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<th>Computations for Coxeter arrangements and Solomon's descent algebra: Groups of rank three and four</th>
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Abstract. In recent papers we have refined a conjecture of Lehrer and Solomon expressing the character of the representation of a finite Coxeter group \( W \) on the \( p \)th graded piece of its Orlik-Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of \( W \). Our refined conjecture relates the character of \( W \) on the \( p \)th graded piece of its Orlik-Solomon algebra with the descent algebra of \( W \). A consequence of our conjecture is that both the regular character of \( W \) and the character of \( W \) acting on its Orlik-Solomon algebra have parallel, graded decompositions as sums of characters induced from linear characters of centralizers of elements of \( W \), one for each conjugacy class of elements of \( W \).

The refined conjectures have been proved for symmetric and dihedral groups. In this paper we develop algorithmic tools to prove the conjectures computationally for a given \( W \) and we use these tools to verify the claim for all finite Coxeter groups of rank three and four. The techniques developed and implemented in this paper provide previously unknown decompositions of the regular characters and the Orlik-Solomon characters of the Coxeter groups of types \( B_3 \), \( H_3 \), \( B_4 \), \( D_4 \), \( F_4 \), and \( H_4 \) as sums of induced representations indexed by the set of conjugacy classes of \( W \).

1. Introduction

Let \( W \) be a finite Coxeter group and let \( V \) be a finite dimensional, complex vector space affording a faithful representation of \( W \) such that each element in a Coxeter generating set \( S \) of \( W \) acts on \( V \) as a reflection. Let \( M \) be the complement in \( V \) of the union of the fixed-point hyperplanes of the reflections in \( W \). Then \( M \) is a \( W \)-stable, open subset of \( V \) and the action of \( W \) on \( M \) determines a representation of \( W \) on \( H^p(M) \), the \( p \)th singular cohomology group of \( M \). Let \( \omega_p^W \) denote the character of the representation of \( W \) on \( H^p(M) \) and let \( \omega_W = \sum_{p \geq 0} \omega_p^W \) denote the character of the representation of \( W \) on the cohomology ring \( H^\bullet(M) = \bigoplus_{p \geq 0} H^p(M) \). The character \( \omega_W \) has been computed by Lehrer and others \([2], [11]\).

Lehrer and Solomon \([16]\) conjectured that \( \omega_p^W \) is a sum of characters induced from linear characters of centralizers of elements of \( W \). Conjecture 2.1 in \([8]\) is a more precise version of the Lehrer-Solomon conjecture that in addition to describing \( \omega_p^W \) as a sum of induced characters, also relates the decomposition \( \omega_W = \sum_{p \geq 0} \omega_p^W \) to a decomposition of the regular character \( \rho_W \) of \( W \) arising from the complete set of primitive orthogonal idempotents of the descent algebra of \( W \) found by Bergeron, Bergeron, Howlett, and Taylor in \([1]\). The main result in \([8]\) is a proof of Conjecture 2.1 for symmetric groups.

In \([10]\) an inductive approach that would lead to a proof of Conjecture 2.1 in \([8]\) was developed. The inductive approach parses Conjecture 2.1 into components known as
Conjectures B and C, which we now describe. If \( n = |S| \), then \( H^n(M) \) is the highest degree non-vanishing cohomology group. For a subset \( L \) of \( S \) we denote the parabolic subgroup \( \langle L \rangle \) of \( W \) by \( W_L \). A conjugacy class \( C \) in \( W \) is said to be cuspidal if \( C \cap W_L = \emptyset \) for every proper subset \( L \) of \( S \). Conjecture B describes the character \( \omega^W_W \) of \( W \) as a sum of characters induced from centralizers of cuspidal conjugacy classes. Furthermore, Conjecture B also relates \( \omega^W_W \) to an appropriate summand in the decomposition of the regular character above, namely the character whose degree is the cardinality of the set of cuspidal elements in \( W \).

Conjecture C is a relative version of Conjecture B for the pair \((W, W_L)\), where the parabolic subgroup \( W_L \) is fixed and the overgroup \( W \) varies. It mirrors Conjecture B for the group \( W_L \), but in place of the character \( \omega^W_L \) it has an extension of \( \omega^W_L \) to the normalizer of \( W_L \) in \( W \), and in place of the characters of centralizers of cuspidal elements in \( W_L \) are the centralizers of the same elements in \( W \). It is shown in [10] that if the parabolic subgroups \( W_L \) satisfy Conjecture C for all \( L \subseteq S \), then Conjecture 2.1 holds for \( W \). Finally, as an application of the method, both conjectures were proved for dihedral groups in [10].

In this paper, we develop algorithms to prove Conjecture B in [10] and consequently Conjecture 2.1 in [8] for a given finite Coxeter group. We have implemented these algorithms using the GAP programming system [20] with the CHEVIE [12] and ZigZag [18] packages. We present the results of our computations for \( W \) of type \( B_3, H_3, B_4, D_4, F_4, \) and \( H_4 \), thus verifying the conjectures for all irreducible Coxeter groups of rank three or four. As a consequence of our computations, we can state Conjecture 2.1 of [8] for groups of rank at most four as the following theorem.

**Theorem 1.1.** Suppose that \( W \) is a finite Coxeter group with rank at most four and that \( R \) is a set of conjugacy class representatives of \( W \). Then for each \( w \in R \) there exists a linear character \( \varphi_w \) of \( C_W(w) \) such that if \( \rho_W \) is the regular character of \( W \), \( \epsilon \) is the sign character of \( W \), and \( \alpha_w \) is the composition of \( \det \) with restriction to the 1-eigenspace of \( w \), then

\[
\rho_W = \sum_{w \in R} \text{Ind}_{C_W(w)}^W \varphi_w \quad \text{and} \quad \omega_W = \epsilon \sum_{w \in R} \text{Ind}_{C_W(w)}^W (\alpha_w \varphi_w).
\]

Moreover, if \( R_p \) is the set of \( w \) in \( R \) such that the codimension in \( V \) of the 1-eigenspace of \( w \) is \( p \), then

\[
\omega_W^p = \epsilon \sum_{w \in R_p} \text{Ind}_{C_W(w)}^W (\alpha_w \varphi_w).
\]

Our current methods are sufficient to treat somewhat larger groups, but are computationally too expensive to be able to handle the largest exceptional Coxeter groups. In future work we hope to develop additional computational techniques to be able to efficiently verify the conjectures for groups with rank up to eight.

The rest of this paper is organized as follows. In §2 we review the constructions from [8] and [10] and show how our computations lead to a proof of Theorem 1.1. In §3 we describe the algorithms we have used and their implementation in GAP. Finally, in §4 we present the results of our computations for rank three and four Coxeter groups. In the appendix we give a table listing all so-called bulky parabolic subgroups of all finite irreducible Coxeter groups.
2. Preliminaries

2.1. Coxeter Groups and the Orlik-Solomon Algebra. In this subsection we briefly review the constructions in [8] and [10], state in Theorem 2.3 the main result verified by our computations, and show how Theorem 2.3 leads to a proof of Theorem 1.1.

Recall that an element of $W$ is called cuspidal if none of its conjugates lies in a proper parabolic subgroup of $W$. A conjugacy class is called cuspidal if its elements are all cuspidal. It follows from the fact that the proper parabolic subgroups of $W$ arise as pointwise stabilizers of proper subspaces of $V$ that an element is cuspidal if and only if its 1-eigenspace has codimension $|S|$ in $V$. It is shown in [13] that to the natural action of $W$, the conjugacy classes in $W$ are parameterized by pairs $(W_1, C_1)$, where $W_1$ is a parabolic subgroup of $W$ and $C_1$ is a cuspidal conjugacy class in $W_1$.

Let $T = \{ w^{-1}sw \mid s \in S, w \in W \}$ be the set of reflections in $W$. For $t$ in $T$ let $H_t$ be the hyperplane in $V$ fixed by $t$. Let $E$ be a $\mathbb{C}$-vector space with basis $\{ e_t \mid t \in T \}$. The Orlik-Solomon algebra $A(W)$ is the quotient of the exterior algebra of $E$ by the ideal generated by elements of the form

$$
\sum_{i=1}^{m} (-1)^i e_{t_1} e_{t_2} \cdots e_{t_i} \cdot e_{t_m}
$$

for every set $\{ H_{t_1}, H_{t_2}, \ldots, H_{t_m} \}$ of linearly dependent hyperplanes. The group $W$ acts on the exterior algebra by $se_t = e_{sts}$ for $s \in S$ and $t \in T$. The ideal generated by elements of the form (2.1) is homogeneous and $W$-stable, and so $A(W) = \bigoplus_{p \geq 0} A^p(M)$ is a graded, skew-commutative $\mathbb{C}$-algebra on which $W$ acts as algebra automorphisms. We denote the image of the generator $e_t$ in $A(W)$ by $a_t$.

It is known that $A(W)$ is isomorphic to the cohomology ring $H^*(M)$ as graded $W$-algebras (see [17, Chapter 3]). It is clear from the definition of $A(W)$ that $A^n(W)$ is the highest degree non-zero component. We refer to $A^n(W)$ as the top component of $A(W)$. Then the character of the top component is $\omega^n_W$. It is shown in [7] that the degree of $\omega^n_W$ is the cardinality of the set of cuspidal elements in $W$.

For a subset $J$ of $S$, let $X_J$ denote the set of minimal length right coset representatives of $W_J$ in $W$ and set $x_J = \sum_{w \in X_J} w$ in the group algebra $\mathbb{C}W$. Solomon has shown that the set $\{ x_J \mid J \subseteq S \}$ is linearly independent and spans a subalgebra of $\mathbb{C}W$ called the descent algebra of $W$ (see [1]).

Bergeron, Bergeron, Howlett, and Taylor [1, §7] define a basis of the descent algebra consisting of quasi-idempotents as follows. For subsets $J$ and $K$ of $S$ define

$$m_{KJ} = | \{ x \in X_J \mid x^{-1}Jx \subseteq K \} | \quad \text{if} \quad J \subseteq K \quad \text{and} \quad m_{KJ} = 0 \quad \text{if} \quad J \nsubseteq K.
$$

Note that $m_{KK} > 0$, since $1_W \in X_K$ for all $K \subseteq S$. Then the $2^n \times 2^n$ matrix with rows and columns indexed by the power set of $S$ and with $(K, J)$-entry $m_{KJ}$ is invertible. Define $n_{KJ}$ to be the $(K, J)$-entry of the inverse matrix and define $e_K = \sum_J n_{KJ} x_J$. Then $e_K e_K = \gamma_K e_K$, where $\gamma_K = | \{ L \subseteq S \mid \exists w \in W, w^{-1}Lw = K \} |$ and so each $e_K$ is a quasi-idempotent in $\mathbb{C}W$. In particular, $e_S = \sum_J n_{SJ} x_J$ is an idempotent. In analogy with $A^n(W)$ we call $\mathbb{C}W e_S$ the top component of $\mathbb{C}W$ and denote the character it affords by $\rho^n_W$. It is shown in [1] that the degree of $\rho^n_W$ is the cardinality of the set of cuspidal elements in $W$. 

Remark 2.2. If $W = W_1 \times W_2$ is reducible, then an element $(w_1, w_2)$ in $W_1 \times W_2$ is cuspidal if and only if $w_1$ is cuspidal in $W_1$ and $w_2$ is cuspidal in $W_2$. It is straightforward to show that the idempotent generating the top component of $CW$ is the product of the idempotents generating the top components of $CW_1$ and $CW_2$. Therefore, the top component of $CW$ is isomorphic to the tensor product of the top components of $CW_1$ and $CW_2$. Similarly, the top component of $A(W)$ is isomorphic to the tensor product of the top components of $A(W_1)$ and $A(W_2)$, by the Künneth theorem.

The content of the next theorem is Conjecture B from [10] for groups with rank at most four.

Theorem 2.3. Suppose that $W$ is a finite Coxeter group with rank $n \leq 4$ and that $C$ is a set of representatives of the cuspidal conjugacy classes of $W$. Then for each $w \in C$ there exists a linear character $\varphi_w$ of $CW(w)$ such that

$$\rho_W^n = \sum_{w \in C} \text{Ind}_{CW(w)}^W \varphi_w = \epsilon \omega_W^n.$$ 

This theorem has been proved with no restriction on the rank of $W$ for symmetric groups in [8] and dihedral groups in [10]. In this paper we prove the theorem for the remaining finite Coxeter groups of rank three or four in §4 by explicitly computing the linear characters $\varphi_w$. A description of the GAP programs used in this calculation is given in §3.

Observe that if $W = W_1 \times W_2$ is reducible, then since induction commutes with tensor products, Remark 2.2 implies that Theorem 2.3 holds for $W$ if and only if it holds for both $W_1$ and $W_2$ where the characters $\varphi_w$ satisfying Theorem 2.3 for $W$ are the tensor products of those satisfying the theorem for $W_1$ and $W_2$. Thus it suffices to consider the case when $W$ is irreducible.

To prove Theorem 1.1 we require a linear character of the centralizer of a representative of every conjugacy class of $W$. For cuspidal classes, we can use the characters satisfying Theorem 2.3. For non-cuspidal conjugacy classes we use a relative version of Theorem 2.3 that takes into account the embedding of a parabolic subgroup of $W$ in its normalizer in $W$, as follows.

Let $L$ be a subset of $S$ of size $r$. Then $WL$ acts on the top components of $A(W_L)$ and $CW_L$ and we denote the characters of $WL$ afforded by these spaces by $\omega_L$ and $\rho_L$ rather than by $\omega_{WL}$ and $\rho_{WL}$ to simplify notation. Suppose that Theorem 2.3 holds for $WL$. This means that for each $w$ in a set $C_L$ of representatives of the cuspidal conjugacy classes of $WL$ we have a linear character $\varphi_w$ of $CW_L(w)$ such that

$$\rho_L^r = \sum_{w \in C_L} \text{Ind}_{CW_L(w)}^{WL} \varphi_w = \epsilon \omega_L^r,$$

where $\epsilon$ is the sign character of $W$. Observe that if $w$ is a cuspidal element in $WL$, then $CW(w)$ is contained in $NW(W_L)$ and that the quotients $CW(w)/CW_L(w)$ and $NW(W_L)/WL$ are isomorphic, by the main result of [15]. It is shown in [8] that the characters $\rho_L^r$ and $\omega_L^r$ of $WL$ extend to characters $\tilde{\rho}_L^r$ and $\tilde{\omega}_L^r$ of $NW(W_L)$. Then Conjecture C of [10] asserts that there is a corresponding extension of the characters $\varphi_w$. We state this as the following theorem, which proves Conjecture C for $W$ of rank at most four.
Theorem 2.4. Suppose that $W$ is a finite Coxeter group of rank at most four, that $L$ is a proper subset of $S$ of size $r$, and that $C_L$ is a set of representatives of the cuspidal conjugacy classes of $W_L$. Then for each $w \in C_L$ the linear character $\varphi_w$ of $C_{W_L}(w)$ in Theorem 2.3 extends to a linear character $\tilde{\varphi}_w$ of $C_{W}(w)$ such that

$$\tilde{\rho}_L = \sum_{w \in C_L} \text{Ind}_{C_{W_L}(w)}^{N_{W}(W_L)} \tilde{\varphi}_w = \epsilon \alpha_L \omega_L,$$

where $\alpha_L$ is the composition of $\det$ with restriction to the subspace of fixed points of $W_L$.

The characters and subgroups in the theorem are summarized in the following diagram.

\[
\begin{array}{ccc}
N_{W}(W_L), \quad \tilde{\varphi}_w & \\ \downarrow & \downarrow & \downarrow \\
C_{W}(w), \quad \tilde{\varphi}_w & W_L, \quad \varphi_{W_L} & C_{W_L}(w), \quad \varphi_w \\
\end{array}
\]

Proof. If $W = W_1 \times W_2$ is reducible, then, by Remark 2.2, the theorem holds for $(W, W_L)$ if and only if it holds for each of $(W_1, W_{L_1})$ and $(W_2, W_{L_2})$, where $L_1, L_2 \subseteq S$ are such that $W_L = W_{L_1} \times W_{L_2}$. Thus, we may assume that $W$ is irreducible.

Recall that $W_L$ is said to be bulky in $W$ if it has a normal complement in $N_{W}(W_L)$ (see [19]). Theorem 2.4 is proved for any $W$ in [10] if $|L| \leq 2$ or if $W_L$ is bulky. It follows that the theorem holds if $W$ has rank three, so we may assume that $W$ has rank four and that $W_L$ is non-bulky of maximal rank. The bulky parabolic subgroups of all finite irreducible Coxeter groups are listed in Appendix A. The eight pairs $(W, W_L)$ where $W$ has rank four and $W_L$ is non-bulky of maximal rank are listed in the following table.

<table>
<thead>
<tr>
<th>$W$</th>
<th>$W_L$</th>
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<tbody>
<tr>
<td>$B_4$</td>
<td>$A_1 A_2$ $A_3$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$A_3$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$A_1 \tilde{A}_2$ $A_2 \tilde{A}_1$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$A_3$ $A_1 A_2$ $I_2(5)A_1$</td>
</tr>
</tbody>
</table>

In this table $\tilde{A}_1$ and $\tilde{A}_2$ denote subgroups of types $A_1$ and $A_2$ generated by reflections orthogonal to short roots.

With no restriction on the rank of $W$, it is shown in [8, §7] that Theorem 2.4 holds in case all factors of $W_L$ are of type $A$. This only leaves the pair $(H_4, I_2(5)A_1)$ to be considered. It is straightforward to verify that Theorem 2.4 holds in this case.

It is shown in [10, Theorem 4.7] that the characters $\varphi_w$ satisfying Theorem 1.1 can be taken to be the union over $L \subseteq S$ of the sets of characters $\tilde{\varphi}_w$ satisfying Theorem 2.4 for $W_L$. Therefore, Theorem 1.1 holds for $W$. 
3. Implementation

The proof of Theorem 2.3 consists of exhibiting the characters $\varphi_w$ satisfying the theorem for each irreducible Coxeter group of rank three or four. These characters are presented in §4. In this section we describe how the characters $\varphi_w$ and the top component characters $\rho^n_W$ and $\omega^n_W$ were calculated.

The calculations were performed using the computer algebra system GAP [20]. As one would expect in a computer algebra system, the user can introduce a specific group and use the system’s built-in commands to calculate information about the group. In this project, we are primarily concerned with conjugacy classes and their representatives, centralizers and normalizers, character tables, linear algebra, and class function manipulation, namely class function sums, the scalar product of class functions, and induced class functions.

The CHEVIE package for GAP [12] provides additional functionality for manipulating a finite Coxeter group $W$ in ways specific to such groups. For example, the package provides the length function for $W$ and a mechanism for expressing elements of $W$ as products of Coxeter generators. It also provides the reflection representation of $W$ through the matrices giving the action of the Coxeter generators on a vector space.

The matrix representation is useful for identifying the cuspidal conjugacy classes of $W$. Recall that an element $w$ is cuspidal if none of its eigenvalues equals 1. Thus, to determine whether $w$ is cuspidal, we only need to inspect the eigenvalues of the matrix representing $w$. The cuspidal classes can also be determined using the CuspidalClasses function supplied by the ZigZag package [18].

3.1. The top component character of $CW$. Let $C_1, C_2, \ldots, C_m$ be the conjugacy classes of $W$ and let $x_i \in C_i$ for $1 \leq i \leq m$. We denote the descent set $\{ s \in S \mid \ell(sw) < \ell(w) \}$ of $w \in W$ by $D(w)$. With this notation $X_J = \{ w \in W \mid D(w) \subseteq S \setminus J \}$.

By Exercise 16 of [6, §9], we have

$$\rho^n_W(x_i^{-1}) = |C_W(x_i)| \sum_{w \in C_i} a_w, \quad \text{where} \quad e_S = \sum_{w \in W} a_w w.$$ 

But since $e_S = \sum_{J \subseteq S} n_{S,J} x_J$, we have $a_w = \sum_{D(w) \subseteq S \setminus J} n_{S,J}$ so that

$$\rho^n_W(x_i^{-1}) = |C_W(x_i)| \sum_{J \subseteq S} n_{S,J} \{ x \in C_i \mid D(x) \subseteq S \setminus J \}.$$ 

This calculation shows that $\rho^n_W$ can be computed as a product of matrices. Namely, let

$$I = (i_{JK}), \quad \text{where} \quad i_{JK} = \begin{cases} 1 & \text{if } J \subseteq K \\ 0 & \text{otherwise} \end{cases}$$

and let

$$D = (d_{ji}), \quad \text{where} \quad d_{ji} = |\{ x \in C_i \mid D(x) = S \setminus J \}|.$$ 

Then $I$ is the incidence matrix of the power set of $S$ and $D$ expresses the distribution of the conjugacy classes into descent classes. Then the $(J,i)$-entry of $ID$ is $|\{ x \in C_i \mid D(x) \subseteq S \setminus J \}|$. Thus multiplying $ID$ on the left by row $S$ of $N$ and on the right by the
diagonal matrix with entries $|C_W(x_i)|$ results in $1 \times m$ matrix whose entries are $\rho_W^i(x_i^{-1})$ for $1 \leq i \leq m$. The ZigZag package provides an implementation of the above calculation through the `Characters` function which returns the characters of $CW_{e_L}$ for $L \subseteq S$ as a list. The last entry in this list is $\rho_W^i$.

### 3.2. Choosing linear characters of the centralizers.

Let $\{w_1, w_2, \ldots, w_r\}$ be a list of representatives of the cuspidal classes of $W$. The selection of the characters $\varphi_{w_i}$ satisfying Theorem 2.3 can in principle be accomplished by testing whether $\rho_W^i = \sum_{j=1}^r \varphi_j^W$ for each tuple $(\varphi_1, \varphi_2, \ldots, \varphi_r)$, where $\varphi_i$ is a linear character of $C_W(w_i)$ for all $1 \leq i \leq r$. While easy to automate, this method is expensive because of the large number of tuples of linear characters to be tested. Therefore, we have adopted a binary search algorithm that we now describe.

Let $\chi_1, \chi_2, \ldots, \chi_m$ be the irreducible characters of $W$. If $\chi$ is any character of $W$, then taking the scalar product of $\chi$ with each of the characters $\chi_i$ we obtain an $m$-tuple of non-negative integers. We call this tuple the *constituency tuple* of $\chi$. We endow that set of constituency tuples with the product partial order, so $(t_1, t_2, \ldots, t_m) \leq (u_1, u_2, \ldots, u_m)$ if and only if $t_i \leq u_i$ for all $1 \leq i \leq m$.

Considering constituency tuples rather than characters converts the problem of selecting a linear character $\varphi_i$ of each $C_W(w_i)$ such that $\rho_W^i = \sum_{j=1}^r \varphi_j^W$ to the problem of selecting a tuple from each of $r$ sets of tuples such that the sum of the selected tuples equals a given tuple.

We select the tuples $\{t^{(i)} \mid 1 \leq i \leq r\}$ in $r$ stages, where each $t^{(i)}$ is the constituency tuple of a linear character of $C_W(w_i)$. At stage $i$ we attempt to select a tuple $t^{(i)}$ for $C_W(w_i)$ such that $\sum_{j=1}^r t^{(j)} \leq \text{the constituency tuple of } \rho_W^i$. Having found such a tuple we proceed to stage $i + 1$. On the other hand, if no tuple for $C_W(w_i)$ satisfies this condition, then we return to stage $i - 1$ and select a different tuple for $C_W(w_{i-1})$.

### 3.3. The top component character of $A(W)$.

We calculate the character $\omega_W^p$, by explicitly calculating the representation of $W$ on the top component of $A(W)$. This calculation is straightforward once we have a basis of the top component and a method for writing an arbitrary product $a_{t_1}a_{t_2} \cdots a_{t_p}$ in $A(W)$ as a linear combinations of basis elements. The solution to both problems is provided by the non-broken circuit basis that we now briefly describe. See §3.1 of [17] for more information.

Let $H$ be a sequence $H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ of hyperplanes in $A$. We call $H$ a *circuit* if $H$ is dependent, but $H_{t_1}, \ldots, H_{t_j}, \ldots, H_{t_p}$ is independent for each $1 \leq j \leq p$. Note that GAP can easily test whether a tuple of vectors is linearly independent using linear algebra functions such as `Rank`. Thus, once a set of linear functionals defining the hyperplanes $H_t$ for $t$ in $T$ has been fixed, it is possible to test whether $H$ is a circuit.

Now fix a total order on the set of reflections $T$ and suppose that $H$ is a sequence $H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ with $t_1 < t_2 < \cdots < t_p$. We call $H$ a *broken circuit* if $H_{t_1}, H_{t_2}, \ldots, H_{t_p}$, $H_t$ is a circuit for some hyperplane $H_t$ with $t > t_p$, and we call $H$ a *non-broken circuit* if no subsequence of $H$ is a broken circuit. Notice that the empty sequence is a non-broken circuit. Then

$$
(3.1) \quad B = \{a_{t_1}a_{t_2} \cdots a_{t_p} \mid H_{t_1}, H_{t_2}, \ldots, H_{t_p} \text{ is a non-broken circuit}\}$$
is a basis of $A(W)$ by Theorem 3.43 of [17]. Clearly, the broken circuits, and hence the basis $B$, depend on the chosen total order on $T$. By construction, $B \cap A^p(W)$ is a basis of $A^p(W)$ for $1 \leq p \leq n$ and so $B \cap A^n(W)$ is a basis of the top component of $A(W)$.

**Remark 3.2.** If $H$ is dependent, then any minimal dependent subsequence of $H$ is a circuit. Removing the last term of such a subsequence results in a broken circuit. This also shows that a non-broken circuit is independent.

We calculate both the broken and the non-broken circuits recursively as follows. Throughout the following procedure we maintain a list $L$ of sequences which remain to be considered. Initially $L$ contains only the empty sequence. Throughout, $L$ has the following properties as a consequence of the way sequences are added to $L$.

(1) Each $H = H_{t_1}, H_{t_2}, \ldots, H_{t_p} \in L$ satisfies $t_1 < t_2 < \cdots < t_p$.

(2) Each $H \in L$ is independent.

(3) For each $p \geq 0$ the sequences of length $p$ occur in $L$ before those of length $p + 1$.

The algorithm consists only of the following loop. While $L$ is not empty we remove the first element $H = H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ from $L$. For each $t > t_p$ let $H_l$ be the sequence obtained from $H$ by appending $H_l$. The following possibilities arise.

(a) Suppose that one of the sequences $H, H_l$ is dependent. Then $H, H_l$ contains a circuit, by Remark 3.2. Note that at this point all the broken circuits of length less than $p$ have been discovered by (3) and the following sentence. If none of the broken circuits identified so far is a subsequence of $H$, then the entire sequence $H, H_l$ is a circuit and $H$ is a broken circuit.

(b) Suppose that $H, H_l$ is independent for all $t > t_p$. If $H$ contained a broken circuit $H'$ as a subsequence, then $H'$ would contain $H_{t_p}$, since otherwise $H$ would not have been added to $L$. But then we would be in case (a) above, since $H, H_l$ would be dependent, where $H_l$ is the hyperplane which completes $H'$ to a circuit. Therefore, $H$ is a non-broken circuit. We add each of the new sequences $H, H_l$ to $L$.

Note that if $H = H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ is any non-broken circuit, then for all $q < p$ and all $t > t_q$ the sequence $H_{t_1}, H_{t_2}, \ldots, H_{t_q}, H_l$ is independent, since otherwise $H$ would contain a broken circuit. Therefore, each of the subsequences $H_{t_1}, H_{t_2}, \ldots, H_{t_q}$ with $q < p$ is added to $L$ in the procedure above, so the procedure eventually discovers that $H$ is a non-broken circuit.

For the purpose of expressing elements of $A(W)$ in terms of $B$, it suffices to identify only the minimal broken circuits, and in fact, this is precisely what the procedure above does. Then by the argument above for non-broken circuits, the procedure also discovers all the minimal broken circuits.

To express an element $a = a_{t_1}a_{t_2} \cdots a_{t_p}$ of $A(W)$ in terms of $B$ we proceed inductively. If $H = H_{t_1}, H_{t_2}, \ldots, H_{t_p}$ is a non-broken circuit, then $a \in B$. Otherwise $H$ has a subsequence $H' = H_{t_{p_1}}, H_{t_{p_2}}, \ldots, H_{t_{p_q}}$ which is a broken circuit. This means that there is a hyperplane $H_l$ with $t > t_{p_q}$ for which $H, H_l$ is a circuit. Observe that it reduces computation to record $t$ in the procedure above when we originally discovered that $H'$ was a broken circuit, as doing so obviates having to search for such a hyperplane at this point. If $H_l$ happens
to be in $H$, then $H$ is dependent so that $a = 0$. Otherwise we use (2.1) to express $a$ in terms of elements corresponding with sequences containing fewer broken circuits as subsequences. Namely, we write
\[
(-1)^q a_{t_{j_1}} \cdots a_{t_{j_q}} = \sum_{k=1}^{q} (-1)^k a_{t_{j_1}} \cdots a_{t_{j_k}} a_t
\]
and multiply both sides by the remaining factors of $a$ resulting in $\pm a$ on the left side and elements of $A(W)$ on the right side which can be expressed in terms of $B$ by induction.

4. Proof of Theorem 2.3

In this section we present the results of our computations for the Coxeter groups of types $B_3$, $H_3$, $B_4$, $D_4$, and $H_4$, thus verifying Theorem 2.3 for irreducible Coxeter groups of rank three and four. For each group we give the following information.

1. For $w$ running through a set of representatives of the cuspidal conjugacy classes of $W$, we derive a generating set for $C_W(w)$ and describe the characters $\varphi_w$ of $C_W(w)$ by giving its values on the generating set. Note that in every case $w_0$ is central and that the character $\varphi_{w_0}$ is always the sign character.

2. We give a table containing the values of the characters $\rho^n_w$ and $\omega^n_w$ as well as $\text{Ind}_{C_W(w)}^W \varphi_w$ for each representative $w$. In all cases we see that
\[
\rho^n_w = \sum_{w \in \mathcal{R}} \text{Ind}_{C_W(w)}^W \varphi_w = \epsilon \omega^n_w
\]
as asserted in Theorem 2.3. In these tables, the rows are indexed by the characters $\text{Ind}_{C_W(w)}^W \varphi_w$ (denoted simply by $\varphi_w$), $\rho^n_w$, and $\omega^n_w$, and the columns are indexed by the conjugacy classes of $W$.

3. For $w$ in $W$ and $\zeta$ an eigenvalue of $w$ on $V$, let $E(\zeta)$ denote the $\zeta$-eigenspace of $w$. Then $C_W(w)$ acts on $E(\zeta)$ and $y \mapsto \det(y|_{E(\zeta)})^p$ defines a linear character of $C_W(w)$ for each natural number $p$. Denote this character of $C_W(w)$ by $(\det|_{E(\zeta)})^p$. The characters $\varphi_w$ do not arise from this construction in general. However, if $w$ is a regular element in $W$ and $E(\zeta)$ is a regular eigenspace of $w$, that is, such that $E(\zeta) \not\subseteq H_t$ for all $t \in T$, then with one exception, the character $\varphi_w$ is equal to $(\det|_{E(\zeta)})^p$ for some $p > 0$. The exception is the class labeled by the partition 22 in type $B_4$ (see §4.1.2). When $w$ is regular we use Springer’s theory of regular elements (see [21]) to identify the complex reflection group given by the action of $C_W(w)$ on a regular eigenspace $E(\zeta)$ and we compare the character $\varphi_w$ with $\det|_{E(\zeta)}$ when possible.

A conjugacy class in $W$ is called regular if it contains a regular element.

In all the groups we consider below, the longest element $w_0$ is central and the character $\varphi_{w_0}$ is the sign character of $W$. Thus $w_0$ is regular, and $\varphi_{w_0} = \det|_{E(-1)}$. The Coxeter class is well-known to be a regular class. If $w$ is a Coxeter element, then it acts on its eigenspace $E(\zeta)$ as a cyclic group of order $|w|$, where $\zeta$ is a primitive $|w|^{th}$ root of unity. It turns out to always be the case that $\varphi_w = (\det|_{E(\zeta)})^p$, but it can happen that $p \neq 1$. 
We use the following notation. The cyclic group of size \( n \) is denoted by \( Z_n \) and the symmetric group on \( n \) letters is denoted by \( S_n \). For \( n \geq 1 \) we denote the primitive complex \( n^{th} \) root of unity \( e^{2\pi i/n} \) by \( \zeta_n \). As in the proof of Theorem 2.4, the labels \( \tilde{A}_1 \) and \( \tilde{A}_2 \) denote subgroups of types \( A_1 \) and \( A_2 \) generated by reflections orthogonal to short roots. The same convention applies to \( \tilde{D}_4 \) in \( W(F_4) \). We denote partitions as strings of numbers without commas written in non-decreasing order.

**Remark 4.1.** Notice that when \( w_0 \) is central in \( W \), multiplication by \( w_0 \) permutes the conjugacy classes of \( W \) and \( C_W(ww_0) = C_W(w) \) for all \( w \in W \).

### 4.1. \( W \) of type \( B \)

Suppose that \( V \) has basis \( \{v_1, \ldots, v_n\} \), where \( n \geq 2 \). We view \( W = W(B_n) \) as acting on \( V \) by signed permutations of \( \{v_1, v_2, \ldots, v_n\} \). Namely, the Coxeter generators \( s_1, s_2, \ldots, s_n \) are given by

\[
\begin{align*}
    s_1(v_k) &= \begin{cases} 
    -v_1, & k = 1 \\
    v_k, & k \neq 1 
    \end{cases} \\
    s_i(v_k) &= \begin{cases} 
    v_i, & k = i - 1 \\
    v_{i-1}, & k = i \\
    v_k, & k \neq i - 1, i 
    \end{cases} 
\end{align*}
\]

and the Dynkin diagram of \( W(B_n) \) is \( \overset{1}{\bullet} \overset{2}{\bullet} \cdots \overset{n-1}{\bullet} \overset{n}{\bullet} \) as in [4] and in CHEVIE. For \( 1 \leq i < j \leq n \) we define elements \( t_i \) and \( s_{i,j} \) by

\[
\begin{align*}
    t_i(v_k) &= \begin{cases} 
    -v_i, & k = i \\
    v_k, & k \neq i 
    \end{cases} \\
    s_{i,j}(v_k) &= \begin{cases} 
    v_j, & k = i \\
    v_i, & k = j \\
    v_k, & k \neq i, j 
    \end{cases} 
\end{align*}
\]

It is well-known that the conjugacy classes in \( W(B_n) \) are indexed by double partitions \( \mu \lambda \) of \( n \). Namely, if \( \mu = \mu_1 \mu_2 \cdots \mu_q \) and \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_p \) are such that \( \sum_{i=1}^q \mu_i + \sum_{j=1}^p \lambda_j = n \), then elements of the conjugacy class indexed by \( \mu \lambda \) have \( q \) “positive” cycles of lengths \( \mu_1, \ldots, \mu_q \) and \( p \) “negative” cycles of lengths \( \lambda_1, \ldots, \lambda_p \). If \( \mu \) or \( \lambda \) is the empty partition, then it is omitted from the notation. With this labeling, the cuspidal conjugacy classes are indexed by the double partitions of the form \( \mu \lambda \) and hence by partitions of \( n \). See [13, §3.4] for more details.

Fix a partition \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_p \) of \( n \). Set \( \tau_1 = 0 \) and for \( i > 1 \) define \( \tau_i = \lambda_1 + \cdots + \lambda_{i-1} \). Then \( \tau_i + \lambda_i = \tau_{i+1} \) and \( \tau_{p+1} = n \). For \( 1 \leq i \leq p \) define

\[
c_i = t_{\tau_i+1}s_{\tau_i+2}s_{\tau_i+3}\cdots s_{\tau_{i+1}}
\]

in \( W \). Then \( c_i \) has order \( 2\lambda_i \) and acts on the set \( \{v_{\tau_i+1}, \ldots, v_{\tau_{i+1}}\} \) as a “negative \( \lambda_i \)-cycle.” Define

\[
w_\lambda = c_1c_2\cdots c_p.
\]

Then \( w_\lambda \) is a representative of the cuspidal conjugacy class labeled by \( \lambda \). For each \( i \) such that \( \lambda_i = \lambda_{i+1} \) define

\[
x_i = s_{\tau_{i+1},\tau_{i+1}+1}s_{\tau_{i+2},\tau_{i+1}+2}\cdots s_{\tau_{i+1},\tau_{i+2}}.
\]

It is straightforward to check that \( x_i \) centralizes \( w_\lambda \) and that \( C_W(w_\lambda) \) is generated by

\[
\{c_i \mid 1 \leq i \leq p\} \cup \{x_i \mid 1 \leq i \leq p, \ \lambda_i = \lambda_{i+1}\}
\]

If \( \lambda \) has \( m_i \) parts equal \( i \), then \( C_W(w_\lambda) = \prod_{m_i > 0} Z_{2i} \wr S_{m_i} \).
The conjugacy class labeled by the partition with all parts equal 1 is central and contains the longest element $w_0$ of $W$. It turns out that the character $\varphi_{w_0}$ is always the sign character. At the other extreme, the conjugacy class labeled by the partition with a single part $n$ is the Coxeter class. To simplify the notation, we denote the character $\varphi_{w_\lambda}$ of $C_W(w_\lambda)$ simply by $\varphi_\lambda$.

4.1.1. $W = W(B_3)$. The cuspidal conjugacy classes are labeled by the partitions 111, 12, and 3. The classes 111 and 3 are regular. The characters $\varphi_\lambda$ satisfying $\rho_\lambda^3 = \sum_{\lambda \vdash 3} \text{Ind}_{C_W(w_\lambda)}^W \varphi_\lambda = \epsilon \omega_\lambda^3$ are given in the following table. For each partition $\lambda$, the table lists the isomorphism type of $C_W(w_\lambda)$ in the second row, the generators of $C_W(w_\lambda)$ using the notation from §4.1 in the third row, and directly below each generator, the value of $\varphi_\lambda$ on that generator. However, when $\varphi_\lambda$ is the sign character, we omit its character values.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>111</th>
<th>112</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_W(w_\lambda)$</td>
<td>$W$</td>
<td>$Z_2 \times Z_4$</td>
<td>$Z_6$</td>
</tr>
<tr>
<td>Generators</td>
<td>$S$</td>
<td>$c_1 \ c_2 \ w_3$</td>
<td></td>
</tr>
<tr>
<td>$\varphi_\lambda$</td>
<td>$\epsilon$</td>
<td>$-1 \ -1 \ \zeta_6$</td>
<td></td>
</tr>
</tbody>
</table>

We see that $\varphi_3 = \det |E(\zeta_6)|$.

The values of the characters $\varphi_\lambda^W$ together with $\rho_\lambda^3$ and $\omega_\lambda^3$ are given in Table 1.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>111. 11.1 1.11 .111 12. 1.2 .12 3. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon = \varphi_{111}$</td>
<td>1 -1 1 -1 -1 1 -1 1</td>
</tr>
<tr>
<td>$\varphi_{12}$</td>
<td>6 -2 2 -6 -2 2 -6 -2</td>
</tr>
<tr>
<td>$\varphi_3$</td>
<td>8 -8 -8 -8 -8 -8 -8 -8</td>
</tr>
<tr>
<td>$\rho_\lambda^3$</td>
<td>15 -3 3 -15 -1 -1 1 -1</td>
</tr>
<tr>
<td>$\omega_\lambda^3$</td>
<td>15 3 3 15 1 -1 1 -1</td>
</tr>
</tbody>
</table>

Table 1. The characters $\varphi_\lambda^W$, $\rho_\lambda^3$, and $\omega_\lambda^3$ for $W(B_3)$

4.1.2. $W = W(B_4)$. The cuspidal conjugacy classes are labeled by the partitions 1111, 1112, 22, 13, and 4. The regular classes are 1111, 22, and 4. The characters $\varphi_\lambda$ satisfying $\rho_\lambda^4 = \sum_{\lambda \vdash 4} \text{Ind}_{C_W(w_\lambda)}^W \varphi_\lambda = \epsilon \omega_\lambda^4$ are given in the following table. The conventions are the same as for $W(B_3)$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>1111</th>
<th>112</th>
<th>22</th>
<th>13</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_W(w_\lambda)$</td>
<td>$W$</td>
<td>$(Z_2 \wr S_2) \times Z_4$</td>
<td>$Z_4 \wr S_2$</td>
<td>$Z_2 \times Z_6$</td>
<td>$Z_8$</td>
</tr>
<tr>
<td>Generators</td>
<td>$S$</td>
<td>$c_1 \ x_1 \ c_3$</td>
<td>$c_1 \ x_1$</td>
<td>$c_1 \ c_2$</td>
<td>$w_4$</td>
</tr>
<tr>
<td>$\varphi_\lambda$</td>
<td>$\epsilon$</td>
<td>$-1 \ -1 \ -1 \ -1 \ -1 \ -1 \ \zeta_6 \ -1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In contrast with the Coxeter class in type $B_3$ where $\varphi_3 = \det |E(\zeta_6)|$, in this case we have $\varphi_4 = (\det |E(\zeta_6)|)^4$. It is easy to compute that if $\zeta$ is any primitive fourth root of unity, then $C_W(w_{22})$ acts on the two-dimensional, regular eigenspace $E(\zeta)$ as the complex reflection group $G(4,1,2) \cong Z_4 \wr S_2$. The eigenvalues of $c_1$ and $x_1$ acting on $E(\zeta)$ are $\{1, \zeta\}$ and
\(\{1, -1\}\), respectively. Thus \(\det c_1|_{E(c)} = \zeta\) and \(\det x_1|_{E(c)} = -1\). It follows that \(\varphi_{22}\) is not equal \((\det|_{E(c)})^p\) for any eigenvalue \(\zeta\) or any power \(p\).

The values of the characters \(\varphi^W_\lambda\) together with \(\rho^W_4\) and \(\omega^W_4\) are given in Table 2.

<table>
<thead>
<tr>
<th>(\epsilon = \varphi_{1111})</th>
<th>1111.</th>
<th>11.11</th>
<th>11.11</th>
<th>1.111</th>
<th>112.</th>
<th>12.1</th>
<th>1.12</th>
<th>2.11</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi_{112})</td>
<td>12</td>
<td>-6</td>
<td>4</td>
<td>-6</td>
<td>12</td>
<td>-2</td>
<td>.</td>
<td>2</td>
</tr>
<tr>
<td>(\varphi_{22})</td>
<td>12</td>
<td>.</td>
<td>-4</td>
<td>4</td>
<td>12</td>
<td>.</td>
<td>-4</td>
<td>.</td>
</tr>
<tr>
<td>(\varphi_{13})</td>
<td>32</td>
<td>-8</td>
<td>.</td>
<td>-8</td>
<td>32</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>(\varphi_4)</td>
<td>48</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>48</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>(\rho^W_4)</td>
<td>105</td>
<td>-15</td>
<td>9</td>
<td>-15</td>
<td>105</td>
<td>-3</td>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>(\omega^W_4)</td>
<td>105</td>
<td>15</td>
<td>9</td>
<td>15</td>
<td>105</td>
<td>3</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\epsilon_{112} = \varphi_{1111})</th>
<th>1</th>
<th>1</th>
<th>-1</th>
<th>1</th>
<th>1</th>
<th>-1</th>
<th>-1</th>
<th>1</th>
<th>-1</th>
<th>1</th>
<th>(\varphi_{1111} = \epsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi_{112})</td>
<td>.</td>
<td>.</td>
<td>2</td>
<td>-4</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>(\varphi_{22})</td>
<td>-4</td>
<td>-4</td>
<td>.</td>
<td>.</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>.</td>
<td>(\varphi_{13})</td>
<td></td>
</tr>
<tr>
<td>(\varphi_4)</td>
<td>.</td>
<td>.</td>
<td>8</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>-4</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>(\rho^W_4)</td>
<td>-3</td>
<td>-3</td>
<td>1</td>
<td>5</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>-1</td>
<td>-1</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>(\omega^W_4)</td>
<td>-3</td>
<td>-3</td>
<td>-1</td>
<td>5</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>-1</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

**Table 2.** The characters \(\varphi^W_\lambda\), \(\rho^W_4\), and \(\omega^W_4\) for \(W(B_4)\)

#### 4.2. \(W\) of type \(D\)

Suppose \(V\) has basis \(\{v_1, \ldots, v_n\}\) where \(n \geq 4\). We consider elements of \(W = W(D_n)\) as acting as signed permutations of the basis of \(V\) having an even number of sign changes. Then \(W(D_n)\) is a normal subgroup of \(W(B_n)\) of index 2. The Coxeter generators of \(W(D_n)\) are \(s'_1, s'_2, \ldots, s'_n\) where \(s_2, \ldots, s_n\) are the last \(n - 1\) Coxeter generators of \(W(B_n)\) defined in §4.1 and \(s'_1\) is given by

\[
s'_1(v_k) = \begin{cases} 
-v_2, & k = 1 \\
-v_1, & k = 2 \\
v_k, & k \neq 1, 2 
\end{cases}
\]

so that \(s'_1 = s_1 s_2 s_1\), where \(s_1\) is the first Coxeter generator of \(W(B_n)\). The Dynkin diagram of \(W = W(D_n)\) is

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & \cdots & n-1 & n \\
\end{array}
\]

as in [4] and in CHEVIE.

Because \(W(D_n)\) is a normal subgroup of \(W(B_n)\), it is a union of conjugacy classes of \(W(B_n)\). The conjugacy class of \(W(B_n)\) labeled by the double partition \(\mu, \lambda\) of \(n\) lies in \(W(D_n)\) if and only if \(\lambda\) has an even number of parts. If the conjugacy class of \(W(B_n)\)
labeled by the double partition \( \mu, \lambda \) lies in \( W(D_n) \), then it is a single \( W(D_n) \)-conjugacy class except in the case when \( \lambda \) is the empty partition and all parts of \( \mu \) are even. In that case, the \( W(B_n) \)-conjugacy class splits into two classes in \( W(D_n) \) labeled by \( \mu, + \) and \( \mu, - \).

An element in \( W(D_n) \) is cuspidal if and only if it is cuspidal in \( W(B_n) \), so the cuspidal conjugacy classes of \( W(D_n) \) are labeled by partitions of \( n \) with an even number of parts. For such a partition \( \lambda \) we take \( w_\lambda \) to be the representative of the conjugacy class of \( W(B_n) \) chosen in §4.1. Then the centralizer in \( W(D_n) \) of \( w_\lambda \) is the intersection of \( W(D_n) \) and the centralizer of \( w_\lambda \) in \( W(B_n) \). See [13, §3.4] for more details.

4.2.1. \( W = W(D_4) \). The cuspidal conjugacy classes are labeled by the partitions 1111, 22, and 13. All three classes are regular. Each of these conjugacy classes is also a conjugacy class in the larger group \( W(B_4) \), and as remarked above, \( C_W(w_\lambda) = W \cap C_{W(B_4)}(w_\lambda) \).

One might conjecture that the character of \( \varphi^D_\lambda \) of \( C_{W(B_4)}(w_\lambda) \) is the restriction of the character of \( \varphi^B_\lambda \) of \( C_{W(B_4)}(w_\lambda) \). This turns out to be the case for the class of \( w_6 \) labeled by 1111 and for the Coxeter class labeled by 13, but not for the class labeled by 22.

Let \( w_{22} = s'_1 s_3 s'_2 s_2 s_3 s_4 \). Using the notation introduced in §4.1 we have \( w_{22} = c_1 c_2 \). The centralizer of \( w_{22} \) in \( W(D_4) \) contains the generator \( x_1 \) of the centralizer of \( w_{22} \) in \( W(B_4) \), but not the generator \( c_1 \). Rather, \( C_{W(D_4)}(w_{22}) \) is generated by \( w_{22}, x_1 \), and the involution \( s'_1 s_2 \). The character \( \varphi_{22} \) maps each of these generators to \(-1\). Notice that \( \varphi_{22}(w_{22}) = (\varphi_{22}^D(w_{22}))^2 \).

The characters \( \varphi_\lambda \) satisfying Theorem 2.3 are summarized in the following table, where the conventions are the same as for \( W(B_3) \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>1111</th>
<th>22</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_W(w_\lambda) )</td>
<td>( W )</td>
<td>((Z_4 \times Z_2) \rtimes S_2)</td>
<td>( Z_6 )</td>
</tr>
<tr>
<td>Generators</td>
<td>( S )</td>
<td>( w_{22}, s'_1 s_2, x_1 )</td>
<td>( w_{13} )</td>
</tr>
<tr>
<td>( \varphi_\lambda )</td>
<td>( \epsilon )</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
</tbody>
</table>

For the Coxeter class 13 we have \( \varphi_{13} = \det |E(\zeta_6)\). Let \( \zeta \) be a primitive fourth root of unity. Using [21, Theorem 4.2] it is easy to compute that if \( \zeta \) is any primitive fourth root of unity, then \( C_W(w_{22}) \) acts on the two-dimensional, regular eigenspace \( E(\zeta) \) as the complex reflection group \( G(4,2,2) \cong (Z_4 \times Z_2) \rtimes S_2 \). The eigenvalues of \( w_{22}, s'_1 s_2 \), and \( x_1 \) acting on \( E(\zeta) \) are \( \{\zeta, \zeta\}, \{1, -1\} \) and \( \{1, -1\} \), respectively. Thus \( \varphi_{22} = \det |E(\zeta)\).

The values of characters \( \varphi^W_\lambda \) together with \( \rho^W_\lambda \) and \( \omega^W_\lambda \) are shown in Table 3.

4.3. \( W = W(F_4) \). The Dynkin diagram of \( W \) is \( \frac{\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}}{\begin{array}{cc}
0 & 1 \\
2 & 3
\end{array}} \). We label the conjugacy classes of \( W \) using Carter’s labeling [5]. This is also the labeling used by the CHEVIE package. There are nine cuspidal conjugacy classes. Their Carter labels are

\[ 4A_1, \ D_4, \ D_4(a_1), \ C_3A_1, \ A_2\bar{A}_2, \ F_4(a_1), \ F_4, \ A_3\bar{A}_1, \text{ and } B_4. \]

The regular classes are \( 4A_1, \ D_4(a_1), \ C_3A_1, \ F_4(a_1), \text{ and } B_4 \). For each label \( d \) above we denote the representative of the class labeled \( d \) by \( w_d \) and the character of \( C_W(w_d) \) satisfying Theorem 2.3 by \( \varphi_d \). The classes labeled by \( 4A_1, \ F_4, \text{ and } B_4 \) are self-centralizing and we consider them first.
\[ \epsilon = \varphi_{1111} \]

<table>
<thead>
<tr>
<th>(4.2)</th>
<th>(\epsilon = \varphi_{1111})</th>
<th>(\varphi_{22})</th>
<th>(\varphi_{13})</th>
<th>(\rho^i_W)</th>
<th>(\omega^i_W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111. 11.11 .1111 211. 1.21 2.11 22+ 22− 32. .31 .31 4+. 4−</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
</tr>
</tbody>
</table>

**Table 3.** The characters \(\varphi^W, \rho^i_W, \omega^i_W\) for \(W(D_4)\)

4\(A_1\): The class labeled by 4\(A_1\) is \(\{w_0\}\). We take \(\varphi_{4A_1} = \epsilon\).

\(F_4\): This is the Coxeter class. Coxeter elements have order 12. We take \(w_{F_4}\) to be any Coxeter element and define \(\varphi_{F_4}\) by \(\varphi_{F_4}(w_{F_4}) = \zeta_3\). Clearly, \(\varphi_{F_4} = (\det |E(\zeta_3)|)^4\).

\(B_3\): This class contains the Coxeter elements in the maximal rank subgroups of \(W\) of type \(B_3\) and is regular. Then \(C_W(w_{B_3}) = C_{W(B_3)}(w_{B_3})\) is cyclic of order 8, where \(w_{B_3}\) denotes the element \(w_3\) from §4.1.2. Define \(\varphi_{B_4}\) by \(\varphi_{B_4}(w_{B_3}) = \zeta_4\). We see from §4.1.2 that \(\varphi_{B_4}^2 = (\varphi_{B_4})^2\). In addition, \(\varphi_{B_3} = (\det |E(\zeta_3)|)^2\).

\(D_4\): This class contains the Coxeter elements in the maximal rank subgroups of \(W\) of type \(D_4\) and \(B_3A_1\) (see [9]). It also contains the class labeled by the partition 13 in the maximal rank subgroups of \(W\) of types \(B_4\) and \(D_4\). In \(W(D_4)\) the centralizer of \(w_{13}\) is isomorphic to \(Z_6\) while in \(W(B_3A_1)\) and \(W(B_4)\) the centralizer of \(w_{13}\) is isomorphic to \(Z_6 \times Z_2\).

Recall that multiplication by \(w_0\) permutes the conjugacy classes of \(W\), by Remark 4.1. In this case multiplication by \(w_0\) sends the class labeled \(D_4\) to the class labeled by \(A_2\) containing \(s_1s_2\). Thus we can take \(w_{D_4} = s_1s_2w_0\). Extending the Dynkin diagram of \(W\) as in the Borel-De Siebenthal algorithm [3] by adjoining the reflection \(s_{21}\) corresponding to the highest short root results in the diagram

(4.2)

The subgroup generated by \(\{s_1, s_{21}\}\) is a parabolic subgroup of type \(\tilde{A}_2\) and

\[ C_W(w_{D_4}) = \langle s_1, s_{21} \rangle \times \langle w_{D_4}\rangle \cong S_3 \times Z_6. \]

We define \(\varphi_{D_4}\) by \(\varphi_{D_4}|_{S_3} = \epsilon_{S_3}\) and \(\varphi_{D_4}(w_{D_4}) = \zeta_6\). Notice that using §4.1.2 and §4.2.1 we have

\[ \zeta_6 = \varphi_{D_4}(w_{D_4}) = -\varphi_{D_4}(w_{D_4}) = (\varphi_{D_4}(w_{D_4}))^2. \]

\(D_4(a_1)\): This class contains the conjugacy classes labeled by the partition 22 in the maximal rank subgroups of \(W\) of types \(B_4\) and \(D_4\) and is regular. It also contains the Coxeter elements in the maximal rank subgroup of \(W\) of type \(B_2B_2\). To find a representative of this class and compute its centralizer we extend the Dynkin diagram of \(W\) by adjoining the reflection \(s_{24}\) corresponding to the highest long root. The resulting diagram is
and we consider the maximal rank subgroup $W(B_4)$ generated by \{s_{24}, s_1, s_2, s_3\}. Recall from §4.1.2 that the centralizer of $w_{22} = \in W(B_4)$ is generated by $c_1$ and $x_1$. Translating from the $B_3$ labeling to our current labeling, we set

$$w_{22} = s_3s_2s_1s_3s_2s_1s_{24}, \quad c_1 = s_3s_2, \quad \text{and} \quad x_1 = s_1s_2s_{24}s_1.$$

Then $w_{22}$ lies in the conjugacy class we are considering and $C_W(w_{22}) = \langle c_1, x_1 \rangle$. Although $w_{22}$ would be a natural representative of $D_4(a_1)$, it is more convenient to define $w_{D_4(a_1)}$ to be the conjugate $s_1s_2s_1w_{22}s_1s_2s_1$ of $w_{22}$ because then $w_{D_4(a_1)}$ will commute with the representative $w_{\tilde{A}_2A_2}$ chosen below. Define $c_1' = s_1s_2s_1c_1s_1s_2s_1$ and $x_1' = s_1s_2s_1x_1s_1s_2s_1$. Then $C_W(w_{D_4(a_1)})$ is generated by $\{c_1', x_1', w_{\tilde{A}_2A_2}\}$.

Define $\varphi_{D_4(a_1)}$ by

$$\varphi_{D_4(a_1)}(c_1') = -1, \quad \varphi_{D_4(a_1)}(x_1') = 1, \quad \text{and} \quad \varphi_{D_4(a_1)}(w_{\tilde{A}_2A_2}) = 1.$$

Notice that using §4.1.2 and §4.2.1 we have

$$\varphi_{D_4(a_1)}^{B_2}(w_{D_4(a_1)}) = \varphi_{D_4(a_1)}^{B_2}(w_{22}) = (\varphi_{D_4(a_1)}(w_{22}))^2.$$

Using [21, Theorem 4.2] it is easy to see that $C_W(w_{D_4(a_1)})$ acts on its $\zeta_4$-eigenspace as the complex reflection group $G_8$. This group may be described by the diagram

$$\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & c & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}$$

where $c = c_1'$ and $d = s_2s_3s_4s_3$. Then $\varphi_{D_4(a_1)}(c) = \varphi_{D_4(a_1)}(d) = -1$ and $\varphi_{D_4(a_1)} = (\det |E(\zeta_4)|)^2$.

$C_3A_1$: This class contains the Coxeter elements of the maximal rank subgroups of $W$ of types $C_3A_1$ and $\tilde{D}_4$ (see [9]). In particular, this class is the image under the graph automorphism of $W$ of the class labeled by $D_4$ so that the centralizers of elements of both classes are isomorphic to $Z_6 \times S_3$. We put $w_{C_3A_1} = s_3s_4s_0$ and compute its centralizer using the same technique used for the class labeled by $D_4$. We define $\varphi_{C_3A_1}$ by $\varphi_{C_3A_1}(w_{C_3A_1}) = \zeta_3$ and $\varphi_{C_3A_1}|_{S_3} = \epsilon$.

$A_2\tilde{A}_2$: This class contains the Coxeter elements of the reflection subgroups of $W$ of type $A_2\tilde{A}_2$ and is regular. We take $w_{A_2\tilde{A}_2} = s_1s_2s_4s_{21}$ (see (4.2)). We noted above that $w_{D_4(a_1)}$ was chosen so that $w_{A_2\tilde{A}_2}$ and $w_{D_4(a_1)}$ commute. Obviously $s_1s_2$ and $s_4s_{21}$ centralize $w_{A_2\tilde{A}_2}$ and generate an elementary abelian subgroup of $C_W(w_{A_2\tilde{A}_2})$ of order 9. Then $C_W(w_{A_2\tilde{A}_2})$ is generated by $\{s_1s_2, s_4s_{21}, w_{D_4(a_1)}\}$. Define $\varphi_{A_2\tilde{A}_2}$ by

$$\varphi_{A_2\tilde{A}_2}(s_1s_2) = \zeta_3, \quad \varphi_{A_2\tilde{A}_2}(s_4s_{21}) = \zeta_3, \quad \text{and} \quad \varphi_{A_2\tilde{A}_2}(w_{D_4(a_1)}) = 1.$$

Then $\varphi_{A_2\tilde{A}_2}(w_{A_2\tilde{A}_2}) = \zeta_3^2$.

Using [21, Theorem 4.2] it is easy to see that $C_W(w_{A_2\tilde{A}_2})$ acts on its $\zeta_3$-eigenspace as the complex reflection group $G_5$. This group may be described by the diagram

$$\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & a & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}$$

where $a = s_1s_2$ and $b = s_2s_3s_2s_3s_4s_3$. Then $\varphi_{A_2\tilde{A}_2}(a) = \varphi_{A_2\tilde{A}_2}(b) = \zeta_3$ and $\varphi_{A_2\tilde{A}_2} = \det |E(\zeta_3)|$.

$F_4(a_1)$: This class does not contain elements that lie in any proper reflection subgroup of $W$. It is regular. The image of this class under multiplication by
$w_0$ is the class labeled by $A_2\tilde{A}_2$. Thus, $C_W(w_{F_4(a_1)}) = C_W(w_{A_2\tilde{A}_2})$. We take $w_{F_4(a_1)} = w_{A_2\tilde{A}_2}w_0$ and $\varphi_{F_4(a_1)} = \varphi_{A_2\tilde{A}_2}$.

Because $w_{F_4(a_1)} = w_{A_2\tilde{A}_2}w_0$ and $w_{A_2\tilde{A}_2}$ has order three, the $\zeta_6$-eigenspace of $w_{F_4(a_1)}$ is equal to the $\zeta_3$-eigenspace of $w_{A_2\tilde{A}_2}$. Therefore, $C_W(w_{F_4(a_1)})$ acts on its $\zeta_6$-eigenspace as the complex reflection group $G_5$ and $\varphi_{F_4(a_1)} = \det |E(\zeta_6)|$

$A_3\tilde{A}_1$: This class contains the Coxeter elements in the reflection subgroups of $W$ of types $A_3\tilde{A}_1$, $A_1\tilde{A}_3$, $2A_1B_2$, and $2A_1B_2$ (see [9]). In addition, the image of this class under multiplication by $w_0$ is the class labeled by $B_2$ and contains $s_2s_3$. The reflections $s_{21}$ and $s_{24}$ corresponding to the highest short and long roots generate a subgroup of $W$ of type $B_2$. Taking $w_{A_3}\tilde{A}_1 = s_2s_3w_0$, we have

$$C_W(w_{A_3}\tilde{A}_1) = \langle w_{A_3}\tilde{A}_1 \rangle \times \langle s_{21}, s_{24} \rangle \cong Z_4 \times W(B_2)$$

Define $\varphi_{A_3}\tilde{A}_1$ by $\varphi_{A_3}\tilde{A}_1(w_{A_3}\tilde{A}_1) = -1$ and $\varphi_{A_3}\tilde{A}_1|W(B_2) = \epsilon W(B_2)$.

The characters $\varphi_d$ satisfying $\rho_W^d = \sum_d \text{Ind}_{C_W(w_d)}^{W} \varphi_d = \omega_W^d$ are summarized in the following table. The conventions are the same as for $W(B_3)$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$4A_1$</th>
<th>$D_4$</th>
<th>$D_4(a_1)$</th>
<th>$C_3A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_W(w_d)$</td>
<td>$W$</td>
<td>$Z_6 \times S_3$</td>
<td>$G_8$</td>
<td>$Z_6 \times S_3$</td>
</tr>
<tr>
<td>Generators</td>
<td>$S$</td>
<td>$w_{D_4} \ast$</td>
<td>$c \ d$</td>
<td>$w_{C_3A_1} \ast$</td>
</tr>
<tr>
<td>$\varphi_d$</td>
<td>$\epsilon$</td>
<td>$\zeta_3$</td>
<td>$\epsilon$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

$A_2\tilde{A}_2$ | $F_4(a_1)$ | $F_4$ | $A_3\tilde{A}_1$ | $B_4$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_5$</td>
<td>$G_5$</td>
<td>$Z_{12}$</td>
<td>$Z_4 \times W(B_2)$</td>
<td>$Z_8$</td>
</tr>
<tr>
<td>$a \ b$</td>
<td>$a \ b$</td>
<td>$w_{F_4}$</td>
<td>$w_{A_3}\tilde{A}_1$</td>
<td>$\ast$</td>
</tr>
<tr>
<td>$\zeta_3 \ \zeta_3$</td>
<td>$\zeta_3 \ \zeta_3$</td>
<td>$\zeta_3 \ \zeta_3$</td>
<td>$-1$</td>
<td>$\epsilon$</td>
</tr>
</tbody>
</table>

The values of the characters $\varphi_d^W$ together with $\rho_W^d$ and $\omega_W^d$ are shown in Table 4.

4.4. $W$ of type $H$. When $W$ is a non-crystallographic group it can happen that for a given positive integer $d$ there is more than one regular conjugacy class of elements of order $d$. In this case, if $\zeta$ is a fixed primitive $d^{th}$ root of unity and $w$ in $W$ is regular with order $d$, then $\zeta$ might not be an eigenvalue of $w$ (see [21, §5]). Thus, some care must be taken when describing the determinant of the character of $C_W(w)$ acting on a regular eigenspace of $w$. Similar considerations apply to the characters $\varphi_w$.

We label the conjugacy classes $C_1, C_2, \ldots$ and choose representatives $w_1, w_2, \ldots$ as in [13] and CHEVIE. When $n$ is fixed, we frequently denote $|w_n|$ by $d$.

4.4.1. $W = W(H_3)$. The cuspidal classes are $C_6$, $C_8$, $C_9$, and $C_{10}$. All these classes are regular.

$C_6$: $d = 10$ and $C_W(w_6) = \langle w_6 \rangle$. Define $\varphi_6(w_6) = \zeta_{10}$. Then $\varphi_6 = \det |E(\zeta_{10})|$. 

\[ \begin{array}{cccccccccccc}
\epsilon = \varphi_{A_1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
\varphi_{D_4} & 32 & 32 & -1 & -1 & -2 & 2 & -4 & -4 & -4 & -4 \\
\varphi_{D_4(a_1)} & 12 & 12 & 4 & -2 & 8 & -2 & 6 & 6 & 2 & 2 \\
\varphi_{C_3A_1} & 12 & 12 & -2 & 2 & -1 & -1 & -4 & -4 & -8 & -8 \\
\varphi_{A_2\tilde{A}_2} & 16 & 16 & -2 & -2 & 8 & -2 & -2 & 7 & 7 & 1 & 1 \\
\varphi_{F_4(a_1)} & 16 & 16 & -2 & -2 & 8 & -2 & -2 & 7 & 7 & 1 & 1 \\
\varphi_{F_4} & 36 & 36 & 4 & -12 & 6 & -6 & 6 & 6 & -6 & -6 \\
\varphi_{A_3\tilde{A}_1} & 36 & 36 & 4 & -12 & 6 & -6 & 6 & 6 & -6 & -6 \\
\omega^d_W & 385 & 385 & 9 & -2 & -2 & 5 & -2 & -2 & 7 & 7 & -1 & 15 & 15 \\
\end{array} \]

\[ \begin{array}{cccccccccccc}
A_1\tilde{A}_2 & C_3 & A_3 & \tilde{A}_1 & 2A_1\tilde{A}_1 & A_2\tilde{A}_1 & B_3 & B_2A_1 & A_1\tilde{A}_1 & B_2 & A_3\tilde{A}_1 & B_4 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & \varphi_{A_4A_1} = \epsilon \\
& & & & & & & & & & & & & \varphi_{D_4} \\
& & & & & & & & & & & & & \varphi_{D_4(a_1)} \\
1 & 1 & & & & & & & & & & & & \varphi_{C_3A_1} \\
& & & & & & & & & & & & & \varphi_{A_2\tilde{A}_2} \\
& & & & & & & & & & & & & \varphi_{F_4(a_1)} \\
& & & & & & & & & & & & & \varphi_{F_4} \\
& & & & & & & & & & & & & \varphi_{A_3\tilde{A}_1} \\
& & & & & & & & & & & & & \varphi_{B_4} \\
& & & & & & & & & & & & & \varphi_{B_4} \\
& & & & & & & & & & & & & \varphi_{B_4} \\
\end{array} \]

Table 4. The characters \( \varphi^W_d, \rho^d_W, \) and \( \omega^d_W \) for \( W(F_4) \)

\[ \begin{array}{cccccccccccc}
C_6: & d = 6 & \text{and } C_W(w_8) = \langle w_8 \rangle. & \text{Define } \varphi_8(w_8) = \zeta_6. & \text{Then } \varphi_8 = \det |_{E(6_0)}. \\
C_9: & d = 10 & \text{and } C_W(w_9) = \langle w_9 \rangle. & \text{Define } \varphi_9(w_9) = \zeta_{10}. & \text{The elements } w_6 \text{ and } w_9 & \text{are related by } w_9 = w_6^3. & \text{Thus the } \zeta_{10}^3 \text{-eigenspace of } w_9 \text{ coincides with the } \zeta_{10} \text{-eigenspace of } w_6, & \text{and we have } \varphi_9 = (\det |_{E(3_{10})})^7. \\
C_{10}: & \text{The element } w_{10} = w_0 \text{ is central} & \text{and we take } \varphi_{10} = \epsilon. \\
\end{array} \]

The values of the characters \( \varphi^W_i \) together with \( \rho^3_W \) and \( \omega^3_W \) are given in Table 5.

4.4.2. \( W = W(H_4) \). The cuspidal classes, the order of their elements, and the sizes of their centralizers are listed in the next table. Only the five classes \( C_{19}, C_{21}, C_{25}, C_{27}, \) and \( C_{31} \) are not regular.
For \( n = 11, 14, 17, 23, 28 \) each of the elements \( w_n \) is self-centralizing and regular. We define \( \varphi_n(w_n) = \zeta_2^4 \) in all cases. Then, \( \varphi_n = (\det |E(\zeta_2)|)^2 \) for \( n = 11, 14, 17 \). For \( n = 23 \) we have \( d = 20 \), \( E(\zeta_2^{15}) \) is a regular eigenspace, and \( \varphi_{23} = (\det |E(\zeta_2^{15})|)^{14} \). For \( n = 28 \) we have \( d = 30 \), \( E(\zeta_2^{15}) \) is a regular eigenspace, and \( \varphi_{28} = (\det |E(\zeta_2^{15})|)^{26} \).

For \( n = 15, 22 \) we have \( d = 15 \) and \( C_W(w_n) = \langle w_0w_n \rangle \cong Z_2 \times \langle w_n \rangle \). These classes are regular. We define \( \varphi_n(w_0w_n) = \zeta_{15} \) in both cases. For \( n = 15 \), \( E(\zeta_{15}) \) is a regular eigenspace. Notice that, since \( \zeta_{15} = (\zeta_2^{15})^{16} \), the \( \zeta_{15} \)-eigenspace of \( w_{15} = (w_0w_{15})^{16} \) is equal to the \( \zeta_2^{15} \)-eigenspace of \( w_0w_{15} \). Thus \( \varphi_{15} = (\det |E(\zeta_{15})|)^{16} \). For \( n = 22 \), \( E(\zeta_2^{15}) \) is a regular eigenspace and \( \varphi_{22} = (\det |E(\zeta_{15})|)^8 \).

For \( n = 18, 26, 30, 33 \) we have \( C_W(w_n) = \langle w_{18}, w_{19}, w_{29} \rangle \). These are regular classes and in all cases \( C_W(w_n) \) acts on a regular, two-dimensional eigenspace of \( w_n \) as the complex reflection group \( G_{16} \). Define

\[
\varphi_{18} = \varphi_{33} \quad \text{by} \quad (w_{18}, w_{19}, w_{29}) \mapsto \left( \zeta_5^2, \zeta_5^4, 1 \right)
\]

and

\[
\varphi_{26} = \varphi_{30} \quad \text{by} \quad (w_{18}, w_{19}, w_{29}) \mapsto \left( \zeta_5^4, \zeta_5^2, 1 \right).
\]

- For \( n = 18 \) we have \( d = 10 \), \( E(\zeta_{10}) \) is a regular eigenspace, and \( \varphi_{18} = (\det |E(\zeta_{10})|)^2 \).
- For \( n = 26 \) we have \( d = 5 \), \( E(\zeta_5) \) is a regular eigenspace, and \( \varphi_{26} = (\det |E(\zeta_5)|)^4 \).
- For \( n = 30 \) we have \( d = 10 \), \( E(\zeta_{10}) \) is a regular eigenspace, and \( \varphi_{30} = (\det |E(\zeta_{10})|)^4 \).
- For \( n = 33 \) we have \( d = 5 \), \( E(\zeta_5^2) \) is a regular eigenspace, and \( \varphi_{33} = (\det |E(\zeta_5^2)|)^2 \).

For \( n = 19, 27 \) we have \( w_{27} = w_{29}^2 \) and \( C_W(w_n) = \langle w_{18} \rangle \times \langle w_{27} \rangle \cong Z_{10} \times Z_5 \). Define

\[
\varphi_{19} = \varphi_{27} \quad \text{by} \quad (w_{18}, w_{27}) \mapsto \left( \zeta_5^3, \zeta_5^4 \right).
\]

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( C_7 )</th>
<th>( C_8 )</th>
<th>( C_9 )</th>
<th>( C_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_6 )</td>
<td>12</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>-\nu</td>
<td>\nu</td>
<td>\cdot</td>
<td>-\mu</td>
</tr>
<tr>
<td>( \varphi_8 )</td>
<td>20</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>1</td>
<td>\cdot</td>
<td>\cdot</td>
<td>-20</td>
</tr>
<tr>
<td>( \varphi_9 )</td>
<td>12</td>
<td>\cdot</td>
<td>\nu</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>-\mu</td>
<td>\mu</td>
<td>\cdot</td>
<td>-\nu</td>
</tr>
<tr>
<td>( \epsilon = \varphi_{10} )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \rho_{\uparrow} )</td>
<td>45</td>
<td>-1</td>
<td>\cdot</td>
<td>1</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>-45</td>
</tr>
<tr>
<td>( \omega_{\uparrow} )</td>
<td>45</td>
<td>1</td>
<td>\cdot</td>
<td>1</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>45</td>
</tr>
</tbody>
</table>

**Table 5.** Induced characters \( W(H_3) \): \( \mu = \zeta_5 + \zeta_5^4 \), \( \nu = \zeta_5^2 + \zeta_5^3 \)
For $n = 21, 31$ we have $w_{31} = w_{31}^2$ and $C_W(w_n) = \langle w_{18} \rangle \times \langle w_{27} \rangle \times \langle s_2 \rangle \cong Z_{10} \times Z_5 \times Z_2$. Define
\[
\varphi_{21} \quad \text{by} \quad (w_{18}, w_{27}, s_2) \mapsto (\zeta_5, \zeta_5^2, -1) \quad \text{and} \quad \varphi_{31} = \varphi_{21}^3.
\]
For $n = 24, 32$ we have $C_W(w_n) = \langle w_{24}, w_{25}, w_{29} \rangle$. Denote this group simply by $Z$. The classes $C_{24}$ and $C_{32}$ are regular. The representatives $w_{24}$ and $w_{32}$ have order $d = 6$ and $d = 3$, respectively and are related by $w_{32} = w_{24}^2$. Thus, the $\zeta_6$-eigenspace of $w_{24}$ is equal to the $\zeta_3$-eigenspace of $w_{32}$. Denote this vector space simply by $E$. Then $E$ is a regular, two-dimensional eigenspace for $w_{24}$ and $w_{32}$, and $Z$ acts on $E$ as the complex reflection group $G_{20}$. Define
\[
\varphi_{24} = \varphi_{32} \quad \text{by} \quad (w_{24}, w_{25}, w_{29}) \mapsto (\zeta_3^2, \zeta_3, 1).
\]
In both cases we have $\varphi_n = (\det |E(\zeta_d)|)^2$.
For $n = 25$ we have $C_W(w_{25}) = \langle w_{w_0w_{24}} \rangle \times \langle w_{25} \rangle \times \langle s_2 \rangle \cong Z_6 \times Z_6 \times Z_2$. Define
\[
\varphi_{25} \quad \text{by} \quad (w_{0w_{24}}, w_{25}, s_2) \mapsto (\zeta_3^2, \zeta_3^2, -1).
\]
For $n = 29$ we have $C_W(w_{29}) = \langle w_{18}, w_{24}, w_{29} \rangle$. This class is regular. We have $d = 4$, $E(\zeta_4)$ is a regular, two-dimensional eigenspace, and $C_W(w_{29})$ acts on $E(\zeta_4)$ as the complex reflection group $G_{22}$. Define
\[
\varphi_{29} \quad \text{by} \quad (w_{18}, w_{24}, w_{29}) \mapsto (1, 1, -1).
\]
Then $\varphi_{29} = \det |E(\zeta_4)|$.
Finally, for $n = 34$ we have $w_{34} = w_0$ and we define $\varphi_{34} = \epsilon$.
The values of the characters $\varphi^W_i$ together with $\rho^4_W$ and $\omega^4_W$ are given in Table 6.

Appendix A. Bulky Parabolic Subgroups

For each finite irreducible Coxeter group $W$ the following table lists the types of all bulky parabolic subgroups of $W$ other than $W$ itself, the trivial subgroup, and the subgroup of type $A_1$. This information has been extracted from the results in [14].

<table>
<thead>
<tr>
<th>$W$</th>
<th>Bulky Parabolic Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$A_{n_1}A_{n_2} \cdots A_{n_k}$ with $n_i$ distinct and $\sum_{i=1}^k n_i \leq n + k - 1$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$B_j$ with $1 \leq j \leq n - 1$, $A_1B_j$ with $1 \leq j \leq n - 2$</td>
</tr>
<tr>
<td>$D_n$, $n$ even</td>
<td>$A_1D_{n-2}$</td>
</tr>
<tr>
<td>$D_n$, $n$ odd</td>
<td>$A_1D_{n-2}$, $A_1A_{n-3}$, $A_{n-1}$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$A_1A_2$, $A_1A_3$, $A_4$, $A_1A_4$, $A_5$, $D_5$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$D_6$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$\tilde{A}_1$, $A_1\tilde{A}_1$, $B_2$, $B_3$, $C_3$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$A_1^2$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$H_3$</td>
</tr>
<tr>
<td>$I_2(m)$, $m$ even</td>
<td>$\tilde{A}_1$</td>
</tr>
</tbody>
</table>
Table 6. Induced characters for \( W(H_4) \): \( \mu = \zeta_5 + \zeta_5^4, \nu = \zeta_5^2 + \zeta_5^3 \)

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References


