Title	Computations for Coxeter arrangements and Solomon's descent algebra: Groups of rank three and four
Author(s)	Bishop, Marcus; Douglass, J. Matthew; Pfeiffer, Götz; Röhrle, Gerhard
Publication Date	2012-06-03
Publication Information	Bishop, Marcus, Douglass, J. Matthew, Pfeiffer, Götz, & Röhrle, Gerhard. (2013). Computations for Coxeter arrangements and Solomons descent algebra: Groups of rank three and four. Journal of Symbolic Computation, 50(Supplement C), 139-158. doi: https://doi.org/10.1016/j.jsc.2012.06.001
Publisher	Elsevier
Link to publisher's version	https://doi.org/10.1016/j.jsc.2012.06.001
Item record	http://hdl.handle.net/10379/7009
DOI	http://dx.doi.org/10.1016/j.jsc.2012.06.001

COMPUTATIONS FOR COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA: GROUPS OF RANK THREE AND FOUR

MARCUS BISHOP, J. MATTHEW DOUGLASS, GÖTZ PFEIFFER, AND GERHARD RÖHRLE

Abstract. In recent papers we have refined a conjecture of Lehrer and Solomon expressing the character of the representation of a finite Coxeter group W on the pth graded piece of its Orlik-Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of W. Our refined conjecture relates the character of W on the pth graded piece of its Orlik-Solomon algebra with the descent algebra of W. A consequence of our conjecture is that both the regular character of W and the character of W acting on its Orlik-Solomon algebra have parallel, graded decompositions as sums of characters induced from linear characters of centralizers of elements of W, one for each conjugacy class of elements of W.

The refined conjectures have been proved for symmetric and dihedral groups. In this paper we develop algorithmic tools to prove the conjectures computationally for a given W and we use these tools to verify the claim for all finite Coxeter groups of rank three and four. The techniques developed and implemented in this paper provide previously unknown decompositions of the regular characters and the Orlik-Solomon characters of the Coxeter groups of types B_3, H_3, B_4, D_4, F_4, and H_4 as sums of induced representations indexed by the set of conjugacy classes of W.

1. Introduction

Let W be a finite Coxeter group and let V be a finite dimensional, complex vector space affording a faithful representation of W such that each element in a Coxeter generating set S of W acts on V as a reflection. Let M be the complement in V of the union of the fixed-point hyperplanes of the reflections in W. Then M is a W-stable, open subset of V and the action of W on M determines a representation of W on $H^p(M)$, the p^{th} singular cohomology group of M. Let ω_W^p denote the character of the representation of W on $H^p(M)$ and let $\omega_W = \sum_{p\geq 0} \omega_W^p$ denote the character of the representation of W on the cohomology ring $H^\bullet(M) = \bigoplus_{p\geq 0} H^p(M)$. The character ω_W has been computed by Lehrer and others [2], [11].

Lehrer and Solomon [16] conjectured that ω_W^p is a sum of characters induced from linear characters of centralizers of elements of W. Conjecture 2.1 in [8] is a more precise version of the Lehrer-Solomon conjecture that in addition to describing ω_W^p as a sum of induced characters, also relates the decomposition $\omega_W = \sum_{p\geq 0} \omega_W^p$ to a decomposition of the regular character ρ_W of W arising from the complete set of primitive orthogonal idempotents of the descent algebra of W found by Bergeron, Bergeron, Howlett, and Taylor in [1]. The main result in [8] is a proof of Conjecture 2.1 for symmetric groups.

In [10] an inductive approach that would lead to a proof of Conjecture 2.1 in [8] was developed. The inductive approach parses Conjecture 2.1 into components known as

Conjectures B and C, which we now describe. If $n = |S|$, then $H^n(M)$ is the highest degree non-vanishing cohomology group. For a subset L of S we denote the parabolic subgroup $\langle L \rangle$ of W by W_L. A conjugacy class $\mathcal{C}$ in W is said to be *cuspidal* if $\mathcal{C} \cap W_L = \emptyset$ for every proper subset L of S. Conjecture B describes the character ω_W^n of W as a sum of characters induced from centralizers of cuspidal conjugacy classes. Furthermore, Conjecture B also relates ω_W^n to an appropriate summand in the decomposition of the regular character above, namely the character whose degree is the cardinality of the set of cuspidal elements in W.

Conjecture C is a relative version of Conjecture B for the pair (W, W_L), where the parabolic subgroup W_L is fixed and the overgroup W varies. It mirrors Conjecture B for the group W_L, but in place of the character ω_{W_L} it has an extension of ω_{W_L} to the normalizer of W_L in W, and in place of the characters of centralizers of cuspidal elements in W_L are the centralizers of the same elements in W. It is shown in [10] that if the parabolic subgroups W_L satisfy Conjecture C for all $L \subseteq S$, then Conjecture 2.1 holds for W. Finally, as an application of the method, both conjectures were proved for dihedral groups in [10].

In this paper, we develop algorithms to prove Conjecture B in [10] and consequently Conjecture 2.1 in [8] for a given finite Coxeter group. We have implemented these algorithms using the GAP programming system [20] with the CHEVIE [12] and ZigZag [18] packages. We present the results of our computations for W of type B_3, H_3, B_4, D_4, F_4, and H_4, thus verifying the conjectures for all irreducible Coxeter groups of rank three or four. As a consequence of our computations, we can state Conjecture 2.1 of [8] for groups of rank at most four as the following theorem.

Theorem 1.1. *Suppose that W is a finite Coxeter group with rank at most four and that $\mathcal{R}$ is a set of conjugacy class representatives of W. Then for each $w \in \mathcal{R}$ there exists a linear character φ_w of $C_W(w)$ such that if ρ_W is the regular character of W, ϵ is the sign character of W, and α_w is the composition of* det *with restriction to the* 1*-eigenspace of w, then*

$$\rho_W = \sum_{w \in \mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \varphi_w \quad \text{and} \quad \omega_W = \epsilon \sum_{w \in \mathcal{R}} \operatorname{Ind}_{C_W(w)}^W (\alpha_w \varphi_w).$$

Moreover, if $\mathcal{R}_p$ is the set of w in $\mathcal{R}$ such that the codimension in V of the 1*-eigenspace of w is p, then*

$$\omega_W^p = \epsilon \sum_{w \in \mathcal{R}_p} \operatorname{Ind}_{C_W(w)}^W (\alpha_w \varphi_w).$$

Our current methods are sufficient to treat somewhat larger groups, but are computationally too expensive to be able to handle the largest exceptional Coxeter groups. In future work we hope to develop additional computational techniques to be able to efficiently verify the conjectures for groups with rank up to eight.

The rest of this paper is organized as follows. In §2 we review the constructions from [8] and [10] and show how our computations lead to a proof of Theorem 1.1. In §3 we describe the algorithms we have used and their implementation in GAP. Finally, in §4 we present the results of our computations for rank three and four Coxeter groups. In the appendix we give a table listing all so-called bulky parabolic subgroups of all finite irreducible Coxeter groups.

2. Preliminaries

2.1. **Coxeter Groups and the Orlik-Solomon Algebra.** In this subsection we briefly review the constructions in [8] and [10], state in Theorem 2.3 the main result verified by our computations, and show how Theorem 2.3 leads to a proof of Theorem 1.1.

Recall that an element of W is called *cuspidal* if none of its conjugates lies in a proper parabolic subgroup of W. A conjugacy class is called *cuspidal* if its elements are all cuspidal. It follows from the fact that the proper parabolic subgroups of W arise as pointwise stabilizers of proper subspaces of V that an element is cuspidal if and only if its 1-eigenspace has codimension $|S|$ in V. It is shown in [13] that up to the natural action of W, the conjugacy classes in W are parameterized by pairs $(W_1, \mathcal{C}_1)$, where W_1 is a parabolic subgroup of W and $\mathcal{C}_1$ is a cuspidal conjugacy class in W_1.

Let $T = \{\, w^{-1}sw \mid s \in S,\, w \in W \,\}$ be the set of reflections in W. For t in T let H_t be the hyperplane in V fixed by t. Let E be a $\mathbb{C}$-vector space with basis $\{\, e_t \mid t \in T \,\}$. The *Orlik-Solomon algebra* $A(W)$ is the quotient of the exterior algebra of E by the ideal generated by elements of the form

$$\sum_{i=1}^{m} (-1)^i e_{t_1} e_{t_2} \cdots \widehat{e_{t_i}} \cdots e_{t_m} \tag{2.1}$$

for every set $\{H_{t_1}, H_{t_2}, \dots, H_{t_m}\}$ of linearly dependent hyperplanes. The group W acts on the exterior algebra by $se_t = e_{sts}$ for $s \in S$ and $t \in T$. The ideal generated by elements of the form (2.1) is homogeneous and W-stable, and so $A(W) = \bigoplus_{p\geq 0} A^p(M)$ is a graded, skew-commutative $\mathbb{C}$-algebra on which W acts as algebra automorphisms. We denote the image of the generator e_t in $A(W)$ by a_t.

It is known that $A(W)$ is isomorphic to the cohomology ring $H^\bullet(M)$ as graded W-algebras (see [17, Chapter 3]). It is clear from the definition of $A(W)$ that $A^n(W)$ is the highest degree non-zero component. We refer to $A^n(W)$ as the *top component* of $A(W)$. Then the character of the top component is ω_W^n. It is shown in [7] that the degree of ω_W^n is the cardinality of the set of cuspidal elements in W.

For a subset J of S, let X_J denote the set of minimal length right coset representatives of W_J in W and set $x_J = \sum_{w\in X_J} w$ in the group algebra $\mathbb{C}W$. Solomon has shown that the set $\{\, x_J \mid J \subseteq S \,\}$ is linearly independent and spans a subalgebra of $\mathbb{C}W$ called the *descent algebra* of W (see [1]).

Bergeron, Bergeron, Howlett, and Taylor [1, §7] define a basis of the descent algebra consisting of quasi-idempotents as follows. For subsets J and K of S define

$$m_{KJ} = \left|\{\, x \in X_J \mid x^{-1}Jx \subseteq K \,\}\right| \quad \text{if} \quad J \subseteq K \quad \text{and} \quad m_{KJ} = 0 \quad \text{if} \quad J \not\subseteq K.$$

Note that $m_{KK} > 0$, since $1_W \in X_K$ for all $K \subseteq S$. Then the $2^n \times 2^n$ matrix with rows and columns indexed by the power set of S and with (K, J)-entry m_{KJ} is invertible. Define n_{KJ} to be the (K, J)-entry of the inverse matrix and define $e_K = \sum_J n_{KJ} x_J$. Then $e_K e_K = \gamma_K e_K$, where $\gamma_K = |\{\, L \subseteq S \mid \exists w \in W,\, w^{-1}Lw = K \,\}|$ and so each e_K is a quasi-idempotent in $\mathbb{C}W$. In particular, $e_S = \sum_J n_{SJ} x_J$ is an idempotent. In analogy with $A^n(W)$ we call $\mathbb{C}We_S$ the *top component* of $\mathbb{C}W$ and denote the character it affords by ρ_W^n. It is shown in [1] that the degree of ρ_W^n is the cardinality of the set of cuspidal elements in W.

Remark 2.2. If $W = W_1 \times W_2$ is reducible, then an element (w_1, w_2) in $W_1 \times W_2$ is cuspidal if and only if w_1 is cuspidal in W_1 and w_2 is cuspidal in W_2. It is straightforward to show that the idempotent generating the top component of $\mathbb{C}W$ is the product of the idempotents generating the top components of $\mathbb{C}W_1$ and $\mathbb{C}W_2$. Therefore, the top component of $\mathbb{C}W$ is isomorphic to the tensor product of the top components of $\mathbb{C}W_1$ and $\mathbb{C}W_2$. Similarly, the top component of $A(W)$ is isomorphic to the tensor product of the top components of $A(W_1)$ and $A(W_2)$, by the Künneth theorem.

The content of the next theorem is Conjecture B from [10] for groups with rank at most four.

Theorem 2.3. *Suppose that W is a finite Coxeter group with rank $n \leq 4$ and that $\mathcal{C}$ is a set of representatives of the cuspidal conjugacy classes of W. Then for each $w \in \mathcal{C}$ there exists a linear character φ_w of $C_W(w)$ such that*

$$\rho_W^n = \sum_{w\in\mathcal{C}} \operatorname{Ind}_{C_W(w)}^W \varphi_w = \epsilon\omega_W^n.$$

This theorem has been proved with no restriction on the rank of W for symmetric groups in [8] and dihedral groups in [10]. In this paper we prove the theorem for the remaining finite Coxeter groups of rank three or four in §4 by explicitly computing the linear characters φ_w. A description of the GAP programs used in this calculation is given in §3.

Observe that if $W = W_1 \times W_2$ is reducible, then since induction commutes with tensor products, Remark 2.2 implies that Theorem 2.3 holds for W if and only if it holds for both W_1 and W_2 where the characters φ_w satisfying Theorem 2.3 for W are the tensor products of those satisfying the theorem for W_1 and W_2. Thus it suffices to consider the case when W is irreducible.

To prove Theorem 1.1 we require a linear character of the centralizer of a representative of every conjugacy class of W. For cuspidal classes, we can use the characters satisfying Theorem 2.3. For non-cuspidal conjugacy classes we use a relative version of Theorem 2.3 that takes into account the embedding of a parabolic subgroup of W in its normalizer in W, as follows.

Let L be a subset of S of size r. Then W_L acts on the top components of $A(W_L)$ and $\mathbb{C}W_L$ and we denote the characters of W_L afforded by these spaces by ω_L^r and ρ_L^r rather than by $\omega_{W_L}^r$ and $\rho_{W_L}^r$ to simplify notation. Suppose that Theorem 2.3 holds for W_L. This means that for each w in a set $\mathcal{C}_L$ of representatives of the cuspidal conjugacy classes of W_L we have a linear character φ_w of $C_{W_L}(w)$ such that

$$\rho_L^r = \sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_{W_L}(w)}^{W_L} \varphi_w = \epsilon\omega_L^r,$$

where ϵ is the sign character of W. Observe that if w is a cuspidal element in W_L, then $C_W(w)$ is contained in $N_W(W_L)$ and that the quotients $C_W(w)/C_{W_L}(w)$ and $N_W(W_L)/W_L$ are isomorphic, by the main result of [15]. It is shown in [8] that the characters ρ_L^r and ω_L^r of W_L extend to characters $\widetilde{\rho_L^r}$ and $\widetilde{\omega_L^r}$ of $N_W(W_L)$. Then Conjecture C of [10] asserts that there is a corresponding extension of the characters φ_w. We state this as the following theorem, which proves Conjecture C for W of rank at most four.

Theorem 2.4. *Suppose that W is a finite Coxeter group of rank at most four, that L is a proper subset of S of size r, and that $\mathcal{C}_L$ is a set of representatives of the cuspidal conjugacy classes of W_L. Then for each $w \in \mathcal{C}_L$ the linear character φ_w of $C_{W_L}(w)$ in Theorem 2.3 extends to a linear character $\widetilde{\varphi_w}$ of $C_W(w)$ such that*

$$\widetilde{\rho_L^r} = \sum_{w \in \mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \widetilde{\varphi_w} = \epsilon \alpha_L \widetilde{\omega_L^r},$$

where α_L is the composition of $\det$ *with restriction to the subspace of fixed points of W_L.*

The characters and subgroups in the theorem are summarized in the following diagram.

$$\begin{tikzcd}[column sep=small]
& N_W(W_L),\ \widetilde{\varphi_w}^{N_W(W_L)} \arrow[dl, dash] \arrow[dr, dash] & \\
C_W(w),\ \widetilde{\varphi_w} \arrow[dr, dash] & & W_L,\ \varphi_w^{W_L} \arrow[dl, dash] \\
& C_{W_L}(w),\ \varphi_w &
\end{tikzcd}$$

Proof. If $W = W_1 \times W_2$ is reducible, then, by Remark 2.2, the theorem holds for (W, W_L) if and only if it holds for each of (W_1, W_{L_1}) and (W_2, W_{L_2}), where $L_1, L_2 \subseteq S$ are such that $W_L = W_{L_1} \times W_{L_2}$. Thus, we may assume that W is irreducible.

Recall that W_L is said to be *bulky* in W if it has a normal complement in $N_W(W_L)$ (see [19]). Theorem 2.4 is proved for any W in [10] if $|L| \leq 2$ or if W_L is bulky. It follows that the theorem holds if W has rank three, so we may assume that W has rank four and that W_L is non-bulky of maximal rank. The bulky parabolic subgroups of all finite irreducible Coxeter groups are listed in Appendix A. The eight pairs (W, W_L) where W has rank four and W_L is non-bulky of maximal rank are listed in the following table.

W	W_L		
B_4	$\widetilde{A}_1 A_2$	A_3	
D_4	A_3		
F_4	$A_1 \widetilde{A}_2$	$A_2 \widetilde{A}_1$	
H_4	A_3	$A_1 A_2$	$I_2(5) A_1$

In this table $\widetilde{A}_1$ and $\widetilde{A}_2$ denote subgroups of types A_1 and A_2 generated by reflections orthogonal to short roots.

With no restriction on the rank of W, it is shown in [8, §7] that Theorem 2.4 holds in case all factors of W_L are of type A. This only leaves the pair $(H_4, I_2(5)A_1)$ to be considered. It is straightforward to verify that Theorem 2.4 holds in this case. ◻

It is shown in [10, Theorem 4.7] that the characters φ_w satisfying Theorem 1.1 can be taken to be the union over $L \subseteq S$ of the sets of characters $\widetilde{\varphi_w}$ satisfying Theorem 2.4 for W_L. Therefore, Theorem 1.1 holds for W.

3. Implementation

The proof of Theorem 2.3 consists of exhibiting the characters φ_w satisfying the theorem for each irreducible Coxeter group of rank three or four. These characters are presented in §4. In this section we describe how the characters φ_w and the top component characters ρ_W^n and ω_W^n were calculated.

The calculations were performed using the computer algebra system GAP [20]. As one would expect in a computer algebra system, the user can introduce a specific group and use the system's built-in commands to calculate information about the group. In this project, we are primarily concerned with conjugacy classes and their representatives, centralizers and normalizers, character tables, linear algebra, and class function manipulation, namely class function sums, the scalar product of class functions, and induced class functions.

The CHEVIE package for GAP [12] provides additional functionality for manipulating a finite Coxeter group W in ways specific to such groups. For example, the package provides the length function for W and a mechanism for expressing elements of W as products of Coxeter generators. It also provides the reflection representation of W through the matrices giving the action of the Coxeter generators on a vector space.

The matrix representation is useful for identifying the cuspidal conjugacy classes of W. Recall that an element w is cuspidal if none of its eigenvalues equals 1. Thus, to determine whether w is cuspidal, we only need to inspect the eigenvalues of the matrix representing w. The cuspidal classes can also be determined using the `CuspidalClasses` function supplied by the ZigZag package [18].

3.1. **The top component character of $\mathbb{C}W$.** Let $C_1, C_2, \dots, C_m$ be the conjugacy classes of W and let $x_i \in C_i$ for $1 \leq i \leq m$. We denote the *descent set* $\{ s \in S \mid \ell(sw) < \ell(w) \}$ of $w \in W$ by $\mathcal{D}(w)$. With this notation $X_J = \{ w \in W \mid \mathcal{D}(w) \subseteq S \setminus J \}$.

By Exercise 16 of [6, §9], we have

$$\rho_W^n(x_i^{-1}) = |C_W(x_i)| \sum_{w \in C_i} a_w, \qquad \text{where} \qquad e_S = \sum_{w \in W} a_w w.$$

But since $e_S = \sum_{J \subseteq S} n_{SJ} x_J$, we have $a_w = \sum_{\mathcal{D}(w) \subseteq S \setminus J} n_{SJ}$ so that

$$\rho_W^n(x_i^{-1}) = |C_W(x_i)| \sum_{J \subseteq S} n_{SJ} |\{ x \in C_i \mid \mathcal{D}(x) \subseteq S \setminus J \}|.$$

This calculation shows that ρ_W^n can be computed as a product of matrices. Namely, let

$$I = (i_{JK}), \quad \text{where } i_{JK} = \begin{cases} 1 & \text{if } J \subseteq K \\ 0 & \text{otherwise} \end{cases}$$

and let

$$D = (d_{Ji}), \quad \text{where } d_{Ji} = |\{ x \in C_i \mid \mathcal{D}(x) = S \setminus J \}.$$

Then I is the incidence matrix of the power set of S and D expresses the distribution of the conjugacy classes into descent classes. Then the (J,i)-entry of ID is $|\{ x \in C_i \mid \mathcal{D}(x) \subseteq S \setminus J \}|$. Thus multiplying ID on the left by row S of N and on the right by the diagonal matrix with entries $|C_W(x_i)|$ results in $1 \times m$ matrix whose entries are $\rho_W^n(x_i^{-1})$ for $1 \leq i \leq m$. The ZigZag package provides an implementation of the above calculation through the `ECharacters` function which returns the characters of $\mathbb{C}We_L$ for $L \subseteq S$ as a list. The last entry in this list is ρ_W^n.

3.2. **Choosing linear characters of the centralizers.** Let $\{w_1, w_2, \ldots, w_r\}$ be a list of representatives of the cuspidal classes of W. The selection of the characters φ_{w_i} satisfying Theorem 2.3 can in principle be accomplished by testing whether $\rho_W^n = \sum_{i=1}^r \varphi_i^W$ for each tuple $(\varphi_1, \varphi_2, \ldots, \varphi_r)$, where φ_i is a linear character of $C_W(w_i)$ for all $1 \leq i \leq r$. While easy to automate, this method is expensive because of the large number of tuples of linear characters to be tested. Therefore, we have adopted a binary search algorithm that we now describe.

Let $\chi_1, \chi_2, \ldots, \chi_m$ be the irreducible characters of W. If χ is any character of W, then taking the scalar product of χ with each of the characters χ_i we obtain an m-tuple of non-negative integers. We call this tuple the *constituency tuple* of χ. We endow that set of constituency tuples with the product partial order, so $(t_1, t_2, \ldots, t_m) \leq (u_1, u_2, \ldots, u_m)$ if and only if $t_i \leq u_i$ for all $1 \leq i \leq m$.

Considering constituency tuples rather than characters converts the problem of selecting a linear character φ_i of each $C_W(w_i)$ such that $\rho_W^n = \sum_{i=1}^r \varphi_i^W$ to the problem of selecting a tuple from each of r sets of tuples such that the sum of the selected tuples equals a given tuple.

We select the tuples $\{t^{(i)} \mid 1 \leq i \leq r\}$ in r stages, where each $t^{(i)}$ is the constituency tuple of a linear character of $C_W(w_i)$. At stage i we attempt to select a tuple $t^{(i)}$ for $C_W(w_i)$ such that $\sum_{j=1}^i t^{(j)} \leq$ the constituency tuple of ρ_W^n. Having found such a tuple we proceed to stage $i+1$. On the other hand, if no tuple for $C_W(w_i)$ satisfies this condition, then we return to stage $i-1$ and select a different tuple for $C_W(w_{i-1})$.

3.3. **The top component character of** $A(W)$**.** We calculate the character ω_W^n by explicitly calculating the representation of W on the top component of $A(W)$. This calculation is straightforward once we have a basis of the top component and a method for writing an arbitrary product $a_{t_1} a_{t_2} \cdots a_{t_p}$ in $A(W)$ as a linear combinations of basis elements. The solution to both problems is provided by the non-broken circuit basis that we now briefly describe. See §3.1 of [17] for more information.

Let H be a sequence $H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ of hyperplanes in $\mathcal{A}$. We call H a *circuit* if H is dependent, but $H_{t_1}, \ldots, \widehat{H_{t_j}}, \ldots, H_{t_p}$ is independent for each $1 \leq j \leq p$. Note that GAP can easily test whether a tuple of vectors is linearly independent using linear algebra functions such as `Rank`. Thus, once a set of linear functionals defining the hyperplanes H_t for t in T has been fixed, it is possible to test whether H is a circuit.

Now fix a total order on the set of reflections T and suppose that H is a sequence $H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ with $t_1 < t_2 < \cdots < t_p$. We call H a *broken circuit* if $H_{t_1}, H_{t_2}, \ldots, H_{t_p}, H_t$ is a circuit for some hyperplane H_t with $t > t_p$, and we call H a *non-broken circuit* if no subsequence of H is a broken circuit. Notice that the empty sequence is a non-broken circuit. Then

$$\mathcal{B} = \{a_{t_1} a_{t_2} \cdots a_{t_p} \mid H_{t_1}, H_{t_2}, \ldots, H_{t_p} \text{ is a non-broken circuit}\} \tag{3.1}$$

is a basis of $A(W)$ by Theorem 3.43 of [17]. Clearly, the broken circuits, and hence the basis $\mathcal{B}$, depend on the chosen total order on T. By construction, $\mathcal{B} \cap A^p(W)$ is a basis of $A^p(W)$ for $1 \leq p \leq n$ and so $\mathcal{B} \cap A^n(W)$ is a basis of the top component of $A(W)$.

Remark 3.2. If H is dependent, then any minimal dependent subsequence of H is a circuit. Removing the last term of such a subsequence results in a broken circuit. This also shows that a non-broken circuit is independent.

We calculate both the broken and the non-broken circuits recursively as follows. Throughout the following procedure we maintain a list L of sequences which remain to be considered. Initially L contains only the empty sequence. Throughout, L has the following properties as a consequence of the way sequences are added to L.

(1) Each $H = H_{t_1}, H_{t_2}, \dots, H_{t_p} \in L$ satisfies $t_1 < t_2 < \cdots < t_p$.

(2) Each $H \in L$ is independent.

(3) For each $p \geq 0$ the sequences of length p occur in L before those of length $p + 1$.

The algorithm consists only of the following loop. While L is not empty we remove the first element $H = H_{t_1}, H_{t_2}, \dots, H_{t_p}$ from L. For each $t > t_p$ let H, H_t be the sequence obtained from H by appending H_t. The following possibilities arise.

(a) Suppose that one of the sequences H, H_t is dependent. Then H, H_t contains a circuit, by Remark 3.2. Note that at this point all the broken circuits of length less than p have been discovered by (3) and the following sentence. If none of the broken circuits identified so far is a subsequence of H, then the entire sequence H, H_t is a circuit and H is a broken circuit.

(b) Suppose that H, H_t is independent for all $t > t_p$. If H contained a broken circuit H' as a subsequence, then H' would contain H_{t_p}, since otherwise H would not have been added to L. But then we would be in case (a) above, since H, H_t would be dependent, where H_t is the hyperplane which completes H' to a circuit. Therefore, H is a non-broken circuit. We add each of the new sequences H, H_t to L.

Note that if $H = H_{t_1}, H_{t_2}, \dots, H_{t_p}$ is any non-broken circuit, then for all $q < p$ and all $t > t_q$ the sequence $H_{t_1}, \dots, H_{t_q}, H_t$ is independent, since otherwise H would contain a broken circuit. Therefore, each of the subsequences $H_{t_1}, \dots, H_{t_q}$ with $q < p$ is added to L in the procedure above, so the procedure eventually discovers that H is a non-broken circuit.

For the purpose of expressing elements of $A(W)$ in terms of $\mathcal{B}$, it suffices to identify only the *minimal* broken circuits, and in fact, this is precisely what the procedure above does. Then by the argument above for non-broken circuits, the procedure also discovers all the minimal broken circuits.

To express an element $a = a_{t_1} a_{t_2} \cdots a_{t_p}$ of $A(W)$ in terms of $\mathcal{B}$ we proceed inductively. If $H = H_{t_1}, \dots, H_{t_p}$ is a non-broken circuit, then $a \in \mathcal{B}$. Otherwise H has a subsequence $H' = H_{t_{j_1}}, \dots, H_{t_{j_q}}$ which is a broken circuit. This means that there is a hyperplane H_t with $t > t_{j_q}$ for which H, H_t is a circuit. Observe that it reduces computation to record t in the procedure above when we originally discovered that H' was a broken circuit, as doing so obviates having to search for such a hyperplane at this point. If H_t happens to be in H, then H is dependent so that $a = 0$. Otherwise we use (2.1) to express a in terms of elements corresponding with sequences containing fewer broken circuits as subsequences. Namely, we write

$$(-1)^q a_{t_{j_1}} \cdots a_{t_{j_q}} = \sum_{k=1}^{q} (-1)^k a_{t_{j_1}} \cdots \widehat{a_{t_{j_k}}} \cdots a_{t_{j_q}} a_t$$

and multiply both sides by the remaining factors of a resulting in $\pm a$ on the left side and elements of $A(W)$ on the right side which can be expressed in terms of $\mathcal{B}$ by induction.

4. Proof of Theorem 2.3

In this section we present the results of our computations for the Coxeter groups of types B_3, H_3, B_4, D_4, F_4, and H_4, thus verifying Theorem 2.3 for irreducible Coxeter groups of rank three and four. For each group we give the following information.

(1) For w running through a set of representatives of the cuspidal conjugacy classes of W, we derive a generating set for $C_W(w)$ and describe the characters φ_w of $C_W(w)$ by giving its values on the generating set. Note that in every case w_0 is central and that the character φ_{w_0} is always the sign character.

(2) We give a table containing the values of the characters ρ_W^n and ω_W^n as well as $\operatorname{Ind}_{C_W(w)}^W \varphi_w$ for each representative w. In all cases we see that

$$\rho_W^n = \sum_{w \in \mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \varphi_w = \epsilon \omega_W^n$$

as asserted in Theorem 2.3. In these tables, the rows are indexed by the characters $\operatorname{Ind}_{C_W(w)}^W \varphi_w$ (denoted simply by φ_w), ρ_W^n, and ω_W^n, and the columns are indexed by the conjugacy classes of W.

(3) For w in W and ζ an eigenvalue of w on V, let $E(\zeta)$ denote the ζ-eigenspace of w. Then $C_W(w)$ acts on $E(\zeta)$ and $y \mapsto \det(y|_{E(\zeta)})^p$ defines a linear character of $C_W(w)$ for each natural number p. Denote this character of $C_W(w)$ by $(\det|_{E(\zeta)})^p$. The characters φ_w do not arise from this construction in general. However, if w is a regular element in W and $E(\zeta)$ is a regular eigenspace of w, that is, such that $E(\zeta) \not\subseteq H_t$ for all t in T, then with one exception, the character φ_w is equal $(\det|_{E(\zeta)})^p$ for some $p > 0$. The exception is the class labeled by the partition 22 in type B_4 (see §4.1.2). When w is regular we use Springer's theory of regular elements (see [21]) to identify the complex reflection group given by the action of $C_W(w)$ on a regular eigenspace $E(\zeta)$ and we compare the character φ_w with $\det|_{E(\zeta)}$ when possible.

A conjugacy class in W is called regular if it contains a regular element.

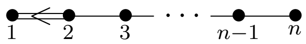
In all the groups we consider below, the longest element w_0 is central and the character φ_{w_0} is the sign character of W. Thus w_0 is regular, and $\varphi_{w_0} = \det|_{E(-1)}$. The Coxeter class is well-known to be a regular class. If w is a Coxeter element, then it acts on its eigenspace $E(\zeta)$ as a cyclic group of order $|w|$, where ζ is a primitive $|w|^{\text{th}}$ root of unity. It turns out to always be the case that $\varphi_w = (\det|_{E(\zeta)})^p$, but it can happen that $p \neq 1$.

We use the following notation. The cyclic group of size n is denoted by Z_n and the symmetric group on n letters is denoted by S_n. For $n \geq 1$ we denote the primitive complex n^{th} root of unity $e^{2\pi i/n}$ by ζ_n. As in the proof of Theorem 2.4, the labels $\widetilde{A}_1$ and $\widetilde{A}_2$ denote subgroups of types A_1 and A_2 generated by reflections orthogonal to short roots. The same convention applies to $\widetilde{D}_4$ in $W(F_4)$. We denote partitions as strings of numbers without commas written in non-decreasing order.

Remark 4.1. Notice that when w_0 is central in W, multiplication by w_0 permutes the conjugacy classes of W and $C_W(ww_0) = C_W(w)$ for all $w \in W$.

4.1. W **of type** B**.** Suppose that V has basis $\{v_1, \dots, v_n\}$, where $n \geq 2$. We view $W = W(B_n)$ as acting on V by signed permutations of $\{v_1, v_2, \dots, v_n\}$. Namely, the Coxeter generators $s_1, s_2, \dots, s_n$ are given by

$$s_1(v_k) = \begin{cases} -v_1, & k = 1 \\ v_k, & k \neq 1 \end{cases} \quad \text{and} \quad s_i(v_k) = \begin{cases} v_i, & k = i-1 \\ v_{i-1}, & k = i \\ v_k, & k \neq i-1, i \end{cases} \quad \text{for } i > 1$$

and the Dynkin diagram of $W(B_n)$ is  as in [4] and in CHEVIE. For $1 \leq i < j \leq n$ we define elements t_i and $s_{i,j}$ by

$$t_i(v_k) = \begin{cases} -v_i, & k = i \\ v_k, & k \neq i \end{cases} \quad \text{and} \quad s_{i,j}(v_k) = \begin{cases} v_j, & k = i \\ v_i, & k = j \\ v_k, & k \neq i, j. \end{cases}$$

It is well-known that the conjugacy classes in $W(B_n)$ are indexed by double partitions $\mu.\lambda$ of n. Namely, if $\mu = \mu_1\mu_2\cdots\mu_q$ and $\lambda = \lambda_1\lambda_2\cdots\lambda_p$ are such that $\sum_{i=1}^q \mu_i + \sum_{j=1}^p \lambda_j = n$, then elements of the conjugacy class indexed by $\mu.\lambda$ have q "positive" cycles of lengths $\mu_1, \dots, \mu_q$ and p "negative" cycles of lengths $\lambda_1, \dots, \lambda_q$. If μ or λ is the empty partition, then it is omitted from the notation. With this labeling, the cuspidal conjugacy classes are indexed by the double partitions of the form $.\lambda$ and hence by partitions of n. See [13, §3.4] for more details.

Fix a partition $\lambda = \lambda_1\lambda_2\cdots\lambda_p$ of n. Set $\tau_1 = 0$ and for $i > 1$ define $\tau_i = \lambda_1 + \cdots + \lambda_{i-1}$. Then $\tau_i + \lambda_i = \tau_{i+1}$ and $\tau_{p+1} = n$. For $1 \leq i \leq p$ define

$$c_i = t_{\tau_i+1} s_{\tau_i+2} s_{\tau_i+3} \cdots s_{\tau_{i+1}}$$

in W. Then c_i has order $2\lambda_i$ and acts on the set $\{v_{\tau_i+1}, \dots, v_{\tau_{i+1}}\}$ as a "negative λ_i-cycle." Define

$$w_\lambda = c_1 c_2 \cdots c_p.$$

Then w_λ is a representative of the cuspidal conjugacy class labeled by λ. For each i such that $\lambda_i = \lambda_{i+1}$ define

$$x_i = s_{\tau_i+1, \tau_{i+1}+1} s_{\tau_i+2, \tau_{i+1}+2} \cdots s_{\tau_{i+1}, \tau_{i+2}}.$$

It is straightforward to check that x_i centralizes w_λ and that $C_W(w_\lambda)$ is generated by

$$\{\, c_i \mid 1 \leq i \leq p \,\} \cup \{\, x_i \mid 1 \leq i \leq p,\ \lambda_i = \lambda_{i+1} \,\}.$$

If λ has m_i parts equal i, then $C_W(w_\lambda) = \prod_{m_i > 0} Z_{2i} \wr S_{m_i}$.

The conjugacy class labeled by the partition with all parts equal 1 is central and contains the longest element w_0 of W. It turns out that the character φ_{w_0} is always the sign character. At the other extreme, the conjugacy class labeled by the partition with a single part n is the Coxeter class. To simplify the notation, we denote the character φ_{w_λ} of $C_W(w_\lambda)$ simply by φ_λ.

4.1.1. $W = W(B_3)$. The cuspidal conjugacy classes are labeled by the partitions 111, 12, and 3. The classes 111 and 3 are regular. The characters φ_λ satisfying $\rho_W^3 = \sum_{\lambda \vdash 3} \mathrm{Ind}_{C_W(w_\lambda)}^W \varphi_\lambda = \epsilon\omega_W^3$ are given in the following table. For each partition λ, the table lists the isomorphism type of $C_W(w_\lambda)$ in the second row, the generators of $C_W(w_\lambda)$ using the notation from §4.1 in the third row, and directly below each generator, the value of φ_λ on that generator. However, when φ_λ is the sign character, we omit its character values.

λ	111	12		3
$C_W(w_\lambda)$	W	$Z_2 \times Z_4$		Z_6
Generators	S	c_1	c_2	w_3
φ_λ	ϵ	-1	-1	ζ_6

We see that $\varphi_3 = \det|_{E(\zeta_6)}$.

The values of the characters φ_λ^W together with ρ_W^3 and ω_W^3 are given in Table 1.

	111.	11.1	1.11	.111	12.	1.2	2.1	.12	3.	.3
$\epsilon = \varphi_{111}$	1	-1	1	-1	-1	1	1	-1	1	-1
φ_{12}	6	-2	2	-6	·	-2	·	2	·	·
φ_3	8	·	·	-8	·	·	·	·	-1	1
ρ_W^3	15	-3	3	-15	-1	-1	1	1	·	·
ω_W^3	15	3	3	15	1	-1	1	-1	·	·

Table 1. The characters φ_w^W, ρ_W^3, and ω_W^3 for $W(B_3)$

4.1.2. $W = W(B_4)$. The cuspidal conjugacy classes are labeled by the partitions 1111, 112, 22, 13, and 4. The regular classes are 1111, 22, and 4. The characters φ_λ satisfying $\rho_W^4 = \sum_{\lambda \vdash 4} \mathrm{Ind}_{C_W(w_\lambda)}^W \varphi_\lambda = \epsilon\omega_W^4$ are given in the following table. The conventions are the same as for $W(B_3)$.

λ	1111	112			22		13		4
$C_W(w_\lambda)$	W	$(Z_2 \wr S_2) \times Z_4$			$Z_4 \wr S_2$		$Z_2 \times Z_6$		Z_8
Generators	S	c_1	x_1	c_3	c_1	x_1	c_1	c_2	w_4
φ_λ	ϵ	-1	-1	-1	-1	-1	-1	ζ_6	-1

In contrast with the Coxeter class in type B_3 where $\varphi_3 = \det|_{E(\zeta_6)}$, in this case we have $\varphi_4 = (\det|_{E(\zeta_8)})^4$. It is easy to compute that if ζ is any primitive fourth root of unity, then $C_W(w_{22})$ acts on the two-dimensional, regular eigenspace $E(\zeta)$ as the complex reflection group $G(4,1,2) \cong Z_4 \wr S_2$. The eigenvalues of c_1 and x_1 acting on $E(\zeta)$ are $\{1, \zeta\}$ and

$\{1, -1\}$, respectively. Thus $\det c_1|_{E(\zeta)} = \zeta$ and $\det x_1|_{E(\zeta)} = -1$. It follows that φ_{22} is not equal $(\det|_{E(\zeta)})^p$ for any eigenvalue ζ or any power p.

The values of the characters φ_λ^W together with ρ_W^4 and ω_W^4 are given in Table 2.

	1111.	111.1	11.11	1.111	.1111	112.	11.2	12.1	1.12	2.11
$\epsilon = \varphi_{1111}$	1	-1	1	-1	1	-1	1	1	-1	-1
φ_{112}	12	-6	4	-6	12	-2	·	·	2	-2
φ_{22}	12	·	4	·	12	·	-4	·	·	·
φ_{13}	32	-8	·	-8	32	·	·	·	·	·
φ_4	48	·	·	·	48	·	·	·	·	·
ρ_W^4	105	-15	9	-15	105	-3	-3	1	1	-3
ω_W^4	105	15	9	15	105	3	-3	1	-1	3

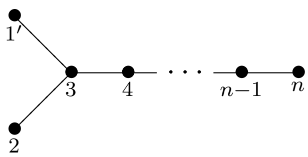
.112	22.	2.2	.22	13.	1.3	3.1	.13	4.	.4	
1	1	-1	1	1	-1	-1	1	-1	1	$\varphi_{1111} = \epsilon$
·	·	2	-4	·	·	·	·	·	·	φ_{112}
-4	-4	·	·	·	·	·	·	·	2	φ_{22}
·	·	·	·	-1	1	1	-1	·	·	φ_{13}
·	·	·	8	·	·	·	·	·	-4	φ_4
-3	-3	1	5	·	·	·	·	-1	-1	ρ_W^4
-3	-3	-1	5	·	·	·	·	1	-1	ω_W^4

TABLE 2. The characters φ_w^W, ρ_W^4, and ω_W^4 for $W(B_4)$

4.2. W **of type** D**.** Suppose V has basis $\{v_1, \dots, v_n\}$ where $n \geq 4$. We consider elements of $W = W(D_n)$ as acting as signed permutations of the basis of V having an even number of sign changes. Then $W(D_n)$ is a normal subgroup of $W(B_n)$ of index 2. The Coxeter generators of $W(D_n)$ are $s_1', s_2, \dots, s_n$ where $s_2, \dots, s_n$ are the last $n-1$ Coxeter generators of $W(B_n)$ defined in §4.1 and s_1' is given by

$$s_1'(v_k) = \begin{cases} -v_2, & k = 1 \\ -v_1 & k = 2 \\ v_k, & k \neq 1, 2 \end{cases}$$

so that $s_1' = s_1 s_2 s_1$, where s_1 is the first Coxeter generator of $W(B_n)$. The Dynkin diagram of $W = W(D_n)$ is



as in [4] and in CHEVIE.

Because $W(D_n)$ is a normal subgroup of $W(B_n)$, it is a union of conjugacy classes of $W(B_n)$. The conjugacy class of $W(B_n)$ labeled by the double partition $\mu.\lambda$ of n lies in $W(D_n)$ if and only if λ has an even number of parts. If the conjugacy class of $W(B_n)$

labeled by the double partition $\mu.\lambda$ lies in $W(D_n)$, then it is a single $W(D_n)$-conjugacy class except in the case when λ is the empty partition and all parts of μ are even. In that case, the $W(B_n)$-conjugacy class splits into two classes in $W(D_n)$ labeled by $\mu.+$ and $\mu.-$.

An element in $W(D_n)$ is cuspidal if and only if it is cuspidal in $W(B_n)$, so the cuspidal conjugacy classes of $W(D_n)$ are labeled by partitions of n with an even number of parts. For such a partition λ we take w_λ to be the representative of the conjugacy class of $W(B_n)$ chosen in §4.1. Then the centralizer in $W(D_n)$ of w_λ is the intersection of $W(D_n)$ and the centralizer of w_λ in $W(B_n)$. See [13, §3.4] for more details.

4.2.1. $W = W(D_4)$. The cuspidal conjugacy classes are labeled by the partitions 1111, 22, and 13. All three classes are regular. Each of these conjugacy classes is also a conjugacy class in the larger group $W(B_4)$, and as remarked above, $C_W(w_\lambda) = W \cap C_{W(B_4)}(w_\lambda)$. One might conjecture that the character of $\varphi_\lambda^{D_4}$ of $C_{W(D_4)}(w_\lambda)$ is the restriction of the character of $\varphi_\lambda^{B_4}$ of $C_{W(B_4)}(w_\lambda)$. This turns out to be the case for the class of w_0 labeled by 1111 and for the Coxeter class labeled by 13, but not for the class labeled by 22.

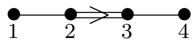
Let $w_{22} = s_1' s_3 s_1' s_2 s_3 s_4$. Using the notation introduced in §4.1 we have $w_{22} = c_1 c_2$. The centralizer of w_{22} in $W(D_4)$ contains the generator x_1 of the centralizer of w_{22} in $W(B_4)$, but not the generator c_1. Rather, $C_{W(D_4)}(w_{22})$ is generated by w_{22}, x_1, and the involution $s_1' s_2$. The character φ_{22} maps each of these generators to -1. Notice that $\varphi_{22}^{B_4}(w_{22}) = \left(\varphi_{22}^{D_4}(w_{22})\right)^2$.

The characters φ_λ satisfying Theorem 2.3 are summarized in the following table, where the conventions are the same as for $W(B_3)$.

λ	1111	22			13
$C_W(w_\lambda)$	W	$(Z_4 \times Z_2) \rtimes S_2$			Z_6
Generators	S	w_{22}	$s_1' s_2$	x_1	w_{13}
φ_λ	ϵ	-1	-1	-1	ζ_6

For the Coxeter class 13 we have $\varphi_{13} = \det|_{E(\zeta_6)}$. Let ζ be a primitive fourth root of unity. Using [21, Theorem 4.2] it is easy to compute that if ζ is any primitive fourth root of unity, then $C_W(w_{22})$ acts on the two-dimensional, regular eigenspace $E(\zeta)$ as the complex reflection group $G(4,2,2) \cong (Z_4 \times Z_2) \rtimes S_2$. The eigenvalues of w_{22}, $s_1' s_2$, and x_1 acting on $E(\zeta)$ are $\{\zeta, \zeta\}$, $\{1, -1\}$ and $\{1, -1\}$, respectively. Thus $\varphi_{22} = \det|_{E(\zeta)}$.

The values of characters φ_λ^W together with ρ_W^4 and ω_W^4 are shown in Table 3.

4.3. $W = W(F_4)$**.** The Dynkin diagram of W is 

. We label the conjugacy classes of W using Carter's labeling [5]. This is also the labeling used by the CHEVIE package. There are nine cuspidal conjugacy classes. Their Carter labels are

$$4A_1, \quad D_4, \quad D_4(a_1), \quad C_3A_1, \quad A_2\widetilde{A}_2, \quad F_4(a_1), \quad F_4, \quad A_3\widetilde{A}_1, \quad \text{and} \quad B_4.$$

The regular classes are $4A_1$, $D_4(a_1)$, C_3A_1, $F_4(a_1)$, F_4, and B_4. For each label d above we denote the representative of the class labeled d by w_d and the character of $C_W(w_d)$ satisfying Theorem 2.3 by φ_d. The classes labeled by $4A_1$, F_4, and B_4 are self-centralizing and we consider them first.

	1111.	11.11	.1111	211.	1.21	2.11	22.+	22.−	.22	31.	.31	4.+	4.−
$\epsilon = \varphi_{1111}$	1	1	1	−1	−1	−1	1	1	1	1	1	−1	−1
φ_{22}	12	−4	12	·	·	·	−4	−4	4	·	·	·	·
φ_{13}	32	·	32	·	·	·	·	·	·	−1	−1	·	·
ρ_W^4	45	−3	45	−1	−1	−1	−3	−3	5	·	·	−1	−1
ω_W^4	45	−3	45	1	1	1	−3	−3	5	·	·	1	1

TABLE 3. The characters φ_λ^W, ρ_W^4, and ω_W^4 for $W(D_4)$

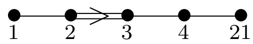
$4A_1$**:** The class labeled by $4A_1$ is $\{w_0\}$. We take $\varphi_{4A_1} = \epsilon$.

F_4**:** This is the Coxeter class. Coxeter elements have order 12. We take w_{F_4} to be any Coxeter element and define φ_{F_4} by $\varphi_{F_4}(w_{F_4}) = \zeta_3$. Clearly, $\varphi_{F_4} = (\det|_{E(\zeta_{12})})^4$.

B_4**:** This class contains the Coxeter elements in the maximal rank subgroups of W of type B_4 and is regular. Then $C_W(w_{B_4}) = C_{W(B_4)}(w_{B_4})$ is cyclic of order 8, where w_{B_4} denotes the element w_4 from §4.1.2. Define φ_{B_4} by $\varphi_{B_4}(w_{B_4}) = \zeta_4$. We see from §4.1.2 that $\varphi_{B_4}^{B_4} = \left(\varphi_{B_4}^{F_4}\right)^2$. In addition, $\varphi_{B_4} = (\det|_{E(\zeta_8)})^2$.

D_4**:** This class contains the Coxeter elements in the maximal rank subgroups of W of type D_4 and $B_3\widetilde{A}_1$ (see [9]). It also contains the class labeled by the partition 13 in the maximal rank subgroups of W of types B_4 and D_4. In $W(D_4)$ the centralizer of w_{13} is isomorphic to Z_6 while in $W(B_3\widetilde{A}_1)$ and $W(B_4)$ the centralizer of w_{13} is isomorphic to $Z_6 \times Z_2$.

Recall that multiplication by w_0 permutes the conjugacy classes of W, by Remark 4.1. In this case multiplication by w_0 sends the class labeled D_4 to the class labeled by A_2 containing s_1s_2. Thus we can take $w_{D_4} = s_1s_2w_0$. Extending the Dynkin diagram of W as in the Borel-De Siebenthal algorithm [3] by adjoining the reflection s_{21} corresponding to the highest short root results in the diagram

(4.2) 

.

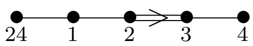
The subgroup generated by $\{s_4, s_{21}\}$ is a parabolic subgroup of type $\widetilde{A}_2$ and

$$C_W(w_{D_4}) = \langle s_4, s_{21}\rangle \times \langle w_{D_4}\rangle \cong S_3 \times Z_6.$$

We define φ_{D_4} by $\varphi_{D_4}|_{S_3} = \epsilon_{S_3}$ and $\varphi_{D_4}(w_{D_4}) = \zeta_3$. Notice that using §4.1.2 and §4.2.1 we have

$$\zeta_6 = \varphi_{D_4}^{D_4}(w_{D_4}) = -\varphi_{D_4}^{B_4}(w_{D_4}) = (\varphi_{D_4}^{F_4}(w_{D_4}))^2.$$

$D_4(a_1)$**:** This class contains the conjugacy classes labeled by the partition 22 in the maximal rank subgroups of W of types B_4 and D_4 and is regular. It also contains the Coxeter elements in the maximal rank subgroup of W of type B_2B_2. To find a representative of this class and compute its centralizer we extend the Dynkin diagram of W by adjoining the reflection s_{24} corresponding to the highest long root. The resulting diagram is



and we consider the maximal rank subgroup $W(B_4)$ generated by $\{s_{24}, s_1, s_2, s_3\}$. Recall from §4.1.2 that the centralizer of $w_{22} =$ in $W(B_4)$ is generated by c_1 and x_1. Translating from the B_4 labeling to our current labeling, we set

$$w_{22} = s_3s_2s_1s_2s_3s_2s_1s_{24}, \quad c_1 = s_3s_2, \quad \text{and} \quad x_1 = s_1s_2s_{24}s_1.$$

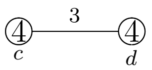
Then w_{22} lies in the conjugacy class we are considering and $C_W(w_{22}) = \langle c_1, x_1\rangle$. Although w_{22} would be a natural representative of $D_4(a_1)$, it is more convenient to define $w_{D_4(a_1)}$ to be the conjugate $s_1s_2s_1w_{22}s_1s_2s_1$ of w_{22} because then $w_{D_4(a_1)}$ will commute with the representative $w_{\widetilde{A}_2A_2}$ chosen below. Define $c_1' = s_1s_2s_1c_1s_1s_2s_1$ and $x_1' = s_1s_2s_1x_1s_1s_2s_1$. Then $C_W(w_{D_4(a_1)})$ is generated by $\{c_1', x_1', w_{\widetilde{A}_2A_2}\}$.

Define $\varphi_{D_4(a_1)}$ by

$$\varphi_{D_4(a_1)}(c_1') = -1, \quad \varphi_{D_4(a_1)}(x_1') = 1, \quad \text{and} \quad \varphi_{D_4(a_1)}(w_{\widetilde{A}_2A_2}) = 1.$$

Notice that using §4.1.2 and §4.2.1 we have

$$\varphi^{F_4}_{D_4(a_1)}(w_{D_4(a_1)}) = \varphi^{B_4}_{22}(w_{22}) = (\varphi^{D_4}_{22}(w_{22}))^2.$$

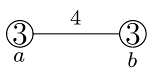
Using [21, Theorem 4.2] it is easy to see that $C_W(w_{D_4(a_1)})$ acts on its ζ_4-eigenspace as the complex reflection group G_8. This group may be described by the diagram  where $c = c_1'$ and $d = s_2s_3s_4s_3$. Then $\varphi_{D_4(a_1)}(c) = \varphi_{D_4(a_1)}(d) = -1$ and $\varphi_{D_4(a_1)} = (\det|_{E(\zeta_4)})^2$.

C_3A_1: This class contains the Coxeter elements of the maximal rank subgroups of W of types C_3A_1 and $\widetilde{D}_4$ (see [9]). In particular, this class is the image under the graph automorphism of W of the class labeled by D_4 so that the centralizers of elements of both classes are isomorphic to $Z_6 \times S_3$. We put $w_{C_3A_1} = s_3s_4w_0$ and compute its centralizer using the same technique used for the class labeled by D_4. We define $\varphi_{C_3A_1}$ by $\varphi_{C_3A_1}(w_{C_3A_1}) = \zeta_3$ and $\varphi_{C_3A_1}|_{S_3} = \epsilon$.

$A_2\widetilde{A}_2$: This class contains the Coxeter elements of the reflection subgroups of W of type $A_2\widetilde{A}_2$ and is regular. We take $w_{A_2\widetilde{A}_2} = s_1s_2s_4s_{21}$ (see (4.2)). We noted above that $w_{D_4(a_1)}$ was chosen so that $w_{A_2\widetilde{A}_2}$ and $w_{D_4(a_1)}$ commute. Obviously s_1s_2 and s_4s_{21} centralize $w_{A_2\widetilde{A}_2}$ and generate an elementary abelian subgroup of $C_W(w_{A_2\widetilde{A}_2})$ of order 9. Then $C_W(w_{A_2\widetilde{A}_2})$ is generated by $\{s_1s_2, s_4s_{21}, w_{D_4(a_1)}\}$. Define $\varphi_{A_2\widetilde{A}_2}$ by

$$\varphi_{A_2\widetilde{A}_2}(s_1s_2) = \zeta_3, \quad \varphi_{A_2\widetilde{A}_2}(s_4s_{21}) = \zeta_3, \quad \text{and} \quad \varphi_{A_2\widetilde{A}_2}(w_{D_4(a_1)}) = 1.$$

Then $\varphi_{A_2\widetilde{A}_2}(w_{A_2\widetilde{A}_2}) = \zeta_3^2$.

Using [21, Theorem 4.2] it is easy to see that $C_W(w_{A_2\widetilde{A}_2})$ acts on its ζ_3-eigenspace as the complex reflection group G_5. This group may be described by the diagram  where $a = s_1s_2$ and $b = s_2s_3s_2s_3s_4s_3$. Then $\varphi_{A_2\widetilde{A}_2}(a) = \varphi_{A_2\widetilde{A}_2}(b) = \zeta_3$ and $\varphi_{A_2\widetilde{A}_2} = \det|_{E(\zeta_3)}$.

$F_4(a_1)$: This class does not contain elements that lie in any proper reflection subgroup of W. It is regular. The image of this class under multiplication by

w_0 is the class labeled by $A_2\widetilde{A}_2$. Thus, $C_W\left(w_{F_4(a_1)}\right) = C_W\left(w_{A_2\widetilde{A}_2}\right)$. We take $w_{F_4(a_1)} = w_{A_2\widetilde{A}_2} w_0$ and $\varphi_{F_4(a_1)} = \varphi_{A_2\widetilde{A}_2}$.

Because $w_{F_4(a_1)} = w_{A_2\widetilde{A}_2} w_0$ and $w_{A_2\widetilde{A}_2}$ has order three, the ζ_6-eigenspace of $w_{F_4(a_1)}$ is equal to the ζ_3-eigenspace of $w_{A_2\widetilde{A}_2}$. Therefore, $C_W\left(w_{F_4(a_1)}\right)$ acts on its ζ_6-eigenspace as the complex reflection group G_5 and $\varphi_{F_4(a_1)} = \det|_{E(\zeta_6)}$.

$A_3\widetilde{A}_1$**:** This class contains the Coxeter elements in the reflection subgroups of W of types $A_3\widetilde{A}_1$, $A_1\widetilde{A}_3$, $2A_1B_2$, and $2\widetilde{A}_1B_2$ (see [9]). In addition, the image of this class under multiplication by w_0 is the class labeled by B_2 and contains s_2s_3. The reflections s_{21} and s_{24} corresponding to the highest short and long roots generate a subgroup of W of type B_2. Taking $w_{A_3\widetilde{A}_1} = s_2s_3w_0$, we have

$$C_W(w_{A_3\widetilde{A}_1}) = \langle w_{A_3\widetilde{A}_1} \rangle \times \langle s_{21}, s_{24} \rangle \cong Z_4 \times W(B_2).$$

Define $\varphi_{A_3\widetilde{A}_1}$ by $\varphi_{A_3\widetilde{A}_1}(w_{A_3\widetilde{A}_1}) = -1$ and $\varphi_{A_3\widetilde{A}_1}|_{W(B_2)} = \epsilon_{W(B_2)}$.

The characters φ_d satisfying $\rho_W^4 = \sum_d \mathrm{Ind}_{C_W(w_d)}^W \varphi_d = \epsilon\omega_W^4$ are summarized in the following table. The conventions are the same as for $W(B_3)$.

d	$4A_1$	D_4		$D_4(a_1)$		C_3A_1		$A_2\widetilde{A}_2$		$F_4(a_1)$		F_4	$A_3\widetilde{A}_1$		B_4
$C_W(w_d)$	W	$Z_6 \times S_3$		G_8		$Z_6 \times S_3$		G_5		G_5		Z_{12}	$Z_4 \times W(B_2)$		Z_8
Generators	S	w_{D_4}	$*$	c	d	$w_{C_3A_1}$	$*$	a	b	a	b	w_{F_4}	$w_{A_3\widetilde{A}_1}$	$*$	w_{B_4}
φ_d	ϵ	ζ_3	ϵ	-1	-1	ζ_3	ϵ	ζ_3	ζ_3	ζ_3	ζ_3	ζ_3	-1	ϵ	ζ_4

The values of the characters φ_d^W together with ρ_W^4 and ω_W^4 are shown in Table 4.

4.4. W **of type** H**.** When W is a non-crystallographic group it can happen that for a given positive integer d there is more than one regular conjugacy class of elements of order d. In this case, if ζ is a fixed primitive d^{th} root of unity and w in W is regular with order d, then ζ might not be an eigenvalue of w (see [21, §5]). Thus, some care must be taken when describing the determinant of the character of $C_W(w)$ acting on a regular eigenspace of w. Similar considerations apply to the characters φ_w.

We label the conjugacy classes $C_1, C_2, \dots$ and choose representatives $w_1, w_2, \dots$ as in [13] and CHEVIE. When n is fixed, we frequently denote $|w_n|$ by d.

4.4.1. $W = W(H_3)$. The cuspidal classes are C_6, C_8, C_9, and C_{10}. All these classes are regular.

C_6**:** $d = 10$ and $C_W(w_6) = \langle w_6 \rangle$. Define $\varphi_6(w_6) = \zeta_{10}$. Then $\varphi_6 = \det|_{E(\zeta_{10})}$.

	A_0	$4A_1$	$2A_1$	A_2	D_4	$D_4(a_1)$	$\widetilde{A}_2$	C_3A_1	$A_2\widetilde{A}_2$	$F_4(a_1)$	F_4	A_1	$3A_1$
$\epsilon = \varphi_{4A_1}$	1	1	1	1	1	1	1	1	1	1	1	-1	-1
φ_{D_4}	32	32	$\cdot$	-1	-1	$\cdot$	2	2	-4	-4	$\cdot$	$\cdot$	$\cdot$
$\varphi_{D_4(a_1)}$	12	12	4	$\cdot$	$\cdot$	8	$\cdot$	$\cdot$	6	6	2	$\cdot$	$\cdot$
$\varphi_{C_3A_1}$	32	32	$\cdot$	2	2	$\cdot$	-1	-1	-4	-4	$\cdot$	-8	-8
$\varphi_{A_2\widetilde{A}_2}$	16	16	$\cdot$	-2	-2	8	-2	-2	7	7	-1	$\cdot$	$\cdot$
$\varphi_{F_4(a_1)}$	16	16	$\cdot$	-2	-2	8	-2	-2	7	7	-1	$\cdot$	$\cdot$
φ_{F_4}	96	96	$\cdot$	$\cdot$	$\cdot$	16	$\cdot$	$\cdot$	-6	-6	-2	$\cdot$	$\cdot$
$\varphi_{A_3\widetilde{A}_1}$	36	36	4	$\cdot$	$\cdot$	-12	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	-6	-6
φ_{B_4}	144	144	$\cdot$	$\cdot$	$\cdot$	-24	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
ρ_W^4	385	385	9	-2	-2	5	-2	-2	7	7	-1	-15	-15
ω_W^4	385	385	9	-2	-2	5	-2	-2	7	7	-1	15	15

	$A_1\widetilde{A}_2$	C_3	A_3	$\widetilde{A}_1$	$2A_1\widetilde{A}_1$	$A_2\widetilde{A}_1$	B_3	B_2A_1	$A_1\widetilde{A}_1$	B_2	$A_3\widetilde{A}_1$	B_4
$\varphi_{4A_1} = \epsilon$	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
φ_{D_4}	$\cdot$	$\cdot$	$\cdot$	-8	-8	1	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\varphi_{D_4(a_1)}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	-4	-4	-2
$\varphi_{C_3A_1}$	1	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\varphi_{A_2\widetilde{A}_2}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\varphi_{F_4(a_1)}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
φ_{F_4}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\varphi_{A_3\widetilde{A}_1}$	$\cdot$	$\cdot$	2	-6	-6	$\cdot$	$\cdot$	2	$\cdot$	$\cdot$	$\cdot$	$\cdot$
φ_{B_4}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
ρ_W^4	$\cdot$	$\cdot$	1	-15	-15	$\cdot$	$\cdot$	1	1	-3	-3	-1
ω_W^4	$\cdot$	$\cdot$	-1	15	15	$\cdot$	$\cdot$	-1	1	-3	-3	-1

TABLE 4. The characters φ_d^W, ρ_W^4, and ω_W^4 for $W(F_4)$

C_8: $d = 6$ and $C_W(w_8) = \langle w_8 \rangle$. Define $\varphi_8(w_8) = \zeta_6$. Then $\varphi_8 = \det|_{E(\zeta_6)}$.

C_9: $d = 10$ and $C_W(w_9) = \langle w_9 \rangle$. Define $\varphi_9(w_9) = \zeta_{10}$. The elements w_6 and w_9 are related by $w_9 = w_6^3$. Thus the ζ_{10}^3-eigenspace of w_9 coincides with the ζ_{10}-eigenspace of w_6, and we have $\varphi_9 = (\det|_{E(\zeta_{10}^3)})^7$.

C_{10}: The element $w_{10} = w_0$ is central and we take $\varphi_{10} = \epsilon$.

The values of the characters φ_i^W together with ρ_W^3 and ω_W^3 are given in Table 5.

4.4.2. $W = W(H_4)$. The cuspidal classes, the order of their elements, and the sizes of their centralizers are listed in the next table. Only the five classes C_{19}, C_{21}, C_{25}, C_{27}, and C_{31} are not regular.

w	$\lvert w\rvert$	$\lvert C_W(w)\rvert$	w	$\lvert w\rvert$	$\lvert C_W(w)\rvert$	w	$\lvert w\rvert$	$\lvert C_W(w)\rvert$	w	$\lvert w\rvert$	$\lvert C_W(w)\rvert$
w_{11}	30	30	w_{19}	10	50	w_{25}	6	36	w_{30}	10	600
w_{14}	20	20	w_{21}	10	100	w_{26}	5	600	w_{31}	10	100
w_{15}	15	30	w_{22}	15	30	w_{27}	5	50	w_{32}	3	360
w_{17}	12	12	w_{23}	20	20	w_{28}	30	30	w_{33}	5	600
w_{18}	10	600	w_{24}	6	360	w_{29}	4	240	w_{34}	2	14400

For $n = 11, 14, 17, 23, 28$ each of the elements w_n is self-centralizing and regular. We define $\varphi_n(w_n) = \zeta_d^2$ in all cases. Then, $\varphi_n = (\det|_{E(\zeta_d)})^2$ for $n = 11, 14, 17$. For $n = 23$ we have $d = 20$, $E(\zeta_{20}^3)$ is a regular eigenspace, and $\varphi_{23} = (\det|_{E(\zeta_{20}^3)})^{14}$. For $n = 28$ we have $d = 30$, $E(\zeta_{30}^7)$ is a regular eigenspace, and $\varphi_{28} = (\det|_{E(\zeta_{30}^7)})^{26}$.

For $n = 15, 22$ we have $d = 15$ and $C_W(w_n) = \langle w_0 w_n\rangle \cong Z_2 \times \langle w_n\rangle$. These classes are regular. We define $\varphi_n(w_0 w_n) = \zeta_{15}$ in both cases. For $n = 15$, $E(\zeta_{15})$ is a regular eigenspace. Notice that, since $\zeta_{15} = (\zeta_{30}^{17})^{16}$, the ζ_{15}-eigenspace of $w_{15} = (w_0 w_{15})^{16}$ is equal to the ζ_{30}^{17}-eigenspace of $w_0 w_{15}$. Thus $\varphi_{15} = (\det|_{E(\zeta_{15})})^{16}$. For $n = 22$, $E(\zeta_{15}^2)$ is a regular eigenspace and $\varphi_{22} = (\det|_{E(\zeta_{15}^2)})^8$.

For $n = 18, 26, 30, 33$ we have $C_W(w_n) = \langle w_{18}, w_{19}, w_{29}\rangle$. These are regular classes and in all cases $C_W(w_n)$ acts on a regular, two-dimensional eigenspace of w_n as the complex reflection group G_{16}. Define

$$\varphi_{18} = \varphi_{33} \quad \text{by} \quad (w_{18}, w_{19}, w_{29}) \mapsto \left(\zeta_5^2, \zeta_5^4, 1\right)$$

and

$$\varphi_{26} = \varphi_{30} \quad \text{by} \quad (w_{18}, w_{19}, w_{29}) \mapsto \left(\zeta_5^4, \zeta_5^3, 1\right).$$

- For $n = 18$ we have $d = 10$, $E(\zeta_{10})$ is a regular eigenspace, and $\varphi_{18} = (\det|_{E(\zeta_{10})})^2$.
- For $n = 26$ we have $d = 5$, $E(\zeta_5)$ is a regular eigenspace, and $\varphi_{26} = (\det|_{E(\zeta_5)})^4$.
- For $n = 30$ we have $d = 10$, $E(\zeta_{10}^3)$ is a regular eigenspace, and $\varphi_{30} = (\det|_{E(\zeta_{10}^3)})^4$.
- For $n = 33$ we have $d = 5$, $E(\zeta_5^2)$ is a regular eigenspace, and $\varphi_{33} = (\det|_{E(\zeta_5^2)})^2$.

For $n = 19, 27$ we have $w_{27} = w_{19}^2$ and $C_W(w_n) = \langle w_{18}\rangle \times \langle w_{27}\rangle \cong Z_{10} \times Z_5$. Define

$$\varphi_{19} = \varphi_{27} \quad \text{by} \quad (w_{18}, w_{27}) \mapsto \left(\zeta_5^3, \zeta_5^4\right).$$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
φ_6	12	$\cdot$	μ	$\cdot$	$\cdot$	$-\nu$	ν	$\cdot$	$-\mu$	-12
φ_8	20	$\cdot$	$\cdot$	$\cdot$	-1	$\cdot$	$\cdot$	1	$\cdot$	-20
φ_9	12	$\cdot$	ν	$\cdot$	$\cdot$	$-\mu$	μ	$\cdot$	$-\nu$	-12
$\epsilon = \varphi_{10}$	1	-1	1	1	1	-1	1	-1	-1	-1
ρ_W^3	45	-1	$\cdot$	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	-45
ω_W^3	45	1	$\cdot$	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	45

TABLE 5. Induced characters $W(H_3)$: $\mu = \zeta_5 + \zeta_5^4$, $\nu = \zeta_5^2 + \zeta_5^3$

For $n = 21, 31$ we have $w_{31} = w_{21}^2$ and $C_W(w_n) = \langle w_{18}\rangle \times \langle w_{27}\rangle \times \langle s_2\rangle \cong Z_{10} \times Z_5 \times Z_2$. Define

$$\varphi_{21} \quad \text{by} \quad (w_{18}, w_{27}, s_2) \mapsto \left(\zeta_5, \zeta_5^2, -1\right) \quad \text{and} \quad \varphi_{31} = \varphi_{21}^3.$$

For $n = 24, 32$ we have $C_W(w_n) = \langle w_{24}, w_{25}, w_{29}\rangle$. Denote this group simply by Z. The classes C_{24} and C_{32} are regular. The representatives w_{24} and w_{32} have order $d = 6$ and $d = 3$, respectively and are related by $w_{32} = w_{24}^2$. Thus, the ζ_6-eigenspace of w_{24} is equal to the ζ_3-eigenspace of w_{32}. Denote this vector space simply by E. Then E is a regular, two-dimensional eigenspace for w_{24} and w_{32}, and Z acts on E as the complex reflection group G_{20}. Define

$$\varphi_{24} = \varphi_{32} \quad \text{by} \quad (w_{24}, w_{25}, w_{29}) \mapsto \left(\zeta_3^2, \zeta_3, 1\right).$$

In both cases we have $\varphi_n = (\det|_{E(\zeta_d)})^2$.

For $n = 25$ we have $C_W(w_{25}) = \langle w_{w_0 w_{24}}\rangle \times \langle w_{25}\rangle \times \langle s_2\rangle \cong Z_6 \times Z_6 \times Z_2$. Define

$$\varphi_{25} \quad \text{by} \quad (w_0 w_{24}, w_{25}, s_2) \mapsto \left(\zeta_3^2, \zeta_3^2, -1\right).$$

For $n = 29$ we have $C_W(w_{29}) = \langle w_{18}, w_{24}, w_{29}\rangle$. This class is regular. We have $d = 4$, $E(\zeta_4)$ is a regular, two-dimensional eigenspace, and $C_W(w_{29})$ acts on $E(\zeta_4)$ as the complex reflection group G_{22}. Define

$$\varphi_{29} \quad \text{by} \quad (w_{18}, w_{24}, w_{29}) \mapsto (1, 1, -1).$$

Then $\varphi_{29} = \det|_{E(\zeta_4)}$.

Finally, for $n = 34$ we have $w_{34} = w_0$ and we define $\varphi_{34} = \epsilon$.

The values of the characters φ_i^W together with ρ_W^4 and ω_W^4 are given in Table 6.

Appendix A. Bulky Parabolic Subgroups

For each finite irreducible Coxeter group W the following table lists the types of all bulky parabolic subgroups of W other than W itself, the trivial subgroup, and the subgroup of type A_1. This information has been extracted from the results in [14].

W	Bulky Parabolic Subgroups
A_n	$A_{n_1}A_{n_2}\cdots A_{n_k}$ with n_i distinct and $\sum_{i=1}^k n_i \le n+k-1$
B_n	B_j with $1 \le j \le n-1$, A_1B_j with $1 \le j \le n-2$
$D_n,\ n$ even	A_1D_{n-2}
$D_n,\ n$ odd	A_1D_{n-2}, A_1A_{n-3}, A_{n-1}
E_6	A_1A_2, A_1A_3, A_4, A_1A_4, A_5, D_5
E_7	D_6
E_8	E_7
F_4	$\widetilde{A}_1$, $A_1\widetilde{A}_1$, B_2, B_3, C_3
H_3	A_1^2
H_4	H_3
$I_2(m),\ m$ even	$\widetilde{A}_1$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}
φ_{11}	480	·	·	·	·	·	·	·	·	·	$-\nu$	·	·	·	$-\mu$	·	·
φ_{14}	720	·	·	·	·	·	·	·	·	·	·	·	·	-2ν	·	·	·
φ_{15}	480	·	·	·	·	·	·	·	·	·	$-\mu$	·	·	·	$-\nu$	·	·
φ_{17}	1200	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2
φ_{18}	24	·	2ν	·	·	·	·	·	·	2μ	μ	·	·	μ	ν	·	·
φ_{19}	288	·	-2	·	·	·	·	·	·	-2	·	·	·	·	·	·	·
φ_{21}	144	-12	$1-\mu$	·	·	$-\mu$	$-\nu$	·	·	$1-\nu$	·	·	$-\mu$	·	·	$-\nu$	·
φ_{22}	480	·	·	·	·	·	·	·	·	·	$-\nu$	·	·	·	$-\mu$	·	·
φ_{23}	720	·	·	·	·	·	·	·	·	·	·	·	·	-2μ	·	·	·
φ_{24}	40	·	·	·	-2	·	·	·	·	·	-1	·	·	·	-1	·	-1
φ_{25}	400	-20	·	·	1	·	·	1	·	·	·	1	·	·	·	·	·
φ_{26}	24	·	2μ	·	·	·	·	·	·	2ν	ν	·	·	ν	μ	·	·
φ_{27}	288	·	-2	·	·	·	·	·	·	-2	·	·	·	·	·	·	·
φ_{28}	480	·	·	·	·	·	·	·	·	·	$-\mu$	·	·	·	$-\nu$	·	·
φ_{29}	60	·	·	-4	·	·	·	·	·	·	·	·	·	-2	·	·	-2
φ_{30}	24	·	2μ	·	·	·	·	·	·	2ν	ν	·	·	ν	μ	·	·
φ_{31}	144	-12	$1-\nu$	·	·	$-\nu$	$-\mu$	·	·	$1-\mu$	·	·	$-\nu$	·	·	$-\mu$	·
φ_{32}	40	·	·	·	-2	·	·	·	·	·	-1	·	·	·	-1	·	-1
φ_{33}	24	·	2ν	·	·	·	·	·	·	2μ	μ	·	·	μ	ν	·	·
$\epsilon = \varphi_{34}$	1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	-1	1
ρ_W^4	6061	-45	-4	-3	-2	·	·	·	-1	-4	-1	·	·	-1	-1	·	-1
ω_W^4	6061	45	-4	-3	-2	·	·	·	1	-4	-1	·	·	-1	-1	·	-1

C_{18}	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}	C_{29}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	
20μ	·	·	·	$-\nu$	·	-12	·	20ν	·	$-\mu$	·	20ν	·	-12	20μ	480	φ_{11}
30μ	·	·	·	·	-2μ	·	·	30ν	·	·	-24	30ν	·	·	30μ	720	φ_{14}
20ν	·	·	·	$-\mu$	·	-12	·	20μ	·	$-\nu$	·	20μ	·	-12	20ν	480	φ_{15}
·	·	·	·	·	·	-30	·	·	·	·	-40	·	·	-30	·	1200	φ_{17}
$11-\mu$	-1	·	2μ	μ	ν	12	·	$11-\nu$	-1	ν	12	$11-\nu$	2ν	12	$11-\mu$	24	φ_{18}
-12	3	·	-2	·	·	·	·	-12	3	·	·	-12	-2	·	-12	288	φ_{19}
12μ	-1	-12	$1-\nu$	·	·	·	·	12ν	-1	·	·	12ν	$1-\mu$	·	12μ	144	φ_{21}
20μ	·	·	·	$-\nu$	·	-12	·	20ν	·	$-\mu$	·	20ν	·	-12	20μ	480	φ_{22}
30ν	·	·	·	·	-2ν	·	·	30μ	·	·	-24	30μ	·	·	30ν	720	φ_{23}
20	·	·	·	-1	·	19	-2	20	·	-1	20	20	·	19	20	40	φ_{24}
·	·	-20	·	·	·	-20	1	·	·	·	·	·	·	-20	·	400	φ_{25}
$11-\nu$	-1	·	2ν	ν	μ	12	·	$11-\mu$	-1	μ	12	$11-\mu$	2μ	12	$11-\nu$	24	φ_{26}
-12	3	·	-2	·	·	·	·	-12	3	·	·	-12	-2	·	-12	288	φ_{27}
20ν	·	·	·	$-\mu$	·	-12	·	20μ	·	$-\nu$	·	20μ	·	-12	20ν	480	φ_{28}
30	·	·	·	·	-2	30	·	30	·	·	28	30	·	30	30	60	φ_{29}
$11-\nu$	-1	·	2ν	ν	μ	12	·	$11-\mu$	-1	μ	12	$11-\mu$	2μ	12	$11-\nu$	24	φ_{30}
12ν	-1	-12	$1-\mu$	·	·	·	·	12μ	-1	·	·	12μ	$1-\nu$	·	12ν	144	φ_{31}
20	·	·	·	-1	·	19	-2	20	·	-1	20	20	·	19	20	40	φ_{32}
$11-\mu$	-1	·	2μ	μ	ν	12	·	$11-\nu$	-1	ν	12	$11-\nu$	2ν	12	$11-\mu$	24	φ_{33}
1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	φ_{34}
11	1	-45	-4	-1	-1	19	-2	11	1	-1	29	11	-4	19	11	6061	ρ_W^4
11	1	45	-4	-1	-1	19	-2	11	1	-1	29	11	-4	19	11	6061	ω_W^4

TABLE 6. Induced characters for $W(H_4)$: $\mu = \zeta_5 + \zeta_5^4$, $\nu = \zeta_5^2 + \zeta_5^3$

Acknowledgments: We acknowledge support from the DFG-priority program SPP1489 "Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory". The third author also wishes to thank Science Foundation Ireland for its support.

References

[1] F. Bergeron, N. Bergeron, R.B. Howlett, and D.E. Taylor. A decomposition of the descent algebra of a finite Coxeter group. *J. Algebraic Combin.*, 1(1):23–44, 1992.

[2] J. Blair and G.I. Lehrer. Cohomology actions and centralisers in unitary reflection groups. *Proc. London Math. Soc. (3)*, 83(3):582–604, 2001.

[3] A. Borel and J. De Siebenthal. Les sous-groupes fermés de rang maximum des groupes de Lie clos. *Comment. Math. Helv.*, 23:200–221, 1949.

[4] N. Bourbaki. *Éléments de mathématique. Groupes et algèbres de Lie. Chapitre IV, V, VI.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[5] R.W. Carter. Conjugacy classes in the Weyl group. *Compositio Math.*, 25:1–59, 1972.

[6] Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. I.* John Wiley & Sons Inc., New York, 1981.

[7] J.M. Douglass. On the cohomology of an arrangement of type B_l. *J. Algebra*, 146:265–282, 1992.

[8] J.M. Douglass, G. Pfeiffer, and G. Röhrle. Coxeter arrangements and Solomon's descent algebra. Preprint, arXiv:1101.2075.

[9] J.M. Douglass, G. Pfeiffer, and G. Röhrle. On reflection subgroups of finite Coxeter groups. Preprint, arXiv:1101.5893.

[10] J.M. Douglass, G. Pfeiffer, and G. Röhrle. An inductive approach to Coxeter arrangements and Solomon's descent algebra. *J. Algebraic Combin.*, 2011. doi:10.1007/s10801-011-0301-9, arXiv:1104.0551.

[11] G. Felder and A. P. Veselov. Coxeter group actions on the complement of hyperplanes and special involutions. *J. Eur. Math. Soc. (JEMS)*, 7(1):101–116, 2005.

[12] M. Geck, G. Hiß, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE — A system for computing and processing generic character tables. *Appl. Algebra Engrg. Comm. Comput.*, 7:175–210, 1996.

[13] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, volume 21 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 2000.

[14] R. B. Howlett. Normalizers of parabolic subgroups of reflection groups. *J. London Math. Soc. (2)*, 21(1):62–80, 1980.

[15] M. Konvalinka, G. Pfeiffer, and C. Röver. A note on element centralizers in finite Coxeter groups. *J. Group Theory*, 14(5):727–745, 2011.

[16] G.I. Lehrer and L. Solomon. On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes. *J. Algebra*, 104(2):410–424, 1986.

[17] P. Orlik and H. Terao. *Arrangements of Hyperplanes.* Springer-Verlag, 1992.

[18] G. Pfeiffer. ZigZag — A GAP3 Package for Descent Algebras of Finite Coxeter Groups. 2007. Electronically available at `http://schmidt.nuigalway.ie/zigzag`.

[19] G. Pfeiffer and G. Röhrle. Special involutions and bulky parabolic subgroups in finite Coxeter groups. *J. Aust. Math. Soc.*, 79(1):141–147, 2005.

[20] M. Schönert et al. GAP – *Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4.* Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1997.

[21] T. A. Springer. Regular elements of finite reflection groups. *Invent. Math.*, 25:159–198, 1974.

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany

E-mail address: marcus.bishop@rub.de

Department of Mathematics, University of North Texas, Denton TX, USA 76203

E-mail address: douglass@unt.edu

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, University Road, Galway, Ireland

E-mail address: goetz.pfeiffer@nuigalway.ie

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany

E-mail address: gerhard.roehrle@rub.de