our method.

In our experiments, the phantom was immersed in water. The Verasonics V1 System (Verasonics Inc., Kirkland, WA, USA) equipped with a
Figure 2.7: Dispersion curves (discrete points) of guided circumferential waves in a pre-stressed tube made of Fung-Demiray material, as found from FE analysis and dispersion curves (solid lines) of the approximation model given by Eq.(2.25). The pressure $P$ varies from 20 to 100 mmHg and the hardening parameter is $b = 1.0$.

L10-5 transducer (Jiarui, Shenzhen, Guangdong, China, $f_0 = 7.5$ MHz,) was used to generate elastic waves in the phantom, i.e., push the PVA phantom by applying the acoustic radiation force produced by the momentum transfer from the acoustic waves to the PVA phantom (Torr, 1984), and acquire the in-phase and quadrature (IQ) data. A typical B-mode image and the velocity field at 1.224 ms after push are shown in Fig. 2.8(b) and (c), respectively. Fig. 2.8(d) gives the spatio-temporal imaging of the waves along the blue line shown in Fig. 2.8(c). Accordingly, the dispersion curves can be obtained by conducting the two-dimensional Fourier Transformation (2DFT) to Fig. 2.8(d) (Alleyne and Cawley, 1991; Li et al., 2017b). Briefly, from the magnitude map obtained by conducting 2DFT to the spatio-temporal image, i.e., Fig. 2.8(e), the wavenumber can be identified by finding the maximum value of the magnitude at each frequency. We then fitted the experimental dispersion curves with the theoretical dispersion relation Eq.(2.25), using the neo-Hookean (2.7), Fung-Demiray (2.6), and fourth-order elasticity (2.8) models to describe the PVA phantom.

The parameter optimization was done using the curve fitting function `curve_fit` and the root finding function `fsolve` in the Python module SciPy. The curve fitting function `curve_fit` only accepts an explicitly defined function as its argument. Since Eq.(2.25) defines the speed $c$ only implicitly, it was necessary to define a Python function which, given the angular frequency $\omega$ and the material parameters of the chosen hyperelastic model, returns the speed $c$ as the output. This function was established by finding numerically the root of the dispersion equation Eq.(2.25).

In the experiments, three different stretch ratios were prescribed. In the absence of pre-stretch, i.e. $\lambda = 1.0$, the initial shear modulus $\mu_0$ was first inferred by fitting the experimental dispersion curve with Eq.(2.26).
2.4. EXPERIMENTS ON PHANTOM GELS

Figure 2.8: (a) Setup of the experiments on phantom gel. The phantom was stretched with a tensile machine and immersed in water. Grids were marked on its surface to evaluate the strains. (b) A typical B-mode image of the gel. (c) The velocity field at 1.224 ms after the push produced by acoustic radiation force. The wave propagated from the center of the phantom in two opposite directions. (d) Along the blue line (see (c)), the spatio-temporal imaging of the waves is shown. (e) The magnitude map of the results obtained by conducting 2D-Fourier Transforms to (d). At each frequency, the corresponding wavenumber is identified by finding the maximum value of the magnitude.

As shown in Fig. 2.9 (a), the initial shear modulus was found to be \( \mu_0 = 37 \) kPa. This value is consistent with the previous results (Li et al., 2017a; Li et al., 2017b) and we further validated it by conducting a separate tensile test, which yielded \( \mu_0 = 36 \) kPa, see Fig. 2.9(b).

The shear modulus is the only material parameter of the neo-Hookean model; once it has been determined, the dispersion equation Eq.(2.25) can be further used to predict the dispersion curves for different prescribed stretch ratios. Fig. 2.9(a) clearly shows that the theoretical predictions match the experimental dispersion curves well when the phantom has been stretched by 8% and then by 18%. This indicates that the hardening effects of the phantom material are not very significant and for this type of materials the present method can be applied to infer the material parameter of the neo-Hookean model of pre-stressed thin-walled soft materials once the value of the pre-stress is provided or estimated.
For the Fung-Demiray model, we fixed \( \mu_0 = 37 \) kPa as determined from the \( \lambda = 1.0 \) curve and then used the curve with the highest stretch \( \lambda = 1.18 \) to determine the hardening parameter \( b \) from the optimization procedure, and found \( b = 0.22 \), indicating that indeed the hardening effect of the material is not significant. Fig. 2.9(c) shows that the corresponding model provided a good predictive fit for the intermediate curve at \( \lambda = 1.08 \).

A similar approach can be used for the fourth-order elasticity model to determine \( A \) and \( D \). Here, however, we took the point of view that the dispersion curve when the material was unstretched was unknown, as would be the case in situ. Hence we determined all three material parameters (\( \mu, A, \) and \( D \)) by fitting to the \( \lambda = 1.18 \) curve. We found \( \mu_0 = 38 \) kPa, \( A = -126 \) kPa, \( D = 23 \) kPa. These values provided very good predictive fits for both the \( \lambda = 1.00 \) and \( \lambda = 1.08 \) curves, see Fig. 2.9(d).

### 2.5 Discussion

The waves induced by mechanical and acoustical stimuli in thin-walled biological soft tissues such as mitral valve, cornea, artery and bladder, are guided waves (Li and Cao, 2017). These guided waves are dispersive and the mechanical properties of the thin walls cannot be directly inferred from their group velocities, in contrast to the situation for bulk shear waves, travelling at a speed independent of the frequency. Instead, a dispersion analysis has to be carried out to understand the correlation between the phase velocity and the frequency. Moreover, most soft tissues contain pre-stresses in their in vivo state and may undergo finite deformations. Thus it is necessary and important to analyze the dispersion equations within the framework of finite deformation theory and to incorporate the effects of pre-stresses. The theoretical solutions derived here serve this purpose. When the level of pre-stress is known, material parameters such as the initial shear modulus and hardening parameters can be inferred from the dispersion curves.

The method developed here may find some clinical applications. Indeed, in clinics the accurate determination of local blood pressure in a non-invasive manner is an ongoing pursuit. Variation of the blood pressure in the arteries causes clear deformation of the arterial wall, which can be observed from the ultrasound image (Ribbers et al., 2007). However, the knowledge on the deformation alone is not sufficient for determining the local blood pressure without the prior knowledge of the mechanical properties of the arterial wall. The theoretical solutions presented here provide a promising means to deal with this challenging issue. In particular, once the real-time dispersion relations of guided circumferential waves are measured and the local deformation (i.e., the circumferential stretch ratio) is extracted from the ultrasound image, the present theoretical solutions can be used to estimate the variation of the local blood pressure. Briefly, when the blood pressure varies from the diastolic to
systolic period, it is possible to measure the stretch ratio along the cir-
cumferential direction using ultrasound imaging by taking the diastole
state as the reference configuration. When the circumferential stretch
ratio is measured and the dispersion relation of the GCW is obtained
at the same time, the theoretical model presented in this study can be
invoked to determine the effective modulus of the arterial wall in the di-
astole. Then from the known effective modulus and the circumferential
stretch ratio, variation in the blood pressure can be estimated.

To realize the measurements in practice, we may however have to
consider the features of real arteries. Indeed, it has been long recognized
that arterial walls are multi-layered structures and that each layer may
be considered as a fiber-reinforced composite (Holzapfel et al., 2000). If
the artery is modeled as such, the incremental theory (Ogden, 2007) can
still be used to write down the differential equations of motion, but they
will have variable coefficients and an analytical dispersion relation similar
to Eq. (2.25) is impossible to obtain. If the theoretical solution given by
Eq. (2.25) is used, then the arterial modulus determined using the guided
wave elastography represents the effective modulus of the arterial wall
and the averaged stretch ratio along the circumferential direction should
be used in the first approximation. Finally, it must be pointed out that
real arteries are surrounded by other soft tissues, and not necessarily by
a fluid on both sides as here. Such a simplification is reasonable when
the surrounding tissues are much softer than the arterial walls (Couade
et al., 2010, Li et al., 2017b), but here the critical modular ratio between
the arterial wall and the surrounding soft tissue has not been determined,
which deserves further investigation.

2.6 Conclusion

Thin-walled soft tissues (including mitral valve, artery, cornea and blad-
der) and thin-walled artificial soft biomaterials in their working state
usually contain pre-stresses and measuring their mechanical properties
in a non-invasive and non-destructive manner remains a challenging is-
sue. This study aimed at addressing this challenge and the following key
results were obtained.

First, we performed a theoretical analysis based on nonlinear elasticity
time and incremental motion theory to investigate the propagation
of guided waves generated by focused acoustic radiation force in pre-
stressed, fluid-loaded hyperelastic plates. For flat plates we derived the
dispersion relations analytically. An important insight gained from our
analytical solution is that the parameter $\zeta$ which is difficult to determine
in ultrasound elastography has negligible effects on the $A_0$ mode.

Second, we built a finite element model to simulate the propagation
of the acoustic wave in both flat and curved plates (tubes). The FE
results show that the theoretical solutions for plane plates are valid and
can be applied to tubular or curved plates, at least when the initial ratio
of the curvature radius to the wall thickness of the tube is such that
$R_0/(2h_0) \geq 2$.

Third, our theoretical and finite element results enabled the development of an ultrasound-based elastography method to infer the elastic and hyperelastic parameters of pre-stressed thin-walled soft biomaterials. The method may be used to characterize in vivo the mechanical properties of soft tissues e.g., artery and bladder, and measure in situ the elastic properties of artificial thin-walled soft biomaterials in their working state.

Finally, to validate the theoretical solutions and demonstrate the usefulness of the proposed method in practical measurements, we performed experiments on polyvinyl alcohol (PVA) cryogel phantoms using the Verasonics V1 System equipped with a L10-5 transducer. The results show that the method is valid when the pre-strain in the soft biomaterial is up to at least 18%.

Acknowledgements

We acknowledge support from the National Natural Science Foundation of China (Grant Nos. 11572179, 11172155, 11432008, and 81561168023) and from the Irish Research Council.

References


CHAPTER 2. GUIDED WAVES IN PLATES AND TUBES


REFERENCES


Figure 2.9: Experimental (points) and theoretical (lines) dispersion curves at different stretch ratios for the PVA gel. The initial thickness and mass density of the plate are $2h_0 = 2$ mm and $\rho = 1.1 \times 10^3$ kg/m$^3$. The bulk modulus and mass density of the fluid are $\kappa = 2.2$ GPa and $\rho_F = 1.0 \times 10^3$ kg/m$^3$. (a) By fitting the dispersion curve of the sample without pre-stretch, the initial shear moduli of the phantom gel is obtained as $\mu_0 = 37$ kPa. The neo-Hookean model (the only material parameter is $\mu_0$) then provides good predictions for the other dispersion curves with stretches of $\lambda = 1.08, 1.18$. (b) A destructive tensile test on the gel gives $\mu_0 = 36$ kPa. (c) By fitting the stiffening parameter $b$ of the Fung model on the curve obtained with stretch $\lambda = 1.18$, a very good agreement is found for the prediction of the curve with stretch $\lambda = 1.08$. (d) By fitting all three material parameters of the fourth-order elasticity model on the $\lambda = 1.18$ curve, a very good agreement is found for the prediction of the curves with stretches $\lambda = 1.00$ and $\lambda = 1.08$. 
Chapter 3
Wrinkles in the opening angle method

Michel Destrade†‡, Irene Lusetti*, Robert Mangan†, and Taisiya Sigaeva¶§

Abstract
We investigate the stability of the deformation modeled by the opening angle method, often used to give a measure of residual stresses in arteries and other biological soft tubular structures. Specifically, we study the influence of stiffness contrast, dimensions and inner pressure on the onset of wrinkles when an open sector of a soft tube, coated with a stiffer film, is bent into a full cylinder. The tube and its coating are made of isotropic, incompressible, hyperelastic materials. We provide a full analytical exposition of the governing equations and the associated boundary value problem for the large deformation and for the superimposed small-amplitude wrinkles. For illustration, we solve them numerically with a robust algorithm in the case of Mooney-Rivlin materials. We compare the results to experimental data that we collected for soft silicone sectors. We study the influence of axial stretch and inner pressure on the stability of closed-up coated tubes with material parameters comparable with those of soft biological tubes such as arteries and veins, although we do not account for anisotropy. We find that the large deformation described in the opening angle method does not always exist, as it can become unstable for certain combinations of dimensions and material parameters.

†School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, University Road, Galway, Ireland
‡School of Mechanical & Materials Engineering, University College Dublin, Belfield, Dublin 4, Ireland
*Politecnico di Milano, piazza Leonardo da Vinci 32, 20133 Milano, Italy
‡Department of Mechanical and Manufacturing Engineering, University of Calgary, Calgary, AB, Canada
§Department of Mechanical Engineering, Lassonde School of Engineering, York University, Toronto, ON, Canada
CHAPTER 3. WRINKLES IN THE OPENING ANGLE METHOD

3.1 Introduction

One of the most effective ways to demonstrate the existence of residual stresses in biological structures is to isolate a cylindrical shape and cut it axially. Invariably it will open up, revealing that the cylinder was under a large circumferential stress, see Fig.3.1.

Figure 3.1: Cutting biological cylindrical structures radially reveals that they were under circumferential residual stresses. Left: slice of an Irish Ash tree; Middle: a green chilli pepper; Right: Equatorial slice of rat heart (taken from [10]).

In turn, one of the most successful advances of non-linear elasticity is the modeling of this stress through the so-called opening angle method. By measuring how much a tube opens up into a sector, one can reconstitute a backward scenario whereby the structure was initially an open circular sector, subsequently bent into a complete tube by the action of what can now be identified as a residual stress, see Fig.3.2. Hence the opening angle gives a measure of the level of residual stress for an assumed model of material behavior.

Of course many questions remain open at the end of the process and here we address the following: Is the bending deformation always possible, or is it limited by loss of stability with respect to small-amplitude static wrinkles? Moreover, can the instability be overcome by pressurization of the reconstituted tube? These issues are most relevant to Finite Element simulations of residually-stressed tubes, where buckling should be avoided as much as possible.

Here we first formulate in Section 3.2 the equations governing the large deformation of a coated circular sector into an intact tube, which is possibly subjected to an internal hydrostatic pressure and a uniform axial stretch. We then specialize the analysis to the case when the coating and the substrate are made of different Mooney-Rivlin materials, because the stress components can then be computed analytically. We pay particular attention to writing the boundary conditions properly (hydrostatic pressure on inner face, perfect contact at the interface, traction-free on outer face).
3.2 THE OPENING ANGLE METHOD

In Section 3.3 we present the algorithm implemented to solve the incremental problem of static wrinkles superimposed onto large bending, axial stretch, and pressurizing. It relies on the Stroh formulation and the Surface Impedance Matrix method, and is robust and unaffected by numerical stiffness.

Finally, Section 3.4 presents experimental and numerical results: first our own, achieved by gluing a silicone coating on a urethane substrate; and second those coming from the literature on soft biological tubes, although of course those cannot be accurately modeled as isotropic. In our experiments, we show that no wrinkles form when a sector of opening angle $120^\circ$ is closed, while wrinkles form before a sector of opening angle $240^\circ$ is closed. Applying the aforementioned algorithm, we show numerically that the critical opening angle at which wrinkles form is $209^\circ$ and that four wrinkles should appear along the circumference, which is consistent with the experimental results. Applying the algorithm for dimensions and material parameters comparable (with the limitation that anisotropy is not accounted for) to those of a rabbit artery, we show that, in the absence of internal pressure, wrinkles form for an opening angle of $320^\circ$, but that these wrinkles can be eliminated by applying an internal pressure or can be delayed by the presence of an axial stretch. These results are in line with intuition and experiments made on biological tubes.

3.2 The opening angle method

Consider the sector of a soft cylindrical tube with geometry delimited in the cylindrical coordinate system $\{R, \Theta, Z\}$ (and orthonormal basis $\{E_R, E_\Theta, E_Z\}$) in its natural state $B_0$ by the region

$$A \leq R \leq C, \quad -(2\pi - \alpha_0)/2 \leq \Theta \leq (2\pi - \alpha_0)/2, \quad 0 \leq Z \leq L,$$

(3.1)

where $A, C$ are the radii of the inner and outer faces of the sector, respectively, $L$ is its height, and $\alpha_0 \in (0, 2\pi)$ is the opening angle. The stress-free circular sector consists of a stiff thin layer placed at the inner side ($A \leq R \leq B$), glued onto a thicker and softer layer located in the outer region $B \leq R \leq C$, where $B$ is the radius of the interface between two layers, as shown on Fig.3.2a. From now on, the superscripts ($c$) and ($s$) refer to the coating and the substrate, respectively.

The sector is deformed into an intact (circular cylindrical) tube with respect to a cylindrical coordinate system $\{r, \theta, z\}$ (with orthonormal basis $\{e_r, e_\theta, e_z\}$) by the following mapping [4]

$$r = r(R), \quad \theta = k\Theta, \quad z = \lambda_z Z,$$

(3.2)

where

$$k = \frac{2\pi}{2\pi - \alpha_0} > 1$$

(3.3)

is a measure of the opening angle and $\lambda_z \geq 1$ is the uniform axial stretch. We denote this configuration by $B_r$ and refer to it as the residually-
CHAPTER 3. WRINKLES IN THE OPENING ANGLE METHOD

Figure 3.2: The opening angle method: (a) An initially stress-free coated sector is subject to axial stretch and bent into (b) a residually-stressed full tube. It can also be subject to (c) an internal pressure. But is that large deformation stable?

stressed configuration. The geometry of the tube is now

\[ a \leq r \leq c, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq \ell, \quad (3.4) \]

where \( a = r(A), \ b = r(B), \ c = r(C) \) and \( \ell = \lambda_z L \) is the current tube length as shown on Fig.3.2.

The associated deformation gradient \( \mathbf{F} \) is

\[ \mathbf{F} = \frac{dr}{dR} \mathbf{e}_r \otimes \mathbf{E}_R + \frac{kr}{R} \mathbf{e}_\theta \otimes \mathbf{E}_\theta + \lambda_z \mathbf{e}_z \otimes \mathbf{E}_z. \quad (3.5) \]

The incompressibility condition, \( \det \mathbf{F} = 1 \), and one of the geometric requirements, e.g. \( r(A) = a \), impose

\[ r(R) = \sqrt{\frac{R^2 - A^2}{k \lambda_z}} + a^2. \quad (3.6) \]

Taking into account the diagonal form of deformation gradient, we introduce the principal stretches

\[ \lambda_1 = \frac{R}{k \lambda_z}, \quad \lambda_2 = \frac{kr}{R}, \quad \lambda_3 = \lambda_z, \quad (3.7) \]

such that \( \lambda_1 \lambda_2 \lambda_3 = 1 \) to satisfy incompressibility.

We take both coating and substrate to be made of isotropic hyperelastic materials with strain energy densities \( W^{(c)}, W^{(s)} \), respectively, so that the Cauchy stress \( \mathbf{\sigma} \) is diagonal in the \( \mathbf{e}_i \otimes \mathbf{e}_j \) basis, with components

\[ \sigma_{rr}^{(l)} = -q^{(l)} + \lambda_1 \frac{\partial W^{(l)}}{\partial \lambda_1}, \quad \sigma_{\theta\theta}^{(l)} = -q^{(l)} + \lambda_2 \frac{\partial W^{(l)}}{\partial \lambda_2}, \quad \sigma_{zz}^{(l)} = -q^{(l)} + \lambda_3 \frac{\partial W^{(l)}}{\partial \lambda_3}. \quad (3.8) \]
Here \( l = c, s \) and \( q^{(l)} \) are the Lagrange multipliers arising from the incompressibility condition.

In the absence of body forces the only non-trivial equation of equilibrium is
\[
\frac{\partial \sigma_{rr}^{(l)}}{\partial r} + \frac{\sigma_{rr}^{(l)} - \sigma_{\theta\theta}^{(l)}}{r} = 0 \quad (l = s, c). \tag{3.9}
\]
For the boundary conditions, we assume that the inner (coated) face of the tube at \( r = a \) is under internal pressure \( P \), that there is perfect bonding between the two layers at the interface \( r = b \), and that the outer face at \( r = c \) is free of traction:
\[
\sigma_{rr}^{(c)}(a) = -P, \quad \sigma_{rr}^{(s)}(b) = \sigma_{rr}^{(c)}(b), \quad \sigma_{rr}^{(s)}(c) = 0. \tag{3.10}
\]
By introducing the following quantities \([4]\),
\[
x \equiv k\lambda_z \frac{r^2}{R^2}, \quad x_a \equiv k\lambda_z \frac{a^2}{A^2}, \quad x_b \equiv k\lambda_z \frac{b^2}{B^2}, \quad x_c \equiv k\lambda_z \frac{c^2}{C^2},
\]
we may rewrite the principal stretches in terms of \( x \) as \( \lambda_1 = 1/\sqrt{k\lambda_z x} \), \( \lambda_2 = \sqrt{kx/\lambda_z} \) so that the energy density for fixed \( \lambda_3 = \lambda_z \) may be seen as a function of \( x \) only: \( \hat{W}^{(l)}(x) = W^{(l)}(1/\sqrt{k\lambda_z x}, \sqrt{kx/\lambda_z}, \lambda_z) \) for \( l = s, c \).

Noting that
\[
\sigma_{\theta\theta}^{(l)} - \sigma_{rr}^{(l)} = 2x\hat{W}_x^{(l)}(x) \quad (l = s, c), \tag{3.12}
\]
integrating equilibrium equations (3.9) for each layer, and using boundary conditions (3.10), we find that the inflating pressure \( P \) is
\[
P = \int_{x_a}^{x_b} \frac{\hat{W}_x^{(c)}(x)}{1 - x} \, dx + \int_{x_b}^{x_c} \frac{\hat{W}_x^{(s)}(x)}{1 - x} \, dx \quad (l = s, c). \tag{3.13}
\]
We can also determine the stress components throughout the wall, as
\[
\sigma_{rr}^{(c)}(x) = -\int_x^{x_c} \frac{\hat{W}_t^{(c)}(t)}{1 - t} \, dt, \quad \sigma_{rr}^{(s)}(x) = -\int_x^{x_b} \frac{\hat{W}_t^{(s)}(t)}{1 - t} \, dt - \int_{x_b}^{x_c} \frac{\hat{W}_t^{(s)}(t)}{1 - t} \, dt,
\]
\[
\sigma_{\theta\theta}^{(l)} = \sigma_{rr}^{(l)} + 2x\hat{W}_x^{(l)}(x), \quad \sigma_{zz}^{(l)} = \sigma_{rr}^{(l)} + \lambda_3 \frac{\partial W^{(l)}}{\partial \lambda_3} - \lambda_1 \frac{\partial W^{(l)}}{\partial \lambda_1}. \tag{3.14}
\]

For a given geometry of an undeformed coated sector in \( B_0 \), the following quantities are prescribed,
\[
\epsilon_B = B^2/A^2 - 1, \quad \epsilon_C = C^2/A^2 - 1. \tag{3.15}
\]
Then the physics of the stretched and pressurized closed-up cylinder in \( B_r \) are prescribed by the given strain energy densities \( \hat{W}^{(l)} \) for coating and substrate, the given axial stretch \( \lambda_z \) and the given inner pressure \( P \). The new geometry is entirely determined by solving the system of three equations for the three unknowns \( x_a, x_b, x_c \) composed by Eq.(3.13) and the two relations
\[
x_b(\epsilon_B + 1) = \epsilon_B + x_a, \quad x_c(\epsilon_C + 1) = \epsilon_C + x_a. \tag{3.16}
\]
CHAPTER 3. WRINKLES IN THE OPENING ANGLE METHOD

Then the state of stress is entirely determined by Eqs. (3.14).

For illustration, in this paper we model the substrate and coating using the Mooney-Rivlin energy density; it reads

$$W(l) = \frac{1}{2} C_1^{(l)} (\text{tr}(C) - 3) + \frac{1}{2} C_2^{(l)} (\text{tr}(C^{-1}) - 3), \quad (l = s, c), \quad (3.17)$$

where $C_1^{(l)} > 0$ and $C_2^{(l)} > 0$ are material constants and $C = F^T F$ is the right Cauchy-Green deformation tensor. This model is quite general because it recovers, at the same level of approximation [3], the most general model of isotropic, incompressible, third-order weakly non-linear elasticity,

$$W = \mu \text{tr}(E^2) + \frac{1}{3} A \text{tr}(E^3), \quad (3.18)$$

where $E = 2C + I$ is the Green-Lagrange strain tensor, $\mu$ is the Lamé coefficient of linear elasticity, and $A$ is the Landau coefficient of third-order elasticity (The connections between the constants are $\mu = C_1 + C_2$, $A = -4C_1 - 8C_2$. ) For the Mooney-Rivlin material (3.17), we have

$$\hat{W}(x) = \frac{1}{2} (C_1^{(s)} + C_2^{(s)} \frac{x}{x_c}) \left( \frac{kx}{\lambda z} + \frac{1}{k \lambda z x} \right) + \text{constant}, \quad (3.19)$$

which provides explicit expressions for the stress components in Eq. (3.14). Hence

$$\sigma_{rr}^{(s)} = \frac{C_1^{(s)} \lambda_z^{-1} + C_2^{(s)} \lambda_z}{2k} \left[ (1 - k^2) \ln \left( \frac{x - 1}{x_c - 1} \right) - \ln \left( \frac{x}{x_c} \right) + \frac{1}{x - x_c} \right], \quad (3.20)$$

and so on for the other components.

For an example, assume that the coating is $\Gamma$ times stiffer than the substrate, in the sense that $C_1^{(c)} = \Gamma C_1^{(s)}$, $C_2^{(c)} = \Gamma C_2^{(s)}$, where $\Gamma \geq 1$ is the stiffness contrast factor. Then we consider how the stresses are distributed along the radial axis for different stiffness factors $\Gamma$. We take the case where there is no inner pressure ($P = 0$) and the opening angle is $139^\circ$. In the undeformed geometry we take $A = 13\text{mm}$, $B = 14.5\text{mm}$, $C = 18\text{mm}$. Fig.3.3 illustrates the distribution of stresses along the thickness of the wall of closed-up cylinders, for a uniform material ($\Gamma = 1.0$), and for two-layered solids with moderately ($\Gamma = 3.0$) and significantly ($\Gamma = 7.0$) stiffer coatings compared to substrates. We clearly observe the jump in the circumferential stresses at the interface between coating and substrate, as expected.

### 3.3 Wrinkling of a coated sector

Here we study the stability of a coated sector closed into a pressurized cylinder. We signal the onset of instability by the existence of small-amplitude wrinkles, solutions to the incremental equations of equilibrium.

From experimental observations, we know that they should be varying sinusoidally along the circumference of the tube, with amplitude decay from the inner face to the outer face. The analysis for the existence of
Figure 3.3: Non-dimensional radial $\sigma_{rr}$ and circumferential $\sigma_{\theta\theta}$ stresses through two-layered wall of coating (red) and substrate (blue) modeled as Mooney-Rivlin materials with corresponding material constants $C_i^{(c)}$ and $C_i^{(s)}$ related by $C_i^{(c)} = \Gamma C_i^{(s)}$ ($i = 1, 2, j = 0, 1, 2$), where $\Gamma \geq 1$ is the stiffness contrast between the coating and the substrate.

such wrinkles can be put together from the results of the previous section and those of Destrade et al. [4] and we omit the details to save space.

In short, the wrinkles exist when the following boundary value problem is solved for $z^{(l)} = z^{(l)}(x)$, ($l = s, c$), the $2 \times 2$ Hermitian surface impedance matrix [2].

(i) Initial condition: $z^{(s)}(x_c) = 0$;

(ii) Numerical integration of the differential Riccati matrix equation

$$
\frac{d}{dx} z^{(l)} = \frac{1}{2x(1-x)} \left[ z^{(l)} G_2^{(l)} z^{(l)} + i \left( G_1^{(l)} \right)^\dagger z^{(l)} - i z^{(l)} G_1^{(l)} + G_3^{(l)} \right],
$$

in the substrate ($l = s$), from $x_c$ to $x_b$;

(iii) Interfacial condition: $z^{(c)}(x_b) = z^{(s)}(x_b)$;

(iv) Numerical integration of the differential Riccati matrix equation (3.21) in the coating ($l = c$), from $x_b$ to $x_a$; and
CHAPTER 3. WRINKLES IN THE OPENING ANGLE METHOD

(v) Target condition:
\[
\det \left( z^{(c)}(x_a) + \bar{P} \begin{bmatrix} 1 & \text{in} \\ -\text{in} & 1 \end{bmatrix} \right) = 0.
\] (3.22)

In Eq.(3.21), † denotes the Hermitian transpose and the Stroh submatrices \(G_i\) have components [4],
\[
G_1 = \begin{bmatrix} i & -n \\ -n(1 - \sigma) & -i(1 - \sigma) \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1/\alpha \end{bmatrix}, \quad G_3 = \begin{bmatrix} \kappa_{11} & i\kappa_{12} \\ -i\kappa_{12} & \kappa_{22} \end{bmatrix},
\] (3.23)
where the superscript “(l)” is understood, \(n\) denotes the wrinkling mode (number of wrinkles in the circumference), and
\[
\begin{align*}
\kappa_{11} &= 2\beta + 2\alpha(1 - \sigma) + n^2[\gamma - \alpha(1 - \sigma)^2], \\
\kappa_{12} &= n[2\beta + \gamma + \alpha(1 - \sigma^2)], \\
\kappa_{22} &= \gamma - \alpha(1 - \sigma)^2 + 2n^2[\beta + \alpha(1 - \sigma)].
\end{align*}
\] (3.24)

Here, in general,
\[
\alpha = \frac{2x\hat{W}_x(x)}{k^2 x^2 - 1}, \quad \gamma = k^2 x^2 \alpha, \quad \beta = 2x^2\hat{W}_{xx}(x) + x\hat{W}_x(x) - \alpha, \quad \sigma = \sigma_{rr}/\alpha,
\] (3.25)
and in particular for the Mooney-Rivlin model,
\[
\alpha = \frac{(C_1\lambda_x^{-1} + C_2\lambda_x)}{k_x}, \quad \gamma = (C_1\lambda_x^{-1} + C_2\lambda_x)k_x, \quad \beta = \frac{1}{2}(\alpha + \gamma).
\] (3.26)

Finally, the derivation of the target condition (3.22) is detailed in the appendix.

3.4 Experimental & numerical results

Here we implement the stability analysis described in the previous section for two cases: polymers and biological tissues. The algorithm is illustrated in Fig.3.7(a). Essentially, we implement the steps (i)-(iv) and iterate over \(\alpha_0\) until the target condition (v) is reached. We denote by \(\alpha_{cr} = \alpha_0\) the critical opening angle at which wrinkles form when the sector is closed into an intact tube, i.e., the value of \(\alpha_0\) when the target condition is reached.

3.4.1 Results for polymers

For our first experiment, we used artificial materials, namely relatively stiff silicone (red) for the coating, urethane (black) and very soft silicone (white) for the substrate. We subjected each material to a tensile test using a MTS electromechanical material characterization machine. We then determined the Mooney-Rivlin constants by curve-fitting over a usable range of data, and found that \(C_1^{(c)} = 0.98, C_2^{(c)} = 0.021\) (MPa) for
the red silicone, and $C_1^{(s)} = 0.14$, $C_2^{(s)} = 0.41$ (MPa) for the black urethane, see Fig.3.4(a). We then glued a 1.6mm thick red silicone layer onto a 26.9mm thick black urethane sector ($B = 23.93$mm, $C = 50.83$mm) and produced two coated sectors, one with opening angle $120^\circ$, the other with opening angle $240^\circ$, see Fig.3.4(b) and (c). We produced similar sectors using white urethane as the substrate, see Fig.3.4(d).

We found that, for the black urethane substrate, no wrinkles formed when the former sector was closed (Fig.3.4(b)), while for the latter sector six wrinkles formed shortly before the sector became intact (Fig.3.4(c)). Thus we would expect the critical opening angle at which wrinkles form when the sector becomes intact to be somewhere between $120^\circ$ and $240^\circ$.

To check this assertion, we performed the stability analysis described in the previous section for the same dimensions and material parameters as in the experiments. We found that the critical opening angle was $209^\circ$ with corresponding mode number $n = 4$, which supports our previous hypothesis.

![Figure 3.4: (a) Tensile tests for red silicone and black urethane. The early part of the data for silicone was discarded as unreliable and the curve-fitting to the Mooney-Rivlin models was done over the $1.5 \leq \lambda \leq 5.0$ range indicated by the dashed lines, yielding a relative error of less than 5%. (b) Sector with opening angle $120^\circ$, black urethane substrate and red silicone coating. No wrinkles form when the sector is closed into an intact tube. (c) Sector with opening angle $240^\circ$, black urethane substrate and red silicone coating. Six wrinkles form shortly before the sector is closed into an intact tube. (d), (e) Similar results for sectors with white silicone substrate and red silicone coating, and opening angles $120^\circ$ and $240^\circ$, respectively.](image)

### 3.4.2 Results for soft tissues

Here we perform the stability analysis using the dimensions and material parameters which are of the same order of magnitude as those of a rabbit carotid artery, as collected by Holzapfel et al. [1].

51
The artery consists of three layers: the intima, the media and the adventitia. However, the intima is very thin and not very stiff (at least in healthy young individuals), and so we can use our two-layer model with the dimensions \([1] \ B - A = 0.26 \text{mm}, \ C - B = 0.12 \text{mm}, \ A = 1.43 \text{mm},\) along with an axial stretch \(\lambda_z = 1.695.\)

For the material parameters, Holzapfel et al. [1] used an anisotropic model. Here we have only considered isotropic models, and so we set to zero Holzapfel et al.’s anisotropic parameters to make a (somewhat arbitrary) connection with their measurements. Moreover, Holzapfel et al. [1] did not consider a dependence of \(W\) on the second invariant of strain \(\text{tr}(C^{-1}),\) so here we take \(C_2^{(c)} = 0, \ C_2^{(s)} = 0.\) For the other (neo-Hookean) parameters, we have \(C_1^{(c)} = 3\text{kPa}, \ C_1^{(s)} = 0.3\text{kPa},\) in line with Holzapfel et al.’s [1] values of the shear modulus for the artery’s elastin matrix.

We perform the stability analysis over a physiological pressure range [5] of 0-170 mmHg. We plot the results in Fig.3.7(c) for the non-dimensional measure of pressure \(\tilde{P} = P/C_1^{(s)}.\) Then the physiological pressure range corresponds to \(0 \leq \tilde{P} \leq 75.5.\)

First we plot the curves giving the critical opening angle \(\alpha_{cr}\) against the pressure \(\tilde{P}\) for increasing values of the mode number \(n = 2, 3, 4, \ldots.\) Each curve is a bifurcation plot: at a given pressure \(\tilde{P},\) a tube with opening angle larger than \(\alpha_{cr}\) will buckle when it is bent into an intact closed tube; in order not to buckle, a sector must have an opening angle which is less than the smallest critical angle from all curves. Here we find that all curves for mode numbers \(n \geq 5\) are all below those for \(n = 2, 3, 4\) and are virtually indistinguishable one from another, see Fig.3.5. Hence our analysis does not allow us to determine the mode number precisely here, in contrast to the scenario of Section 3.4.1.

From the plots we see that when there is no internal pressure \((\tilde{P} = 0),\) only sectors with an opening angle greater than \(\alpha_{cr} \approx 320^\circ\) will buckle when closed into an intact tube. This value is significantly above the recorded opening angle for the rabbit artery [1], which was 160°. Hence we would expect (provided the crudeness of our modelling arteries here is overlooked) that the rabbit artery is smooth when it is not subject to internal pressure.

We also observe that as the internal pressure increases, the critical opening angle increases, with asymptotic behaviour \(\alpha_{cr} \to 360^\circ\) as \(\tilde{P} \to \infty.\) Hence buckling can be eliminated by applying an internal pressure, which is in line with our intuition and with, for example, experiments on a rat’s pulmonary artery [6], see Fig.3.7(b).

For comparison, we also plot the curves obtained in the case of no axial stretch, \(\lambda_z = 1,\) see Fig.3.6. We find that the axial stretch makes the sector more stable with respect to bending into an intact tube (the values of \(\alpha_{cr}\) are higher when \(\lambda_z > 1\) than when \(\lambda_z = 1)).\) To complete the picture, we also provide the plots of the variations of the critical circumferential stretch \(\lambda_a\) (contraction stretch on the inner face of the intact tube).
3.5 Discussion

Figure 3.5: (a) Plots of the critical opening angle $\alpha_{cr}$ for several mode numbers $n$ versus the non-dimensional pressure $\hat{P}$ using the material parameters and dimensions comparable with those of a rabbit artery [1], when it is subject to an axial stretch $\lambda_z = 1.695$. (b) Plots of the critical circumferential stretch $\lambda_n$ on the inner face of the intact tube at buckling versus the pressure $\hat{P}$.

Figure 3.6: (a) Plots of the critical opening angle $\alpha_{cr}$ for several mode numbers $n$ versus the non-dimensional pressure $\hat{P}$ using the material parameters and dimensions comparable with those of a rabbit artery [1], when it is not subject to an axial stretch ($\lambda_z = 1.0$). (b) Plots of the critical circumferential stretch $\lambda_n$ on the inner face of the intact tube at buckling versus the pressure $\hat{P}$.

3.5 Discussion

Often it is assumed that a stable deformation of a sector into an intact tube exists. These “opening angle” deformations are then used to
estimate the residual stresses in the material [7]. Here we have shown that, depending on the material properties and dimensions, wrinkling may occur before the sector becomes intact, which would be followed by further buckling and creases when the sector is closed. Our results have important implications for finite element reconstructions of the opening angle method. First, a stiffer coating will lead to instabilities in finite element simulations, earlier than for a homogeneous sector [4, 7]. Second, if the wrinkles occur, then our analysis is a first step towards providing meaningful precursors to creases (see Fig.3.1 and Fig.3.7(b)).

We also showed that wrinkles can be eliminated by applying an internal pressure, as has been confirmed in experiments.

Our method could also be applied to other tissues such as the esophagus, which is often modeled as a two-layered structured, and in which wrinkles and creases have been observed [12]. However, it is important to consider the limitations of our model. For example, in the iliac artery of an 81 year old human, buckling of the intima in the zero-pressure state leading to delamination has been observed [9]. As has been noted, the intima, one of the three layers of the artery, becomes thicker and stiffer with age. Evidently, there are residual stresses present leading to buckling, but clearly a three-layer model would be necessary to investigate such an occurrence. Furthermore, each layer of the artery is highly anisotropic.
due to the presence of collagen fibers [1], and so a more realistic model would reflect this fact.

Acknowledgments

MD and RM are grateful to the Irish Research Council for support through a Government of Ireland Postgraduate Scholarship. MD and IL thank the NUI Galway College of Science for support with the Summer Internship Program. TS is grateful to Aleksander Czekanski and IDEA-Lab research group from York University for their valuable support in performing the experiments. Finally, we are grateful to Valentina Balbi (Galway) for helpful discussions on the incremental problem.

Appendix: Derivation of the target condition

At the coating/vacuum interface, the incremental nominal traction is [11]

\[ \sigma^* + P (\text{grad } u)^T \mathbf{e}_r, \]  

where \( u \) is the incremental mechanical displacement, and \( \sigma^* \) is the Cauchy incremental stress in the \( 0 \leq r \leq a \) region. But that space is under constant hydrostatic pressure \( P \) and has no constitutive law to speak of, being the vacuum, so that \( \sigma^* \equiv 0 \). Also, the displacement gradient has components [4]

\[ \text{grad } u = \begin{bmatrix} \frac{\partial u}{\partial r} \frac{1}{r} \left( \frac{\partial u}{\partial \theta} - v \right) \\ \frac{\partial v}{\partial r} \frac{1}{r} \left( u + \frac{\partial v}{\partial \theta} \right) \end{bmatrix}, \]

in the \( \mathbf{e}_i \otimes \mathbf{e}_j \) basis.

For displacements of the form

\[ \{ u, v \} = \{ U(r)e^{i\theta}, V(r)e^{i\theta} \}, \]

\[ \{ s_{rr}, s_{r\theta} \} = \{ S_{rr}(r)e^{i\theta}, S_{r\theta}(r)e^{i\theta} \}, \]

where \( U, V, S_{rr}, S_{r\theta} \) are functions of \( r \) only. Then (3.27) reads

\[ r \begin{bmatrix} S_{rr} \\ S_{r\theta} \end{bmatrix} = P \begin{bmatrix} rU' \\ \text{inU} - V \end{bmatrix} = P \begin{bmatrix} -U - \text{inV} \\ \text{inU} - V \end{bmatrix}, \]

at \( r = a \), where for the second equality we used the incremental incompressibility equation,

\[ \text{div } u = \frac{\partial u}{\partial r} + \frac{1}{r} \left( u + \frac{\partial v}{\partial \theta} \right) = (rU' + U + \text{inV}) \frac{e^{i\theta}}{r} = 0. \]
On the other hand, the traction is related to the displacement by the surface impedance matrix [2]:

\[ r \begin{bmatrix} S_{rr} \\ S_{r\theta} \end{bmatrix} = z^{(c)} \begin{bmatrix} U \\ V \end{bmatrix}. \tag{3.33} \]

In particular, at the \( r = a \) interface, we have by (3.31)

\[ P \begin{bmatrix} -U(a) - inV(a) \\ inU(a) - V(a) \end{bmatrix} = z^{(c)}(a) \begin{bmatrix} U(a) \\ V(a) \end{bmatrix}, \tag{3.34} \]

from which the target condition (3.22) follows (see Balbi and Ciarletta [1] for an early, but not entirely correct, derivation of the target condition).

References


REFERENCES


Chapter 4

Wrinkles and creases in the bending, unbending and eversion of soft sectors

Taisiya Sigaeva†, Robert Mangan‡, Luigi Vergori§, Michel Destrade‡, and Les Sudak†

Abstract

We study what is clearly one of the most common modes of deformation found in nature, science and engineering, namely the large elastic bending of curved structures, as well as its inverse, unbending, which can be brought beyond complete straightening to turn into eversion. We find that the suggested mathematical solution to these problems always exists and is unique when the solid is modelled as a homogeneous, isotropic, incompressible hyperelastic material with a strain-energy satisfying the strong ellipticity condition. We also provide explicit asymptotic solutions for thin sectors. When the deformations are severe enough, the compressed side of the elastic material may buckle and wrinkles could then develop. We analyse in detail the onset of this instability for the Mooney-Rivlin strain energy, which covers the cases of the neo-Hookean model in exact non-linear elasticity and of third-order elastic materials in weakly non-linear elasticity. In particular the associated theoretical and numerical treatment allows us to predict the number and wavelength of the wrinkles. Guided by experimental observations we finally look at the development of creases, which we simulate through advanced finite element computations. In some cases the linearised analysis allows us to predict correctly the number and the wavelength of the creases, which

†Department of Mechanical and Manufacturing Engineering, University of Calgary, Calgary, AB, Canada
‡School of Mathematics, Statistics and Applied Mathematics, National University of Ireland Galway, University Road, Galway, Ireland
§Dipartimento di Ingegneria, Università degli studi di Perugia, 06125 Perugia, Italy
turn out to occur only a few percent of strain earlier than the wrinkles.

### 4.1 Introduction

Bending and unbending are without a doubt the two most common modes of deformation for the elastic curved structures found in nature and engineering. Mathematically, large bending and unbending are actually exact solutions for incompressible, isotropic, non-linearly elastic circular sectors, as shown by Rivlin [1]. Over the years there have been a good number of studies investigating the bending of a rectangular block into a sector of circular cylinder up to, and including the possible appearance of wrinkles on the inner face of the resulting sector [2, 3, 4, 5, 6]. Very few works have looked at the stability of the converse problem, the straightening of a sector into a rectangular block [7, 8], or at the stability of the bending into a closed full cylinder [9]. The questions of existence, uniqueness and stability for the continuous problem of bending and large unbending that can go all the way to eversion (when the inner and outer faces swap roles) remain scarcely investigated (only a few studies related to the deformation itself exist [11, 10, 12, 13]).

![Image of aortic valve leaflet](image)

**Figure 4.1:** Commissural region of an aortic valve leaflet (porcine heart): undeformed (left), bent during valve opening (middle), and unbent during valve closure (right). The bottom pictures are the zooms indicated by the dotted squares in the top pictures. They show that wrinkles develop in both modes, eventually evolving into creases. Reprinted with permission from [18].

Many works looking at large bending take their motivation from biological applications. An example can be found in the recent work by Rudykh and Boyce [6] on the super flexibility of elasmoid fish in bending, due to the multilayered structure of their imbricated scale tissue. Similarly, researchers wishing to model residual stresses in tubular soft
tissues often use the so-called “opening angle method”, where the bending of a cylindrical sector into an intact tube creates large residual stresses – see, for instance, the textbook by Taber [14] for the modelling of residual stresses in arteries, in the left ventricle and in the embryonic heart.

Wrinkles, in turn, signal the onset of instability, and are often precursors to the development of creases, which are ubiquitous in nature, see the deformation of a heart valve leaflet in Figure 4.1, or the deep creases developed on the inner face of a depressurised pulmonary artery [15]. These latter creases would considerably alter the blood flow during a low pressure episode due to an upstream blockage and alter the geometry of an artery for a planned surgery. In order to model creases, we must first discover when the sector buckles on its way to be closed into a full cylinder. With this ultimate goal in mind, we now embark on a complete resolution of the titular problem.

In the next section we recall the exact solution of non-linear elasticity for the flexure of circular sectors made of incompressible isotropic solids. We pay particular attention to the unbending mode, because it can be brought to go beyond the stage where the sector is deformed into a straight rectangular block. Then the sector becomes an everted sector and the inner and outer faces exchange roles.

In Section 4.3 we present analytical results for the existence and uniqueness of the deformation. It turns out that bending, unbending and eversion of a cylindrical sector are always possible (and the solution is unique), provided its strain energy function satisfies the strong ellipticity condition. We also manage to provide an explicit thin-wall expansion of the results, valid for all strain energy functions up to the third order in the thickness. Details of the associated calculations are given in Appendix A.

In Section 4.4 and Appendix B we summarise our strategy to write down and solve numerically the boundary value problem of small-amplitude wrinkles superimposed on large bending, unbending or eversion, leaving the curved faces free of incremental traction. Within the framework of incremental elasticity [17], we formulate the governing equations and the boundary conditions using the Stroh formalism. This formulation allows us to implement robust numerical procedures (surface impedance matrix method, compound matrix method) to overcome the numerical stiffness arising here.

In Section 4.5 we present the results of those numerical procedures for sectors made of Mooney-Rivlin materials or equivalently, of weakly non-linear, third-order elastic solids. The results turn out to be independent of the material constants, and are thus universal to these families of models. Our analysis of the number of wrinkles forming at the onset of instability is quite detailed and is consistent with, and thus generalises, the previously studied special cases of bending of a rectangular block into a sector, unbending of a sector into a rectangular block, and bending of a sector into a full cylinder.

In Section 4.6 we present the results of table-top and finite element (FE) experiments of bending, unbending and everting a cylindrical sec-
Both types of experiments reveal the formation of creases rather than sinusoidal wrinkles, in line with previous results for deforming homogeneous solids. In both cases, we get period-doubling due to the merging of some creases. The FE simulations show that the creases appear a bit earlier (a couple of percent less strain) than the wrinkles, which are thus not expressed. Nevertheless, the wrinkles analysis still proves useful, because we find that the number and the wavelength of the creases (counting the creases which would exist in the absence of period-doubling) predicted by the FE simulations is the same or close to the number and wavelength of wrinkles predicted by the numerical procedures of Section 4.5. It follows that the linearised analysis can be used to approximate the more computationally expensive FE simulations of creases, by predicting within a few percents the bifurcation strain, and the number and wavelength of creases. It also generates the best shape possible for the perturbation introduced in the numerical creasing analysis. Finally it forms the basis for the study of the stability of coated sectors, for which sinusoidal wrinkles are the dominant mode.

4.2 Large bending, unbending and eversion

We consider a right cylinder sector, initially undeformed and placed in the following region,

\[ A \leq R \leq B, \quad -\alpha_r \leq \Theta \leq \alpha_r, \quad 0 \leq Z \leq L, \quad (4.1) \]

where \((R, \Theta, Z)\) are the cylindrical coordinates in the reference configuration, with orthonormal basis \((E_R, E_\Theta, E_Z)\). Here \(A, B\) are the inner and outer radii of the undeformed sector, respectively, \(L\) its axial length, and \(2\alpha_r\) its undeformed or referential angle, related to its opening angle \(\alpha_o\) through the relation \(\alpha_o = 2(\pi - \alpha_r)\).

By applying appropriate moments and forces (determined later), the sector can be deformed into a more closed (bending) or more open (unbending) sector, with current axial length \(\ell\) and deformation angle \(2\alpha_d\).

Hereon we exclude the possibility of scenarios when \(\alpha_r\) and \(\alpha_d\) are exact zeros, which are to be treated separately using Cartesian coordinate system and different universal solutions as in [7, 3], for instance. Case \(\alpha_r = 0\) describes bending of a rectangular block, while case \(\alpha_d = 0\) corresponds to the problem of the sector straightening. Still, we encompass these cases in the limits where \(\alpha_r \ll 1\) and \(|\alpha_d| \ll 1\). Hence we have

\[ \alpha_r \in (0, \pi], \quad \alpha_d \in [-\pi, \pi] - \{0\}. \quad (4.2) \]

We now introduce \(\kappa\), a measure of the change in the angles, as

\[ \kappa = \frac{\alpha_d}{\alpha_r} \in \left[ \frac{\pi}{\alpha_r}, \frac{\pi}{\alpha_r} \right] - \{0\}. \quad (4.3) \]

Hence \(\kappa > 1\) corresponds to bending, \(\kappa < 1\) corresponds to unbending, and further, \(\kappa < 0\) corresponds to unbending beyond the straight rectangular...
configuration, a deformation which we call *eversion* from now on. Figure 4.2 shows sketches of these deformations and where wrinkling is going to take place.

The deformation can be modelled as

\[ r = r(R), \quad \theta = \kappa \Theta, \quad z = \lambda_z Z, \quad (4.4) \]

where \( \lambda_z = \ell/L \) is the axial stretch, and \( (r, \theta, z) \) are the cylindrical coordinates in the current configuration, with orthonormal basis \( (e_r, e_\theta, e_z) \).

We define the current radii \( a \equiv r(A) \) and \( b \equiv r(B) \). When \( \kappa > 0 \), the inner and outer faces remain the respective inner and outer faces of the deformed sector, which occupies the following region,

\[ a \leq r \leq b, \quad -\alpha_d \leq \theta \leq \alpha_d, \quad 0 \leq z \leq \ell. \quad (4.5) \]

But when \( \kappa < 0 \) (eversion), the inner face of the undeformed sector becomes the outer face of the deformed sector, and vice versa. The sector then occupies the region

\[ b \leq r \leq a, \quad \alpha_d \leq \theta \leq -\alpha_d, \quad 0 \leq z \leq \ell. \quad (4.6) \]

Note that the deformation does not account for slanting surfaces that can appear in large bending, especially in eversion [10].

The corresponding deformation gradient \( \mathbf{F} \) has components

\[ \mathbf{F} = \frac{dr}{dR} e_r \otimes e_r + \frac{\kappa r}{R} e_\theta \otimes e_\theta + \lambda_z e_z \otimes e_z. \quad (4.7) \]
For incompressible solids, \( \det \mathbf{F} = 1 \) at all times, from which we deduce that
\[
r = \sqrt{\frac{R^2 - A^2}{\kappa \lambda_z} + a^2}, \quad b = \sqrt{\frac{B^2 - A^2}{\kappa \lambda_z} + a^2}.
\] (4.8)

Then we find the principal stretches (the square roots of the eigenvalues of \( \mathbf{F} \mathbf{F}^T \)) as
\[
\lambda_r = \frac{R}{|\kappa| \lambda_z r}, \quad \lambda_\theta = \frac{|\kappa| r}{R}, \quad \lambda_z.
\] (4.9)

Notice that in the special case where \( \kappa \lambda_z = A^2/a^2 \), the deformation is homogeneous. It then reads
\[
r = \frac{R}{\sqrt{\kappa \lambda_z}}, \quad \theta = \kappa \Theta, \quad z = \lambda_z Z,
\] (4.10)
with constant principal stretches,
\[
\lambda_r = \frac{1}{\sqrt{\kappa \lambda_z}}, \quad \lambda_\theta = \sqrt{\frac{\kappa}{\lambda_z}}, \quad \lambda_z.
\] (4.11)

Now we compute the forces and moments required to effect the deformation. For an incompressible, isotropic and hyperelastic material with strain energy density \( W = W(\lambda_r, \lambda_\theta, \lambda_z) \), the Cauchy stress tensor \( \mathbf{\sigma} \) has components
\[
\mathbf{\sigma} = \sigma_{rr} \mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\theta\theta} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{zz} \mathbf{e}_z \otimes \mathbf{e}_z, \quad \sigma_{qq} = -p + \lambda_q \frac{\partial W}{\partial \lambda_q} \quad (q = r, \theta, z),
\] (4.12)
where \( p \) is the Lagrange multiplier introduced by the constraint of incompressibility. Because the principal stretches do not depend on \( \theta \) and \( z \), we readily deduce from the equilibrium equations that \( p = p(r) \) only, and that
\[
\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0,
\] (4.13)
which must be solved subject to the boundary conditions of traction-free inner and outer faces:
\[
\sigma_{rr}(a) = \sigma_{rr}(b) = 0.
\] (4.14)

To non-dimensionalise the equations, we use the scaled circumferential stretch \( \lambda \) (and its values \( \lambda_a, \lambda_b \) on the faces), and the radii ratio \( \rho \), defined as
\[
\lambda = \sqrt{\lambda_z |\kappa| r} / R, \quad \lambda_a = \sqrt{\lambda_z |\kappa| a} / A, \quad \lambda_b = \sqrt{\lambda_z |\kappa| b} / B, \quad \rho = A / B \in ]0, 1[.
\] (4.15)
According to (4.8) they are linked as follows,
\[
\lambda_b = \sqrt{\rho^2 \lambda_a^2 + (1 - \rho^2) \kappa}.
\] (4.16)
4.3 Existence, uniqueness and thin-wall expansion

Now we implement the change of variables from $r$ to $\lambda$ through

$$r \frac{d\lambda}{dr} = \frac{\lambda}{\kappa} (\kappa - \lambda^2), \quad \text{or, equivalently,} \quad dr = \frac{a\lambda_a \kappa}{\kappa - \lambda^2} \sqrt{\frac{\kappa - \lambda_a^2}{\kappa - \lambda^2}} d\lambda. \quad (4.17)$$

Then, introducing the single variable strain energy function $\widehat{W} = \widehat{W}(\lambda)$ as

$$\widehat{W}(\lambda) \equiv W \left( 1/(\sqrt{\kappa\lambda}), \lambda/\sqrt{\kappa\lambda}, \lambda \zeta \right), \quad (4.18)$$

we deduce that

$$\lambda \widehat{W}'(\lambda) = -\left( \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2} \right) = -\left( \sigma_{rr} - \sigma_{\theta\theta} \right), \quad (4.19)$$

so that the governing equation (4.13) and the boundary conditions (4.14) become, respectively,

$$\frac{d\sigma_{rr}}{d\lambda} = \frac{\kappa}{\kappa - \lambda^2}, \quad \sigma_{rr}(\lambda_a) = 0, \quad \sigma_{rr}(\lambda_b) = 0. \quad (4.20)$$

These can be integrated to give

$$\sigma_{rr} = \kappa \int_{\lambda_a}^{\lambda} \frac{\widehat{W}'(s)}{\kappa - s^2} ds, \quad \int_{\lambda_a}^{\lambda_b} \frac{\widehat{W}'(s)}{\kappa - s^2} ds = 0. \quad (4.21)$$

Notice that the latter equation and (4.16) form a system of two equations for $\lambda_a$, $\lambda_b$ (of course, it must be checked first that the equation has a solution, see next section). Hence, if a given material is prescribed by the choice of its strain energy $W$, and the original dimensions $A$, $B$, $\alpha_r$ are prescribed, and the target deformation angle $\alpha_d$ is prescribed, then $\lambda_a$, $\lambda_b$ are found from these two equations, and the new radii $a$, $b$ follow.

Now that the radial stress $\sigma_{rr}$ is determined, we deduce the circumferential stress from (4.19) as $\sigma_{\theta\theta} = \sigma_{rr} + \lambda \widehat{W}'(\lambda)$. Finally, we find that the stresses on the end surfaces $\theta = \pm \alpha_d$ are equivalent to couples with moments

$$M_{\theta=\pm\alpha_d} = \pm \left\{ A^2 L \lambda_a^4 (\kappa - \lambda_a^2) \int_{\lambda_a}^{\lambda_b} \frac{\widehat{W}'(s)}{\kappa - s^2} ds \right\} e_z. \quad (4.22)$$

4.3 Existence, uniqueness and thin-wall expansion

We investigated the existence and uniqueness of a positive root to (4.21)$_2$, and found that they are always guaranteed for materials with a strain-energy function $W$ satisfying the strong ellipticity condition. This condition simply puts constraints on the material parameters of many widely used models. For example it is satisfied by the neo-Hookean, Mooney-Rivlin, Fung, Gent, and one-term Ogden models, as long as all parameters are positive [7]. We relegate the details of this proof to Appendix A.
For thin sectors, we were also able to establish some general conclusions about the deformed configuration. For our asymptotic analysis we introduced the following small thickness parameter $\varepsilon > 0$ defined as

$$\varepsilon = 1 - \rho = (B - A)/B \ll 1.$$ (4.23)

Then we found the following expansion of $\lambda_a$ up to order $\varepsilon^4$:

$$\lambda_a = 1 + \frac{1}{2}(1 - \kappa)\varepsilon + \frac{1}{24}(1 - \kappa)(13 - 3\kappa)\varepsilon^2 - \frac{1}{48}(1 - \kappa)(3\kappa^2 + 8\kappa - 27)\varepsilon^3 + \frac{1}{5760}(1 - \kappa)\left[45\kappa^3 - 363\kappa^2 - 1813\kappa + 3667\right] \varepsilon^4 + O(\varepsilon^5).$$ (4.24)

In particular, note that the results are independent of the form of strain-energy function up to order $\varepsilon^3$. Again the details are collected in Appendix A.

### 4.4 Wrinkles

Incremental instability is triggered by the apparition of small-amplitude wrinkles on the compressed face of the deformed sector. For bending ($\kappa > 1$) and eversion ($\kappa < 0$), this is the inner face; for unbending with $0 < \kappa < 1$, it is the outer face, see the last column of Figure 4.2.

The existence of small-amplitude wrinkles itself is governed by the incremental equations of incompressibility and of equilibrium. These equations can be formatted into the so-called Stroh formulation, a first-order system of linear equations with variable coefficients. We do not present the details of this derivation, which can be found in Destrade et al. [9].

It suffices to recall that the incremental mechanical displacements $u$ are sought in the form

$$u = \Re\{[U(r)e_r + V(r)e_\theta]e^{in\theta}\}, \quad n = \frac{m\pi}{\alpha_d} = \frac{m\pi}{\kappa\alpha_r} \quad (m \in \mathbb{N}),$$ (4.25)

where the amplitudes $U$ and $V$ are functions of $r$ only, and $n$ is a real number to be determined from the condition of no incremental normal tractions on the end faces $\theta = \pm \alpha_d$ of a sector; $m$ is an integer, which we call the circumferential mode number, giving the number of wrinkles on the contracted face. Then the components of the incremental nominal traction $\hat{S}$ have the same structure:

$$\hat{S}^T e_r = \Re\{[S_{rr}(r)e_r + S_{r\theta}(r)e_\theta]e^{in\theta}\}. \quad (4.26)$$

We can readily obtain the equations for the displacement-traction Stroh vector $\eta = [U, V, irS_{rr}, irS_{r\theta}]^T$ in the form [9]

$$\frac{d}{dr} \eta(r) = \frac{i}{r} G(r) \eta(r). \quad (4.27)$$
4.5. NUMERICAL RESULTS FOR WRINKLES

where $G$ is the Stroh matrix:

$$
G = \begin{pmatrix}
i & -n & 0 & 0 \\
-n(1 - \sigma_{rr}/\alpha) & -i(1 - \sigma_{rr}/\alpha) & 0 & -1/\alpha \\
\kappa_{11} & i\kappa_{12} & -i & -n(1 - \sigma_{rr}/\alpha) \\
-i\kappa_{12} & \kappa_{22} & -n & i(1 - \sigma_{rr}/\alpha)
\end{pmatrix}.
$$

(4.28)

Here

$$
\begin{align*}
\alpha &= \lambda \hat{W}'(\lambda)/\lambda^4 - 1, \\
\gamma &= \lambda^4 \alpha, \\
\beta &= \lambda^2 \hat{W}''(\lambda) - \alpha, \\
\kappa_{11} &= 2(\alpha + \beta - \sigma_{rr}) + n^2[\gamma - \alpha(1 - \sigma_{rr}/\alpha)^2], \\
\kappa_{12} &= n[2\beta + \alpha + \gamma - \sigma_{rr}^2/\alpha], \\
\kappa_{22} &= \gamma - \alpha(1 - \sigma_{rr}/\alpha)^2 + 2n^2(\alpha + \beta - \sigma_{rr}).
\end{align*}
$$

(4.29)

Now the system (4.27) needs to be integrated numerically, subject to the boundary conditions that the incremental traction vanish on the inner and outer faces, i.e.

$$
S_{rr}(a) = S_{rr}(b) = 0, \quad S_{r\theta}(a) = S_{r\theta}(b) = 0.
$$

(4.30)

4.5 Numerical results for wrinkles

The numerical techniques described in Appendix B can be implemented to predict the onset of instability in sectors made of any hyperelastic material. From now on we specialise our discussion to Mooney-Rivlin solids, for which

$$
W = \frac{C}{2} \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right) + \frac{D}{2} \left( \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 - 3 \right),
$$

(4.31)

where $C \geq 0$ and $D \geq 0$ are material constants.

For this class of nonlinearly elastic materials, the single variable function $\hat{W}$ reads

$$
\hat{W}(\lambda) = \frac{1}{2} \left[ (C\lambda_z^{-1} + D\lambda_z) (\lambda^2 + \lambda^{-2}) + C(\lambda_z^2 - 3) + D(\lambda_z^{-2} - 3) \right],
$$

(4.32)

from which the parameters $\alpha$, $\beta$ and $\gamma$ in (4.29), and the radial stress $\sigma_{rr}$ are found to be

$$
\begin{align*}
\alpha &= (C\lambda_z^{-1} + D\lambda_z) \lambda^{-2}, \\
\gamma &= (C\lambda_z^{-1} + D\lambda_z) \lambda^2, \\
\beta &= (\alpha + \gamma)/2, \\
\sigma_{rr} &= - (C\lambda_z^{-1} + D\lambda_z) \left[ \frac{\kappa^2 - 1}{2\kappa} \ln \left( \frac{\lambda_z^2 - \kappa}{\lambda_z\kappa - \kappa} \right) + \frac{1}{\kappa} \ln \left( \frac{\lambda}{\lambda_a} \right) + \frac{\lambda^2 - \lambda_a^2}{2\lambda_z^2\lambda_a^2} \right].
\end{align*}
$$

(4.33)

(4.34)
The scaled circumferential stretch $\lambda_a$ in (4.34) is the unique root of the equation
\[
\frac{\kappa^2 - 1}{\kappa} \ln \rho + \frac{1}{\kappa} \ln \sqrt{\rho^2 \lambda_a^2 + (1 - \rho^2)\kappa} + \frac{(1 - \rho^2)(\kappa - \lambda_a^2)}{2\lambda_a^2 [\rho^2 \lambda_a^2 + (1 - \rho^2)\kappa]} = 0,
\]
(4.35)
which is derived from (4.16) and (4.21) for the Mooney-Rivlin $\hat{W}$ of (4.32).

It can be easily shown that, in agreement with the thin-wall expansion (4.24), the unique solution to (4.35) tends to 1 as $\rho \to 1$ for any value of $\kappa$. For sectors with small radii ratio $\rho = A/B$ we distinguish the bending ($\kappa > 1$), unbending ($0 < \kappa < 1$) and eversion ($\kappa < 0$) cases and obtain the following respective approximations for the scaled circumferential stretch on the side under compression (i.e. $\lambda_a$ in bending, $\lambda_b$ in unbending and eversion):
\[
\begin{align*}
\lambda_a &\approx \sqrt{\frac{\kappa}{W_0(e\rho^2(1-\kappa^2))}} & \text{if } \kappa > 1, \\
\lambda_b &\approx \sqrt{\frac{\kappa}{1 + W_0(-\rho^{2\kappa^2}/e)}} & \text{if } 0 < \kappa < 1, \\
\lambda_b &\approx \sqrt{\frac{\kappa}{1 + W_{-1}(-\rho^{2\kappa^2}/e)}} & \text{if } \kappa < 0,
\end{align*}
\]
(4.36)
where $W_0$ and $W_{-1}$ are, respectively, the upper and lower branches of the real-valued Lambert-W function. From (4.36)$_1$ (resp., (4.36)$_3$) we deduce that for bending (respectively, eversion) the scaled circumferential stretch $\lambda_a$ (respectively, $\lambda_b$) is an infinitesimal quantity of the same order as $1/\sqrt{|\ln \rho|}$ when $\rho \to 0$ and thus it tends abruptly to 0 as $\rho \to 0$. For unbending, $\lambda_b \to \sqrt{\kappa}$ as $\rho \to 0$ and
\[
\frac{d\lambda_b}{d\rho} \to \begin{cases} 
+\infty & \text{if } \kappa \in \left(0, \frac{1}{\sqrt{2}}\right), \\
\frac{1}{2e\sqrt{2}} & \text{if } \kappa = \frac{1}{\sqrt{2}}, \\
0 & \text{if } \kappa \in \left(\frac{1}{\sqrt{2}}, 1\right).
\end{cases}
\]
(4.37)
Once again, for $\kappa \in (0, 1/\sqrt{2})$ the stretch on the side under compression changes rapidly near $\rho = 0$. Because of the asymptotic behaviour of the circumferential stretches $\lambda_a$ and $\lambda_b$, the stability problem for a sector with a very small $\rho$ is numerically stiff.

Now from (4.33) and (4.34) we can readily compute all the coefficients (4.29) of the Stroh matrix (4.28). As shown by Destrade et al. [9] and from (4.33)-(4.34), the incremental governing equations and boundary conditions can then be normalised in such a way that $C$, $D$ and $\lambda_z$ disappear (simply by dividing all equations across by $C\lambda_z^{-1} + D\lambda_z$). These quantities thus play no role in the stability analysis, and the following results are thus valid for all values of $C$, $D$, $\lambda_z$. This flexibility makes
4.5. NUMERICAL RESULTS FOR WRINKLES

the results quite general, because the Mooney-Rivlin model recovers not only the neo-Hookean model of exact non-linear elasticity \((D = 0)\) but also, at the same order of approximation, the general form of strain energy function for weakly non-linear third-order (isotropic, incompressible) elasticity [19, 20]:

\[
W = \mu \text{tr} (E^2) + \frac{A}{3} \text{tr} (E^3),
\]

(4.38)

where \(\mu > 0\) is the second-order Lamé coefficient, \(A\) is the third-order Landau constant and \(E = (F^T F - I)/2\) is the Green-Lagrange strain tensor.

Figures 4.3 and 4.4 report the critical values of the bending angles \(\alpha_d\) and the critical circumferential stretches on the corresponding contracted faces as functions of the radii ratio \(\rho = A/B\) for a sector with an undeformed angle \(\alpha_r = \pi/6\). In bending (Figure 4.3) the critical thresholds for \(\alpha_d\) and \(\lambda_a\) are plotted for \(\rho \in (0, 0.7619)\) because in this range, \(\alpha_d \leq \pi\). As \(\rho\) approaches 0.7619, \(\alpha_d\) approaches \(\pi\). Hence a circular cylindrical sector with \(\rho \in (0, 0.7619, 1)\) can be closed to form an intact tube without experiencing wrinkles on the inner face \(r = a\). In unbending/eversion (Figure 4.4) the critical thresholds are plotted for \(\rho \in (0, 0.8079)\). Here we see that a sector with \(\rho \in (0.8079, 1)\) can be completely everted to form an intact tube without the appearance of wrinkles on the inner side \(r = b\).

Figures 4.3 and 4.4 display curves corresponding to different circumferential mode numbers \(m = 1, \ldots, N\), which define the number of prismatic wrinkles appearing on the contracted side of a deformed sector. However, only the modes corresponding to the highest critical stretches \(\lambda_a\) and \(\lambda_b\) (correspondingly, the lowest critical angles \(\alpha_d\) in bending and the highest critical angles \(\alpha_d\) in unbending and eversion) are meaningful, as the lower stretches cannot be reached once a sector has buckled. We call these mode numbers the acute mode numbers.

For example, Figure 4.3a-c shows that the acute mode number for a sector with \(\alpha_r = \pi/6\) and \(\rho \in (0, 0.21)\) is \(m = 2\); for a sector with \(\rho \in (0.21, 0.39)\), it is \(m = 3\); and so on. We use circle markers to highlight the transitions from one acute mode number to another as shown in Figures 4.3a-c and 4.4a-c.

Now we provide a more in-depth examination of how the critical deformations and number of wrinkles in bending and eversion depend on the referential geometry; in particular, how they differ for the same sector in bending and eversion (unbending \(\alpha_r > \alpha_d > 0\) is not that noticeable in Figure 4.4a for \(\alpha_r = \pi/6\) and, thus, will be illustrated in the subsequent discussion for another \(\alpha_r\)). To this end, we pick the geometries labeled \(\Box\), \(\Box\), \(\Box\) as shown in Figures 4.3b, 4.4b, denoting sectors with initial radii ratios \(\rho \simeq 0.1, 0.486, 0.67\), respectively, and referential angle \(\alpha_r = \pi/6\). In Figures 4.3d and 4.4d we show an \((r, \theta)\) plane view of Sector \(\Box\) in the reference configuration and at buckling (the lengths are normalised with respect to the initial thickness \(H = B - A\)). Table 4.1 collects the results of the incremental stability analysis. We note that these sectors \(\Box\), \(\Box\), \(\Box\) present one more wrinkle in bending than in eversion, although this
Figure 4.3: Critical deformation angles $\alpha_d$ (a) and critical stretches $\lambda_a$ [(b): scaled and (c): regular] plotted versus radii ratios $\rho = A/B$ for sectors with $\alpha_r = \pi/6$ and mode numbers $m = 1, \cdots, 10$ undergoing plane strain bending. $(r, \theta)$ plane view (d) of the sectors indicated in (b) in the undeformed configuration and at buckling. The numerical results for the specific sectors $\text{A, B, C}$ are shown in Table 4.1; in (d), the lengths are normalised with respect to the initial thickness $H = B - A$, so that $A = \rho H/(1 - \rho)$ and $B = H/(1 - \rho)$.

is not always the case for other referential geometries.

To validate our results here, we connect with the analysis of Haughton [2] for the flexure of rectangular blocks. Even though we have an initial curvature for our sectors (as opposed to the rectangular geometry from [2]), we similarly discover that the different acute mode numbers effectively form an envelope predicting loss of stability at the critical stretch of approximately $\lambda = 0.563$ (Figures 4.3c and 4.4c), see also [4].

The discussion above was conducted for sectors with $\alpha_r = \pi/6$. In Figure 4.5 we provide the critical deformation angles $\alpha_d$ as functions
4.5. NUMERICAL RESULTS FOR WRINKLES

Figure 4.4: Critical deformation angles $\alpha_d$ (a) and critical stretches $\lambda_b$ (b – scaled and c – regular) plotted versus radii ratios $\rho = A/B$ for sectors with $\alpha_r = \pi/6$ and mode numbers $m = 1, \cdots, 11$ undergoing plane strain unbending ($\alpha_r > \alpha_d > 0$) and eversion ($\alpha_d < 0$). $(r, \theta)$ plane view (d) of the sectors indicated in (b) in the undeformed configuration and at buckling. The numerical results for the specific sectors A–C are shown in Table 4.1; in (d), the lengths are normalised with respect to the initial thickness $H = B - A$, so that $A = \rho H/(1 - \rho)$ and $B = H/(1 - \rho)$.

of the radii ratio $\rho = A/B$ for sectors corresponding to other relevant undeformed angles $\alpha_r$ (labeled by $\square$). Figure 4.5a illustrates critical deformations for bending and Figure 4.5b for unbending ($\alpha_r > \alpha_d > 0$) and eversion ($\alpha_d < 0$). Only the curves corresponding to the acute mode numbers and points of transition between them are displayed, while the other information is not reported (compare, for illustration, critical deformation angles $\alpha_d$ for sectors with $\alpha_r = \pi/6$ in Figures 4.3a, 4.4a and sectors with $\alpha_r = \pi/6$ in Figure 4.5). Labels $\bigcirc_1$, $\bigcirc_2$, $\bigcirc_3$ are used to indicate buckling states of sectors illustrated in Figure 4.2.
CHAPTER 4. WRINKLES AND CREASES IN BENDING, UNBENDING

<table>
<thead>
<tr>
<th>Sector</th>
<th>$\rho = A/B$</th>
<th>$\alpha_r$</th>
<th>$m$</th>
<th>$\alpha_d$</th>
<th>$\lambda_a$</th>
<th>$m$</th>
<th>$\alpha_d$</th>
<th>$\lambda_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.1</td>
<td>$\pi/6$ 2</td>
<td>0.2425$\pi$</td>
<td>0.5604</td>
<td>1</td>
<td>0.0049$\pi$</td>
<td>0.5654</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>0.486</td>
<td>$\pi/6$ 4</td>
<td>0.4562$\pi$</td>
<td>0.5607</td>
<td>3</td>
<td>0.0022$\pi$</td>
<td>0.5614</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0.67</td>
<td>$\pi/6$ 7</td>
<td>0.7175$\pi$</td>
<td>0.5609</td>
<td>6</td>
<td>0.0082$\pi$</td>
<td>0.5619</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>0.486</td>
<td>$\pi/3$ 9</td>
<td>0.9124$\pi$</td>
<td>0.5607</td>
<td>6</td>
<td>0.0043$\pi$</td>
<td>0.5626</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Critical parameters for bending and eversion of three sectors from Figures 4.3-4.4 and one sector from the following section.

For each referential angle $\alpha_r$, there exist limiting radii ratios $\rho^*_\alpha, \rho^{**}_\alpha \in (0, 1)$ such that $\alpha_d > \pi$ for all $\rho \in (\rho^*_\alpha, 1)$ in bending, and $\alpha_d < -\pi$ for all $\rho \in (\rho^{**}_\alpha, 1)$ in eversion. This means that a sector with $\rho \in (\rho^*_\alpha, 1)$ (resp., $\rho \in (\rho^{**}_\alpha, 1)$) can be closed (respectively, completely everted) to form an intact tube without the appearance of wrinkles on the inner side. Table 4.2 reports the radii ratios $\rho^*_\alpha$ and $\rho^{**}_\alpha$ for all the referential angles in Figure 4.5 as well as the corresponding acute mode numbers in bending and eversion.

Figure 4.5 and Table 4.2 contain then all the required information to form a theoretical prediction on whether, when and how a given sector will wrinkle. Notice that the overall largest acute number was found to be $m = 14$, in eversion, as shown in Figure 4.5b. For shorter wavelengths (larger $m$), the buckling occurs for sectors with deformed angles such that $|\alpha_d| > \pi$, which is physically impossible.

<table>
<thead>
<tr>
<th>$\alpha_r$</th>
<th>$\rho^*_\alpha$</th>
<th>$m$</th>
<th>$\rho^{**}_\alpha$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>-</td>
<td>-</td>
<td>0.423</td>
<td>14</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>0.1193</td>
<td>8</td>
<td>0.518</td>
<td>13</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0.3092</td>
<td>9</td>
<td>0.5866</td>
<td>12</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>0.5295</td>
<td>9</td>
<td>0.6787</td>
<td>12</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>0.6445</td>
<td>10</td>
<td>0.7374</td>
<td>11</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>0.7619</td>
<td>10</td>
<td>0.8079</td>
<td>11</td>
</tr>
<tr>
<td>$\pi/12$</td>
<td>0.8808</td>
<td>10</td>
<td>0.8936</td>
<td>11</td>
</tr>
<tr>
<td>$\pi/36$</td>
<td>0.9603</td>
<td>11</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.2: Values of the limiting radii ratios $\rho^*_\alpha$ and $\rho^{**}_\alpha$ for different values of the referential angle $\alpha_r$. The acute mode numbers $m$ indicate the number of wrinkles appearing on the inner face of the intact tube obtained by bending (resp., evert ing) a sector with exact radii ratio $\rho^*_\alpha$ (resp., $\rho^{**}_\alpha$). Sectors with radii ratio greater than $\rho^*_\alpha$ (resp., $\rho^{**}_\alpha$) can be completely closed (everted) into an intact tube without wrinkling.

Finally, to illustrate the generality of our wrinkling analysis, we make the connection with three special cases already reported in the literature: closing of a cylindrical sector into an intact tube [9], straightening of a cylindrical sector into a rectangular block [7, 8], and bending of a rectangular block into a sector of a circular cylinder [4].
In the case of closing of a sector into an intact tube on the onset of instability, we recover the critical deformations from [9] by simply looking at the limiting values of $\alpha_d = \pi$ for different $\alpha_r$ (Figure 4.5a). What is novel here is that our solution can address the situation when the sector buckles before the full closing (Figure 4.5b, $\alpha_d < \pi$).

Similar observations can be made for the straightening of a cylindrical sector. Critical deformations from [7] can be retrieved when we take $\alpha_d = 0$ for different $\alpha_r$ (Figure 4.5b) and again, our treatment is able to predict how sectors buckle before the exact straightening (Figure 4.5b, $0 < \alpha_d < \alpha_r$).

Next, when making the link with stability results for bending of a rectangular block, we are not able to derive the solution for $\alpha_r = 0$ due to characteristic singularity, but we can simply consider a small referen-
CHAPTER 4. WRINKLES AND CREASES IN BENDING, UNBENDING

tial angle $\alpha_r = \pi/36$, say (Figure 4.5a), and the corresponding critical deformations then match the results from [4] very well.

To sum up, from the problem of bending, unbending and eversion of a cylindrical sector we are able to recover critical deformations of three classical universal deformations of incompressible non-linear elasticity. Moreover, it allows us to obtain new results on critical deformations of a circular sector bent, unbent or everted (Figure 4.5b, $\alpha_d < 0$) into another sector.

4.6 Numerical results for creases

Our treatment of small-amplitude wrinkles superimposed on large bending and unbending presented in the previous section is rigorous and complete. However, it fails to predict the behaviour actually observed in the laboratory when sectors are bent or unbent too severely: their compressed side does buckle, but earlier than predicted by the incremental theory, and creases develop instead of smooth sinusoidal wrinkles, see Figures 4.1 and 4.6.

![Figure 4.6: Bending (left) and eversion (right) of a cylindrical sector made of silicone, with reference angle $\alpha_r = \pi/3$ and radii ratio $\rho = A/B = 0.486$. Notice how creases appear on the contracted face, not wrinkles. We denote this physical sector as Sector $D$. In bending, we count about six or seven creases; in eversion, only one.](image)

This observation is well known and documented for the buckling of homogeneous solids, see for example experimental pictures for the bending of blocks [21, 22, 5, 23, 24], of a cylinder [25] and of a sector [9], the torsion of a cylinder [26] and of a tube [24], the eversion of a tube [27, 24] and the shear-box deformation of a block [24]. It has also been successfully captured by Finite Element (FE) simulations, see the seminal articles by Hong et al. [22], Hohlfeld and Mahadevan [25], and Cao and Hutchinson [28, 29] (the latter include a nonlinear post-bifurcation analysis and imperfection sensitivity). Note that there are very few FE simulations of creases in cylindrical coordinates [30, 27, 31].

For our table-top experiments, we prepared a circular sector (Sector $D$) of soft silicone of height 58 mm, inner radius $A = 35$ mm, outer radius
4.6. NUMERICAL RESULTS FOR CREASES

$B = 72$ mm (so that $\rho = A/B = 0.486$), and reference angle $\alpha_r = \pi/3$. We used superglue to attach two $70 \times 70$ mm$^2$ squares of acrylic glass to the end faces of the sectors to bend or unbend the sector by applying torques mostly, and as little normal forces as possible. Here the sector is not stretched axially ($\lambda_z = 1$) and the scaled circumferential stretch $\lambda$ defined in equation (4.15) coincides with $\lambda_\theta$ of equation (4.9). According to the incremental analysis summarised in Figure 4.5, for this sector we should expect nine wrinkles to form in bending when $\alpha_d = 0.9124 \pi$, and six in eversion when $\alpha_d = -0.4372 \pi$. We collected the results of the incremental stability analysis for Sector $[\square]$ on the last line of Table 4.1.

In practice, we do not observe the formation of sinusoidal wrinkles in bending, but instead the formation of about eight creases. Moreover, although these creases are regularly spaced, some of them sometimes merge (period-doubling), depending on a given bending event. In unbending the surface of the sector does not buckle. In eversion, it buckles with a single deep crease in the middle of the everted sector, see Figure 4.6. However, we note that a perfect unbending so that the everted block has a circular shape is very hard to effect in practice.

To investigate numerically the formation of wrinkles/creases, we implemented FE models in ABAQUS/Explicit. For computational efficiency, we considered a 2D sector only, as in any case we are primarily interested in prismatic buckling. We chose a long time for the analysis to ensure a quasi-static deformation. Because perfect incompressibility is not possible in ABAQUS/Explicit, we used the neo-Hookean model with an initial bulk modulus 100 larger than the initial shear modulus to achieve near-incompressibility. We used linear reduced integration quadrilateral elements (CPE4R).

When a displacement is prescribed (say of a side of the sector from one location to another), ABAQUS implements it as taking place along a straight line. Thus, if one applies a displacement which deforms an undeformed sector into a closed tube (through bending or unbending), the intermediate deformation is not in agreement with the deformation described by (4.4), (4.8) for deforming an undeformed sector into a bent or unbent sector. To solve this problem, we thus implemented a sequence of small displacements rather than a single large displacement. Here, to ensure the sector was deformed along the ‘correct’ path, that is the path closest to that of the exact solution given by (4.4), (4.8), we considered $N$ deformations:

$$r = r(R), \quad \theta = i\frac{\kappa}{N}\Theta, \quad z = \lambda_z Z,$$

(4.39)

for $i = 1, 2, 3, \ldots, N$, where $\kappa = \pi/\alpha_r$ for bending and $\kappa = -\pi/\alpha_r$ for unbending. Here, the $N$-th deformation is the deformation which closes the sector completely, while the deformation for $i < N$ refers to an intermediate deformation. For each of these deformations, we calculated the deformed geometry at each node of the two end faces and the non-buckling face using (4.8), (4.16) and (4.21). From here, we calculated the displacements necessary to go from the $i$-th deformation to the $(i+1)$-th deformation, resulting in a set of $N$ displacements for each node. Then
we created $N$ steps in ABAQUS, in each step imposing the calculated
displacements at each node of the two end faces and on the non-buckling
face. In practice, we used about 100 steps. Zero tractions boundary
condition was used on the remaining surface where the buckling will
occur.

To initiate the buckling of the contracted face, we added a sinusoidal
game perturbation of very small amplitude along that contour [28,
30]. Effectively, the amplitude was three orders of magnitude smaller
than the radii.

The simulations revealed the spontaneous formation of creases, with
no smooth transition from the sinusoidal perturbation. The creases
deeper quickly as the deformation progresses, and their sides come into
contact, consistent with the analysis for compression of a half-space [28].
There was also a spontaneous merging of some adjacent creases to form
period-doubling patterns, so that the number of final creases was often
less than the number $n$ of wrinkles that they emerged from.

We conducted a mesh sensitivity analysis with respect to the onset
of buckling, which was identified by a drop in the elastic energy per unit
thickness of the creased sector compared to that in the smooth body
[22]. We generally found that buckling occurred earlier every time the
mesh was made finer. The solution started to converge as the number of
elements increased, but eventually a very large number of elements was
required to ascertain the limit, up to 240,000 elements, see Figure 4.7. We
had to use a gradient of mesh refinement near the compressed face and
to book time on a high-end computer to perform these calculations. An
alternative would have been to add a thin coating of vanishing stiffness
[25].

We also conducted a spectral analysis with respect to the linear si-
inusoidal perturbation, that is, we performed the simulations for $m =
2, 3, 4, \ldots$ wrinkles, and kept the one for which the buckling occurred the
earliest. For most cases (with the exception of sector $D$ in unbending),
the number of periods in the sinusoidal perturbation corresponding to
earliest buckling was the same or close to the number of wrinkles pre-
picted by the linear analysis. We also note that the number $m$ is evident
through the development of stress concentrations along the buckling faces
in the lead up to buckling, see the images on the left-hand side of Figure
4.8. We conclude that the incremental analysis tends to provide a good
indication of the optimal shape for the perturbation.

For the simulation of the physical Sector $D$ we used 183,750 elements.
We found that buckling occurred at its earliest in bending when $m = 9$
for the deformed angle $\alpha_d = 0.842\pi$, and $m = 9$ in unbending when
$\alpha_d = -0.374\pi$, with period-doubling occurring at several locations, see
Figure 4.8.

For Sector $C$, we used 240,000 elements. We found that buckling
responded to $m = 7$ in bending when $\alpha_d = 0.633\pi$, and to $m = 5$ in
unbending when $\alpha_d = -0.392\pi$, with period-doubling occurring again.

The crease formations which occurred in our finite element simula-
tions are consistent with our table-top experiments (at least with the
4.6. NUMERICAL RESULTS FOR CREASES

Figure 4.7: Red point line: convergence with the refinement of the mesh of the critical circumferential stretch of compression for the formation of creases, here when bending Sector C. For comparison, the level of critical stretch is given for wrinkles (red dashed line below). We see that creases occur at about 4% compression earlier than wrinkles. In contrast, creases appear on the surface of a half-space compressed in plane strain more than 10% compression earlier (upper black full line) than wrinkles (lower dashed black line) [22].

bending experiment; the eversion experiment with its single crease is not well captured by the modelling, which assumed a circular everted sector, a geometry which is impossible to obtain in practice). But there are differences with the non-linear stability analyses of crease formation conducted previously.

For example, Hong et al. [22] showed that a semi-infinite body of neo-Hookean material creases in plane strain at a critical amount of stretch equal to 0.65, which is 11% strain earlier than 0.54, the critical stretch for wrinkles found by Biot [32]. Here, we found that for our sectors the difference between crease onset and wrinkle onset was 4% or less. Also, half-space crease analysis does not provide a wavelength for the crease, since there is no characteristic length in that context. Here we note the agreement (or near agreement) between the number of creases in the finite element simulations and the number of wrinkles predicted by the linear analysis.

In conclusion, the incremental stability provides valuable information on the loss of stability for the large bending or unbending of a circular sector. It will also be quite straightforward to extend it to material models other than Mooney-Rivlin. Finally, it provides the basis for the study of coated materials where sinusoidal wrinkles are expected to dominate [28].
CHAPTER 4. WRINKLES AND CREASES IN BENDING, UNBENDING

Acknowledgements

MD and RM gratefully acknowledge the financial support of the Irish Research Council. LS and TS are grateful for the support received from the Schulich School of Engineering and Zymetrix Biomaterials & Tissue Engineering Technology Development Centre, University of Calgary. LV would like to thank the Carnegie Trust for the financial support (R&E Project Code: 67954/1). This publication has also been made possible by a James M Flaherty Research Scholarship from the Ireland Canada University Foundation, with the assistance of the Government of Canada/avec l’appui du gouvernement du Canada. The authors gratefully acknowledge the SFI/HEA Irish Centre for High-End Computing (ICHEC) for the provision of computational facilities and support.

Appendix A: Proofs of existence and uniqueness; Thin-wall expansions

4.6.1 Existence and uniqueness

We assume that the strain-energy function $W$ satisfies the strong ellipticity condition. As shown by Ogden [17], it amounts to

$$\frac{\lambda}{\lambda^2 - 1} \hat{W}'(\lambda) > 0, \quad \lambda^2 \hat{W}'' + \frac{2\lambda}{\lambda^2 + 1} \hat{W}'(\lambda) > 0. \quad (4.40)$$

From these inequalities, respectively, we deduce that

$$\hat{W}'(\lambda) \geq 0 \quad (\text{as } \lambda \geq 1) \quad \text{and} \quad \hat{W}''(1) > 0. \quad (4.41)$$

Finally, by integrating the second inequality (4.40) we deduce that, in a right neighbourhood of 0,

$$\frac{\lambda^2}{\lambda^2 + 1} |\hat{W}'(\lambda)| > c, \quad (4.42)$$

for some positive constant $c$, from which follows that

$$\lim_{\lambda \to 0^+} \lambda^2 |\hat{W}'(\lambda)| \geq c > 0. \quad (4.43)$$

We now investigate the existence and uniqueness of a positive root to (4.21), which we recall here:

$$\int_{\lambda_a}^{\lambda_b} \frac{\hat{W}'(s)}{\kappa - s^2} ds = 0. \quad (4.44)$$

First, assume that $\kappa \in [-2\pi/\alpha_r, 0]$. Then $\lambda_a^2 > (1 - \rho^2)|\kappa|/\rho^2$ and $\lambda \in [\lambda_b, \lambda_a]$. It follows that if $\lambda_b \geq 1$ or $\lambda_a \leq 1$, then the integrand has the same sign over the entire range of integration and hence (4.44) does not admit a solution. Hence we must have $\lambda_b < 1 < \lambda_a$, or, using (4.16),

$$\frac{\sqrt{1 - \rho^2}|\kappa|}{\rho} < \lambda_a < \frac{\sqrt{1 + (1 - \rho^2)|\kappa|}}{\rho}. \quad (4.45)$$
4.6. NUMERICAL RESULTS FOR CREASES

To prove the existence of a root for (4.44) we set

$$\lambda^* = \max \left\{ \sqrt{\frac{(1 - \rho^2)\kappa}{\rho}} , 1 \right\},$$

(4.46)

and define the function $f$ as

$$f : y \in \left[ \lambda^*, \frac{\sqrt{1 + (1 - \rho^2)\kappa}}{\rho} \right] \to \int_y^1 \frac{\hat{W}'(s) - \hat{W}' \left( \sqrt{\rho^2 y^2 + (1 - \rho^2)\kappa} \right)}{\kappa - s^2} \, ds.$$  (4.47)

Next, since $\sqrt{\rho^2 y^2 + (1 - \rho^2)\kappa} < y$ for all $y \in \text{Dom}(f)$ and $\kappa - y^2$ is negative, from (4.40) we conclude that

$$f'(y) = \frac{1}{\kappa - y^2} \left[ \hat{W}'(y) - \hat{W}' \left( \sqrt{\rho^2 y^2 + (1 - \rho^2)\kappa} \right) \right] < 0,$$  (4.48)

whence $f$ is a decreasing function. For $|\kappa| \leq \rho^2 / (1 - \rho^2)$, from (4.41) we deduce that

$$f(\lambda^*) = f(1) = \int_1^1 \frac{\hat{W}'(s)}{\kappa - s^2} \, ds > 0,$$  (4.49)

and for $|\kappa| > \rho^2 / (1 - \rho^2)$, from (4.43) we deduce that

$$f(\lambda^*) = f \left( \sqrt{\frac{(1 - \rho^2)\kappa}{\rho}} \right) = \int_0^{\sqrt{\frac{(1 - \rho^2)\kappa}{\rho}}} \frac{\hat{W}'(s)}{\kappa - s^2} \, ds = +\infty.$$  (4.50)

At the other end of the interval where $f$ is defined we have, as a consequence of (4.41),

$$f \left( \sqrt{\frac{(1 - \rho^2)\kappa}{\rho}} \right) = \int_1^{\sqrt{\frac{(1 - \rho^2)\kappa}{\rho}}} \frac{\hat{W}'(\lambda)}{\kappa - s^2} \, ds < 0.$$  (4.51)

In conclusion, $f$ has exactly one zero in its domain and so, when $\kappa \in [-2\pi/\alpha_r, 0[$, equation (4.44) admits a unique root in the range $x^*, \frac{\sqrt{1 + (1 - \rho^2)\kappa}}{\rho}$.

We assume now that $0 < \kappa < 1$. If $0 < \lambda_a < \sqrt{\kappa}$, then the integrand in (4.44) is strictly negative and the equation does not admit a solution. Thus we must have $\sqrt{\kappa} < \lambda_b < 1 < \lambda_a$, or using (4.16),

$$\kappa < 1 < \lambda_a < \sqrt{1 - (1 - \rho^2)\kappa}/\rho.$$  (4.52)

Consider the function $g$ defined as

$$g : y \in \left[ 1, \frac{\sqrt{1 - (1 - \rho^2)\kappa}}{\rho} \right] \to \int_y^1 \frac{\hat{W}'(s)}{\kappa - s^2} \, ds.$$  (4.53)
With the aid of (4.41)_1, we deduce in turn that
\[
g'(y) = \frac{1}{\kappa - y^2} \left[ \hat{W}'(y) - \hat{W}' \left( \sqrt{\rho^2 y^2 + (1 - \rho^2) \kappa} \right) \frac{y}{\sqrt{\rho^2 y^2 + (1 - \rho^2) \kappa}} \right] < 0,
\]
\[
g \left( \frac{\sqrt{1 - (1 - \rho^2) \kappa}}{\rho} \right) = \int_1^{\sqrt{1 - (1 - \rho^2) \kappa}} \frac{\hat{W}'(s)}{\kappa - s^2} ds < 0,
\]
\[
g(1) = \int_1^{\sqrt{\rho^2 + (1 - \rho^2) \kappa}} \frac{\hat{W}'(s)}{\kappa - s^2} ds > 0.
\]

From these we conclude that \( g \) has exactly one zero in its domain and so (4.44) admits a unique solution
\[
\lambda_a \in \left[ 1, \frac{\sqrt{1 - (1 - \rho^2) \kappa}}{\rho} \right].
\]

Finally in the case \( \kappa > 1 \) it is easy to show that (4.44) admits a solution only if \( \lambda_a < 1 < \lambda_b < \sqrt{\kappa} \), or using (4.16),
\[
\lambda^* < \lambda_a < 1,
\]
where
\[
\lambda^* = \sqrt{\max \{0, 1 - (1 - \rho^2) \kappa / \rho^2\}}.
\]

Then we consider the function \( h \) defined as
\[
h : y \in [\lambda^*, 1] \mapsto \int_y^{\sqrt{\rho^2 y^2 + (1 - \rho^2) \kappa}} \frac{\hat{W}'(s)}{\kappa - s^2} ds.
\]

Assume first that \( 0 < \kappa < 1/(1 - \rho^2) \). Then, with the aid of (4.41)_1, we find that
\[
h(\lambda^*) = h \left( \frac{\sqrt{1 - (1 - \rho^2) \kappa}}{\rho} \right) = \int_1^{\sqrt{1 - (1 - \rho^2) \kappa} / \rho} \frac{\hat{W}'(s)}{\kappa - s^2} ds < 0.
\]

Conversely, when \( \kappa \geq 1/(1 - \rho^2) \), we have \( \lambda^* = 0 \) and, in view of (4.43), we deduce that
\[
\lim_{y \to 0^+} h(y) = \int_0^{\sqrt{1 - \rho^2} / \kappa} \frac{\hat{W}'(s)}{\kappa - s^2} ds = -\infty.
\]

Thanks to (4.41)_1 we have
\[
h(1) = \int_1^{\sqrt{\rho^2 + (1 - \rho^2) \kappa}} \frac{\hat{W}'(s)}{\kappa - s^2} ds > 0,
\]
and
\[
h'(y) = \frac{1}{\kappa - y^2} \left[ \hat{W}' \left( \sqrt{\rho^2 y^2 + (1 - \rho^2) \kappa} \right) \frac{y}{\sqrt{\rho^2 y^2 + (1 - \rho^2) \kappa}} - \hat{W}'(y) \right] > 0.
\]
4.6. NUMERICAL RESULTS FOR CREASES

It follows that \( h \) has exactly one zero in its domain and thus (4.44) admits a unique solution

\[
\lambda_n \in \left[ \sqrt{\max \left\{ 0, \frac{1 - (1 - \rho^2)\kappa}{\rho^2} \right\}}, 1 \right].
\]  

(4.63)

Finally, it is worth noting that from (4.55) and (4.63) \( \lambda_n \to 1 \) as \( \kappa \to 1 \). On the other hand, from (4.16) and (4.40) we readily deduce that the unique root to (4.44) with \( \lambda_n = 1 \) is \( \kappa = 1 \). In other words, the axial stretching (4.10) with \( \kappa = 1 \) is the only admissible homogeneous deformation.

### 4.6.2 Thin-walled sectors

For thin sectors, we perform an asymptotic analysis in the small thickness parameter \( \varepsilon > 0 \) defined as: \( \varepsilon = 1 - \rho \ll 1 \).

First we rewrite the left-hand side of (4.44) as a function \( F \) of \( \varepsilon \), specifically

\[
F(\varepsilon) = \int_{\lambda_n}^{\sqrt{(1-\varepsilon)^2\lambda^2 + \varepsilon(2-\varepsilon)\kappa}} \frac{W''(\lambda)}{\kappa - \lambda^2} d\lambda.
\]  

(4.64)

Expanding \( F(\varepsilon) \) as a Maclaurin series in \( \varepsilon \) up to the fifth order, substituting into the equation \( f(\lambda_n) = 0 \) and dropping a common factor \( \varepsilon \), yields the equation

\[
\hat{W}'(\lambda_n) + F^{(1)}(\varepsilon) + F^{(2)}(\varepsilon)^2 + F^{(3)}(\varepsilon)^3 + F^{(4)}(\varepsilon)^4 + O(\varepsilon^5) = 0,
\]  

(4.65)

where

\[
F^{(1)} = \frac{1}{\lambda_n^2} \left[ (2\lambda_n^2 - \kappa)\hat{W}'(\lambda_n) - \lambda_n(\lambda_n^2 - \kappa)\hat{W}''(\lambda_n) \right],
\]

\[
F^{(2)} = \frac{1}{\lambda_n^4} \left[ (6\lambda_n^4 - 7\lambda_n^2\kappa + 3\kappa^2)\hat{W}'(\lambda_n) - \lambda_n(4\lambda_n^4 - 7\lambda_n^2\kappa + 3\kappa^2)\hat{W}''(\lambda_n) + \lambda_n^2(\lambda_n^2 - \kappa)^2\hat{W}'''(\lambda_n) \right],
\]

\[
F^{(3)} = \frac{1}{24\lambda_n^8} \left[ 3(8\lambda_n^6 - 16\lambda_n^4\kappa + 15\lambda_n^2\kappa^2 - 5\kappa^3)\hat{W}'(\lambda_n) - 3\lambda_n(6\lambda_n^6 - 16\lambda_n^4\kappa + 15\lambda_n^2\kappa^2 - 5\kappa^3)\hat{W}''(\lambda_n) + 6\lambda_n^2(\lambda_n^2 - \kappa)^2\hat{W}'''(\lambda_n) - \lambda_n^3(\lambda_n^2 - \kappa)(2\lambda_n^2 - \kappa)^3\hat{W}^{(iv)}(\lambda_n) \right],
\]

\[
F^{(4)} = \frac{1}{120\lambda_n^{10}} \left[ 3(40\lambda_n^8 - 120\lambda_n^6\kappa + 183\lambda_n^4\kappa^2 - 130\lambda_n^2\kappa^3 + 35\kappa^4)\hat{W}'(\lambda_n) - \lambda_n(96\lambda_n^8 - 360\lambda_n^6\kappa + 539\lambda_n^4\kappa^2 - 390\lambda_n^2\kappa^3 + 105\kappa^4)\hat{W}''(\lambda_n) + 3\lambda_n^2(12\lambda_n^8 - 50\lambda_n^6\kappa + 79\lambda_n^4\kappa^2 - 56\lambda_n^2\kappa^3 + 9\kappa^4)\hat{W}'''(\lambda_n) - 2\lambda_n^3(\lambda_n^2 - 17\lambda_n^6\kappa + 27\lambda_n^4\kappa^2 - 19\lambda_n^2\kappa^3 + 5\kappa^4)\hat{W}^{(iv)}(\lambda_n) + \lambda_n^4(\lambda_n^8 - 4\lambda_n^6\kappa + 6\lambda_n^4\kappa^2 - 4\lambda_n^2\kappa^3 + \kappa^4)\hat{W}^{(v)}(\lambda_n) \right].
\]  

(4.67)
Next, we expand $\lambda_a$ in terms of $\varepsilon$ to the fourth order,

$$\lambda_a = \lambda_a^{(0)} + \lambda_a^{(1)}\varepsilon + \lambda_a^{(2)}\varepsilon^2 + \lambda_a^{(3)}\varepsilon^3 + \lambda_a^{(4)}\varepsilon^4 + \mathcal{O}(\varepsilon^5), \quad (4.68)$$

where the $\lambda^{(i)}$ are determined in turn as follows.

Substituting the expansion of $\lambda_a$ into the previous expansion (4.65) and equating to zero the coefficients of each power in the resulting expression, we obtain first, at zero order, that

$$\hat{W}'(\lambda_a^{(0)}) = 0,$$

and hence, by (4.41), that $\lambda_a^{(0)} = 1$.

Using this result in the first-order term, we then obtain

$$\left[ \frac{1}{2}(\kappa - 1) + \lambda_a^{(1)} \right] \hat{W}''(1) = 0,$$

and because $\hat{W}''(1) > 0$, we deduce that $\lambda_a^{(1)} = (1 - \kappa)/2$.

Then, the second-order term yields

$$\left[ \lambda_a^{(2)} + \frac{5}{12}(\kappa - 1) \right] \hat{W}''(1) + \frac{(\kappa - 1)^2}{24} \hat{W}'''(1) = 0. \quad (4.70)$$

The resulting expression for $\lambda_a$, up to the second order in $\varepsilon$, is therefore

$$\lambda_a = 1 + \frac{1}{2}(1 - \kappa)\varepsilon + \frac{1 - \kappa}{24} \left[ 10 - \frac{\hat{W}''(1)}{\hat{W}'(1)} (1 - \kappa) \right] \varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (4.71)$$

However, in virtue of the universal result: $\hat{W}'''(1)/\hat{W}''(1) = -3$ (see for example [16]), the above formula reduces to

$$\lambda_a = 1 - \frac{1}{2}(1 - \kappa)\varepsilon + \frac{1}{24} (1 - \kappa)(13 - 3\kappa)\varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (4.72)$$

Proceeding in a similar way (and omitting the lengthy details), we obtain, up to the fourth order in $\varepsilon$,

$$\lambda_a = 1 + \frac{1}{2}(1 - \kappa)\varepsilon + \frac{1}{24} (1 - \kappa)(13 - 3\kappa)\varepsilon^2 - \frac{1}{48} (1 - \kappa)(3\kappa^2 + 8\kappa - 27)\varepsilon^3$$

$$+ \frac{1}{5760} (1 - \kappa) \left[ 45\kappa^3 - 363\kappa^2 - 1813\kappa + 3667 \right] \varepsilon^4 \mathcal{O}(\varepsilon^5). \quad (4.73)$$

Note, in particular, that the results are independent of the strain energy function up to order $\varepsilon^3$.

**Appendix B: Algorithms for the analysis of the Stroh problem**

Here we outline two numerically robust methods to obtain the numerical solution of the Stroh problem (4.27).
The first one is called the compound matrix method. In this method, we let $\eta^{(1)}$, $\eta^{(2)}$ be two linearly independent solutions of (4.27), and use them to generate the six compound functions $\phi_1 = \langle \eta_1, \eta_2 \rangle$, $\phi_2 = \langle \eta_1, \eta_3 \rangle$, $\phi_3 = i \langle \eta_1, \eta_4 \rangle$, $\phi_4 = i \langle \eta_2, \eta_3 \rangle$, $\phi_5 = \langle \eta_2, \eta_4 \rangle$, $\phi_6 = \langle \eta_3, \eta_4 \rangle$, where $\langle \eta_i, \eta_j \rangle \equiv \eta_i^{(1)} \eta_j^{(2)} - \eta_i^{(2)} \eta_j^{(1)}$. Now, computing the derivatives of $\phi_i$ with respect to $r$ yields the so-called compound equations

$$\frac{d\phi}{dr} = \frac{1}{r} A(r) \phi(r),$$

(4.1)

where $\phi = (\phi_1, \ldots, \phi_6)^T$ and $A$, the compound matrix, has the form

$$A = \begin{pmatrix}
-\sigma_{rr}/\alpha & 0 & -1/\alpha & 0 & 0 & 0 \\
-\kappa_{12} & 0 & -n(1 - \sigma_{rr}/\alpha) & -n & 0 & 0 \\
-\kappa_{22} & n & -2(2 - \sigma_{rr}/\alpha) & n(1 - \sigma_{rr}/\alpha) & -1/\alpha & 0 \\
\kappa_{11} & n(1 - \sigma_{rr}/\alpha) & 0 & (2 - \sigma_{rr}/\alpha) n(1 - \sigma_{rr}/\alpha) & -1/\alpha & 0 \\
-\kappa_{12} & 0 & -n(1 - \sigma_{rr}/\alpha) & -n & 0 & 0 \\
0 & -\kappa_{12} & \kappa_{11} & -\kappa_{22} & -\kappa_{12} & \sigma_{rr}/\alpha
\end{pmatrix}.
$$

(4.2)

The compound equations (4.1) must be integrated numerically, starting with the initial condition $\phi(a) = \phi_1(a)[1, 0, 0, 0, 0, 0]^T$, and aiming at the target condition $\phi_6(b) = 0$, according to (4.30). In passing, note that $A$ is clearly singular, as in the straightening problem (noticed by [7, 8]) and in the bending of a straight block (unnoticed [2, 3, 4, 5]). However, it turns out that the singularity of the matrix does not affect the efficiency of the integration scheme.

The second approach is called the surface impedance matrix method. In this method, we define the matricant solution matrix $M(r, r_c) = \begin{pmatrix} M_1(r, r_c) & M_2(r, r_c) \\ M_3(r, r_c) & M_4(r, r_c) \end{pmatrix}$ of (4.27) such as $\eta(r) = M(r, r_c) \eta(r_c)$ (clearly $M(r_c, r_c)$ is the identity matrix). Here $r_c$ can be either $r_a$ or $r_b$, depending on what is most convenient. This allows us to write the initial boundary conditions in the simple form $z^c(r_c) = 0$, where $z^c = -i M_3(r, r_c) M_4^{-1} r_c$ is called the conditional impedance matrix. Now from the Stroh formalism (4.27) we can derive two relevant equations

$$\frac{dz^c}{dr} = \frac{1}{r} \left[ z^c G_2 z^c + i \left( G_1 \right)^\dagger z^c - i z^c G_1 + G_3 \right],$$

$$\frac{dU}{dr} = \frac{1}{r} \left[ i G_1 U - G_2 z^c U \right],$$

(4.3)

where $G_i$ ($i = 1, 2, 3$) are subblocks of matrix $G$ from (4.27). The numerical integration of (4.3), a differential Riccati equation, for $r_c = r_a$ with initial condition $z^c(r_a) = 0$, allows us to find the critical eigenvalues, i.e. critical deformation angles and stretches, of the Stroh problem (4.27) upon satisfaction of the boundary condition on the other face of the sector, which is $\det z^c(r_b) = 0$ (the latter one is equivalent to $V(r_b)/U(r_b) = -z_{11}^a(r_b)/z_{12}^a(r_b) = -z_{21}^a(r_b)/z_{22}^a(r_b)$). Next, the corresponding eigenvectors of the Stroh problem (4.27) are obtained through the simultaneous numerical integration of the two (4.3) for $r_c = r_b$, with initial condition $z^b(r_b) = 0$ and $U(r_b) = U(r_b)[1, -z_{11}^a(r_b)/z_{12}^a(r_b)]^T$.

Table 4.3 gives a detailed numerical algorithm to solve the impedance and compound matrix equations.
Table 4.3: Numerical implementation of the impedance and compound matrix methods.

Define reference geometry, e.g., $\alpha_r$ and $\rho$

DO For different mode numbers $m=1,2,3...$

DO For all deformations $\kappa < \pi/\alpha_r$ in bending (or $\kappa > -\pi/\alpha_r$ in unbending and eversion)

$n = m\pi/(\kappa \alpha_r)$;

Find $\lambda_a$ and $\lambda_b$;

*************** In case of the compound matrix method approach***************

Integrate the compound matrical differential equation $\frac{d\phi}{dr} = \frac{1}{r}A\phi$ on $r \in (a, b)$ with the boundary condition $\phi(a) = \phi_1(a)[1,0,0,0,0,0]^T$;

IF $\phi(b) = 0$ (OR $\phi(b)$ is monotonic function of $\kappa$ and changes its sign) THEN

Obtain critical values of $\kappa$, $\alpha_d$ and $\lambda_a$ in bending (or $\lambda_b$ in unbending and eversion);

BREAK

END IF

*************** In case of the impedance matrix method approach***************

Integrate the Riccati equation $\frac{dz}{dr} = \frac{1}{r}[z^*G_2z + i(G_1)z^* - iz^*G_1 + G_3]$ on $r \in (a, b)$ with the boundary condition $z(a) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$;

IF $\det z(b) = 0$ (OR $\det z(b)$ is monotonic function of $\kappa$ and start to plummet) THEN

Obtain critical values of $\kappa$, $\alpha_d$ and $\lambda_a$ in bending (or $\lambda_b$ in unbending and eversion);

Integrate the Riccati equation $\frac{dz}{dr} = \frac{1}{r}[z^*G_2z + i(G_1)z^* - iz^*G_1 + G_3]$ together with the boundary conditions $z(b) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $U(b) = U(b)\begin{pmatrix} 1, -x_1^*(b)/z_3^*(b) \end{pmatrix}^T$ to obtain the mechanical displacement field across the thickness of a given sector;

BREAK

END IF

END DO

Determine the acute mode number among all considered modes $m = 1, 2, 3...$, for which critical stretches $\lambda_a$ (bending) or $\lambda_b$ (unbending and eversion) are the highest. This allows to predict for a given sector when the buckling will occur and in how many wrinkles it will result.

References


REFERENCES


CHAPTER 4. WRINKLES AND CREASES IN BENDING, UNBENDING


Figure 4.8: Finite Element solutions of bending and unbending immediately before and after buckling. From top to bottom: bending sector $\mathcal{C}$, unbending sector $\mathcal{C}$, bending sector $\mathcal{D}$, unbending sector $\mathcal{D}$. The colours correspond to the von Mises stress level, from blue (low) to red (high).
Conclusion

In Chapter 1, we found the class of materials which guarantees a linear stress response in simple shear and torsion. We came up with several example models and found that they offered an improvement over the neo-Hookean and Mooney-Rivlin models in uni-axial tension by capturing non-linear effects (e.g. strain-stiffening).

In Chapter 2, we studied guided waves in fluid-loaded plates and tubes. We performed experiments on a plate and showed that our model can be used to characterise the mechanical properties of the solid. We also obtained dispersion curves for guided waves in a tube and showed that our analytical dispersion curve for the plate can be used as an approximation for the tube, as long as the radius is sufficiently large. Thus the method could be used in principle to characterise the mechanical properties of fluid-loaded tubes such as arteries.

In Chapter 3, we investigated wrinkle formation during the opening angle method, i.e. when a cylindrical (two-layered) sector is deformed into an intact cylinder. Our results have implications for finite element simulations of the opening angle method, where it is sometimes assumed that the deformation is stable. We also show analytically that wrinkles can be eliminated by applying an internal pressure, which has been observed in experiments.

In Chapter 4, we studied the bending, unbending and eversion of a homogeneous cylindrical sector. We proved existence and uniqueness of the solution and we investigated semi-analytically the loss of stability of this deformation, i.e. the formation of wrinkles. We also investigated numerically the formation of creases using finite element simulations.

In conclusion, our results aid in modelling and understanding the behaviour of soft solids subject to large deformations and characterising their mechanical properties. Our work could be extended in multiple ways. Many biological tissues consist of multiple layers and are anisotropic, due to the presence of collagen fibers. Arteries, for example, can be modelled as two-layered structures in which each layer is treated as a fiber-reinforced material, consisting of two families of fibers symmetrically distributed about the axis of the tube [1]. Thus, our work on guided waves in plates and tubes, and on bending, unbending and eversion of cylindrical sectors, could be extended to take account of these physiological complexities. We could also derive analytically the dispersion equations for guided waves in a cylindrical tube, in order to eliminate the need to rely on the dispersion equations for a plate. Finally, instability of a soft solid may be driven by growth or residual stresses [2] [3].
Thus we could investigate wrinkle and crease formation in cylindrical sectors under these circumstances.

References

