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ON REFLECTION SUBGROUPS OF FINITE COXETER GROUPS

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Abstract. Let W be a finite Coxeter group. We classify the reflection subgroups of W up to conjugacy and give necessary and sufficient conditions for the map that assigns to a reflection subgroup R of W the conjugacy class of its Coxeter elements to be injective, up to conjugacy.

1. Introduction

Throughout, let (W, S) be a finite Coxeter system with distinguished set of generators S and let E be the real reflection representation of W. Define $T = \{ wsw^{-1} \mid w \in W,\ s \in S \}$ to be the set of elements of W that act on E as reflections. By a *reflection subgroup of* W we mean a subgroup of W generated by a subset of T. Reflection subgroups of W play an important role in the theory of Coxeter groups; for instance, by a fundamental theorem due to Steinberg, [14, Thm. 1.5], the stabilizer of any subspace of E is a reflection subgroup of W.

Our first aim in this note is to give a complete classification of all reflection subgroups of W up to conjugacy. In case W is a Weyl group, Carter [5, p. 8] has already outlined a procedure which leads to the classification based on the algorithm of Borel–De Siebenthal [3]. Here, we recast slightly Carter's construction and give the classification for non-crystallographic Coxeter groups as well. Similar classifications have been described by Felikson and Tumarkin [15], and by Dyer and Lehrer [8]. Our methods differ from those used in the sources cited above in that we use the notion of a parabolic closure of a reflection subgroup as an inductive tool in our analysis.

Every reflection subgroup of W is a maximal rank reflection subgroup of some parabolic subgroup of W. Thus, classifying conjugacy classes of reflection subgroups may be done recursively and reduces to first classifying conjugacy classes of parabolic subgroups and then classifying maximal rank subgroups of irreducible Coxeter groups. Conjugacy classes of parabolic subgroups of an irreducible finite Coxeter group are described in Chapter 2 and Appendix A of [10]. Classifying maximal rank reflection subgroups of W amounts to listing, up to the action of W, all subsets Y of T whose fixed point set in E is trivial and which are closed in the sense that $\langle Y \rangle \cap T = Y$. In case W is a Weyl group, the algorithm of Borel–De Siebenthal [3] is computationally much more efficient than classifying subsets of T with the two required properties.

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We have implemented the classification algorithms in the computer algebra system GAP [12] with the aid of the package CHEVIE [9]. Thus, it is feasible to actually compute the classification explicitly for W of a fixed rank. Indeed, we present the classification in cases W is a Weyl group of exceptional type, or a non-crystallographic Coxeter group of type H_3 and H_4, in the form of explicit lists.

Our second aim in this note is to study the map which assigns to a given conjugacy class of reflection subgroups the conjugacy class of its Coxeter elements. It is well known [11, Lem. 3.5] that if R and R' are parabolic subgroups containing Coxeter elements c and c' respectively, then R and R' are conjugate subgroups if and only if c and c' are conjugate in W. Thus, conjugacy classes of parabolic subgroups are parametrized by a distinguished set of conjugacy classes of elements in W. For general reflection subgroups this need not be the case. However, the following somewhat surprising result shows that when T is a single conjugacy class, conjugacy classes of reflection subgroups are still parametrized by the conjugacy classes of their Coxeter elements in all but one case.

Theorem 1.1. *Suppose that T is a single conjugacy class. Let R and R' be reflection subgroups containing Coxeter elements c and c', respectively. Then R and R' are conjugate if and only if c and c' are conjugate in W; unless W is of type E_8 and R and R' are of types A_1A_7 and A_3D_5, respectively. In this case, c and c' are conjugate, while R and R' are not.*

This theorem is an immediate consequence of the classification of the reflection subgroups of W and our computation of the map γ, which is defined as follows. Denote by $\mathcal{R}$ the set of conjugacy classes of reflection subgroups of W and by $\mathcal{C}$ the set of conjugacy classes of elements of W. Then, denote by

$$\gamma\colon \mathcal{R} \to \mathcal{C}$$

the map defined by $\gamma([R]) = [c]$, which associates to the conjugacy class $[R]$ of a reflection subgroup R of W the conjugacy class $[c]$ in W of a Coxeter element c in R. This map γ is well-known to be a bijection for Coxeter groups W of type A_n, with both the conjugacy classes of reflection subgroups and the conjugacy classes of elements of W labeled by the partitions of n. In §2 we state necessary and sufficient conditions for the map γ to be an injection (note that if γ is an injection, then the statement of Theorem 1.1 holds). The image of γ is computed explicitly for each type of irreducible Coxeter group in §3 - §5 and Tables 3 - 9. The classes in the image of γ are also known in the literature as *semi-Coxeter classes*, e.g., see [6]. Properties of the map γ have not been considered in the earlier literature on the subject.

The rest of this note is organized as follows. In §2 we recall some definitions, give some preliminary results, and state precisely when the map γ is injective or surjective. §3 contains explicit combinatorial rules that describe the map γ for classical types, and demonstrate that γ is surjective but not injective for Coxeter groups of type B_n ($n \geq 2$), and that γ is injective but not surjective for Coxeter groups of type D_n ($n \geq 4$). The classification of conjugacy classes of reflection subgroups and the explicit computation of the map γ is given for classical Weyl groups in §3 (with the examples of $W(B_5)$ and $W(D_6)$ in Tables 1 and 2 respectively); for exceptional Weyl groups in §4 and Tables 3 - 7; and for non-crystallographic Coxeter groups in §5 and Tables 8 and 9.

2. Preliminaries

For general information on Coxeter groups, root systems, and groups generated by reflections, we refer the reader to Bourbaki [4].

For the rest of this note we fix a W-invariant, positive definite, bilinear form on E.

Notice first that if $R = \langle Y \rangle$ is a reflection subgroup of W, then R is a Coxeter group in its own right. Moreover, the orthogonal complement of the space of fixed points of R in E is an R-stable subspace that affords the reflection representation of R.

Recall that a *parabolic subgroup* of W is a subgroup of the form

$$W_V = \{\, w \in W \mid w(v) = v \ \ \forall v \in V \,\},$$

where V is a subspace of E. By Steinberg's Theorem [14, Thm. 1.5], parabolic subgroups are generated by the reflections they contain and so are reflection subgroups.

For a subset X of W let

$$\mathrm{Fix}(X) = \{\, v \in E \mid x(v) = v \ \ \forall x \in X \,\}$$

denote the set of fixed points of X in E. Following Solomon [13] and Bergeron et al. [2], we define the *parabolic closure* of X to be the parabolic subgroup $\mathrm{A}(X) = W_{\mathrm{Fix}(X)}$ of W. Obviously $X \subseteq \mathrm{A}(X)$ and it follows from Steinberg's Theorem that $\mathrm{A}(\mathrm{A}(X)) = \mathrm{A}(X)$. When $X = \{w\}$ we simply write $\mathrm{Fix}(w)$ and $\mathrm{A}(w)$ instead of $\mathrm{Fix}(\{w\})$ and $\mathrm{A}(\{w\})$, respectively. For a discussion of parabolic closures of finitely generated subgroups of arbitrary Coxeter systems, see the recent paper by Dyer [7].

For $w, x \in W$ we denote the w-conjugate $w^{-1}xw$ of x by x^w and for a subset X of W let $X^w = \{x^w \mid x \in X\}$ denote the w-conjugate of X.

The *rank* of a Coxeter group is the cardinality of a Coxeter generating set, or equivalently, the dimension of its reflection representation. It follows from the next lemma that every reflection subgroup is a maximal rank reflection subgroup of its parabolic closure.

Lemma 2.1. *Let R be a reflection subgroup of W. Then R and its parabolic closure $\mathrm{A}(R)$ have the same rank as Coxeter groups.*

Proof. The rank of R is the codimension of its fixed point space $\mathrm{Fix}(R)$. The rank of $\mathrm{A}(R)$, as stabilizer of $\mathrm{Fix}(R)$, is not larger than the rank of R, and, since $R \subseteq \mathrm{A}(R)$, not smaller than the rank of R either. □

As noted in the Introduction, the classification of conjugacy classes of reflection subgroups of W reduces to (1) classifying conjugacy classes of parabolic subgroups of W and (2) classifying maximal rank reflection subgroups of irreducible Coxeter groups.

The conjugacy classes of parabolic subgroups of an irreducible finite Coxeter group are described in Chapter 2 and Appendix A of [10] (see also [1, Prop. 6.3]). In most cases, two parabolic subgroups are conjugate if and only if they have the same type. However, in type D_{2m} there are two conjugacy classes of parabolic subgroups of type $A_{k_1} \times A_{k_2} \times \cdots \times A_{k_r}$

with all k_i odd so that $2m = \sum(k_i+1)$ and in type E_7 there are two classes of parabolic subgroups for each of the types A_1^3, A_1A_3, and A_5.

For a given W, classifying the maximal rank reflection subgroups of W up to conjugacy amounts to listing, up to conjugacy in W, all subsets Y of T such that

$$\langle Y\rangle \cap T = Y \qquad \text{and} \qquad \mathrm{Fix}(Y) = \{0\}.$$

For a Coxeter group of small rank (including the non-crystallographic types H_3 and H_4) these sets can be systematically enumerated. As described below, for crystallographic Coxeter groups, that is, Weyl groups, using the algorithm of Borel–De Siebenthal [3] is computationally more efficient than classifying subsets of T.

By a *root system* in E we mean a reduced root system in the sense of Bourbaki [4, Ch. VI]. Suppose Φ is a root system in E. The Weyl group of Φ, $W(\Phi)$, is the group of linear transformations of E generated by the reflections through the hyperplanes orthogonal to the roots in Φ. The *dual* of Φ is the root system $\tilde\Phi = \{\, \frac{1}{|\alpha|^2}\alpha \mid \alpha \in \Phi\}$. Note that $W(\Phi) = W(\tilde\Phi)$. By a *Weyl group* or a *crystallographic Coxeter group* we mean the Weyl group of a root system in E.

Suppose that $W = W(\Phi) = W(\tilde\Phi)$ is a Weyl group. We may extract a classification of the maximal rank reflection subgroups of W from the arguments in [5]. Each maximal rank reflection subgroup of W is again a Weyl group and thus is the Weyl group of a maximal rank subsystem of Φ or $\tilde\Phi$. By work of Dynkin, two maximal rank subsystems are isomorphic if and only if they are equivalent under the action of W; see [5, Prop. 32] or [4, Ch. VI, §4, ex. 4]. By the classification of root systems, two root systems are isomorphic if and only if they have the same Dynkin diagram. We have already observed that a root system and its dual have the same Weyl group. Thus, the conjugacy classes of maximal rank reflection subgroups of W are in one-one correspondence with the set of Coxeter graphs arising from Dynkin diagrams of maximal rank subsystems of Φ and $\tilde\Phi$.

The Borel–De Siebenthal algorithm produces all maximal rank subsystems of Φ and $\tilde\Phi$ as follows (see [5, p. 8]).

(1) Add a node to the Dynkin diagram of Φ corresponding to the negative of the highest root of Φ. Take the extended Dynkin diagram and remove one node in all possible ways.

(2) Add a node to the Dynkin diagram of $\tilde\Phi$ corresponding to the negative of the highest root of $\tilde\Phi$. Take the extended Dynkin diagram and remove one node in all possible ways.

(3) Repeat steps (1) and (2) with each of the resulting Dynkin diagrams until no new diagrams appear.

This algorithm does not apply to the non-crystallographic groups $W(H_3)$, $W(H_4)$ and $W(I_2(m))$, but these groups are sufficiently small that all relevant information can be calculated directly

We now turn to the map γ which assigns to a given conjugacy class of reflection subgroups of W the conjugacy class of its Coxeter elements.

Recall [4, Ch. V, §6, no. 1] that a *Coxeter element* in W is the product of the elements of some Coxeter generating set of W taken in some order. All Coxeter elements of W are conjugate in W.

Suppose that R is a reflection subgroup of W. Then R is a Coxeter group and so we may consider Coxeter elements in R. If c is a Coxeter element in R and w is in W, it is easy to see that c^w is a Coxeter element in R^w. Thus, conjugate reflection subgroups of W have conjugate Coxeter elements and the map γ is well-defined.

The proof of Theorem 1.1 follows immediately from the classification of reflection subgroups and the explicit computation of the map γ in Theorems 3.1 and 3.3 and Tables 3 - 5. It would be interesting to have a conceptual explanation of why the single exception occurs in Theorem 1.1. More generally, from Theorems 3.1, 3.2, and 3.3 along with Tables 3 - 9, we derive necessary and sufficient conditions for the map γ to be injective.

Theorem 2.2. *Suppose that W is irreducible and not of type E_8. Then the map $\gamma\colon \mathcal{R} \to \mathcal{C}$ is injective if and only if T is a single conjugacy class in W.*

If W is of type E_8, then γ is not injective: the conjugacy classes of reflection subgroups of types A_1A_7 and A_3D_5 both map to the same conjugacy class of elements of W.

Hence, we conclude that the map γ is injective if and only if W has type A_n, D_n, E_6, E_7, H_3, H_4 and $I_2(m)$ with m odd. Moreover, it follows from the computations in §3 - §5 that, if W is irreducible, then γ is surjective when W has type A_n, B_n, or G_2. Notice that when the map γ is surjective, every conjugacy class of W contains a representative that is a Coxeter element in some reflection subgroup of W.

It is easy to see that $\mathrm{A}(X^w) = \mathrm{A}(X)^w$ when $X \subseteq W$ and $w \in W$. In particular, conjugate reflection subgroups have conjugate parabolic closures. Similarly, conjugate elements in W have conjugate parabolic closures. In particular, if $c \in R$ and $c' \in R'$ are Coxeter elements in reflection subgroups R and R', and c and c' are conjugate in W, then $\mathrm{A}(c)$ and $\mathrm{A}(c')$ are conjugate in W.

Lemma 2.3. *Suppose R is a reflection subgroup of W and $x \in R$ is not contained in any proper parabolic subgroup of R. Then $\mathrm{A}(x) = \mathrm{A}(R)$. Consequently, if V is a subspace of E and c is a Coxeter element of R that is conjugate to an element of W_V, then R is conjugate to a subgroup of W_V.*

Proof. It is shown in [5, §2] that $\mathrm{Fix}(x) = \mathrm{Fix}(R)$. It follows immediately that $\mathrm{A}(x) = \mathrm{A}(R)$.

For the second statement, let $w \in W$ be such that c^w is in W_V. Then $\mathrm{A}(c^w) \subseteq W_V$ and so $R^w \subseteq \mathrm{A}(R)^w = \mathrm{A}(c)^w = \mathrm{A}(c^w) \subseteq W_V$. □

Now suppose that R and R' are reflection subgroups of W containing Coxeter elements c and c' respectively. Then, if c and c' are conjugate in W, $\mathrm{A}(R)$ and $\mathrm{A}(R')$ are conjugate.

In other words, even if γ is not injective, reflection subgroups with non-conjugate parabolic closures must have non-conjugate Coxeter elements. This observation shows that conjugacy classes containing Coxeter elements of reflection subgroups are separated by the parabolic closures of reflection subgroups that contain them.

Note that Lemma 2.3 generalizes [13, Lem. 7] which is the special case of Lemma 2.3 when R is a parabolic subgroup of W. In the same way, Theorem 1.1 generalizes the forward implication of [11, Lem. 3.5].

3. The classical Weyl groups

A *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is a non-increasing finite sequence of positive integers $\lambda_1 \geq \dots \geq \lambda_k > 0$. The integers λ_i are called the *parts* of the partition λ. If $\sum_{i=1}^k \lambda_i = n$, then λ is a partition of n and we write $\lambda \vdash n$. The unique partition of $n = 0$ is the *empty partition*, denoted by $\varnothing$. We denote by $\ell(\lambda) = k$ the length of the partition $\lambda = (\lambda_1, \dots, \lambda_k)$, e.g., $\ell(\varnothing) = 0$. A partition of n is *even*, if all its parts are even, i.e., if it has the form $\lambda = (2\mu_1, \dots, 2\mu_k)$ for some partition μ of $n/2$. The *join* $\lambda^1 \cup \lambda^2$ of two partitions $\lambda^1 \vdash n_1$ and $\lambda^2 \vdash n_2$ is the partition of $n_1 + n_2$ consisting of the parts of both λ^1 and λ^2, suitably arranged. The *sum* of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ and an integer m is the partition $\lambda + m = (\lambda_1 + m, \dots, \lambda_k + m)$. We write $\lambda > m$ if $\lambda_i > m$ for all parts λ_i of λ. Note that, vacuously, $\varnothing > m$ for all m.

The symmetric group $\mathfrak{S}_n$ on n points is a Coxeter group of type A_{n-1} with Coxeter generators $s_i = (i, i+1)$, $i = 1, \dots, n-1$. The *cycle type* of $w \in \mathfrak{S}_n$ is the partition λ of n which has the lengths of the cycles of w on $\{1, \dots, n\}$ as its parts (here a fixed point contributes a cycle of length 1). Of course, two permutations in $\mathfrak{S}_n$ are conjugate if and only if they have the same cycle type. The next theorem is well-known, and can easily be deduced from Bourbaki [4, Ch. VI, §4, ex. 4].

Theorem 3.1. *Let W be a Coxeter group of type A_n. Then every reflection subgroup of W is a parabolic subgroup. Moreover, the map γ from conjugacy classes of reflection subgroups to conjugacy classes of W is a bijection. Both sets are in one-to-one correspondence with the set of all partitions of n.*

An *r-partition* is a sequence $\lambda = (\lambda^1, \dots, \lambda^r)$ of r partitions $\lambda^1, \dots, \lambda^r$. We say that λ is an r-partition of the integer n, and write $\lambda \vdash^r n$, if $\lambda^1 \cup \dots \cup \lambda^r \vdash n$. We call λ a *double partition* if $r = 2$, and a *triple partition* if $r = 3$.

The Coxeter group $W(B_n)$ acts faithfully as a group of signed permutations on the set of long roots $\{\pm e_i \mid i = 1, \dots, n\}$, permuting the lines $\langle e_i \rangle$, $i = 1, \dots, n$. A cycle of w in $W(B_n)$ is either *positive* or *negative*, depending on whether the number of positive roots e_i with $\langle e_i \rangle$ in the cycle that are mapped to negative roots is even or odd. The *cycle type* of w in $W(B_n)$ is a double partition $\lambda = (\lambda^1, \lambda^2)$ of n, where λ^1 records the lengths of the positive cycles of w and λ^2 records the lengths of the negative cycles. Again, two elements of $W(B_n)$ are conjugate if and only if they have the same cycle type and, in this way, the double partitions of n naturally parametrize the conjugacy classes of $W(B_n)$.

According to [10, Prop. 2.3.10], the parabolic subgroups of $W(B_n)$ are of the form $W(B_{n-m}) \times \prod_i W(A_{\lambda_i - 1})$, one conjugacy class for each partition $\lambda \vdash m$, $0 \leq m \leq n$. By Borel–De Siebenthal, the maximal rank reflection subgroups of $W(B_n)$ are of type $\prod_i W(B_{\lambda^1_i}) \times \prod_i W(D_{\lambda^2_i})$, one class for each double partition $\lambda \vdash^2 n$ with $\lambda^2 > 1$ (or $\lambda^2 = \varnothing$). It follows that the reflection subgroups of $W(B_n)$ are direct products of Coxeter groups of types A, B and D, and their classes are naturally labeled by triple partitions of n.

Theorem 3.2. *Let W be a Coxeter group of type B_n, $n \geq 2$. Then the conjugacy classes of reflection subgroups of W are represented by*

$$\{W_\lambda \mid \lambda \vdash^3 n,\ \lambda^3 > 1\},$$

where $W_\lambda = \prod_i W(A_{\lambda^1_i - 1}) \times \prod_i W(B_{\lambda^2_i}) \times \prod_i W(D_{\lambda^3_i})$. The parabolic closure of W_λ has type $W(B_{n-m}) \times \prod_i W(A_{\lambda^1_i - 1})$, where $\lambda^1 \vdash m$. The Coxeter elements of W_λ have cycle type $(\lambda^1, \lambda^2 \cup (\lambda^3 - 1) \cup 1^{\ell(\lambda^3)})$. In particular, the map $\gamma \colon \mathcal{R} \to \mathcal{C}$ is surjective, but not injective.

We illustrate the classification in type B_n in Table 1 below, where we list all conjugacy classes of reflection subgroups of $W(B_5)$ according to Theorem 3.2. Clearly, it follows from the data in Table 1 that γ is not injective in this case.

The Coxeter group $W(D_n)$ is a normal subgroup of index 2 in $W(B_n)$, and as such it is a union of $W(B_n)$-conjugacy classes of elements. In fact, the class of elements of cycle type $\lambda = (\lambda^1, \lambda^2)$ is contained in $W(D_n)$ if and only if $\ell(\lambda^2)$ is even, and it is a single conjugacy class in $W(D_n)$, unless $\lambda^2 = \varnothing$ and λ^1 is even. In the latter case, the $W(B_n)$-class splits into two $W(D_n)$-classes, labelled $(\lambda^1, +)$ and $(\lambda^1, -)$. In this way, the conjugacy classes of $W(D_n)$ are parametrized by certain double partitions of n.

According to [10, Prop. 2.3.13], $W(D_n)$ has three distinct kinds of parabolic subgroups: one class of subgroups of type $W(D_{n-m}) \times \prod_i W(A_{\lambda_i - 1})$ for each partition $\lambda \vdash m$, $0 \leq m \leq n-2$, two classes of subgroups of type $\prod_i W(A_{\lambda_i - 1})$ for each even partition $\lambda \vdash n$, and one class of subgroups of type $\prod_i W(A_{\lambda_i - 1})$ for each non-even partition $\lambda \vdash n$. By Borel–De Siebenthal, the maximal rank reflection subgroups of $W(D_n)$ are of type $\prod_i W(D_{\lambda_i})$, one class for each partition $\lambda \vdash n$ with $\lambda > 1$. It follows that reflection subgroups of $W(D_n)$ are direct products of Coxeter groups of types A and D, and their classes are naturally labeled by double partitions of n. This yields the following classification of the conjugacy classes of reflection subgroups of $W(D_n)$, in terms of double partitions of n.

Theorem 3.3. *Let W be a Coxeter group of type D_n, $n \geq 4$. Then the conjugacy classes of reflection subgroups of W are represented by*

$$\{W_\lambda \mid \lambda \vdash^2 n,\ \lambda^2 > 1\}$$

if n is odd, and by

$$\{W_\lambda \mid \lambda \vdash^2 n,\ \lambda^2 > 1 \text{ and } \lambda^1 \text{ non-even in case } \lambda^2 = \varnothing\} \cup \{W^\pm_\lambda \mid \lambda \vdash n \text{ and } \lambda \text{ even}\}$$

if n is even, where $W_\lambda = \prod_i W(A_{\lambda^1_i - 1}) \times \prod_i W(D_{\lambda^2_i})$ and $W^\epsilon_\lambda = \prod_i W(A_{\lambda_i - 1})$, where $\epsilon = \pm$. The parabolic closure of W_λ has type $W(D_{n-m}) \times \prod_i W(A_{\lambda^1_i - 1})$, where $\lambda^1 \vdash m$; the parabolic closure of W^ϵ_λ is W^ϵ_λ itself. The Coxeter elements of W_λ have cycle type $(\lambda^1, (\lambda^2 - 1) \cup 1^{\ell(\lambda^2)})$;

the Coxeter elements of W_λ^ϵ have cycle type (λ, ϵ). In particular, the map $\gamma\colon \mathcal{R} \to \mathcal{C}$ is injective, but not surjective.

We illustrate the classification in type D_n from Theorem 3.3 for $n = 6$ in Table 2 below.

4. The exceptional Weyl groups

For the exceptional Weyl groups all results are obtained by following the recursive procedure outlined in §2, using the Borel–De Siebenthal algorithm for the various factors of each standard parabolic subgroup of W. The calculations were carried out with the use of GAP [12] and CHEVIE [9]. Here the conjugacy classes of the elements in W are labeled as in Carter [5].

In Tables 3 - 7 we list all reflection subgroups in case W is of exceptional type up to conjugacy. In the cases when W has only a single class of reflections, it is readily checked that γ is injective, as required for Theorem 1.1.

Table 5 contains the results for $W(E_8)$. Here the two maximal rank reflection subgroups of types A_1A_7 and A_3D_5 have Coxeter elements that are conjugate in W. Hence γ is not injective.

In Tables 6 and 7 we list all conjugacy classes of reflection subgroups of $W(F_4)$ and $W(G_2)$, respectively. In both instances we see that γ is not injective.

5. The non-crystallographic cases

As in the exceptional cases, the non-crystallographic instances were computed using GAP [12] and CHEVIE [9]. The Borel–De Siebenthal algorithm does not apply, but these groups are sufficiently small that all the relevant information can be calculated directly.

In Tables 8 and 9 we list all conjugacy classes of reflection subgroups of $W(H_3)$ and $W(H_4)$, respectively. Here we see that γ is injective in both cases. The labeling of the conjugacy classes is the one used by CHEVIE.

The reflection subgroups of the dihedral group $W(I_2(m))$ can be described as follows.

Theorem 5.1. *Let W be of type $I_2(m)$, $m = 5$ or $m > 6$.*

(1) *If m is odd then the classes of reflection subgroups of W are of types $\varnothing$, A_1, and $I_2(d)$ where $d > 1$ is a divisor of m. The map γ is injective, but not surjective.*

(2) *If m is even then the classes of reflection subgroups of W are of types $\varnothing$, A_1, $\tilde{A}_1$, $I_2(d)$ where $d > 1$ is a divisor of m and $\tilde{I}_2(d)$ where $2d > 2$ is a divisor of m. The map γ is neither injective nor surjective.*

The subgroups of a dihedral group are determined by a straightforward computation. The theorem follows by filtering out those subgroups that are generated by reflections.

6. Tables

In Tables 1 - 9 we present the classification of the reflection subgroups of W in various cases. The tables provide the following information. In the first column of each table we list the types of the reflection subgroups R of W. In the second column in Tables 1 and 2 we also give the partition representing R according to Theorems 3.2 and 3.3, respectively. The next two columns give the cardinality of R and the cardinality of the class $[R]$ of R (that is, $|W : N_W(R)|$). Finally, in the last column we list the image of γ, i.e. the class $[c]$ of a Coxeter element c of R in W. For the classical types, conjugacy classes are labeled by cycle type. For the exceptional types, conjugacy classes are labeled as in Carter's classification [5].

Conjugacy classes of reflection subgroups with distinct parabolic closures are separated by horizontal lines. For a given parabolic subgroup P of W, the row for P is preceded by a horizontal line and followed by the rows for reflection subgroups R of W with $\mathrm{A}(R) = P$.

Table 1. Reflection subgroups of $W(B_5)$.

Type of R	λ	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
$\varnothing$	$1^5..$	1	1	$1^5.$
B_1	$1^4.1.$	2	5	$1^4.1$
A_1	$21^3..$	2	20	$21^3.$
B_1A_1	$21^2.1.$	4	60	$21^2.1$
A_1^2	$2^21..$	4	60	$2^21.$
A_2	$31^2..$	6	40	$31^2.$
B_2	$1^3.2.$	8	10	$1^3.2$
B_1^2	$1^3.1^2.$	4	10	$1^3.1^2$
D_2	$1^3..2$	4	10	$1^3.1^2$
$B_1A_1^2$	$2^2.1.$	8	60	$2^2.1$
B_1A_2	$31.1.$	12	80	31.1
A_1A_2	$32..$	12	80	$32.$
B_2A_1	$21.2.$	16	60	21.2
$B_1^2A_1$	$21.1^2.$	8	60	21.1^2
D_2A_1	$21..2$	8	60	21.1^2
A_3	$41..$	24	40	$41.$
B_3	$1^2.3.$	48	10	$1^2.3$
B_1B_2	$1^2.21.$	16	30	$1^2.21$
D_3	$1^2..3$	24	10	$1^2.21$
D_2B_1	$1^2.1.2$	8	30	$1^2.1^3$
B_1^3	$1^2.1^3.$	8	10	$1^2.1^3$
B_2A_2	$3.2.$	48	40	3.2
$B_1^2A_2$	$3.1^2.$	24	40	3.1^2
D_2A_2	$3..2$	24	40	3.1^2
B_1A_3	$4.1.$	48	40	4.1
B_3A_1	$2.3.$	96	20	2.3
$B_1B_2A_1$	$2.21.$	32	60	2.21
D_3A_1	$2..3$	48	20	2.21
$D_2B_1A_1$	$2.1.2$	16	60	2.1^3
$B_1^3A_1$	$2.1^3.$	16	20	2.1^3

Type of R	λ	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
A_4	$5..$	120	16	$5.$
B_4	$1.4.$	384	5	1.4
B_1B_3	$1.31.$	96	20	1.31
B_2^2	$1.2^2.$	64	15	1.2^2
D_4	$1..4$	192	5	1.31
D_3B_1	$1.1.3$	48	20	1.21^2
D_2B_2	$1.2.2$	32	30	1.21^2
$B_1^2B_2$	$1.21^2.$	32	30	1.21^2
$D_2B_1^2$	$1.1^2.2$	16	30	1.1^4
B_1^4	$1.1^4.$	16	5	1.1^4
D_2^2	$1..2^2$	16	15	1.1^4
B_5	$.5.$	3840	1	$.5$
B_1B_4	$.41.$	768	5	$.41$
B_2B_3	$.32.$	384	10	$.32$
D_5	$..5$	1920	1	$.41$
D_4B_1	$.1.4$	384	5	$.31^2$
D_3B_2	$.2.3$	192	10	$.2^21$
D_2B_3	$.3.2$	192	10	$.31^2$
$B_1^2B_3$	$.31^2.$	192	10	$.31^2$
$B_1B_2^2$	$.2^21.$	128	15	$.2^21$
$D_3B_1^2$	$.1^2.3$	96	10	$.21^3$
$D_2B_1B_2$	$.21.2$	64	30	$.21^3$
$B_1^3B_2$	$.21^3.$	64	10	$.21^3$
$D_2B_1^3$	$.1^3.2$	32	10	$.1^5$
$D_2^2B_1$	$.1.2^2$	32	15	$.1^5$
D_2D_3	$..23$	96	10	$.21^3$
B_1^5	$.1^5.$	32	1	$.1^5$

Table 2. Reflection subgroups of $W(D_6)$.

Type of R	λ	$\|R\|$	$\|[R]\|$	Class
$\varnothing$	$1^6.$	1	1	$1^6.$
A_1	$21^4.$	2	30	$21^4.$
D_2	$1^4.2$	4	15	$1^4.1^2$
A_1^2	$2^21^2.$	4	180	$2^21^2.$
A_2	$31^3.$	6	80	$31^3.$
A_1^3	$2^3.+$	8	60	$2^3.+$
A_1^3	$2^3.-$	8	60	$2^3.-$
D_2A_1	$21^2.2$	8	180	$21^2.1^2$
A_1A_2	$321.$	12	480	$321.$
D_3	$1^3.3$	24	20	$1^3.21$
A_3	$41^2.$	24	120	$41^2.$
$D_2A_1^2$	$2^2.2$	16	180	$2^2.1^2$
D_2A_2	31.2	24	240	31.1^2
A_2^2	$33.$	36	160	$33.$
D_3A_1	21.3	48	120	21.21
A_1A_3	$42.+$	48	120	$42.+$
A_1A_3	$42.-$	48	120	$42.-$

Type of R	λ	$\|R\|$	$\|[R]\|$	Class
A_4	$51.$	120	96	$51.$
D_4	$1^2.4$	192	15	$1^2.31$
D_2^2	$1^2.2^2$	16	45	$1^2.1^4$
D_2A_3	4.2	96	120	4.1^2
D_3A_2	3.3	144	80	3.21
D_4A_1	2.4	384	30	2.31
$D_2^2A_1$	2.2^2	32	90	2.1^4
A_5	$6.+$	720	16	$6.+$
A_5	$6.-$	720	16	$6.-$
D_5	1.5	1920	6	1.41
D_2D_3	1.32	96	60	1.21^3
D_6	$.6$	23040	1	$.51$
D_2D_4	$.42$	768	15	$.31^3$
D_3^2	$.3^2$	576	10	$.2^21^2$
D_2^3	$.2^3$	64	15	$.1^6$

Table 3. Reflection subgroups of $W(E_6)$.

Type of R	$\|R\|$	$\|[R]\|$	Class
$\varnothing$	1	1	1
A_1	2	36	A_1
A_1^2	4	270	$2A_1$
A_2	6	120	A_2
A_1^3	8	540	$3A_1$
A_1A_2	12	720	A_2+A_1
A_3	24	270	A_3
$A_1^2A_2$	24	1080	A_2+2A_1
A_2^2	36	120	$2A_2$
A_1A_3	48	540	A_3+A_1
A_4	120	216	A_4

Type of R	$\|R\|$	$\|[R]\|$	Class
D_4	192	45	D_4
A_1^4	16	135	$4A_1$
$A_1A_2^2$	72	360	$2A_2+A_1$
A_1A_4	240	216	A_4+A_1
A_5	720	36	A_5
D_5	1920	27	D_5
$A_1^2A_3$	96	270	A_3+2A_1
E_6	51840	1	E_6
A_1A_5	1440	36	A_5+A_1
A_2^3	216	40	$3A_2$

Table 4. Reflection subgroups of $W(E_7)$.

Type of R	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
$\varnothing$	1	1	1
A_1	2	63	A_1
A_1^2	4	945	$2A_1$
A_2	6	336	A_2
A_1^3	8	315	$3A_1'$
A_1^3	8	3780	$3A_1''$
A_1A_2	12	5040	A_2+A_1
A_3	24	1260	A_3
A_1^4	16	3780	$4A_1'$
$A_1^2A_2$	24	15120	A_2+2A_1
A_2^2	36	3360	$2A_2$
A_1A_3	48	1260	A_3+A_1'
A_1A_3	48	7560	A_3+A_1''
A_4	120	2016	A_4
D_4	192	315	D_4
A_1^4	16	945	$4A_1''$
$A_1^3A_2$	48	5040	A_2+3A_1
$A_1A_2^2$	72	10080	$2A_2+A_1$
$A_1^2A_3$	96	7560	A_3+2A_1'
A_2A_3	144	5040	A_3+A_2
A_1A_4	240	6048	A_4+A_1
A_1D_4	384	945	D_4+A_1
A_1^5	32	2835	$5A_1$

Type of R	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
A_5	720	336	A_5'
A_5	720	1008	A_5''
D_5	1920	378	D_5
$A_1^2A_3$	96	3780	A_3+2A_1''
$A_1A_2A_3$	288	5040	$A_3+A_2+A_1$
A_2A_4	720	2016	A_4+A_2
A_1A_5	1440	1008	A_5+A_1'
A_1D_5	3840	378	D_5+A_1
$A_1^3A_3$	192	3780	A_3+3A_1
A_6	5040	288	A_6
D_6	23040	63	D_6
$A_1^2D_4$	768	945	D_4+2A_1
A_3^2	576	630	$D_4(a_1)+2A_1$
A_1^6	64	945	$6A_1$
E_6	51840	28	E_6
A_1A_5	1440	1008	A_5+A_1''
A_2^3	216	1120	$3A_2$
E_7	2903040	1	E_7
A_1D_6	46080	63	D_6+A_1
A_7	40320	36	A_7
A_2A_5	4320	336	A_5+A_2
$A_1A_3^2$	1152	630	$2A_3+A_1$
$A_1^3D_4$	1536	315	D_4+3A_1
A_1^7	128	135	$7A_1$

Table 5. Reflection subgroups of $W(E_8)$.

Type of R	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
$\varnothing$	1	1	1
A_1	2	120	A_1
A_1^2	4	3780	$2A_1$
A_2	6	1120	A_2
A_1^3	8	37800	$3A_1$
A_1A_2	12	40320	A_2+A_1
A_3	24	7560	A_3
A_1^4	16	113400	$4A_1''$
$A_1^2A_2$	24	302400	A_2+2A_1
A_2^2	36	67200	$2A_2$
A_1A_3	48	151200	A_3+A_1
A_4	120	24192	A_4
D_4	192	3150	D_4
A_1^4	16	9450	$4A_1'$
$A_1^3A_2$	48	604800	A_2+3A_1
$A_1A_2^2$	72	403200	$2A_2+A_1$
$A_1^2A_3$	96	453600	A_3+2A_1''
A_2A_3	144	302400	A_3+A_2
A_1A_4	240	241920	A_4+A_1
A_1D_4	384	37800	D_4+A_1
A_1^5	32	113400	$5A_1$
A_5	720	40320	A_5
D_5	1920	7560	D_5
$A_1^2A_3$	96	75600	A_3+2A_1'
$A_1^2A_2^2$	144	604800	$2A_2+2A_1$
$A_1A_2A_3$	288	604800	$A_3+A_2+A_1$
$A_1^2A_4$	480	362880	A_4+2A_1
A_3^2	576	151200	$2A_3''$
A_2A_4	720	241920	A_4+A_2
A_2D_4	1152	50400	D_4+A_2
$A_1^4A_2$	96	151200	A_2+4A_1
A_1A_5	1440	120960	A_5+A_1''
A_1D_5	3840	45360	D_5+A_1
$A_1^3A_3$	192	453600	A_3+3A_1
A_6	5040	34560	A_6
D_6	23040	3780	D_6
$A_1^2D_4$	768	56700	D_4+2A_1
A_3^2	576	37800	$2A_3'$
A_1^6	64	56700	$6A_1$
E_6	51840	1120	E_6
A_1A_5	1440	40320	A_5+A_1'
A_2^3	216	44800	$3A_2$
$A_1A_2A_4$	1440	241920	A_4+A_2+1
A_3A_4	2880	120960	A_4+A_3
A_1A_6	10080	34560	A_6+A_1
A_2D_5	11520	30240	D_5+A_2
$A_1^2A_2A_3$	576	302400	$A_3+A_2+2A_1$
A_7	40320	8640	A_7''
A_1E_6	103680	3360	E_6+A_1
$A_1^2A_5$	2880	120960	A_5+2A_1
$A_1A_2^3$	432	134400	$3A_2+A_1$
D_7	322560	1080	D_7
$A_1^2D_5$	7680	22680	D_5+2A_1
A_3D_4	4608	37800	D_4+A_3
$A_1^4A_3$	384	113400	A_3+4A_1
E_7	2903040	120	E_7
A_1D_6	46080	7560	D_6+A_1
A_7	40320	4320	A_7'
A_2A_5	4320	40320	A_5+A_2
$A_1A_3^2$	1152	75600	$2A_3+A_1$
$A_1^3D_4$	1536	37800	D_4+3A_1
A_1^7	128	16200	$7A_1$
E_8	696729600	1	E_8
D_8	5160960	135	D_8
A_8	362880	960	A_8
A_1A_7	80640	4320	A_7+A_1
$A_1A_2A_5$	8640	40320	$A_5+A_2+A_1$
A_4^2	14400	12096	$2A_4$
A_3D_5	46080	7560	A_7+A_1
A_2E_6	311040	1120	E_6+A_2
A_1E_7	5806080	120	E_7+A_1
$A_1^2D_6$	92160	3780	D_6+2A_1
D_4^2	36864	1575	$2D_4$
$A_1^2A_3^2$	2304	37800	$2A_3+2A_1$
A_2^4	1296	11200	$4A_2$
$A_1^4D_4$	3072	9450	D_4+4A_1
A_1^8	256	2025	$8A_1$

Table 6. Reflection subgroups of $W(F_4)$.

Type of R	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
$\varnothing$	1	1	1
A_1	2	12	A_1
$\tilde{A}_1$	2	12	$\tilde{A}_1$
$A_1\tilde{A}_1$	4	72	$A_1+\tilde{A}_1$
A_2	6	16	A_2
$\tilde{A}_2$	6	16	$\tilde{A}_2$
B_2	8	18	B_2
$\tilde{A}_1^2$	4	18	$2A_1$
A_1^2	4	18	$2A_1$
$A_2\tilde{A}_1$	12	48	$A_2+\tilde{A}_1$
$A_1\tilde{A}_2$	12	48	$\tilde{A}_2+A_1$
B_3	48	12	B_3
A_3	24	12	A_3
$\tilde{A}_1B_2$	16	36	A_3
$A_1^2\tilde{A}_1$	8	36	$2A_1+\tilde{A}_1$
$\tilde{A}_1^3$	8	12	$2A_1+\tilde{A}_1$
C_3	48	12	C_3
$\tilde{A}_3$	24	12	B_2+A_1
A_1B_2	16	36	B_2+A_1
$A_1\tilde{A}_1^2$	8	36	$3A_1$
A_1^3	8	12	$3A_1$

Type of R	$\lvert R\rvert$	$\lvert[R]\rvert$	Class
F_4	1152	1	F_4
B_4	384	3	B_4
C_4	384	3	B_4
$\tilde{D}_4$	192	1	C_3+A_1
D_4	192	1	D_4
$\tilde{A}_1B_3$	96	12	D_4
A_1C_3	96	12	C_3+A_1
B_2^2	64	9	$D_4(a_1)$
$A_1\tilde{A}_3$	48	12	$A_3+\tilde{A}_1$
$A_3\tilde{A}_1$	48	12	$A_3+\tilde{A}_1$
$A_2\tilde{A}_2$	36	16	$A_2+\tilde{A}_2$
$\tilde{A}_1^2B_2$	32	18	$A_3+\tilde{A}_1$
$A_1^2B_2$	32	18	$A_3+\tilde{A}_1$
$A_1^2\tilde{A}_1^2$	16	18	$4A_1$
$\tilde{A}_1^4$	16	3	$4A_1$
A_1^4	16	3	$4A_1$

Table 7. Reflection subgroups of $W(G_2)$.

Type of R	$\|R\|$	$\|[R]\|$	Class
$\varnothing$	1	1	1
A_1	2	3	A_1
$\tilde{A}_1$	2	3	$\tilde{A}_1$
G_2	12	1	G_2
$\tilde{A}_2$	6	1	A_2
$A_1\tilde{A}_1$	4	3	$A_1+\tilde{A}_1$
A_2	6	1	A_2

Table 8. Reflection subgroups of $W(H_3)$.

Type of R	$\|R\|$	$\|[R]\|$	Class
$\varnothing$	1	1	1
A_1	2	15	2
A_1^2	4	15	4
A_2	6	10	5
$I_2(5)$	10	6	3
H_3	120	1	6
A_1^3	8	5	10

Table 9. Reflection subgroups of $W(H_4)$.

Type of R	$\|R\|$	$\|[R]\|$	Class
$\varnothing$	1	1	1
A_1	2	60	2
A_1^2	4	450	4
A_2	6	200	5
$I_2(5)$	10	72	3
A_1A_2	12	600	8
$I_2(5)A_1$	20	360	7
A_3	24	300	9

Type of R	$\|R\|$	$\|[R]\|$	Class
H_3	120	60	6
A_1^3	8	300	20
H_4	14400	1	11
H_3A_1	240	60	21
$I_2(5)^2$	100	36	26
A_4	120	60	27
A_2^2	36	100	32
D_4	192	25	25
A_1^4	16	75	34

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