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Edge-minimal graphs of exponent 2

A thesis submitted

by

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to

The School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland, Galway

in fulfillment of the requirements for the degree of

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August 2017

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Declaration

This thesis is presented in fulfillment of the requirements for the degree of Doctor of Philosophy. It is entirely my own work and has not been submitted to any other university or institute of higher education, or for any other academic award in this university. Where use has been made of the work of other people it has been fully acknowledged and referenced.

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Olga O’Mahony
August 2017
I would like to thank God, to whom all glory and honour belongs. I can do all things through Christ who strengthens me.

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Abstract

A simple undirected finite graph $G$ has the $\text{me}_2$-property if every pair of distinct vertices of $G$ is connected by a path of length 2, but this property does not survive the deletion of an edge. If $u$ and $v$ are adjacent vertices in an $\text{me}_2$-graph $G$, then either $u$ is the unique common neighbour in $G$ of $v$ and another vertex $w$, or $v$ is the unique common neighbour in $G$ of $u$ and another vertex $w'$. If both of these properties hold for every pair of adjacent vertices in $G$, then we say that $G$ has the strong-$\text{me}_2$-property. The $\text{me}_2$- and strong-$\text{me}_2$-properties can be viewed as relaxations of the friendship property, and this thesis investigates graphs with the $\text{me}_2$- and strong-$\text{me}_2$-properties. The relationship between these properties is discussed, and particular classes of graphs with these properties are described. We also discuss the behaviour of the $\text{me}_2$- and strong-$\text{me}_2$-properties under certain graph products.

It is shown that every graph of order $n$ is an induced subgraph of an $\text{me}_2$-graph of order at most $3n + 2$. The problem of which graphs can be embedded as induced subgraphs of strong-$\text{me}_2$-graphs is considered, and a construction for complete graphs is presented. The problem of embedding a given graph as an induced subgraph of an $\text{me}_2$-graph or strong-$\text{me}_2$-graph with no edges amongst the additional vertices is studied in detail for trees. Not all graphs can be embedded in this manner. This thesis initiates a study of edge-minimal graphs of exponent 2 and poses some open problems on this subject.
Chapter 1

Background

This chapter presents some background theory about graphs that will be relevant to the reading of this thesis. We introduce the concept of a graph, the adjacency matrix of a graph and some of the connections between the two. We discuss some properties of cycles of odd length and bipartite graphs. Since this thesis deals with particular classes of primitive graphs of exponent 2, we then focus on the concept of primitivity of a graph and the connection between primitive graphs and non-bipartite graphs. This chapter consists entirely of standard theory, and can be found in [4]. Further background information on graphs can be found in [3], [9], and [18].

Definition 1.0.1. A graph \( G = (V, E) \) consists of a finite set \( V \) of vertices and a set \( E \) of 2-element subsets of \( V \) called edges. A pair of vertices \( u \) and \( v \) are adjacent if \( \{u, v\} \in E \) and in this case we say \( uv \) is an edge of \( G \).

A graph as defined above is typically referred to as a simple undirected graph. In the wider context of graph theory, multiedge and directed graphs are also defined. However, all graphs considered in this thesis are simple and undirected and therefore we only define these. In order to give a visual representation of a graph, it is useful to think of a graph as a picture with dots used to represent the vertices of the graph and lines representing edges connecting adjacent vertices.

Example 1.0.2. A graph with 5 vertices.
We now present some standard terminology that will be needed to facilitate our discussion.

- A graph with \( n \) vertices is said to be of order \( n \).

- A walk in a graph \( G \) is a sequence \( v_0v_1v_2...v_n \) of vertices with \( v_{i-1}v_i \in E(G) \) for each \( i \). The length of a walk is its number of edges (counting repetitions). A closed walk is a walk that starts and ends at the same vertex. Note that if \( uv \) is an edge in \( G \), then the walk \( uvu \) is a closed walk of length 2. A path is a walk with no repeated vertices. A cycle is a closed walk in which the only repeated vertices are the first and the last.

- We will refer to a cycle of length 3 in a graph as a triangle.

- We say that a graph \( G \) is connected if for every pair of vertices \( u \) and \( v \) in \( G \), there exists a path in \( G \) between \( u \) and \( v \).

- The neighbours of a vertex \( v \) of a graph \( G \) are the vertices that are adjacent to \( v \). The degree of \( v \), \( \deg(v) \), is the number of neighbours of \( v \).

- An isolated vertex of a graph \( G \) is a vertex that has no neighbours in \( G \).

- A graph \( G \) is called regular if every vertex of \( G \) has the same degree.

- If there exists at least one path between two vertices \( u \) and \( v \) in a graph \( G \), then we can define the distance between \( u \) and \( v \), denoted \( d(u,v) \), as the number of edges in a shortest path from \( u \) to \( v \).

- The diameter of a connected graph \( G \), \( \text{diam}(G) \), is the maximum distance between a pair of vertices in \( G \).

- A subgraph of a graph \( G \) is a graph whose vertex set is a subset of \( V(G) \) and whose edge set is a subset of \( E(G) \). An induced subgraph of \( G \) is a subgraph \( H \) such that two vertices of \( H \) are adjacent in \( H \) if and only if they are adjacent in \( G \).
• Two graphs $G$ and $H$ are isomorphic, denoted by $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that any pair of vertices $u$ and $v$ in $G$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$.

**Definition 1.0.3.** Let $G$ be a graph with vertices $v_1, \ldots, v_n$. The adjacency matrix of $G$, with respect to this ordering of the vertices, is the $n \times n$ matrix $A$ where the rows and columns of $A$ are labelled by the vertices of $G$ and

$$A_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

From Definition 1.0.1 we can see that $A$ is a symmetric matrix, i.e. $A_{ij} = A_{ji}$ for all $i$ and $j$, and that $A$ has all zeros on the main diagonal.

In order to write down the adjacency matrix of a graph, we must give an ordering to the vertices of the graph. Below we present a graph and its corresponding adjacency matrix with respect to a particular ordering of the vertices.

**Example 1.0.4.** A graph and its adjacency matrix.

![Graph](image)

$$A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & 0 & 0 & 1 & 1 \\ v_2 & 0 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \\ v_5 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The adjacency matrix of a graph $G$ depends on the way that we order the vertices of $G$. Suppose $\sigma$ is a permutation in $S_n$ and let $A$ be the adjacency matrix of $G$ with respect to the ordering $v_1, \ldots, v_n$. Let $A'$ be the adjacency matrix of $G$ with respect to the ordering $v_{\sigma(1)}, \ldots, v_{\sigma(n)}$. Then $A'$ can be obtained from $A$ by first reordering the columns of $A$ by replacing column $j$ with column $\sigma(j)$. This is essentially multiplying $A$ on the right by the permutation matrix $P_\sigma$ in which each column $j$ has a 1 as its $\sigma(j)^{th}$-entry and 0 otherwise. Now we replace row $i$ in $A$ with row $\sigma(i)$, which means that we are multiplying $A$ on the left by $(P_\sigma)^T$. Thus $A$ and $A'$ represent the same graph if $A' = P^T A P$ for a permutation matrix $P$. 

3
We now recall an important relationship between walks in a graph and entries of powers of its adjacency matrix.

**Theorem 1.0.5.** Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$ and adjacency matrix $A$ and let $k$ be a positive integer. Then the $(i,j)$ entry of $A^k$ is the number of walks of length $k$ between $v_i$ and $v_j$ in $G$.

**Proof.** We will use induction on $k$ in order to prove this theorem.

**Base case:** When $k = 1$ the theorem is true by the definition of the adjacency matrix since an edge is a walk of length 1 between $v_i$ and $v_j$ in $G$, and the $(i,j)$ entry of $A$ is 1 if there is an edge between $v_i$ and $v_j$ and 0 otherwise.

**Inductive step:** Assume that the theorem is true for all positive integers up to $k - 1$. We note that

$$(A^k)_{ij} = \sum_{r=1}^{n} (A^{k-1})_{ir} A_{rj}$$

So now we need to show that this is the number of walks of length $k$ from $v_i$ to $v_j$ in $G$. By the induction hypothesis, $(A^{k-1})_{ir}$ is the number of walks of length $k - 1$ from $v_i$ to $v_r$ in $G$. We know that $A_{rj}$ is the number of walks of length 1 from $v_r$ to $v_j$ in $G$. For a vertex $v_r$ of $G$, we can consider the number of walks of length $k$ from $v_i$ to $v_j$ that have $v_r$ as their second-last vertex. If $v_r$ is adjacent to $v_j$ this is the number of walks of length $k - 1$ from $v_i$ to $v_r$. If $v_r$ is not adjacent to $v_j$, it is zero. Either way, the number is $(A^{k-1})_{ir} A_{rj}$, since $A_{rj}$ is either 1 or 0 depending on whether $v_r$ and $v_j$ are adjacent. So the total number of walks of length $k$ from $v_i$ to $v_j$ is the sum of $(A^{k-1})_{ir} A_{rj}$ over all vertices $v_r$ of $G$, which is $(A^k)_{ij}$. \qed

The following elementary observation will be used later.

**Lemma 1.0.6.** Let $G$ be a graph and $u, v$ be vertices of $G$. If a walk of length $m$ exists from $u$ to $v$ in $G$, then walks of length $m'$ exist for all $m'$ with $m' \geq m$ and $m' \equiv m \mod 2$.

**Proof.** Let $u = u_0 u_1 \ldots u_m = v$ be a walk of length $m$ between $u$ and $v$ in $G$. We can extend this walk by an even number of steps by going back and forth between two neighbours $u_i$ and $u_{i+1}$ in the walk, to get a walk of length $m' \geq m$ where $m' \equiv m \mod 2$. \qed
The powers of the adjacency matrix of a graph $G$ tell us about the number of walks between pairs of vertices of $G$. We now introduce the concept of a \textit{primitive} graph and its connection to matrices.

\textbf{Definition 1.0.7.} A graph $G$ is \textit{primitive} if there exists a positive integer $k$ with the property that for every pair of vertices $u$ and $v$ of $G$, there exists a walk of length $k$ from $u$ to $v$ in $G$. The least positive integer $k$ with this property is called the \textit{exponent} of $G$.

We provide an example of a family of graphs on $n$ vertices that are primitive and a family of graphs of even order that are not. We demonstrate these properties for the graph in each family when $n = 4$, but the reasoning applies to all graphs within the given family.

\textbf{Example 1.0.8.} The complete graphs $K_n$ comprise a family of graphs that are primitive of exponent 2. Below is $K_4$. It can be easily seen that there are paths of length two between every pair of vertices. While it is true that there exist paths of length 1 between every pair of distinct vertices, there are no loops in the complete graph and hence the exponent is not 1.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (2,0) -- (2,2) -- (0,2) -- cycle;
\end{tikzpicture}
\end{center}

\textbf{Example 1.0.9.} The cycles on $n$ vertices where $n$ is an even integer form a family of graphs which are not primitive. We take the example $C_4$ below. There exist walks of only odd length between vertex 1 and vertex 2 and walks of only even length between vertex 1 and vertex 4, hence there will never be walks of the same length between vertex 1 and 2, and 1 and 4. A similar argument works for all $C_n$ where $n = 2m$ for $m \in \mathbb{Z}$.

\begin{center}
1 \quad 2
\begin{tikzpicture}
\draw (0,0) -- (2,0) -- (2,2) -- (0,2) -- cycle;
\end{tikzpicture}
\end{center}

We remark that cycles of odd order are primitive, and this will be shown later in the chapter.
The concept of primitivity may be interpreted in terms of adjacency matrices. It follows from Theorem 1.0.5 that a graph $G$ is primitive of exponent $k$ if and only if its adjacency matrix $A$ has the property that $k$ is the least positive integer for which every entry of $A^k$ is positive. This motivates the following definition for non-negative square matrices. A matrix is non-negative if all of its entries are non-negative, and positive if all of its entries are positive.

**Definition 1.0.10.** A non-negative square matrix $A$ is called primitive if $A^k$ is positive for some positive integer $k$; the least such $k$ is called the exponent of $A$.

Thus a graph is primitive of exponent $k$ if and only if its adjacency matrix is a primitive matrix of exponent $k$.

The zero-nonzero patterns of positive integer powers of a non-negative matrix depend only on its zero-nonzero pattern, and not on the values of its positive entries. If $A$ is a non-negative matrix with zeros on its main diagonal and a symmetric pattern of positive entries, let $A'$ be the matrix obtained from $A$ by replacing every positive entry with 1. Then $A$ is primitive of exponent $k$ if and only if the graph whose adjacency matrix is $A'$ is primitive of exponent $k$. This interpretation may easily be extended to general non-negative matrices by allowing directed edges and loops.

The concept of primitivity for non-negative matrices is a convenient tool for proving the following lemma.

**Lemma 1.0.11.** Let $G$ be a primitive graph of exponent $k$. Then there exists a path of length $l$ between every pair of vertices of $G$, for every integer $l \geq k$.

**Proof.** Let $A$ be the adjacency matrix of $G$. Then $A^k$ is positive. Since $G$ is primitive it has no isolated vertex, hence every column of $A$ has at least one positive entry. Since every entry of $A^k$ is positive, it follows that $A^{k+1} = A^k A$ is positive also, and hence so are all subsequent powers of $A$. This completes the proof (by Theorem 1.0.5). 

We now present an important observation about cycles of odd length that will be referred to again at a later stage.

**Lemma 1.0.12.** Let $G$ be a graph that has a closed walk of odd length. Then $G$ has a cycle of odd length.
Proof. Let $C = uv_1...v_m u$ be a closed walk of shortest odd length in $G$. If $u = v_k$ for some $k$, then $uv_1...v_k$ and $v_kv_{k+1}...v_m u$ are closed walks, one of which has odd length shorter than that of $C$. This contradiction shows that none of the $v_k$ is equal to $u$. Since we may take any vertex of $C$ to be the start of $C$, the same argument shows that the vertices $u, v_1, ..., v_m$ are distinct, and that $C$ is a cycle of odd length. \qed

We now introduce bipartite graphs, and present an important characterization of bipartite graphs that will be used in our discussion of primitivity.

**Definition 1.0.13.** We say that a graph $G$ is **bipartite** if the vertex set of $G$ can be partitioned into two disjoint sets such that no two vertices within the same set are adjacent.

**Lemma 1.0.14.** Let $G$ be a graph. Then $G$ is bipartite if and only if $G$ has no cycle of odd length.

Proof. ($\implies$) Suppose $G$ is a bipartite graph. Then the vertex set of $G$ can be partitioned into two disjoint sets $A$ and $B$ such that the only adjacencies are between vertices in $A$ and vertices in $B$. Let $C$ be a cycle in $G$. Since vertices of $C$ are alternately in $A$ and in $B$, the length of $C$ must be even.

($\impliedby$) If $G$ is disconnected then it is bipartite only if all of its connected components are bipartite. So it is sufficient to prove the theorem for a connected graph. Suppose $G$ is connected and has no cycle of odd length. Let $u$ be a vertex in $G$ and partition the vertex set of $G$ as follows:

- Let $A$ be the set of vertices $v$ of $G$ such that $d(u, v)$ is even.
- Let $B$ be the set of vertices $w$ of $G$ such that $d(u, w)$ is odd.

We can see that $A$ and $B$ are disjoint, every vertex of $G$ is either in $A$ or $B$ and that $u \in A$. We show that no adjacencies exist among the vertices of $A$, and no adjacencies exist among the vertices of $B$.

Suppose two vertices $v_1$ and $v_2$ of $A$ are adjacent. Since $d(u, v_1)$ and $d(u, v_2)$ are even and $d(v_1, v_2)$ is odd, it follows that there is a closed walk of odd length in $G$ and by Lemma 1.0.12 there is a cycle of odd length in $G$, which contradicts the hypothesis. Hence no two vertices in $A$ are adjacent. The same argument applies to $B$. We conclude that $G$ is bipartite. \qed
Lemma 1.0.15. Let $G$ be a connected graph. If $G$ has a cycle of odd length, then for every pair of vertices $u, v$ in $G$, there exist walks of even and odd lengths between $u$ and $v$ in $G$.

Proof. Suppose $G$ has a cycle $C$ of odd length. We first show that for any vertex $u$ in $G$, there is a walk of odd length from $u$ to $u$ in $G$.

If $u$ is a vertex of $C$ then $C$ is a walk of odd length from $u$ to $u$. If $u$ is not in $C$, then let $W$ be a walk from $u$ to a vertex $x$ of $C$. Such a walk exists because $G$ is connected. We can construct a walk of odd length from $u$ to $u$ consisting of the walk $W$, followed by the cycle $C$ from $x$ to $x$, followed by the walk $W$ in the opposite direction. Therefore for any vertex $u$ of $G$, there exists a walk of odd length from $u$ to $u$ in $G$.

Now let $u$ and $v$ be vertices of $G$. Choose a shortest walk $P_{u,v}$ between $u$ and $v$ in $G$. We show that we can extend $P_{u,v}$ to a walk of the required parity.

- If $P_{u,v}$ is even, then we can extend $P_{u,v}$ to a walk of odd length by combining a walk of odd length from $u$ to $u$ with $P_{u,v}$.

- Similarly, if $P_{u,v}$ is odd, we can extend $P_{u,v}$ to a walk of even length by combining a walk of odd length from $u$ to $u$ with $P_{u,v}$.

We conclude that there exist walks of even and odd lengths between $u$ and $v$ in $G$. □

We observe that the cycles of even length in Example 1.0.9 are bipartite.

Theorem 1.0.16. Let $G$ be a connected graph. Then $G$ is primitive if and only if $G$ is non-bipartite.

Proof. ($\implies$) Let $G$ be a primitive graph. We know from Lemma 1.0.11 that in $G$, paths of both even and odd lengths must exist between all pairs of vertices and therefore $G$ is non-bipartite.

($\impliedby$) Let $G$ be a non-bipartite graph. Then by Lemma 1.0.14, $G$ has at least one cycle of odd length, and we know by Lemma 1.0.15 that for every pair of vertices in $G$ there exist walks of even and odd length between them. For every pair $u,v$ of vertices of $G$, let $t_{uv}$ be the length of some walk of odd length between $u$ and $v$ in $G$, and let $s_{uv}$ be the length of some walk of even length between $u$ and $v$ in $G$. Let $M$ be the maximum over all pairs $u$ and $v$ of the numbers $s_{uv}$ and $t_{uv}$. We show that for every pair of vertices $p$ and $q$ in $G$, there exists a walk of length $M$ between $p$ and $q$ in $G$. 


If $M$ is even, we can extend $s_{pq}$ to a walk of length $M$ by Lemma [1.0.6]. If $M$ is odd then we can extend $t_{pq}$ by an even number of steps to get a walk of length $M$ by Lemma [1.0.6]. We conclude that there exists a walk of length $M$ between every pair of vertices in $G$ and therefore $G$ is primitive.
Chapter 2

Introduction to the me$_2$-property

This thesis is concerned with a specific class of primitive graphs of exponent 2, namely those that are edge-minimal with this property. In this chapter, we present examples of such graphs, some of which arise frequently in other contexts in graph theory. We discuss the motivation for this research and make some preliminary observations on this class of graphs.

2.1 The me$_2$-property

A graph $G$ is primitive of exponent 2 if there exists a walk of length 2 between every pair of vertices in $G$. Equivalently, every pair of distinct vertices in $G$ must share a common neighbour and that every edge of $G$ belongs to at least one triangle in $G$. We observe that this means that $G$ must have order at least 3. We now present the minimal exponent 2 (me$_2$) property.

**Definition 2.1.1.** We say a graph $G$ has the me$_2$-property if $G$ is primitive of exponent 2, and deleting any edge in $G$ results in a graph that does not have exponent 2.

A graph $G$ has the me$_2$-property if and only if $G$ has exponent 2 and every edge of $G$ is part of the unique path of length 2 between some pair of vertices. In this case we refer to $G$ as an me$_2$-graph. If $uw$ and $vw$ are edges of $G$ (with $u \neq w$), we will say that $uvw$ is a unique 2-path in $G$ if $v$ is the only mutual neighbour of $u$ and $w$. 
We introduce the following terminology to aid us in our discussions of me$_2$-graphs, and present an example of an me$_2$-graph.

**Definition 2.1.2.** Let $G$ be a graph. We say that an edge $uv$ in $G$ is 2-required if there exists a vertex $w$ in $G$ such that $u$ is the unique common neighbour of $v$ and $w$ in $G$, or there exists a vertex $w'$ in $G$ such that $v$ is the unique common neighbour of $u$ and $w'$ in $G$.

To say that an edge $uv$ in a graph $G$ is 2-required means that either there exists a vertex $w$ for which $vuw$ is a unique 2-path in $G$, or there exists a vertex $w'$ for which $uvw'$ is a unique 2-path in $G$. Thus the edge $uv$ is required for the existence of a 2-path between some pair of vertices in $G$. It follows from Definition 2.1.2 that $G$ is an me$_2$-graph if and only if $G$ has exponent 2 and every edge of $G$ is 2-required.

**Example 2.1.3.** An me$_2$-graph $G$ on four vertices.

![Diagram of a me$_2$-graph](image)

$G$

It is straightforward to see that every pair of distinct vertices of $G$ has a common neighbour and therefore $G$ has exponent 2. The edge $v_1v_2$ is 2-required for $v_1$ to $v_4$, $v_1v_4$ is 2-required for $v_1$ to $v_2$, $v_2v_3$ is 2-required for $v_3$ to $v_4$, $v_2v_4$ is 2-required for $v_2$ to $v_3$ and $v_3v_4$ is 2-required for $v_3$ to $v_2$. Every edge of $G$ is 2-required and $G$ has exponent 2 therefore $G$ is an me$_2$-graph. Note that by symmetry, it is necessary only to check that $v_1v_2$ and $v_2v_4$ are 2-required in $G$.

The me$_2$-property can be interpreted in terms of non-negative matrices as follows. We say that a square non-negative matrix $A$ has the me$_2$-property if its pattern of zero and non-zero entries coincides with that of the adjacency matrix of an me$_2$-graph. In matrix terms this means that $A^2$ is positive, but that the replacement of any symmetric pair of positive entries of $A$ with zeros would result in a matrix whose square has at least one zero entry. If $A$ is an me$_2$-matrix, then by replacing all positive entries of $A$ with ones and leaving any zero entries of $A$ unchanged, we can obtain an me$_2$-matrix.
$A'$ such that $A'$ is the adjacency matrix of some $\text{me}_2$-graph $G$, since the zero-nonzero pattern of $(A')^k$ is the same as that of $A^k$ for any $k \in \mathbb{N}$. The $\text{me}_2$-property can be investigated in terms of matrices and/or graphs. This thesis focuses mainly on the $\text{me}_2$-property for graphs.

The $\text{me}_2$-property is defined in terms of graphs being *edge-minimal* of exponent 2. We say that a graph $G$ is *vertex-minimal* of exponent 2 if $G$ has exponent 2 and the deletion of any vertex of $G$ and its incident edges results in a graph that no longer has exponent 2. This is equivalent to the condition that $G$ has exponent 2 and every vertex of $G$ is the unique common neighbour of some pair of vertices in $G$. We observe that, when considering whether the deletion of a vertex leaves a graph of exponent 2, it is sufficient to consider walks of length 2 between neighbours of the vertex that has been deleted. We introduce the following notation and present an example of a graph of exponent 2 that is vertex-minimal but not edge-minimal.

Let $G$ be a graph and let $u, v$ and $w$ be vertices of $G$. We write $u = \text{ucn}(v, w)$ if $u$ is the unique common neighbour of $v$ and $w$ in $G$.

**Example 2.1.4.** A graph $G$ of exponent 2 that is vertex-minimal but not edge-minimal.

![Diagram of a graph](image)

It is straightforward to see that $G$ has exponent 2. We show that every vertex in $G$ is the unique common neighbour of some pair of vertices in $G$, but $G$ does not have the $\text{me}_2$-property.

We observe:

- $v_1 = \text{ucn}(v_2, v_3)$
Therefore, every vertex of $G$ is the unique common neighbour of some pair of vertices in $G$. However, the edge $v_4v_5$ is not 2-required in $G$ since $v_4$ has paths of length 2 to all vertices in $G$ via $v_3$ or $v_7$, and $v_5$ has paths of length 2 to all vertices in $G$ via $v_1$, $v_6$ or $v_7$. We conclude that $G$ does not have the me$_2$-property.

The focus of this thesis is on edge-minimal graphs of exponent 2, but there is potential for investigation of graphs that are vertex-minimal of exponent 2. We return in Chapter 4 to graphs that are both edge and vertex minimal of exponent 2, or have the \textit{double-me$_2$-property}.

We now discuss the motivation for investigating graphs with the me$_2$-property and how it relates to some well-known themes in graph theory.

### 2.2 Motivation

Primitive graphs and their corresponding adjacency matrices have been extensively studied. General background on primitive graphs and matrices can be found for example in [4].

A motivating factor for the study of me$_2$-graphs comes from the Friendship Theorem due to Paul Erdős, Alfred Rényi and Vera Sós [7] which states that if $G$ is a finite graph with the property that every pair of distinct vertices have exactly one common neighbour then $G$ has a vertex that is adjacent to all others. The condition that every pair of distinct vertices of $G$ have exactly one common neighbour is referred to as the
friendship property, and finite graphs with the friendship property are completely determined by the Friendship Theorem; such graphs are referred to as windmills. We write $W_r$ for the "windmill with $r$ blades", which has $2r + 1$ vertices and consists of $r$ triangles all having a single vertex in common and being otherwise disjoint. We present an example below.

**Example 2.2.1.** $W_3$ - the windmill with 3 blades

![Windmill with 3 blades](image)

We state the Friendship Theorem more formally below.

**Theorem 2.2.2.** *(Erdős-Rényi-Sós)*

Suppose that $G$ is a finite graph in which every pair of vertices have exactly one common neighbour. Then $G$ is a windmill.

The Friendship Theorem classifies all finite graphs with the friendship property, however it is not true that all infinite graphs with the friendship property are windmills (see [5]). One example of an infinite graph with the friendship property that is not a windmill arises as follows (see [11]).

Let $G_1 \cong C_5$ and denote by $V_1$ the vertex set of $G_1$. Let $G_2$ be the extension of $G_1$ whereby a common neighbour is adjoined to every pair of $V_1$ that do not have a common neighbour in $G_1$. We denote by $V_2$ the vertex set of $G_2$. We continue this process to obtain $G = \bigcup_{i=1}^{\infty} G_i$, where $G_i$ for $i = 2, \ldots, \infty$ has vertex set $V_i$ and $G_i$ is the extension of $G_{i-1}$ whereby a common neighbour is adjoined to every pair of vertices of $V_{i-1}$ that do not have a common neighbour in $G_{i-1}$. Since the only vertices that are added to $G_{i-1}$ are a single common neighbour for each pair of vertices in $V_{i-1}$ that don’t have a common neighbour in $G_{i-1}$, we do not have a pair of vertices in $G$ that has more than one common neighbour, and there is no vertex in $G$ that is adjacent to all others in $G$. Hence $G$ is an example of an infinite graph with the friendship property and $G$ is not an infinite windmill.
There are many proofs of the Friendship Theorem in the literature, both algebraic and combinatorial. Proofs of this theorem can be found in [7] or [10].

The windmills are examples of me_2-graphs. The friendship property requires a unique 2-path between every pair of distinct vertices. The me_2-property allows the possibility of multiple 2-paths between some pairs of distinct vertices, but requires unique 2-paths to be sufficiently plentiful that every edge belongs to one. Therefore, the me_2-property can be viewed as a relaxation of the friendship property. Example 2.1.3 shows that windmills are not the only examples of me_2-graphs having a vertex adjacent to all others, and we discuss such graphs shortly. We first introduce the following notation and definitions to aid us in our discussion.

Let \( G \) be a graph and \( u \) a vertex of \( G \). We denote by \( N_G(u) \) (or just \( N(u) \) if there is no risk of ambiguity) the neighbour set of \( u \) in \( G \).

**Definition 2.2.3.** A connected graph \( T \) is said to be a tree if there are no cycles in \( T \).

It follows from Definition 2.2.3 that every pair of vertices in a tree are connected by a unique path. A leaf is a vertex of degree 1 in a tree. A forest is a disjoint union of trees.

**Definition 2.2.4.** The star \( S_n \) of order \( n \geq 2 \) is the tree of order \( n \) with a vertex of degree \( n - 1 \).

It follows from Definition 2.2.4 that \( S_n \) has a vertex \( u \) that is adjacent to all other vertices in \( S_n \) and the degree of each of the neighbours of \( u \) in \( S_n \) is one. Therefore, the subgraph of \( S_n \) induced on \( N(u) \) is a null subgraph.

**Example 2.2.5.** The star on 9 vertices - \( S_9 \)

Stars play an important role in our discussion of me_2-graphs that have a vertex adjacent to all other vertices in the graph, and we discuss such me_2-graphs now.
Proposition 2.2.6. Suppose $G$ is an $me_2$-graph of order $n$ and let $u$ be a vertex of degree $n-1$ in $G$. Let $G_1$ be the subgraph induced on $N(u)$. Then every vertex of $G_1$ has degree at least 1 and $G_1$ cannot have two adjacent vertices whose degrees are both at least 2.

Proof. Since $G$ is an $me_2$-graph, every vertex of $G_1$ has a common neighbour with $u$. Suppose $G_1$ has two adjacent vertices whose degrees are both at least 2. It follows that the order of $G$ must be at least 4. Let $x_1, x_2, x_3, x_4$ be vertices of $G_1$ such that $\deg_{G_1}(x_2) \geq 2$, $\deg_{G_1}(x_3) \geq 2$ and $x_1x_2, x_2x_3, x_3x_4$ (with $x_1 = x_4$ possibly) are edges of $G_1$. The edge $x_2x_3$ must be 2-required in $G$. There exists a walk of length 2 from $x_2$ to $u$ via $x_1$ and from $x_2$ to any other vertex of $G_1$ via $u$, so $x_2x_3$ is not 2-required in $G$ for a walk of length 2 that begins at $x_2$. Similar reasoning shows that $x_2x_3$ is not 2-required in $G$ for any walk of length 2 beginning at $x_3$. We conclude that $x_2x_3$ is not 2-required in $G$ and we reach a contradiction. □

It follows from Proposition 2.2.6 that if any vertex of $G_1$ has degree at least 2, then all of its neighbours in $G_1$ have degree exactly one in $G_1$. This leads us to conclude that every component of $G_1$ is a star. We present the following theorem.

Theorem 2.2.7. Let $G$ be a graph of order $n \geq 3$ and suppose $G$ has a vertex $u$ of degree $n-1$. Then $G$ is an $me_2$-graph if and only if every component of the subgraph of $G$ induced on $N(u)$ is a star.

Proof. ($\implies$) This direction is immediate from Proposition 2.2.6.

($\impliedby$) Suppose that every component of the subgraph of $G$ induced on $N(u)$ is a star. Since $u$ is adjacent to all vertices in $G$, there exist paths of length 2 between all vertices in $N(u)$ via $u$. Since every component of the subgraph induced on $N(u)$ is a star, every vertex of $G$ has degree at least 2 in $G$ and therefore every vertex in $N(u)$ shares a common neighbour with $u$. Therefore, $G$ has exponent 2. We now show every edge of $G$ is 2-required.

There are two types of edge in $G$; edges incident with $u$ and edges incident with two neighbours of $u$. Let $uv$ be an edge of $G$. Since $v$ has a neighbour $w$ in $G$, other than $u$, and every component of the subgraph induced on $N(u)$ is a star, it follows that $u = \text{ucn}(v, w)$ in $G$ and therefore the edge $uv$ is 2-required for $v$ to $w$ in $G$. Now consider an edge $xy$ of $G$, not incident with $u$, such that $\deg(x) \geq \deg(y)$. Therefore, $\deg(y) = 2$
by Proposition 2.2.6 and the only neighbours of \( y \) in \( G \) are \( u \) and \( x \), and the edge \( xy \) is 2-required for \( y \) to \( u \) in \( G \). We conclude that \( G \) is an \( m_{e_2} \)-graph.

Theorem 2.2.7 provides a characterization of all \( m_{e_2} \)-graphs of order \( n \) that have a vertex of degree \( n - 1 \).

We note that \( m_{e_2} \)-graphs need not possess a vertex that is adjacent to all others. We provide an example below.

**Example 2.2.8.** An \( m_{e_2} \)-graph \( G \) of order 7. The maximum degree of a vertex of \( G \) is 4.

![Diagram of a graph G](image)

The Friendship Theorem is frequently presented as the statement that if \( G \) is a finite graph with the friendship property, then there is a vertex in \( G \) that is adjacent to all others. From there it is not difficult to show that \( G \) is a windmill. Since there exist \( m_{e_2} \)-graphs of order \( n \) that do not possess a vertex of degree \( n - 1 \), this version of the Friendship Theorem does not hold if the friendship property is replaced with the \( m_{e_2} \)-property. The problem of classifying all \( m_{e_2} \)-graphs up to isomorphism appears to be extremely intricate, and this thesis reports on some investigations of the \( m_{e_2} \)-property.

We now note the minimum possible number of edges in a graph \( G \) of exponent 2 and order \( n \). It is clear that a graph \( G \) that attains this minimum is an \( m_{e_2} \)-graph, since every edge is 2-required. Otherwise, the deletion of any edge that is not 2-required in \( G \) would result in a graph of exponent 2 with fewer edges than that of \( G \), which is a contradiction.

Kim, Song and Hwang [12] determined the minimum possible number of edges in a graph of exponent 2 and order \( n \) as being \( \frac{1}{2}(3n - 3) \) if \( n \) is odd and \( \frac{1}{2}(3n - 2) \) if \( n \) is even. The bound is uniquely attained by the windmill \( W_{\frac{n-1}{2}} \) if \( n \) is odd, and by a graph closely related to \( W_{\frac{n-2}{2}} \) if \( n \) is even, and these graphs all have a vertex of degree \( n - 1 \).
We now describe those graphs of even order which attain this minimum bound as characterized in [12].

Let $G$ be a graph of exponent 2 and even order $n$. Then $G$ has $\frac{1}{2}(3n - 2)$ edges if and only if there exists a vertex $u$ of $G$ of degree $n - 1$, and the subgraph induced on $N(u)$ consists of $\frac{n-2}{2}$ components, one of which is a star with two leaves and the remaining $\frac{n-4}{2}$ components consist of single edges. We present an example below.

**Example 2.2.9.** The unique graph on 8 vertices which attains the minimum number of edges in a graph of exponent 2.

![Graph example](image)

We now present some further examples of me$_2$-graphs that arise in other areas of research, and present some preliminary results on me$_2$-graphs.

### 2.3 Further examples and preliminary observations

We begin this section by introducing the following two classes of graphs that have been widely studied and show that some of these graphs occur as me$_2$-graphs.

A graph $G$ of order $n$ is called strongly regular with parameters $(n, k, \lambda, \mu)$, abbreviated by writing that $G$ is a srg$(n, k, \lambda, \mu)$, if

- $G$ is regular of degree $k$
- Every pair of adjacent vertices of $G$ share $\lambda$ common neighbours
- Every pair of non-adjacent vertices of $G$ share $\mu$ common neighbours
- $1 \leq k < n - 1$ and therefore the complete graph and null graph on $n$ vertices are not considered to be strongly regular.
Example 2.3.1. The Petersen graph is a srg(10, 3, 0, 1).

We now discuss the values for $\lambda$ and $\mu$ in order for a srg($n, k, \lambda, \mu$) to be an me$_2$-graph. We first note that $\mu$ and $\lambda$ cannot both be equal to one, since that is the friendship property, and the graph $W_r$ is not strongly regular.

Lemma 2.3.2. Let $G$ be a srg($n, k, \lambda, \mu$) with $\lambda = 1$ and $\mu > 1$. Then $G$ is an me$_2$-graph.

Proof. Since $\lambda = 1$ and $\mu > 1$ every pair of distinct vertices have at least one common neighbour. Therefore $G$ has exponent 2. Let $uv$ be an edge of $G$. Since $\lambda$ is equal to 1, $u$ and $v$ have a unique common neighbour $w$, and it follows that $v = \text{ucn}(u, w)$ also, so $uv$ is 2-required for $u$ to $w$ in $G$. We conclude that $G$ is an me$_2$-graph. $\square$

Strongly regular graphs with parameters $(n, k, 1, \mu)$ with $\mu > 1$ exist. Examples can be found in [2], where a complete description of strongly regular graphs with parameters $((n^2 + 3n - 1)^2, n^2(n + 3), 1, n(n + 1))$ is given. It is proved in [2] that the only such graphs are the lattice graph $L_{3,3}$ with parameters $(9, 4, 1, 2)$, the Brouwer-Haemers graph with parameters $(81, 20, 1, 6)$ and the Games graph with parameters $(729, 112, 1, 20)$. We note that these are just examples of strongly regular graphs with parameters $(n, k, 1, \mu)$ with $\mu > 1$, and are not a classification. Below is the graph $L_{3,3}$.

Example 2.3.3. The graph $L_{3,3}$. 

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We now consider the case where $G$ is a $\text{srg}(n, k, \lambda, \mu)$ and $\lambda > 1$ and $\mu = 1$. There are some observations made in [6] about strongly regular graphs with $\mu = 1$. To our knowledge, there are no known examples of a strongly regular graph where $\lambda > 1$ and $\mu = 1$. However, we can show that if such graphs do exist and $\lambda + 1 \neq k$, then these graphs have the $\text{me}_2$-property. We first present the following proposition.

**Proposition 2.3.4.** Let $G$ be a connected $\text{srg}(n, k, \lambda, \mu)$ with $\lambda > 1$ and $\mu = 1$ and let $u$ be a vertex of $G$. Then the subgraph of $G$ induced on $N(u)$ is a disjoint union of complete graphs of order $\lambda + 1$ [6].

**Proof.** Let $xu$ be an edge of $G$ and consider the $\lambda$ common neighbours of $x$ and $u$. Label these vertices as $x_1, \ldots, x_\lambda$. We want to show that all the $x_i$ are adjacent to each other. Suppose there exists a pair of common neighbours of $x$ and $u$, $x_i$ and $x_j$ with $i \neq j$ and $i, j \in \{1, 2, \ldots, \lambda\}$, such that $x_ix_j$ is not an edge in $G$. Then $x_i$ and $x_j$ have at least two common neighbours in $G$, namely $x$ and $u$ which is a contradiction since $\mu = 1$. Therefore, the subgraph of $G$ induced on $x$ and the $\lambda$ common neighbours of $x$ and $u$ forms a copy of $K_{\lambda+1}$.

If $\lambda + 1 = k$ then $G \cong K_n$ and $K_n$ is not a strongly regular graph. So suppose $\lambda + 1 \neq k$. Let $y$ be a neighbour of $u$ that is not adjacent to $x$. Then $u = \text{ucn}(x, y)$ since $\mu = 1$, and it follows that no neighbour of $x$, other than $u$, is also a neighbour of $y$. Also, since any vertex that is adjacent to $u$ but not $x$, and the $\lambda$ common neighbours of this vertex and $u$, form a copy of $K_{\lambda+1}$, it follows that the subgraph induced on $N(u)$ is a disjoint union of complete graphs of order $\lambda + 1$.

Since $\lambda > 1$ and $\mu = 1$ it follows that $G$ has exponent 2. Also, we can see that every edge in $G$ is 2-required as follows. Since every edge incident with $u$ is 2-required in $G$
for a path of length 2 from a vertex in one copy of $K_{\lambda+1}$ to a vertex in another copy of $K_{\lambda+1}$, and $u$ is an arbitrary vertex of $G$, it follows that $G$ is an $me_2$-graph.

There are no other values for $\lambda$ and $\mu$ for which a $srg(n, k, \lambda, \mu)$ can be an $me_2$-graph as we show below. Recall that $\lambda$ and $\mu$ cannot both be equal to one since that is the friendship property, and the graph $W_\nu$ is not strongly regular.

**Lemma 2.3.5.** Let $G$ be a $srg(n, k, \lambda, \mu)$ with $\mu \neq 1$ and $\lambda \neq 1$. Then $G$ is not an $me_2$-graph.

**Proof.** If $\mu = 0$ or $\lambda = 0$ then $G$ does not have exponent 2 and $G$ is not an $me_2$-graph. If $\mu > 1$ and $\lambda > 1$ in $G$ then every pair of distinct vertices of $G$ have at least 2 common neighbours. It follows that there are no unique 2-paths in $G$ and $G$ is not an $me_2$-graph.

The Kneser graph $K(s, r)$ where $s$ and $r$ are positive integers and $s \geq r$, is the graph of order $\binom{s}{r}$ whose vertices correspond to the $r$-element subsets of an $s$-element set, and two vertices in $K(s, r)$ are adjacent if and only if the subsets that they represent are disjoint.

**Example 2.3.6.** $K(5, 2)$ is isomorphic to the Petersen graph. The vertices in the graph below are labelled as the 2-element subsets of the set $\{1, 2, 3, 4, 5\}$.

\[ K(5, 2) \]

We note that the Petersen graph is both a $srg(10, 3, 0, 1)$ and is isomorphic to $K(5, 2)$. We observe that $K(s, 2)$ is a strongly regular graph with parameters $\left(\binom{s}{2}, \binom{s-2}{2}, \binom{s-4}{2}, \binom{s-3}{2}\right)$.

We now present some elementary observations of Kneser graphs. We begin by showing that if $K(s, r)$ is strongly regular, then $r = 2$. Further information on Kneser graphs can be found in [9].
Suppose not. Suppose $K(s, r)$ is strongly regular and $r \neq 2$. If $r = 1$ then $K(s, r) \cong K_s$, the complete graph on $s$ vertices, and $K_s$ is not strongly regular. So suppose $r > 2$. Let $u$ and $v$ be non-adjacent vertices in $K(s, r)$. Hence $u$ and $v$ represent $r$-element subsets of $s$ that intersect. Let $A$ be the set represented by $u$ and let $B$ be the set represented by $v$.

Suppose $A$ and $B$ represent subsets that intersect in only one element. We consider the number of $r$-element subsets of $s$ that are disjoint from both $A$ and $B$, and obtain an expression for the number of mutual neighbours of $u$ and $v$ in $K(s, r)$. Since $A$ and $B$ intersect in exactly one element, it follows that the number of distinct elements in $A$ and $B$ is $2r - 1$. So there are $s - (2r - 1)$ elements that are not in $A$ and not in $B$. Hence the number of mutual neighbours of $u$ and $v$ in $K(s, r)$ is \(\binom{s-(2r-1)}{r}\).

Now suppose $A$ and $B$ intersect in exactly $r - 1$ elements. Then the number of distinct elements in $A$ and $B$ is $r + 1$. So there are $s - (r + 1)$ elements that are in neither $A$ nor $B$. Hence the number of mutual neighbours of $u$ and $v$ in $K(s, r)$ is \(\binom{s-(r+1)}{r}\). Since $K(s, r)$ is a strongly regular graph, this means that \(\binom{s-(2r-1)}{r} = \binom{s-(r+1)}{r}\). However, it follows that $2r - 1 = r + 1$ and hence $r = 2$ is the only possibility.

We now discuss when $K(s, r)$ has exponent 2 and when $K(s, r)$ is an me$_2$-graph.

**Lemma 2.3.7.** The Kneser graph $K(s, r)$ is primitive of exponent 2 if and only if $s \geq 3r$.

**Proof.** $(\implies)$ Suppose $K(s, r)$ is primitive of exponent 2. Then every pair of distinct vertices in $K(s, r)$ share a common neighbour and $s > 2r$. In particular, every pair of adjacent vertices in $K(s, r)$ share a common neighbour, and it follows that for every pair of disjoint $r$-element subsets of $s$, there is at least one $r$-element subset that is disjoint from both of them. Hence $s \geq 3r$.

$(\impliedby)$ Let $u$ and $v$ be vertices of $K(s, r)$ and suppose $s \geq 3r$. Suppose $u$ and $v$ are adjacent in $K(s, r)$. Since $s \geq 3r$, it follows that there exists at least one $r$-element subset that is disjoint from both of the sets represented by $u$ and $v$. Let such a set be represented by the vertex $w$. Then $uvw$ is a 2-path in $K(s, r)$ and every pair of adjacent vertices have a path of length 2 between them in $K(s, r)$. Now suppose $u$ and $v$ are not adjacent in $K(s, r)$. Then since $s \geq 3r$, it follows that there is at least one $r$-element subset of $K(s, r)$ that is disjoint from both of the sets represented by $u$ and $v$, and the vertex that represents this $r$-element subset of $K(s, r)$ is a common neighbour of both
and \( v \) in \( K(s, r) \). Hence there exists a path of length 2 between every pair of non-adjacent vertices in \( K(s, r) \). We conclude that \( K(s, r) \) is primitive of exponent 2 if and only if \( s \geq 3r \).

**Lemma 2.3.8.** The Kneser graph \( K(s, r) \) is an \( m_{e_2} \)-graph if and only if \( s = 3r \).

**Proof.** (\( \implies \)) Suppose \( K(s, r) \) is an \( m_{e_2} \)-graph and suppose \( s \neq 3r \). By Lemma 2.3.7 \( s \geq 3r \) so we can assume \( s > 3r \). Then every pair of vertices in \( K(s, r) \) has multiple common neighbours, and there are no unique 2-paths in \( K(s, r) \) which contradicts that \( K(s, r) \) is an \( m_{e_2} \)-graph. Hence \( s = 3r \).

(\( \impliedby \)) Suppose \( s = 3r \). Then by Lemma 2.3.7 \( K(3r, r) \) has exponent 2. Let \( uv \) be an edge of \( K(3r, r) \). Then \( u \) and \( v \) have a unique common neighbour \( w \) and it follows that \( v = \text{ucn}(u, w) \). Therefore \( uv \) is 2-required in \( K(3r, r) \) for \( u \) to \( w \). We conclude that \( K(3r, r) \) is an \( m_{e_2} \)-graph.

We present an example of \( K(3r, r) \) below.

**Example 2.3.9.** The graph \( K(6, 2) \), which is also a \( \text{srg}(15, 6, 1, 3) \). The vertices are labelled as the 2-element subsets of the set \( \{1, 2, 3, 4, 5, 6\} \).

We now discuss some preliminary general observations on \( m_{e_2} \)-graphs, and use these to obtain more examples.

**Lemma 2.3.10.** Let \( G \) be an \( m_{e_2} \)-graph and suppose there exists a vertex \( v \) in \( G \) with the property that every edge incident with \( v \) is 2-required for a path from \( v \) to some other vertex in \( G \). Then we can obtain an \( m_{e_2} \)-graph \( G' \) by introducing a vertex \( v' \) to \( G \) with \( N_G(v') = N_G(v) \).
Proof. We first show that $G'$ has exponent 2. Let $x$ and $y$ be two vertices of $G'$. If $x$ and $y$ are also both vertices of $G$, then since $G$ has exponent 2 and $G$ is a subgraph of $G'$, it follows that there exists a path of length 2 between $x$ and $y$ in $G'$. Now suppose $x = y$. If $y = v$ then as $v$ and $v'$ have the same neighbour set in $G'$, there exist paths of length 2 between $v$ and $v'$ in $G'$. If $y \neq v$ then since for a vertex $z$ in $G$ there exists a 2-path $vy$ in $G$, it follows that there exists a 2-path $v'y$ in $G'$. Therefore $G'$ has exponent 2.

We now show that every edge of $G'$ is 2-required. Let $xy$ be an edge of $G'$. Suppose first that neither $x$ nor $y$ is equal to $v'$ or $v$. Then $xy$ is an edge in $G$ and since $G$ is an me$_2$-graph, without loss of generality we can assume that $xy$ is 2-required for $x$ to some vertex $z$ in $G$. Since $v$ is not a common neighbour of $x$ and $z$ in $G$, it follows that $v'$ is not a common neighbour of $x$ and $z$ in $G'$ and $xy$ is 2-required for $x$ to $z$ in $G'$. Now suppose $x = v'$ in $G'$. Since $vy$ is 2-required in $G$ for $v$ to some vertex $p$ in $G$, the edge $v'y$ is 2-required for $v'$ to $p$ in $G'$. We conclude that every edge of $G'$ is 2-required and $G'$ is an me$_2$-graph. \qed

**Lemma 2.3.11.** Let $G$ be an me$_2$-graph and suppose there exist vertices $v$ and $v'$ in $G$ such that $N(v) = N(v')$. Then $G \setminus \{v'\}$ is an me$_2$-graph.

**Proof.** Let $G' = G \setminus \{v'\}$. We first show that $G'$ has exponent 2. Let $x$ and $y$ be two vertices of $G'$. Since $G$ has exponent 2, it follows that there exists a 2-path $xzy$ in $G$ for some vertex $z$ in $G$. If $z \neq v'$ in $G$ then the 2-path $xzy$ exists in $G'$. If $z = v'$ in $G$ then there is a 2-path $xvy$ in $G'$ and therefore $G'$ has exponent 2.

We now show that every edge of $G'$ is 2-required. Let $xy$ be an edge of $G'$. Since $N(v) = N(v')$ in $G$, it follows that every edge incident with $v'$ in $G$ is 2-required only for $v'$ to some vertex in $G'$. We note that if $xy$ is 2-required in $G$ for $x$ to $v'$, then $xy$ is also 2-required in $G$ for $x$ to $v$. Therefore, the edge $xy$ is 2-required in $G'$ for the same reason it is in $G$ and $G'$ is an me$_2$-graph. \qed

We can extend windmills to obtain new me$_2$-graphs of all orders at least 4 using Lemma 2.3.10 as follows.

Let $u$ be the vertex in $W_k$ of degree $2k$. Since every vertex in the subgraph of $W_k$ induced on $N(u)$ belongs to exactly one of the $k$ edges, it follows that for a vertex $x$ in $N(u)$, $x = \text{ucn}(u, y)$ for some vertex $y$ in $W_k$, and therefore every edge incident with $u$ is
2-required in $W_k$ for $u$ to some other vertex in $W_k$. By Lemma 2.3.10, for any $n \geq 2k + 1$, we can define a graph $G_{n,k}$ by adjoining $n - 2k - 1$ vertices to $W_k$, each with the same neighbour set of $u$ in $W_k$. Then the order of $G_{n,k}$ is $n$ and the degree of each of these $n - 2k - 1$ vertices in $G_{n,k}$ is $2k$. The subgraph of $G_{n,k}$ induced on the neighbour set of the vertices of degree $2k$ is comprised of $k$ disjoint edges. The graphs $G_{n,k}$ play an important role in our investigation of the maximum possible number of edges in an $me_2$-graph of order $n$, which we will return to later in the chapter.

The example below shows the graphs $G_{5,1}$ and $G_{7,2}$.

**Example 2.3.12.** The graphs $G_{5,1}$ and $G_{7,2}$

We note that the graphs $G_{n,k}$ are unions of isomorphic windmills. We now use Lemma 2.3.10 to produce examples of $me_2$-graphs.

**Example 2.3.13.** Let $G$ be the following graph on 6 vertices

It can be checked that $G$ is an $me_2$-graph. Let $u$ be a vertex of $G$ of degree 2. We observe that every edge incident with $u$ is 2-required in $G$ for $u$ to $v_i$ for some $i \in \{1, 2, 3\}$. By
Lemma 2.3.10, we can extend \( G \) by adding vertices with the same neighbour set as \( u \) and the resulting graph is an \( \text{me}_2 \)-graph.

We observe that Lemma 2.3.10 applies to any vertex of degree 2 in any \( \text{me}_2 \)-graph. Let \( G \) be an \( \text{me}_2 \)-graph and let \( u \) be a vertex of degree 2 in \( G \). Let \( x \) and \( y \) be the neighbours of \( u \). Since \( G \) has exponent 2, \( u, x, y \) and the edges between them form a triangle, and \( ux \) is 2-required for \( u \) to \( y \) in \( G \), and \( uy \) is 2-required for \( u \) to \( x \) in \( G \). Therefore, by Lemma 2.3.10, we can add vertices to \( G \) that have the same neighbour set as \( u \) in \( G \). □

2.4 The maximum possible number of edges in an \( \text{me}_2 \)-graph of order \( n \)

Recall that Kim et al. showed in [12] that the minimum possible number of edges in a graph of exponent 2 and order \( n \) is \( \frac{1}{2}(3n - 3) \) if \( n \) is odd and \( \frac{1}{2}(3n - 2) \) if \( n \) is even, and the graphs that uniquely attain these bounds were determined in [12]. This provides a lower bound for the number of edges in an \( \text{me}_2 \)-graph of specified order. This motivates the idea of considering an upper bound for the number of edges in an \( \text{me}_2 \)-graph of specified order. We pose the following problem.

**Problem:** What is the maximum possible number of edges in an \( \text{me}_2 \)-graph of order \( n \), and what are the \( \text{me}_2 \)-graphs that attain the maximum?

We present some partial progress on this problem. We begin by considering the maximum possible number of edges in the graph \( G_{n,k} \) for a given \( n \).

We first make the following observations about the graph \( G_{n,k} \) of order \( n \).

- The vertex set of \( G_{n,k} \) is the disjoint union \( A \cup B \) where \( |A| = n - 2k \geq 1 \) and \( |B| = 2k \).

- The edge set of \( G_{n,k} \) includes an edge joining every vertex of \( A \) to every vertex of \( B \). Additionally, each vertex of \( B \) is adjacent to exactly one other vertex of \( B \).

**Lemma 2.4.1.** The number of edges in \( G_{n,k} \) is \( 2nk - 4k^2 + k \), and for a fixed \( n \), this number is maximized when \( k \) is the nearest integer to \( \frac{2n+1}{8} \).
Proof. We first note that $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and therefore there are $\left\lfloor \frac{n-1}{2} \right\rfloor$ graphs $G_{n,k}$ of order $n$. We count the number of edges in the graph $G_{n,k}$ of order $n$.

Since every vertex in $B$ is adjacent to exactly one other vertex in $B$, there are $k$ edges between the $2k$ vertices in $B$. There are $n - 2k$ vertices in $A$ and these are all adjacent precisely to each of the $2k$ vertices in $B$, and therefore there are $2k(n - 2k)$ edges incident with a vertex of $A$ and a vertex of $B$. The number of edges in total is

$$2k(n - 2k) + k = 2nk - 4k^2 + k.$$ 

We now compute the maximum number of edges that can occur in a graph $G_{n,k}$ of order $n$. To find the maximum value of $2nk - 4k^2 + k$, we consider $f(k) = 2nk - 4k^2 + k$ as a function of $k$ and set $f'(k) = 0$. We obtain $k = \frac{2n+1}{8}$. We conclude that the graph $G_{n,k}$ of order $n$ attains the maximum number of edges in a graph of this type when $k$ is the nearest integer to $\frac{2n+1}{8}$. In this case, it follows that $|B|$ is the nearest even integer to $\frac{n}{4}$, and $|A| = n - |B|$.

It follows from Lemma 2.4.1 that for the graphs $G_{n,k}$ of order $n$, if $n \equiv 0$ or $1 \mod 4$, then $G_{n,k}$ attains the maximum number of edges in a graph of this type when $k = \left\lfloor \frac{n}{4} \right\rfloor$, and if $n \equiv 2$ or $3 \mod 4$, then $G_{n,k}$ attains the maximum number of edges in a graph of this type when $k = \left\lfloor \frac{n}{4} \right\rfloor + 1$.

It has been verified by a computer programme written by Niall Madden (private communication), that the unique graphs that attain the maximum number of edges in an $me_2$-graph of order $n \leq 9$, are the graphs $G_{n,k}$ of order $n$ as described above. The question of whether this is true in general is open, and we present some partial progress below.

We now consider the complement of an $me_2$-graph, and show that the graph $G_{n,k}$ of order $n$ where $k$ is the nearest integer to $\frac{2n+1}{8}$ has the maximum number of edges amongst all $me_2$-graphs of order $n$ whose complement is disconnected.

We introduce the definition of the complement of a graph, and discuss the translation of the $me_2$-property from a graph to its complement.

**Definition 2.4.2.** Let $G$ be a graph. Then the complement of $G$ is the graph $\overline{G}$ with the same vertex set as $G$, in which two vertices are adjacent if and only if they are not adjacent in $G$. 

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Let $G$ be an me$_2$-graph. Since $G$ has no isolated vertices, it follows that every vertex of $G$ has at least one neighbour. Hence every vertex in $\overline{G}$ has at least one non-neighbour. For all pairs of distinct vertices $u$ and $v$ in $G$, there exists a vertex which is a common neighbour of $u$ and $v$ in $G$. Therefore for every pair of distinct vertices $u$ and $v$ in $\overline{G}$, there exists a vertex that is a non-neighbour of both $u$ and $v$ in $\overline{G}$. Since $G$ has the me$_2$-property, the deletion of any edge of $G$ results in a graph where at least one pair of distinct vertices no longer share a common neighbour. In $\overline{G}$, this is equivalent to saying that the addition of an edge to $\overline{G}$ results in a graph for which there exists a pair of distinct vertices with no common non-neighbour, hence every other vertex in this augmented graph is adjacent to at least one of the vertices in this pair.

We now discuss when the complement of an me$_2$-graph is disconnected.

Let $G$ be an me$_2$-graph of order $n$. If $G$ has a vertex $u$ of degree $n - 1$, then $u$ is an isolated vertex in $\overline{G}$ and it follows that $\overline{G}$ is disconnected. The me$_2$-graphs of order $n$ with a vertex of degree $n - 1$ have been described in Section 2.2. We now show that every other me$_2$-graph with disconnected complement is a $G_{n,k}$.

We first note that if $H$ is a graph with $|V(H)| = n$, then $\overline{H}$ is disconnected if and only if $V(H)$ is the disjoint union of $A$ and $B$ where $|A| = k$, $|B| = n - k$ and every vertex of $A$ is adjacent to every vertex of $B$ in $H$. Since $V(G_{n,k})$ is the disjoint union of $A$ and $B$ with $|A| = n - 2k$ and $|B| = 2k$ and every vertex in $A$ is adjacent to every vertex in $B$, it follows that $\overline{G_{n,k}}$ is disconnected. \[\square\]

**Lemma 2.4.3.** Let $G$ be an me$_2$-graph of order $n \geq 4$ and suppose that $G$ does not have a vertex of degree $n - 1$. If $\overline{G}$ is disconnected then $G \cong G_{n,k}$.

**Proof.** Suppose $\overline{G}$ is disconnected. Let $\overline{G}$ be the disjoint union of two subgraphs $C_1$ and $C_2$ with $|V(C_1)| = k$ and $|V(C_2)| = n - k$. Since $G$ does not have a vertex of degree $n - 1$ and $n \geq 4$, it follows that $k \geq 2$ and $n - k \geq 2$. At least one of $C_1$ and $C_2$ must have no vertex of full degree, otherwise there would be a distinct pair of vertices in $\overline{G}$ with no common non-neighbour. We can assume $C_1$ has no vertex of full degree and it follows that $C_1$ is not complete. Let $x$ be a vertex of $C_1$ and let $y$ be a vertex of $C_2$. We note that there is a vertex in $C_1$ that is not a neighbour of $x$. Add the edge $xy$ to $\overline{G}$. Now there is a pair of vertices with no common non-neighbour in this augmented graph of $\overline{G}$, i.e., no
common neighbour in $G$, and such a pair must include either $x$ or $y$. Such a pair cannot consist of both $x$ and $y$, as if it did then it would mean $x$ and $y$ already had no common non-neighbour in $G$. Now suppose $x$ is included in such a pair. We note that after the addition of the edge $xy$, there is still a vertex in $C_1$ that is not adjacent to $x$. Then, since $|V(C_2)| \geq 2$ and the only edge between vertices of $C_1$ and vertices of $C_2$ is the edge $xy$, the other vertex in this pair would have to be one that is adjacent to the non-neighbour of $x$ in $C_1$, and to all vertices in $V(C_2) \setminus \{y\}$ which is a contradiction. We conclude that $y$ must be included in the pair of vertices with no common non-neighbour in $G$. Then, and using the fact that the number of vertices in $C_2$ is at least $2$, $y$ is adjacent to all vertices in $C_2$, and since $y$ is an arbitrary vertex in $C_2$, $C_2$ is complete.

Since $G$ is disconnected and there are at least 2 vertices in $C_1$, it follows that the other vertex in the pair must be a vertex $z$ in $C_1$ that is adjacent to every other vertex in $C_1$, except for $x$ (and $z$ is not adjacent to any vertices in $C_2$). Therefore $x$ is the unique non-neighbour of $z$ in $C_1$ and since $x$ is an arbitrary vertex in $C_1$, every vertex in $C_1$ is the unique non-neighbour of some other vertex in $C_1$. Since the only non-neighbour of $z$ is $x$, and $z$ must be the unique non-neighbour of some vertex of $C_1$, it follows that $z$ is the unique non-neighbour of $x$ in $C_1$. Therefore, every vertex of $C_1$ has a unique non-neighbour and $C_1$ is regular of degree $k - 2$. Hence $k$ is even and $C_1$ is the graph obtained from a complete graph of order $k$ by deleting $\frac{k}{2}$ disjoint edges. Therefore $G$ is the union of two graphs $D_1$ and $D_2$, where $\overline{D_1}$ is $C_1$ and $\overline{D_2}$ is $C_2$, and since $C_2$ is complete, $D_2$ comprises $n - k$ isolated vertices. So $|V(D_2)| = n - k$ and $|V(D_1)| = k$, every vertex of $D_1$ is adjacent to every vertex of $D_2$ and $D_1$ has exactly $\frac{k}{2}$ edges hence every vertex of $D_1$ is adjacent to exactly one other vertex of $D_1$. We conclude that $G \cong G_{n,k}$.

We now compute the maximum possible number of edges in an $m_{\mathbb{E}_2}$-graph $G$ of order $n$ with a vertex of degree $n - 1$, and show that this maximum is attained only when $G \cong G_{n,1}$. We use this to show that the graphs $G_{n,k}$ of order $n$ where $k$ is the nearest integer to $\frac{2n+1}{8}$ has the maximum number of edges in an $m_{\mathbb{E}_2}$-graph of order $n$ with disconnected complement.

Let $u$ be a vertex of $G$ of degree $n - 1$. Since the subgraph of $G$ induced on $N(u)$ is a forest by Theorem 2.2.7, its number of edges is maximized (at $n - 2$) when it consists of
a single star on $n - 1$ vertices. In this case the graph is $G_{n,1}$. The number of edges in this graph is $2n - 3$. Now consider the number of edges in the graph $G_{n,k}$ of order $n$. This number is $2nk - 4k^2 + k$, and if $k$ is the nearest integer to $\frac{2n+1}{8}$, then $2nk - 4k^2 + k \geq 2n - 3$ with equality only when $3 \leq n \leq 5$. \qed
Chapter 3

The Strong-Me$_2$-Property

3.1 Introduction and examples

The focus of this chapter is on a refinement of the me$_2$-property that we will refer to as the strong-me$_2$-property. A graph with the strong-me$_2$-property is referred to as a strong-me$_2$-graph. We discuss certain characteristics of strong-me$_2$-graphs and investigate small examples. We also note some examples of infinite families of strong-me$_2$-graphs.

Definition 3.1.1. Let $G$ be a primitive graph of exponent 2. We say that $G$ has the strong-me$_2$-property if for every edge $uv$ of $G$ there exists a vertex $w$ such that $v$ is the unique common neighbour of $u$ and $w$ and there exists a vertex $w'$ such that $u$ is the unique common neighbour of $v$ and $w'$.

Example 3.1.2. The windmills $W_r$ comprise a family of graphs with the strong me$_2$-property.

![Diagram of a windmill graph]

$W_3$
Let $c$ be the vertex in $W_r$ that is adjacent to all other vertices. There are two types of edges in $W_r$: edges incident with $c$ and edges incident with two neighbours of $c$. Let $cuv$ be a triangle in $W_r$. For the edge $cu$, $c$ is the unique common neighbour of $u$ and $v$ and $u$ is the unique common neighbour of $c$ and $v$. For the edge $uv$, $u$ is the unique common neighbour of $v$ and $c$ and $v$ is the unique common neighbour of $u$ and $c$. Hence $W_r$ has the strong $me_2$-property.

The windmill graphs provide examples of strong-$me_2$-graphs of all odd orders from 3 onward.

We observe that for a graph $G$ of exponent 2, $G$ has the $me_2$-property if for every edge $uv$ of $G$ there exists a vertex $w$ such that $v$ is the unique common neighbour of $u$ and $w$ or there exists a vertex $w'$ such that $u$ is the unique common neighbour of $v$ and $w'$. However, the strong $me_2$-property requires that for every edge $uv$, both of these conditions hold. So it can be noted that the strong-$me_2$-property implies the $me_2$-property. The following example shows that the converse is not true.

**Example 3.1.3.** The following graph is an example of an $me_2$-graph that is not a strong-$me_2$-graph.

![Diagram](attachment:/example.png)

The highlighted edge $uv$ demonstrates the failure of the graph to be a strong-$me_2$-graph as there does not exist a vertex $w$ for which $u$ is the unique common neighbour of $v$ and $w$.

The strong-$me_2$-property can be interpreted in terms of non-negative matrices as follows.

Given a non-negative $n \times n$ matrix $A$ with a symmetric pattern of zero entries and zeros on the main diagonal, we say that $A$ is a strong-$me_2$-matrix if $A$ has the strong-$me_2$-property. In matrix terms this means that $A^2$ is positive, but the replacement of symmetric positive entries in the $(i, j)$ and $(j, i)$ positions of $A$ with zeros would result
in a matrix whose square has at least one zero in row \(i\) and at least one zero in row \(j\).

If \(A\) is the adjacency matrix of a graph \(G\), the replacement of positive entries in the \((i,j)\) and \((j,i)\) positions of \(A\) with zeros is equivalent to the deletion of the edge incident with vertices \(i\) and \(j\) in \(G\). If \(G\) is a strong-me\(_2\)-graph, then this results in the deletion of two unique 2-paths in \(G\) - one starting at the vertex \(i\) and one starting at the vertex \(j\). Therefore, in the square of the adjacency matrix corresponding to this graph, there exists at least one zero entry in row \(i\), and at least one zero entry in row \(j\).

We introduce the next definition in order to deal with our discussion of the strong-me\(_2\)-property more efficiently.

**Definition 3.1.4.** Let \(G\) be a graph. We say that an edge \(uv\) in \(G\) is strongly-2-required if there exists a vertex \(w\) in \(G\) and a vertex \(w'\) in \(G\) such that \(u\) is the unique common neighbour of \(v\) and \(w\) in \(G\) and \(v\) is the unique common neighbour of \(u\) and \(w'\) in \(G\).

To say that an edge \(uv\) in a graph \(G\) is strongly-2-required means that there exist vertices \(w\) and \(w'\) in \(G\) for which \(uw'\) and \(vuw\) are unique 2-paths in \(G\), so that \(uw\) and \(vu\) are respectively 2-required for paths from \(u\) to \(w'\) and from \(v\) to \(w\). We observe that \(G\) is a strong-me\(_2\)-graph if and only if \(G\) has exponent 2 and every edge of \(G\) is strongly-2-required.

We recall the following notation from Chapter 2.

Let \(G\) be a graph of exponent 2 and let \(u, v\) and \(w\) be vertices of \(G\). We write \(u = ucn(v, w)\) if \(u\) is the unique common neighbour of \(v\) and \(w\) in \(G\).

We observe the following lemmas which highlight some of the differences between the me\(_2\)-property and the strong-me\(_2\)-property. We first recall Theorem 2.2.7 which says that if \(G\) is an me\(_2\)-graph of order \(n\) with a vertex \(u\) of degree \(n - 1\) then every component of the subgraph of \(G\) induced on \(N(u)\) is a star. We now classify all strong-me\(_2\)-graphs of order \(n\) with a vertex of degree \(n - 1\).

**Lemma 3.1.5.** Let \(G\) be a strong-me\(_2\)-graph of order \(n\) and suppose \(G\) has a vertex of degree \(n - 1\). Then \(G\) is a windmill.

**Proof.** Suppose \(u\) is a vertex in \(G\) of degree \(n - 1\). Let \(x\) be a neighbour of \(u\). Then \(x = ucn(u, y)\) for some \(y\) in \(G\), and \(y\) is also a neighbour of \(u\). Since \(x\) is the only
common neighbour of $u$ and $y$ in $G$, it follows that $u$ and $x$ are the only neighbours of $y$, and $\deg_G(y) = 2$. Finally, $y = \text{ucn}(u, w)$ for some vertex $w$, and since $\deg_G(y) = 2$ it must be that $w = x$ and the subgraph of $G$ induced on $N(u)$ is a collection of disjoint edges, as required. \hfill \Box

Another difference between $\text{me}_2$-graphs and strong-$\text{me}_2$-graphs is observed in the following lemma.

**Lemma 3.1.6.** Let $G$ be a strong-$\text{me}_2$-graph and let $u$ and $v$ be distinct vertices of $G$. Then $N(u) \nsubseteq N(v)$.

**Proof.** Suppose $N(u) \subseteq N(v)$ in $G$. Let $x$ be a vertex in $G$ adjacent to both $u$ and $v$. Since $G$ is a strong-$\text{me}_2$-graph, there exists a vertex $y$ in $G$ such that $u = \text{ucn}(x, y)$. However $N(u) \subseteq N(v)$ so $v$ is also a common neighbour of $x$ and $y$ which is a contradiction. Therefore $N(u) \nsubseteq N(v)$. \hfill \Box

Lemma 3.1.6 is in contrast to Lemma 2.3.10 and Lemma 2.3.11. There may be at least one vertex in an $\text{me}_2$-graph of order $n$ that has the same neighbour set as another vertex in the graph, for example this occurs in the graphs $G_{n, k}$ where $n - 2k \geq 2$.

We now consider a refinement of the strong-$\text{me}_2$-property and use it to identify some collections of strong-$\text{me}_2$-graphs.

**Definition 3.1.7.** Let $G$ be a graph. We say that an edge $uv$ in $G$ is unitriangular if it belongs to exactly one triangle in $G$.

**Lemma 3.1.8.** Let $G$ be a graph of exponent two in which every edge is unitriangular. Then $G$ has the strong-$\text{me}_2$-property.

**Proof.** Let $uv$ be an edge of $G$. Since $uv$ is unitriangular, there exists a vertex $x$ in $G$ such that $x = \text{ucn}(u, v)$ in $G$. Since $xu$ is also unitriangular there exists a unique common neighbour of $x$ and $u$ in $G$. This vertex must be $v$. Similarly, $u = \text{ucn}(x, v)$ in $G$. Hence every edge in $G$ is strongly-2-required and $G$ is a strong-$\text{me}_2$-graph. \hfill \Box

The windmill graphs are examples of strong-$\text{me}_2$-graphs in which every edge is unitriangular. The following two families of examples demonstrate that a strong-$\text{me}_2$-graph in which every edge is unitriangular need not be a windmill.
Example 3.1.9. The Kneser graph $K(3r, r)$ is an $\text{me}_2$-graph as shown in Chapter [2]. We recall that every edge in $K(3r, r)$ is unitriangular and hence by Lemma [3.1.8], $K(3r, r)$ is a strong-$\text{me}_2$-graph.

Let $uv$ be an edge of $K(3r, r)$. The vertices $u$ and $v$ in $K(3r, r)$ represent disjoint $r$-element subsets of a $3r$-element set. Hence there is a unique $r$-element subset, represented by the vertex $w$ in $K(3r, r)$, that is disjoint from both of them. So $w = \text{ucn}(u, v)$ in $K(3r, r)$ which means the edge $uv$ is unitriangular.

Example 3.1.10. Strongly regular graphs with parameters $(n, k, \lambda, \mu)$ where $\lambda = 1$ and $\mu > 1$ are $\text{me}_2$-graphs as noted in Chapter [2]. We show every edge of such a graph is unitriangular.

Let $G$ be a $\text{srg}(n, k, \lambda, \mu)$ with $\lambda = 1$ and $\mu > 1$. Recall that $\mu$ and $\lambda$ cannot both be equal to one, since that is the friendship property, and the graph $W_r$ is not strongly regular. Since $\lambda$ is equal to 1, every pair of adjacent vertices has a unique common neighbour and therefore every edge is unitriangular, and since $\mu > 1$, every pair of non-adjacent vertices has at least one common neighbour therefore the graph has exponent 2.

We provided some examples of strongly regular graphs for which $\lambda = 1$ and $\mu > 1$ in Chapter [2] as described in [2].

We now show that any strongly regular graph with parameters $(n, k, \lambda, \mu)$ where $\lambda > 1$, $\mu = 1$ and $\lambda + 1 \neq k$ must be a strong-$\text{me}_2$-graph. As noted in Chapter [2], to our knowledge, it is not known whether such graphs exist.

Recall from Chapter [2] that if $G$ is a connected $\text{srg}(n, k, \lambda, \mu)$ with $\lambda > 1$, $\mu = 1$ and $\lambda + 1 \neq k$, then $G$ is an $\text{me}_2$-graph. Recall that for a vertex $u$ of $G$, the subgraph of $G$ induced on $N(u)$ is a disjoint union of complete graphs of order $\lambda + 1$ [6]. Let $uv$ be an edge of $G$. Let $x \in N(u)$ such that $xv$ is not an edge in the subgraph of $G$ induced on $N(u)$ and let $y \in N(v)$ such that $uy$ is not an edge in the subgraph of $G$ induced on $N(v)$. We observe that, in the subgraph of $G$ induced on $N(u)$, $x$ and $v$ are in two different copies of $K_{\lambda+1}$, and similarly for $u$ and $y$ in the subgraph of $G$ induced on $N(v)$. Then $uv$ is strongly-2-required in $G$ for $u$ to $y$ and for $v$ to $x$. Therefore, $G$ is a strong-$\text{me}_2$-graph.

For a graph $G$ of exponent 2, the property that every edge is unitriangular is sufficient for $G$ to be a strong-$\text{me}_2$-graph, but it is not necessary as the family of examples
Example 3.1.11. We provide a construction of examples of strong-me$_2$-graphs of odd order at least 7 in which there exist edges that are not unitriangular:

For $m \geq 1$ we define the graph $G_m$ of order $2m + 5$ as follows:

- $V(G_m) = \{a, b, c, d, e, f_1, g_1, \ldots, f_m, g_m\}$.
- $a$ is adjacent to all vertices of $G_m$ except for $b$ and $d$.
- $b$ is adjacent to all vertices of $G_m$ except for $a$ and $c$.
- $c$ is adjacent only to $a$ and $e$ in $G_m$.
- $d$ is adjacent only to $b$ and $e$ in $G_m$.
- $e$ is adjacent only to $a, b, c$ and $d$ in $G_m$.
- For all $i \in \{1, \ldots, m\}$, $f_i$ and $g_i$ are adjacent to both $a$ and $b$ in $G_m$.
- For each $i \in \{1, \ldots, m\}$, $f_i$ and $g_i$ are adjacent in $G_m$.

It can be easily verified that $G_m$ is a strong-me$_2$-graph. The edges $f_i g_i$ for all $i \in \{1, \ldots, m\}$ are not unitriangular.

Below is the graph $G_1$. We can see that the edge $f_1 g_1$ is not unitriangular.

Example 3.1.11 also provides a family of strong-me$_2$-graphs that achieve every odd order at least 7, that are not windmills.
Most of the examples of strong-me\(_2\)-graphs that have been discussed so far have odd order, with the exception of \(K(3r, r)\) which has even order for some values of \(r\), for example when \(r = 3\). We will now look at strong-me\(_2\)-graphs of even order.

### 3.2 Strong-me\(_2\)-graphs of even order

The following graph will play a role in our discussions. We will see later that it is an example of a strong-me\(_2\)-graph of minimum even order.

**Example 3.2.1.** A strong-me\(_2\)-graph \(G\) on 10 vertices.

![Graph](image)

We leave to the reader to verify \(G\) has exponent 2. We show that every edge of \(G\) is strongly-2-required and therefore \(G\) is a strong-me\(_2\)-graph:

- The edge \(v_1v_2\) is strongly-2-required for \(v_1\) to \(a_2\) and \(v_2\) to \(a_1\). The edges \(v_1v_3\) and \(v_2v_3\) are strongly-2-required by similar reasoning.

- The edge \(v_1a_1\) is strongly-2-required for \(v_1\) to \(b_1\) and for \(a_1\) to \(v_2\). The edges \(v_2a_2\) and \(v_3a_3\) are strongly-2-required by similar reasoning. So are \(v_1b_1\), \(v_2b_2\) and \(v_3b_3\) by symmetry.

- The edge \(a_1b_1\) is strongly-2-required for \(a_1\) to \(w\) and for \(b_1\) to \(w\). The edges \(a_2b_2\) and \(a_3b_3\) are strongly-2-required by similar reasoning.

- The edge \(wa_1\) is strongly-2-required for \(w\) to \(b_1\) and for \(a_1\) to \(a_2\). The edges \(wa_2\) and \(wa_3\) are strongly-2-required by similar reasoning. So are \(wb_1\), \(wb_2\) and \(wb_3\) by symmetry.
Hence every edge of $G$ is strongly-2-required and $G$ is a strong-me$_2$-graph.

We now show that if $G$ is a strong-me$_2$-graph of even order $n$ then $G$ cannot have a vertex of degree $n - 1$, and we discuss a consequence of this.

**Lemma 3.2.2.** Let $G$ be a graph of exponent 2 and of even order $n \geq 4$. If $G$ has a vertex of degree $n - 1$, then $G$ is not a strong-me$_2$-graph.

**Proof.** Suppose $x$ is a vertex in $G$ of degree $n - 1$. We know that every vertex of $G$ must have degree at least 2. Since $x$ has degree $n - 1$ which is odd and the sum of the vertex degrees in $G$ must be even, there exists a vertex $v_1$ in $G$ other than $x$, such that $\text{deg}(v_1) > 2$. Therefore, $v_1$ is adjacent to at least 2 vertices of $V(G) \setminus \{x\}$. Now $v_1$ must be the unique common neighbour of $x$ and some other vertex in $G$. Let $v_2$ be such a vertex. Then $v_2$ is adjacent to $x$ and $v_1$ and to no other vertex of $G$ since $x$ has degree $n - 1$. However, this means that $v_2$ cannot be the unique common neighbour of $x$ and any other vertex as the degree of $v_2$ is 2 and we have a path of length 2 from $x$ to $v_1$ via a neighbour of $v_1$ different from $v_2$. Hence $G$ is not a strong-me$_2$-graph. \qed

We know that if $G$ is a graph of exponent 2 and even order $n$ then the minimum number of edges in $G$ is $\frac{3n}{2} - 1$ [12] and the only graphs which attain this minimum are described in [12]. Such graphs have a vertex of full degree and hence are not strong-me$_2$-graphs. We note however that if $n$ is odd, the graphs of exponent 2 that attain the minimum number of edges are the windmill graphs and these are strong-me$_2$-graphs.

We now observe that there does not exist a strong-me$_2$-graph on 4 vertices. Every vertex in such a graph would have degree at least 2, and by Lemma 3.2.2 no vertex could have degree 3. Hence every vertex must have degree exactly 2. However, the only graph on 4 vertices in which every vertex is of degree 2 is $C_4$ and $C_4$ is not a strong-me$_2$-graph.

The windmill graphs show that vertices of degree 2 and $n - 1$ can exist together in a strong-me$_2$-graph of odd order $n$. We now determine the maximum and minimum possible vertex degrees in a strong-me$_2$-graph of even order.

**Theorem 3.2.3.** Let $G$ be a strong-me$_2$-graph of even order $n$. Then $G$ cannot have a vertex whose degree exceeds $n - 4$. 

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Proof. We showed in Lemma [2.2] that $G$ cannot have a vertex of degree $n - 1$. So we must show that $G$ cannot have a vertex of degree $n - 2$ or $n - 3$.

Suppose $G$ has a vertex of degree $n - 2$ and label this vertex $x$. So there exists a unique vertex $y$ in $G$ such that $xy$ is not an edge in $G$. Now $x$ must have a path of length 2 to $y$ so there exists a vertex $v_1$ such that $xv_1$ and $v_1y$ are edges in $G$. However, since $x$ is adjacent in $G$ to every other vertex in $G$ except $y$, $y$ cannot be the unique common neighbour of $v_1$ and any other vertex in $G$. Hence $v_1y$ is not strongly-2-required and $G$ is not a strong-me$_2$-graph.

Now suppose $G$ has a vertex of degree $n - 3$ and label this vertex $x$. We know that $G$ has even order at least 6. Therefore $x$ is adjacent to at least 3 vertices in $G$, and there exist two vertices $y$ and $z$ not adjacent to $x$ in $G$. Since $n - 3$ is odd and every edge must be on a triangle, there exists a vertex adjacent to $x$ such that the degree of this vertex in the subgraph induced on $x$ and its neighbours is at least 3. Suppose this vertex is $v_1$ and $v_1v_2$ and $v_1v_3$ are edges in $G$. Now the edge $xv_2$ must be strongly-2-required in $G$ so we have two possibilities:

1. $v_2 = \text{ucn}(x, y)$ or $v_2 = \text{ucn}(x, z)$. Without loss of generality we can assume $v_2 = \text{ucn}(x, y)$. The edge $v_2y$ must be strongly-2-required hence $yz$ is an edge. Now there must be a path of length 2 from $v_2$ to $y$ and $y$ cannot be adjacent to any other vertices that are adjacent to $x$ so $v_2z$ must be an edge. However, now the edge $v_1v_3$ is not strongly-2-required for $v_1$ to any other vertex in $G$.

2. $v_2$ is the unique common neighbour of $x$ and some other vertex adjacent to $x$. This vertex can’t be $v_1$ or $v_3$. Suppose $v_2 = \text{ucn}(x, v_4)$ and hence $v_4$ has no more neighbours amongst the neighbours of $x$. Now $v_4$ must have paths of length 2 to $y$ and $z$. We have two possibilities for this:

   - $v_4$ has paths of length 2 to $y$ and $z$ via $v_2$, hence $v_2y$ and $v_2z$ are edges. Then $y$ must be the unique common neighbour of $v_2$ and another vertex and the only possibility for this vertex is $z$. Hence $yz$ is an edge and the edge $v_2v_1$ is not strongly-2-required.

   - Now suppose we have a path of length 2 from $v_4$ to $z$ via $y$. Hence $yz$ is an edge. In order for $v_4$ to have a path of length 2 to $y$, either $v_4z$ is an edge or $v_2y$
is an edge. In either case we get that the edge $v_2v_1$ is not strongly-2-required. Hence $G$ cannot have a vertex of degree $n-1$, $n-2$ or $n-3$. The graph of Example §3.2.1 has order 10 and a vertex $w$ of degree 6. Therefore the maximum degree of a vertex in a strong-me$_2$-graph of even order $n$ is at most $n-4$. □

**Theorem 3.2.4.** Let $G$ be a strong-me$_2$-graph of even order $n$. Then $G$ cannot have a vertex of degree 2.

**Proof.** We know that $G$ cannot have a vertex of degree 1, so we must show that $G$ cannot have a vertex of degree 2.

Suppose $G$ has a vertex $x$ of degree 2 and label the neighbours of $x$ as $v_1$ and $v_2$. Hence $v_1v_2$ is an edge. Let $X = V(G) \setminus \{x, v_1, v_2\}$. Now $x$ must have a path of length 2 to every vertex in $X$ and the edge $v_1x$ must be strongly-2-required for $v_1$ to $v_2$. Therefore, $v_1$ and $v_2$ cannot have a common neighbour in $X$ and the disjoint union of $N_X(v_1)$ and $N_X(v_2)$ is all of $X$.

Since the maximum degree of a vertex in $G$ is at most $n-4$, every vertex has at least 3 non-neighbours, and it follows that $|N_X(v_1)| \geq 3$ and $|N_X(v_2)| \geq 3$. Since $|V(X)| = n-3$ which is odd, it follows that exactly one of $v_1, v_2$ is adjacent to an odd number of vertices of $X$, greater than 1.

Suppose this vertex is $v_1$. Then $|N_X(v_1)|$ is odd and $|N_X(v_1)| \geq 3$. There must exist paths of length 2 from $v_1$ to each of its neighbours, hence there exists a vertex $v_3$ in $X$ adjacent to $v_1$ and adjacent to at least two neighbours of $v_1$ in $X$. The edge $v_1v_3$ must be strongly-2-required so $v_3$ is the unique common neighbour of $v_1$ and a neighbour of $v_1$ in $X$, say $v_4$. Hence $v_4$ cannot share any more neighbours with $v_1$ and $v_1v_4$ is not strongly-2-required as we have a path of length 2 from $v_1$ to $v_3$ via another common neighbour of $v_1$ and $v_3$ in $X$. We conclude that a strong-me$_2$-graph $G$ of even order $n$ cannot have a vertex of degree 2 and therefore the minimum degree of a vertex in $G$ is at least 3. □

The graph of Example §3.2.1 has a vertex of degree 3 so we know that the minimum degree of a vertex in a strong-me$_2$-graph of even order can be equal to 3. In fact, the graph of Example §3.2.1 includes vertices of degree $n-4$ and degree 3, so both extremes can occur in the same graph.
Using Theorems 3.2.3 and 3.2.4 we will now show that the minimum number of vertices for which a strong-me$_2$-graph of even order exists is 10.

**Lemma 3.2.5.** There does not exist a strong-me$_2$-graph $G$ on 6 vertices.

*Proof.* Suppose $G$ is a strong-me$_2$-graph on 6 vertices. Since the order of $G$ is even, it follows from Theorem 3.2.3 that the maximum degree of a vertex in $G$ is at most 2, and using Theorem 3.2.4 we know that the minimum degree of a vertex of $G$ is at least 3. We reach a contradiction and therefore there cannot exist a strong-me$_2$-graph on 6 vertices.

**Lemma 3.2.6.** There does not exist a strong-me$_2$-graph on 8 vertices.

*Proof.* Suppose $G$ is a strong-me$_2$-graph on 8 vertices. By Theorem 3.2.3, the maximum vertex degree in $G$ is at most 4, and we also know by Theorem 3.2.4 the minimum vertex degree is at least 3. We will now show that $G$ cannot have a vertex of degree 3 or 4 and hence there does not exist a strong-me$_2$-graph on 8 vertices.

Suppose $G$ has a vertex $v_1$ of degree 3 and suppose that $v_1$ is adjacent to $v_2$, $v_3$ and $v_4$. One of $v_2$, $v_3$, $v_4$ must be adjacent to the other two for paths of length 2 to $v_1$. Suppose $v_3v_2$ and $v_3v_4$ are edges in $G$. Now $v_2$ must be the unique common neighbour of $v_1$ and another vertex in $G$ and similarly for $v_4$. So there are distinct vertices $v_5$ and $v_6$ in $G$, different from the other vertices mentioned so far in $G$, for which $v_2 = \text{ucn}(v_1, v_5)$ and $v_4 = \text{ucn}(v_1, v_6)$.

We also note that $v_4v_1$ must be required for a path of length 2 from $v_4$ to $v_3$ and similarly for the edge $v_2v_1$. Therefore, $v_3$ and $v_4$ cannot share any common neighbour other than $v_1$, and neither can $v_2$ and $v_3$.

Now there must be a path of length 2 from $v_2$ to $v_5$ (note that this cannot be via $v_1$ since $v_1$ has degree 3 in $G$ and the neighbours of $v_1$ are $v_2$, $v_3$ and $v_4$), and $v_2v_6$ cannot be an edge since $v_4 = \text{ucn}(v_1, v_6)$, so there must be a vertex $v_7$, different from $v_3$, $v_4$ or $v_6$, such that $v_2v_7$ and $v_5v_7$ are edges. Also, $v_4$ must have a path of length 2 to $v_6$, so both $v_4$ and $v_6$ must be adjacent to either $v_7$ or a vertex $v_8$ which is distinct from the other vertices mentioned so far in $G$. However $v_2$ now has degree 4 in $G$ which is maximal and therefore $v_2$ has no more neighbours in $G$, and $v_5$ must have a path of length 2 to
\(v_8\), so it must be that \(v_4v_8\) is an edge and hence \(v_6v_8\) must be an edge also. We now note that \(v_4\) has maximal degree 4 in \(G\) and therefore has no more neighbours in \(G\).

We also note that \(v_3\) has only \(v_1, v_2\) and \(v_4\) as neighbours, otherwise the edge \(v_2v_1\) or \(v_4v_1\) is not strongly-2-required. Therefore at this point, the only remaining adjacencies possible are between \(v_5, v_6, v_7\) and \(v_8\).

There must be paths of length 2 from \(v_5\) to \(v_4\) and from \(v_7\) to \(v_4\). So \(v_5\) must be adjacent to at least one of \(v_6, v_8\) and similarly for \(v_7\). We have two possibilities:

1. \(v_5\) and \(v_7\) are mutually adjacent to one of \(v_6, v_8\). Suppose \(v_5v_6\) and \(v_7v_6\) are edges. There must be a path of length 2 from \(v_2\) to \(v_8\) so \(v_5v_8\) or \(v_7v_8\) is an edge, say \(v_5v_8\). Now the graph has exponent 2 but \(v_6v_7\) is not strongly-2-required, so we reach a contradiction and we conclude that \(v_5\) and \(v_7\) cannot both be adjacent to one of \(v_6, v_8\).

2. \(v_5v_6, v_7v_8\) are edges and \(v_5v_8, v_6v_7\) are not edges. Now there is no path of length 2 from \(v_5\) to \(v_6\) and \(G\) does not have exponent 2.

We conclude that we reach a contradiction and therefore \(G\) cannot have a vertex of degree 3.

It remains to consider the possibility that \(G\) is a 4-regular graph on 8 vertices. Assume this and let \(v_1\) be a vertex of \(G\) with neighbours \(v_2, v_3, v_4\) and \(v_5\). There must be paths of length 2 from \(v_1\) to each of its neighbours so we have the following two possibilities:

- There is a vertex of degree at least 2 in the subgraph of \(G\) induced on the neighbours of \(v_1\). Suppose that \(v_3\) is adjacent to both \(v_2\) and \(v_4\). At most one of \(v_1v_2\) and \(v_1v_4\) can be strongly-2-required for \(v_1\) to \(v_5\). This means that the other must be strongly-2-required for \(v_1\) to a vertex \(v_6\) in \(G\), different from \(v_2, v_3, v_4\) and \(v_5\). Suppose \(v_1v_4\) is strongly-2-required for \(v_1\) to \(v_6\). Then \(v_2, v_3\) or \(v_5\) are not adjacent to \(v_6\). However, this means that \(v_6\) has degree at most 3 in \(G\).

- The degree of each vertex in the graph induced on the neighbours of \(v_1\) is 1. So suppose \(v_2v_3\) and \(v_4v_5\) are the only edges among the neighbours of \(v_1\). There are 3 more vertices in \(G\) which we call \(v_6, v_7\) and \(v_8\), and each of \(v_2, v_3, v_4, v_5\) must each
be adjacent to exactly 2 of them since the degree of each of $v_2, v_3, v_4$ and $v_5$ must be 4. Suppose $v_2v_6$ and $v_2v_7$ are edges. Now $v_3, v_4$ and $v_5$ must each be adjacent to at least one of $v_6, v_7$ and hence the edge $v_2v_1$ is not strongly-2-required for $v_2$ to any other vertex of $G$.

Therefore, there does not exist a strong-me$_2$-graph on 8 vertices.

We know that there does not exist a strong-me$_2$-graph on 2 or 4 vertices and by Lemma 3.2.5 and Lemma 3.2.6 we know that there does not exist a strong-me$_2$-graph on 6 or 8 vertices. Example 3.2.1 demonstrates a strong-me$_2$-graph on 10 vertices and this is the minimum number of vertices on which a strong-me$_2$-graph of even order exists.

In fact, we can show that there exists a strong-me$_2$-graph $G$ of order $n$ for every even $n \geq 10$.

**Lemma 3.2.7.** Let $n$ be a positive even integer greater than or equal to 10. Then there exists a strong-me$_2$-graph $G$ of order $n$.

**Proof.** Let $n = 2k$ such that $k \in Z^+$ and $k \geq 5$. We construct a strong-me$_2$-graph $G$ of order $n$ as follows:

1. There are 3 vertices in $G$ labelled $v_1, v_2$ and $v_3$ such that $v_1v_2, v_1v_3$ and $v_2v_3$ are edges.

2. There are two vertices $a_1$ and $b_1$ such that $a_1b_1, a_1v_1$ and $b_1v_1$ are edges. Similarly, there are two vertices $a_2$ and $b_2$ such that $a_2b_2, a_2v_2$ and $b_2v_2$ are edges.

3. There are $2k - 8$ vertices $a_i, b_i$ for $i = 3, ..., k - 2$ that are adjacent to $v_3$ such that $a_ib_i$ is an edge for each $i = 3, ..., k - 2$.

4. There is a vertex $w$ adjacent to all $a_i$ and $b_i$ for $i = 1, ..., k - 2$.

When $n = 10$ this construction produces the graph of Example 3.2.1. The argument used to show that the graph of Example 3.2.1 is a strong-me$_2$-graph extends here to show that $G$ is a strong-me$_2$-graph.
We remark that the graphs described in Lemma 3.2.7 can be considered as an extension of Example 3.2.1, where we are adding on multiple copies of $a_3$ and $b_3$. In fact, once we begin with Example 3.2.1, we can add on multiple copies of the $a_i, b_i$ adjacent to any of $v_1, v_2$ and $v_3$ and obtain a strong-me$_2$-graph on an even number of vertices.

We now introduce some notation describing these graphs, which we then show are the only strong-me$_2$-graphs of even order attaining the bound of $n - 4$ for their maximum vertex degree.

For every non-decreasing triple $(k_1, k_2, k_3)$ of natural numbers we define a strong-me$_2$-graph $H(k_1, k_2, k_3)$ of order $2(k_1 + k_2 + k_3) + 4$ as follows:

- $V(H(k_1, k_2, k_3)) = \{x, y, z, w, a_1, b_1, ..., a_{k_1}, b_{k_1}, c_1, d_1, ..., c_{k_2}, d_{k_2}, e_1, f_1, ..., e_{k_3}, f_{k_3}\}$.
- The subgraph induced on $\{x, y, z\}$ is a copy of $K_3$.
- The $a_i, b_i$ are all adjacent to $x$, the $c_i, d_i$ are all adjacent to $y$, and the $e_i, f_i$ are all adjacent to $z$.
- Each $a_i$ is adjacent to the corresponding $b_i$, each $c_i$ is adjacent to the corresponding $d_i$, and each $e_i$ is adjacent to the corresponding $f_i$.
- $w$ is adjacent to everything except $x, y$ and $z$.

**Theorem 3.2.8.** Let $G$ be a strong-me$_2$-graph of even order $n$. Then $G \cong H(k_1, k_2, k_3)$ for some $k_1, k_2, k_3$ with $2(k_1 + k_2 + k_3) + 4 = n$ if and only if $G$ has a vertex of degree $n - 4$.

**Proof.** ($\implies$) Suppose $G \cong H(k_1, k_2, k_3)$. It is clear that the vertex $w$ in the above construction has degree $n - 4$.

($\impliedby$) Let $u$ be a vertex of $G$ of degree $n - 4$ and label the non-neighbours of $u$ as $y, z, w$. We first show that every neighbour of $u$ must be adjacent to exactly one of $y, z, w$.

- We show there cannot be a neighbour of $u$ adjacent to all three of $y, z, w$. Suppose on the contrary that $x_1$ is such a vertex. The edge $x_1 y$ must be strongly-2-required for $x_1$ to $w$ or for $x_1$ to $z$. We can assume $x_1 y$ is strongly-2-required for $x_1$ to $w$. Therefore $yw$ is an edge. However, now the edge $x_1 z$ is not strongly-2-required as there are paths of length 2 from $x_1$ to the neighbours of $u$ via $u$ and there are paths
of length 2 from \( x_1 \) to \( w \) and \( y \) via \( y \) and \( w \) respectively. We conclude that \( u \) cannot have a neighbour adjacent to all three of \( y, z, w \).

- We show that there cannot be a neighbour of \( u \) that is adjacent to exactly two of \( y, z, w \). Suppose there is. Suppose \( x_1 \) is a neighbour of \( u \) and \( x_1y, x_1w \) are edges but \( x_1z \) is not an edge. We first show that no common neighbour of \( x_1 \) and \( u \) in \( G \) can be adjacent to any of \( y, z, w \).

Let \( x_2 \) be a common neighbour of \( u \) and \( x_1 \).

- Suppose \( x_2z \) is an edge in \( G \). Then \( x_2z \) is not required for a path of length 2 beginning at \( x_2 \) and \( G \) is not a strong-me\(_2\)-graph.

- Suppose \( x_2y \) is an edge in \( G \). Then \( x_2y \) must be strongly-2-required in \( G \) and the only possibility is for \( x_2 \) to \( z \). However, now the edge \( x_1w \) is not strongly-2-required in \( G \). A similar argument shows that \( x_2w \) is not an edge in \( G \).

We conclude that \( x_2 \) is not adjacent to any of \( y, w, z \).

Now \( x_2 \) must have a path of length 2 to \( z \) so there exists a vertex \( x_3 \) that is a common neighbour of \( u \) and \( x_2 \) and is also a neighbour of \( z \) in \( G \). Also, \( x_1 \) must have paths of length 2 to \( y, w \) and \( z \) (and these paths cannot be via a neighbour of \( u \)), so we can assume \( yw \) and \( yz \) are edges in \( G \). We show that \( x_1, x_2 \) and \( x_3 \) have no more common neighbours with \( u \) which will contradict the hypothesis that \( G \) is a strong-me\(_2\)-graph. The edge \( x_1x_2 \) can only be strongly-2-required for \( x_1 \) to \( u \) and so \( x_2 = \text{ucn}(x_1, u) \), so \( x_1 \) has no more common neighbours with \( u \) in \( G \). By a similar argument \( x_2 = \text{ucn}(x_3, u) \) in \( G \). Note that the only common neighbours of \( x_2 \) and \( u \) are \( x_1 \) and \( x_3 \), since if there is another such neighbour \( x_4 \), then the edge \( x_2x_4 \) is not strongly-2-required in \( G \). We conclude that \( x_1, x_2 \) and \( x_3 \) have no more common neighbours with \( u \) in \( G \).

Let \( X = V(G) \setminus \{u, x_1, x_2, x_3, y, z, w\} \). Since \( G \) is of even order at least 10, it follows that the number of vertices in \( X \) is odd. Note that every vertex in \( X \setminus \{u\} \) is a neighbour of \( u \) in \( G \), and every vertex in \( X \setminus \{u\} \) has at least one neighbour in \( X \setminus \{u\} \). This means that there exists a vertex \( x_4 \) in \( X \) that is adjacent to at least two distinct neighbours \( x_5 \) and \( x_6 \) of \( u \) in \( G \). Now the edge \( ux_5 \) must be
strongly-2-required in $G$, and there exist paths of length 2 from $u$ to $y, w$ and $z$ via $x_1$ and $x_3$, so there exists a vertex $x_7$, distinct from those already encountered in $G$, such that $x_5 = \text{ucn}(u, x_7)$. However, the edge $ux_7$ is not strongly-2-required in $G$ and therefore $G$ is not a strong-me$_2$-graph. We conclude that $u$ cannot have a neighbour adjacent to exactly two of $y, z, w$.

- We now show that every neighbour of $u$ must be adjacent to at least one of $y, z, w$. Suppose on the contrary that $x_1$ is a neighbour of $u$ that is not adjacent to any of $y, z, w$. The edge $ux_1$ must be strongly-2-required so there is a neighbour $x_2$ of $u$ in $G$ such that $x_1$ is the unique common neighbour of $u$ and $x_2$. Hence the degree of $x_2$ in the subgraph induced on $u$ and its neighbours is 2. Now $x_2$ must have paths of length 2 to $y, w$ and $z$, and $x_1$ is not adjacent to any of $y, z, w$. This means that $x_2$ must be adjacent to at least 2 of $y, z, w$ in order to have paths of length 2 to all three vertices, but we have previously shown that a neighbour of $u$ cannot be adjacent to two or more of $y, z, w$. Therefore, there cannot be a neighbour of $u$ that is adjacent to none of $y, z, w$.

We conclude that every neighbour of $u$ must be adjacent to exactly one of $y, z, w$. We now use this to show that this means $G \cong H_n(k_1, k_2, k_3)$.

Let $x_1$ be a neighbour of $u$. Then $x_1$ must be adjacent to exactly one of $y, z, w$, so we can assume $x_1y$ is an edge. There must be a path of length 2 from $x_1$ to $y$ so let $x_2$ be a common neighbour of $x_1$ and $y$. We note that $x_2$ is different from $w, z$ and $u$, so this means that $x_2u$ is an edge. The edge $x_1y$ must be strongly 2-required for $x_1$ to $w$ or $x_1$ to $z$, so we can assume $y = \text{ucn}(x_1, w)$. Now $ux_1$ can only be strongly-2-required for $u$ to a neighbour of $u$. We show that this vertex is $x_2$.

Suppose not. Suppose $ux_1$ is strongly-2-required for $u$ to a neighbour $x_3$ of $u$, other than $x_2$. So $x_1 = \text{ucn}(u, x_3)$. Hence $x_3$ has no more neighbours amongst the neighbours of $u$. Now $x_1x_3$ can only be strongly-2-required for $x_1$ to $z$ and therefore $x_3z$ is an edge. But now there is no path of length 2 from $x_3$ to $z$ as $x_3$ cannot have any more neighbours amongst the neighbours of $u$ and $x_3$ is only adjacent to one of $y, z, w$. We conclude $ux_1$ is strongly-2-required for $u$ to $x_2$, and similarly, $ux_2$ is strongly-2-required for $u$ to $x_1$. So this means that, in the subgraph induced on $N(u)$, the degree of each vertex must be exactly 1, and each pair of neighbours in this subgraph must both be adjacent to the
same vertex $y, z$ or $w$.

Now $x_1$ must also have a path of length 2 to $z$, and the only possibility for this is via $y$ so $yz$ must be an edge. There must be a path of length 2 from $u$ to $w$, so there is a neighbour of $u$, say $x_3$, such that $x_3w$ is an edge and there exists a vertex $x_4$ such that $x_4u, x_4x_3$ and $x_4w$ are edges. There must be a path of length 2 from $x_3$ to $z$ and the only possibility for this is via $w$ so $wz$ must be an edge. We now observe that the subgraph induced on the non-neighbours of $u$ is a copy of $K_3$.

There must be a path of length 2 from $u$ to $z$ so there is a vertex $x_5$ such that $x_5z$ is an edge, and a vertex $x_6$ such that $ux_6, x_6x_5$ and $x_6z$ are edges.

In the subgraph induced on $V(G) \setminus \{u, x_1, x_2, x_3, x_4, x_5, y, z, w\}$, each vertex is of degree 1, and each pair of adjacent vertices in this subgraph are adjacent to exactly one of $y, z, w$ in $G$. Furthermore, each pair of adjacent vertices are adjacent to exactly the same vertex $y, z, w$. Hence $G \cong H(k_1, k_2, k_3)$.  

We remark that a consequence of Theorem 3.2.8 is that a strong-me$_2$-graph of even order $n$ can possess at most one vertex of degree $n - 4$.

### 3.3 The complement of a strong-me$_2$-graph

We now discuss the complement of a strong-me$_2$-graph. We show that the complement of a strong-me$_2$-graph is connected and has diameter at most 3, except in the case of windmills.

We first recall Lemma 3.1.5 that says if $G$ is a strong-me$_2$-graph of order $n$ with a vertex of degree $n - 1$ then $G$ is a windmill. We use this to show that any strong-me$_2$-graph with disconnected complement is a windmill.

**Lemma 3.3.1.** Let $G$ be a strong-me$_2$-graph. Then $\overline{G}$ is disconnected if and only if $G$ is a windmill.

**Proof.** ($\Longrightarrow$) Suppose $\overline{G}$ is disconnected and $|V(G)| = n$. Then $V(G)$ is the disjoint union of $A$ and $B$ where $|A| = k, |B| = n - k$ and every vertex of $A$ is adjacent to every vertex of $B$ in $G$.

Suppose $k$ and $n - k$ are both at least 2 and let $a \in A$ and $b \in B$. Since $G$ is a strong-me$_2$-graph and $|B| \geq 2$, the edge $ab$ can only be strongly-2-required for $a$ to a vertex $b'$
in $B$, since there are at least 2 paths of length 2 between $a$ and every other vertex of $A$ via vertices of $B$.

Similarly, the edge $ba$ must be strongly-2-required for $b$ to a vertex $a'$ in $A$. However there exists a path of length 2 from $b$ to $a'$ via $b'$ so we reach a contradiction. We conclude that either $k$ or $n-k$ must be 1 and therefore $G$ has a vertex of degree $n-1$. By Lemma 3.1.5 this means that $G$ is a windmill.

( $\iff$ ) If $G$ is a windmill then $G$ has a vertex $x$ of degree $n-1$. This means that $x$ is an isolated vertex in $\overline{G}$ and therefore $\overline{G}$ is disconnected. \hfill $\square$

We observe that the proof of Lemma 3.3.1 follows from Lemma 2.4.3, since the only strong-me$_2$-graphs of order $n$ with a vertex of degree $n-1$ are windmills, and the only $G_{n,k}$ that are strong-me$_2$-graphs are those which are windmills for the following reason. First recall that the vertex set of $G_{n,k}$ is the disjoint union $A \cup B$ where $|A| = n - 2k$ and $|B| = 2k$. If $G_{n,k}$ is not a windmill, then $|A| \geq 2$ and there are two vertices in $G_{n,k}$ with the same neighbour set. By Lemma 3.1.6, this cannot happen in a strong-me$_2$-graph and therefore if $G_{n,k}$ is not a windmill, then $G_{n,k}$ is not a strong-me$_2$-graph.

**Lemma 3.3.2.** Let $G$ be a strong-me$_2$-graph of even order $n$. Then $\overline{G}$ is connected and the diameter of $\overline{G}$ is 2.

**Proof.** We know that $\overline{G}$ is connected by Lemma 3.3.1. The diameter of $\overline{G}$ must be greater than 1, since $\overline{G} \not= K_n$. Suppose $u$ and $v$ are vertices of $\overline{G}$ and that $uv$ is not an edge in $\overline{G}$. We must show that there exists a path of length 2 between $u$ and $v$ in $\overline{G}$.

Suppose $u$ and $v$ are adjacent vertices in $G$ with no mutual non-neighbour, which means that there is no path of length 1 or 2 between $u$ and $v$ in $\overline{G}$. Let $G_u$ be the subgraph of $G$ induced on $N_G(u) \setminus (N_G(v) \cup \{v\})$ and let $G_v$ be the subgraph of $G$ induced on $N_G(v) \setminus (N_G(u) \cup \{u\})$. Since the maximum possible degree of a vertex in $G$ is $n-4$ and since $N_G(u) \cup N_G(v) = V(G)$, the sets $V(G_u)$ and $V(G_v)$ are non-empty.

Let $y \in G_u$. Then $y = \text{ucn}(u, y')$ for some vertex $y'$ in $G$ with $y' \not\in G_v$. Thus $y$ is the only neighbour of $y'$ in $G_u$. Furthermore, $y'$ is the unique common neighbour of $u$ and some vertex in $G_u$, and the only candidate for this vertex is $y$. It follows that every vertex in $G_u$ has degree 1. Hence $|G_u|$ is even and so is $|G_v|$ by the same argument.

Since $G$ has even order and $V(G) \setminus \{u, v\}$ is the disjoint union of $V(G_u), V(G_v)$ and $N_G(u) \cap N_G(v)$, it follows that $|N_G(u) \cap N_G(v)|$ is a positive even number. In particular
u and v have at least 2 common neighbours in G. If \( p \in N_G(u) \cap N_G(v) \), it follows that the edge \( up \) is not required for a 2-path in G starting from u, contrary to the hypothesis that G is a strong-me_2-graph.

\[ \square \]

**Lemma 3.3.3.** Let G be a strong-me_2-graph of odd order. If \( \overline{G} \) is connected, then the maximum possible diameter of \( \overline{G} \) is 3.

**Proof.** Since \( \overline{G} \) is connected, there is no vertex of degree \( n - 1 \) in G. We know \( \overline{G} \not\cong K_n \), therefore the diameter of \( \overline{G} \) is not 1. We will show that if \( u \) and \( v \) are vertices of G, and there is no path of length 2 between \( u \) and \( v \) in \( \overline{G} \), then there must be a path of length 3 between \( u \) and \( v \) in \( \overline{G} \).

Assume \( uv \) is not an edge in \( \overline{G} \) and that there does not exist a path of length 2 from \( u \) to \( v \) in \( \overline{G} \). Therefore, \( uv \) is an edge in G. Let \( G_u \) be the subgraph of G induced on \( N_G(u) \setminus (N_G(v) \cup \{v\}) \) and let \( G_v \) be the subgraph of G induced on \( N_G(v) \setminus (N_G(u) \cup \{u\}) \). Since there is no vertex of degree \( n - 1 \) in G and since \( N_G(u) \cup N_G(v) = V(G) \), the sets \( V(G_u) \) and \( V(G_v) \) are non-empty. To show that there is a path of length 3 between \( u \) and \( v \) in \( \overline{G} \), we must show that there exists a vertex in \( G_u \) and a vertex in \( G_v \) that are not adjacent to each other in G.

Suppose that every vertex in \( G_u \) is adjacent to every vertex in \( G_v \) in G. The argument of Lemma 3.3.2 shows that every vertex in \( G_u \) has degree 1, similarly for \( G_v \), and also that \( |G_u| \) and \( |G_v| \) are both even, hence both are at least 2. Let \( x \in G_v \) and \( y \in G_u \) and so \( xy \) is an edge in G. The edge \( xy \) must be strongly-2-required for \( x \) to a vertex \( y' \) in \( G \) and for \( y \) to a vertex \( x' \) in G. Note that since \( uv \) is an edge in \( G \), it follows that \( y' \neq u \) and \( x' \neq v \). Hence \( y = \text{ucn}(x, y') \) and \( x = \text{ucn}(y, x') \) in G and so \( yy' \) is an edge in \( G_u \) and \( xx' \) is an edge in \( G_v \). However, since \( x' y' \) is an edge in \( G \), \( y' \) is also a common neighbour of \( y \) and \( x' \) in G and \( xy \) is not strongly-2-required in \( G \). Therefore \( G \) is not a strong-me_2-graph. We conclude that there exists a vertex in \( G_u \) and a vertex in \( G_v \) that are not adjacent to each other, and therefore there exists a path of length 3 between \( u \) and \( v \) in \( \overline{G} \). \[ \square \]

The graph \( L_{3,3} \) as shown in Example 2.3.3 is a strong-me_2-graph G on 9 vertices for which \( \overline{G} \) is connected and the diameter of \( \overline{G} \) is 2. It can be easily seen that every pair of
adjacent vertices of $L_{3,3}$ has 2 mutual non-neighbours and hence every pair of vertices in the complement of $L_{3,3}$ will be adjacent or have a path of length 2 between them.

The graph $G_1$ shown in Example 3.1.11 is an example of a strong-me$_2$-graph of order 7 such that $\overline{G_1}$ is connected and the diameter of $\overline{G_1}$ is 3. For example $d(a,e)$ and $d(b,e)$ in $\overline{G_1}$ are both 3, and the distance in $\overline{G_1}$ between every other pair of vertices is at most 2.
Chapter 4

Graph Products and $\text{me}_2$-type properties

In this chapter, we will begin by discussing a gradation of properties that are related to the $\text{me}_2$-property. We recall the Friendship Theorem, which states that the finite graphs with the property that every two vertices have exactly one neighbour in common are exactly the windmill graphs $W_r$, and we say that such graphs have the friendship property. We will recall and introduce some graph properties that are intermediate between the $\text{me}_2$-property and the friendship property and explore how these properties behave under different graph products.

4.1 Intermediate properties

We first recall and introduce some terminology and some intermediate properties between the $\text{me}_2$-property and the friendship property.

**Definition 3.1.7.** Let $G$ be a graph. We say that an edge $uv$ in $G$ is unitriangular if it belongs to exactly one triangle in $G$.

**Definition 3.1.1.** Let $G$ be a graph of exponent 2. We say that $G$ has the strong-$\text{me}_2$-property if for every edge $uv$ of $G$ there exists a vertex $w$ such that $v$ is the unique common neighbour of $u$ and $w$ and there exists a vertex $w'$ such that $u$ is the unique common neighbour of $v$ and $w'$.

**Definition 4.1.1.** Let $G$ be a graph of exponent 2. We say that $G$ has the double-$\text{me}_2$-property if $G$ is an $\text{me}_2$-graph and every vertex of $G$ is the unique common neighbour of some pair of vertices of $G$. 
A graph $G$ has the double-me$_2$-property if $G$ has exponent 2 and the deletion of any edge of $G$, or the deletion of any vertex of $G$ and its incident edges results in a graph that no longer has exponent 2. Hence $G$ is a graph of exponent 2 that is both edge-minimal and vertex-minimal.

We recall from Chapter 3 that any graph of exponent 2 in which every edge is unitriangular has the strong-me$_2$-property, but not every strong-me$_2$-graph has this feature. In the remainder of this section we focus attention on the double-me$_2$-property.

Recall from Chapter 2 that the property that every vertex is the unique common neighbour of some pair of vertices in a graph $G$ of exponent 2 does not imply that $G$ is an me$_2$-graph. There may be edges in such a graph $G$ that are not 2-required in $G$ as the following example from Chapter 2 shows.

**Example 2.1.4.** A graph $G$ of exponent 2 that is vertex-minimal but not edge-minimal.

$G$ has exponent 2 and every vertex of $G$ is the unique common neighbour of some pair of vertices in $G$. However, the edge $v_4v_5$ is not 2-required in $G$ and therefore $G$ is not an me$_2$-graph.

The following example shows that not every me$_2$-graph has the double-me$_2$-property.

**Example 4.1.2.** An example of a graph that has the me$_2$-property, but not the double-me$_2$-property. We observe that vertex $v_1$ is not the unique common neighbour of any pair of vertices in the graph.
From Definition 3.1.1 we can see that, in a strong-me$_2$-graph, every vertex is the unique common neighbour of some pair of vertices. It follows that the strong-me$_2$-property implies the double-me$_2$-property. The following example demonstrates that the converse is false.

**Example 4.1.3.** An example of a graph that has the double-me$_2$-property, but not the strong-me$_2$-property. It is easily confirmed that every vertex in $G$ is the unique common neighbour of some pair of vertices in $G$, but the edge $v_2v_1$ is not strongly-2-required in $G$.

The following relationships exist between the properties that we have discussed.

The friendship property $\Rightarrow$ every edge is unitriangular $\Rightarrow$ the strong-me$_2$-property $\Rightarrow$ the double-me$_2$-property $\Rightarrow$ the me$_2$-property.

### 4.2 Graph products

The main theme of this section is the interesting behaviour of the me$_2$-property and its variants under the Kronecker product of graphs. We also consider some other well-
known graph products in this context.

4.2.1 The Cartesian product

Definition 4.2.1. Let $G_1$ and $G_2$ be two graphs. The Cartesian product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the graph with vertex set $V$ and edge set $E$, where

- $V = \{(u, v) : u \in V(G_1), v \in V(G_2)\} = V(G_1) \times V(G_2),$  \\
- $E = \{((u, v)(u', v')) : u = u' \text{ and } vv' \in E(G_2), \text{ or } v = v' \text{ and } uu' \in E(G_1)\}.$

We note that the Cartesian product is commutative, i.e. $G \times H \cong H \times G$.

Example 4.2.2. This example demonstrates the Cartesian product of $P_2$ and $K_3$. Note $P_2 \times K_3$ consists of two copies of $K_3$ connected by three copies of $P_2$, and does not have exponent 2.

We want to consider how graphs with the me$_2$-property behave under the Cartesian product. We begin by presenting the following lemma which establishes when a Cartesian product has exponent 2.

Lemma 4.2.3. Let $G_1$ and $G_2$ be two graphs both of order at least 3. Then $G_1 \times G_2$ has exponent 2 if and only if $G_1$ and $G_2$ are complete graphs.
Proof. ( \( \implies \) ) Suppose \( G_1 \times G_2 \) has exponent two. Let \( u, w \) be distinct vertices of \( G_1 \) and let \( v, z \) be distinct vertices of \( G_2 \). Since \( G_1 \times G_2 \) has exponent 2, it follows that the vertices \( (u, v) \) and \( (w, z) \) of \( G_1 \times G_2 \) have a common neighbour in \( G_1 \times G_2 \) and therefore \( uw \) is an edge of \( G_1 \) and \( vz \) is an edge of \( G_2 \). Hence \( G_1 \) and \( G_2 \) both have order at least 2. We conclude that if \( G_1 \times G_2 \) has exponent 2, then \( G_1 \) and \( G_2 \) are both complete graphs.

( \( \iff \) ) Suppose \( G_1 \) and \( G_2 \) are complete graphs. Let \( (u, v) \) and \( (w, z) \) be vertices of \( G_1 \times G_2 \). We will show \( (u, v) \) and \( (w, z) \) have a common neighbour in \( G_1 \times G_2 \). We consider two possibilities:

- If \( u \neq w \) and \( v \neq z \), then \( (u, v) \) and \( (w, z) \) have a common neighbour \( (w, v) \) in \( G_1 \times G_2 \) since \( uw \) is an edge in \( G_1 \) and \( vz \) is an edge in \( G_2 \).
- If \( u = w \) then \( (u, v) \) and \( (w, z) \) have a common neighbour \( (u, p) \) where \( p \in V(G_2) \setminus \{v, z\} \). The case where \( v = z \) is similar. Hence \( G_1 \) and \( G_2 \) both must have order at least 3.

We conclude that every pair of vertices in \( G_1 \times G_2 \) have a common neighbour and so \( G_1 \times G_2 \) has exponent 2. \( \square \)

We note that it is necessary that both graphs have order at least 3 in Lemma 4.2.3 for the following reasons. If \( G_1 \) consists of an isolated vertex and \( G_2 \) is a graph of exponent 2, then \( G_1 \times G_2 \) has exponent 2, but it does not follow that \( G_2 \) is complete. The graphs in Example 4.2.2 are both complete graphs, one of which is of order 2, but the product does not have exponent 2.

From Lemma 4.2.3 it follows that for two non-complete graphs \( G_1 \) and \( G_2 \), \( G_1 \times G_2 \) is not an me\(_2\)-graph. We now show that, in general, the Cartesian product of two complete graphs is not an me\(_2\)-graph. The only exceptions to this are \( K_1 \times K_3 \cong K_3 \) and \( K_3 \times K_3 \cong L_{3,3} \). We have already encountered the graph \( L_{3,3} \) in Chapter 3 and we know that it is a strong-me\(_2\)-graph. We remark that \( K_2 \times K_2 \cong C_4 \), \( K_2 \times K_3 \) is the graph of Example 4.2.2 which is not an me\(_2\)-graph, and it is clear that \( K_1 \times K_n \) for \( n \neq 3 \) is not an me\(_2\)-graph. The following lemma shows that the Cartesian product of two complete graphs can have the me\(_2\)-property only if both have order at most 3.

**Lemma 4.2.4.** Let \( l \) and \( m \) be positive integers with \( m \geq 4 \). Then \( K_l \times K_m \) is not an me\(_2\)-graph.
Proof. Let \((u,v)(u,z)\) be an edge in \(K_l \times K_m\). We show that it is not 2-required in \(K_l \times K_m\). Suppose it is. Without loss of generality we can assume \((u,v)(u,z)\) is 2-required in \(K_l \times K_m\) for \((u,v)\) to some other vertex of \(K_l \times K_m\). We have two possibilities.

- \((u,v)(u,z)\) is 2-required for \((u,v)\) to a vertex \((u,x)\) in \(K_l \times K_m\). However, since \(m \geq 4\), there exists a vertex \((u,y)\) in \(K_l \times K_m\) such that \((u,v)(u,y)\) and \((u,y)(u,x)\) are edges in \(K_l \times K_m\).

- \((u,v)(u,z)\) is 2-required for \((u,v)\) to a vertex \((w,z)\) in \(K_l \times K_m\). However, then there exists a vertex \((w,v)\) in \(K_l \times K_m\) such that \((u,v)(w,v)\) and \((w,v)(w,z)\) are edges in \(K_l \times K_m\).

We conclude that the edge \((u,v)(u,z)\) is not 2-required in \(K_l \times K_m\) and therefore \(K_l \times K_m\) is not an me_2-graph. \(\square\)

Using Lemma 4.2.3 and Lemma 4.2.4 it is clear that, except in the cases noted above, the Cartesian product of two graphs is not an me_2-graph. \(\square\)

### 4.2.2 The Kronecker product

We now state the definition of the Kronecker product of matrices and present an example. This definition, applied to adjacency matrices, motivates the concept of the Kronecker product for graphs.

**Definition 4.2.5.** Let \(A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}\). Then the Kronecker product of \(A\) and \(B\), denoted \(A \otimes B\), is defined as the matrix

\[
A \otimes B = \begin{bmatrix}
    a_{11}B & \ldots & a_{1n}B \\
    \vdots & \ddots & \vdots \\
    a_{m1}B & \ldots & a_{mn}B \\
\end{bmatrix} \in \mathbb{R}^{mp \times nq}.
\]

**Example 4.2.6.** The following is the Kronecker product of the adjacency matrices of \(P_2\) and \(K_3\).

Let \(A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\), \(B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\)
Then \( A \otimes B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \)

Then Kronecker product \( P_2 \otimes K_3 \) is defined to be the graph whose adjacency matrix is \( A \otimes B \).

As this example suggests, the definition of the graph Kronecker product is as follows.

**Definition 4.2.7.** Let \( G_1 \) and \( G_2 \) be two graphs. The Kronecker product of \( G_1 \) and \( G_2 \), denoted \( G_1 \otimes G_2 \), is the graph with vertex set \( V \) and edge set \( E \), where

- \( V = \{(u,v) : u \in V(G_1), v \in V(G_2)\} = V(G_1) \times V(G_2) \),
- \( E = \{(u,v)(u',v') : uu' \in E(G_1), vv' \in E(G_2)\} \).

The Kronecker product of two graphs \( G_1 \) and \( G_2 \) is a graph whose vertices are the ordered pairs whose entries are vertices of \( G_1 \) and \( G_2 \) respectively, and in which there
is an edge connecting two vertices \((u, v)\) and \((u', v')\) in \(G_1 \otimes G_2\) precisely when \(uu'\) is an edge in \(G_1\), and \(vv'\) is an edge in \(G_2\).

We can see that Definition 4.2.5 and Definition 4.2.7 are equivalent when we consider the matrices \(A\) and \(B\) in Definition 4.2.5 to be adjacency matrices of graphs \(G_1\) and \(G_2\) respectively. Let \(V(G_1) = \{u_1, ..., u_m\}\) and \(V(G_2) = \{v_1, ..., v_n\}\) and let \(A\) and \(B\) be the adjacency matrices of \(G_1\) and \(G_2\) with respect to these orderings. Then \(G_1 \otimes G_2\) is the graph with vertex set \(V(G_1) \times V(G_2)\), whose adjacency matrix with respect to the corresponding lexicographic ordering on \(V(G_1) \times V(G_2)\) is \(A \otimes B\).

The Kronecker product is also known as the tensor product or direct product, among others [15]. The Kronecker product is a useful device to generate realistic network models and analyse network properties, see [16]. The Kronecker product is an important mechanism in the study of certain applied problems involving large scale Markov chains, see [15]. Further reading on the Kronecker product and applications of its properties can be found in [15], [16] and [19].

We now discuss some relevant properties of the Kronecker product of graphs.

1. Let \(G_1\) and \(G_2\) be graphs and let \(H = G_1 \otimes G_2\). Then:

   (a) \(|V(H)| = |V(G_1)||V(G_2)|\)

   (b) \(|E(H)| = 2|E(G_1)||E(G_2)|\)

   (c) for every \((u, v) \in V(H)\), \(\deg_H(u, v) = \deg_{G_1}(u) \deg_{G_2}(v)\)

We give a brief explanation of the above properties.

- The first property is straightforward since \(V(H) = V(G_1) \times V(G_2)\).
- Let \(e_1 \in E(G_1)\) and \(e_2 \in E(G_2)\). Then the pair \((e_1, e_2)\) contributes two edges to \(G_1 \otimes G_2\). Say \(e_1 = u_1u_2\) and \(e_2 = v_1v_2\). Then \((u_1, v_1)(u_2, v_2)\) and \((u_1, v_2)(u_2, v_1)\) are edges in \(G_1 \otimes G_2\).
- For every pair consisting of an edge in \(G_1\) incident with \(u\) and an edge in \(G_2\) incident with \(v\), there exists an edge in \(H\) incident with \((u, v) \in V(H)\). So the number of edges incident with a vertex \((u, v)\) in \(H\) is the product of the number of edges incident with \(u\) in \(G_1\), and the number of edges incident with \(v\) in \(G_2\).
2. The Kronecker product of graphs is commutative, i.e. \( G_1 \otimes G_2 \) and \( G_2 \otimes G_1 \) are isomorphic. Note that the Kronecker product of matrices is not commutative. However if \( A \) and \( B \) are two square matrices then \( A \otimes B \) and \( B \otimes A \) are permutation similar.

We will be considering connectedness and primitivity of Kronecker products. We first recall the following relevant facts about graphs from Chapter 1 before proceeding to a discussion of the behaviour of graph properties under the Kronecker product.

**Theorem 1.0.16.** A graph \( G \) is primitive if and only if it is connected and non-bipartite.

**Lemma 1.0.11.** If \( G \) is primitive of exponent \( k \) then there exists a walk of every length \( \geq k \) between every pair of vertices in \( G \).

**Lemma 1.0.6.** If a walk of length \( m \) exists from \( u \) to \( v \) in a graph \( G \), then walks of length \( m' \) exist from \( u \) to \( v \) for all \( m' \) with \( m' \geq m \) and \( m' \equiv m \mod 2 \).

Our discussion of connectedness of Kronecker products begins with the following observations.

1. Let \( G_1 \) and \( G_2 \) be graphs. If there exists a walk \( W_1 \) of length \( k \) between vertices \( u \) and \( v \) in \( G_1 \) and there exists a walk \( W_2 \) of length \( k \) between vertices \( w \) and \( z \) in \( G_2 \), then there exists a walk of length \( k \) between \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \) whose projections on \( G_1 \) and \( G_2 \) are \( W_1 \) and \( W_2 \) respectively.

2. If \( G_1 \otimes G_2 \) is connected, then there is a walk \( W \) between vertices \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \). This means that there is a walk between vertices \( u \) and \( v \) in \( G_1 \) and there is a walk between vertices \( w \) and \( z \) in \( G_2 \) both of the same length as \( W \). Therefore \( G_1 \) and \( G_2 \) are connected.

We conclude that \( G_1 \otimes G_2 \) is connected if and only if for all pairs \((u, v)\) of vertices of \( G_1 \), and all pairs \((w, z)\) of vertices in \( G_2 \) there exist walks of the same length from \( u \) to \( v \) in \( G_1 \) and from \( w \) to \( z \) in \( G_2 \).

**Theorem 4.2.8.** \( G_1 \otimes G_2 \) is connected if and only if \( G_1 \) and \( G_2 \) are connected and at least one of \( G_1 \) and \( G_2 \) is primitive.
Proof. \( \implies \) Assume \( G_1 \otimes G_2 \) is connected and suppose neither \( G_1 \) nor \( G_2 \) is primitive. Then \( G_1 \) and \( G_2 \) are both connected by Observation 2 above, and since neither \( G_1 \) nor \( G_2 \) is primitive, it follows from Theorem 1.0.16 that both \( G_1 \) and \( G_2 \) are bipartite. Let \( A, B \) be a bipartition of \( V(G_1) \) and \( C, D \) be a bipartition of \( V(G_2) \). Let \( u \) and \( v \) be vertices of \( A \), and \( w, z \) be vertices of \( C \) and \( D \) respectively. Then every path from \( u \) to \( v \) in \( G_1 \) has even length, and every path from \( w \) to \( z \) in \( G_2 \) has odd length. So there is no path between the vertices \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \), contradicting the assumption that \( G_1 \otimes G_2 \) is connected.

\( \iff \) Now suppose that \( G_1 \) is primitive with exponent \( k \) and \( G_2 \) is connected. Let \((u, w)\) and \((v, z)\) be a pair of vertices in \( G_1 \otimes G_2 \). We want to show there exists a walk between them. Since \( G_1 \) is primitive of exponent \( k \), there exist walks of all lengths at least \( k \) between every pair of vertices in \( G_1 \). Since \( G_2 \) is connected there exists a walk of some length between \( w \) and \( z \) in \( G_2 \), and by Lemma 1.0.6 there exists a walk from \( w \) to \( z \) in \( G_2 \) whose length is at least \( k \). We conclude that there exist walks of the same length from \( u \) to \( v \) in \( G_1 \) and from \( w \) to \( z \) in \( G_2 \), so \( G_1 \otimes G_2 \) is connected.

Theorem 4.2.9. \( G_1 \otimes G_2 \) is primitive if and only if both \( G_1 \) and \( G_2 \) are primitive. In this case, the exponent of \( G_1 \otimes G_2 \) is \( \max(\exp(G_1), \exp(G_2)) \).

Proof. \( \implies \) Suppose \( G_1 \otimes G_2 \) is primitive of exponent \( k \). So there exist walks of all lengths at least \( k \) between every pair of vertices in \( G_1 \otimes G_2 \). It follows that there exists walks of all lengths at least \( k \) between every pair of vertices in \( G_1 \) and between every pair of vertices in \( G_2 \). So \( G_1 \) and \( G_2 \) are primitive and \( \max(\exp(G_1), \exp(G_2)) \leq \exp(G_1 \otimes G_2) \).

\( \iff \) Suppose \( G_1 \) and \( G_2 \) are primitive and \( \exp(G_1) = k \), \( \exp(G_2) = m \) with \( m \leq k \). Then there exist walks of length \( k \) between all pairs of vertices in \( G_1 \) and \( G_2 \), hence there exist walks of length \( k \) between every pair of vertices in \( G_1 \otimes G_2 \). We conclude that \( G_1 \otimes G_2 \) is primitive of exponent at most \( k \), i.e., \( \exp(G_1 \otimes G_2) \leq \max(\exp(G_1), \exp(G_2)) \). \( \square \)

Corollary 4.2.10. \( G_1 \otimes G_2 \) is connected and imprimitive if and only if both \( G_1 \) and \( G_2 \) are connected, and exactly one of \( G_1 \) and \( G_2 \) is primitive.
Proof. The result follows directly from Theorem 4.2.8 and Theorem 4.2.9.

We now look at how the me$_2$-property and strong-me$_2$-property behave under the Kronecker product and what conditions are necessary and sufficient in order for the Kronecker product of two graphs to have the me$_2$-property/strong-me$_2$-property.

**Theorem 4.2.11.** Suppose that $G_1$ and $G_2$ are graphs for which $G_1 \otimes G_2$ has the me$_2$-property. Then $G_1$ and $G_2$ are me$_2$-graphs.

Proof. Since $G_1 \otimes G_2$ is an me$_2$-graph, it follows from Theorem 4.2.9 that $G_1$ and $G_2$ have exponent 2. Let $uv$ be an edge of $G_1$ and let $wz$ be an edge of $G_2$. The edge $(u, w)(v, z)$ is 2-required in $G_1 \otimes G_2$ since $G_1 \otimes G_2$ is an me$_2$-graph, so there exists a vertex $(x, y)$ in $G_1 \otimes G_2$ such that either $(u, w)$ is the unique common neighbour of $(v, z)$ and $(x, y)$ in $G_1 \otimes G_2$, or $(v, z)$ is the unique common neighbour of $(u, w)$ and $(x, y)$ in $G_1 \otimes G_2$. It follows that either $u = \text{ucn}(v, x)$ or $v = \text{ucn}(u, x)$ in $G_1$ and therefore the edge $uv$ is 2-required in $G_1$, and either $w = \text{ucn}(y, z)$ or $z = \text{ucn}(y, w)$ in $G_2$ and therefore the edge $wz$ is 2-required in $G_2$. Hence $G_1$ and $G_2$ are me$_2$-graphs.

Theorem 4.2.11 shows that $G_1$ and $G_2$ being me$_2$-graphs is a necessary condition for $G_1 \otimes G_2$ to be an me$_2$-graph, but it is not sufficient as the example below shows.

**Example 4.2.12.** Let the graphs below be $G_1$ and $G_2$ respectively. Note that these are two me$_2$-graphs.

$$
\begin{array}{ccc}
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
\end{array} & \begin{array}{c}
p_2 \\
q_1 \\
p_3 \\
q_4 \\
\end{array} & \begin{array}{c}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
\end{array}
\end{array}
\quad
G_1 \quad \otimes \quad G_2
$$

$G_1 \otimes G_2$ does not have the me$_2$-property. The product $G_1 \otimes G_2$ has exponent 2, but the edge $(p_2, q_3)(p_3, q_4)$ is not 2-required in $G_1 \otimes G_2$. The reason for this is that $p_2p_3$ is required for a path of length 2 only from $p_2$ to $p_1$ in $G_1$, and $q_3q_4$ is required for a path of length 2 only from $q_4$ to $q_1$ in $G_2$. Informally, we can think of the edges $p_2p_3$ and $q_3q_4$ as being 2-required in two opposing directions. In order for the edge $(p_2, q_3)(p_3, q_4)$ to
be 2-required in \( G_1 \otimes G_2 \), \( p_2p_3 \) must be 2-required in \( G_1 \) for a unique 2-path that begins with \( p_2 \) and \( q_3q_4 \) must be 2-required in \( G_2 \) for a unique 2-path that begins with \( q_3 \).

We use the following lemma to aid us in determining the conditions under which the Kronecker product of two graphs has the me\(_2\)-property.

**Lemma 4.2.13.** Let \((x, y)\) be a common neighbour of \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \). Then \((x, y) = ucn((u, w), (v, z))\) if and only if \(x = ucn(u, v)\) in \( G_1 \) and \(y = ucn(w, z)\) in \( G_2 \).

**Proof.** ( \( \implies \) ) Assume \((x, y) = ucn((u, w), (v, z))\) in \( G_1 \otimes G_2 \), and suppose \(x\) is not the unique common neighbour of \(u\) and \(v\) in \( G_1 \). So there exists another vertex \(x'\) that is a common neighbour of \(u\) and \(v\) in \( G_1 \). It follows that the vertex \((x', y)\) is a neighbour of \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \), contradicting our assumption that \((x, y) = ucn((u, w), (v, z))\) in \( G_1 \otimes G_2 \). Therefore, \(x = ucn(u, v)\) in \( G_1 \), and by a similar argument, \(y = ucn(w, z)\) in \( G_2 \).

( \( \iff \) ) Assume \(x = ucn(u, v)\) in \( G_1 \) and \(y = ucn(w, z)\) in \( G_2 \). Then the only path of length 2 between \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \) is via the vertex \((x, y)\) in \( G_1 \otimes G_2 \). \(\square\)

**Theorem 4.2.14.** The product \( G_1 \otimes G_2 \) is an me\(_2\)-graph if and only if \( G_1 \) and \( G_2 \) are me\(_2\)-graphs and at least one of \( G_1 \) and \( G_2 \) is a strong-me\(_2\)-graph.

**Proof.** ( \( \implies \) ) Assume that \( G_1 \otimes G_2 \) is an me\(_2\)-graph. By Theorem [4.2.11] \( G_1 \) and \( G_2 \) are me\(_2\)-graphs. If \( G_1 \) is a strong-me\(_2\)-graph then we are done. Suppose \( G_1 \) is not a strong-me\(_2\)-graph and let \( uv \) be an edge of \( G_1 \) that is 2-required only for a path from \( u \) to a vertex in \( G_1 \). We show that \( G_2 \) must be a strong-me\(_2\)-graph.

Suppose that \( G_2 \) is not a strong-me\(_2\)-graph. Let \( wz \) be an edge in \( G_2 \) that is 2-required only for a path from \( z \) to a vertex in \( G_2 \). From Lemma [4.2.13] it follows that \((u, w)\) is not the unique common neighbour of \((v, z)\) and any other vertex in \( G_1 \otimes G_2 \), and \((v, z)\) is not the unique common neighbour of \((u, w)\) and any other vertex in \( G_1 \otimes G_2 \). We conclude that the edge \((u, w)(v, z)\) is not 2-required in \( G_1 \otimes G_2 \), contradicting the assumption that \( G_1 \otimes G_2 \) is an me\(_2\)-graph.

( \( \iff \) ) Assume \( G_1 \) and \( G_2 \) are both me\(_2\)-graphs and suppose \( G_2 \) is strong-me\(_2\)-graph. Let \((u, w)(v, z)\) be an edge in \( G_1 \otimes G_2 \). Since \( G_1 \) is an me\(_2\)-graph, we know the edge \(uv\) is 2-required in \( G_1 \). Without loss of generality we can assume that the edge \(uv\)
is 2-required in \( G_1 \) for a path from \( u \) to a vertex \( x \) in \( G_1 \). Since \( G_2 \) is a strong-me\(_2\)-graph, the edge \( wz \) is strongly-2-required in \( G_2 \) for \( w \) to a vertex \( y \) in \( G_2 \). We conclude that the edge \((u, w)(v, z)\) in \( G_1 \otimes G_2 \) is required for a path of length 2 from \((u, w)\) to \((x, y)\) in \( G_1 \otimes G_2 \), and therefore \( G_1 \otimes G_2 \) is an me\(_2\)-graph.

We now present necessary and sufficient conditions for the Kronecker product of two graphs to be a strong-me\(_2\)-graph.

**Theorem 4.2.15.** \( G_1 \otimes G_2 \) is a strong-me\(_2\)-graph if and only if \( G_1 \) and \( G_2 \) are strong-me\(_2\)-graphs.

**Proof.** \(( \Rightarrow )\) Suppose \( G_1 \otimes G_2 \) is a strong-me\(_2\)-graph. Let \( uv \) be an edge of \( G_1 \) and \( wz \) be an edge of \( G_2 \). Since \( G_1 \otimes G_2 \) is a strong-me\(_2\)-graph, then for the edge \((u, w)(v, z)\) in \( G_1 \otimes G_2 \), \((u, w)\) is the unique common neighbour of \((v, z)\) and a vertex \((x, y)\) in \( G_1 \otimes G_2 \), and \((v, z)\) is the unique common neighbour of \((u, w)\) and a vertex \((x', y')\) in \( G_1 \otimes G_2 \). It follows from Lemma 4.2.13 that \( u = \text{ucn}(v, x) \) in \( G_1 \) and \( v = \text{ucn}(u, x') \) in \( G_1 \). Therefore, the edge \( uv \) is strongly-2-required in \( G_1 \) and \( G_1 \) is a strong-me\(_2\)-graph. A similar argument shows that \( G_2 \) is a strong-me\(_2\)-graph.

\(( \Leftarrow )\) On the other hand assume \( G_1 \) and \( G_2 \) are two strong-me\(_2\)-graphs. Let \((u, w)(v, z)\) be an edge of \( G_1 \otimes G_2 \). Since \( u \) is the unique common neighbour of \( v \) and a vertex \( x \) in \( G_1 \), \( v \) is the unique common neighbour of \( u \) and a vertex \( x' \) in \( G_1 \), and \( w \) is the unique common neighbour of \( z \) and a vertex \( y \) and \( z \) is the unique common neighbour of \( w \) and a vertex \( y' \) in \( G_2 \), we conclude that \((u, w)\) is the unique common neighbour of \((v, z)\) and the vertex \((x, y)\) in \( G_1 \otimes G_2 \), and \((v, z)\) is the unique common neighbour of \((u, w)\) and the vertex \((x', y')\) in \( G_1 \otimes G_2 \). Hence the edge \((u, w)(v, z)\) is strongly-2-required in \( G_1 \otimes G_2 \) and \( G_1 \otimes G_2 \) is a strong-me\(_2\)-graph.

We present two tables below that summarize some of the items discussed so far. The tables show an analogy between the behaviour of the properties of connectedness and primitivity under the Kronecker product, and that of the me\(_2\) and strong-me\(_2\) properties.
We now focus on how the other properties we have discussed behave under the Kronecker product.

**Theorem 4.2.16.** $G_1 \otimes G_2$ is a double-me$_2$-graph if and only if $G_1$ and $G_2$ are double-me$_2$-graphs and at least one of $G_1$ and $G_2$ is a strong-me$_2$-graph.

**Proof.** ($\implies$) Suppose that $G_1 \otimes G_2$ is a double-me$_2$-graph. By Theorem 4.2.14 it follows that both $G_1$ and $G_2$ are me$_2$-graphs and at least one of $G_1$ and $G_2$ is a strong-me$_2$-graph. Assume $G_1$ is a strong-me$_2$-graph. We must show that $G_2$ is a double-me$_2$-graph. Let $w$ be a vertex of $G_2$. Since $G_1 \otimes G_2$ has the double-me$_2$-property, for any vertex $u$ of $G_1$ we have that the vertex $(u, w)$ is the unique common neighbour of a pair of vertices $(v, z)$ and $(x, y)$ in $G_1 \otimes G_2$. It follows that $w = \text{ucn}(z, y)$ in $G_2$. Therefore, $G_2$ has the double-me$_2$-property.

($\impliedby$) Suppose that $G_1$ and $G_2$ are double-me$_2$-graphs and that $G_1$ is a strong-me$_2$-graph. By Theorem 4.2.14 the product $G_1 \otimes G_2$ is an me$_2$-graph. We must show that every vertex in $G_1 \otimes G_2$ is the unique common neighbour of some pair of vertices in $G_1 \otimes G_2$. Let $(u, w)$ be a vertex in $G_1 \otimes G_2$. Since $G_1$ is a double-me$_2$-graph, there exist vertices $v$ and $x$ such that $u = \text{ucn}(v, x)$ in $G_1$. Since $G_2$ is a double-me$_2$-graph, there exist vertices $y$ and $z$ such that $w = \text{ucn}(y, z)$ in $G_2$. Therefore, $(u, w) = \text{ucn}((v, z)(x, y))$ in $G_1 \otimes G_2$. We conclude that $G_1 \otimes G_2$ is a double-me$_2$-graph. \qed

**Theorem 4.2.17.** $G_1 \otimes G_2$ is a graph in which every edge is unitriangular if and only if $G_1$ and $G_2$ are graphs in which every edge is unitriangular.
Proof. ( \( \rightarrow \) ) Suppose every edge in \( G_1 \otimes G_2 \) is unitriangular. Let \( uv \) be an edge of \( G_1 \) and \( wz \) be an edge of \( G_2 \). Since every edge of \( G_1 \otimes G_2 \) is unitriangular, the vertices \((u, w)\) and \((v, z)\) have a unique common neighbour \((x, y)\) in \( G_1 \otimes G_2 \). It follows that \( x = \text{ucn}(u, v) \) in \( G_1 \) and \( y = \text{ucn}(w, z) \) in \( G_2 \). We conclude that every edge in \( G_1 \) and every edge in \( G_2 \) is unitriangular.

( \( \leftarrow \) ) Let \( G_1 \) and \( G_2 \) be graphs in which every edge is unitriangular. Let \((u, w)\) and \((v, z)\) be a pair of adjacent vertices in \( G_1 \otimes G_2 \). Since every edge in \( G_1 \) and \( G_2 \) are unitriangular, there exists a unique common neighbour, \( x \), of \( u \) and \( v \) in \( G_1 \), and there exists a unique common neighbour, \( y \), of \( w \) and \( z \) in \( G_2 \). Thus the only common neighbour of \((u, w)\) and \((v, z)\) in \( G_1 \otimes G_2 \) is \((x, y)\). Therefore every edge in \( G_1 \otimes G_2 \) is unitriangular.

\[ \text{Lemma 4.2.18.} \] Let \( G_1 \) and \( G_2 \) be two graphs of order \( n \) and \( m \) respectively. Then the maximum possible degree of a vertex in \( G_1 \otimes G_2 \) is \( nm - (n + m - 1) \).

Proof. The maximum possible degree of a vertex in \( G_1 \) is \( n - 1 \) and the maximum possible degree of a vertex in \( G_2 \) is \( m - 1 \). Since the degree of a vertex \((u, v)\) in \( G_1 \otimes G_2 \) is the product of the degree of \( u \) in \( G_1 \) and the degree of \( v \) in \( G_2 \), it follows that the maximum possible degree of a vertex in \( G_1 \otimes G_2 \) is \( nm - (n + m - 1) \).

A consequence of Lemma 4.2.18 is that the Kronecker product of two graphs will never be a windmill, as there will be no vertex that is adjacent to all other vertices in the product. In particular, the friendship property is not preserved under the Kronecker product. We observe however by Theorem 4.2.17 that the Kronecker product of two windmills is a graph in which every edge is unitriangular.

### 4.2.3 The strong and co-normal product

We now introduce two more graph products and consider their behaviour with respect to the properties of interest.

**Definition 4.2.19.** Let \( G_1 \) and \( G_2 \) be two graphs. The **strong product** of \( G_1 \) and \( G_2 \), denoted by \( G_1 \boxdot G_2 \), is the graph with vertex set \( V \) and edge set \( E \), where

- \( V = \{(u, v) : u \in V(G_1), v \in V(G_2)\} = V(G_1) \times V(G_2) \),


We observe that the edge set of the strong product is the union of the edge sets of the Cartesian and Kronecker products.

**Definition 4.2.20.** Let $G_1$ and $G_2$ be two graphs. The co-normal product of $G_1$ and $G_2$, denoted by $G_1 * G_2$, is the graph with vertex set $V$ and edge set $E$, where

- $V = \{(u,v) : u \in V(G_1), v \in V(G_2)\} = V(G_1) \times V(G_2)$,
- $E = \{(u,v)(u',v') : uu' \in E(G_1), or vv' \in E(G_2)\}$.

The Kronecker product of two $me_2$-graphs always has exponent 2 by Theorem 4.2.9, and it follows that the strong product and co-normal product of two $me_2$-graphs $G$ and $H$ have exponent 2. We observe that both the strong product and co-normal product of two graphs $G$ and $H$ have the same vertex set as the Kronecker product of $G$ and $H$, and their edge sets are strict supersets of $E(G \otimes H)$. To see this let $G_1$ and $G_2$ be two graphs of order at least 2 where $u$ and $u'$ are distinct vertices of $G_1$ and $uu'$ is an edge in $G_1$, and $v$, $v'$ are distinct vertices of $G_2$ and $vv'$ is an edge of $G_2$. Then $(u,v)(u',v')$ is an edge in the Kronecker, strong, and co-normal product of $G_1$ and $G_2$. However, $(u,v)(u,v')$ is also an edge in the strong and co-normal product but not the Kronecker product of $G_1$ and $G_2$. Therefore, if $G$ and $H$ are two $me_2$-graphs, there are edges in $G \boxtimes H$ and edges in $G * H$ that are not 2-required in these products. We conclude that the $me_2$-property, the double-$me_2$-property and the strong-$me_2$-property are not preserved under the strong product or the co-normal product.

We now investigate when the strong product of two graphs is connected. Some results that we present below can be found in [8] and [14].

**Lemma 4.2.21.** $G_1 \boxtimes G_2$ is connected if and only if $G_1$ and $G_2$ are connected.

**Proof.** ($\implies$) Suppose that $G_1 \boxtimes G_2$ is connected. Let $u$ and $v$ be vertices of $G_1$ and let $w$ be a vertex of $G_2$. The first components of the vertices on a shortest path from $(u,w)$ to $(v,w)$ in $G_1 \boxtimes G_2$ comprise a path from $u$ to $v$ in $G_1$. Thus $G_1$ is connected, and so is $G_2$ by the same reasoning.
Suppose that $G_1$ and $G_2$ are connected and let $(u, x)$ and $(v, y)$ be vertices of $G_1 \boxtimes G_2$. Suppose we have a path $W_1$ of length $k$ from $u$ to $v$ in $G_1$ and a path $W_2$ of length $m$ from $x$ to $y$ in $G_2$, where $k \leq m$. The sequence of vertices of $G_1 \boxtimes G_2$ whose first components are the vertices of $W_1$ (with $v$ as the first component of the last $m - k$ terms if $k < m$) and whose second components are the vertices of $W_2$, is a path from $(u, v)$ to $(x, y)$ in $G_1 \boxtimes G_2$.

\[\]

**Lemma 4.2.22.** Let $G_1$ and $G_2$ be two graphs such that neither $G_1$ nor $G_2$ has an isolated vertex. Then every edge of $G_1 \boxtimes G_2$ belongs to at least two triangles.

**Proof.** Let $(u, w)(v, z)$ be an edge in $G_1 \boxtimes G_2$.

- If $uw$ is an edge in $G_1$ and $wz$ is an edge in $G_2$, then $(u, z)$ and $(v, w)$ are distinct common neighbours of $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$.

- If $uw$ is not an edge in $G_1$ then $u = v$ in $G_1$ and $wz$ is an edge of $G_2$. Since $G_1$ has no isolated vertices, it follows that there exists a vertex $x$ in $G_1$ such that $ux$ is an edge in $G_1$. Then $(x, w)$ and $(x, z)$ are distinct common neighbours of $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$. A similar argument applies when $uv$ is an edge in $G_1$ and $w = z$ in $G_2$.

We conclude that every edge of $G_1 \boxtimes G_2$ belongs to at least two triangles. \[\]

Since $G_1 \boxtimes G_2$ is connected if and only if $G_1$ and $G_2$ are connected, it follows from Lemma 4.2.22 that $G_1 \boxtimes G_2$ is never a windmill since no edge of $G_1 \boxtimes G_2$ is unitriangular.

We now investigate when $G_1 \boxtimes G_2$ is primitive, and in the positive case, determine the exponent of $G_1 \boxtimes G_2$. We present the following lemma to aid us with this.

**Lemma 4.2.23.** Let $G_1 \boxtimes G_2$ be a connected graph. Then $\text{diam}(G_1 \boxtimes G_2) = \max(\text{diam}(G_1), \text{diam}(G_2))$.

**Proof.** Let $(u, x)$ and $(v, y)$ be vertices of $G_1 \boxtimes G_2$. The proof of Lemma 4.2.21 shows that $d_{G_1 \boxtimes G_2}((u, x), (v, y)) = \max(d_{G_1}(u, v), d_{G_2}(x, y))$. It follows that the greatest distance between a pair of vertices in $G_1 \boxtimes G_2$ is equal to the maximum of the distances between all pairs of vertices in $G_1$ or $G_2$. \[\]
Suppose $G_1 \boxtimes G_2$ is connected and there exists a walk of length $m$ between vertices $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$. Since every edge in $G_1 \boxtimes G_2$ belongs to at least one triangle in $G_1 \boxtimes G_2$ by Lemma 4.2.22, we can replace any single edge in the walk between $(u, w)$ and $(v, z)$ by two edges to get a walk of length $m + 1$ between $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$. We can repeatedly do this in order to get walks of all lengths greater than $m$ between $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$. We now determine when $G_1 \boxtimes G_2$ is primitive and what the exponent of $G_1 \boxtimes G_2$ is in this case.

**Lemma 4.2.24.** If $G_1 \boxtimes G_2$ is connected then $G_1 \boxtimes G_2$ is primitive. In this case, $\exp(G_1 \boxtimes G_2) = \max(\text{diam}(G_1), \text{diam}(G_2))$.

**Proof.** By Lemma 4.2.23, $\text{diam}(G_1 \boxtimes G_2) = \max(\text{diam}(G_1), \text{diam}(G_2))$, so we show that if $G_1 \boxtimes G_2$ is connected and $\text{diam}(G_1 \boxtimes G_2) = k$ for some positive integer $k$, then there exist walks of length $k$ between every pair of vertices in $G_1 \boxtimes G_2$.

Suppose that $d((u, w)(v, z)) = m$ in $G_1 \boxtimes G_2$ for $m \leq k$. If $m = k$ we are done. Let $m < k$ and let $W$ be a walk of length $m$ between $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$. Since $G_1 \boxtimes G_2$ is connected and by Lemma 4.2.22 every edge in $G_1 \boxtimes G_2$ belongs to a triangle, we can extend $W$ until we obtain a walk of length $k$ between $(u, w)$ and $(v, z)$ in $G_1 \boxtimes G_2$. □

By Lemma 4.2.24, the strong product of two graphs has exponent 2 if and only if both factors have diameter at most 2, and at least one factor has diameter exactly 2. We now show that the strong product of two graphs is never an $m_{e_2}$-graph.

**Lemma 4.2.25.** Let $G_1$ and $G_2$ be graphs both of order at least 2. Then the product $G_1 \boxtimes G_2$ is never an $m_{e_2}$-graph.

**Proof.** Suppose $G_1 \boxtimes G_2$ is an $m_{e_2}$-graph. We can assume $G_1$ and $G_2$ are connected by Lemma 4.2.21 and therefore both $G_1$ and $G_2$ have no isolated vertices.

Let $(u, w)(v, z)$ be an edge in $G_1 \boxtimes G_2$ such that $uw$ is an edge of $G_1$ and $wz$ is an edge of $G_2$. We only need consider this case since, in order to show a graph is not an $m_{e_2}$-graph, we only need to show that there is at least one edge in the graph that is not 2-required. Without loss of generality we can assume $(u, w)(v, z)$ is 2-required in $G_1 \boxtimes G_2$ for $(u, w)$ to a vertex $(x, y)$ in $G_1 \boxtimes G_2$. We have three possibilities for the edge $(v, z)(x, y)$ in $G_1 \boxtimes G_2$. 

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• $v = x$ and $zy$ is an edge in $G_2$. However this means that the vertex $(u, z)$ in $G_1 \boxtimes G_2$ is a common neighbour of $(u, w)$ and $(x, y)$ and the edge $(u, w)(v, z)$ is not 2-required in $G_1 \boxtimes G_2$.

• By a similar argument to above, if $z = y$ and $vx$ is an edge, then $(u, w)(v, z)$ is not 2-required in $G_1 \boxtimes G_2$.

• $vx$ is an edge in $G_1$ and $zy$ is an edge in $G_2$. Then $(v, y)$ is a common neighbour of $(u, w)$ and $(x, y)$ and the edge $(u, w)(v, z)$ is not 2-required in $G_1 \boxtimes G_2$.

We conclude that the edge $(u, w)(v, z)$ is not 2-required in $G_1 \boxtimes G_2$ and therefore $G_1 \boxtimes G_2$ is not an me$_2$-graph.

We now discuss the co-normal product and show that the co-normal product of two graphs, both of which have order at least 2, is never an me$_2$-graph. The following characterization of connected co-normal products appears in [8].

**Lemma 4.2.26.** Let $G_1$ and $G_2$ be graphs both of order at least 2. Then $G_1 \ast G_2$ is connected if and only if one of the following holds:

1. One of $G_1$, $G_2$ is null and the other is connected.

2. $G_1$ and $G_2$ are non-null and do not both have isolated vertices.

We know that the co-normal product does not preserve the me$_2$-property. We will use the following lemmas to show that the co-normal product of two graphs, both of which have order at least 2, is never an me$_2$-graph.

**Lemma 4.2.27.** Let $G_1$ and $G_2$ be two graphs both of order at least 2. Then $G_1 \ast G_2$ has exponent 2 if and only if one of the following holds:

• Neither $G_1$ nor $G_2$ has an isolated vertex.

• One of $G_1$, $G_2$ has an isolated vertex and the other has exponent 2.

Proof. ($\implies$) Assume $G_1 \ast G_2$ has exponent 2. Suppose that $G_1$ has an isolated vertex $u$. We must show $G_2$ has exponent 2. Let $x$ and $y$ be distinct vertices of $G_2$. Since $G_1 \ast G_2$ has exponent 2, it follows that $(u, x)$ and $(u, y)$ have a common neighbour in $G_1 \ast G_2$.
which means that \( x \) and \( y \) must have a common neighbour in \( G_2 \). We conclude that \( G_2 \) has exponent 2.

( \( \iff \) ) First note that if either \( G_1 \) or \( G_2 \) has exponent 2 then it is immediate that \( G_1 * G_2 \) also does. Now suppose that neither \( G_1 \) nor \( G_2 \) has exponent 2 and neither has an isolated vertex. Let \((u, w)\) and \((v, z)\) be distinct vertices in \( G_1 * G_2 \). Since neither \( G_1 \) nor \( G_2 \) has any isolated vertices, then there exists a vertex \( x \) in \( G_1 \) such that \( ux \) is an edge in \( G_1 \) and there exists a vertex \( y \) such that \( zy \) is an edge in \( G_2 \). Hence the vertex \((x, y)\) is a common neighbour of \((u, w)\) and \((v, z)\) in \( G_1 * G_2 \). We conclude that \( G_1 * G_2 \) has exponent 2.

\[ \text{Lemma 4.2.28.} \] \( G_1 \) and \( G_2 \) be graphs both of order at least 2 for which \( G_1 * G_2 \) has exponent 2. Then there exists at least two paths of length 2 between every distinct pair of vertices in \( G_1 * G_2 \).

**Proof.** If either \( G_1 \) or \( G_2 \) has exponent 2 then it is easy to see that there exist multiple paths of length 2 between every pair of distinct vertices in \( G_1 * G_2 \). The alternative is that neither \( G_1 \) nor \( G_2 \) has exponent 2 and then by Lemma 4.2.27, neither has an isolated vertex. Let \((u, w)\) and \((v, z)\) be distinct vertices of \( G_1 * G_2 \). If \( u \) and \( v \) have a common neighbour in \( G_1 \) or if \( w \) and \( z \) have a common neighbour in \( G_2 \), then \((u, w)\) and \((v, z)\) have multiple common neighbours in \( G_1 * G_2 \). To see this we can assume without loss of generality that \( x \) is a common neighbour of \( u \) and \( v \) in \( G_1 \). Then \((x, q)\) is a common neighbour of \((u, w)\) and \((v, z)\) in \( G_1 * G_2 \) for every vertex \( q \) of \( G_2 \). The case where \( w \) and \( z \) have a common neighbour in \( G_2 \) is similar. If \( u \) and \( v \) do not have a common neighbour in \( G_1 \) and \( w, z \) do not have a common neighbour in \( G_2 \), let \( x_1 \) be a neighbour of \( u \) in \( G_1 \) and let \( x_2 \) be a neighbour of \( v \) in \( G_1 \). Let \( y_1, y_2 \) respectively be neighbours of \( w \) and \( z \) in \( G_2 \). Then \((x_1, y_1)\) and \((x_2, y_2)\) are different common neighbours of \((u, v)\) and \((w, z)\) in \( G_1 * G_2 \). We conclude that every pair of distinct vertices in \( G_1 * G_2 \) have at least two paths of length 2 between them.

As a consequence of Lemma 4.2.28, it follows that the co-normal product of two graphs \( G_1 \) and \( G_2 \), both of which have order at least 2, is never an me\(_2\)-graph.

We observe that the co-normal product \( G_1 * G_2 \) is the complement of \( \overline{G_1 * G_2} \).

\( \overline{G_1 * G_2} \) is the graph with vertex set \( V(G_1) \times V(G_2) \), and two vertices \((u, w)(v, z)\)
in \( G_1 \otimes G_2 \) are adjacent if and only if \( uv \) is not an edge of \( G_1 \) and \( wz \) is not an edge of \( G_2 \). The complement of \( G_1 \otimes G_2 \) is the graph where \((u, w)(v, z)\) is an edge only if \( uv \) is an edge in \( G_1 \) or \( wz \) is an edge of \( G_2 \), which is exactly the definition of the co-normal product.
Chapter 5

Simple me$_2$-embeddings

In this chapter, our focus is on the occurrence of given graphs as induced subgraphs of me$_2$-graphs. We introduce the concept of a simple me$_2$-embedding of a graph, which involves the addition of mutually non-adjacent vertices with patterns of incidences that yield the me$_2$-property in the resulting extension. In further sections, we specialize our attention to trees. We characterize trees that admit simple me$_2$-embeddings and consider the minimum number of vertices that must be adjoined in order to achieve such an embedding of a given tree. The last section of this chapter deals with simple strong-me$_2$-embeddings of trees. The content of the first three sections of this chapter will appear as a journal article in a 2017 issue of Linear Algebra and its Applications [17].

5.1 Simple me$_2$-embeddability

We begin by introducing the following definition.

**Definition 5.1.1.** A simple extension of a graph $G$ is a graph $H$ that has $G$ as an induced subgraph and has the property that the subgraph induced on $V(H) \setminus V(G)$ is null (or $V(H) \setminus V(G)$ is empty and $H = G$).

A simple me$_2$-embedding of $G$ is an me$_2$-graph that is a simple extension of $G$.

In the context where $H$ is a simple extension of $G$, we will refer to the elements of $V(H) \setminus V(G)$ as $G$-external vertices in $H$. We say that a graph is simply me$_2$-embeddable if
it admits a simple $me_2$-embedding. Clearly any $me_2$-graph is a simple $me_2$-embedding
of itself and is therefore simply $me_2$-embeddable.

The concept of simple $me_2$-embeddability may be interpreted as a completion prob-
lem for non-negative square matrices as follows. A non-negative $n \times n$ matrix $A$ with
a symmetric pattern of zero entries is simply $me_2$-embeddable if there is some integer
$m \geq n$ and some $m \times m$ $me_2$-matrix $B$ that has $A$ as its upper left $n \times n$ submatrix, and
has a zero matrix in its lower right $(m - n) \times (m - n)$ region.

In this section we present some basic observations about the graph property of sim-
ple $me_2$-embeddability. First we recall that every edge in a graph of exponent 2 must
belong to a triangle, since its two incident vertices must have a common neighbour. An
immediate consequence of this is that a simply $me_2$-embeddable graph cannot contain
an isolated vertex. We note some examples of connected simple graphs that are not
simply $me_2$-embeddable.

**Example 5.1.2.** The complete graph $K_n$ is simply $me_2$-embeddable only if $n = 2$ or $n = 3$.

*Proof.* Both $K_2$ and $K_3$ have $K_3$ as a simple $me_2$-embedding.

Suppose that $n \geq 4$ and let $e = uv$ be an edge of $K_n$. Note that there are no unique
2-paths in $K_n$. If $K_n$ admits a simple $me_2$-embedding $H$ then $e$ must be part of a unique
2-path in $H$ from a vertex of $e$ (say $u$) to an adjoined vertex $x_e$. Since the edge $vx_e$ must
belong to a triangle in $H$ and since $v$ is the unique common neighbour in $H$ of $u$ and
$x_e$, we deduce that $u$ and $v$ are the only neighbours of $x_e$ in $H$. Now let $f$ be an edge of
$K_n$ that is disjoint from $e$. By the same argument, some vertex $x_f$ of $H$ has only the two
vertices of $f$ as neighbours in $H$. However, this means that there is no path of length 2
from $x_e$ to $x_f$ in $H$. Hence $K_n$ is not simply $me_2$-embeddable. \hfill \qed

**Example 5.1.3.** The path $P_6$ on six vertices is not simply $me_2$-embeddable.

*Proof.* Let the vertices of $P_6$ be labelled as in the diagram below, and write $e$ for the edge
$u_1u_2$.

```
    w_1   v_1   u_1   u_2   v_2   w_2
```

Let $H$ be a simple extension of exponent 2 of this copy of $P_6$. Both $v_1$ and $v_2$ must
be adjacent in $H$ to all $P_6$-external vertices, to afford 2-paths from $w_1$ and $w_2$ to these
vertices. Each of $u_1$ and $u_2$ must be adjacent to some $P_6$-external vertex in order for $H$ to have exponent 2, so multiple 2-paths exist in $H$ from $u_1$ to $v_2$ and from $u_2$ to $v_1$. Since $u_1$ and $u_2$ both have 2-paths to all $P_6$-external vertices and these 2-paths do not involve the edge $e$, we conclude that $e$ is not part of a unique 2-path in $H$. Thus $H$ is not an $me_2$-graph and $P_6$ is not simply $me_2$-embeddable.

We use the term leaf to refer to a vertex of degree 1 in any graph. The following elementary observation, which has already been used in the proof of Lemma 5.1.3, will have a key role in the constructions and arguments of Sections 5.2 and 5.3.

**Lemma 5.1.4.** Let $v$ be a leaf in a graph $G$, and let $u$ be the unique neighbour of $v$ in $G$. If $H$ is a simple extension of $G$ of exponent 2, then $u$ is adjacent in $H$ to every $G$-external vertex in $H$.

**Proof.** If $x$ is a $G$-external vertex in $H$, then $x$ and $v$ must have a common neighbour in $H$, and this common neighbour must be a vertex of $G$. The only candidate is $u$, and so $x$ is adjacent to $u$ in $H$. \hfill $\Box$

Lemma 5.1.4 introduces a particular constraint to the consideration of simple $me_2$-embeddings of graphs that possess leaves, and motivates the following definition.

**Definition 5.1.5.** Let $G$ be a graph and let $e = uv$ be an edge of $G$. We say that $e$ is a critical edge if $u$ has a neighbour (other than $v$) that is adjacent to a leaf of $G$ and $v$ has a neighbour (other than $u$) that is adjacent to a leaf of $G$.

Critical edges have a key role in our arguments for the following reason. Let $e = uv$ be a critical edge in a graph $G$, and let $u'$ and $v'$ be two more vertices that are neighbours of $u$ and $v$ respectively, each adjacent to a leaf in $G$. Suppose that $H$ is a simple $me_2$-embedding of $G$ and let $X$ be the set of $G$-external vertices of $H$. Then each of $u'$ and $v'$ is adjacent in $H$ to every vertex of $X$. It follows that the edge $e$ cannot be required in $H$ for a unique 2-path between $u$ or $v$ and any vertex of $X$. Thus a unique 2-path in $H$ that includes $e$ must be a unique 2-path in the original $G$. If a graph $G$ contains a critical edge that is not part of a unique 2-path in $G$ between any pair of vertices, then $G$ is not simply $me_2$-embeddable.

The presence of a critical edge in a graph need not preclude simple $me_2$-embeddability; this depends on the surrounding configurations. However, a critical edge in a simply
me₂-embeddable graph $G$ must belong to at least one unique 2-path in $G$ that has the potential to retain its uniqueness in a simple extension of $G$ of exponent 2. In Example [5.1.3] the 2-paths that contain the critical edge of $P_6$ both have the property that one of their terminal vertices is adjacent to a leaf, and must therefore be adjacent to all adjoined vertices in any simple extension of exponent 2, by Lemma [5.1.4]. Thus the uniqueness of these 2-paths cannot be preserved in a simple extension of exponent 2, and $P_6$ is not simply me₂-embeddable. We observe that all critical edges are related to Example [5.1.3] since a critical edge in a graph is the central edge of a copy of $P_6$ that is a subgraph of $G$ and whose terminal vertices are leaves in $G$. We return in Section [5.2] to the connection between configurations of critical edges and simple me₂-embeddability in the case of trees.

In general, the absence of critical edges does not imply simple me₂-embeddability, as the complete graphs show. However, Theorem [5.1.7] below does give conditions under which a graph without critical edges is simply me₂-embeddable. Before stating the Theorem, we introduce a technical definition to facilitate arguments involving the demonstration of simple me₂-embeddability.

**Definition 5.1.6.** A simple pre-me₂-embedding of a graph $G$ is a simple extension $H$ of $G$ that has exponent 2 and has the property that every edge of $G$ belongs to a unique 2-path between some pair of vertices in $H$.

If $H$ is a simple pre-me₂-embedding of $G$, then any edge of $H$ that does not belong to a unique 2-path in $H$ involves one vertex of $G$ and one vertex external to $G$. The result of deleting such an edge is still a simple pre-me₂-embedding of $G$. The step of deleting an edge of this type can be repeated until what remains is a simple me₂-embedding of $G$. Thus in order to show that a graph $G$ is simply me₂-embeddable, it is enough to show that it admits a simple pre-me₂-embedding.

**Theorem 5.1.7.** Let $G$ be a graph on at least two vertices, and suppose that $G$ contains no critical edge, no isolated vertex and no triangle, and has the property that for vertices $u$ and $v$, $N(u)$ can be a subset of $N(v)$ only if $u$ has degree 1. Then $G$ admits a simple me₂-embedding.

**Proof.** We present an explicit construction of a simple pre-me₂-embedding $H$ of $G$, in which the number of adjoined vertices is $|E(G)|$. 

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Since $G$ possesses no critical edge, every edge of $G$ has at least one incident vertex that is not adjacent to a leaf in $G$. For each edge $e$ of $G$, we assign the labels $v_1(e)$ and $v_2(e)$ to the two vertices incident with $e$, in such a way that no neighbour of $v_1(e)$ is a neighbour of a leaf in $G$ (unless $v_1(e)$ is a leaf). To construct $H$ we introduce a vertex $x_e$ for each edge $e$ of $G$, and make $x_e$ adjacent in $H$ to $v_1(e)$, $v_2(e)$, and all other vertices of $G$ except further neighbours in $G$ of $v_1(e)$. For a particular edge $e$, $v_2(e)$ is the only common neighbour in $H$ of $v_1(e)$ and $x_e$, and so the edge $e$ is involved in the unique 2-path from $v_1(e)$ to $x_e$ in $H$. Note that the construction of $H$ ensures that any vertex that is a neighbour of a leaf in $G$ will be adjacent in $H$ to every $x_e$.

We now show that $H$ has exponent 2.

- First consider a pair of vertices $u$ and $v$ of $G$. If $u$ and $v$ are adjacent in $G$, then there is a 2-path in $H$ from $u$ to $v$ via the vertex $x_e$ where $e = uv$. If $d_G(u, v) = 2$, then there is a path of length 2 between $u$ and $v$ in $G$, hence in $H$. If $d_G(u, v) > 2$, let $e$ be any edge of $G$ that is incident with $v$. Then $x_e$ is adjacent to both $u$ and $v$ in $H$ and $d_H(u, v) = 2$.

- Now suppose that $u \in V(G)$ and $x \in V(H) \setminus V(G)$. Then $x = x_e$ for some edge $e$ of $G$. If $u$ is a leaf in $G$, then its unique $G$-neighbour is not adjacent to $v_1(e)$ and is therefore adjacent to $x$ in $H$. If $u$ is not a leaf in $G$, then by hypothesis $N(u)$ is not a subset of $N(v_1(e))$, so $u$ has some $G$-neighbour that is not adjacent to $v_1(e)$ and is therefore adjacent to $x$ in $H$. Thus $u$ and $x$ have a common neighbour in $H$ as required.

- Now choose two vertices $x_e$ and $x_f$ of $V(H) \setminus V(G)$, where $e$ and $f$ are different edges of $G$. If $e$ and $f$ share a vertex, then this vertex is a common neighbour of $x_e$ and $x_f$ in $H$. If not, then the vertices incident with $e$ cannot both be adjacent to $v_1(f)$ since $G$ contains no triangle, so one of these is a common neighbour of $x_e$ and $x_f$ in $H$.

Thus $H$ is a simple pre-me$_2$-embedding of $G$. □

The problem of determining which graphs admit simple me$_2$-embeddings appears to be reasonably intricate, and we will consider it in further detail only for trees, in
Sections 5.2 and 5.3. It follows from Theorem 5.1.7 that trees and forests without critical edges or isolated vertices are simply me\(_2\)-embeddable.

## 5.2 Trees

The remainder of this chapter is concerned with simple me\(_2\)-embeddings of trees. In this section we give a characterization of simply me\(_2\)-embeddable trees, and in Section 5.3 we investigate the minimum number of vertices that must be adjoined to a tree in order to obtain a simple me\(_2\)-embedding.

It follows from Theorem 5.1.7 that trees without critical edges are simply me\(_2\)-embeddable, and the example of \(P_6\) shows that not all trees are. In this section we investigate when the presence of critical edges in a tree precludes simple me\(_2\)-embeddability. In a path, we refer to the two vertices of degree 1 as terminal vertices and to vertices of degree 2 as internal vertices.

### Definition 5.2.1

Let \(e\) be a critical edge in a tree \(T\). A viable path for \(e\) in \(T\) (or an \(e\)-viable path) is a 2-path in \(T\) that includes \(e\) and has the property that neither of its terminal vertices is adjacent to a leaf in \(T\).

If a tree \(T\) has a critical edge \(e\) and admits a simple me\(_2\)-embedding \(H\), then there is a 2-path in \(T\) containing \(e\) that is a unique 2-path in \(H\). By Lemma 5.1.4, no terminal vertex of this 2-path can be adjacent to a leaf in \(T\). Thus a necessary condition for a tree \(T\) to be simply me\(_2\)-embeddable is that any critical edge of \(T\) belongs to at least one viable path. The critical edge of \(P_6\) belongs to no viable path, since all additional neighbours of its two incident vertices are adjacent to leaves. In general a critical edge \(e\) in a tree \(T\) possesses a viable path if and only if its vertices can be labelled \(v_1(e)\) and \(v_2(e)\) so that \(v_1(e)\) is adjacent to no leaf of \(T\) and \(v_2(e)\) has a neighbour (different from \(v_1(e)\)) that is adjacent to no leaf of \(T\). Note that the internal vertex of a 2-path that is viable for a critical edge always has degree at least 3, since in addition to the two terminal vertices it has at least one neighbour that is adjacent to a leaf.

The following example shows that the existence of a viable path for every critical edge is not enough to guarantee simple me\(_2\)-embeddability of a tree.

### Example 5.2.2

Let \(T\) be the following tree.
There are two critical edges in $T$, and these are highlighted with curved lines. Each critical edge has a single viable path, and these viable paths are highlighted in bold. If $H$ is a simple me$_2$-embedding of $T$, then the viable paths highlighted above must be unique 2-paths in $H$. However, the vertices $y$ and $y'$ must have a common neighbour $v$ in $H$, and $u$ must have a path of length 2 to $v$ in $H$. This means that either $w$ or $w'$ must be adjacent to $v$ in $H$, and the highlighted viable paths cannot both have their uniqueness preserved in a simple extension of exponent 2. Thus $T$ is not simply me$_2$-embeddable.

**Definition 5.2.3.** Let $T$ be a tree with distinct critical edges $e_1$ and $e_2$. Let $P_1$ and $P_2$ be viable paths in $T$ for $e_1$ and $e_2$ respectively. Then $P_1$ and $P_2$ are incompatible if there exists a vertex of degree 2 in $T$ whose two neighbours are terminal vertices of $P_1$ and $P_2$ respectively. If no such vertex exists we say that $P_1$ and $P_2$ are compatible.

We remark that two viable paths that share one or more vertices are always compatible, and that it is possible that a single 2-path could be viable for two different critical edges.

As Example 5.2.2 indicates, a necessary condition for simple me$_2$-embeddability of a tree is the existence of a choice of viable path for every critical edge in such a way that the selected paths are pairwise compatible. The main theorem of this section is to show that this necessary condition is also sufficient.

**Theorem 5.2.4.** Let $T$ be a tree and let $C$ be the set of critical edges in $T$. Then $T$ is simply me$_2$-embeddable if and only if for each $e \in C$ there exists an $e$-viable path $P_e$ with the property that $\{P_e : e \in C\}$ is a set of pairwise compatible paths.

The necessity of the condition in Theorem 5.2.4 has already been established, so we assume the existence of a system $\{P_e : e \in C\}$ of pairwise compatible viable paths for the critical edges of $T$, and demonstrate the existence of a simple pre-me$_2$-embedding.
of $T$ in which each $P_e$ is a unique 2-path. Our argument involves the introduction of an external vertex for every path in $T$ whose length is different from 2, according to the following construction.

Let $u$ and $v$ be distinct vertices in $T$, with $d_T(u, v) \neq 2$, and let $Q$ denote the unique path in $T$ from $u$ to $v$. Construct a partial vertex colouring of $T$ as follows.

- Assign the colour red to $u, v$ and all neighbours of leaves in $T$.
- Apply the following procedure to the vertices $t$ of $Q$, taken one by one in any order (the choice of order has no bearing on the proof mechanism).
  - If $t$ already has a red neighbour, leave all neighbours of $t$ as they are. (Note that this applies to any vertex of $Q$ whose degree in $T$ is 1.)
  - If $t$ has degree 2 in $T$ and does not already have a red neighbour, assign the colour red to a neighbour of $t$ that is not a terminal vertex of any 2-path in the set \{$P_e : e \in C$\}. That such a neighbour exists is essentially the condition of the theorem.
  - If $t$ has degree at least 3 in $T$ and does not already have a red neighbour, assign the colour red to a neighbour of $t$ that does not belong to $Q$.
- For $i \geq 1$, write $V_i = \{x \in V(T) : d_T(x, Q) = i\}$, and let $m$ be the maximum $T$-distance from $Q$ of a vertex of $T$. Apply the following procedure in turn for $i = 1, 2, \ldots, m$.
  For vertex $x \in V_i$, extend the vertex-colouring as follows: if $x$ already has a red neighbour in $T$, leave all neighbours of $x$ as they are. If $x$ has no red neighbour in $T$, assign the colour red to a neighbour $y$ of $x$ with $d_T(y, Q) = i + 1$. Note that such a $y$ exists, since the alternative is that $x$ is a leaf in $T$, in which case its unique neighbour is already coloured red.

**Lemma 5.2.5.** The resulting partial vertex-colouring of $T$ has the following properties.

1. Every vertex of $T$ is adjacent to a red vertex.

2. If $e$ is a critical edge of $T$, then at most one of the terminal vertices of the path $P_e$ is coloured red.
Proof. Item 1 is easily checked, so we focus on 2. Let $w$ and $z$ be red vertices of $T$ for which $d_T(w, z) = 2$ and $d_T(w, Q) \leq d_T(z, Q)$. Let $y$ be the common neighbour of $w$ and $z$ in $T$. We consider three cases.

1. If $d_T(w, Q) < d_T(z, Q)$, then $w$ was already coloured red at the point where neighbours of $y$ were considered in the colouring process. Since $z$ would not have been coloured red at this stage, it must have been already coloured red by virtue of being adjacent to a leaf in $T$. This means that $w$ and $z$ are not the terminal vertices of a viable path for any critical edge of $T$.

2. If $d_T(w, Q) = d_T(z, Q) = 1$, then $y$ is a vertex of $Q$. At the point where neighbours of $y$ were considered in the colouring process, both $w$ and $z$ must have already been coloured red, as at each stage of the colouring process, at most one neighbour of a vertex in $Q$ is coloured red. This means that both $w$ and $z$ are adjacent to leaves in $T$, and neither can be a terminal vertex of a viable path for a critical edge.

3. The remaining possibility is that both $w$ and $z$ are vertices of $Q$. Since $d_T(w, z) = 2$, at least one of $w$ and $z$ is an internal vertex of $Q$. However, an internal vertex of $Q$ can be coloured red only if it is adjacent to a leaf in $T$ or if it is not a terminal vertex of $P_e$ for any critical edge $e$.

It follows that the 2-path between $w$ and $z$ in $T$ does not belong to the set $\{P_e : e \in C\}$ and that every 2-path in this set has at least one terminal vertex uncoloured. \hfill \Box

When the colouring procedure for the path $Q$ is finished, let $T_Q$ be the graph obtained from $T$ by introducing a new vertex $x_Q$ that is adjacent precisely to all red vertices of $T$, and then erasing the colouring. From Lemma 5.2.5 it follows that paths of length 2 exist in $T_Q$ from each vertex of $T$ to $x_Q$, and that for every critical edge $e$, $P_e$ is a unique 2-path in $T_Q$.

Now let $T'$ be the union of the $T_Q$ over all paths $Q$ in $T$ whose length either exceeds 2 or is equal to 1. We complete the proof of Theorem 5.2.4 with the following lemma.

Lemma 5.2.6. The graph $T'$ is a simple pre-me$_2$-embedding of $T$.  

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Proof. Let $X = V(T') \setminus V(T)$. From Lemma 5.2.5, it is clear that paths of length 2 exist in $T'$ between all vertices of $T$ and all vertices of $X$. A pair of vertices of $X$ share any neighbour of a leaf in $T$ as a common neighbour in $T'$. Finally, if $u$ and $v$ are vertices of $T$ and $d_T(u, v) \neq 2$, then $x_{P(u,v)}$ is a common neighbour of $u$ and $v$ in $T$, where $P(u, v)$ denotes the path between $u$ and $v$ in $T$.

It remains to show that every edge $e$ of $T$ is 2-required in $T'$. If $e$ is a critical edge in $T$, then from item 2 of Lemma 5.2.5, it follows that $P_e$ is a unique 2-path in $T'$. If $e$ is not critical in $T$, we write $e = uv$, where $u$ is not adjacent to a neighbour of a leaf in $T$. Now $e$ is the unique path between $u$ and $v$ in $T$, and the vertex $x_e \in X$ corresponds to this path. From the colouring procedure corresponding to the introduction of the vertex $x_e$ it is clear that $v$ is the only neighbour of $u$ in $T$ that is adjacent to $x_e$ in $T'$. Thus $e$ is 2-required for $u$ to $x_e$ in $T'$, and $T'$ is a pre-me$_2$-embedding of $T$ as required. 

5.3 Efficient simple me$_2$-embeddings of trees

In this section we consider the least number of vertices that must be adjoined to a tree in order to obtain a simple me$_2$-embedding, if one exists.

**Definition 5.3.1.** For a simply me$_2$-embeddable graph $G$, we denote by $\alpha(G)$ the minimum over all simple me$_2$-embeddings $H$ of $G$ of the number $|V(H)| - |V(G)|$.

Thus $\alpha(G) = 0$ precisely if $G$ is an me$_2$-graph. We begin this section with a characterization of trees for which $\alpha$ is 1 or 2, and then show that there exist large classes of trees $T$ satisfying $\alpha(T) = 3$. We give a construction to show that the value of $\alpha(T)$ may be arbitrarily high for a simply me$_2$-embeddable tree $T$.

**Lemma 5.3.2.** Let $T$ be a tree of diameter 1 or 2. Then $T$ is simply me$_2$-embeddable and $\alpha(T) = 1$.

**Proof.** The only tree of diameter 1 is $P_2$. Trees of diameter 2 are stars, in which one vertex is adjacent to all the others, which are leaves. It is easily confirmed that an me$_2$-graph is obtained by adjoining a single vertex, incident with every original vertex, to any such tree. 

$\square$
Lemma 5.3.3. Let $T$ be a tree of diameter 3 or 4. Then $T$ is simply me$_2$-embeddable and $\alpha(T) = 2$.

Proof. We show how to construct a simple pre-me$_2$-embedding $T''$ of $T$. Since $T$ has diameter at least 3 it possesses an edge $e$ that is incident with no leaf of $T$. The extension $T'$ of $T$ obtained by introducing a new vertex $x$ that is adjacent to every vertex of $T$ is not a simple pre-me$_2$-embedding of $T$, since $e$ is involved in no unique 2-path in this graph. Thus $\alpha(T) \neq 1$. Let $T''$ be the further extension of $T$ obtained by adjoining a new vertex $y$ to $T'$, whose neighbours in $T''$ are the vertices of $T$ that have degree 2 or greater in $T$. To see that $T''$ has exponent 2, observe that every pair of vertices of $T$ has $x$ as a common neighbour in $T''$, that every vertex of $T$ has a neighbour in $T$ of degree at least 2 and therefore has paths of length 2 to both $x$ and $y$, and that $x$ and $y$ have common neighbours in $T$. Now let $e$ be an edge of $T$. Suppose first that $e$ is not incident with a leaf in $T$. Then, since $T$ contains no path of length 5, we may write $e = uv$ where all $T$-neighbours of $u$, except $v$, are leaves in $T$. Then the edge $e$ belongs to the unique 2-path from $u$ to $y$ in $T''$. On the other hand if $e$ is incident with a leaf in $T$, then $e$ belongs to the unique 2-path from this leaf to $y$ in $T''$. Thus $T''$ is a simple pre-me$_2$-embedding of $T$ and $\alpha(T) = 2$.

Lemma 5.3.4. Let $T$ be a simply me$_2$-embeddable tree of diameter at least 5. Then $\alpha(T) \geq 3$.

Proof. We assume that $T$ is simply me$_2$-embeddable and let $T'$ be a simple me$_2$-embedding of $T$. Suppose first that some $T$-external vertex $x$ of $V(T')$ is adjacent in $T'$ to every vertex of $T$. Then every edge of $T$ that is not incident with a leaf must be part of the unique 2-path in $T'$ from a vertex of $T$ to a vertex of $V(T') \setminus V(T)$, other than $x$. Let $u_1, u_2, \ldots$ be a path of length at least 5 in $T$, where $u_1$ is a leaf in $T$. Since $u_3$ has a path via $u_2$ to every vertex of $V(T') \setminus V(T)$, the edge $u_3u_4$ must be part of the unique 2-path in $T'$ from $u_4$ to a vertex $y$ of $V(T') \setminus V(T)$, with $y \neq x$. This means that $u_5$ is not adjacent to $y$. If the edge $u_4u_5$ is part of a unique 2-path in $T'$ from $u_5$ to $y$, then $u_6$ is not adjacent to $y$ and $u_5u_6$ must be part of the unique 2-path in $T'$ to a third vertex of $V(T') \setminus V(T)$. If $u_4u_5$ is not part of a unique 2-path involving $y$, then it must be part of a unique 2-path in $T'$ involving a third vertex of $V(T') \setminus V(T)$. Thus $T'$ must have at least three vertices not belonging to $T$, if all vertices of $T$ have a common neighbour in $T'$.  

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Now assume that no vertex of \( T' \) is adjacent to all vertices of \( T \), and suppose that \( V(T') \setminus V(T) = \{ x, y \} \). Let \( S_x \) be the set of vertices of \( T \) that are adjacent to \( x \) and not \( y \) in \( T' \), and let \( S_y \) be the set of vertices of \( T \) that are adjacent to \( y \) and not \( x \) in \( T' \). Neither \( S_x \) nor \( S_y \) is empty, and every element of \( S_x \) must be at distance 2 in \( T \) from every element of \( S_y \), since \( T' \) has exponent 2. It follows that one of \( S_x \) and \( S_y \), say \( S_x \), consists of a single vertex \( u \). Every vertex of \( T \) except \( u \) is adjacent to \( y \) in \( T' \). Choose a leaf \( v \) of \( T \) for which \( d(T, u, v) \) is maximal. Since \( \text{diam}(T) \geq 5 \), \( d(T, u, v) \geq 3 \). First consider the case where \( d(T, u, v) = 3 \). In this case \( u \) is not a leaf in \( T \), since \( \text{diam}(T) \geq 5 \). Let \( uv_1v_2v \) be the path in \( T \) from \( u \) to \( v \). The vertex \( v_2 \) cannot belong to \( S_y \) by Lemma 5.1.4, so both \( v_1 \) and \( v_2 \) are adjacent to both \( x \) and \( y \) in \( T' \). If the edge \( v_1v_2 \) is part of a unique 2-path in \( T' \), this path must be between \( v_1 \) and \( y \), in which case \( v_1 \) has only \( u \) and \( v_2 \) as neighbours in \( T \). Then since \( u \) is not a leaf in \( T \), it has a neighbour other than \( v_1 \) which is adjacent to both \( x \) and \( y \) in \( T' \). It follows that the edge \( uv_1 \) is not part of any unique 2-path in \( T' \), contrary to the hypothesis that \( T' \) is an me_2-graph.

It remains to consider the possibility that \( d(T, u, v) \geq 4 \). Let the last four vertices of the path from \( u \) to \( v \) in \( T \) be \( v_1, v_2, v_3, v \). If \( d(T, u, v) = 4 \) it is possible that \( v_2 \in S_y \), but in any case \( v_1, v_3 \) and \( v \) are all adjacent to both \( x \) and \( y \) in \( T' \). It follows that the edge \( v_2v_3 \) is not part of a unique 2-path in \( T' \), another contradiction. We conclude that \( V(T') \setminus V(T) \) must have at least three elements, as required.

We next consider circumstances under which \( \alpha(T) = 3 \) for a tree \( T \). We need some terminology.

**Definition 5.3.5.** A **bough** in a tree \( T \) is a subgraph \( B \) with the following properties:

- \( B \) is isomorphic to the path graph \( P_k \) for some \( k \);

- The two terminal vertices of \( B \) have degree different from 2 in \( T \), and any internal vertices of \( B \) have degree 2 in \( T \).

A vertex of degree exceeding 2 in a tree is referred to as a **junction**, and two junctions are called **neighbouring** if there are no more junctions on the unique path between them. A bough may consist of two neighbouring junctions and the path between them, or of a leaf together with its nearest junction and the path between them, or of the entire tree.
in the case of a path graph. A bough that includes a leaf is called a terminal bough, and a bough that does not include a leaf is called an internal bough. Every tree is the union of its boughs, and boughs intersect only at junctions. Every edge of a tree belongs to a unique bough. The length of a bough is the number of edges in it.

Theorem 5.3.6 below, and its extension Theorem 5.3.7 establish some very general conditions under which a tree $T$ of diameter at least 5 satisfies $\alpha(T) = 3$.

**Theorem 5.3.6.** Let $T$ be a tree of diameter at least 5, in which every bough has even length. Then $T$ is simply me$_2$-embeddable and $\alpha(T) = 3$.

**Proof.** Since all boughs in $T$ have even length, $T$ cannot contain a copy of $P_6$ as a maximal path, so $T$ contains no critical edge and is simply me$_2$-embeddable by Theorem 5.1.7. That $\alpha(T) \geq 3$ is immediate from Lemma 5.3.4. We remark that the hypotheses imply that the diameter of $T$ must be at least 6. We describe a construction of a simple pre-me$_2$-embedding of $T$ with three adjoined vertices.

Choose any leaf $w$ of $T$. For each edge $e$ of $T$, let $v_e$ be the vertex of $e$ that is further from $w$ in $T$. Colour each edge of $T$ either blue or red according to the following scheme.

- The edge $e$ is coloured blue if $d_T(w, v_e) \equiv 0 \mod 4$ or if $d_T(w, v_e) \equiv 1 \mod 4$;

- The edge $e$ is coloured red if $d_T(w, v_e) \equiv 2 \mod 4$ or if $d_T(w, v_e) \equiv 3 \mod 4$.

Thus the edge incident with $w$ is coloured blue, and the edges of paths in $T$ emanating from $w$ are coloured in the sequence blue, red, red, blue, blue, red, red, . . . , with the pattern of alternating pairs of red and blue edges continuing until a leaf is encountered. Since all boughs of $T$ have even length, every junction of $T$ is incident with edges of only one colour, and every vertex of $T$ that is incident with edges of both colours has degree 2 in $T$. Every edge of $T$ has one vertex that is incident with edges of both colours, and one vertex that is incident with edges of one colour only.

We now extend $T$ by introducing two vertices $x$ and $y$ and edges from $x$ to every vertex of $T$ that is incident with a blue edge in $T$, and from $y$ to every vertex of $T$ that is incident with a red edge in $T$. The result of this construction is a graph $T'$ in which every edge of $T$ is involved in the unique 2-path between some vertex of degree 2 in $T$ and either $x$ or $y$. Let $e = uv$ be an edge of $T$, and suppose that $u$ is incident in $T$ with
edges of both colours and that \( v \) is incident in \( T \) with edges of one colour only. Then \( u \) has degree 2 in \( T \), and either \( uvx \) or \( uvy \) is a unique 2-path in \( T' \), according as \( e \) is coloured blue or red. Therefore, every edge of \( T \) is 2-required in \( T' \).

The graph \( T' \) does not have exponent 2, since \( T \) certainly contains a pair of vertices at distance 6 apart, one of which is incident only with blue edges and the other only with red edges. These vertices have no common neighbour in \( T' \). We now extend \( T' \) by adjoining a vertex \( z \) that is adjacent to every vertex of the original \( T \). The resulting graph \( T'' \) has exponent 2 and has the property that each edge of \( T \) is 2-required for a vertex of \( T \) to either \( x \) or \( y \). Thus \( T'' \) is a simple pre-me\(_2\)-embedding of \( T \) and \( \alpha(T) = 3 \).

The construction in the proof of Theorem 5.3.6 can be adapted to obtain the following stronger version.

**Theorem 5.3.7.** Let \( T \) be a tree of diameter at least 5, with no bough of length 1 and no terminal bough of length 3, and suppose that \( T \) is not isomorphic to \( P_6 \), the path on six vertices. Then \( T \) is simply me\(_2\)-embeddable and \( \alpha(T) = 3 \).

**Proof.** We first note that the hypotheses imply that \( T \) contains no copy of \( P_6 \) as a maximal path, so \( T \) has no critical edge and \( T \) is simply me\(_2\)-embeddable by Theorem 5.1.7.

By Lemma 5.3.4 \( \alpha(T) \geq 3 \).

We adapt the construction of Theorem 5.3.6 to construct a simple pre-me\(_2\)-embedding of \( T \) with three adjoined vertices. Choose a leaf \( w \) of \( T \). As before we construct a colouring of the edges of \( T \), this time with three colours. If the bough to which \( w \) belongs has odd length (and therefore has odd length at least 5 by the hypothesis), collapse the fourth edge of this bough that is encountered along a path from \( w \), by identifying its two incident vertices. For every other bough of odd length in \( T \) (and every other bough of odd length has odd length at least 3 by the hypothesis), collapse the second edge of the bough that is encountered along a path from \( w \), by identifying its two incident vertices. Every edge that is collapsed involves two vertices of degree 2 in \( T \). The result of this process of collapsing edges is a tree \( T_1 \) in which every bough has even length. We colour each edge of \( T_1 \) either blue or red as in the proof of Theorem 5.3.6, so that every path emanating from \( w \) in \( T_1 \) has edges coloured in the sequence blue, red, red,
blue, blue..., and every junction in $T_1$ is incident with edges of one colour only. We now restore the collapsed edges and colour them green. At this stage we have a 3-colouring of the edges of the original $T$.

Note that every vertex that is incident with edges of different colours has degree 2 in $T$, and that every green edge involves two vertices of degree 2, one of which is also incident with a red edge and the other with a blue edge. Thus every vertex of degree 2 has a neighbour incident with a blue edge and a neighbour incident with a red edge.

Let $v$ be a junction in $T$, so $v$ is incident with at least three edges of $T$ and all have the same colour, either red or blue. It is possible that every neighbour of $v$ is otherwise incident only with a green edge; this occurs if the last bough on the path from $w$ to $v$ in $T$ has length 3 (or 5 if it is the bough to which $w$ belongs), and every bough to which $v$ belongs has odd length. In this situation $v$ has no neighbour incident with a red edge or no neighbour incident with a blue edge, and we say that $v$ is a troublesome junction.

We resolve troublesome junctions by repeating the following adaptation to the edge colouring until none remain.

**Adaptation:** Let $v$ be a troublesome junction that is closest to $w$ in $T$, and let $e$ be the last edge on the path from $w$ to $v$ in $T$. Let $T_v$ be the subtree of $T$ consisting of all vertices and edges that belong to the connected component of $v$ in the graph $T \setminus \{e\}$. In $T_v$ only, we switch every red edge to blue and every blue edge to red, leaving green edges unchanged.

After the adaptation, the edges incident with $v$ are all coloured blue or red, with one of one colour and the remainder of the other. Thus $v$ is no longer a troublesome junction, but the partial recolouring does not change the status of any other junction in $T$.

After successively applying the adaptation until no troublesome junctions remain, we construct a simple extension $T'$ of $T$ as follows:

- we adjoin a vertex $x$ that is adjacent to every vertex of $T$ that is incident with a blue edge;
- we adjoin a vertex $y$ that is adjacent to every vertex of $T$ that is incident with a red edge;
we adjoin a vertex $z$ that is adjacent to every vertex of $T$.

We claim that $T'$ is a simple pre-me$_2$-embedding of $T$. Certainly $T'$ has exponent 2. This follows from the fact that all vertices of $T$ share $z$ as a common neighbour and the fact that every vertex of $T$ has a neighbour in $T$ incident with a red edge and a neighbour in $T$ incident with a blue edge.

Let $e = uv$ be an edge of $T$.

- If $e$ is coloured green then both $u$ and $v$ have degree 2 in $T$, and the other two edges of $T$ that share a vertex with $e$ are respectively coloured blue and red. In this case $e$ is part of a unique 2-path from one of its vertices to $x$, and a unique 2-path from its other vertex to $y$.

- If $e$ is coloured blue and one of its vertices (say $v$) is incident with blue edges only, then $u$ has degree 2 and its other incident edge is either red or green. In either case, $uvx$ is the unique 2-path from $u$ to $x$ in $T'$.

If $e$ is coloured blue and both of its vertices are incident with edges of more than one colour, then one of its vertices (say $v$) is an adapted troublesome junction. Then $u$ has degree 2 in $T$ and its other incident edge is green. The unique 2-path from $u$ to $x$ in $T'$ is via $v$.

- The case where $e$ is coloured red is the same as the case where it is blue.

Hence every edge of $T$ is 2-required in $T'$ and $T'$ is a simple pre-me$_2$-embedding of $T$ and $\alpha(T) = 3$. \qed

A simple characterization of trees $T$ satisfying $\alpha(T) = 3$ would be of interest. While the trees of Theorem 5.3.7 do not have critical edges, the following example shows that the absence of critical edges in a tree $T$ is not enough to guarantee that $\alpha(T) \leq 3$.

**Example 5.3.8.** Let $T$ be the following tree.

```
     v'  v
    /   /   \
   /   /     /
  v   v
```

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Since $T$ has no critical edge it is simply $\text{me}_2$-embeddable by Theorem 5.1.7 and $\alpha(T) \geq 3$ by Lemma 5.3.4. Suppose that $T'$ is a simple extension of $T$ of exponent 2 with $X = V(T') \setminus V(T) = \{x, y, z\}$. We will show that $T'$ cannot be an $\text{me}_2$-graph. To see this, consider the three edges incident with the vertex $v$ in $T$. By Lemma 5.1.4 none of these can be part of a unique 2-path in $T'$ from a neighbour of $v$ to a vertex of $X$, or from $v$ to a vertex of $T$.

If all three of these edges are required for 2-paths in $T'$ from $v$ to vertices of $X$, then each of the $T$-neighbours of $v$ has exactly one neighbour in $X$, and each of the three elements of $X$ is adjacent in $T'$ to exactly one $T$-neighbour of $v$.

If an edge incident with $v$ is part of a unique 2-path in $T'$ between two $T$-neighbours of $v$, it means that these two vertices share no neighbour in $X$ and hence one of them is adjacent in $T'$ only to a single vertex of $X$.

In either case, if all three edges incident with $v$ are involved in unique 2-paths in $T'$, then $v$ has a neighbour $u$ in $T$ that is adjacent to only one of $x, y, z$, say $x$. In addition $v$ has two $T$-neighbours whose sets of $T$-external neighbours in $T'$ are disjoint.

Now since every $T$-neighbour of $v'$ has a path of length 2 to $u$ in $T'$, it must be adjacent to $x$. This means that $v'$ does not have a pair of $T$-neighbours whose $T$-external neighbour sets in $T'$ are disjoint. By the above argument applied to $v'$ instead of $v$, it follows that the edges of $T$ incident with $v'$ cannot all be involved in unique 2-paths in $T'$. Thus $T'$ is not an $\text{me}_2$-graph and $\alpha(T) \geq 4$.

In the final theorem of this section, we show that the parameter $\alpha$ is unbounded on the class of simply $\text{me}_2$-embeddable trees.

**Theorem 5.3.9.** Let $n$ be any positive integer. There exists a simply $\text{me}_2$-embeddable tree $T$ with $\alpha(T) = n$.

**Proof.** We may assume that $n \geq 4$, since the cases $n \leq 3$ have already been considered. We present an explicit construction of a family $\{T_k : k \geq 2\}$ of simply $\text{me}_2$-embeddable trees all having diameter 12. The tree $T_k$ has $8k + 1$ vertices and $k$ critical edges, and $\alpha(T_k)$ is an unbounded increasing function of $k$. For $k \geq 2$, the tree $T_k$ is the union of $k$ copies of the graph $T_1$ shown below, all sharing the vertex $a$ but otherwise disjoint. We refer to these copies of $T_1$ as the branches of $T_k$. The copies of the vertices $s, u, x, v, w, y, z, t$ in the $i$th branch will be denoted respectively by $s_i, u_i, x_i, v_i, w_i, y_i, z_i, t_i$.  

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We denote the edge \( x_i v_i \) in \( T_k \) by \( e_i \). Each \( e_i \) is a critical edge of \( T_k \), and these are the only critical edges. The only viable path for \( e_i \) in \( T_k \) is \( u_i x_i v_i \). Since there are no incompatibilities between these viable paths, \( T_k \) is simply \( \text{me}_2 \)-embeddable by Theorem 5.2.4. Fix \( k \) and let \( T' \) be a simple \( \text{me}_2 \)-embedding of \( T_k \) with \( X = V(T') \setminus V(T_k) \) and \( |X| = \alpha(T_k) \). For any \( i \), the vertices \( u_i \) and \( v_i \) in the graph \( T' \) cannot have a common neighbour in \( X \). However, both \( u_i \) and \( v_i \) must share neighbours in \( X \) with all those \( u_j \) and \( v_j \) for which \( j \neq i \).

For any vertex \( v \) of \( T_k \), let \( N_X(v) \) denote the set of neighbours of \( v \) in \( X \). Let \( \mathcal{N} \) denote the collection of the \( 2k \) subsets \( N_X(u_i), N_X(v_i), i = 1, \ldots, k \). Then \( \mathcal{N} \) has the following intersection properties.

- For each \( i \), \( N_X(u_i) \) and \( N_X(v_i) \) are disjoint.
- The intersection of two elements of \( \mathcal{N} \) corresponding to different indices \( i \) and \( j \) is non-empty.
- Let \( N_i \) and \( N_j \) be two elements of \( \mathcal{N} \) corresponding to different indices \( i \) and \( j \). Then \( N_i \) intersects both \( N_j \) and the complement of \( N_j \) in \( X \), hence \( N_i \) cannot be a subset of \( N_j \).

Thus no element of \( \mathcal{N} \) can be a subset of any other. Write \( r = \left\lfloor \frac{|X|}{2} \right\rfloor \), and note that for each \( i \) either \( N_X(u_i) \) or \( N_X(v_i) \) has at most \( r \) elements. Let \( \mathcal{N}' \) be a subset of \( \mathcal{N} \) which includes exactly one of \( N_X(u_i) \) and \( N_X(v_i) \) for each \( i \), selected so that no element of \( \mathcal{N}' \) has more than \( r \) elements. Since the elements of \( \mathcal{N}' \) are pairwise non-disjoint, the Erdös-Ko-Rado Theorem states that the number of them is at most \( \binom{|X| - 1}{r - 1} \). Thus \( |X| = \alpha(T_k) \) must be great enough that \( \left( \frac{|X| - 1}{\left\lfloor |X|/2 \right\rfloor - 1} \right) \) is at least equal to \( k \). This establishes
that $\alpha(T_k)$ is unbounded as a function of $k$, which is part of the content of Theorem 5.3.9.

To complete the proof we need to exhibit for every positive integer $n \geq 4$ a value of $k$ for which $\alpha(T_k) = n$. Define $\theta : \mathbb{N} \to \mathbb{N}$ by

$$\theta(n) = \left(\frac{n - 1}{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right).$$

It is easily checked that $\theta$ is strictly increasing for $n \geq 3$ and satisfies the following recursions

$$\theta(n + 1) = \begin{cases} 
\theta(n) \times \frac{2n}{n+2} & \text{if } n \text{ is even} \\
\theta(n) \times \frac{2n}{n-1} & \text{if } n \text{ is odd}
\end{cases}.$$

For a given $n \geq 4$, choose an integer $k$ for which $\theta(n - 1) < k \leq \theta(n)$. We will show that $\alpha(T_k) = n$. Certainly $\alpha(T_k) > n - 1$, since $n - 1$ adjoined vertices can accommodate no more than $\theta(n - 1)$ branches. We show that $\alpha(T_k) = n$ by demonstrating an explicit pre-me$_2$-embedding of $T_k$. Let $X$ be a set of $n$ vertices to be adjoined to $T_k$. Since $k \leq \theta(n)$, we may choose $k$ distinct subsets of $X$ so that each has exactly $\left\lfloor \frac{n}{2} \right\rfloor$ elements and all have one particular element in common. Assign these (in some order) to be the external neighbour sets of $u_1, \ldots, u_k$, and for each $i$ assign $N_X(v_i)$ to be the complement in $X$ of $N_X(u_i)$. Complete this to a pre-me$_2$-embedding $T'$ of $T_k$ as follows.

- Each $w_i$ and $z_i$, being a neighbour of a leaf in $T_k$, must be adjacent to all elements of $X$.

- For each $i$, assign $N_X(y_i) = N_X(u_i)$. Choose $N_X(t_i), N_X(x_i)$ and $N_X(s_i)$ so that each has $n - 1$ elements and $N_X(x_i) \cup N_X(t_i) = N_X(x_i) \cup N_X(s_i) = X$. Since $n \geq 4$, this ensures that paths of length 2 exist from $x_i, t_i$ and $s_i$ to all vertices of $T_k$. Moreover, it ensures that the edges $z_ix_i$ and $u_is_i$ are 2-required for $z_i$ and $u_i$ respectively to vertices of $X$.

- Assign $N_X(a) = X$.

It is easily confirmed that the resulting graph $T'$ is a pre-me$_2$-embedding of $T_k$, hence $\alpha(T_k) = n$. \qed
5.4 Simple strong-me$_2$-embeddings of trees

We now focus our attention on simple strong-me$_2$-embeddings of trees. We provide a classification of trees that admit a simple strong-me$_2$-embedding. Furthermore, we show that the given embeddings are unique.

**Definition 5.4.1.** Let $G$ be a graph. A simple strong-me$_2$-embedding of $G$ is a strong-me$_2$-graph that is a simple extension of $G$.

We introduced the concept of a simple pre-me$_2$-embedding $H$ of a graph $G$ in Definition 5.1.6. We recall that in order to show that a graph $G$ is simply me$_2$-embeddable, it is sufficient to show that $G$ admits a simple pre-me$_2$-embedding. This is because deleting edges in succession that are not 2-required in a graph of exponent 2 eventually results in an me$_2$-graph. However, this process does not necessarily result in a strong-me$_2$-graph and hence there is no easy analogue for the pre-me$_2$-embedding concept for simple strong-me$_2$-embeddings.

**Definition 5.4.2.** For a strongly-me$_2$-embeddable graph $G$, we denote by $\beta(G)$ the minimum over all simple strong-me$_2$-embeddings $H$ of $G$ of the number $|V(H)| - |V(G)|$.

It is clear that any tree that does not admit a simple me$_2$-embedding does not admit a simple strong-me$_2$-embedding. If a tree $T$ admits a simple strong-me$_2$-embedding, $\beta(T) \geq \alpha(T)$.

We note an example of a path which is not simply strongly-me$_2$-embeddable, and one which is. These will lead us to important observations regarding simple strong-me$_2$-embeddability of trees.

**Example 5.4.3.** The path $P_4$ is not simply strongly-me$_2$-embeddable.

**Proof.** Let the vertices of $P_4$ be labelled as in the diagram below, and write $e$ for the edge $u_3u_4$.

![Diagram of P4](https://example.com)

If $H$ is a simple extension of exponent 2 of this copy of $P_4$, both $u_2$ and $u_3$ must be adjacent in $H$ to all $P_4$-external vertices by Lemma 5.1.4. Then since $u_3$ has a path of length
2 to all $P_1$-external vertices via $u_2$ in $H$, we conclude that $e$ is not strongly-2-required in $H$. Thus $H$ is not an $me_2$-graph and $P_4$ is not simply strongly-$me_2$-embeddable. \hfill $\Box$

This example leads us to the following observations.

**Lemma 5.4.4.** Let $T$ be a tree of order at least 3 and let $H$ be a simple strong-$me_2$-embedding of $T$. Then there are no two adjacent vertices in $T$ that are adjacent to all $T$-external vertices in $H$.

**Proof.** Suppose there are two adjacent vertices $u_1$ and $u_2$ in $T$ that are adjacent to all $T$-external vertices in $H$. Since $T$ has order at least 3, $u_1$ and $u_2$ cannot both have degree 1 in $T$ and we may assume that there exists a vertex $u_3$ in $T$ such that $u_2u_3$ is an edge in $T$. However, since every vertex of $T$ is adjacent to at least one $T$-external vertex in $H$, and $u_2$ has a path of length 2 to all $T$-external vertices via $u_1$, the edge $u_2u_3$ is not strongly-2-required in $H$, and $H$ is not a simple strong-$me_2$-embedding of $T$. \hfill $\Box$

**Lemma 5.4.5.** Let $T$ be a tree with a maximal path of length 3. Then $T$ is not simply strongly-$me_2$-embeddable.

**Proof.** The proof is immediate from Lemma 5.1.4 and Lemma 5.4.4. \hfill $\Box$

We give an example of a path that is simply strongly-$me_2$-embeddable.

**Example 5.4.6.** The path $P_5$ is simply strongly-$me_2$-embeddable with $\beta(P_5) = 2$.

**Proof.** We note that by Lemma 5.3.3 if $P_5$ is simply strongly-$me_2$-embeddable then $\beta(P_5) \geq 2$. The graph below is a simple strong-$me_2$-embedding $H$ of $P_5$ in which the number of $P_5$-external vertices is 2. We note that a pre-$me_2$-embedding of $P_5$ was presented in Lemma 5.3.3, and the $me_2$-embedding of $P_5$ obtained by deleting one redundant edge is the graph presented below. The vertices of the original copy of $P_5$ are labelled $u_1, \ldots, u_5$ and the $P_5$-external vertices in $H$ are labelled $x$ and $y$. 

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We show that every edge of $H$ is strongly-2-required.

• $u_1u_2$ is strongly-2-required for $u_1$ to $x$ and for $u_2$ to $x$.

• $u_2u_3$ is strongly-2-required for $u_2$ to $y$ and for $u_3$ to $x$.

• $xu_1$ is strongly-2-required for $x$ to $u_2$ and for $u_1$ to $u_2$.

• $xu_2$ is strongly-2-required for $x$ to $u_1$ and for $u_2$ to $u_1$.

• $yu_2$ is strongly-2-required for $y$ to $u_1$ and for $u_2$ to $u_3$.

• $yu_3$ is strongly-2-required for $y$ to $u_2$ and for $u_3$ to $u_4$.

The remaining edges are strongly-2-required by symmetry. Hence $H$ is a strong-me$_2$-graph and $P_5$ admits a simple strong-me$_2$-embedding. Furthermore, $\beta(P_5) = 2$. \hfill \square

We observe that in Example 5.4.6, since $u_2$ is a neighbour of a leaf in $P_5$, $u_2$ is adjacent to both $P_5$-external vertices $x$ and $y$. Therefore, edges incident with $u_2$ in $P_5$ must be strongly-2-required in $H$ for $u_2$ to $P_5$-external vertices in $H$ (since $u_2$ has a path of length 2 to all vertices of $P_5$ in $H$ via $x$ or $y$). This means that every neighbour of $u_2$ in $P_5$ is adjacent to a $P_5$-external vertex that is adjacent to no other neighbour of $u_2$. These observations apply to any leaf neighbour in a simple strong-me$_2$-embedding of any graph.

We introduce the following terminology.

Suppose $T$ is a tree. Then $\gamma(T) = \max \{ \deg(v) | v \text{ is a neighbour of a leaf in } T \}$.

Lemma 5.4.7. Let $T$ be a tree that admits a simple strong-me$_2$-embedding. Then $\beta(T) \geq \gamma(T)$. 

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Proof. Let $H$ be a simple strong-me$_2$-embedding of $T$ and let $v$ be a neighbour of a leaf of $T$ that has degree $\gamma(T)$. Therefore, $v$ is adjacent to all $T$-external vertices in $H$ and every edge incident with $v$ in $T$ is strongly-2-required for $v$ to a $T$-external vertex in $H$. This means that every neighbour of $v$ in $T$ is adjacent to a $T$-external vertex that is adjacent to no other neighbour of $v$. We conclude that $\beta(T) \geq \gamma(T)$. \hfill $\Box$

Let $T$ be a simple strongly-me$_2$-embeddable tree. We show that $\beta(T) = \gamma(T)$ and furthermore, that the number of $T$-external vertices in any simple strong-me$_2$-embedding of $T$ is exactly $\gamma(T)$.

**Theorem 5.4.8.** Let $T$ be a tree that admits a simple strong-me$_2$-embedding $H$. Then the number of $T$-external vertices in $H$ is $\gamma(T)$.

**Proof.** Suppose $\gamma(T) = m$ and let $v$ be a neighbour of a leaf of $T$ with $\deg_T(v) = m$. If $m = 1$ then $T \cong P_2$ and the theorem holds since $K_3$ is the only simple strong-me$_2$-embedding of $P_2$.

So we may assume $m \geq 2$. By Lemma 5.4.7 there are at least $m$ $T$-external vertices in $H$. We will show that there are exactly $m$ $T$-external vertices in $H$.

Label the neighbours of $v$ in $T$ as $v_1, ..., v_m$. By Lemma 5.4.7 every neighbour of $v$ in $T$ is adjacent to a $T$-external vertex that is adjacent to no other neighbour of $v$. Label the $T$-external vertices of $H$ as $x_1, ..., x_k$ for some $k \geq m$, such that $x_i$ is adjacent to $v_i$ and no other neighbour of $v$ for $i = 1, ..., m$. Now, since $vx_i$ is an edge for all $i$ in $H$, $vx_i$ is strongly-2-required in $H$ for all $i$. This means that for each $x_i$ in $H$, there exists a vertex $u_i$ in $T$, not at distance 2 from $v$ in $T$, such that $x_i$ is the unique $T$-external neighbour of $u_i$ in $H$. The $u_i$ for $i = 1, ..., k$ must share a common neighbour in $T$, to afford 2-paths between them in $H$. We show that this common neighbour must be $v$ and therefore that the number of $T$-external vertices in $H$ is $m$.

Suppose not. Suppose the $u_i$ share a common neighbour $u$, different from $v$, in $T$. Now $u$ has a neighbour $u_1$ whose only $T$-external neighbour in $H$ is $x_1$, and $u$ has a neighbour $u_2$ whose only $T$-external neighbour in $H$ is $x_2$. Since $H$ is a strong-me$_2$-graph, if $u \neq v$ it follows that $u_1$ is at distance 2 from $v_2$ in $T$ and $u_2$ is at distance 2 from $v_1$ in $T$. These cannot both hold in a tree so it must be that $u = v$. We conclude that $k \leq m$. Since $k \geq m$ by Lemma 5.4.7 it follows that $k = m$. Therefore, the number of $T$-external vertices in $H$ is exactly $m$. \hfill $\Box$
Suppose $T$ is a tree that admits a simple strong-me$_{2}$-embedding $H$. Then Theorem 5.4.8 and its proof not only give the precise number of $T$-external vertices in $H$, but also tell us that for a neighbour $v$ of a leaf in $T$ with $\deg_T(v) = \gamma(T)$, each neighbour of $v$ in $T$ must be adjacent to a unique $T$-external vertex in $H$, and no two neighbours of $v$ can share a common $T$-external neighbour in $H$. This will enable us to show that $H$ is the unique simple strong-me$_{2}$-embedding of $T$.

We now classify all trees that admit a simple strong-me$_{2}$-embedding, and show that in each case the embedding is unique.

**Theorem 5.4.9.** Let $T$ be a tree and let $v$ be a neighbour of a leaf of $T$ such that $\deg_T(v) = \gamma(T) \geq 2$. Then $T$ admits a simple strong-me$_{2}$-embedding if and only if all leaves of $T$ are at distance 1 or 3 from $v$ in $T$, and every vertex at distance 2 from $v$ in $T$ is adjacent to exactly one leaf. Furthermore, such an embedding of $T$ is unique.

The remainder of this section consists of a proof of Theorem 5.4.9. First suppose $T$ is simply strongly-me$_{2}$-embeddable. We write $k$ for $\gamma(T)$. We make the following observations regarding leaves of $T$.

**Lemma 5.4.10.** There is no leaf of $T$ at distance 2 from $v$.

*Proof.* If any leaf of $T$ is at distance 2 from $v$, then $T$ has a maximal path of length 3 and by Lemma 5.4.5, $T$ is not simply strongly-me$_{2}$-embeddable. We conclude that every leaf of $T$ is at distance 1, or distance at least 3 from $v$. □

**Lemma 5.4.11.** There is no leaf of $T$ at distance 4 or more from $v$.

*Proof.* Let $H$ be a simple strong-me$_{2}$-embedding of $T$ and suppose there exists a leaf $u$ in $T$ at distance 4 or more from $v$. Label the neighbours of $v$ as $v_1, ..., v_k$. By Lemma 5.4.8, there exist exactly $k$ $T$-external vertices in $H$ such that each neighbour of $v$ in $T$ is adjacent to a unique $T$-external vertex, and no two neighbours of $v$ are adjacent to the same $T$-external vertex in $H$.

If $u$ is at distance 4 or more from $T$, then $u$ is adjacent to all $T$-external vertices in $H$ in order to afford 2-paths between $u$ and $v_i$ in $H$ for all $i$. Since $u$ is a leaf, it follows by Lemma 5.1.4 that the neighbour of $u$ in $T$ is also adjacent to all $T$-external vertices in $H$; contrary to Lemma 5.4.4. This completes the proof. □
We observe that by Lemma 5.4.10 and Lemma 5.4.11 every leaf of $T$ must be at distance 1 or 3 from $v$. We now show that any leaf at distance 3 from $v$ in $T$ has the property that its unique neighbour in $T$ has degree 2.

**Lemma 5.4.12.** Every vertex at distance 2 from $v$ in $T$ is adjacent to exactly one leaf of $T$.

*Proof.* Suppose not. Suppose $H$ is a simple strong-me$_2$-embedding of $T$ and suppose that $u$ is at distance 2 from $v$ in $T$. Label the neighbours of $v$ as $v_1, ..., v_k$ such that $v_k$ is a neighbour of $u$ in $T$. Label the $T$-external vertices of $H$ as $x_1, ..., x_k$ such that $x_k$ is the unique $T$-external neighbour of $v_k$ in $H$. If $u_1$ and $u_2$ are leaves of $T$ at distance 3 from $v$ that share $u$ as a common neighbour, then $u_1 x_i$ for $i \neq k$ and $u_2 x_i$ for $i \neq k$ are edges in $H$. It follows that $uu_1$ and $uu_2$ are not strongly-2-required for 2-paths from $u$ in $H$. Hence $H$ is not a strong-me$_2$-graph. Thus any vertex at distance 2 from $v$ is adjacent to at most one leaf of $T$. By Lemma 5.4.10 and Lemma 5.4.11 we conclude that every vertex at distance 2 from $v$ in $T$ is adjacent to exactly one leaf of $T$. □

Lemmas 5.4.10, 5.4.11 and 5.4.12 provide us with necessary conditions for a tree to admit a simple strong-me$_2$-embedding. We now show that these conditions are also sufficient.

**Lemma 5.4.13.** Let $T$ be a tree and let $v$ be a neighbour of a leaf of $T$ such that $\deg_T(v) = \gamma(T) \geq 2$. Then $T$ admits a unique simple strong-me$_2$-embedding if all leaves of $T$ are at distance 1 or 3 from $v$ in $T$, and every vertex at distance 2 from $v$ in $T$ is adjacent to exactly one leaf.

*Proof.* Label the neighbours of $v$ as $v_1, ..., v_k$ where $k = \gamma(T)$. We construct a strong-me$_2$-embedding $H$ of $T$ as follows:

1. Introduce $k$ vertices $x_1, ..., x_k$ to $H$ such that the unique $T$-external neighbour of $v_i$ is $x_i$ in $H$.

2. All neighbours of leaves in $T$ are adjacent to all $k$ $T$-external neighbours in $H$.

3. If $u$ is a leaf of $T$, not adjacent to $v$, let $v_j$ be the unique neighbour of $v$ at distance 2 from $u$. Then $ux_i$ are edges for all $i \neq j$ in $H$.

We now show that $H$ is a strong-me$_2$-graph. We first show that $H$ has exponent 2.
• There are paths of length 2 between $v$ and each $x_i$ via $v_i$ in $H$, and there exist paths of length 2 between $v$ and every vertex in $T$ via the $T$-external vertices in $H$.

• Let $v_i$ be a neighbour of $v$ in $T$. Then $v_i$ has a path of length 2 to all $T$-external vertices in $H$ via $v$. There exist paths of length 2 between $v_i$ and $v_j$ via $v$ in $H$. There exist paths of length 2 between $v_i$ and any neighbour of a leaf in $T$ via $x_i$ in $H$. Let $u$ be a leaf in $T$ that is not a neighbour of $v$ in $T$, and let $p$ be the neighbour of $u$ in $T$. Then $v_i$ has a path of length 2 to $u$ either via $x_i$, or via $p$ in $H$.

• Let $p$ be a vertex of $T$ at distance 2 from $v$ in $T$. Then since $p$ is adjacent to all $T$-external vertices in $H$, it follows that $p$ has a path of length 2 in $H$ to all vertices of $T$. The only vertices adjacent to $p$ in $T$ are $v_j$ for some $j$ and a leaf $u$ that is not a neighbour of $v$ in $T$. So $p$ has a path of length 2 to $x_j$ via $v_j$ in $H$, and $p$ has a path of length 2 in $H$ to each $x_i$ for $i \neq j$ via $u$.

• The only case left to consider for vertices of $T$ are paths of length 2 between leaves of $T$. Let $u_1$ and $u_2$ be two leaves of $T$ such that neither $u_1$ or $u_2$ are neighbours of $v$ in $T$. Then either $u_1$ and $u_2$ are both at distance 2 from the same $v_i$ for some $i$, and hence they have the same $T$-external neighbours in $H$ and therefore a path of length 2 between them in $H$, or $u_1$ is at distance 6 from $u_2$ in $T$ and $k \geq 3$. Since $k \geq 3$ and there only exists one $T$-external vertex of $H$ that is not a neighbour of $u_1$, and similarly for $u_2$, it follows that $u_1$ and $u_2$ have a path of length 2 between them in $H$ via some $T$-external vertex. Now let $u_1$ be a leaf of $T$ that is not a neighbour of $v$ in $T$, and let $v_i$ be a leaf of $T$ that is a neighbour of $v$. Then $u_1$ has a path of length 2 to $v_i$ in $H$ via $x_i$.

• We have shown that all vertices of $T$ have paths of length 2 to all other vertices of $T$ and to each $x_i$ in $H$. Now $x_i$ has a path of length 2 to $x_j$ via $v$ in $H$.

Thus $H$ has exponent 2. We now show that every edge of $H$ is strongly-$2$-required.

• The edge $vv_i$ is strongly-$2$-required in $H$ for $v$ to $x_i$ and for $v_i$ to $v_j$ for $i \neq j$.

• The edge $vx_i$ is strongly-$2$-required in $H$ for $v$ to $v_i$ and for $x_i$ to $v_i$.

• The edge $v_i x_i$ is strongly-$2$-required in $H$ for $v_i$ to $v$ and for $x_i$ to $v$. 

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Now let \( p \) be a vertex of \( T \) at distance 2 from \( v \) in \( T \) and let \( u \) be the leaf adjacent to \( p \) in \( T \). Let \( v_j \) be the common neighbour of \( p \) and \( v \) in \( T \).

- Then \( pv_j \) is strongly-2-required in \( H \) for \( p \) to \( x_j \) and for \( v_j \) to \( u \).
- The edges \( px_i \) are strongly-2-required in \( H \) for \( p \) to \( v_i \) and for \( x_i \) to \( u \).
- The edge \( up \) is strongly-2-required in \( H \) for \( u \) to \( v_j \) and for \( p \) to \( x_i \) where \( i \neq j \).
- The edges \( ux_i \) for \( i \neq j \) are strongly-2-required in \( H \) for \( u \) to \( v_i \) and for \( x_i \) to \( p \)

We conclude that \( H \) is a strong-me_2-graph. We now show that \( H \) is the unique simple strong-me_2-embedding of \( H \).

Let \( K \) be a simple strong-me_2-embedding of \( T \). By Theorem 5.4.8, there are \( k \) external vertices \( x_1, ..., x_k \) in \( K \), and each \( v_i \) is adjacent to a unique \( x_i \) in \( K \). All other vertices in \( T \) are either neighbours of leaves, or are leaves that are not adjacent to \( v \) in \( T \). By Lemma 5.1.4 all neighbours of leaves are adjacent to all \( T \)-external vertices in \( K \). Let \( u \) be a leaf that is not adjacent to \( v \) in \( T \), and let \( p \) be the neighbour of \( u \) in \( T \). Then \( pv_i \) is an edge for some vertex \( v_i \) that is a neighbour of \( v \) in \( T \). In order for \( pv_i \) to be strongly-2-required in \( K \), \( v_i \) must be the unique common neighbour of \( p \) and \( x_i \) in \( K \). Therefore \( ux_i \) is not an edge in \( K \). However, there must be a path of length 2 from \( u \) to all \( v_j \) in \( K \) for \( i \neq j \), so \( ux_j \) must be edges in \( K \). We conclude that \( K \cong H \). \( \square \)

We present an example below of the embedding \( H \) as described in Lemma 5.4.13.

**Example 5.4.14.** The unique simple strong-me_2-embedding of a tree \( T \).

Let \( T \) be the tree shown below with vertex set \( \{v, v_1, v_2, v_3, p_1, p_2, p_3, u_1, u_2, u_3\} \).
Then $\deg(v) = \gamma(T) = 3$. Therefore, in a simple strong-me$_2$-embedding $H$ of $T$, there are three $T$-external vertices $x_1, x_2$ and $x_3$. All neighbours of leaves in $T$ are adjacent to all of $x_1, x_2, x_3$. In order to facilitate the reading of the graph, we write the labels of the $T$-external neighbours of vertices of $T$ in $H$.

Each neighbour of $v$ in $T$ is adjacent to exactly one of $x_1, x_2, x_3$, and no two neighbours of $v$ are adjacent to the same $T$-external vertex.

The only $T$-external neighbours of $u_1$ are $x_1$ and $x_3$. The only $T$-external neighbours of $u_2$ and $u_3$ are $x_1$ and $x_2$. 
Chapter 6

Me₂-embeddings

This focus of this chapter is on general me₂-embeddings. Let $G$ be a graph; we consider whether $G$ can be embedded as an induced subgraph of an me₂-graph, and the number of additional vertices in such an embedding. We present a construction that embeds any graph on $n$ vertices as an induced subgraph of an me₂-graph with $2n + 2$ extra vertices. We then turn our attention to embedding $K_n$ as an induced subgraph of a strong-me₂-graph.

6.1 General me₂-Embeddings

Definition 6.1.1. Let $G$ be a graph. Then an me₂-embedding of $G$ is an me₂-graph $H$ that has $G$ as an induced subgraph. If $G$ admits an me₂-embedding $H$, then we say $G$ is me₂-embeddable.

Definition 6.1.2. A pre-me₂-embedding of a graph $G$ is a graph $H$ that has exponent 2, has $G$ as an induced subgraph, and has the property that every edge of $G$ is 2-required in $H$.

If $H$ is a (pre) me₂-embedding of a graph $G$, we refer to the vertices of $V(H) \setminus V(G)$ as $G$-external vertices.

We discussed simple pre-me₂-embeddings in Chapter 5. Unlike in simple me₂-embeddings, in an me₂-embedding of a graph $G$, there may exist adjacencies among the $G$-external vertices. If $H$ is a pre-me₂-embedding of $G$, then any edge of $H$ that is not 2-required in $H$ involves either one vertex of $G$ and one vertex external to $G$, or two vertices external to $G$. Therefore, similarly to simple pre-me₂-embeddings, in
order to show that a graph is me$_2$-embeddable, it is enough to show that it admits a pre-me$_2$-embedding.

We note that no vertices are deleted when transitioning from a pre-me$_2$-embedding $H$ of $G$ to an me$_2$-embedding $H'$ of $G$ and hence $|V(H) \setminus V(G)| = |V(H') \setminus V(G)|$.

We provide a construction below that allows us to embed any graph $G$ of order $n$ as an induced subgraph of an me$_2$-graph with $2n + 2$ $G$-external vertices, and we will show how this number can be reduced given specific constraints on $G$.

**Theorem 6.1.3.** Let $G$ be a graph of order $n$. Then $G$ admits an me$_2$-embedding $H$ with $|V(H) \setminus V(G)| = 2n + 2$.

*Proof.* Label the vertices of $G$ as $v_1, \ldots, v_n$. We extend $G$ to a pre-me$_2$-embedding $H$ as follows:

1. Introduce two vertices $x$ and $y$ such that $xy$ is an edge and $xv_i$ and $yv_i$ are edges for all $i$.

2. For each $v_i$ in $G$, introduce two new vertices $a_i$ and $b_i$ and edges $v_i a_i$, $v_i b_i$ and $a_i b_i$.

3. For all $i > 1$, $a_1 a_i$ and $a_1 b_i$ are edges in $H$.

4. If $v_i v_j$ is not an edge in $G$ then $a_i a_j$ and $b_i b_j$ are edges in $H$.

5. $H$ has no more edges than the ones described above.

We now show that $H$ is a pre-me$_2$-embedding of $G$. We will first show that $H$ has exponent 2 and then show that every edge in $G$ is 2-required.

- There is a path of length 2 between every pair of vertices $v_i, v_j$ in $G$ via $x$.
- There are paths of length 2 from $x$ and $y$ to $a_i, b_i$ via $v_i$.
- There are paths of length 2 from $v_i$ to $x$ via $y$ and from $v_i$ to $y$ via $x$.
- There is a path of length 2 from $x$ to $y$ via $v_i$ for any $i$.
- Now choose $v_i \in V(G)$. We want to show $v_i$ has 2-paths to all $a_j$ and $b_j$.
- If \( i = j \) then \( v_i, a_i, b_i \) and the edges between them form a triangle and there exist paths of length 2 between \( v_i, a_i \) and \( b_i \).
- If \( i \neq j \), then either \( v_i v_j \) is an edge in which case there are paths of length 2 from \( v_i \) to \( a_j \) and \( b_j \) via \( v_j \), or \( v_i v_j \) is not an edge and there exist paths of length 2 from \( v_i \) to \( a_j \) and \( b_j \) via \( a_i, b_i \) respectively.

- Since \( a_1 \) is adjacent to all \( a_i, b_i \) for \( i \neq a_1 \) in \( H \), there exist paths of length 2 between all \( a_i, a_j, b_j, b_i \) for \( i \neq j \) and \( a_i \neq a_1, a_j \neq a_1 \) via \( a_1 \).
- There exist paths of length 2 between \( a_1 \) and \( a_i \) via \( b_i \) for all \( i > 1 \) and there exist paths of length 2 between \( a_1 \) and \( b_i \) via \( a_i \) for all \( i > 1 \).

Therefore there exist paths of length 2 between all pairs of vertices in \( H \) and so \( H \) has exponent 2. We now must show that every edge in \( G \) is 2-required.

- Let \( v_i v_j \) be an edge of \( G \) such that \( v_i \neq v_1 \). Then \( v_i v_j \) is required for a path of length 2 from \( v_i \) to \( b_j \).

Hence \( H \) has exponent 2 and every edge of \( G \) is 2-required in \( H \) so \( H \) is a pre-me\(_2\) embedding of \( G \) and the number of \( G \)-external vertices is \( 2n + 2 \). \( \square \)

We remark that Theorem 6.1.3 shows that every graph of order \( n \) is an induced subgraph of an me\(_2\)-graph of order at most \( 3n + 2 \). It can be easily checked that if \( G \) has any isolated vertices, then the construction in Theorem 6.1.3 produces an me\(_2\)-graph \( H \). The external edges \( xv_i \) and \( yv_i \) are 2-required for all \( i \) for paths of length 2 from \( x \) and \( y \) to \( a_i \) respectively, and the edge \( xy \) is 2-required in order to afford a 2-path from an isolated vertex of \( G \) to \( x \) in \( H \). If \( G \) has no isolated vertices, then by deleting the edge \( xy \) in \( H \), we obtain an me\(_2\)-graph \( H' \) for which \( G \) is an induced subgraph and the number of \( G \)-external vertices is \( 2n + 2 \). This is because the edges \( xv_i \) and \( yv_i \) are still 2-required for all \( i \) for paths of length 2 from \( x \) and \( y \) to \( a_i \) respectively, and the edge \( xy \) is not 2-required as we have paths of length 2 from vertices \( v_i \) in \( G \) to \( x \) and \( y \) via a neighbour of \( v_i \) in \( G \).

We observe that in some cases, \( 2n + 2 \) greatly overestimates the number of additional vertices required to be adjoined to a graph \( G \) in order for \( G \) to admit an me\(_2\)-embedding. Examples include when \( G \) is already an me\(_2\)-graph and in the case of
simply me₂-embeddable trees in Chapter 5. We now focus on some conditions on the graph $G$ under which we can reduce the number of $G$-external vertices from $2n + 2$.

If $G$ has any isolated vertices, then the $G$-external vertex $y$ is required in the construction above in order to have paths of length 2 between isolated vertices and $x$. However, if $G$ has no isolated vertices then vertices $v_i$ on $G$ have paths of length 2 to $x$ via a neighbour of $v_i$ on $G$ and hence $y$ is not required in the construction and therefore $G$ can be embedded as an induced subgraph of an me₂-graph $F$ with $2n + 1$ $G$-external vertices.

Suppose that $G$ has a vertex $v_2$ that has a path of length 2 in $G$ to all other vertices of $G$. Note that this means $G$ cannot have an isolated vertex. We can embed $G$ as an induced subgraph of $F$ as above. We can reduce the number of $G$-external vertices to obtain an me₂-graph $F'$ that has $G$ as an induced subgraph as follows: Delete the edge $xv_2$ and delete the vertices $a_2, b_2$ and incident edges. Now for all $i$ such that $v_2v_i$ is not an edge, insert the edges $v_2a_i$ and $v_2b_i$. The graph $F'$ has exponent 2 and the edges in $F \cap F'$ are still required for the same reasons that they were in $F$. The edges $v_2a_i$ are required for paths of length 2 from $v_2$ to $b_i$ and $v_2b_i$ are required for paths of length 2 from $v_i$ to $a_i$. Hence $F'$ is an me₂-graph where the number of $G$-external vertices is $2n - 1$.

Now suppose that $G$ has exponent 2 (and hence does not have any isolated vertices) and that $G$ is embedded in $F'$ as above. Now the vertex $x$ and its incident edges are not needed for paths of length 2 between vertices $v_i$ and $v_j$ and hence we can delete $x$ and its incident edges to obtain a graph $F''$. The deletion of $x$ and its incident edges does not change the exponent and leaves an me₂-embedding $F''$ of $G$ with $2n - 2$ $G$-external vertices.

We can note that, for the complete graph on $n$ vertices, $K_1, K_2$ and $K_3$ have $K_3$ as an me₂-embedding. Now we show that, for $n \geq 4$, $K_n$ can be embedded using only $2n - 3$ $G$-external vertices.

**Lemma 6.1.4.** Let $G$ be the complete graph on $n$ vertices for $n \geq 4$. Then $G$ admits an me₂-embedding $H$ with $|V(H) \setminus V(G)| = 2n - 3$.

**Proof.** Label the vertices of $G$ as $v_1, ..., v_n$. We extend $G$ to an me₂-embedding $H$ as
follows:

1. For each $v_i$ such that $i > 2$, introduce two vertices $a_i$ and $b_i$ and edges $a_ib_i, v_ia_i$ and $v_ib_i$. Introduce a vertex $x$ adjacent to both $v_1$ and $v_2$.

2. For $i \geq 4$, $a_3a_i$ are edges in $H$ and for $i \geq 3$, $a_3b_i$ are edges in $H$.

3. $a_3x$ and $a_4x$ are edges in $H$.

4. $H$ has no more edges than the ones described above.

We now show that $H$ is an $m_{e_2}$-graph. First we show that $H$ has exponent 2 and then that every edge in $H$ is 2-required.

- Since $G = K_n$ for $n \geq 4$, there exists a path of length 2 between $v_i$ and $v_j$ in $H$.

- There exist paths of length 2 between $v_i$ and $a_j, b_j$ for $i \neq j$ and $j \geq 3$ via $v_j$ and there exist paths of length 2 between $v_i$ and $x$ for $i \geq 2$ via $v_1$. There exists a path of length 2 from $v_1$ to $x$ via $v_2$.

- Since $v_i, a_i, b_i$ and the edges between them form a triangle for all $i \geq 3$, there exists paths of length 2 between $v_i, a_i$ and $b_i$ for $i \geq 3$.

- For $a_i, a_j, b_i, b_j$ such that $a_i \neq a_3 \neq a_j$ and $i, j \geq 3$, there exist paths of length 2 between them via $a_3$.

- There exist paths of length 2 between $a_3$ and $a_i$ via $b_i$ for $i > 3$ and similarly for $a_3$ and $b_i$ for $i > 3$.

- Since $a_3x, a_4x$ and $a_3a_4$ are edges in $H$, there exists a path of length 2 from $x$ to $a_3$ via $a_4$, and from $x$ to all other $G$-external vertices in $H$ via $a_3$.

Hence $H$ has exponent 2 and now we show every edge in $H$ is 2-required.

- The edge $v_1v_2$ is 2-required for $v_1$ to $x$.

- The edges $v_1v_j$ for $j \geq 3$ are 2-required for $v_1$ to $b_j$ for $j \geq 3$. Similarly for the edges $v_2v_j$ for $j \geq 3$.

- The edges $v_iv_3$ are 2-required for $v_i$ to $b_3$. 

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• The edges $v_iv_j$ for $i, j > 3$ are 2-required for $v_i$ to $b_j$.

• The edges $v_ia_i$ and $a_ib_i$ are 2-required for $v_i$ to $b_i$. Similarly for the edge $v_ib_i$.

• The edge $v_1x$ is 2-required for $v_2$ to $x$. Similarly for the edge $v_2x$.

• The edge $xa_4$ is 2-required for $x$ to $a_3$.

• The edges $xa_3$ and $a_3a_i, a_3b_i$ for $a_i \neq a_3$ are 2-required for $x$ to $a_i$ and $b_i$.

Hence $H$ is an $me_2$-graph and the number of $G$-external vertices in $H$ is $2n - 3$. □

6.2 Strong-me$_2$-embeddings

In this section we discuss embedding a graph as an induced subgraph of a strong-me$_2$-graph. The me$_2$-embeddings as constructed in the last section are not strong-me$_2$-graphs; it is easily checked that there are edges that are not strongly-2-required in each of the graphs. Presently, we do not know if every graph can be embedded as an induced subgraph of a strong-me$_2$-graph. The situation for complete graphs however is known, and demonstrates the complexity of embedding graphs in this manner. We will show that $K_n$ for $n \geq 4$ can be embedded as an induced subgraph of a strong-me$_2$-graph with $2n + 1$ external vertices and that the minimum number of external vertices required for such an embedding is at least $2n$.

**Definition 6.2.1.** Let $G$ be a graph. Then a strong-me$_2$-embedding of $G$ is a strong-me$_2$-graph $H$ that has $G$ as an induced subgraph. If $G$ admits a strong-me$_2$-embedding $H$, then we say $G$ is strongly-me$_2$-embeddable.

We first show that $K_n$ for $n \geq 3$ can be extended to a strong-me$_2$-graph with $2n + 1$ extra vertices.

**Theorem 6.2.2.** Let $n \geq 3$. Then $K_n$ admits a strong-me$_2$-embedding in which the number of $K_n$-external vertices is $2n + 1$.

**Proof.** We extend $K_n$ to a strong-me$_2$-embedding $H$ as follows.

Label the vertices of $K_n$ as $v_1, ..., v_n$. 

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1. For each $v_i$ in $K_n$, introduce two external vertices $a_i$ and $b_i$ and edges $v_ia_i, v_ib_i$ and $a_ib_i$ in $H$.

2. Introduce an external vertex $w$ and edges $wa_i$ and $wb_i$ for all $i$.

Now we claim $H$ is a strong-me$_2$-embedding of $K_n$. It is clear to see that $H$ has exponent 2, so we show that every edge in $H$ is strongly-2-required.

- Every edge $v_iv_j$ on $K_n$ is strongly-2-required for $v_i$ to $a_j$ and $v_j$ to $a_i$.

- The edge $v_ia_i$ is strongly-2-required for $v_i$ to $b_i$ and $a_i$ to $v_j$ for all $i \neq j$. Similarly, the edge $v_ib_i$ is strongly-2-required for $v_i$ to $a_i$ and $b_i$ to $v_j$ for all $i \neq j$. The edge $a_ib_i$ is strongly-2-required for $a_i$ to $v_i$ and $b_i$ to $v_i$.

- The edges $wa_i$ are strongly-2-required to get from $w$ to $b_i$ and from $a_i$ to $a_j$ for $i \neq j$. Similar reasoning applies to the edges $wb_i$.

Hence $H$ is a strong-me$_2$-embedding of $K_n$ in which the number of $K_n$-external vertices is $2n + 1$.

We provide an example below of the construction as in Theorem 6.2.2 for $n = 3$. We note that $K_3$ is already a strong-me$_2$-graph and hence a strong-me$_2$-embedding of itself, but for the purpose of neatly demonstrating the mechanism of the proof in Theorem 6.2.2, we use $n = 3$.

**Example 6.2.3.** A strong-me$_2$-embedding of $K_3$ with 7 $K_n$-external vertices. The vertices of the original $K_3$ are $v_1, v_2$ and $v_3$. We have encountered this graph in Chapter 3 as an example of a strong-me$_2$-graph on 10 vertices.
The majority of the remainder of this section is dedicated to proving the following theorem.

**Theorem 6.2.4.** For \( n \geq 4 \), let \( H \) be a strong-me\(_2\)-embedding of \( K_n \). Then \( H \) has at least \( 2n \) \( K_n \)-external vertices.

We will return later in this section to the case \( n = 4 \), but for now we concentrate on the proof of Theorem 6.2.4 when \( n \geq 5 \). Let \( H \) be a strong-me\(_2\)-embedding of \( K_n \) for \( n \geq 5 \). We consider the possible patterns of adjacencies among vertices of \( K_n \) and \( K_n \)-external vertices in \( H \), and use this to show that the number of \( K_n \)-external vertices in \( H \) with a neighbour in \( K_n \) is at least \( 2n \). We begin by considering the vertices in \( V(H) \setminus V(K_n) \) which belong to unique 2-paths involving edges of \( K_n \).

**Lemma 6.2.5.** If \( V(H) \setminus V(K_n) \) has a vertex \( x \) that is adjacent to three or more vertices of \( K_n \), then no edge of \( K_n \) is 2-required in \( H \) for a vertex of \( K_n \) to \( x \).

**Proof.** Label the vertices of \( K_n \) as \( v_1, \ldots, v_n \). Suppose there exists a \( K_n \)-external vertex \( x \) such that \( xv_1, xv_2 \) and \( xv_3 \) are edges in \( H \). Now no edge \( v_i v_j \) of \( K_n \) is 2-required for a vertex \( v_i \) in \( K_n \) to \( x \) via \( v_j \), as we have a path of length 2 from \( v_i \) to \( x \) via \( v_k \) on \( K_n \) for some \( k \neq j \), with \( k \in \{1, 2, 3\} \). \( \square \)

We observe that, due to Lemma 6.2.5, each edge \( uv \) in \( K_n \) is strongly-2-required in \( H \) for \( u \) to a \( K_n \)-external vertex of \( H \) that is adjacent to at most two vertices of \( K_n \), and for \( v \) to a \( K_n \)-external vertex of \( H \) that is adjacent to at most two vertices of \( K_n \). These \( K_n \)-external vertices of \( H \) are of significance when considering the minimum possible number of \( K_n \)-external vertices in \( H \). This motivates the following definition.

**Definition 6.2.6.** For a \( K_n \)-external vertex \( x \) in \( H \), let \( \delta(x) \) be the number of vertices of \( K_n \) that are neighbours of \( x \) in \( H \). For a vertex \( v \) of \( K_n \), let \( N_{ex}(v) \) be the set of \( K_n \)-external neighbours of \( v \) in \( H \), and write

\[
\theta(v) = \sum_{\substack{x \in N_{ex}(v) \\colon 1 \leq \delta(x) \leq 2}} \frac{1}{\delta(x)}.
\]

We note that \( \theta(v) \) is a positive integer multiple of \( \frac{1}{2} \). Note that the number of \( K_n \)-external vertices that are adjacent in \( H \) to one or two vertices of \( K_n \) is

\[
\sum_{v \in V(K_n)} \theta(v).
\]
We will show that this number is at least $2n$.

We now make some observations regarding the number of $K_n$-external neighbours of a vertex $v \in V(K_n)$, and the value $\theta(v)$.

**Lemma 6.2.7.** Let $v$ be a vertex of $K_n$. Then either $v$ has a $K_n$-external neighbour $x$ in $H$ such that $\delta(x) = 1$, or $v$ has $n - 1$ $K_n$-external neighbours $x_i$ in $H$ with $\delta(x_i) = 2$ for $i = 1, \ldots, n - 1$.

**Proof.** Consider an edge $uv$ in $K_n$. Since $H$ is a strong-me$_2$-graph, $uv$ is strongly-$2$-required in $H$ for $u$ to some vertex in $H$. There are multiple paths of length 2 between vertices of $K_n$, so $uv$ is strongly-$2$-required in $H$ for $u$ to some $K_n$-external vertex in $H$. Then there exists a $K_n$-external vertex in $H$ which has $v$ as its only $K_n$-neighbour, or has $u$ and $v$ as its only $K_n$-neighbours. If $v$ has no $K_n$-external neighbour $x$ with $\delta(x) = 1$, then for every vertex $u$ of $K_n$ with $u \neq v$, there is a $K_n$-external vertex of $H$ that is adjacent to $u$ and $v$ and no other vertex of $K_n$. \hfill \Box

**Lemma 6.2.8.** Let $v$ be a vertex of $K_n$. Then $\theta(v) \geq \frac{n}{2}$. If $\theta(v) = \frac{n}{2}$, then $v$ has adjacent $K_n$-external neighbours $x$ and $y$ with $\delta(x) = 1$ and $\delta(y) = 2$, and any other $K_n$-external neighbour of $v$ has $\delta$-value at least 3.

**Proof.** First suppose $v$ does not have a $K_n$-external neighbour $x$ such that $\delta(x) = 1$. Then by Lemma 6.2.7, $\theta(v) \geq \frac{n - 1}{2}$, so $\theta(v) \geq 2$.

Now suppose $v$ does have a $K_n$-external neighbour $x$ such that $\delta(x) = 1$. Then $v$ must share a common neighbour with $x$ in $H$, so there exists a $K_n$-external vertex $y$ such that $xy$ and $vy$ are edges in $H$. We have three possibilities for $\delta(y)$:

1. $\delta(y) = 1$ in which case $\theta(v) \geq 2$.

2. $\delta(y) = 2$ in which case $\theta(v) \geq \frac{3}{2}$.

3. $\delta(y) \geq 3$. Let $u$ and $w$ be vertices of $K_n$, different from $v$, and suppose $y$ is adjacent to $v, u$ and $w$ in $H$. Then $uv$ is not strongly-$2$-required for $u$ to $x$ nor for $u$ to $y$, so there must exist a distinct $K_n$-external vertex $x_1$ in $H$ such that the only $K_n$-neighbours of $x_1$ in $H$ are $v$ and possibly $u$. If $v$ is the only $K_n$-neighbour of $x_1$ then $\theta(v) \geq 2$. If $v$ and $u$ are the only $K_n$-neighbours of $x_1$, then in order for $uv$ to be strongly-$2$-required there must exist a $K_n$-external vertex $x_2$ such that $v$ is the
only $K_n$-neighbour of $x_2$ in $H$, or $v$ and $w$ are the only $K_n$-neighbours of $x_2$ in $H$. In either case, $\theta(v) \geq 2$.

We conclude that for every vertex $v$ of $K_n$, $\theta(v) \geq \frac{3}{2}$. We observe that if $\theta(v) = \frac{3}{2}$, then $v$ has two adjacent $K_n$-external neighbours $x$ and $y$ with $\delta(x) = 1$ and $\delta(y) = 2$, and any other $K_n$-external neighbour of $v$ has $\delta$-value at least 3. \hfill \Box

Showing that $\sum_{v \in V(K_n)} \theta(v) \geq 2n$ is equivalent to showing that the average value of $\theta$ over the vertices of $K_n$ is at least 2. If not, then there exists a vertex of $K_n$ whose $\theta$-value is less than 2 and hence is $\frac{3}{2}$ by Lemma 6.2.8. Let $U = \{v \in V(K_n) : \theta(v) = \frac{3}{2}\}$ and let $u \in U$.

We define a walk in $K_n$, which we call the critical walk $W_u$ determined by $u$ as follows. Since $\theta(u) = \frac{3}{2}$, $u$ has $K_n$-external neighbours $x_0$ and $y_0$, where $x_0y_0$ is an edge, $\delta(x_0) = 1$, $\delta(y_0) = 2$, and $x_0$ and $y_0$ are the only neighbours of $u$ that contribute to $\theta(u)$. Let $f_1(u)$ be the other neighbour of $y_0$ in $K_n$; the edge $uf_1(u)$ is the first edge of $W_u$. Now $uf_1(u)$ is 2-required in $H$ for $u$ to a vertex $x_1$ with $\delta(x_1) = 1$. If $x_1$ is the only external neighbour of $f_1(u)$ with $\delta$-value 1, and if $x_1$ and $f_1(u)$ have a unique common neighbour $y_1$, and if $\delta(y_1) = 2$, we define $f_2(u)$ to be the other neighbour of $y_1$ in $K_n$. In all other circumstances we declare $f(u) = f_1(u)$ and define $W_u$ to consist of the vertices $u$ and $f_1(u)$ and the edge connecting them.

The construction of $W_u$ continues in this manner. When the distinct vertices $u, f_1(u), \ldots, f_k(u)$ of $K_n$ have been identified, we either conclude the list at $f_k(u)$, write $f(u) = f_k(u)$ and define $W_u$ to be the path $uf_1(u)\ldots f(u)$, or extend it to $f_{k+1}(u)$ according to the following criterion. If $f_k(u)$ has exactly one external neighbour $x$ in $H$ that satisfies $\delta(x) = 1$, and if $f_k(u)$ and $x$ have exactly one common neighbour $y$ in $H$, and if $\delta(y) = 2$, let $f_{k+1}(u)$ be the other neighbour of $y$ in $H$.

- If $f_{k+1}(u)$ coincides with $f_j(u)$ for some $j \leq k$, define $W_u$ to be the subgraph of $K_n$ with vertices

  $$u = f_0(u), f_1(u), \ldots, f_k(u)$$

  and edges $f_i(u)f_{i+1}(u)$ for $i = 0, \ldots, k - 1$ and $f_k(u)f_j(u)$ (if $j \neq k - 1$). In this case $W_u$ has exactly one cycle and we note that $k \geq 2$. If $f_{k+1}(u) = f_{k-1}(u)$, then $W_u$ is
a path with vertices
\[ u = f_0(u), f_1(u), \ldots, f_k(u) \]
and edges \( f_i(u)f_{i+1}(u) \) for \( i = 0, \ldots, k - 1 \).

- If \( f_{k+1}(u) \) is distinct from all of the \( f_j(u) \) with \( j \leq k \), the process continues.

The process must end since \( K_n \) has a finite number of vertices, and if the process does not end before we reach a repetition of a vertex, it ends at the first repetition. This result is a subgraph \( W_u \) of \( K_n \) which is either a path or has exactly one cycle. We observe that each vertex \( f_i(u) \) for \( i \geq 1 \) of \( W_u \) is completely determined by \( f_{i-1}(u) \). This is because \( f_i(u) \) is the unique vertex of \( K_n \) that is the other neighbour of the \( K_n \)-external vertex \( y \), where \( f_{i-1}(u) \) has a unique \( K_n \)-external neighbour \( x \) with \( \delta(x) = 1 \), and a unique \( K_n \)-external neighbour \( y \) with \( \delta(y) = 2 \) such that \( xy \) is an edge in \( H \). Let \( W_u \) and \( W_{u'} \) be two distinct critical walks in \( K_n \) and suppose that \( V(W_{u'}) \) intersects \( V(W_u) \). Consider the least \( i \) such that \( f_i(u') = f_j(u) \) for some \( j \leq k \). From the remarks above, the remainder of \( W_{u'} \) from \( f_i(u') \) onwards coincides with that of \( W_u \) from \( f_j(u) \).

It follows that the relation \( \sim \) on \( U \) that relates \( u \) to \( u' \) if \( W_u \) and \( W_{u'} \) share a vertex is an equivalence relation, in particular that it is transitive.

Now we define the critical graph \( G_u \) of \( u \) as the union of all the graphs \( W_u' \) whose vertex sets have non-empty intersection with that of \( W_u \). If \( W_u \) is a tree then each \( W_{u'} \) that intersects \( W_u \) is a tree since \( W_{u'} \) coincides with \( W_u \) from the least \( i \) such that \( f_i(u') \in V(W_u) \), and hence \( G_u \) is a tree. If \( W_u \) has a cycle then \( W_{u'} \) has the same cycle and no other, and hence \( G_u \) has exactly one cycle. Let \( G_u \) and \( G_v \) be two critical graphs for \( u, v \in U \). Then either \( G_u \) and \( G_v \) are equal if \( W_u \) and \( W_v \) share a vertex, or \( G_u \) and \( G_v \) are vertex-disjoint if not. If the vertex sets of \( W_u \) and \( W_v \) are disjoint, then \( V(W_v) \) is also disjoint from \( V(W_{u'}) \) for any \( u' \) for which \( u' \sim u \).

Now consider \( G_u \) for some \( u \in U \). Write \( m \) for the order of \( G_u \) and let \( w \) be a vertex in \( G_u \). We note that for every \( w, \theta(w) \geq 1 + \frac{1}{2}(\deg_{G_u}(w)) \) since \( w \) is adjacent to exactly one \( K_n \)-external vertex with \( \delta \)-value 1, and for every neighbour \( w' \) of \( w \) in \( G_u \), there exists a \( K_n \)-external vertex adjacent to \( w' \) and \( w \) with \( \delta \)-value equal to 2; each of these vertices contributes \( \frac{1}{2} \) to \( \theta(w) \).
This means
\[
\sum_{w \in V(G_u)} \theta(w) \geq m + \frac{1}{2} \sum \deg_{G_u}(w).
\]

We have three possibilities for \( G_u \).

1. If \( G_u \) contains a cycle then \( G_u \) has \( m \) edges and \( \sum \deg_{G_u}(w) = 2m \), so
\[
\sum_{w \in V(G_u)} \theta(w) \geq 2m \text{ as required.}
\]

2. Let \( G_u \) be a tree where \( f_{k+1}(u) \neq f_{k-1}(u) \), and consider \( f(u) \).

   - If \( f(u) \) is adjacent to at least two distinct \( K_n \)-external vertices each with \( \delta \)-value equal to 1, and for every vertex in \( G_u \) adjacent to \( f(u) \) there exists a \( K_n \)-external vertex adjacent to \( f(u) \) with \( \delta \)-value equal to 2, then
\[
\theta(f(u)) \geq 2 + \frac{1}{2}(\deg_{G_u}(f(u))).
\]

   - Suppose that \( f(u) \) has exactly one \( K_n \)-external neighbour \( x \) with \( \delta(x) = 1 \), and that \( f(u) \) and \( x \) have at least two common neighbours \( y \) and \( z \), each with \( \delta \)-value at least 2. If both \( y \) and \( z \) have \( \delta \)-value equal to 2, then
\[
\theta(f(u)) \geq 2 + \frac{1}{2}(\deg_{G_u}(f(u))). \text{ Suppose that } \delta(y) \geq 3 \text{ and that } y \text{ has further distinct neighbours } p \text{ and } q \text{ in } K_n. \text{ Then since } z \text{ has at least one neighbour in } K_n \text{ different from } f(u), \text{ at least one of } pf(u), qf(u) \text{ is not required for a path of length } 2 \text{ from } p \text{ or } q \text{ to } x, y \text{ or } z. \text{ Suppose first that } pf(u) \text{ is the only such edge. This means that } \delta(z) = 2 \text{ and the other neighbour of } z \text{ in } K_n \text{ is } q. \text{ Then, since } pf(u) \text{ is strongly-2-required and } f(u) \text{ has exactly one } K_n \text{-external neighbour } x \text{ with } \delta(x) = 1, \text{ there is a } K_n \text{-external vertex } x_1 \text{ such that } f(u) \text{ and } p \text{ are the only neighbours of } x_1 \text{ in } K_n. \text{ Therefore } \theta(f(u)) \geq 2 + \frac{1}{2}(\deg_{G_u}(f(u))).
\]

   - If both \( pf(u) \) and \( qf(u) \) are not required for paths of length 2 to \( x, y \) or \( z \), then there exists a \( K_n \)-external vertex \( x_1 \) such that \( f(u) \) and \( p \) are the only neighbours of \( x_1 \) in \( K_n \) and there exists a \( K_n \)-external vertex \( x_2 \) whose only neighbours in \( K_n \) are \( f(u) \) and \( q \). Again we have that \( \theta(f(u)) \geq 2 + \frac{1}{2}(\deg_{G_u}(f(u))).
\]

   - Suppose \( f(u) \) has no \( K_n \)-external neighbour with \( \delta \)-value equal to 1. Then by Lemma 6.2.7 for each vertex \( z \) in \( K_n \) different from \( f(u) \), the edge \( zf(u) \) must be 2-required in \( H \) for \( z \) to a \( K_n \)-external vertex \( p \) that is adjacent only
to \( z \) and \( f(u) \) in \( K_n \). If \( zf(u) \) is an edge of \( G_u \), note that \( p \) cannot be the unique common neighbour of \( z \) and \( x \), where \( x \) is the unique \( K_n \)-external neighbour of \( z \) with \( \delta(x) = 1 \). So for every vertex \( z \) adjacent to \( f(u) \) in \( K_n \), there exists a vertex \( p \) adjacent only to \( f(u) \) and \( z \) in \( K_n \), so each of these external vertices contributes \( \frac{1}{2} \) to \( \theta(f(u)) \), and there exists a \( K_n \)-external vertex \( y \) that is the unique common neighbour of \( z \) and \( x \), where \( x \) is the unique \( K_n \)-external neighbour of \( z \) with \( \delta(x) = 1 \). We note that if \( n = 4 \), then \( f(u) \) would only have 3 neighbours in \( K_n \), and \( \theta(f(u)) \geq \frac{3}{2} + \frac{1}{2}(\deg_{G_u}(f(u))) \).

Since \( n \geq 5 \), it follows that \( f(u) \) has at least 4 neighbours in \( K_n \) and therefore \( \theta(f(u)) \geq 2 + \frac{1}{2}(\deg_{G_u}(f(u))) \).

In each case, \( f(u) \) satisfies \( \theta(f(u)) \geq 2 + \frac{1}{2}(\deg_{G_u}(f(u))) \), and \( \sum_{w \in V(G_u)} \deg_{G}(w) = 2(m - 1) \), so again

\[
\sum_{w \in V(G_u)} \theta(w) \geq m + (m - 1) + 1 = 2m,
\]
as required.

3. If \( f_{k+1}(u) = f_{k-1}(u) \), let \( x_k \) be the unique \( K_n \)-external neighbour of \( f_k(u) \) with \( \delta(x_k) = 1 \). The notation used here is as in the initial construction on page 109. Let \( y_k \) be the unique common neighbour of \( f_k(u) \) and \( x_k \) with \( \delta(y_k) = 2 \). The other \( K_n \)-neighbour of \( y_k \) is \( f_{k-1}(u) \). Consider \( \theta(f_{k-1}(u)) + \theta(f_k(u)) \). Since \( f_{k-1}(u) \) has \( K_n \)-external neighbours \( y_{k-2}, y_{k-1} \) and \( y_k \) each with \( \delta \)-value equal to 2, and \( f_{k-1}(u) \) has a \( K_n \)-external neighbour \( x_{k-1} \) with \( \delta(x_{k-1}) = 1 \), it follows that \( \theta(f_{k-1}(u)) \geq \frac{5}{2} \). Since \( f_k(u) \) has \( K_n \)-external neighbours \( y_{k-1} \) and \( y_k \), both with \( \delta \)-value equal to 2, and \( f_k(u) \) has a \( K_n \)-external neighbour \( x_k \) with \( \delta(x_k) = 1 \), it follows that \( \theta(f_k(u)) \geq 2 \). Now the sum of the \( \theta \)-values of all other vertices in \( G_u \) is at least \((m - 2) + (m - 2) = 2m - 4\), so

\[
\sum_{w \in V(G_u)} \theta(w) \geq 2m - 4 + \frac{9}{2} > 2m.
\]

Since every vertex in \( K_n \) either has \( \theta \)-value at least 2, or belongs to a critical graph in which the average value of \( \theta \) over all vertices is at least 2, and since different critical
graphs are vertex-disjoint, we conclude that the average value of \( \theta \) over all vertices of \( K_n \) is at least 2. Thus
\[
\sum_{v \in V(K_n)} \theta(v) \geq 2n,
\]
and the number of \( K_n \)-external vertices in \( H \) with one or two neighbours in \( K_n \) is at least \( 2n \). This concludes the proof of Theorem 6.2.4 when \( n \geq 5 \). \(\square\)

We now consider whether this bound of \( 2n \) can be attained. Suppose \( H \) is a strong-
\( \text{me}_2 \)-embedding of \( K_n \) and suppose that the number of \( K_n \)-external vertices in \( H \) is \( 2n \). We note that, if \( u \) is a vertex of \( K_n \) and \( x \) is a \( K_n \)-external neighbour of \( u \) with \( \delta(x) = 1 \), then the edge \( ux \) in \( H \) must be required for a path of length 2 from \( u \) to a \( K_n \)-external vertex \( y \) with \( \delta(y) \leq 1 \). Since the number of \( K_n \)-external vertices in \( H \) is exactly \( 2n \), it follows that \( u \) is the unique \( K_n \)-neighbour of \( y \) in \( H \), and therefore \( u \) is the unique \( K_n \)-neighbour of exactly two adjacent \( K_n \)-external vertices, namely \( x \) and \( y \). Since \( u \) is an arbitrary vertex of \( K_n \), it follows that every vertex of \( K_n \) is the unique \( K_n \)-neighbour of exactly two adjacent \( K_n \)-external vertices. If \( n > 5 \) then every vertex of \( K_n \) must be adjacent to at least one \( K_n \)-external vertex \( x \) with \( \delta(x) = 1 \); otherwise, by Lemma 6.2.7 the number of \( K_n \)-external vertices in \( H \) is greater than \( 2n \).

If \( n = 5 \) and the number of \( K_5 \)-external vertices in \( H \) is \( 2n = 10 \), then we have to consider the possibility that there is a vertex of \( K_5 \) that is not adjacent to a \( K_5 \)-external vertex with \( \delta \)-value equal to 1. We leave to the reader to verify that if there exists a vertex of \( K_5 \) that is not adjacent to a \( K_5 \)-external vertex with \( \delta \)-value equal to 1, then since the number of \( K_5 \)-external vertices is equal to 10, it must be that no vertex of \( K_5 \) is adjacent to \( K_5 \)-external vertex with \( \delta \)-value equal to 1. In this case, for every edge \( uv \) in \( K_5 \), there exists a unique \( K_5 \)-external vertex \( y \) in \( H \) such that the only \( K_5 \)-neighbours of \( y \) in \( H \) are \( u \) and \( v \). We do not need to consider the case where at least one vertex of \( K_5 \) is not adjacent to any \( K_5 \)-external vertex with \( \delta \)-value equal to one and at least one vertex of \( K_5 \) is adjacent to a \( K_5 \)-external vertex with \( \delta \)-value equal to 1, as the number of \( K_5 \)-external vertices will be greater than 10. Now let \( u'v' \) be an edge in \( K_5 \), disjoint from \( uv \), and let \( y' \) be the common \( K_n \)-external neighbour of \( u' \) and \( v' \) in \( H \). Then there must be a path of length 2 from \( y \) to \( y' \) in \( H \), so \( y \) and \( y' \) must both be adjacent to a distinct \( K_n \)-external vertex \( y'' \), where the only \( K_n \)-neighbours of \( y'' \) in \( K_n \) are \( u'' \) and \( v'' \). Since \( n = 5 \), the edge \( u''v'' \) is not disjoint from both \( uv \) and \( u'v' \) in \( K_n \). Suppose
$u = u''$ in $K_n$ (and $v \neq v''$, otherwise $u''v''$ would have at least two common $K_n$-external neighbours). Then the edge $u''v''$ is not strongly-2-required in $H$ since there is a path of length 2 from $u$ to $y''$ via $y$ in $H$, and $y''$ is the unique common $K_n$-external neighbour of $u''$ and $v''$ in $H$. We conclude that $H$ is not a strong-me$_2$-graph. Therefore, if $H$ is a strong-me$_2$-embedding of $K_n$ for $n \geq 5$ and the number of $K_n$-external vertices in $H$ is $2n$, then every vertex of $K_n$ is the unique $K_n$-neighbour of exactly two adjacent $K_n$-external vertices. \hfill \Box

We now consider the case where $n = 4$. We note that Theorem 6.2.4 does not apply to the cases where $n = 2$ or $n = 3$, since $K_2$ can be embedded as an induced subgraph of a strong-me$_2$-graph with exactly one $K_2$-external vertex, and $K_3$ is a strong-me$_2$-graph.

The reason why $K_4$ does not fit into the full proof of Theorem 6.2.4 is that there are two ways in which a vertex of $K_4$ can have $\theta$-value equal to $\frac{3}{2}$. Let $H$ be a strong-me$_2$-embedding of $K_4$. If for every vertex $v$ in $K_4$ with $\theta(v) = \frac{3}{2}$ there is a $K_4$-external vertex $x$ in $H$ whose unique $K_4$-neighbour is $v$ and there is a $K_4$-external vertex $y$ whose only neighbours in $K_4$ are $v$ and some other vertex, then the proof of Theorem 6.2.4 also applies to $K_4$.

The alternative is that there is no $K_4$-external vertex whose unique $K_4$-neighbour is $v$. Label the neighbours in $K_4$ of $v$ as $v_1$, $v_2$ and $v_3$. Then by Lemma 6.2.7 there exist $K_4$-external vertices $x_1$, $x_2$ and $x_3$ in $H$ where $v$ and $v_1$ are the only $K_4$-neighbours of $x_1$, $v$ and $v_2$ are the only $K_4$-neighbours of $x_2$, and $v$ and $v_3$ are the only $K_4$-neighbours of $x_3$. Now $vx_i$ must be strongly-2-required for all $i$ in $H$, so for each $i$ there exists a $K_4$-external vertex $y_i$ that is a neighbour of $x_i$ and possibly $v$, and has no further $K_4$-neighbours. Now $vv_i$ is not 2-required for $v$ to $x_i$ for $i \in 1, 2, 3$, so there exist at least three more $K_4$-external vertices in $H$. Therefore, the number of $K_4$-external vertices is at least $9 = 2(4) + 1$. \hfill \Box

We currently do not know whether there are values of $n$ for which $2n$ additional vertices suffice for a strong-me$_2$-embedding of $K_n$, however through analysis of the cases where $n = 4$ and $n = 5$, we conclude that the number of $K_n$-external vertices required to embed $K_4$ and $K_5$ is $2n + 1$. Theorem 6.2.2 provides a construction for a strong-me$_2$-embedding $H$ of $K_n$ for $n \geq 3$ with $2n+1$ $K_n$-external vertices, hence for $n \geq 6$ the minimum possible number of $K_n$-external vertices in a strong-me$_2$-embedding of

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$K_n$ is either $2n$ or $2n+1$. We remark that we do not know if $H$ as constructed in Theorem 6.2.2 is the only example of a strong-me$_2$-graph of order $3n+1$ that has $K_n$ as an induced subgraph.
Chapter 7

Conclusion

The aim of this thesis is to study the me$_2$-property and the strong-me$_2$-property for simple undirected finite graphs. The following contributions are noted.

1. We have introduced the concepts of the me$_2$-property and the strong-me$_2$-property, and provided examples of graphs with these properties, some of which also arise in other contexts in graph theory. We have classified me$_2$-graphs that have a vertex that is adjacent to all others, and classified all strong-me$_2$-graphs of even order with a vertex of maximum possible degree.

2. We have investigated the behaviour of graphs with the me$_2$-property, and variations of this property, under the Kronecker product of graphs. Precise conditions under which the Kronecker product of two graphs has the me$_2$-property or the strong-me$_2$-property have been identified.

3. We have explored the problem of embedding a given graph as an induced subgraph of an me$_2$-graph with no edges amongst the additional vertices, with particular attention to the case of trees. A criterion for the failure of simple me$_2$-embeddability for trees was discussed, and the minimum number of additional vertices required to simply embed a tree in an me$_2$-graph was investigated. It was shown that this parameter is unbounded on the class of simply me$_2$-embeddable trees.

4. We showed that every graph of order $n$ occurs as an induced subgraph of an me$_2$-graph of order at most $3n + 2$. We showed that if $H$ is a strong-me$_2$-embedding of
$K_n$, then the order of $H$ is at least $3n$, and provided a construction for $H$ where $H$ is of order $3n + 1$.

7.1 Open questions and further work

We now mention some open questions raised by this thesis, and suggest some possible directions for further research.

In Chapter 2, we discussed the maximum number of possible edges in an $me_2$-graph of order $n$, and described those $me_2$-graphs with disconnected complement whose edge count is maximal. It would be of interest to determine the maximum possible number of edges in an $me_2$-graph of order $n$, and to identify examples attaining this maximum. The problem of determining the maximum possible number of edges in a strong-$me_2$-graph of order $n$ is also open to investigation, as is the question of the minimum possible number of edges in a strong-$me_2$-graph of even order.

Chapter 6 deals with embedding graphs as induced subgraphs of $me_2$-graphs and strong-$me_2$-graphs, and presents a construction that embeds any graph of order $n$ as an induced subgraph of an $me_2$-graph of order $3n + 2$. We do not know at present whether there exist graphs of order $n$ that require the addition of $2n + 2$ vertices in order to be embedded in this manner. The question of whether every graph occurs as an induced subgraph of a strong-$me_2$-graph is currently unresolved, as is the question of whether $K_n$ can be embedded as an induced subgraph of a strong-$me_2$-graph with exactly $2n$ $K_n$-external vertices.

The definition of the $me_2$-property can be extended to directed graphs in a natural way. The analogue of the strong-$me_2$-property for directed graphs is that the digraph has exponent 2, and every arc $uv$ is required for a unique 2-path beginning at $u$, and is required for a unique 2-path from a vertex $w$ in the digraph to $v$. This thesis may serve as a starting point for investigation into such digraphs. There is also potential for investigating infinite graphs with the $me_2$- and strong-$me_2$-properties.

The $me_2$-property can be adapted to primitive graphs of exponent $n$. For a given $n$, the question of what can be said about primitive graphs that are edge-minimal (or vertex-minimal) of given exponent can be explored. The minimum possible number
of edges in a simple graph of order \( n \) and exponent \( k \geq 3 \) is determined by Kim et al. in [13]. Primitive simple graphs of order \( n \) and exponent 3, and even exponent \( k \) for \( 4 \leq k \leq 2n - 4 \), that achieve the minimum are characterised in the same article.

This thesis initiates the study of the \( \text{me}_2 \)- and strong-\( \text{me}_2 \)-properties for graphs, and raises some questions of potential interest for future research.
Bibliography


