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AN INDUCTIVE APPROACH TO COXETER ARRANGEMENTS
AND SOLOMON’S DESCENT ALGEBRA

J. MATTHEW DOUGLASS, GÖTZ PFEIFFER, AND GERHARD RÖHRLE

Abstract. In our recent paper [4], we claimed that both the group algebra of a finite Coxeter group $W$ as well as the Orlik-Solomon algebra of $W$ can be decomposed into a sum of induced one-dimensional representations of centralizers, one for each conjugacy class of elements of $W$, and gave a uniform proof of this claim for symmetric groups. In this note we outline an inductive approach to our conjecture. As an application of this method, we prove the inductive version of the conjecture for finite Coxeter groups of rank up to 2.

1. Introduction

Let $W$ be a finite Coxeter group, generated by a set $S$ of simple reflections. If $|S| = r$, then $W$ acts as a reflection group on Euclidean $r$-space $V$. The reflection arrangement of $W$ is the hyperplane arrangement consisting of the reflecting hyperplanes in $V$ of all the reflections in $W$. The Orlik-Solomon algebra $A(W)$ of $W$ is the cohomology ring of the complement of the complexified reflection arrangement. It follows from a result of Brieskorn [3] that the algebra $A(W)$ is a $W$-module of dimension $|W|$. For some history of the computation of $A(W)$ as a $W$-module, see the introduction of our recent paper [4].

In [4], we claimed that both the group algebra $CW$ of $W$ (affording the regular character $\rho_W$) as well as the Orlik-Solomon algebra $A(W)$ (affording the Orlik-Solomon character $\omega_W$) can be decomposed into a sum of induced one-dimensional representations of centralizers, one for each conjugacy class of elements of $W$, in the following interlaced way.

Conjecture A. Let $R$ be a set of representatives of the conjugacy classes of $W$. Then, for each $w \in R$, there are linear characters $\tilde{\varphi}_w$ and $\tilde{\psi}_w$ of $CW(w)$ such that

$$\rho_W = \sum_{w \in R} \text{Ind}^{W}_{CW(w)} \tilde{\varphi}_w, \quad \omega_W = \sum_{w \in R} \text{Ind}^{W}_{CW(w)} \tilde{\psi}_w$$

are sums of induced linear characters. Moreover, for each $w \in R$, the characters $\tilde{\varphi}_w$ and $\tilde{\psi}_w$ can be chosen so that

$$\tilde{\psi}_w = \tilde{\varphi}_w \epsilon \alpha_w,$$

where $\epsilon$ is the sign character of $W$, and $\alpha_w$ is the determinant on the 1-eigenspace of $w$.

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When $W$ is a symmetric group, the formula for $\rho_W$ has been proved independently by Bergeron, Bergeron, and Garsia [1], Hanlon [6], and Schocker [14]. The formula for $\omega_W$ follows from work of Lehrer and Solomon [9], who also checked the identity for $\omega_W$ in the case of a dihedral group $W$. Conjecture 2.1 in [4] is a graded refinement of Conjecture A and the main result in [4] is a uniform proof of this refined conjecture for symmetric groups.

The details of the proof of Conjecture 2.1 in [4] for symmetric groups rely on properties of these groups not shared by other finite Coxeter groups. However, the underlying strategy of the proof using induced characters both generalizes and admits a “relative” version, for pairs $(W, W_L)$, where $W_L$ is a parabolic subgroup of $W$. In Section 4, we formalize this notion in Conjecture C, show how it leads to a proof of Conjecture A, and describe a two-step procedure that can be used to prove this relative conjecture. Prior to that, in Sections 2 and 3 we review some notation and basic facts about the descent algebra $\Sigma(W)$ and the Orlik-Solomon algebra $A(W)$. In the final section we apply the methods from Section 4 and prove Conjecture C for all pairs $(W, W_L)$ where $W$ is arbitrary and $W_L$ has rank at most 2. As a consequence, we deduce that Conjecture A holds for Coxeter groups of rank 2 or less.

2. Minimal Length Transversals of Parabolic Subgroups

The descent algebra of a finite Coxeter group $W$ encodes many aspects of the combinatorics of the minimal length coset representatives of the standard parabolic subgroups of $W$. In this section, we provide notation and summarize useful properties of these distinguished coset representatives following Pfeiffer [12].

For $J \subseteq S$, let

$$X_J = \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$ 

Then $X_J$ is a right transversal of the parabolic subgroup $W_J = \langle J \rangle$ of $W$, consisting of the unique elements of minimal length in their cosets. If we set

$$x_J = \sum_{x \in X_J} x^{-1} \in \mathbb{C}W,$$

then, by Solomon’s Theorem [15], the subspace

$$\Sigma(W) = \langle x_J : J \subseteq S \rangle_{\mathbb{C}}$$

is a $2^r$-dimensional subalgebra of the group algebra $\mathbb{C}W$, called the descent algebra of $W$.

For $J \subseteq S$, denote

$$X_J^x = \{x \in X_J : J^x \subseteq S\}.$$
The action of $W$ on itself by conjugation partitions the power set of $S$ into equivalence classes of $W$-conjugate subsets. We call the class

$$[J] = \{ J^x : x \in X_J \}$$

of a subset $J \subseteq S$ the \textit{shape} of $J$, and denote by

$$\Lambda = \{ [J] : J \subseteq S \}$$

the set of shapes of $W$. The shapes parametrize the conjugacy classes of parabolic subgroups of $W$, since two subsets $J, K \subseteq S$ are conjugate if and only if the corresponding parabolic subgroups $W_J$ and $W_K$ are conjugate. We say that a parabolic subgroup of $W$ has shape $[J]$ if it is conjugate to $W_J$ in $W$.

Furthermore, for $J \subseteq S$, we define

$$N_J = \{ x \in X_J : J^x = J \}.$$ 

Then $N_J$ is a subgroup of $W$ and by results of Howlett [7], the normalizer of $W_J$ in $W$ is a semi-direct product $N_W(W_J) = W_J \rtimes N_J$.

An element $w \in W$ is called \textit{cuspidal} in case $w$ has no fixed points in the reflection representation of $W$. For $J \subseteq S$, an element $w \in W_J$ is cuspidal in the parabolic subgroup $W_J$ if $w$ has no fixed points in the orthogonal complement of $\text{Fix}(W_J)$ in $V$. If $w$ is a cuspidal element in $W_J$, then the quotient $C_W(w)/C_{W_J}(w)$ is isomorphic to $N_J$ (see [8]).

We consider the character $\alpha_J$ of $N(W_J)$, defined, for $w \in N_W(W_J)$, as

$$\alpha_J(w) = \det(w|_{\text{Fix}(W_J)}),$$

where $\text{Fix}(W_J)$ is the fixed point subspace of $W_J$ in $V$. Note that $W_J$ is contained in the kernel of $\alpha_J$ and so $\alpha_J(un) = \alpha_J(n)$ for $u \in W_J, n \in N_J$.

**Lemma 2.1.** Let $J \subseteq S$. For $n \in N_J$ denote by $\sigma_J(n)$ the sign of the permutation induced on $J$ by conjugation with $n$. Then

$$\sigma_J(n) = \epsilon(n)\alpha_J(n),$$

for all $n \in N_J$.

**Proof.** Denote by $V_J$ the orthogonal complement of $\text{Fix}(W_J)$ in $V$. Then $V_J$ affords the reflection representation of the parabolic subgroup $W_J$, and the decomposition $V = V_J \oplus \text{Fix}(W_J)$ is $N_W(W_J)$-stable. For $n \in N_J$, the matrix of $n$ on $V_J$ is equivalent to the permutation matrix of the conjugation action of $n$ on $J$ and thus has determinant $\sigma_J(n)$. The matrix of $n$ on $\text{Fix}(W_J)$ has determinant $\alpha_J(n)$, by definition. Consequently, the determinant of $n$ on $V$ is $\epsilon(n) = \sigma_J(n)\alpha_J(n)$. \hfill $\square$

Pfeiffer and Röhrle [13] call $W_J$ a \textit{bulky} parabolic subgroup of $W$ if $N_W(W_J)$ is isomorphic to the direct product $W_J \times N_J$, or equivalently, if $N_J$ centralizes $W_J$. Notice that $W_J$
is bulky whenever $W_J$ is a self-normalizing subgroup of $W$. Suppose $W_J$ is bulky in $W$. Then $\sigma_J(n) = 1$ for all $n \in N_J$. Consequently, for $u \in W_J$ and $n \in N_J$, we have

\begin{equation}
\epsilon(un)\alpha_J(un) = \epsilon(u).
\end{equation}

Thus, the character $\epsilon\alpha_J = \epsilon_1 \times 1_{N_J}$ of $N_W(W_J) = W_J \times N_J$ is the trivial extension of the sign character of $W_J$.

Here and in the remainder of the paper we denote the restrictions of the trivial and the sign character of $W$ to a subgroup $U$ of $W$ by $1_U$ and $\epsilon_U$, respectively, or by $1_J$ and $\epsilon_J$, if $U = W_J$ for some $J \subseteq S$. If no confusion can arise, we denote the restrictions of the characters $1_S$ and $\epsilon_S$ of $W$ to any of its subgroups simply by $1$ and $\epsilon$, respectively.

Following Bergeron et al. [2], we decompose $\Sigma(W)$ into projective indecomposable modules, using a basis of quasi-idempotents, that naturally arise as follows. For $L, K \subseteq S$, we define

\[ m_{KL} = \begin{cases} |X_K \cap X_L^2|, & \text{if } L \subseteq K, \\ 0, & \text{otherwise}. \end{cases} \]

Then $(m_{KL})_{K,L \subseteq S}$ is an invertible matrix, and consequently, there is a basis $(e_L)_{L \subseteq S}$ of $\Sigma(W)$ such that

\[ x_K = \sum_{L \subseteq S} m_{KL} e_L \]

for $K \subseteq S$. Define, for $\lambda \in \Lambda$, elements

\[ e_\lambda = \sum_{L \in \lambda} e_L. \]

Then $\{e_\lambda : \lambda \in \Lambda\}$ is a set of primitive, pairwise orthogonal idempotents in $\Sigma(W)$. In particular,

\[ \sum_{\lambda \in \Lambda} e_\lambda = 1 \in CW. \]

Thus, if we set

\[ E_\lambda = e_\lambda CW, \]

then

\begin{equation}
CW = \bigoplus_{\lambda \in \Lambda} E_\lambda
\end{equation}

is a decomposition of the group algebra into right ideals. We call the right ideal $E_{[S]}$ the top component of $CW$.

For $\lambda \in \Lambda$, denote by $\Phi_\lambda$ the character of the $W$-module $E_\lambda$. Furthermore, for $L \subseteq S$, denote by $\Phi_L$ the character of the top component of the group algebra $CW_L$. Notice that for $\lambda = [L]$, $\Phi_{[L]}$ is a character of $W$ whereas $\Phi_L$ is a character of $W_L$. If $L = S$, then $W_L = W$ and $\Phi_{[S]} = \Phi_S$. In general, the characters $\Phi_{[L]}$ and $\Phi_L$ are related in the following way.
Proposition 2.4 ([4, Prop. 3.6(a)]). Let \( L \subseteq S \). Then the character \( \Phi_L \) of \( W_L \) extends to a character \( \tilde{\Phi}_L \) of the normalizer \( N_W(W_L) = W_L \rtimes N_L \) such that

\[
\Phi_{\lfloor L \rfloor} = \text{Ind}_{N_W(W_L)}^{W_L} \tilde{\Phi}_L.
\]

Remark 2.5. The argument in the proof of [4, Prop. 3.6(a)] shows that if \( W_L \) is a bulky parabolic subgroup of \( W \), then \( \tilde{\Phi}_L \) is the character \( \Phi_L \times 1_{N_L} \) of \( N_W(W_L) = W_L \rtimes N_L \) and so \( \Phi_{\lfloor L \rfloor} = \text{Ind}_{W_L \rtimes N_L}^{W_L} (\Phi_L \times 1_{N_L}) \).

3. The Reflection Arrangement and the Orlik-Solomon Algebra \( A(W) \)

A finite Coxeter group of rank \( r \) acts as a reflection group on Euclidean space \( \mathbb{R}^r \). Here it is convenient to regard this as an action on the complex space \( V_C = \mathbb{C}^r \). Let

\[
T = \{ s^w : s \in S, w \in W \}
\]

be the set of reflections of \( W \). For \( t \in T \), denote by \( H_t \) the reflecting hyperplane of \( t \), i.e., the 1-eigenspace of \( t \). The set of hyperplanes \( \mathcal{A} = \{ H_t : t \in T \} \) is called the reflection arrangement of \( W \); for details see [11, Ch. 6]. Examples of (the real part of) reflection arrangements in dimension 2 are shown in Figures 1 and 2 below.

**Figure 1.** Hyperplane Arrangements of Type \( I_2(m), m = 3, 5, 7, 9 \).

**Figure 2.** Hyperplane Arrangements of Type \( I_2(m), m = 4, 6, 8, 10 \).

The lattice of \( \mathcal{A} \) is the set of all possible intersections of hyperplanes

\[
L(\mathcal{A}) = \{ H_{t_1} \cap \cdots \cap H_{t_p} : t_1, \ldots, t_p \in T \}.
\]

For \( X \in L(\mathcal{A}) \), the pointwise stabilizer

\[
W_X = \{ w \in W : x.w = x \text{ for all } x \in X \}
\]
is a parabolic subgroup of $W$. We define the shape $\text{sh}(X)$ of $X$ to be the shape of $W_X$, i.e., $\text{sh}(X) = [L] \in \Lambda$ if $W_X$ is conjugate to $W_L$ in $W$ for some $L \subseteq S$. The group $W$ acts on $T$ by conjugation and the $W$-action on $T$ induces actions of $W$ on $A$ and $L(A)$. Orlik and Solomon [10] have shown that the normalizer of $W_X$ in $W$ is the setwise stabilizer of $X$ in $W$, that is

$$N_W(W_X) = \{ w \in W : X.w = X \}.$$ 

Consequently, the orbits of $W$ on the lattice $L(A)$ are parametrized by the shapes of $W$. We denote by $\alpha_X : N_W(W_X) \to \mathbb{C}$ the linear character of $N_W(W_X)$ defined by

$$\alpha_X(w) = \det(w|_X)$$

for $w \in N_W(W_X)$. Then, for $w \in W$, we have $\alpha_w = \alpha_X$, where $X = \text{Fix}(w)$, the fixed point subspace of $w$ in $V$. Moreover, for $L \subseteq S$, we have $\alpha_L = \alpha_X$, where $X = \text{Fix}(W_L)$.

The Orlik-Solomon algebra of $W$ is the associative $\mathbb{C}$-algebra $A(W)$, generated as an algebra by elements $a_t, t \in T$, subject to the relations

$$a_t a_{t'} = -a_{t'} a_t$$

for all $t, t' \in T$, and

$$\sum_{i=1}^{p} (-1)^i a_{t_1} \cdots a_{t_{i-1}} \hat{a}_{t_i} a_{t_{i+1}} \cdots a_{t_p} = 0,$$

where the hat denotes omission, whenever $\{H_{t_1}, \ldots, H_{t_p}\}$ is linearly dependent. The action of $W$ on the hyperplanes extends to an action on $A(W)$ via

$$a_t.w = a_{t.w}$$

for $t \in T, w \in W$. The algebra $A(W)$ is a skew-commutative, graded algebra

$$A(W) = \bigoplus_{p \geq 0} A^p,$$

where the degree $p$ subspace $A^p$ is spanned by those monomials $a_{t_1} \cdots a_{t_p}$ in $A(W)$ with $\dim H_{t_1} \cap \cdots \cap H_{t_p} = r - p$. Clearly, $A^p = 0$ for $p > r$. We call $A^r$ the top component of $A(W)$. We need a refinement of this decomposition, due to Brieskorn [3]. For a subspace $X \in L(A)$ of codimension $p$, define a subspace

$$A_X = \langle a_{t_1} \cdots a_{t_p} : H_{t_1} \cap \cdots \cap H_{t_p} = X \rangle$$

of $A(W)$. Then $A_{\{0\}} = A^r$ is the top component of $A(W)$. Note that $A_X$ is an embedding of the top component of $A(W_X)$ into $A(W)$. For $w \in W$, we have $A_X.w = A_{X.w}$ and so $A_X$ is an $N_W(W_X)$-stable subspace.

We have

$$A(W) = \bigoplus_{X \in L(A)} A_X$$
and if we set
\[ A_\lambda = \bigoplus_{\text{sh}(X) = \lambda} A_X, \]
for \( \lambda \in \Lambda \), then
\[ A(W) = \bigoplus_{\lambda \in \Lambda} A_\lambda \]
is a decomposition of \( A(W) \) into \( W \)-modules \( A_\lambda \). Note that \( A_{[S]} = A_{\{0\}} \) is the top component of \( A(W) \).

For \( \lambda \in \Lambda \), denote by \( \Psi_\lambda \) the character of the component \( A_\lambda \) of the Orlik-Solomon algebra \( A(W) \). Furthermore, for \( L \subseteq S \), denote by \( \Psi_L \) the character of the top component of the Orlik-Solomon algebra \( A(W_L) \) of the parabolic subgroup \( W_L \) of \( W \). Notice that for \( \lambda = [L] \), \( \Psi_{[L]} \) is a character of \( W \) whereas \( \Psi_L \) is a character of \( W_L \). If \( L = S \), then \( \Psi_{[S]} = \Psi_S \). In general, the characters \( \Psi_{[L]} \) and \( \Psi_L \) are related in the following way, analogous to Proposition 2.4.

**Proposition 3.1** ([9, §2]). Let \( L \subseteq S \). Then the character \( \Psi_L \) of \( W_L \) extends to a character \( \widetilde{\Psi}_L \) of the normalizer \( N_W(W_L) = W_L \rtimes N_L \) such that
\[ \Psi_{[L]} = \text{Ind}^W_{N_W(W_L)} \widetilde{\Psi}_L. \]

**Remark 3.2.** Suppose that \( W_L \) is a bulky parabolic subgroup of \( W \) and set \( X = \text{Fix}(W_L) \). If \( \text{codim } X = p \) and \( t_1, \ldots, t_p \) are in \( T \) with \( X = H_{t_1} \cap \cdots \cap H_{t_p} \), then \( t_1, \ldots, t_p \) are in \( W_L \) and so, since \( N_L \) centralizes \( W_L \), we have \( a_{t_1} \cdots a_{t_p} \cdot n = a_{t_1} a_{t_2} \cdots a_{t_p} = a_{t_1} \cdots a_{t_p} \), for \( n \in N_L \). Thus, \( \widetilde{\Psi}_L \) is the character \( \Psi_L \times 1_{N_L} \) of \( N_W(W_L) = W_L \times N_L \) and so \( \Psi_{[L]} = \text{Ind}^W_{W_L \times N_L} (\Psi_L \times 1_{N_L}) \).

4. The Inductive Strategy

Before stating our relative Conjecture C, we briefly review the proof of Conjecture 2.1 in [4] and describe how it leads to a proof of Conjecture A. We first showed that the characters of the top components of \( CW \) and \( A(W) \) are related as described in the following conjecture which makes sense for any finite Coxeter group. To this end, let \( C \) be the set of cuspidal conjugacy classes of \( W \) and, for \( L \subseteq S \), let \( C_L \) denote the set of cuspidal conjugacy classes in \( W_L \). For a class \( C \) in \( C \) or \( C_L \), we denote by \( w_C \in C \) a fixed representative.

**Conjecture B.** For each class \( C \in C \), there exist linear characters \( \varphi_{w_C} \) and \( \psi_{w_C} \) of the centralizer \( C_W(w_C) \) such that the following hold:

(i) \( \Phi_S = \sum_{C \in C} \text{Ind}^W_{C_W(w_C)} \varphi_{w_C} \);

(ii) \( \Psi_S = \sum_{C \in C} \text{Ind}^W_{C_W(w_C)} \psi_{w_C} \);

(iii) \( \psi_{w_C} = \varphi_{w_C} e \) for all \( C \in C \).
Remark 4.1. If it is known that $\Psi_S = \Phi_S \epsilon_S$, then choosing $\psi_{wC}$ or $\varphi_{wC}$ in such a way that $\psi_{wC} = \varphi_{wC} \epsilon$, we have that part (iii) in the above Conjecture B holds and that (i) and (ii) are equivalent statements.

When $W$ is a symmetric group, every parabolic subgroup $W_L$ of $W$ is a product of symmetric groups and so Conjecture B holds for the group $W_L$. Thus, for $w_C \in C \in \mathcal{C}_L$, we obtained linear characters $\varphi_{wC}$ and $\psi_{wC}$ of $C_{W_L}(w_C)$ such that the characters $\Phi_L$ and $\Psi_L$ of $W_L$ decompose as

$$\Phi_L = \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_{W_L}(w_C)}^{W_L} \varphi_{wC} \quad \text{and} \quad \Psi_L = \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_{W_L}(w_C)}^{W_L} \psi_{wC}. $$

We know from Propositions 2.4 and 3.1 that $\Phi_L$ and $\Psi_L$ extend to characters $\bar{\Phi}_L$ and $\bar{\Psi}_L$ of $N_W(W_L)$. The next step in [4] was to show that each $\varphi_{wC}$ and $\psi_{wC}$ extend to characters $\bar{\varphi}_{wC}$ and $\bar{\psi}_{wC}$ of $C_W(w_C)$ in such a way that

$$\bar{\Phi}_L = \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(w_C)}^{N_W(W_L)} \bar{\varphi}_{wC} \quad \text{and} \quad \bar{\Psi}_L = \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(w_C)}^{N_W(W_L)} \bar{\psi}_{wC}, $$

and moreover that $\bar{\psi}_{wC} = \bar{\varphi}_{wC} \epsilon_S \alpha_L$ for all $C \in \mathcal{C}_L$. Finally, we applied $\text{Ind}_{N_W(W_L)}^{W_L}$ to (4.2) and summed over the set of shapes $[L] \in \Lambda$. Conjecture A then follows immediately by transitivity of induction.

Motivated by (4.2) we make the following general conjecture.

Conjecture C. Let $L \subseteq S$. Then, for each $C \in \mathcal{C}_L$, there exist linear characters $\bar{\varphi}_{wC}$ and $\bar{\psi}_{wC}$ of $C_W(w_C)$ such that the following hold:

1. $\bar{\Phi}_L = \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(w_C)}^{N_W(W_L)} \bar{\varphi}_{wC}$;
2. $\bar{\Psi}_L = \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(w_C)}^{N_W(W_L)} \bar{\psi}_{wC}$;
3. $\bar{\psi}_{wC} = \bar{\varphi}_{wC} \epsilon_S \alpha_L$ for all $C \in \mathcal{C}_L$.

Remark 4.3. If it is known that $\bar{\Psi}_L = \bar{\Phi}_L \epsilon_S \alpha_L$, then choosing $\bar{\psi}_{wC}$ or $\bar{\varphi}_{wC}$ in such a way that $\bar{\psi}_{wC} = \bar{\varphi}_{wC} \epsilon_S \alpha_L$, we have that part (iii) in the above Conjecture C holds and that (i) and (ii) are equivalent statements.

Conjecture B is known to hold in the following cases:

1. $W$ of type $A$ (see [4, Thm. 4.1]);
2. $W$ has rank 2 or less (see Lemmas 5.1 and 5.2, Theorem 5.11).

Conjecture C is known to hold in the following cases:

1. $W$ of type $A$; all $L$ (see [4, Thm. 5.2]);
2. $W$ arbitrary; $W_L$ is bulky and satisfies Conjecture B (by Theorem 4.7);
3. $W$ arbitrary; $|L| \leq 2$ (see Corollary 5.3, Theorem 5.17).
If Conjecture C holds for all \( L \subseteq S \), then Conjecture A is true for \( W \).

**Theorem 4.4.** Suppose that Conjecture C holds for all subsets \( L \subseteq S \). Then for each \( w \) in a set \( R \) of representatives of the conjugacy classes of \( W \), there are linear characters \( \tilde{\varphi}_w \) and \( \tilde{\psi}_w \) of \( C_W(w) \) such that

(i) the regular character of \( W \) is given by \( \rho_W = \sum_{w \in R} \text{Ind}_{C_W(w)}^W \tilde{\varphi}_w \),

(ii) the Orlik-Solomon character of \( W \) is given by \( \omega_W = \sum_{w \in R} \text{Ind}_{C_W(w)}^W \tilde{\psi}_w \), and

(iii) \( \tilde{\psi}_w = \tilde{\varphi}_w e_{\alpha_w} \) for all \( w \in R \).

**Proof.** For \( L \subseteq S \), let \( \mathcal{R}_L \) be a set of minimal length representatives of the classes \( \mathcal{C}_L \). For a class \( C \in \mathcal{C}_L \), denote by \( w_C \in \mathcal{R}_L \) its representative. Let \( \mathcal{L} \) be a set of representatives of shapes, so \( \Lambda = \{ [L] \mid L \in \mathcal{L} \} \). Then, by [5, Thm. 3.2.12], we may assume without loss that

\[
\mathcal{R} = \coprod_{L \in \mathcal{L}} \mathcal{R}_L = \{ w_C : C \in \mathcal{C}_L, L \in \mathcal{L} \}.
\]

Then the equality in (iii) holds. By (2.3) and Proposition 2.4, we have

\[
\rho_W = \sum_{\lambda \in \Lambda} \Phi_\lambda = \sum_{L \in \mathcal{L}} \text{Ind}_{N_W(W_L)}^W \Phi_L = \sum_{L \in \mathcal{L}} \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(w_C)}^W \tilde{\varphi}_{w_C},
\]

as desired. The formula for \( \omega_W \) follows in the same way. \( \Box \)

Notice that in the case when \( L = S \), Conjecture C for \( L \subseteq S \) is simply a restatement of Conjecture B. In general, Conjecture C for \( L \subseteq S \) implies the validity of Conjecture B for the group \( W_L \), as follows.

**Proposition 4.5.** Suppose that Conjecture C holds for a subset \( L \subseteq S \). Then the restrictions

\[
\varphi_{wc} = \text{Res}_{C_W(W_L)}^W \tilde{\varphi}_{w_C} \quad \text{and} \quad \psi_{wc} = \text{Res}_{C_W(W_L)}^W \tilde{\psi}_{w_C}
\]

are linear characters that satisfy Conjecture B for \( W_L \).

**Proof.** By Mackey’s theorem, we have

\[
\text{Res}_{W_L}^{N_W(W_L)} \text{Ind}_{C_W(W_L)}^{N_W(W_L)} \tilde{\varphi}_{w_C} = \text{Ind}_{C_W(W_L)}^{W_L} \text{Res}_{C_W(W_L)}^W \tilde{\varphi}_{w_C},
\]

since \( N_W(W_L) = W_L C_W(w_C) \) (see [8]), and therefore,

\[
\Phi_L = \text{Res}_{W_L}^{N_W(W_L)} \Phi_L = \sum_{C \in \mathcal{C}_L} \text{Res}_{W_L}^{N_W(W_L)} \text{Ind}_{C_W(W_L)}^{N_W(W_L)} \tilde{\varphi}_{w_C}
\]

\[
= \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(W_L)}^{W_L} \text{Res}_{C_W(W_L)}^W \tilde{\varphi}_{w_C}
\]

\[
= \sum_{C \in \mathcal{C}_L} \text{Ind}_{C_W(W_L)}^{W_L} \varphi_{w_C}.
\]
The formula for $\Psi_L$ follows in the same way. The conclusion that $\psi_{wc} = \varphi_{wc} e$ for $C \in \mathcal{C}_L$ is easily seen to hold.

\begin{proof}

Remark 4.6. Although Conjecture B for $W_L$ formally follows from Conjecture C, as in [4], the characters $\tilde{\varphi}_{wc}$ and $\psi_{wc}$ of $C_W(\omega_C)$ arise in practice as extensions of characters $\varphi_{wc}$ and $\psi_{wc}$ of $C_{W_l}(\omega_C)$ that satisfy Conjecture B for $W_l$. In particular, if Conjecture B is known to hold for $W_l$, then using Remark 4.3, to prove Conjecture C for $L \subseteq S$, it suffices to prove that each $\varphi_{wc}$ extends to $C_W(\omega_C)$ in such a way that Conjecture C (i) holds and that $\tilde{\Psi}_L = \tilde{\Phi}_L e_S e_L$.

When $L \subseteq S$ is such that $W_L$ is a self-normalizing subgroup of $W$ (e.g., if $L = S$), then $N_L$ is the trivial group and Conjecture B for the group $W_L$ vacuously implies Conjecture C for the subset $L$ in this case. More generally, whenever the complement $N_L$ centralizes $W_L$, i.e., when $W_L$ is bulky in $W$, Conjecture B for $W_L$ implies Conjecture C for $L \subseteq S$, as follows.

\textbf{Theorem 4.7.} Let $L \subseteq S$. Suppose that Conjecture B holds for the group $W_L$ and that $W_L$ is a bulky parabolic subgroup of $W$. Then Conjecture C holds with $\tilde{\varphi}_{wc} = \varphi_{wc} \times 1_{N_L}$ and $\tilde{\psi}_{wc} = \psi_{wc} \times 1_{N_L}$ for each cuspidal class $C$ of $W_L$.

Proof. As observed in the remark above, it suffices to show that each $\varphi_{wc}$ extends to $C_W(\omega_C)$ in such a way that Conjecture C (i) holds and that $\tilde{\Psi}_L = \tilde{\Phi}_L e_S e_L$.

Because $N_L$ centralizes $W_L$, we have that the centralizer $C_{W}(\omega_C)$ is the direct product of $C_{W_l}(\omega_C)$, and $N_L$ and so $\tilde{\varphi}_{wc}$ is indeed a linear character of $C_{W}(\omega_C)$ that extends $\varphi_{wc}$. Thanks to Remark 2.5, $\tilde{\Phi}_L = \Phi_L \times 1_{N_L}$. Thus, by Conjecture B (i) we have,

$$\tilde{\Phi}_L = \Phi_L \times 1_{N_L} = \sum_{C \in \mathcal{C}_L} \text{Ind}^{W_L}_{C_{W}(\omega_C)} \varphi_{wc} \times 1_{N_L}$$

$$= \sum_{C \in \mathcal{C}_L} \text{Ind}^{W_L \times N_L}_{C_{W_l}(\omega_C) \times N_L} (\varphi_{wc} \times 1_{N_L})$$

$$= \sum_{C \in \mathcal{C}_L} \text{Ind}^{N_{W}(\omega_L)}_{C_{W}(\omega_C)} \varphi_{wc}.$$ 

Hence Conjecture C (i) holds.

By Remark 3.2, Conjecture B (iii), Remark 2.5, and Lemma 2.1, we have

$$\tilde{\Psi}_L = \Psi_L \times 1_{N_L} = \Phi_L e_L \times 1_{N_L} e_L = (\Phi_L \times 1_{N_L}) e \sigma_L = \tilde{\Phi}_L e_S e_L,$$

using the fact that $W_L \subseteq \ker \sigma_L$, whence we are done.

\end{proof}

Combining Theorem 4.7 with the results in [4], we see that if $W_L$ is a product of Coxeter groups of type $A$ and is a bulky parabolic subgroup of $W$, then Conjecture C holds for $L \subseteq S$. For example, if $W_L$ is of type $A_1 \times A_3$ and $W$ is of type $E_6$, then the characters $\varphi_{wc}$ and $\psi_{wc}$ constructed in [4] satisfy Conjecture B and so, by Theorem 4.7, they extend to $C_W(\omega_C)$ and Conjecture C holds. Note however, that the property of being a
bulky parabolic subgroup depends in a fundamental way on the embedding of $W_L$ in $W$. If $W_L$ is of type $A_1 \times A_3$ and $W$ is of type $E_7$, then $W_L$ is not bulky and Theorem 4.7 cannot be applied.

5. Conjectures A, B and C for Coxeter Groups of Rank up to 2

In this section we show that Conjecture C holds for $L \subseteq S$ for any $S$ as long as $|L| \leq 2$. Note that because the type of the ambient Coxeter group $W$ is arbitrary, even for types $A_1 \times A_1$ and $A_2$ Conjecture C is a stronger statement than is proved in [4] for such parabolic subgroups. The strategy we use is to first prove that Conjecture B holds for $W$ when the rank of $W$ is at most 2 and then use the procedure outlined in Remark 4.6. Combining Conjecture C with Theorem 4.7 we conclude that Conjectures A, B, and C all hold in case the rank of $W$ is at most two.

The top components of Coxeter groups of rank 0 or 1 almost trivially satisfy Conjecture B. For later reference, we record this explicitly in the following lemmas.

**Lemma 5.1.** The top component characters of $W_{\emptyset}$ are $\Phi_{\emptyset} = 1_\emptyset$ and $\Psi_{\emptyset} = 1_\emptyset$. Moreover, $W_{\emptyset}$ satisfies Conjecture B with $\varphi_1 = 1_\emptyset$ and $\psi_1 = 1_\emptyset$.

**Lemma 5.2.** Suppose $W$ is a Coxeter group of rank 1, generated by $S = \{s\}$. Then the top component characters of $W$ are $\Phi_S = \epsilon_S$ and $\Psi_S = 1_S$. Moreover, $W$ satisfies Conjecture B with $\varphi_s = \epsilon_S$ and $\psi_s = 1_S$.

*Proof.* In this case, the non-trivial conjugacy class $\{s\}$ is the unique cuspidal conjugacy class in $W$. From the definitions we have $e_{|S|} = e_S = \frac{1}{2}(1 - s)$ and it follows that $W$ acts on the top component $E_{|S|} = e_{|S|}CW$ with character $\Phi_{|S|} = \epsilon_S$. Moreover, $W$ acts trivially on the basis $\{a_s\}$ of the top component $A_{|S|}$ of $A(W)$, which therefore affords the trivial character. Thus, $\Psi_{|S|} = 1_S$ and so $\Phi_{|S|} = \Psi_{|S|} \epsilon_S$. Set $\varphi_s = \epsilon_S$ and $\psi_s = 1_S$. Then $\varphi_s$ and $\psi_s$ obviously satisfy the conclusions of Conjecture B. \qed

In any finite Coxeter group $W$, parabolic subgroups of rank 0 and 1 are always bulky. We may thus conclude from Lemmas 5.1 and 5.2 and Theorem 4.7 that Conjecture C holds for $L \subseteq S$ with $|L| \leq 1$.

**Corollary 5.3.** Suppose that $L \subseteq S$ has size $|L| \leq 1$. Then Conjecture C holds.

As a consequence of the corollary, $W$ acts trivially on both the component $E_{|\emptyset|}$ of the group algebra $CW$ (with character $\Phi_{|\emptyset|} = \Phi_{\emptyset} = 1_\emptyset$) and the component $A_{|\emptyset|}$ of the Orlik-Solomon algebra $A(W)$ (with character $\Psi_{|\emptyset|} = \Psi_{\emptyset} = 1_\emptyset$), as one can easily establish directly.

Moreover, the degree 1 component of $A(W)$ is a direct sum of transitive permutation modules, one for each conjugacy class of reflections of $W$. This agrees with the description
of the degree 1 component of $A(W)$ as the permutation representation of $W$ on its reflections, that can easily be obtained directly.

Next we consider the case when $W$ has rank 2. Until further notice, we assume that

$$W = \langle s, t : s^2 = t^2 = (st)^m = 1 \rangle.$$ 

Then $W$ is a Coxeter group of rank two and is of type $A_1 \times A_1$, or $I_2(m)$ for $m \geq 3$, with Coxeter generators $S = \{s, t\}$. For convenience, we regard type $A_1 \times A_1$ as type $I_2(2)$, noting that the general results of this section remain true for $m = 2$.

To prove Conjecture B for $W$, we first compute the character $\Phi_S$ of the top component $E_{[S]}$ of the group algebra $CW$, and verify that it is a sum of induced linear characters. Then we compute the character $\Psi_S$ of the top component $A_{[S]}$ of the Orlik-Solomon algebra $A(W)$ and verify that $\Psi_S = \Phi_S e_S$. Conjecture B then follows as observed in Remark 4.1.

As usual, denote by $w_0$ the longest element of $W$. Furthermore, we denote

$$Av(U) = \frac{1}{|U|} \sum_{u \in U} u$$

for a subgroup $U$ of $W$. Recall that $Av(U)u = Av(U)$ for all $u \in U$ and that $Av(U)CW$ is the permutation module of $W$ on the cosets of $U$.

**Lemma 5.4.** $e_S = Av(\langle w_0 \rangle) - Av(W)$.

**Proof.** By Solomon’s theorem [15], the elements

$$x_\varnothing = 1 + s + t + st + ts + \cdots + w_0, \quad x_s = 1 + t + st + t^2 + \cdots + w_0 s,$$

$$x_{st} = 1, \quad x_t = 1 + s + ts + sts + \cdots + w_0 t$$

form a basis of the descent algebra $\Sigma(W)$. Note that $x_t + x_s = x_{\varnothing} + 1 - w_0$.

For $L \subseteq K \subseteq S$, the numbers $m_{KL} = |X_K \cap X_L|$ are easily determined as

$$(m_{KL})_{K,L \subseteq S} = \begin{bmatrix} 2m & \cdots & \cdots \\ m & 2 & \cdots \\ m & \cdots & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad (m_{KL})^{-1} = \begin{bmatrix} \frac{1}{2m} & \cdots & \cdots \\ -\frac{1}{4} & \frac{1}{2} & \cdots \\ -\frac{1}{4} & \frac{1}{2} & \cdots \\ m-1 & \frac{1}{2m} & \frac{1}{2} & \cdots \end{bmatrix}.$$

Hence the idempotents $e_L$ are (cf. [2])

$$e_\varnothing = \frac{1}{2m} x_\varnothing, \quad e_s = \frac{1}{2} x_s - \frac{1}{4} x_\varnothing, \quad e_{st} = 1 - \frac{1}{2} x_s - \frac{1}{2} x_t + \frac{m-1}{2m} x_\varnothing, \quad e_t = \frac{1}{2} x_t - \frac{1}{4} x_\varnothing.$$

From $x_t + x_s = 1 + x_\varnothing - w_0$, it follows that $e_s + e_t = \frac{1}{2}(1 - w_0)$, and hence that $e_S = \frac{1}{2}(1 + w_0) - e_\varnothing = Av(\langle w_0 \rangle) - Av(W)$, as required. \(\square\)

As an immediate consequence we obtain the character of the top component of $CW$. 

Corollary 5.5. The $W$-module $E_{[S]}$ affords the character $\Phi_S = \text{Ind}^W_{\langle w_0 \rangle}(1) - I_S$.

Next we identify linear characters of centralizers of cuspidal elements. Note that the group $W$ consists of $m$ reflections and $m$ rotations. The centralizer of a rotation $w$ is the rotation subgroup $W^+ = \langle \text{st} \rangle$ of $W$, unless $w$ is central in $W$. The cuspidal classes of $W$ are exactly the classes of nontrivial rotations, represented by the set $\{(\text{st})^j : j = 1, \ldots, \lfloor \frac{m}{2} \rfloor \}$, containing $w_0 = (\text{st})^{m/2}$ in case $m$ is even. The group $W^+$ is a cyclic group of order $m$ and it has $m$ linear characters $\chi_j$, $j = 0, \ldots, m - 1$, defined by

$$\chi_j(\text{st}) = \zeta_m^j$$

for a primitive $m$th root of unity $\zeta_m$. In the following arguments, we make frequent use of the fact that the sum of all the nontrivial characters $\chi_j$ of $W^+$ equals the difference of its regular and its trivial character,

$$\sum_{j=1}^{m-1} \chi_j = \text{Ind}^W_{\langle w_0 \rangle}(1) - I_{W^+},$$

which obviously follows from $\sum_{j=0}^{m-1} \chi_j = \text{Ind}^W_{\langle w_0 \rangle}(1)$ and $\chi_0 = 1_{W^+}$.

We distinguish two cases, depending on the parity of $m$.

Proposition 5.6. Suppose that $m = 2k$ with $k > 0$. Let

$$\varphi_{(\text{st})^j} = \begin{cases} \chi_{2j}, & 0 < j < k, \\ \epsilon_S, & j = k. \end{cases}$$

Then $\varphi_{(\text{st})^j}$ is a linear character of $C_W((\text{st})^j)$, for $j = 1, \ldots, k$, and

$$\sum_{j=1}^{k} \text{Ind}^W_{C_W((\text{st})^j)}(\varphi_{(\text{st})^j}) = \epsilon_S + \sum_{j=1}^{k-1} \text{Ind}^W_{W^+}(\chi_{2j}) = \Phi_S.$$ 

Proof. Note that $C_W((\text{st})^j) = W^+$ and $w_0$ lies in the kernel of the characters $\varphi_{(\text{st})^j} = \chi_{2j}$, for all $j = 1, \ldots, k - 1$. Hence the $\chi_{2j}$ can be regarded as a full set of nontrivial irreducible characters of the quotient group $W^+/\langle w_0 \rangle$, whence their sum $\sum_{j=1}^{k-1} \chi_{2j}$ equals the difference of its regular and its trivial characters. Thus, as a character of $W^+$, we have

$$\sum_{j=1}^{k-1} \chi_{2j} = \text{Ind}^W_{\langle w_0 \rangle}(1) - I_{W^+}.$$

Thus

$$\epsilon_S + \text{Ind}^W_{W^+} \left( \sum_{j=1}^{k-1} \chi_{2j} \right) = \epsilon_S + \text{Ind}^W_{\langle w_0 \rangle}(1) - \text{Ind}^W_{W^+}(1) = \text{Ind}^W_{\langle w_0 \rangle}(1) - I_S = \Phi_S,$$

where the penultimate equality follows from $\text{Ind}^W_{W^+}(1) = 1_S + \epsilon_S$. $\square$
Proposition 5.7. Suppose that $m = 2k + 1$ for some $k > 0$. For $j = 1, \ldots, k$, let

$$\varphi_{(st)^j} = \chi_j.$$ 

Then $\varphi_{(st)^j}$ is a linear character of $C_W((st)^j)$, for $j = 1, \ldots, k$, and

$$\sum_{j=1}^{k} \text{Ind}_{C_W((st)^j)}^{C_W((st)^j)}(\varphi_{(st)^j}) = \sum_{j=1}^{k} \text{Ind}_{W^+}^W(\chi_j) = \Phi_S.$$ 

Proof. We have $C_W((st)^j) = W^+$ and $\text{Res}_{W^+}^W(\text{Ind}_{W^+}^W(\chi_j)) = \chi_j + \chi_{m-j}$ for all $j = 1, \ldots, k$. Hence

$$\text{Res}_{W^+}^W\left(\sum_{j=1}^{k} \text{Ind}_{W^+}^W(\chi_j)\right) = \sum_{j=1}^{m-1} \chi_j = \text{Ind}_{[1]}^W(1) - 1_{W^+} = \text{Res}_{W^+}^W(\text{Ind}_{(w_0)}^W(1) - 1_S) = \text{Res}_{W^+}^W(\Phi_S).$$ 

It follows that

$$\Phi_S = \sum_{j=1}^{k} \text{Ind}_{W^+}^W(\chi_j),$$

since the restrictions of both characters to the subgroup $\langle w_0 \rangle$ of $W$ also coincide. \ 

Proposition 5.8. Let $\pi_A$ be the character of the permutation action of $W$ on the hyperplane arrangement $A$. Then $W$ acts on the degree 1 component of $A(W)$ with character $\pi_A$, and $W$ acts on the component $A_{[S]}$ of $A(W)$ with character

$$\Psi_S = \pi_A - 1_S.$$ 

Consequently, $W$ acts on $A(W)$ with character $2\pi_A$.

Proof. The degree 1 component of $A(W)$ has basis $\{a_t : t \in T\}$ and $W$ acts on it by permuting the basis vectors. In order to analyze the top component of $A(W)$, we make this permutation action explicit as follows.

Label the hyperplanes $H_0, \ldots, H_{m-1}$, so that the hyperplane $H_j$ is spanned by $\zeta_j^{2m}$, where $\zeta_2 = e^{2\pi i / 2m}$ is a primitive $2m$th root of unity, as shown in Figures 1 and 2.

Let $s$ be the reflection about $H_0$ (the x-axis) and $ts = (st)^{-1}$ the (anti-clockwise) rotation about the angle $2\pi/m$. Then $t$ is the reflection about $H_{m-1}$.

The reflection $s$ then permutes the hyperplanes according to the rule

$$H_j.s = H_{m-j},$$

for $j = 0, \ldots, m-1$, fixing $H_0$. The rotation $ts$ acts as

$$H_j.ts = H_{j+2},$$

for $j = 0, \ldots, m-1$, where the indices are reduced mod $m$ if necessary.
The top component $A_{[S]}$ has a basis $\{a_0a_j : j = 1, \ldots, m-1\}$, where $W$ acts on the indices as indicated above, subject to the relation $a_0a_j - a_0a_k + a_ia_k = 0$, i.e.,

$$a_ia_k = a_0a_k - a_0a_j.$$  

The reflection $s$ fixes $H_0$ and thus maps $a_0a_j$ to 

$$a_0a_j.s = a_0a_{m-j},$$ 

for $j = 1, \ldots, m-1$. The rotation $ts$ maps $a_0a_j$ to 

$$a_0a_j.ts = a_2a_{j+2} = \begin{cases} a_0a_{j+2} - a_0a_2, & j \neq m-2, \\ -a_0a_2, & j = m-2. \end{cases}$$  

Now define vectors 

$$b_0 = -\frac{1}{m} \sum_{j=1}^{m-1} a_0a_j$$  

and, for $j = 1, \ldots, m-1$,

$$b_j = a_0a_j + b_0.$$  

Then $b_0.s = b_0$ and $b_j.s = b_{m-j}$ for $j = 1, \ldots, m-1$. Moreover, $b_j.ts = b_{j+2}$ for $j = 0, \ldots, m-1$, with indices reduced mod $m$ if necessary. Hence the map $a_j \mapsto b_j$ is a $W$-equivariant bijection from the basis $\{a_j : j = 0, \ldots, m-1\}$ of the degree 1 component to a generating set $\{b_j : j = 0, \ldots, m-1\}$ of $A_{[S]}$, and since $\sum_{j=0}^{m-1} b_j = 0$ in $A_{[S]}$, the character of $W$ on $A_{[S]}$ is $\pi_A - 1_S$.  

\begin{lemma}
 The element $a_0a_{m-1}$ generates the top component $A_{[S]}$ as $CW$-module.
\end{lemma}

\begin{proof}
 Let $M = a_0a_{m-1}.CW$. Then $M$ contains the elements

$$a_0a_1 = a_0a_{m-1}.s, \quad a_1a_2 = -a_0a_{m-1}.ts, \quad \text{and} \quad a_0a_2 = a_0a_1 + a_1a_2,$$

and, by induction, the elements

$$a_{j-1}a_j = a_{j-3}a_{j-2}.ts, \quad \text{and} \quad a_0a_j = a_0a_{j-1} + a_{j-1}a_j,$$

for $j > 2$. Consequently, $M$ contains the basis $\{a_0a_j : j = 1, \ldots, m-1\}$ of $A_{[S]}$, whence $M = A_{[S]}$.  
\end{proof}

\begin{proposition}
 $\psi_S = \phi_s \psi_s$.
\end{proposition}

\begin{proof}
 We distinguish two cases.

If $m$ is odd, then $\pi_A = \text{Ind}_W^W(1)$, since $C_W(s) = \langle s \rangle$ and all reflections are conjugates of $s$. Hence

$$\psi_S = \text{Ind}_W^W(1) - 1_S = \text{Ind}_W^W(1) - 1_S = \phi_s$$

and $\phi_s = \phi_s \psi_s$, since $\phi_s(w) = 0$ for all $w \in W$ with $\psi_s(w) = -1$.
If $m$ is even, then $\text{Ind}_{(w_0)}^W(1)e_S = \text{Ind}_{(w_0)}^W(1)$ and
\[
\Phi_S e_S = (\text{Ind}_{(w_0)}^W(1) - 1_S)e_S = \text{Ind}_{(w_0)}^W(1) - e_S = \pi_A - 1_S = \Psi_S,
\]
since $\pi_A - \text{Ind}_{(w_0)}^W(1) = 1_S - e_S$, as can be easily verified. \hfill $\square$

We can now conclude that Conjecture B holds for $W$ of rank 2.

**Theorem 5.11.** Let $W$ be a Coxeter group of rank 2, generated by $S = \{s, t\}$. Then, with notation as above, the top component characters of $W$ are $\Phi_S = \text{Ind}_{(w_0)}^W(1) - 1_S$ and $\Psi_S = \pi_A - 1_S = \Phi_S e_S$. Moreover, $W$ satisfies Conjecture B with $\varphi_{(st)^i} = \chi_j$ in case $m$ odd, while $\varphi_{w_0} = e_S$ and $\varphi_{(st)^i} = \chi_2$ in case $m$ even.

**Proof.** Apply Propositions 5.6, 5.7, and 5.10, and Remark 4.3. \hfill $\square$

**Corollary 5.12.** Suppose that $W$ is a Coxeter group with rank at most 2. Then Conjecture A holds for $W$.

**Proof.** By Lemmas 5.1 and 5.2, and Theorem 5.11, Conjecture B holds for all parabolic subgroups of $W$. By Theorem 4.4 it suffices to show that Conjecture C holds for all subsets $L \subseteq S$. If $|L| = 0, 1$, this follows from Corollary 5.3. It follows from Theorem 5.11 that Conjecture C holds when the rank of $W$ and $|L|$ are both equal 2. \hfill $\square$

It follows in particular from Corollary 5.12 that every Coxeter group of type $I_2(m)$ satisfies Conjecture A. We list the corresponding decomposition of the regular character $\rho_W$ into characters $\Phi_{[L]} = \text{Ind}_{N_W(W_L)}^N \Phi_L$ and the decomposition of the Orlik-Solomon character $\omega_W$ into characters $\Psi_{[L]} = \text{Ind}_{N_W(W_L)}^N \Psi_L$ in Table 1 below. In Table 1, the

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>s</th>
<th>t</th>
<th>$w_0$</th>
<th>$(st)^i$</th>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
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<td>-k</td>
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<td>-1</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_{[t]}$</td>
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<td>.</td>
<td>1</td>
<td>2</td>
<td>k</td>
</tr>
<tr>
<td>$\psi_{[s]}$</td>
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<td>1</td>
<td>m-1</td>
<td>-1</td>
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<tr>
<td>$\omega_W$</td>
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<td>4</td>
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<tr>
<td>$\Phi_{[t]}$</td>
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<tr>
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<tr>
<td>$\psi_{[s]}$</td>
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<tr>
<td>$\omega_W$</td>
<td>2m</td>
<td>2</td>
<td>.</td>
</tr>
</tbody>
</table>

**Table 1.** The characters $\Phi_A$ and $\Psi_A$ for $I_2(m)$; $m = 2k$, $m = 2k + 1$. The left character table covers the case $m = 2k$ and the right character table covers the case $m = 2k + 1$. The columns of the character tables are labelled by representatives of the
conjugacy classes of \( W \), where the parameter in \((st)^i\) is \( i = 1, \ldots, k - 1 \) for \( m = 2k \), and \( i = 1, \ldots, k \) for \( m = 2k + 1 \). An entry ‘.’ in the table stands for the value 0. As observed in Proposition 5.8, the rank 1 component of \( \omega_W \) is the permutation character of the action of \( W \) on the set \( A \) of hyperplanes. In case \( m = 2k \), the constituent \( \Psi_{\{st\}} \) corresponds to the action on the \( W \)-orbit of the hyperplane \( H_s \), and whether the element \( s \) has 2 or 1 fixed points in this action depends on whether \( k \) is even or odd. In such a situation, an entry of the form ‘\( x \mid y \)’ in the table stands for ‘\( x \) if \( k \) is even and \( y \) if \( k \) is odd’.

We saw in Theorem 5.11 that Conjecture B holds when \( W \) has rank 2 and we saw in Corollary 5.3 that Conjecture C holds when the subset \( L \subseteq S \) has size \(|L| \leq 1\). In the rest of this section, we prove that if the parabolic subgroup \( W_L \) has rank two, then Conjecture C holds for any ambient group \( W \). A similar result when \( W_L \) is a product of symmetric groups would reduce the proof of Conjecture A to considering only a small number of cases.

From now on, \( W \) is a finite Coxeter group, generated by \( S \) with \(|S| \geq 3\) and \( W_L \) is a rank 2 parabolic subgroup of \( W \) with \( L = \{s, t\} \subseteq S \). The elements \( x_K \) and \( e_K \) are defined relative to the ambient set \( S \). We use a superscript to indicate this ambient set when it is not equal to \( S \). Thus, for \( K \subseteq L \), \( x_K^L \) denotes a basis element of the descent algebra of \( W_L \).

If \( W_L \) is bulky, then \( W_L \) satisfies Conjecture C, by Theorem 4.7.

If \( W_L \) is not bulky, then \( N_L \) does not centralize \( W_L \) and so \( N_L \) contains an element inducing the nontrivial graph automorphism \( \gamma \) on \( W_L \), interchanging \( s \) and \( t \). In this case \( s \) and \( t \) are conjugate in \( W \) and so \( W_L \) is either of type \( A_1 \times A_1 \) or of type \( I_2(m) \) for odd \( m > 2 \). We distinguish two cases accordingly.

First, suppose that \( W_L \) is of type \( A_1 \times A_1 \). Then \( W_L \) has exactly one cuspidal element \( w = st = ts \), which is central in \( W_L \) and invariant under \( N_L \), hence central in \( N_W(W_L) \). We have

\[
\varphi_w = \Phi_L = e_L, \quad \text{and} \quad \psi_w = \Psi_L = 1_L,
\]

by Corollary 5.5 and Proposition 5.8. Parts (i) and (ii) of Conjecture C are therefore trivially satisfied, with

\[
\tilde{\varphi}_w = \tilde{\Phi}_L, \quad \text{and} \quad \tilde{\psi}_w = \tilde{\Psi}_L,
\]

which exist by Propositions 2.4 and 3.1.

For part (iii) of Conjecture C, note that the idempotent

\[
f = \frac{1}{4}(1 - s - t + st)
\]

spans a subspace of \( \mathbb{C}W_L \) affording the character \( \Phi_L \). As in the proof of Lemma 5.4,

\[
e_L^L = 1 - \frac{1}{2}x_s^L - \frac{1}{2}x_t^L + \frac{1}{4}x_{st}^L = \frac{1}{4}(1 + st) - \frac{1}{4}(s + t) = f,
\]
and thus $e^L_1 f = e^L_1$ is a basis of the top component of $W_L$ which is centralized by $N_L$. Hence $\tilde{\varphi}_w(u) = \varphi_w(u)$, for $u \in W_L$ and $n \in N_L$. Moreover, note that $\alpha_L = a_s a_t$ spans the top component of $\mathcal{A}(W_L)$, and that $e_1 n = e_1$, whereas $a_1 n = \sigma(n) a_1$ for $n \in N_L$. It follows that $\tilde{\varphi}_L(u) = \psi_L(u) \sigma_L(n) = \varphi_L(u) e(u) e(n) \alpha_L(n) = \tilde{\varphi}_L(u) e(u) e(n) \alpha_L(u)$, for $u \in W_L$ and $n \in N_L$, as desired. This proves the following proposition.

**Proposition 5.13.** Suppose $L = \{s, t\} \subseteq S$ is such that $W_L$ is of type $A_1 \times A_1$. Then Conjecture C holds for $L \subseteq S$.

Second, suppose that $W_L$ is of type $I_2(m)$ for $m$ odd. Recall that the character $\chi_i: st \mapsto \zeta^i_m$ is afforded by the subspace of $CW^+$ spanned by the idempotent

\[
(5.14) \quad f_j = \frac{1}{m} \sum_{k=0}^{m-1} \zeta^j_m(st)^{-k},
\]

for $j = 1, \ldots, m - 1$. As usual, denote by $w_L$ the longest element of $W_L$. Note that $f^w_l = f_{m-j}$, for $j = 1, \ldots, m - 1$, since $(st)^w = (st)^{-1}$, and that

\[
e^L_1 f_j = Av((w_L)) f_j,
\]

by Lemma 5.4, since $Av(W_L) f_j = \sum_{k=0}^{m-1} \zeta^j_m(st)^{-k}$, for $j = 1, \ldots, m - 1$. Obviously, the graph automorphism $\gamma$ swaps $e^L_1 f_j$ and $e^L_1 f_{m-j}$, and so does right multiplication by $w_L$:

\[
e^L_1 f_j w_L = Av((w_L)) f_j w_L = Av((w_L)) w_L f^w_l = Av((w_L)) f_{m-j} = e^L_1 f_{m-j}.
\]

Moreover, if $n \in N_L$ induces the automorphism $\gamma$ on $W_L$, then $w_L n \in C_W(st)$. Therefore, if we write $N_L = N^+_L \cup N^-_L$, where $N^+_L = N_L \cap C_W(st)$ and $N^-_L = N_L \setminus N^+_L$, then we have

\[
C_W(st) = C_W(st) N^+_L \cup C_W(st) w_L N^-_L.
\]

It follows that we can naturally extend the characters $\varphi_{(st)}$ to characters $\tilde{\varphi}_{(st)}$ of the full centralizer $C_W(st)$ via

\[
\tilde{\varphi}_{(st)}(c) = \varphi_{(st)}(v),
\]

where either $c = v n$ for some $v \in C_W(st)$ and $n \in N^+_L$, or $c = v w_L n$ for some $v \in C_W(st)$ and $n \in N^-_L$.

We are now in a position to prove that Conjecture C holds in this case.

**Proposition 5.15.** Suppose $L = \{s, t\} \subseteq S$ is such that the order $m$ of $st$ is odd. Then Conjecture C holds for $L \subseteq S$.

**Proof.** We have that $\Phi_L = \sum_{j=1}^{k} \text{Ind}^{W_L}_{C_W(st)} \varphi_{(st)}$, by Theorem 5.11.

Recall from [4, Sec. 3] that left multiplication by $x_L$ defines an isomorphism of the right $W_L$-modules $e^L_1 CW_L$ and $e_L CW_L$. Therefore, the elements $e_L f_j = x_L e^L_1 f_j$, for
j = 1, \ldots, m-1, form a $C$-basis of $e_L CW_L$, which as $N_W(W_L)$-module affords the character $\Phi_L$, and as $W_L$-module is isomorphic to the top component $E_L$ with character $\Phi_L$.

Moreover, if we denote $M_j = e_L f_j \cap N_W(W_L)$, then the $N_W(W_L)$-module $M_j$ has $C$-basis \{e_L f_j, e_L f_{m-j}\}, due to the nontrivial action of $\gamma$ and $w_L$, and the direct sum $\bigoplus_{j=1}^k M_j$ is a decomposition of $e_L CW_L$ as $N_W(W_L)$-module. Consequently, part (i) of Conjecture C follows from the observation that as $W_L$-module $M_j$ affords the character $\text{Ind}^{W_L}_{C_W(st)} \varphi_{(st)}$, and as $N_W(W_L)$-module it affords the character $\text{Ind}^{N_W(W_L)}_{C_{W(st)}} \widetilde{\varphi}_{(st)}$, i.e.,

$$\widetilde{\Phi}_L = \sum_{j=1}^k \text{Ind}^{N_W(W_L)}_{C_{W(st)}} \widetilde{\varphi}_{(st)}.$$ 

By Remark 4.3, it now suffices to show that $\widetilde{\Psi}_L = \widetilde{\Phi}_L e_s \alpha_L$. For this, denote $a_L = a_s a_L$, and recall from Lemma 5.9 that $a_L CW_L$ is isomorphic to the top component of $A(W_L)$.

Since $m$ is odd, we have $W_L = \langle st \rangle \cup w_L \langle st \rangle$ and thus

$$a_L C \langle st \rangle = a_L CW_L,$$

since $a_L w_L = a_s a_L w_L = a_t a_s = -a_s a_t = -a_L$.

Since the idempotents $f_j$ from equation (5.14) form a Wedderburn basis of the group algebra $C \langle st \rangle$, the module $a_L C \langle st \rangle$ is spanned by the elements \{a_L f_j : j = 0, \ldots, m-1\}, and since

$$a_L f_0 = \sum_{k=0}^{m-1} a_0 a_{m-1} (ts)^k = \sum_{k=0}^{m-1} a_{k+1} a_k = a_0 a_{m-1} - a_0 a_1 + \sum_{k=1}^{m-2} a_0 a_k - a_0 a_{k+1} = 0,$$

we also have that the set \{a_L f_j : j = 1, \ldots, m-1\} is a $C$-basis of $a_L CW_L$. Conjecture C (iii) now follows if we can show that

(5.16) $$a_L f_j w = \epsilon(w) a_L (w) e_L f_j w,$$

for all $w \in N_W(W_L)$. It suffices to show this for $w = st$, $w = w_L$, and for $w = n \in N_L$.

For $w = st$, (5.16) follows, since $f_j st = e_{m} f_j$ and $\epsilon(st) = \alpha_L(st) = 1$. For $w = w_L$, (5.16) follows, since $f_j w_L = w_L f_{m-j}$ and $e_L w_L = e_L$, $a_L w_L = -a_L$, and $\epsilon(w_L) = -1$ and $\alpha_L(w_L) = 1$. Finally, for $w = n \in N_L$, (5.16) holds, since $f_j n = n f_j^n$ and $e_L n = e_L$, $a_L n = \sigma_L(n) a_L$ and $\sigma_L(n) = \epsilon(n) \alpha_L(n)$, by Lemma 2.1. \qed

We summarize Propositions 5.13, 5.15, and Theorem 4.7 for rank 2 parabolic subgroups as follows.

**Theorem 5.17.** Suppose that $W_L$ is a rank 2 parabolic subgroup of $W$. Then Conjecture C holds for $W_L$. 

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