Title: Polarisation of synchrotron radiation from isolated neutron stars

Author(s): de Burca, Diarmaid

Publication Date: 2016-03-10

Item record: http://hdl.handle.net/10379/5622
Polarisation of Synchrotron Radiation from Isolated Neutron Stars

Author: Diarmaid de Búrca
Supervisor: Prof. Andrew Shearer

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy in the Centre for Astronomy, School of Physics

March 2016
## Contents

Declaration of Authorship xii

Abstract xiv

Acknowledgements xv

Abbreviations xvi

Formulae xvii

1 Introduction to Pulsars 1
   1.1 Introduction .............................................. 1
   1.2 Pulsar Introduction ....................................... 2
      1.2.1 Canonical Model ..................................... 4
      1.2.2 Standard Model ...................................... 5
   1.3 Emission Mechanism ....................................... 7
   1.4 High Energy Models ....................................... 9
      1.4.1 Polar Cap Models .................................... 10
      1.4.2 Outer Gap Model ..................................... 12
      1.4.3 Slot Gap Model ....................................... 18
      1.4.4 Striped Wind Model .................................. 22
      1.4.5 Other High Energy Emission Models ................. 25
      1.4.6 Other High Energy Considerations .................. 26
   1.5 Optical Emission Models .................................. 26
   1.6 Conclusions ................................................ 32

2 Incoherent Synchrotron Radiation from High Magnetic Fields 35
   2.1 Introduction ................................................ 35
5 Discussion

5.1 Problems with the DS Formulation ................................. 87
5.1.1 Integrating with Respect to Infinity ........................... 88
5.1.2 Expanding the Particle PAD Function to the Next Order .... 89
5.1.3 Restraints on the Particle PAD Function ...................... 92
5.2 POREC2.0 Results and the Application to Current Pulsar Theories ... 95
5.2.1 Explaining the Polarisation Results? ............................ 95
5.2.2 POREC2.0 and the Slot Gap Model ............................ 97
5.2.3 POREC2.0 and the Outer Gap Model .......................... 97
5.2.4 Pulsars as Two-pole Emitters? ................................. 98
5.3 Computational Artefacts? ............................................ 98
5.4 Relativistic Rotating Vector Model ................................. 101
5.5 Conclusions .......................................................... 102

6 Future Work

6.1 POREC2.0 ............................................................ 103
6.1.1 Emission location .................................................. 104
6.1.2 Magnetic Field ..................................................... 105
6.1.3 Different Emission Components ................................. 105
6.1.4 Output .............................................................. 106
6.1.5 Statistical Component ............................................. 107
6.2 Synchrotron Formulæ .................................................. 108
6.2.1 PAD Problems ..................................................... 108
6.2.2 Realistic Energy Limits ......................................... 108
6.3 Conclusions .......................................................... 109

A Airy Functions

A.1 Airy Functions ........................................................ 111
A.1.1 Converting integrals to Airy functions ......................... 112
A.1.1.1 \( \int_{-\infty}^{\infty} y^3 \exp \left[ i \frac{3}{2} \eta \left( y + \frac{1}{3} y^3 \right) \right] dy \) ... 112
A.1.1.2 \( \int_{-\infty}^{\infty} y^2 \exp \left[ i \frac{3}{2} \eta \left( y + \frac{1}{3} y^3 \right) \right] dy \) ... 113
A.1.1.3 \( \int_{-\infty}^{\infty} y \exp \left[ i \frac{3}{2} \eta \left( y + \frac{1}{3} y^3 \right) \right] dy \) ... 113
A.1.1.4 \[ \int_{-\infty}^{\infty} \exp \left[ i \frac{3}{2} \eta \left( y + \frac{1}{4} y^3 \right) \right] dy \] \hspace{1cm} 114

B General Theta Integrals \hspace{1cm} 115

B.1 General Theta Integrals \hspace{1cm} 115
B.1.1 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 115
B.1.2 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) \left( \frac{3}{2} \theta_8^2 \right) d\theta \] \hspace{1cm} 116
B.1.3 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 117

C Theta Integrals \hspace{1cm} 119

C.1 Theta Integrations \hspace{1cm} 119
C.2 Integrating over \( \phi \) \hspace{1cm} 119
C.3 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 119
C.4 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 120
C.5 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 121
C.6 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 121
C.7 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 122
C.8 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 122
C.9 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 123
C.10 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_2^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 123
C.11 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 123
C.12 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) d\theta \] \hspace{1cm} 124
C.13 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_1^2 \left( \frac{3}{2} \theta_7^2 \right) \left( \frac{3}{2} \theta_8^2 \right) d\theta \] \hspace{1cm} 124
C.14 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) \left( \frac{3}{2} \theta_8^2 \right) d\theta \] \hspace{1cm} 125
C.15 \[ \int_{-\infty}^{\infty} \theta_7^2 \theta_8^2 K_3^2 \left( \frac{3}{2} \theta_7^2 \right) \left( \frac{3}{2} \theta_8^2 \right) d\theta \] \hspace{1cm} 125
C.16  ∫  _−∞^∞_ θ^5 θ^4 K_3^1 (\frac{x}{2} \theta^3) K_3^2 (\frac{x}{2} \theta^3) .......................................................... 126

C.17  ∫  _−∞^∞_ θ^3 θ^4 K_4^1 (\frac{x}{2} \theta^3) K_4^2 (\frac{x}{2} \theta^3) .......................................................... 126

C.18  ∫  _−∞^∞_ θ^3 K_3^1 (\frac{x}{2} \theta^3) K_3^2 (\frac{x}{2} \theta^3) .......................................................... 127

C.19  ∫  _−∞^∞_ θ^3 K_3^1 (\frac{x}{2} \theta^3) K_3^2 (\frac{x}{2} \theta^3) .......................................................... 127

D  Integral Conversions  129

D.1  ∫  _x^∞_ K_3^1 (\mu) d\mu .......................................................... 129

D.2  ∫  _x^∞_ (\frac{\mu}{x})^\frac{3}{2} K_3^1 (\mu) d\mu .......................................................... 130

D.3  ∫  _x^∞_ (\frac{\mu}{x})^\frac{4}{3} K_3^1 (\mu) .......................................................... 130

D.4  ∫  _x^∞_ (\frac{\mu}{x})^6 K_3^1 (\mu) .......................................................... 131

D.5  ∫  _x^∞_ (\frac{\mu}{x})^8 K_3^1 (\mu) d\mu .......................................................... 132

D.6  ∫  _x^∞_ (\frac{\mu}{x})^\frac{16}{3} K_3^1 (\mu) d\mu .......................................................... 133

D.7  ∫  _x^∞_ (\frac{\mu}{x})^\frac{1}{3} K_3^2 (\mu) d\mu .......................................................... 133

D.8  ∫  _x^∞_ (\frac{\mu}{x}) K_3^2 (\mu) d\mu .......................................................... 134

D.9  ∫  _x^∞_ (\frac{\mu}{x})^\frac{5}{3} K_3^2 (\mu) d\mu .......................................................... 134

D.10 ∫  _x^∞_ (\frac{\mu}{x})^\frac{7}{3} K_3^2 (\mu) d\mu .......................................................... 135

D.11 ∫  _x^∞_ (\frac{\mu}{x})^3 K_3^2 (\mu) d\mu .......................................................... 136

D.12 K_{n,\mu} = ∫  _0^∞_ x^n K_{\mu}(x) dx = \frac{1}{\mu+1} ∫  _0^∞_ x^{n+1} K_{\mu-1}(x) dx .......................................................... 136

E  Bessel Function Relations  139

Bibliography  141
List of Figures

1.1 Here the geometry associated with a pulsar can be seen. The magnetic axis $\mu$ is inclined to the rotational axis $\Omega$ at an angle $\alpha$, while the observer views the pulsar at an angle $\chi$ to $\Omega$. Here LOFL stands for the Last Open Field Lines (see text). ........................................................... 5

1.2 There are three different types of outer gaps that can form, but only type A is stable, as if gap A forms it will emit charged particles which will redistribute the charge in gaps B and C. Image taken from [10]. . . 13

2.1 The WL model predicts that the percentage circular polarisation will increase linearly with the magnetic field, regardless of the power law index of the electrons used. At some point this model fails and predicts clearly non-physical results. This is dependant on the pitch angle ($\alpha$), the frequency ($5.212 \times 10^{14}$ Hz), and the power law index (1.42), but in the area of interest the WL model fails long before the predicted surface magnetic field of a pulsar ($\approx 10^{12}$ Gauss). While the corrections introduced in [34] do change the point at which the model fails, the WLG model still fails before reaching the surface magnetic field of a pulsar. Here the minimum energy corresponded to a $\gamma$ factor of 10 and the maximum energy to a $\gamma$ factor of $10^{9}$. ................................. 43

2.2 The geometry used in order to calculate the synchrotron emission. Firstly, define the $x$-$y$ plane as the instantaneous plane of orbit of the particle. Then define the origin as the point at which the velocity $v$ and the vector to the observer $n$ are both in the $x$-$z$ plane. Define $\varepsilon_{\perp}$ to be along the $y$ axis, and $\varepsilon_{\parallel}$ as $n \times \varepsilon_{\perp}$. This gives a natural frame of reference for the polarisation of the emission. ........................................................... 45

2.3 The circular polarisation for a power law distribution of particles with a power law index of 1.42, at a frequency of $5.212 \times 10^{14}$ Hz. Here WL stands for Westfold and Legg, the original emission theory, and theta stands for the particle pitch angle (in degrees). The PAD used was the Inversely Linear PAD. Here This Work stands for the DS model. Figure reproduced from [25]. ........................................................... 50
2.4 The linear polarisation change with regard to the magnetic field for a particle power law index of 1.42 and frequency $5.212 \times 10^{14}$ Hz and theta for the particle pitch angle (in degrees). As can be seen, the linear polarisation is steady at low magnetic fields and at high magnetic fields, with the linear polarisation changing smoothly between the two values in the intermediate range of magnetic field values. Here This Work stands for the DS model. The PAD used here was the Inverse Linear PAD. Figure reproduced from [25].

3.1 The various stages of the design of POREC. By splitting the design into separate areas, each part of the design can be worked on independently of the whole code. This modular approach means that individual components (e.g. radiation function) are capable of being changed without changing the structure of POREC itself. This figure is reproduced from [63].

3.2 The four original pitch angle distribution functions programmed into POREC. Here the cut-off point for each function was 20 degrees (for the Gaussian function this results in particles spread out to 1.5 times the cut-off).

3.3 The pitch angle distribution function used with the DS model. This distribution predicts non-zero emission in high magnetic fields.

3.4 An example of how the different emission points in POREC are calculated. On the left is a representation of how emission from a single pitch angle is visible over a range of viewing angles and phase. The centre shows how the different points emission is calculated from are arranged in concentric spheres. Finally, on the right there is an example of how the emission is calculated for each point, taking the local conditions into account. This figure is taken from [63] and from [56].

3.5 The change in the Gaussian PAD as the cut-off is changed.
4.1 Inverse mapping results of the intensity simulations from POREC. Here the emission seen on the right is from the primary peak, while the emission on the left comes from the secondary peak. As can be seen the emission is coming from opposite poles. The blue circles represent the light cylinder, while the green lines give selected LOFL. Figure taken from [56]. Note that the colour here is intensity in units of \( \text{ergs/cm}^2\text{s} \), however as the intensity can be changed by changing the number of particles the absolute intensity is relatively arbitrary. It is only the relative intensity that is of interest. The banding structure seen here is due to the numerics used to calculate the emission points.

4.2 Here the inclination angle decreases as the viewing angle required to obtain a double peaked structure increases. Below an inclination angle of 50° the pulse profile only has a single peak visible at all viewing angles. Here the inclination angle is 50°.

4.3 The simulated pulse profiles are symmetric about a viewing angle of 90° for any inclination angle. Here an inclination angle of 90° was used.

4.4 For higher inclination angles, a lower viewing angle was required in order to see the double peaked structure of the Crab pulsar. At inclination angles higher than 140° the secondary peak was always greater than 0.3 times the primary peak. Here the inclination angle is 140°.

4.5 Q vs U plots predicted by POREC2.0 does not agree with observations. Here an example of one of the predicted QU plots is shown. No combination of inclination angle/viewing angle was found which predicted the correct shape to the QU plot. In general the amount of linear polarisation predicted was too high. Here an inclination angle of 120° was used. The black dots on this diagram are the observation of the linear polarisation from the Crab pulsar [84].

4.6 For any inclination angle and viewing angle where a secondary peak is seen, there is a flip in the sign of the circular polarisation as the secondary peak emission becomes dominant. Here an inclination angle of 90° was used.

4.7 Here, the emission can be broadly split into two main components. There is the component from close to the LOFL, which is brighter, and a component from far out in the magnetosphere. These two components have markedly different linear polarisations. Here the inclination angle is 70° with the viewing angle of 94°, at a phase of 0.15 [14]. Here the intensity given is in units of ergs/s. This is relatively arbitrary, as the total intensity can be manipulated by changing the number of emitting particles.
4.8 Here, the two different components share markedly different circular polarisation properties. Here an inclination angle of 70° and a viewing angle of 94° is used, at a phase of 0.5 [14]. .................................................. 83

4.9 Here the linear polarisation of the slot gap component and the outer gap component are markedly different, though both show a banded structure. Here the inclination angle is 70°, the viewing angle is 94° and the phase is 0.5 [14]. .................................................. 84

4.10 As can be seen here, the linear polarisation values are in bands stretching from near the pulsar to far out in the magnetosphere. Here an inclination angle of 70° and a viewing angle of 10° is used, at an initial phase of 0 [14]. .................................................. 84

4.11 The amount of linear polarisation at any particular point does not remain constant over the complete phase. Instead, the bands of polarised emission move around. Here an inclination angle of 70° and a viewing angle of 10° were chosen to illustrate this. In this particular case the second half of the phase is approximately the same as the first half, but this is not the case in general [14]. .................................................. 85

4.12 The percentage values for U over the course of a single phase. As can be seen the banded structures move around over the course of a phase. Here an inclination angle of 70° and a viewing angle of 10° is used [14]. 86

5.1 Here the behaviour of the linear polarisation modes are compared against phase. As can be seen, for the majority of the pulse there is no linear polarisation to be seen. During the primary peak, the U component increases until the peak reaches approximately its FWHM. At that point the polarisation switches sign and increases in magnitude. The Q component, on the other hand, behaves in the opposite manner, initially being large and negative, and becoming smaller and switching sign in the middle of the pulse peak. In the inter-pulse, the Q component peaks before the pulse peak, while the U component peaks after the pulse peak. Here the observations used were taken by [84]. ................................. 96

5.2 A comparison between runs where the distance between the points was changed. In the top image on the left, the full resolution of 800,000 points was used. In the image in the centre the φ distance between the points was doubled, whereas in the image on the bottom the θ distance between the points was doubled. As can be seen changing the resolution of the points used had little effect on the banding structure that is seen [14]. Here the left row shows the %Q while the right row shows the %U seen. .................................................. 99
5.3 On the left is the simulated emission seen. On the right is the simulated random values sent through the computational pipeline. As can be seen, there is still a small amount of patterning to be seen on the random emission, however, it is nowhere near the patterns that are seen in the actual simulated emission. Here the inclination angle is 70°, the viewing angle is 86°. The phase is at 0.14 [14]. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 100
Declaration of Authorship

I, Diarmaid de Búrca, declare that this thesis titled, ‘Polarisation of Synchrotron Radiation from Isolated Neutron Stars’ and the work presented in it are my own. I confirm that:

■ This work was done wholly or mainly while in candidature for a research degree at this University.

■ Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

■ Where I have consulted the published work of others, this is always clearly attributed.

■ Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

■ I have acknowledged all main sources of help.

■ Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: Diarmaid de Búrca

Date: March 2016
“There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory, which states that this has already happened.”

Douglas Adams

“Give a man a fire and he is warm for a day. Set a man on fire and he is warm for the rest of his life.”

Terry Pratchett
Abstract

Centre for Astronomy
School of Physics

Doctor of Philosophy

Polarisation of Synchrotron Radiation from Isolated Neutron Stars

by Diarmuid de Búrca

Pulsars are enigmatic neutron stars which have defied efforts to understand them for over 40 years. POREC (Pulsar Optical Reverse Engineering Code) was first written to constrain pulsar emission theories through constraining the emission location. POREC originally used the Westfold, Legg and Gleeson (WLG) model to calculate the incoherent optical emission that would be seen.

In this work it is shown that the WLG model fails for optical (and higher) wavelengths in high magnetic fields ($> 10^6$ G). A new model for incoherent synchrotron radiation in high magnetic fields (the DS model) is proposed. While this does solve the original problems associated with the WLG model, the DS model has its own weaknesses. A full discussion of the derivation of the DS model and of the various weaknesses of the DS model will be covered.

POREC is updated to the DS model. A new pitch angle distribution (PAD) for POREC is proposed, and the problems with this new PAD discussed. The new simulations are compared against the latest observations of the Crab nebula. While these new simulations give a better fit to the observations than previous models using the WLG model, there are still several serious problems with the results from POREC. These problems will also be covered in more detail.
Acknowledgements

This project has taken me many years to complete, and I could not have completed it without the help of a large number of different people. It is impossible for me to thank everyone who has helped me over my journey, but I am grateful for any help and advice that I have gotten over the years.

I would like to thank my family for supporting me through my degree and on through my PhD, their support is always unswerving and appreciated. I would particularly like to thank my mother for the help she gave in proof reading my various projects.

I would like to thank my supervisor, Prof. Shearer, for putting up with my idiotic astronomy questions and complete lack of ability to make graphs.

I would like to thank my office mates - Lisa, Paul, Susan, Mike S., Laura and Eoin - for ignoring my increasingly insane mutterings as the deadlines grew closer.

I would like to thank my friends outside the department - Bill, Paddy, Padraic, Steve, Fintan and Miley - for providing distractions and support whenever it was needed.

I would like to thank the other postgraduate students and staff - Deirdre, James, Mike M., Aonghus, Nev, Nav, Gillian, Kirsten, Ali, Eamonn, Ariel (honourary), Brandon, Conor, Ollie, and everyone else - for helping to make my breaks interesting and informative through informal discussions in the tea room.

Finally I would like to thank my external examiner, Prof. Kirk, and my internal examiner, Dr. Redman, whose comments have improved this work.
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOFL</td>
<td>Last Open Field Line</td>
</tr>
<tr>
<td>WL</td>
<td>Westfold and Legg (see [49])</td>
</tr>
<tr>
<td>WLG</td>
<td>Westfold, Legg and Gleeson (see [34])</td>
</tr>
<tr>
<td>DS</td>
<td>De Búrca and Shearer (see [25])</td>
</tr>
<tr>
<td>POREC</td>
<td>Pulsar Optical Reverse Engineering Code</td>
</tr>
<tr>
<td>PAD</td>
<td>Pitch Angle Distribution</td>
</tr>
<tr>
<td>PAD_{co}</td>
<td>Pitch Angle Distribution Cut-off Limit</td>
</tr>
<tr>
<td>GASP</td>
<td>Galway Astronomical Stokes Polarimeter</td>
</tr>
<tr>
<td>GLAST</td>
<td>Gamma-ray Large Area Space Telescope</td>
</tr>
<tr>
<td>PC</td>
<td>Polar Cap</td>
</tr>
<tr>
<td>OG</td>
<td>Outer Gap</td>
</tr>
<tr>
<td>SG</td>
<td>Slot Gap</td>
</tr>
<tr>
<td>SCLF</td>
<td>Space Charge Limited Flow</td>
</tr>
<tr>
<td>PFF</td>
<td>Pair Formation Front</td>
</tr>
<tr>
<td>TPC</td>
<td>Two-pole Caustic</td>
</tr>
<tr>
<td>MHD</td>
<td>Magnetohydrodynamics</td>
</tr>
<tr>
<td>CTA</td>
<td>Cherenkov Telescope Array</td>
</tr>
<tr>
<td>I, Q, U, V</td>
<td>Stokes Parameters</td>
</tr>
<tr>
<td>R_{LC}</td>
<td>Light-cylinder Radius</td>
</tr>
<tr>
<td>SVM</td>
<td>Single Vector Model</td>
</tr>
<tr>
<td>RRVM</td>
<td>Relativistic Rotating Vector Model</td>
</tr>
<tr>
<td>MPI</td>
<td>Message Passing Interface</td>
</tr>
</tbody>
</table>
\( K_n(x) \) Modified Bessel Function of the Second Kind of Order \( n \)

\[
J_n = \int_0^\infty x^{n-1} f_x K_{\frac{3}{2}} (\nu) \, d\nu
\]

\[
L_n = \int_0^\infty x^{n-1} K_{\frac{3}{2}} (x) \, dx
\]

\[
R_n = \int_0^\infty x^{n-1} K_{\frac{1}{2}} (x) \, dx
\]

\[
Q_n = \int_0^\infty x^{n-1} \frac{K_{\frac{2}{3}} (x)}{x} \, dx
\]

\[
J_n(p) = \int_0^p x^{n-1} f_x K_{\frac{3}{2}} (\nu) \, d\nu
\]

\[
L_n(p) = \int_0^p x^{n-1} K_{\frac{3}{2}} (x) \, dx
\]

\[
R_n(p) = \int_0^p x^{n-1} K_{\frac{1}{2}} (x) \, dx
\]

\[
\phi(\alpha) \quad \text{Particle Pitch Angle Distribution}
\]

\[
g(\alpha) = 1 + \frac{\phi'(\alpha)}{\phi(\alpha)} \tan \alpha
\]

\[
f(\alpha') = \phi(\alpha') \sin \alpha'
\]

\[
k(\alpha) = \phi(\alpha) \sin \alpha [1 + g(\alpha) \cot \alpha] = \phi(\alpha) \sin \alpha + \phi(\alpha) \cos \alpha
\]

\[
h(\alpha') = \phi''(\alpha') \sin \alpha' + 2\phi'(\alpha') \cos \alpha' - \phi(\alpha') \sin \alpha'
\]

\[
C^1(x_1, x_2) = \left[ \frac{J_n(x)}{J_n} \right]^{x_1}_{x_2}
\]

\[
C^2(x_1, x_2) = \left[ \frac{L_n(x)}{L_n} \right]^{x_1}_{x_2}
\]

\[
C^4(x_1, x_2) = \left[ \frac{R_n(x) + [1 + g(\alpha)](L_n(x) - \frac{1}{2} J_n(x))}{\frac{R_n}{L_n} + [1 + g(\alpha)](L_n - \frac{1}{2} J_n)} \right]^{x_1}_{x_2}
\]

\[
K^1(p) = \frac{2(\alpha + 1) \Gamma\left(\frac{\alpha}{2}\right)}{\pi \Gamma\left(\frac{\alpha + 1}{2}\right)}
\]

\[
K^2(p) = \frac{p + \frac{3}{2} \Gamma\left(\frac{p+1}{2}\right) K^1(p)}{2 \Gamma\left(\frac{p+1}{2}\right)}
\]

\[
K^4(p) = \frac{2\pi^{\frac{p}{2}}}{\pi^{\frac{p+1}{2}}} \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{p+3}{2}\right)
\]

\[
x_i^+(p) = 2 \left[ K^i(p) \right]^{\frac{p-i}{2}} , \quad i = 1, 2
\]

\[
x_i^+(p) = 2 \left[ K^i(p) \right]^{\frac{p-i}{2}}
\]
Chapter 1

Introduction to Pulsars

1.1 Introduction

It has been over 40 years since pulsars have been discovered, but there is still no agreement about where and how the pulsar emission occurs. Pulsars have enormous gravitational fields, with only black holes having greater gravitational fields, but the pulsars’ magnetic field dominates the emission process. By measuring the polarisation of the pulsar emission, new constraints can be placed on the emission theories. To date, there have been instrumental difficulties with measuring the polarisation in the optical regime. Even those studies which have measured the polarisation of the emission have only concentrated on the linear polarisation [84][85]. Circular polarisation measurements would open up a whole new dimension to constrain the pulsar emission theories. However, as the measurement of circular polarisation is difficult, it must first be shown that there would be useful results from the circular polarisation measurements.

In NUI Galway, a reverse engineering approach was taken to constrain pulsar emission zones. To achieve this a code was developed, the Pulsar Optical Reverse Engineering Code (POREC). This code simulates the intensity and polarisation that would be seen if the emission was free to occur from anywhere in the open magnetosphere. The primary assumption used by the code is that the emission mechanism is synchrotron radiation. The remainder of the unknown quantities are treated as separate dimensions in the parameter search space. The code steps through each unknown in finite steps in an attempt to find the best fit with observations.
POREC predicts that the circular polarisation would have a narrow peak that is aligned with the radio pulse from the pulsar (slightly before the main optical pulse), and that the emission comes from 300 km from the Crab [56], which is the only pulsar for which the linear polarisation has been measured. To date, there have been no observations of circularly polarised optical emission from pulsars. However, with new instruments such as the Galway Astronomical Stokes Parameter (GASP) [19] coming online, a test of this model is now possible.

POREC assumes that the emission process is through synchrotron radiation, in line with the current theories of optical emission. However, the emission theory used [95][49][34] fails at high magnetic fields [25]. As there are extremely high magnetic fields surrounding pulsars, a new theory of synchrotron emission is required.

In this work, the new emission theory will be derived, and the changes that this makes to POREC’s predictions will be detailed.

1.2 Pulsar Introduction

Pulsars are the end state of a range of stars after they have had a supernova explosion. If a star is sufficiently large, fusion can occur between all elements up to iron. Above iron elements require an infusion of energy for fusion to occur. In these large stars, an iron core grows at the centre of the star until it reaches a limit which is known as the Chandrasekhar limit. At this point, the core can no longer support itself against its own gravitational force and it collapses down to a very small radius. This transforms it into a neutron star, as the matter that the star contains has been compressed into neutrons. As the remainder of the star no longer has the pressure from the core to counteract its own gravity, it will collapse inwards into a supernova explosion. If the star is sufficiently large, this will cause a black hole to form. For smaller stars a neutron star is formed. Whether a star becomes a black hole or a neutron star is dependent on the star’s initial size.

Pulsars are extremely interesting astronomical objects. There are a number of different reasons for this. The high gravitational field of the pulsars makes them exciting laboratories for general relativity, and they can been used to indirectly measure the presence of gravitational waves [48]. The high density of the material in the pulsar means that
the laws of ordinary matter break down. The electrodynamics surrounding the star give rise to extreme emission processes. Understanding more about the composition and emission processes of pulsars help greatly in understanding how matter behaves at extreme pressures, extreme magnetic fields, and extreme gravitational fields.

A neutron star will typically have a mass of 1-3 solar masses [51], with a radius of approximately 10 kilometers [51]. There is still some disagreement in the literature about the upper limit of the size of neutron stars, with some sources claiming an upper limit of roughly 2 solar masses while other sources claim the upper limit could be as high as 3 solar masses [51]. However, the majority of theories assume that the pulsar mass is approximately 1.4 solar masses, in agreement with the mass measurements from timing observations of binary pulsars [51][86].

Due to the conservation of angular momentum, neutron stars will typically rotate extremely fast when they are born, slowing down over their lifetime [18]. The emission seen from these neutron stars is a narrow beam, which combined with the rotation of the neutron star means that the emission is seen on Earth as a rapidly pulsating star. For this reason, these neutron stars are known as pulsars. It is important to note that there are different types of pulsars, such as millisecond pulsars (knowns as such due to their extremely short period), which do not follow this pattern. Millisecond pulsars will increase in rotation speed, generally due to accretion effects [70].

As mentioned, pulsars are completely dominated by their strong magnetic fields. For a typical pulsar, the magnetic field at the surface of the pulsar will be of the order of $10^{12}$ Gauss. Another class of pulsars, known as magnetars, have stronger magnetic fields of the order of $10^{15}$ Gauss, while millisecond pulsars have weaker magnetic fields of the order of $10^8$ Gauss.

There are two main pulsar observational properties - the period of the pulsar rotation ($P$), and the period derivative ($\dot{P}$). The magnetic field of a pulsar can be derived from these properties. Pulsars can be thought of as rotating magnetic dipoles with moment $|m|$. In that case, the electromagnetic power of the dipole is

$$\dot{E}_{\text{dipole}} = \frac{2}{3}\epsilon c^3 |m|^2 \Omega^4 \sin^2 \alpha$$  \hspace{1cm} (1.1)
where $\alpha$ is the angle of the pulsar’s magnetic field to the rotation axis. The rotational energy loss can be written as

$$\dot{E} = -I\Omega \dot{\Omega} = -4\pi^2 I \dot{P} P^{-3}$$  \hspace{1cm} (1.2)$$

and assuming that the rotational spindown energy loss is supplied by the electromagnetic dipole then it is possible to relate the period and period derivative to the magnetic field. This then gives

$$\frac{2}{3c^3}|m|^2 \Omega^4 \sin^2 \alpha = -I\Omega \dot{\Omega} \implies B_s = \sqrt{\frac{3c^3 I}{2\pi^2 R^6 \sin^2 \alpha} P \dot{P}}$$  \hspace{1cm} (1.3)$$

where $R$ is the pulsar radius, $B_s$ is the surface magnetic field, $I$ is the pulsar moment of inertia, $P$ is the pulsar period, $\Omega$ is the angular velocity and $\alpha$ is the inclination angle.

The above analysis has two inherent assumptions. Firstly, it assumes that the magnetic field of the pulsar is a dipole, although this is generally not the case. Secondly, it assumes that the pulsar is surrounded by a vacuum, which is also not accurate [35]. This introduces some error into the calculation of the magnetic field.

There are two complementary models of pulsars which are currently accepted. One, known as the canonical model of pulsars, only attempts to explain the overall energetics while the other, known as the standard model, then deals with the environment around the pulsar.

### 1.2.1 Canonical Model

The canonical model of pulsars was first proposed in 1967 [65], prior to the discovery of the first pulsar. This model attempts to explain the overall energetics of a pulsar. Pacini advanced the theory that the energy powering nebulae came from rapidly rotating neutron stars in the centre of the nebula, with the rotational energy of the star being transferred to the nebula through the medium of monochromatic electromagnetic waves [65]. The energy of the waves comes from dipole breaking of the neutron star. Calculations showed that the energy from dipole breaking of the Crab pulsar was sufficient to explain the energy output of the Crab nebula [66].
If the energy comes from dipole braking, then there should be an increase in the rotational period of the pulsar. When it was shown that the Crab’s rotation was increasing by one part in 2400 every year [24] [18], this model was accepted as explaining the overall energetics behind a pulsar.

While the canonical model is very important to the understanding of the general energetics of the system, for a complete model it is also necessary to know what is in the environment around the pulsar itself. This is dealt with in the standard model.

### 1.2.2 Standard Model

The standard model (or Goldreich-Julian model) complements the canonical model by trying to explain what occurs in the immediate environment of a pulsar. Prior to explaining the standard model, some definitions are required. The magnetosphere is the volume surrounding the pulsar in which the magnetic fields co-rotate with the pulsar. The magnetosphere extends from the pulsar to the point at which the co-rotation speed of the magnetic field lines approach the speed of light. The surface at which the co-rotating magnetic field reaches the speed of light is known as the light...
cylinder. There are two main parts to the magnetosphere, the closed magnetosphere where the magnetic field lines do not extend beyond the light cylinder (or the magnetic field lines make closed loops) and the open magnetosphere where the magnetic field lines do extend beyond the light cylinder. The boundary between the open and closed magnetosphere is known as the Last Open Field Lines (LOFL). The area of the pulsar which has open field lines leaving its surface is known as the polar cap.

The standard model starts with the assumption that pulsars are rapidly rotating highly magnetised neutron stars. Goldreich and Julian then showed via contradiction that the surroundings of a neutron star would contain a plasma, and not a vacuum [35]. Assuming that the supernova had swept away all of the interstellar material to a distance $D$, Goldreich and Julian then split the area surrounding the magnetosphere into three main parts, the near zone, the wind zone and the boundary zone. The near zone is the volume within the light cylinder (which will be referred to as the magnetosphere in this work) which is bounded in the $z$ direction by $Z = \pm c/\Omega$. The boundary zone is the boundary between the volume which contains the interstellar material and the rest of the volume surrounding the pulsar, and starts at approximately $r \approx D/10$, while the wind zone encloses the near zone and merges into the boundary zone [35].

While Goldreich and Julian did obtain the charge density for each of these volumes, in this work it is the magnetosphere or near zone that is of interest. Within the magnetosphere, classical electrodynamics state that the angular rotation will induce an electric field which obeys

\[ \mathbf{E} + \mathbf{c} (\mathbf{\Omega} \times \mathbf{m}) \times \mathbf{B} = 0 \]  

(1.4)

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field and $\mathbf{\Omega}$ is the angular rotation. If the star were surrounded by a vacuum, and Equation 1.4 determined the field inside the star, then the field just outside the surface of the star would be sufficient to rip ions from the star surface and hence surround the star with a plasma [35]. Ignoring particle inertia, and assuming that a sufficient number of particles were ripped from the star to satisfy Ohm’s law, Goldreich and Julian showed that the resulting charge density could be given by

\[ \rho_{\text{GJ}} = \frac{1}{4\pi} \nabla \cdot \mathbf{E} = -\frac{1}{2\pi c} \frac{\mathbf{\Omega} \cdot \mathbf{B}}{1 - \frac{|\mathbf{\Omega} \times \mathbf{r}|^2}{c^2}} \]  

(1.5)
which is known as the Goldreich-Julian density [35]. This is valid within the light
cylinder. Particles in the closed magnetosphere stream back to the neutron star, while
particles in the open magnetosphere will leave the light cylinder and possibly contribute
to a magnetised wind from the pulsar.

The majority of the magnetosphere is therefore in a force-free state, i.e. there is a charge
distribution such that Equation 1.4 is valid. However, if the entire magnetosphere was
in a force-free state, there would be no emission. As such, it is necessary to look at
the standard model with a view to finding the volumes of the magnetosphere in which
emission can occur. Emission will in general occur when a particle is accelerated - in the
case of pulsars this occurs when there is a electric field which is parallel to the particles’
motion. However, if there is a sufficient density of plasma, this parallel electric field
will be screened and no emission can occur. This is the case in the majority of the
magnetosphere. In places where there is no plasma, known as vacuum gaps, a parallel
electric field forms which accelerates the particles to high energies, causing emission.
The exact location of the emission is still debated, with no single theory of emission
able to explain all of the observations.

1.3 Emission Mechanism

While there are a number of different theories which explain pulsar emission, in general,
the emission mechanism is the same. Emission can only occur when there is a electric
field parallel to the magnetic field, which then accelerates the particles. In the majority
of the pulsar magnetosphere these gaps are screened, i.e. there is a sufficient plasma
to eliminate the parallel electric field (the force-free magnetosphere). Emission will
only occur from volumes where this force-free condition breaks down. These volumes
are where there are vacuum gaps in which a particle can experience acceleration. In
general, these gaps will only occur in the open magnetosphere.

In a vacuum gap there are, by definition, no particles to emit. How then do these
particles occur? Current theories state that emission occurs due to a pair production
cascade effect. When a $\gamma$-ray crosses the vacuum gap, it is converted into a $e^\pm$ pair.
The origin of the initial ray is different in different models, and can occur either due
to the pair production higher in the magnetosphere, due to thermal X-rays from the
neutron star, or from some other mechanism. The $e^\pm$ pair is then accelerated by the parallel electric field, and emits more $\gamma$-rays, which in turn are converted into electron-positron pairs, which emit more $\gamma$-rays, and a cascade effect occurs. Note that this cascade effect also provides a natural explanation for the gap closure, as at some point the electron-positron pairs will become dense enough to merge with the surrounding plasma.

Once the particle has been accelerated, then different emission processes occur for different spectral ranges. These can broadly be split into two different regimes, coherent emission processes and incoherent emission processes.

In order to distinguish between the incoherent and coherent emission processes, consider the radiation flux emitted by a sphere of radius $r$ at a distance $R$. Then

$$ F_\nu = \frac{2\pi\nu^2}{c^2}\kappa T_B \left(\frac{r}{R}\right)^2 $$

(1.6)

where $F_\nu$ is measured in units of flux with 1 f.u. = $10^{-26}$ Watt m$^{-2}$ Hz [32]. Here $T_B$ is the Boltzmann temperature, $\nu$ is the frequency and $\kappa$ is Boltzmann’s constant. Then, for the Crab pulsar ($R = 1500$ pc, $r \sim 5 \times 10^7$ cm), it can be shown that the temperature brightness of the source is of the order of $10^{26}$ K in the radio regime, $10^9$ K in the optical regime and 10 K in the X-ray regime [32]. It is clear that while the optical and X-ray emission can be fully explained by incoherent emission, the radio emission must be coherent, as an incoherent emission process would require the particles to be accelerated to energies greater that $10^{16}$ eV.

In this work, the radio emission process will not be further explored. Interested readers are referred to [32] and [33], and the follow on papers.

For incoherent emission there are three main contenders - synchrotron emission, inverse Compton scattering and curvature emission. Each of these will be detailed below.

For the Crab pulsar, the gyro-frequency (e.g. $qB/m$, where $q$ is the charge, $B$ is the magnetic field, and $m$ is the mass) at the light cylinder is of the order of $10^{12}$ Hz, and of the order of $10^{18}$ Hz at the surface of the pulsar (assuming a dipolar magnetic field) [36]. The brightness temperature of the emission from the Crab pulsar is of the order of $10^9$ K in the optical regime. This brightness temperature is of the same intensity
as that given by the Rayleigh-Jeans tail of the X-ray spectrum \[82\], arguing that the emission is therefore from incoherent synchrotron radiation.

Inverse-Compton scattering (ICS) involves the scattering of low energy photons to higher energies by collisions with ultra-relativistic electrons. The details of the energy gained in ICS will not be described here, but suffice to say that the energy of the scattered photon is proportional to the Lorentz factor squared \[50\]. For high energy particles in the pulsar magnetosphere, the Lorentz factor is expected to be of order \( > 10^3 \), which means that radio would be scattered to the UV range, IR to X-ray, and optical to gamma-ray. There would be very few (if any) photons scattered into the optical domain, and so ICS does not need to be considered for optical emission from pulsars.

Curvature radiation is emitted by charged particles which are moving along pulsar magnetic field lines. These particles will emit at a characteristic frequency \( \mu \sim \gamma^3 c/r_c \), where \( r_c \) is the radius of curvature of the field lines. For the typically assumed \( \gamma \) factors \( (10^6-10^7) \) this places the curvature emission in the \( \gamma \)-ray regime.

While there will be some photons in the optical regime from curvature radiation, the majority of the emission seen will be from incoherent synchrotron radiation. Most pulsar optical emission theories start by assuming that the emission comes from incoherent synchrotron radiation from within the pulsar magnetosphere \[100][69][80][82]\.

### 1.4 High Energy Models

Historically, there are two main types of high energy models, a polar cap model where emission is from within a few polar cap radii of the surface (typically tens of kilometers), and an outer gap model where emission is from high in the magnetosphere, near the light cylinder. However, the polar-cap model is unable to explain the emission from radio quiet gamma-ray pulsars (e.g. Geminga). Since the advent of the Fermi space telescope (previously named Gamma-ray Large Area Space Telescope (GLAST)) a large population of radio quiet gamma-ray pulsars have been discovered \[1\]. In addition, the polar cap model predicts that high energy emission should have a super-exponential cut-off in the 1-10 GeV range \[87\] due to single photon magnetic pair creation from
Chapter 1. Introduction

high energy $\gamma$-rays. This cut-off has not been seen by the Fermi observations [1]. The polar cap model has therefore fallen out of favour in exchange for the slot gap model.

While the outer gap model and the polar cap model were the most popular models for pulsar high energy emission, there are still a large number of other models which attempt to explain the emission with varying degrees of success. The vast majority of these models place the emission from within the light cylinder of the pulsar, but in some models (e.g. in the striped wind model [46]) emission occurs from outside the light cylinder. Most models agree that the radiating particles are created by the pair production process detailed above (Section 1.3), but this is not universally the case, e.g. the striped wind model has emission occurring due to magnetic reconnection.

Even models which agree upon the emission process (e.g. the slot gap models and outer gap models) have different shaped vacuum gaps. The outer gap model predicts a large gap which can cover up to 70 per cent of the outer magnetosphere [99]. The slot gap model predicts an extremely narrow gap which lies along the LOFL [37]. Using the same parameters, these models therefore predict very different results.

One of the problems with simulating pulsar emission is the large number of unconstrained variables that theorists can use to tune their models. For example, with most pulsars the inclination angle of the pulsar is unknown, as is the viewing angle. This means that it is possible to get the same predicted emission from two different theories by using different inclination and viewing angles. While some progress has been made, it is still impossible to state which of the current theories correctly predicts the emission process from pulsars. No model can explain pulsar emission better than its competitors, and each model has its own specific strengths and weaknesses.

1.4.1 Polar Cap Models

Polar cap (PC) models were the first model proposed to explain pulsar emission. It is generally accepted radio emission occurs from within a few pulsar radii of the polar cap, with the exact height being dependant on the frequency. However, as mentioned above, the polar cap model is unsuccessful in accurately predicting high energy emission.

Firstly, the polar cap model predicts a turnover in the 1-10 GeV range [87], after which there should be a super-exponential cut off due to absorption effects. Recently, it has
became possible to measure the emission at such high energies and no such cut-off has been observed.

Secondly, if the high energy emission came from the polar cap, then the high energy and radio emission should overlap, as both types of emission occur in the same area of the magnetosphere. In fact, seeing a $\gamma$-ray pulsar without corresponding radio emission is difficult to explain using the polar cap model. While there was only one example of a radio-quiet $\gamma$-ray pulsar (Geminga) this could be explained as strange geometry. With the launch of the Fermi satellite, a large number of radio quiet $\gamma$-ray pulsars were observed. It is unlikely that a large number of pulsars would be seen with such strange geometry. It is more likely that the emission is coming from a different area of the pulsar surroundings.

The polar cap model was first advanced only a few years after the discovery of pulsars [88], and was worked on until the launch of the Fermi satellite [38][22][23]. There are two main types of polar cap accelerators, which are dependant on the surface temperature of the star [37]. These two types are the vacuum gap type [79][91] and the space charge limited flow (SCLF) type [6][39]. The difference between these two types of models lies in the surface temperature of the star.

There are binding forces on charged particles in the neutron star surface due to the lattice structure in a strong magnetic field. As such, the particles are only free if the surface temperature is greater than the thermionic emission temperature [37]. The electron and ion thermionic emission temperatures are of course different, and can be found in [91]. If the surface temperature is below that of the ion and electron thermionic emission temperatures, then the charges are trapped on the surface and a full vacuum gap develops. On the other hand, if the surface temperature is greater than the ion and electron thermionic emission temperatures, then the charges are ‘boiled off’ [37], and SCLF occurs. In general the measured temperature of pulsars is above the thermionic emission temperatures, however, for some pulsars the full vacuum gap may occur [37].

The full details of the polar cap model are beyond the scope of this work, so the last thing to be mentioned about this model is that the altitude of the pair formation front (PFF) (where the gap is screened) varies with the magnetic colatitude. Field lines which terminate well inside the polar cap have a strong parallel electric field, and the PFF is close to the neutron star surface. However, for field lines which terminate at
the edges of the polar cap, the parallel electric field is weaker, and the particles reach higher altitudes before achieving the Lorentz factor required to cause pair production. The PFF thus curves upwards as the LOFL are reached. When this effect is taken into account, the polar cap model is known as the slot gap model (Section 1.4.3).

### 1.4.2 Outer Gap Model

Outer gap (OG) models have the emission located in the outer gap of the pulsar magnetosphere, as opposed to directly above the polar cap. In the outer gap model, radio emission and high energy/optical emission have different emission locations, and as such it would not be expected that the emission would be coincident. The radio and $\gamma$-ray emission profiles are not normally coincident, which implies different emission regions, in agreement with the OG model. OG models are also preferred on the basis of the emission geometry of the pulsar [96][77].

The outer gap model was first proposed in 1986 [10]. It was further developed by Chiang & Romani [12][13], extended to older pulsars [99], and to magnetars [90]. However, an estimation of the validity of the assumptions which underpin a lot of the OG models was made in 2015 [92], which found that the assumptions used were not valid, and could be off by up to a few orders of magnitude.

The original OG model was mainly concerned with young pulsars [10] which could sustain pair-production from otherwise depleted regions in the outer magnetosphere. The initial formation of the charge region occurred due to a current flow between the star and the light cylinder along charged field lines. However in this model, the Cheng, Ho and Ruderman (CHR) model, the initial formation of the gap is not critical, as the shape and geometry of the gap are sustained by the production rate of charged particles in the gap, not by its initial formation [10]. The CHR model argued that there were only three types of gaps which could form in a pulsar magnetosphere. These gaps are a gap along the LOFL, a gap along the null surface (where the charge density vanishes, i.e. $\Omega \cdot B = 0$), or a gap which is not connected to either the null surface or the LOFL. Cheng, Ho and Ruderman further argued that of these three gaps, only the gap along the LOFL was sustainable, as other gaps would be crossed by a $\gamma$-ray at some point, which in turn would cause a pair-cascade to occur and the gap to vanish (Figure 1.2)
Figure 1.2: There are three different types of outer gaps that can form, but only type A is stable, as if gap A forms it will emit charged particles which will redistribute the charge in gaps B and C. Image taken from [10].

Therefore, the most likely outer gap would be the one which stretched from the null charge surface to the light-cylinder along the LOFL.

In the CHR model, the gap and its surroundings can be split into three regions. Firstly, there is the gap itself where the primary $e^{\pm}$ pairs are created and accelerated to high energies. In this region the created particles will be accelerated in opposite directions to ultra-relativistic energies. The limit to the energies of the particles will come from the loss of energy to primary $\gamma$-rays. These $\gamma$-rays, which will have a different source in the Vela and Crab models, initiate a pair cascade. The electrons and positrons are accelerated in opposite directions, creating a net charge. This process will be repeated until there is a sufficient charge created to close the gap. Close to the gap boundary, the primary $\gamma$-rays will continue to create the secondary pairs [11].

The boundary region is where secondary particles are created by the primary $\gamma$-rays, but are no longer accelerated by the large parallel electric field of the gap. The secondary particles will still be created with large energies, and will also radiate $\gamma$-rays and X-rays. These will then cross the region of the primary $\gamma$-rays (as the primary particles - and hence the primary gamma-rays - will be accelerated in both directions) where they will produce tertiary low-energy pairs [11].

Finally there is the tertiary region, which will cover much of the remaining magnetoosphere. In this region, the pairs created do not have enough energy to radiate gamma-rays or X-rays, and instead will radiate soft synchrotron radiation (in the IR region for
Chapter 1. Introduction

the Vela pulsar) which will then flood the entire outer magnetosphere, including the gap itself [11].

The CHR model may be split into two separate but related models, known as the Crab CHR model and the Vela CHR model. While the outer gap behaves in the same manner in both of these models, the differences lie in the particle pair creation process. The outer gap itself can be sustained in a number of different ways. CHR considered these methods, and proposed a different gap maintenance method for the Crab and the Vela pulsar.

For the Crab pulsar the gap could be maintained by primary-curvature $\gamma$-rays interacting with secondary X-ray emission. However, this model for sustaining the Vela outer gap fails for two reasons. Firstly the small optical depth of the gap means the observed beamed spectrum should be from primary curvature radiation from the gap which is inconsistent with the observations of the Vela $\gamma$-ray spectrum. Secondly, the amount of particles created are two orders of magnitude too low to supply the lower-end of Vela’s $\gamma$-ray spectrum [11].

The control process for the Vela pulsar is therefore proposed to be pair production from inverse Compton scattering of primary $e^\pm$ pairs with secondary or tertiary soft photons. This process could also be responsible for closing the gap in the Crab pulsar model [11].

The CHR model was able to successfully reproduce the optical - gamma-ray spectrum for the Crab pulsar and the Vela pulsar.

Chiang and Romani further developed the OG model in 1992 [12] and 1996 [13]. This model, the CR model, followed on from the CHR model by taking into account that the particle production and radiation emission are varying with position in the outer gap. This model also took into account an approximate treatment of the particle transport mechanism and the photon flux between the gap emission zones [13].

The CR still makes a number of assumptions. It assumes that the incident particles and photons in the inverse Compton scattering and in the pair creation process meet each other head on [13], and justify this assumption by noting that, in the Crab pulsar at least, the emission needs to be beamed. Another approximation used is the description
of the outer limit of the pulsar gap [13]. Here they follow the same parametrization that is used by [43].

The CR model was able to expand the CHR model to cover a wide range of different pulsars. The light curves predicted by this model were able to account for the pulsars which had been observed up to that time by the Compton Gamma-Ray Observatory [13]. Furthermore, this model was able to explain the spectral variation with phase that was seen, as there is a clear mapping of pulse phase to location in the magnetosphere [13]. However, calculations carried out using this model suffered from too little low-energy photon flux, and the spectra reproduced were too hard for all phases of the pulse profile [13]. This model did show that a realistic computation of the pulsar emission would require a three-dimensional treatment of the radiation transport in the outer magnetosphere [13].

An attempt was made in 1995 to obtain a three-dimensional model of the outer gap [96]. The Romani and Yadigaroglu (RY) model was successfully able to reproduce the relationship between the radio pulses and the gamma-ray emission [96]. The model was also able to explain the polarisation profile of the Crab pulsar, which arose naturally [96].

Up to the RY model the outer gap was only concerned with young, energetic pulsars, and the models proposed failed when applied to older pulsars. This was due to the approximation that there is a mono-energetic distribution of particles in the outer gap. While this approximation is reasonable in young, energetic pulsars, which have a thin outer gap which covers at most 10% of the magnetosphere, in older pulsars the outer gap is much larger and the assumption breaks down. For some pulsars, the outer gap can cover up to 70% of the light cylinder radius [99]. So to account for older pulsars a new theory was required.

In 1997 a thick outer gap theory was proposed [99]. In the thick outer gap model the energy of the primary particles was assumed to be a power law across the gap diameter. Using this, Zhang and Cheng were able to calculate the expected spectrum from a thick outer gap in terms of the minimum value for $x$ ($x_{\text{min}}$, the start of the gap) and the maximum value for $x$ ($x_{\text{max}}$, the end of the gap) [99]. The theoretical properties of the geometry of the outer gap are still unclear and for this reason Zhang and Cheng used the $x_{\text{max}}$ as a parameter in their search space.
Chapter 1. Introduction

Zhang and Cheng were able to match the thick outer gap to the spectrum of Geminga and PSR B1055-52, which they estimated as having a gap width of the order of 70%, together with Vela, PSR B1706-44 and PSR B1951+32, which have medium outer gaps [99]. The model results fit the observed data for all of the above pulsars except for the Vela pulsar [99]. For the Vela pulsar the thick outer gap model did not fit the observations very well, as the model expected there to be a spectral break at approximately 100 MeV. Zhang and Cheng put forward two possible reasons why the Vela pulsar did not fit the data. Firstly, their value for $x_{\text{max}}$ was too small, and secondly, the emission from the Vela pulsar can be explained using the thin outer gap approximation [99].

Finally, it should be noted that the parameters $x_{\text{min}}$ and $x_{\text{max}}$ are important in the thick outer gap, especially $x_{\text{max}}$ [99]. These parameters are not very well constrained. However, a mature pulsar will have a larger outer gap, and so a larger $x_{\text{max}}$ is expected. There should also be a correlation between the phase separation and pulse shape and the inner and outer edges of the gap, which can be seen by the fact that Geminga has a larger phase separation than Vela [99].

In 2000, the thick outer gap model was expanded into a 3D view of the outer gap, merging the CR model and the thick outer gap model [9]. However, rather than restrict the model to a single outer-gap with only outgoing current, this model (the CRZ model) assumes various physical properties (such as the radius of curvature of the magnetic field, the surface field structure etc.) to try to restrict the three dimensional geometry of the gap [9]. This model was able to roughly reproduce the observations of the Crab, and naturally gives off-peak (or bridge) emission [9]. Other than that the conclusions drawn were the same as those from the RY model.

The CRZ model does have the problem that as only one surface layer of the magnetic field is used, the model is unable to reproduce the leading edge or trailing edge emission. However, it is possible that if particles with large pitch angles are taken into account this would cause the observed off-peak emission [9].

With the launch of the Fermi satellite, the number of high-energy observed pulsars rose sharply. This allowed the previous models to be tested against new observations, as well as allowing population studies of high energy pulsars to be carried out.
With the new Fermi data, Romani and Watters tested three models for the magnetosphere structure, and two different emission models, the OG model and the Two Pole Caustic (TPC) model (Section 1.4.5).

For the TPC model the authors followed the original definition proposed by Dyks and Rudak in 2003 and 2004 [78]. In the outer gap model, emission occurs on field points which crossed the null charge surface and extended towards the light cylinder. A gap thickness of $w$ is assumed [78]. Models were then computed for all inclination angles. This was compared against the first six months of Fermi observations [78].

Romani and Watters found that models where the emission began above the null charge surface were strongly statistically preferred over the TPC models. However, they did note that the fitted luminosities had some problems, in particular with the fitting to PSR J1709-4429. Furthermore, best fit models were often off by several degrees where the inclination angle and viewing angle were already known. The authors suggested that this could mean that their work had the potential to show perturbations in the simplified field geometries used [78].

Another attempt to improve the fit was made in 2010, when Wang, Takata and Cheng (WTC) considered a 2 layer outer gap model. In the WTC model the spectrum consists of two components, a primary region where the emission would occur from primary $e^\pm$ particles and a screening region, where the emission comes from secondary particles [94]. This model was able to parameterise the gap using only three values, the fractional gap size, the number of particles in the primary region, and the ratio of the thickness of the screening gap to the primary gap [94]. The authors were then able to fit the spectrum from their model to a total of 42 different gamma-ray pulsars [94]. It should be noted that young Crab-like pulsars were not included, as the emission mechanism in young Crab-like pulsars is expected to be from a synchrotron self Compton process, whereas the WTC model considers emission to be due to curvature radiation [94].

The authors also noted that there were some questions about the validity of the electromagnetic picture of the pulsar that they were using. However, the computational success of their simplified PFF model suggests that the magnetosphere is similar to the model used [78].
A recent paper by Viganó, Torres, Hirotani and Pessah has thrown some doubt on the outer gap model. In this paper the authors look at the underlying assumptions that have been made, both in the thin outer gap model and the thick outer gap model. They found that there were a number of issues in previous analytical theories [92]. In particular, there were a number of underlying assumptions which could change the results by orders of magnitude [92]. For the thick outer gap, the parallel electric field relies on a number of approximations and assumptions. These assumptions are not always mutually consistent [92]. Furthermore, the numerically obtained values for the parallel magnetic field are typically a fifth to a tenth of the size of those used in analytical theories [92].

While the OG model is still a promising area of study, the problems raised with the current theories mean that it is not possible to conclude that the OG model is able to explain all of the observed emission from pulsars.

1.4.3 Slot Gap Model

The slot gap (SG) model is a modification of the polar cap model. The polar cap model has difficulties with reproducing the observed pulse profiles, as the predicted beam size of the emitted radiation is too small [3]. It an effort to overcome this problem, Arons proposed the existence of a high-altitude acceleration medium near the polar rim [3], which became known as the slot gap model.

In the polar cap model, the vacuum gap is directly above the polar cap of the pulsar and close to the pulsar surface [6] and the width of the polar cap is quite small. In the slot gap model, which is an extension of the polar cap model, the vacuum gap extends high into the magnetosphere along the LOFL. The width of the slot gap is very narrow (of the order of tens to hundreds of meters) [37]. The existence of the slot-gap model naturally falls out from the boundary conditions that are generally used for the polar-cap model.

In the polar cap model, pair creation occurs due to one of two different mechanisms (Section 1.4.1). Regardless of the specific production mechanism, a vacuum gap is created. This gap in turn creates a parallel electric field. The field is then shorted out by the production of a $e^\pm$ pairs at a well defined, time independent surface (the
Chapter 1. Introduction

PFF). The size or height of the gap is controlled by this pair formation front, which is in general quite low, and is at its lowest at the centre of the gap.

The pair production above the PFF does not completely short out the parallel electric field. In particular, at the edges of the gap, close to the LOFL, the $\gamma$-rays escape from the pulsar rather than converting to an $e^\pm$ plasma. This leads to the formation of a slot-gap, where the parallel electric field stretches out along the LOFL to the light cylinder [3]. While the emission of high energy radiation from the polar cap is unlikely due to a number of reasons (Section 1.4.1), the slot gap allows the high energy emission to occur much higher in the magnetosphere, where there is a lower magnetic field.

In 1983, Arons showed that the amount of energy from particles in the slot gap region is sufficient to explain the emission from the Crab pulsar [4]. This work assumes that the dominant emission process for $\gamma$-ray emission is curvature radiation [4]. This model also shows that for young pulsars with a strong magnetic field, the slot gap dominates the emission [4]. However, this theory was advanced before the idea of an outer gap was proposed. Furthermore, the emission was assumed to be solely from curvature radiation, without taking the inverse Compton scattering into account [39].

The slot gap model Arons’ proposed did not receive any attention for the next 13 years, as the two main competitors for emission from pulsars were the PC and OG models. It is worth noting that some pulsar studies found that the slot gap was the only model that was able to explain the energetics of the observations [54], though they did not favour that explanation for the observed emission. In 1998, observations of J0437-4715 were shown to be in “excellent agreement with the prediction of the slot gap model of radio pulsars” [98].

In 1996, Arons revisited the slot-gap model, this time as a $\gamma$-ray accelerator [5]. At that time, the periods of known pulsars ranged from 1.6 milliseconds to almost 5 seconds, and the period derivative ranged from $10^{-20}$ to $10^{-12}$ seconds, which have corresponding luminosities equivalent to “one hundred thousand to 5 trillion nuclear wars a second” ($10^{31.5} \times 10^{38}$ ergs/sec) [5]. Here Arons noted that the idea of a ‘starvation’ or quasi-vacuum parallel electric field limits the current to $I_\parallel \approx (\mu \Omega^2/c)(A_{gap}/2\pi r^2)$, where $r$ is the radius of the gap from the star, $\mu$ is the magnetic moment inferred from the relativistic spin down and $A_{gap}$ is the area of the gap [5]. If there is pulsed emission, then the area of the gap must be much smaller than $2\pi r^2$, which in turn means the
efficiency of the gap must be small [5]. There are \(\gamma\)-ray pulsars where the efficiency of the emission approaches one, and as such Arons concluded that the slot-gap model was not a viable emission theory [5].

The slot gap model was next revisited in 2003, when Muslimov and Harding proposed a much narrower gap [60] (MH model). This work arose from considerations of the slot gap which were obtained in 1998 [39], where the primary purpose of their paper was to investigate the effects of general relativity on the polar cap model. The MH model took into account the relativistic effects of the accelerating electrical field, as well as the screening of the gap which occurs due to the slot gap geometry [60]. This model also considered the high energy emission which resulted from the pair cascades at the inner edge of the slot gap, whereas the Arons model had only considered the curvature emission.

The effects of considering different emission types significantly changed the parameter space from which emission could occur - in particular with respect to a nearly aligned rotator. In Arons’ original model, for an aligned rotator with a small inclination angle there was no acceleration, however the MH model allows acceleration from nearly aligned rotators due to the frame dragging effects [60]. The MH model was therefore able to explain the \(\gamma\)-ray emission observed from the Crab pulsar. Initially the MH model was restricted to altitudes of several stellar radii [61].

A follow on paper in 2004 extended the MH model to very high altitudes [61]. As the electric field is not screened, emission can still occur at these high altitudes. The authors incorporated the effects of the motion of the charge perpendicular to the magnetic field near the light cylinder, and derived explicit expressions for the accelerating electric field under the SCLF approximation. The model exhibited both the hollow cone emission pattern centered on the magnetic poles, and the caustic emission pattern from high-altitude trailing field lines [61]. However, the model requires that the source of the accelerating electric field be established within a few stellar radii of the pulsar, and remain constant along the magnetic field lines up to high altitudes [61].

Another slot gap model was proposed in 2008 [40], which simulated emission over all wavelengths for the Crab pulsar. This model (the HSDF08 model) was developed prior to the launch of Fermi [40]. The HSDF08 model of the Crab has three different regimes.
Firstly, there is the spectral region which is dominated by synchrotron radiation of primary electrons. Secondly, there is the spectral region which is dominated by curvature radiation of the primary electrons. Finally, there is the region which is dominated by synchrotron radiation of pairs [40]. This model also predicted that the synchrotron component of the high energy emission and the radio emission would be correlated in phase, which is weakly supported by the link between the optical emission and the high energy emission [81]. It should be noted that this model, while able to reproduce the emission from the Crab pulsar, was not applied to any other pulsars. As of the model’s publication, there were only 6 pulsars had been detected by EGRET [101].

A comparison between the outer gap model and the slot gap model in 2008 found that the slot gap model could only produce 20% of the observed flux, whereas the outer gap is able to reproduce all of the observed flux [42]. This is in striking opposition to earlier work which had found that the slot gap was the only pulsar model able to reproduce the observed flux [54], albeit for different pulsars. This study also found that the HSDF08 model overestimated the flux emitted from the slot gap model by a factor of 33 [42]!

The observations from the Fermi satellite effectively ruled out the polar cap model for pulsar high energy emission [1]. In 2012 Kalapotharakos, Harding, Kazanas and Contopoulos made an attempt to distinguish between the slot gap model and the outer gap model [44]. This model, the KHKC model, explored the effects of a resistive magnetosphere for the first time. Two separate approaches were taken. Firstly, they defined the emission regions for the slot gap and the outer gap models, and calculated the resulting \(\gamma\)-ray light curves. Secondly, they defined particle trajectories, and simulated the resulting light curves. These sets of light curves were then compared to the observed emission [44]. The KHKC model showed that the emitted light curves could change as the resistivity changes. High resistivity results in an increase in the ion volume of the magnetosphere, which in turn results in a increase in the pulsar pulse-width [44]. For the particle trajectory model, this assumed emission from all points of the magnetosphere [44].
1.4.4 Striped Wind Model

The striped wind model of pulsar emission is unique in that the emission occurs outside the magnetosphere. This model was first proposed in 1990 [20], and heavily modified in 2002 [46]. The model proposed in 2002 was subsequently refined at later dates [71][72][58].

The motivation behind the striped wind model comes from the interaction between the pulsar and the nebula. The model for the interaction was that the pulsar spin-down luminosity was moved to the nebula by a mixed relativistic wind. The wind is decelerated to subsonic speeds at a strong standing shock, roughly at the boundary of Baade’s under-luminous zone. The radial expansion of the nebula is occurring at approximately 2000 km s$^{-1}$, and in order to get the post-shock flow speed to match, the ratio of the Poynting flux and the particle kinetic energy flux ($\sigma$) must be small [20].

However, theories of pulsar emission at the time did not suggest that the ratio should be small, with the polar cap model suggesting $\sigma \approx 10^4$-$10^5$, and the outer gap model suggesting a ratio of $\approx 10^4$. Hence, in order to get the pulsar emission models to agree with the observations of the nebula an investigation into the conversion of the wind from a Poynting dominated wind to a particle kinetic energy dominated wind is required [20]. If the energy conversion was in the near zone, then the synchrotron losses would be visible in the observations. As these losses are not seen, it is likely that the conversion occurs in the the wind zone.

In the original model, Coroniti assumed that the striped wind began at a constant radius from the star which was outside the light cylinder [20]. Appropriate parameters for a cool relativistic magnetohydrodynamics (MHD) wind were chosen, with the initial Lorentz factor for the flow being much greater than one. Coroniti also assumed that the wind was spherically diverging, and that there was no poloidal component to the wind [20].

Coroniti found that the annihilation of opposing magnetic fields could convert a Poynting dominated wind into a kinetic/thermal energy dominated wind. However, one constraint of the model was that the dipole axis of the pulsar was required to be very oblique. This was not in agreement with the usually assumed geometry of the pulsar.
Coroniti suggested that this could be due to the assumption that there were no poloidal or toroidal current flows in the magnetosphere [20].

The striped wind model was next revisited in 1996, where the reconnection energies just beyond the closed magnetosphere were considered [52]. This model was tested against the Crab pulsar and the Vela pulsar. This magnetic reconnection model could explain the observed spectral characteristics of the pulsar radiation [52]. Lyubarskii also noted that this model was able to explain the emission from PSR 1509-58, which was not explainable using the polar cap model [52].

A critical analysis of the ideal MHD model for Crab-like pulsar winds was carried out in 1998 [15]. This considered the case for a stationary ideal MHD pulsar wind which had a terminal Lorentz factor of $10^6$, and a terminal ratio $\sigma$ of $10^{-2}, 10^{-3}$. With these conditions, they found that the shape of the poloidal field lines ruled out a gradual transition from a high $\sigma$ wind to a low $\sigma$ wind in regions beyond the light cylinder [15]. They also considered an abrupt change, with two different acceleration types, firstly a spontaneous transition and secondly a transition requiring external support. Both of these acceleration types fail to satisfy energy and momentum conservation, nor do they satisfy the MHD flux-freezing condition [15]. Chiueh, Li and Bagelman were able to create a stationary, ideal, low-$\sigma$ MHD wind but this required that the acceleration occur in the immediate vicinity of the pulsar. Having the acceleration occur in the immediate vicinity of the pulsar requires drastic modifications of the accepted pulsar dipole magnetosphere [15].

While Chiueh, Li and Bagelman’s revisions point out serious problems in the striped wind model, it should be noted that these revisions only apply to winds which obey ideal MHD laws, are stationary, and attain a very low terminal $\sigma << 1$ [15]. As such, it did not completely disprove the striped wind model. Furthermore, this model can produce a wind with a terminal $\sigma$ of the order of one without any problems [15] which might be consistent with the data [7].

In 2001, Kirk and Lyubarsky once again considered the problem of a striped wind [45]. Like Coroniti’s model, this model wind had two phases, a hot unmagnetised phase (corresponding to the current sheets) and a cold magnetised phase [45]. This model had two main limitations. Firstly, it implicitly assumed that the particles in the hot phase were moving relativistically [45]. Secondly, the thickness of the current sheet was
described in terms of the density of particles in each phase and the velocity of the wind. The movement of the sheet edge is therefore not coupled with the local wave speed, and can move superluminally. Therefore a physically realistic model would need to take into account the extra energy lost when the velocity of the sheet-edge approached that of the speed of light. As such the model fails beyond a critical radius [45].

Noting these problems, Kirk, Skjæraasen and Gallant proposed a pulsar wind model (KSG model) where the dissipation of magnetic energy to particle energy happens rapidly [46]. Starting with the assumption that the dissipation is triggered at some surface outside the light cylinder, the KSG model shows that for young pulsars the emission is in general pulsed, with a pulse and interpulse which are not symmetrically spaced in phase, and that the predicted synchrotron emission agrees with the observations of the high energy emission from the Crab pulsar to within an order of magnitude [46]. The major uncertainty with the KSG model was the speed at which the dissipation occurs, as this is not constrained by any observations. However, despite this problem the striped wind model proposed is a valid candidate for pulsar emission.

This model was tested in 2005 by Pétri and Kirk [73]. The Stokes parameters for the optical emission from the KSG model were computed and compared against the observations from the Crab pulsar. The main success of this comparison was that the the electric vector of the off-pulse emission was aligned with the projection of the pulsar’s rotation axis on the plane of the sky, in agreement with observations of the Crab [73]. There were still some problems the model did not address.

With the advent of Fermi, there were a large number of new γ-ray pulsars observed. These observations allowed previous models to be tested. In 2011, Pétri attempted to fully explain the pulsar emission by combining the radio emission of the polar-cap model with the high energy emission from the striped wind model [71]. While this model was successful at explaining the lag between the observed gamma-ray emission and the radio emission, large uncertainties in the radio emission height and time-retardation effects could lead to a discrepancy between the proposed model and observations [71]. Furthermore, the model overestimated the time lag by roughly 0.1 in phase for many pulsars. This suggests that the model is not complete [71].

A further development of the model was carried out a year later [72]. Here the author showed that the pulsed emission in the MeV - GeV range could be explained using a
synchrotron model of a striped wind. This model predicted a clear break in the spectra at approximately a few GeV (depending on the Lorentz factors and reconnection rate) [72]. This model could be tested by phase resolved polarisation measurements of high energy emission [72], such as observations taken with GASP [19].

Further simulations of the striped wind model were carried out [58] and found to be consistent with the high energy emission from the Crab and Vela pulsar. Here it was shown that the emission can be explained by synchrotron radiation from current sheets with a relatively low gamma factor ($\leq 100$ for the Crab pulsar and $\leq 50$ for the Vela pulsar [72]). Furthermore, they also suggested using the gamma-ray spectrum as a probe of the physics of the relativistic magnetic reconnection in the striped wind [72]. Finally, the model would be relatively easy to test, as it predicts a new synchrotron self-Compton component. This component should be detectable when the new Cherenkov Telescope Array (CTA) comes online.

### 1.4.5 Other High Energy Emission Models

Another high energy model is the two pole caustic model (TPC), proposed by Dyks and Rudak in 2003 [27]. This is a purely geometrical model, and does not cover how the gaps proposed were formed or maintained. Nevertheless, the two-pole caustic model was able to reproduce the generic features of pulsar light curves. In this model the natural shape of the pulsar light curve is a double peaked structure, with the second peak being separated by approximately 0.4 to 0.5 in phase from the main peak [27]. In addition, the peaks have well developed wings, the model predicts naturally occurring inter-peak emission and off pulse emission. It also predicts that the radio emission peak would occur prior to the high energy emission peak [27].

In the TPC model, the initial assumptions are that the gap extends from the polar cap to the light cylinder, that the gap is thin, and that the gap is fixed to the last open field lines [27]. This closely resembles the slot gap model, so the TPC model can be thought of as a precursor to the slot-gap model, which provides the physical justification for a gap of this shape.

A more exotic model is the annular gap model, first proposed in 2004 [75]. This model considers a pulsar to be a bare strange star as opposed to a neutron star. In this
model an inner annular gap is formed and emission occurs between this gap and the null charge surface [75][74]. This model focuses on pulsars with shorter periods [26] and has successfully reproduced the light curves for 6 different pulsars. It is also able to explain the leading or trailing radio emission, based on where in the magnetosphere the radio emission occurs [26].

1.4.6 Other High Energy Considerations

Most high emission models depend on where in the magnetosphere the emission occurs, and are therefore sensitive to the form of the magnetic field used. Prior to 2005, the majority of the models used the Deutsch solution [62] (which is also used in POREC, see Chapter 3). However, Muslimov and Harding argue that a more realistic model could produce rotational distortion high in the magnetosphere, and for this reason the Deutsch solution should not be used [62]. Rather they proposed their own solution which took these distortions into account. This solution should be accurate up to 0.5-0.7 of the light cylinder radius \( R_{LC} \) [62].

With the launch of the Fermi satellite, the number of observations of \( \gamma \)-ray pulsars increased dramatically. This of course offered a huge influx of data to distinguish between the different high energy models. With this came the idea that there is more than one acceleration region in the pulsar magnetosphere. In 2012, Yuki and Shibata proposed that there were a number of different emission processes which in turn corresponded to a number of different pulsar gaps. They claimed that the bi-modal distribution of the gamma-ray pulsars which had been observed by Fermi (i.e. that the peak to peak seperation in the gamma-ray regime is either 0.2-0.3 or 0.4-0.5 in phase [1]) could be explained by different pulsar emission sites [97].

1.5 Optical Emission Models

The original optical emission model was proposed in 1970, shortly after the discovery of pulsars [82]. This model (the S model) proposes that the radiation is generated by incoherent synchrotron radiation near to but within the light cylinder radius. No mechanism was proposed to give the electrons the energy required to produce the synchrotron radiation [82].
Another model (SZ model) was proposed in 1972 [100]. In this model, pulsars were split into three broad categories. Firstly, there are pulsars which emit in radio frequencies only. Secondly, there are pulsars which emit at higher energy frequencies only. Finally, there are pulsars which emit over the entire spectrum, with the Crab pulsar being the only pulsar of the third type known at the time. The three different pulsar types pointed to a different emission mechanism being responsible for each spectral range.

In order to choose between the different emission types, the authors looked at the pulsars’ brightness temperature. Pulsar radio emission has an extremely high brightness temperature (up to a $T_{\text{eff}} \sim 10^{28}$ K for the Crab pulsar [100][32]) which points to a coherent emission process. On the other hand, the emission in the optical and X-ray ranges has a much smaller effective temperature ($T_{\text{eff}} \sim 10^{11}$ K) which points to an incoherent mechanism. As such, the authors assumed that the IR to X-ray emission is from incoherent synchrotron radiation.

The only pulsar emitting in the optical that was known at the time was the Crab pulsar, so the SZ model attempted to obtain a fit to the observed optical spectrum of the Crab [100]. To do so, they assumed that the particles distribution was an inverse power law, and then modelled this for three different types of particle power law index. Firstly, emission was modelled without a break in the power law index. Secondly, emission was modelled with a break in the power law index. Thirdly, emission was modelled for a different power law index prior to the spectral break occurring.

In order to choose the particle power law index, they used the observation of the Crab pulsar’s spectral index, together with the relationship $\gamma = 2\alpha + 1$, where $\alpha$ is the observed spectral index and $\gamma$ is the power law index of the particle’s energy.

Under these assumptions, the authors obtained the expected lifetime of an emitting particle, and the number of turns that such a particle would make before losing its energy to synchrotron losses. Crucially, they showed that the particle effective lifetime exceeds the electron rotation period in the magnetic field, with particles rotating between $3 \times 10^7$ in the IR to $10^2$ times in the hard X-rays [100]. The authors do note that in general it would be expected that the number of turns that a particle would make in its expected lifetime is lower than the numbers stated, as they do not take into account the Compton losses from relativistic electrons, which may be significant.
The SZ model also took into account the change in the emission that would be expected from the relativistic rotation of the pulsar, and the expected synchrotron re-absorption of the emission.

Finally, the SZ model predicted that the magnetic field of the pulsar would be of the order of $10^{14}$ Gauss at the surface [100], in good agreement with other predictions of the pulsar magnetic field strength. They also noted that the sharp peak observed in the optical spectrum of the Crab pulsar could be explained by the emitting region being close to the pulsar light cylinder, and hence moving with a velocity close to the speed of light. Finally, the authors observed that the expected linear polarisation of their model was much higher than the observed linear polarisation in all three model fits. The authors did not test the circular polarisation of the model, which was likely due to the lack of circular polarisation measurements.

Another model, known as the Single Vector Model (SVM), was proposed in 1969 [76], and a number of authors continued with this model [28][93][16]. This model describes the plane of linear polarisation as always being perpendicular to the projection on the plane of the sky of a magnetic field vector which rotates with the surface of the star. In this model, the polarisation minima occurs when the direction of the vector is nearest to the Earth, and in consequence the projection is smallest. This model does correctly predict the sweep of the polarisation angle through each pulse [28]. However, observations showed that the polarisation peak occurs 2.5 ms before the main peak, and if the polarisation peak occurs when the magnetic field vector is perpendicular to the line of sight, the simple non-relativistic single vector model can not be valid [28][17].

Other models being proposed concurrently with the SVM model and the SZ model were models which put optical emission from the neutron star far from the pulsar surface [82][36]. With that in mind, a model was proposed that modified to non-relativistic single vector model to a relativistic rotating vector model (RRVM) [28]. This model, like the original, was mainly geometrical. The emission regions were assumed to be localised, with the different emission regions following circular paths around the neutron star. The orbital velocities of the emission regions are the co-rotation velocities at the radii of the emission locations [28].

In the RRVM, it is assumed that the magnetic field lies in a plane containing the rotation axis and the emission region. The authors justify this by noting that it is the
simplest geometrical assumption, and also that if the emission came from polar regions
the field lines where the emission takes place would be almost radial, which would also
satisfy this condition [28]. This also gives a very good fit to the observations [17].

While there were a number of uncertainties in the result (the interested reader is di-
rected to [17]), the RRVM gave surprisingly accurate results for the polarisation data
available. This suggested that the emission came from regions that were far from the
neutron star surface, but still well within the neutron star light cylinder [17].

Another model for optical emission was proposed in 1972 [36]. In this model, Goldreich,
Pacini, and Rees argued that the steadiness of the optical pulses, in combination with
the lower brightness temperature in the optical band, meant that the optical emission
from the Crab pulsar was an incoherent process [36]. This model then suggests that
the emission is incoherent synchrotron radiation, which Shklovski had shown to be
valid [82][36]. One thing this model does point out is that if the emission is incoherent
synchrotron radiation, then the expected flux would vary roughly as the tenth power of
Ω, or the tenth power of the pulsar period. This could explain the lack of observations
of any other optical pulsars at the time.

Pacini and Salvati further developed this model (the PS model) in 1983. At this
point, there were two optical pulsars known, the Crab and the Vela pulsars. There
was only an upper bound for the optical emission from other pulsars [68]. This model
assumed that the emitting region is located at some constant fraction of the light-
cylinder distance which is proportional to the pulsar period. The angular extent of
the emitting region was assumed to be constant, and limited by the pulsar duty cycle.
The conversion efficiency of the rotational energy to the optical emission was assumed
to be independent of the period and surface magnetic field of the pulsar. Finally, the
energy distribution of the particles was assumed to follow an inverse power law, i.e
\( N(E) = kE^{-p} \), where \( p = 2\alpha + 1 \), and \( \alpha \) was the observed spectral index in the optical
regime [68]. Using these assumptions, the PS model found that the

\[
F_{\nu,\text{thick}} \propto \dot{P}^{\frac{1}{2}} P^{\frac{13}{4}} \nu^{\frac{5}{2}}
\]
\[
F_{\nu,\text{thin}} \propto \dot{P}^{2-\frac{\alpha}{2}} P^{\frac{5\alpha}{2} - 7} \nu^{-\alpha}
\]

where thick and thin refer to the optically thick and optically thin regimes respectively.
Using this model, the authors were able to get a good agreement with the Crab pulsar
emission. However, it is important to note that the model depended on a large number of assumptions, and as more optical pulsars were discovered, it struggled to accommodate the new observations. The discovery of strong optical flashes from PSR 0540-69 contradicted the scaling laws that were developed in this model [69].

An attempt was made in 1987 to update the PS model to agree with the new observations. In this, it was noted that the original model assumed that pulsars all had the same spectral index $\alpha$ and the same duty cycle $\delta$. This assumption does not hold, with the duty cycle for PSR 0540-69 being ~ 5 times larger than that of the Vela pulsar [69]. The next step to update the PS model is to take these differences explicitly into account. This led to the development of a scaled relationship of [69]

$$\frac{F_{\nu,1}}{F_{\nu,2}} = \left( \frac{\nu_1}{\nu} \right)^{\alpha_2 - \alpha_1} \left( \frac{B_2}{B_1} \right)^{4 - \alpha_2} \left( \frac{P_2}{P_1} \right)^{3 \alpha_2 - 9} \left( \frac{\psi_2}{\psi_1} \right)^{2 - \alpha_2} \left( \frac{\delta_2}{\delta_1} \right)^{1 - n}$$

(1.9)

where quantities with the subscript 1 come from the first pulsar, and quantities with subscript 2 from the second pulsar. Then $\nu$ is the frequency, $B$ is the surface magnetic field, $P$ is the period, $\psi$ is the pitch angle and $\delta$ is the duty cycle. This model was able to successfully explain the difference in the luminosity between the Crab, PSR 0540-69 and the Vela pulsars. While there were a large number of assumptions, the agreement of this scaling law suggested that the basic assumption - that emission was due to an incoherent synchrotron process at a constant fraction of the light cylinder proportional to the period - was correct.

However, it should be noted that the PS model is only valid for young, energetic pulsars. For older pulsars, the assumption that the emission occurs at a constant fraction of the light cylinder is no longer valid [67]. This formulation was revisited in 2001, and found to still be valid, albeit with a few changes [80]. The main change was to consider the scaling to the peak flux, as opposed to the pulsed values considered in 1987. This review took a phenomenological approach, and also showed that an involved model was not necessary to explain the pulsed values. Instead, these can be explained by the viewing angle.

Another model, proposed in 2001 was that of Gil, Khechinashvili and Melidkidze (GKM model) [29]. This model was primarily dealing with the emission from the Geminga pulsar. The spectrum of the Geminga pulsar follows a continuous power law between 370 nm to 800 nm, with a broad absorption feature in the 630-650 nm range [55]
This indicates that the dominant emission is from a non-thermal synchrotron source. The GKM model is a continuation of an earlier radio model showing that the radio emission from the Geminga pulsar is absorbed via cyclotron resonance, and re-emitted in the IR regime. Thus, the optical emission from the Geminga pulsar would be due to the gyration of plasma particles around weak magnetic fields in the remote magnetosphere creating synchrotron emission [29]. The GKM model considered two different observational fits to the Geminga power law, firstly a continuous power law spectrum with $\alpha = 0.8$, and secondly a composite spectrum of a thermal component and a power law with $\alpha = 1.9$. In both cases, the model was able to predict the observed optical luminosities. However, in order to do so low values of the Sturrock multiplication factor ($\kappa < 100$) were needed [29], in sharp contrast with the usual value of the Sturrock multiplication factor ($\kappa \approx 10^2 - 10^6$). The authors do note that the difference in the spectra between middle-aged pulsars, such as the Geminga pulsar, and young pulsars, such as the Crab, likely indicates separate emission processes[29].

The GKM model connects the emission in the radio regime with the optical emission. As such, a weak test of the model would be to see if there is any correlation between the radio and optical emission. One way to test this is to see if the optical emission changes during giant radio bursts from the Crab pulsar. Giant radio bursts are times when the Crab is a few hundred to a few thousand times brighter in the radio than it normally is. A link has been made between the giant radio pulses and the optical emission, with the optical pulses which are coincident with the giant radio pulses being $\approx 3\%$ brighter [81][64]. This model was used in 2008 as a source of synchrotron radiation producing pairs for the slot gap model [41].

Also introduced in 2001 was the model of Cruisius-Watzel, Kunzl and Lesch (CWKL model)[21]. Once again, the mechanism for emission was incoherent synchrotron radiation, although in this case the emission came from an outer-gap type region of the pulsar magnetosphere, rather than a slot-gap or polar-cap model [21]. The CWKL modelled synchrotron emission with a single energy distribution $N(\gamma) = \gamma^{-2}$. They further showed that such a power law of particles naturally occurs during an efficient pair-production cascade. They modelled the synchrotron emission as coming from particles with very small pitch angles, and showed that the spectrum of the Crab pulsar from IR to X-ray frequencies could be reproduced using a single power law distribution. In this model, the cascade process responsible for creating the emitting particles ensures
that there is no net circular polarisation [21], while the linear polarisation becomes a maximum at the outside of the cone, and is zero at the centre of the emission cone. Therefore, if the line of sight does not cut directly across the centre of the emission cone, you would expect there to be a swing in the polarisation angle.

Another model introduced in 2001 was that of Malov [53] Once again, the basic emission process was incoherent synchrotron radiation from the edge of the magnetosphere. This model connected the expected luminosity of the emission with the pulsar period and period derivative, and applied the model to a number of different radio pulsars. The model then predicted that there would be measurable optical emission from pulsars which had a large period-period derivative relationship \( \dot{P} \sim P^{-4} \), where the period derivative is in units of \( 10^{-14} \) seconds. All pulsars which have a period of less than 0.1s fall into this category [53]. Furthermore, the theory predicted another 54 objects that would emit strongly, with all 27 pulsars that had been discovered at he time to emit in the optical to gamma-ray regime being included in the list of strong emitters [53].

Despite the number of different theories that assume synchrotron radiation is the emission source for optical pulsations, there are also theories which suggest that the emission could not come from synchrotron radiation originating within the magnetosphere [8]. Synchrotron radiation can be used to obtain an upper limit to the strength of the magnetic field, and the authors argue that this upper limit precludes there being emission from within the magnetosphere, unless the inclination angle of the pulsar is very small. Instead, the authors argue in favour of a coherent synchrotron emission process, which would keep many of the attractive features of the incoherent emission theories. Alternatively, they suggest that the emission could occur from outside of the magnetosphere [8].

1.6 Conclusions

In a large part due to the problems constraining the inclination angle and the viewing angle, there are a number of different emission theories [40][10][23][75][46]. The majority of the emission models place the emission from within the magnetosphere [10][23][40][75] although some models do place the emission from well outside the magnetosphere [46][8]. No one model has emerged ahead of its competitors in explaining the
observations from pulsars, and choosing between these models is a major component of pulsar studies.

The primary difference distinguishing the models is the emission location. Nearly all optical models assume incoherent synchrotron radiation from somewhere in the magnetosphere. High energy models have attempted to constrain the emission location, however the geometry of a pulsar means that these models have had limited success. One method that could be used to constrain the emission location is an inverse method - allowing emission to occur from anywhere in the magnetosphere (taking into account the retarded magnetic dipole and effects due to the magnetospheric rotation). Then, by comparing the model results to observations it is possible to constrain the emission locations.

To do this with high energy emission (e.g. $\gamma$-ray emission) is complicated by the number of candidates for the emission process - e.g. curvature radiation, inverse Compton scattering etc. In the optical domain, the emission process is nearly universally agreed to be incoherent synchrotron emission. The optical emission location is highly likely to be in the same volume as the high energy emission. Observationally the pulses are coincident which suggests the same emission location and theoretically the optical and high energy emission would occur from the same vacuum gap in the pulsar magnetosphere.

So a natural method that could be used to constrain the pulsar emission location is to simulate the optical emission with no a priori assumptions. A code was developed at NUI Galway in order to do so. The Pulsar Optical Reverse Engineering Code (POREC) was originally written by Dr. O’Connor [63], and simulates emission from everywhere in the open magnetosphere. Rather than make assumptions about the pitch angle or inclination angle, the search space that POREC covers was increased to include the inclination and viewing angles as variables to be iterated over. POREC assumes that the emission in the optical regime will be incoherent synchrotron radiation from a power law series of particles.

This work will concentrate on attempting to constrain the emission theories using a reverse engineering code which has been developed at NUI Galway [63][56]. The underlying mathematical theory for the synchrotron radiation will be covered in Chapter 2. The various code components will be covered in Chapter 3, while the results from POREC will be covered in Chapter 4. Discussions of the problems with the underlying
Chapter 1. *Introduction*

theory, together with the effect that the results from POREC and POREC2.0 will have, will be covered in Chapter 5. Finally, areas that could be improved will be covered in Chapter 6.
Chapter 2

Incoherent Synchrotron
Radiation from High Magnetic Fields

2.1 Introduction

Pulsar emission in the optical regime is generally accepted to be incoherent synchrotron radiation and consequently it should be polarized. To date most attention has been on linear polarisation, in part due to the instrumental limitations of most polarimeters [84, 85]. Optical instrumentation, such as the Galway Astronomical Stokes Polarimeter (GASP) [19], are now in a position to measure all of the Stokes parameters from pulsars on time-scales from milliseconds to hours. Hence the requirement for a fully self-consistent model for synchrotron radiation in high magnetic fields.

The original model for synchrotron emission was published in 1959 [95], with other authors coming to the same conclusions [47][30][31], albeit with slightly different derivation methods. This model was then further developed in 1968 [49], and corrections to the model were applied in 1974 [34]. An error in the derivation used was found 1986 [83], but luckily this error did not affect the results.

The motivation behind the Westfold, Legg and Gleeson (hereafter WLG) model of synchrotron radiation was to study the emission from Jupiter, which has a magnetic
field of approximately one Gauss. As such, the behaviour of the model was never tested in high magnetic fields. There are two main components to the WLG model. Firstly, there is the underlying Westfold and Legg model (the WL model) [49] which calculates the emission. This model is only valid when the frequency of interest is far from the minimum or maximum frequency that the energy range of interest can produce. This is due to the fact that the WL model assumes that the energy range of the particles goes between 0 and $\infty$, rather than between realistic energy ranges. The second part of the model is an attempt to correct for this using correction functions. This extends the range of the model to cover all frequencies.

As pulsars have extremely high magnetic fields ($\sim 10^{12}$ G), it is important to test the model in high magnetic fields before applying it to pulsar emission. When the WLG model is used to calculate emission from high magnetic fields ($\sim 10^6$ G), it predicts a circular polarisation greater than one hundred percent. As this is in clear contradiction to reality, a new model for the incoherent synchrotron emission is required. The most likely source of error was in the expansion of either the velocity of the particle or in the expansion of the particle pitch angle distribution. When the expansion of the velocity was examined it was found that expanding the velocity to the next order did not significantly alter the circular polarisation measurements. In particular the percentage circular polarisation was still greater than one hundred percent in high magnetic fields. Therefore the expansion of the particle pitch angle distribution was examined. It was found that within certain parameters, by expanding the particle pitch angle distribution to the next order the problem of circular polarisation greater than one hundred percent in high magnetic fields could be solved.

Firstly, the Westfold, Legg and Gleeson model will be covered. As mentioned, this can broadly be separated into two parts - the model itself (published by Legg and Westfold in 1968 [49]), and frequency corrections to the model made in 1974 [34]. Then the derivation of a new model by this author will be detailed [25]. The majority of the work which has been previously published in a peer-reviewed journal is presented in this chapter.
Chapter 2. Sync. Rad. from High Mag. Fields

2.2 The WLG Model of Synchrotron Emission

2.2.1 The WL Model

In order to calculate the incoherent synchrotron emission, Westfold and Legg assumed that the particle was travelling at relativistic field in a uniform magnetic field. Furthermore, they assumed that there was no self-absorption, and that the particles were travelling through a volume with a low plasma density. Finally, it was assumed that the reference origin was close to the position of the particle during the time of observation, in comparison with the distance to the observer, and that the angle of the magnetic field with the direction of observation was not close to either 0 or \( \pi \). With this in mind, they calculated the Stokes parameters (I, Q, U and V) for a power law distribution of electrons.

A power law is given by

\[
N(E) = kE^{-p} \quad 0 < E < \infty
\]  

(2.1)

where \( k \) is the constant of proportionality, \( N(E) \) is the number of electron with energy \( E \) and \( p \) is the power law index.

The coordinate system was then defined as follows: the trajectory of the particle will in general describe a helix through space. The the origin be the projection of the initial position of the particle on the axis of the helix. Then a right-handed orthogonal system of vectors \((i, j, k)\) can be defined such that \( i \) is in the direction of the initial position and \( k \) is in the direction of the initial magnetic field direction [95].
The WL model then gives the Stokes parameters as \cite[Equation 33]{LeggW68:SyncRad}

\begin{align}
I &= \frac{k\mu^2 e^2 c}{2\sqrt{2}} \left( \frac{3}{2} \right)^{\frac{3}{2}} \phi(\alpha) (fB_0 \sin \alpha)^{\frac{p+1}{2}} f^{-\frac{p+1}{2}} J_{\frac{p+1}{2}} \\
Q &= \frac{k\mu^2 e^2 c}{2\sqrt{2}} \left( \frac{3}{2} \right)^{\frac{3}{2}} \phi(\alpha) (fB_0 \sin \alpha)^{\frac{p+1}{2}} f^{-\frac{p+1}{2}} L_{\frac{p+1}{2}} \\
U &= 0 \\
V &= \frac{k\mu^2 c}{\sqrt{3}} \left( \frac{3}{2} \right)^{\frac{3}{2}} \phi(\alpha) \cot \alpha (fB_0 \sin \alpha)^{\frac{p+1}{2}} f^{-\frac{p+1}{2}} \left[ R_{\frac{p+1}{2}} + (1 + g(\alpha))(L_{\frac{p+1}{2}} - \frac{1}{2} J_{\frac{p+1}{2}}) \right]
\end{align}

where \( f \) is the frequency, \( \mu \) is the permeability constant, \( p \) is the power law index, \( e \) is the charge of the particle, \( \alpha \) is the particle pitch angle, \( fB_0 = eB(2\pi mc)^{-1} \) is the fundamental gyro-frequency, \( \phi(\theta) \) is the pitch angle distribution function and

\begin{align}
J_n &= \int_0^\infty \int_0^\infty x^{n-1} K^2_\frac{3}{2} (\nu) d\nu dx = \frac{2^{\frac{2}{3}+n}}{n} L_n, \quad n > \frac{2}{3} \\
L_n &= \int_0^\infty x^{n-1} K^2_\frac{3}{2} (x) dx = 2^{n-2} \Gamma \left( \frac{1}{2} n - \frac{1}{3} \right) \Gamma \left( \frac{1}{2} n + \frac{1}{3} \right), \quad n > \frac{2}{3} \\
R_n &= \int_0^\infty x^{n-1} K^2_\frac{3}{2} (x) dx = 2^{n-2} \Gamma \left( \frac{1}{2} n - \frac{1}{6} \right) \Gamma \left( \frac{1}{2} n + \frac{1}{6} \right), \quad n > \frac{1}{3}
\end{align}

This then gives the emission due to incoherent synchrotron radiation in high magnetic fields from a particle power law which goes between 0 and \( \infty \). It is expected that at the frequency considered the emission from particles outside the power-law range of the distribution will be negligible.

\subsection*{2.2.2 The WLG Model}

In order to get a more realistic expression for the incoherent synchrotron radiation from a power law distribution of charged particles, corrections need to be made to the integration over the energy. Gleeson, Legg and Westfold made corrections so that the results would be for a particle power law going between a lower energy, \( E_1 \), and a higher
energy, \( E_2 \), i.e.

\[
N(E) = \begin{cases} 
0 & E < E_1 \\
kE^{-p} & E_1 < E < E_2 \\
0 & E_2 < E 
\end{cases} \tag{2.7}
\]

where \( N(E) \) is the number of particles at energy \( E \), it is assumed that particles with energy less that \( E_1 \) cannot contribute to the emission and that there are no particles with energy greater than \( E_2 \). As before, \( k \) is the constant of proportionality.

As shall be shown later, there is a relationship between the energy and the dimensionless parameter \( x \). Therefore in order to take the new limits of integration into account the only things which need to change in Equation 2.5 are the integrations over \( x \), which are given in Equation 2.6.

There is a relationship between the energy integrations and the integrations over the dimensionless quantity \( x \) such that changing the limits of the energy distribution is analogous to changing the integration over \( x \). In particular

\[
x = \frac{f}{f_c} = \frac{2}{3} \frac{f}{f_B E^2 \sin \alpha} \tag{2.8}
\]

where \( f_c \) is the critical frequency and the other variables are as before.

Gleeson, Legg and Westfold decided that the best method to use to correct for the errors introduced by the integration between 0 and \( \infty \) was to use a correction function on the original WL model. These correction functions can now be easily defined - namely that [34]

\[
C^1(x_1, x_2) = \left[ \frac{J_n(x)}{J_n} \right]^{x_1}_{x_2} \tag{2.9}
\]

\[
C^2(x_1, x_2) = \left[ \frac{L_n(x)}{L_n} \right]^{x_1}_{x_2} \tag{2.10}
\]

\[
C^4(x_1, x_2) = \left[ \frac{R_n(x) + \left[ 1 + g(\alpha) \right] \left( L_n(x) - \frac{1}{2} J_n(x) \right)}{R_n + \left[ 1 + g(\alpha) \right] \left( L_n - \frac{1}{2} J_n \right)} \right]^{x_1}_{x_2} \tag{2.11}
\]
where \( g(\alpha) = 1 + \phi'(\alpha) \tan \alpha/\phi(\alpha) \) and

\[
\mathcal{J}_n(x) = \int_0^x \epsilon^{n-1} \int \epsilon K_\frac{2}{3}(\nu) d\nu d\epsilon 
\]  (2.12)

\[
\mathcal{L}_n(x) = \int_0^x \epsilon^{n-1} K_\frac{2}{3}(\epsilon) d\epsilon 
\]  (2.13)

\[
\mathcal{R}_n(x) = \int_0^x \epsilon^{n-1} K_\frac{4}{3}(\epsilon) d\epsilon 
\]  (2.14)

where \( C^1(x_1, x_2) \) is the correction factor applied to \( I \), \( C^2(x_1, x_2) \) is the correction factor applied to \( Q \) and \( C^4(x_1, x_2) \) is the correction factor applied to \( V \) (noting that the correction factor applied to \( U \) is zero, \( x_1 \) is the dimensionless variable corresponding to the lower energy limit and \( x_2 \) corresponds to the higher energy limit.

Another thing that has to be taken into account is the fundamental emission frequency. There can be no emission below this frequency, and this should be accounted for in the correction factors. To do so, WLG defined a parameter \( x_l \) such that

\[
x_l = \min(x_1, x'_1),
\]

where \( x_1 \) corresponds to the minimum emission frequency of the energy range and \( x'_1 \) corresponds to the minimum energy required for emission to occur and is given by

\[
x'_1 = \frac{x_1^3}{x_1^*} \quad x_1^* = \frac{2}{3} \left( \frac{E_0}{E_1 \sin \alpha} \right)^3
\]  (2.15)

By using \( x_l \) in the correction factors rather than \( x_1 \), the fundamental frequency limit is taken into account.

Finally, Westfold, Legg and Gleeson also showed that there is a relationship between \( x_1 \) and \( x_2 \), namely that

\[
x_2 = \left( \frac{E_1}{E_2} \right)^2 x_1
\]  (2.16)

Using this relationship, it is possible to substitute \( x_1 \) into the correction factors rather than \( x_2 \), leaving only one \( x \) variable to work with. This then gives the correction factors...
Chapter 2. Sync. Rad. from High Mag. Fields

as (reproduced from [34])

\[ C^i(x_1, x_2) = \begin{cases} 
1 - \left( \frac{E_1}{E_2} \right)^{\frac{3p-1}{3}} & \left[ \frac{x_1}{x_1^p(x)} \right]^{\frac{3p-1}{6}}, x_1^* < x_1 < x_1^p(p) \\
1 - \left( \frac{E_1}{E_2} \right)^{\frac{3p-1}{3}} & \left[ \frac{x_1}{x_1^p(x)} \right]^{\frac{3p-1}{6}}, x_1^* < x_1 < x_1^p(p) \\
1 - \left( \frac{E_1}{E_2} \right)^{\frac{3p-1}{3}} & \left[ \frac{x_1}{x_1^p(x)} \right]^{\frac{3p-1}{6}}, x_1^* < x_1 < x_1^p(p) 
\end{cases} \] (2.17)

\[ C^4(x_1, x_2) = \begin{cases} 
1 - \left( \frac{E_1}{E_2} \right)^{\frac{3p-1}{3}} & \left[ \frac{x_1}{x_1^p(x)} \right]^{\frac{5}{6}}, x_1^* < x_1 < x_1^p(p) \\
1 - \left( \frac{E_1}{E_2} \right)^{\frac{3p-1}{3}} & \left[ \frac{x_1}{x_1^p(x)} \right]^{\frac{5}{6}}, x_1^* < x_1 < x_1^p(p) \\
1 - \left( \frac{E_1}{E_2} \right)^{\frac{3p-1}{3}} & \left[ \frac{x_1}{x_1^p(x)} \right]^{\frac{5}{6}}, x_1^* < x_1 < x_1^p(p) 
\end{cases} \]

where [34]

\[ x_1^p(p) = 2 \left[ K^p(p) \right]^{\frac{3}{3p-1}}, i = 1, 2 \] (2.18)

\[ x_1^4(p) = 2 \left[ K^4(p) \right]^{\frac{3}{p}} \] (2.19)

and [34] (Note that the original has a minor typo where 3p – 1 is represented as 3(p – 1) which is fixed here)

\[ K^1(p) = \frac{24 (p + 1) \Gamma \left( \frac{2}{3} \right)}{(3p - 1) \Gamma \left( \frac{3p - 1}{12} \right) \Gamma \left( \frac{3p + 7}{12} \right) (p + 7)} \] (2.20)

\[ K^2(p) = \frac{p+7}{2(p+1)} K^1(p) \] (2.21)

\[ K^4(p) = \frac{2 \pi p + 2}{\sqrt{3} p} \left( \frac{1 + g(\alpha)}{p + 1 + g(\alpha)} \right) \Gamma \left( \frac{3p + 4}{12} \right) \Gamma \left( \frac{3p + 8}{12} \right) (p + 2) \] (2.22)

where p here stands for the The derivation of these results will not be expanded upon here, but interested readers are referred to [34].

The WLG model predicts the emission and polarisation of incoherent synchrotron radiation in low magnetic fields at optical wavelengths. Let us consider the simplified WL model for a moment. This model predicts that the circular polarisation (defined as $VJ^{-1}$) is given by

\[ \frac{V}{J} = \frac{2\sqrt{2}}{3} \cot \alpha (fB_0 \sin \alpha)^{\frac{1}{2}} f^{-\frac{1}{2}} \left[ \frac{R_{\frac{p}{2}} + 1 + g(\alpha)}{J_{\frac{p}{2} + 1}} \right] \propto B^{\frac{1}{2}} \] (2.23)
As the circular polarisation is proportional to the root of the magnetic field, it is clear that as the magnetic field is increased there will come a point at which the percentage circular polarisation exceeds one hundred percent. This is clearly unrealistic. However, all models will only have a certain range of validity, and if the magnetic field at which the circular polarisation fails is extremely high (higher than the magnetic fields of pulsars), then the model can still be used for lower magnetic fields. The percentage circular polarisation was found to be greater than 100% at approximately $10^5$ to $10^7$ Gauss (Figure 2.1) and above, which is below the surface magnetic field strength of pulsars, but above planetary magnetic fields.

The WLG model does not significantly alter the circular polarisation characteristics. In fact, this can be shown as follows. Pick a frequency such that the energy limits do not significantly affect the emission at that frequency (i.e. the correction factors that are given above are approximately 1). Then increase the magnetic field while keeping the frequency constant. At some point, the circularly polarised emission will again be greater than one hundred percent. Because the correction factors are close to 1, the WLG model will predict a circular polarisation greater than 100% at approximately the same magnetic field (for that frequency) as the WL model. The difference between the WLG model and the WL model can be seen in Figure 2.1, but as can be seen, the circularly polarised component is greater than the total emission at approximately the same magnetic field.

### 2.3 Expanding the WL Model

As the model fails at high magnetic fields, a new model is needed to describe the polarisation of synchrotron radiation in those fields. The WLG model uses the WL model as a baseline, and any attempt to fix the problems with the circular polarisation should be made using the WL model rather than the WLG model, as the difference is in the frequency corrections only. Therefore for the rest of the derivation it is the WL model that will be looked at.

The WL model has a number of different assumptions. The main assumptions of the WL model that will be considered here are:

1. The velocity of the particle is close to the speed of light ($\gamma \gg 1$).
Chapter 2. Sync. Rad. from High Mag. Fields

% Circular Polarisation

![Graph showing WL and WLG models with magnetic field and percentage circular polarisation on axes.]

**Figure 2.1:** The WL model predicts that the percentage circular polarisation will increase linearly with the magnetic field, regardless of the power law index of the electrons used. At some point this model fails and predicts clearly non-physical results. This is dependant on the pitch angle ($\alpha$), the frequency ($5.212 \times 10^{14}$ Hz), and the power law index ($1.42$), but in the area of interest the WL model fails long before the predicted surface magnetic field of a pulsar ($\approx 10^{12}$ Gauss). While the corrections introduced in [34] do change the point at which the model fails, the WLG model still fails before reaching the surface magnetic field of a pulsar. Here the minimum energy corresponded to a $\gamma$ factor of 10 and the maximum energy to a $\gamma$ factor of $10^9$.

2. The expansion of the velocity in the chosen frame is only to first order in $1/\gamma$.

3. The expansion of the particle pitch angle is only to first order of the angle $\theta$.

The first assumption is a basic assumption as emission from particles which are travelling at less than the speed of light emit cyclotron radiation. Therefore this investigation concentrated on the second and third assumptions.

When the velocity was expanded to the next order of magnitude, the qualitative degree of the circular polarisation at any particular magnetic field strength did not change. In particular it did not change the relationship between the circular polarisation and the emission. Basically, the expansion of the circularly polarised component and the non-circularly polarised component follow the same shape, such that the circular polarisation follows the same trend. While the specific magnetic field at which the circular polarisation became greater than one hundred percent changed, the circular polarisation still became greater than 100% before the magnetic field strength was as strong as that surrounding a pulsar.
Chapter 2. Sync. Rad. from High Mag. Fields

It should also be noted that the problem with the circular polarisation only arises with a power law spectrum of emitting particles - which points in the direction of expanding the particle pitch angle distribution. Expanding the particle pitch angle distribution does solve the problem with the circular polarisation for a subset of particle pitch angle distributions. However, it does so by putting strong constraints on the shape of the particle pitch angle distribution. These constraints will be discussed in more detail later (Section 5.1).

2.3.1 Coordinate System

In order to solve Equation 2.26, it is necessary to choose a system of coordinates. In this case, the system was constructed as follows: the particle is spiralling around a magnetic field at an angular frequency of \( \omega_B = qB(\gamma mc)^{-1} \), where \( q \) is the charge, \( B \) is the strength of the magnetic field and \( \gamma \) is the Lorentz factor of the particle. The particle maintains a constant pitch angle of \( \alpha \) with respect to the magnetic field direction. At any particular time the orbit has a radius of curvature of \( a \). Now, let the \( x-y \) plane be the instantaneous plane of the orbit of the particle. Take the origin of the \( x \)-axis to be the point where the velocity vector and the observer are in the \( x-z \) plane, and let the \( y \) coordinate be in the direction of the radial vector \( a \), with the \( x \) coordinate being defined as perpendicular to the \( y \) and \( z \) coordinates.

Now, define a new set of coordinates \((n, \varepsilon_\parallel, \varepsilon_\perp)\) such that the origin is at the same point as the \( (x, y, z) \) coordinate system origin, \( n \) is pointing towards the observer, \( \varepsilon_\perp \) is pointing along \( y \), and \( \varepsilon_\parallel = n \times \varepsilon_\perp \). This then gives a natural coordinate system in which to consider the polarisation of the emission, as \( \varepsilon_\perp \) is perpendicular to the magnetic field and \( \varepsilon_\parallel \) is parallel to the magnetic field direction, as seen in projection by an observer, which is illustrated in Figure 2.2. Here \( a \) stands for the radius of curvature of the particle.

2.3.2 Electric Field

Synchrotron radiation from a single particle comes from a source in periodic motion with frequency \( f_{B0}/\gamma \). Following the same formulation as WL, the electric field for
Chapter 2. Sync. Rad. from High Mag. Fields

Figure 2.2: The geometry used in order to calculate the synchrotron emission. Firstly, define the \(x-y\) plane as the instantaneous plane of orbit of the particle. Then define the origin as the point at which the velocity \(v\) and the vector to the observer \(n\) are both in the \(x-z\) plane. Define \(\varepsilon_\parallel\) to be along the \(y\) axis, and \(\varepsilon_\bot\) as \(n \times \varepsilon_\bot\). This gives a natural frame of reference for the polarisation of the emission.

Each harmonic can be shown to be

\[
E_n = \frac{\mu e c (\omega_B b)}{8\pi^2 r} \int_0^{\frac{2\pi b}{\omega_B}} \left[ n \times \left( \frac{(n - \beta) \times \frac{dt}{dt}}{(1 - n \cdot \beta)^3} \right) \right] \exp \left[ in \left( \frac{\omega_B b}{b} \right) t \right] dt
\]  

(2.24)

where the expression in brackets is evaluated at the retarded time

\[
t' = t - \frac{R(t')}{c}
\]

(2.25)

\[
R(t') = r - r(t') \approx |r| + n \cdot r
\]

where \(b = \beta' \sin \alpha \sin (\alpha - \theta)\) \[47][49\]. Changing the integration to an integration over \(t'\), and integrating by parts (remembering that the emission at infinity disappears), gives

\[
E_n = \frac{\mu c e}{8\pi^2 r} \left( \frac{\omega_B}{b} \right)^2 in \exp \left( in \left( \frac{\omega_B}{b} \right) \left| r \right| \right) \int_{-\infty}^{\infty} n \times (n \times \beta) \exp \left[ in \left( \frac{\omega_B}{b} \right) \left( t' - \frac{n \cdot r(t')}{c} \right) \right] dt'
\]  

(2.26)

where \(n\) is a unit vector in the direction of the observer and \(n\) stands for the nth harmonic.
Finally, in this coordinate system the position and velocity are [50]

\[
\mathbf{r}_0(t') = 2a \sin \left(\frac{vt'}{2a}\right) \left[ \sin \left(\frac{vt'}{2a}\right) \mathbf{e}_\perp + \cos \theta \cos \left(\frac{vt'}{2a}\right) \mathbf{n} - \sin \theta \cos \left(\frac{vt'}{2a}\right) \mathbf{e}_\parallel \right]
\] (2.27)

\[
\mathbf{v} = v \left[ \sin \left(\frac{vt'}{a}\right) \mathbf{e}_\perp + \cos \theta \cos \left(\frac{vt'}{a}\right) \mathbf{n} - \sin \theta \cos \left(\frac{vt'}{a}\right) \mathbf{e}_\parallel \right]
\] (2.28)

where \(\theta\) is the angle between the velocity of the particle at \(t_0\) and the observer. This then gives the electric field for the nth harmonic as (splitting it into its component parts parallel and perpendicular to the projection of the magnetic field and dropping the subscript)

\[
E_\parallel = \frac{\mu c e}{8\pi^2 r} \left(\frac{\omega_B}{b}\right)^2 \sin \left(\frac{\omega_B}{b}\right) |r| \gamma \int_{-\infty}^{\infty} \exp \left[i \eta \left(\frac{vt'}{c} + \frac{\theta^2}{2} - \frac{\theta^3}{6 c a^2 + 3}\right)\right] dt'
\] (2.29)

\[
E_\perp = -\frac{\mu c e}{8\pi^2 r} \left(\frac{\omega_B}{b}\right)^2 \sin \left(\frac{\omega_B}{b}\right) |r| \gamma \int_{-\infty}^{\infty} \frac{vt'}{a} \exp \left[i \eta \left(\frac{vt}{c} + \frac{\theta^2}{2} - \frac{\theta^3}{6 c a^2 + 3}\right)\right] dt'
\] (2.30)

\[
\mathcal{H} = i n \left(\frac{\omega_B}{b}\right) \left[t' \left(1 - \frac{v}{c}\right) + \frac{\theta^2}{2} t' + \frac{v^3}{6 c a^2 + 3}\right]
\] (2.31)

A convenient substitution of

\[
\theta_\gamma^2 = (1 + \gamma^2 \theta^2); \quad y = \frac{\gamma c t'}{a \theta_\gamma}; \quad \eta = \frac{n \left(\frac{\omega_B}{b}\right) a \theta_\gamma^3}{3 c \gamma^3}
\] (2.32)

then gives

\[
E_\parallel = \frac{\mu c e}{8\pi^2 r} \left(\frac{\omega_B}{b}\right)^2 \sin \left(\frac{\omega_B}{b}\right) |r| \gamma \int_{-\infty}^{\infty} \exp \left[i \eta \left(y + \frac{1}{3} y^3\right)\right] dy
\] (2.33)

\[
E_\perp = -\frac{\mu c e}{8\pi^2 r} \left(\frac{\omega_B}{b}\right)^2 \sin \left(\frac{\omega_B}{b}\right) |r| \gamma \int_{-\infty}^{\infty} y \exp \left[i \eta \left(y + \frac{1}{3} y^3\right)\right] dy
\] (2.34)
2.3.3 Emission-polarisation Tensor

The emission-polarisation tensor is defined as

\[
\rho = \frac{2\pi r^2}{\mu} \begin{pmatrix} E_{\perp} E_{\perp}^* & E_{\perp} E_{\parallel}^* \\ E_{\parallel} E_{\perp}^* & E_{\parallel} E_{\parallel}^* \end{pmatrix}
\]  

(2.35)

This is equivalent to getting the Stokes parameters for each harmonic, as

\[
I = \rho_{11} + \rho_{22}
\]  

(2.36)

\[
Q = \rho_{11} - \rho_{22}
\]  

(2.37)

\[
U = \rho_{12} + \rho_{21}
\]  

(2.38)

\[
V = \frac{1}{i}(\rho_{12} - \rho_{21})
\]  

(2.39)

2.3.4 Airy Functions

It is possible to convert the electric field exponential into Bessel functions (see Appendix A for full details). This gives

\[
\rho_{11} = \frac{\mu e^2 c}{24\pi^4} \left( \frac{\omega_B}{b} \right)^4 n^2 a^2 \theta^4 \gamma^3 \frac{a^2}{c^2} \frac{K^2}{3} (\eta)
\]  

(2.40)

\[
\rho_{12} = \text{sgn}(e) \frac{\mu e^2 c}{24\pi^4} \left( \frac{\omega_B}{b} \right)^4 n^2 \theta^2 \gamma^3 \frac{a^2}{c^2} K^1_3 (\eta) K^2 (\gamma)
\]  

(2.41)

\[
\rho_{22} = \frac{\mu e^2 c}{24\pi^4} \left( \frac{\omega_B}{b} \right)^4 n^2 \theta^2 \gamma^3 \frac{a^2}{c^2} K^1_3 (\eta)
\]  

(2.42)

This then gives the polarisation tensor for a particular harmonic of the emission.

2.3.5 Converting to Frequency Domain

For large-order harmonics, the radiation becomes quasi-continuous [49] and it is possible to convert the polarisation tensor for a single harmonic to the frequency polarisation tensor using

\[
\rho_f = \rho_n b \frac{\omega_B}{f_B}
\]  

(2.43)
where \( f_B = f_{B_0}/\gamma \), \( b = \beta' \sin \alpha \sin (\alpha - \theta) \), \( \rho_n \) is the polarisation tensor for a single harmonic and \( \rho_f \) is the polarisation emission tensor at a particular frequency, and

\[
f = nf_B b^{-1} \tag{2.44}
\]

This gives the polarisation tensor at a particular frequency. It is convenient to convert from the frequency into a dimensionless parameter \( x \) such that

\[
x = \frac{f}{f_c} = \frac{4\pi a}{3c\gamma^3} f \tag{2.45}
\]

which in turn gives

\[
n = \frac{3c\gamma^3}{2a} \left( \frac{b}{\omega_B} \right) x \quad \eta = \frac{x\theta^2}{2}\gamma \tag{2.46}
\]

This gives

\[
\rho_{x11} = \frac{3}{16} \frac{\mu e^2 c \omega_B}{\pi^3} \frac{x^2 \gamma^2 \theta^4}{b} K^2 \left( \frac{x}{2}\gamma \right) \tag{2.47}
\]
\[
\rho_{x12} = \text{sign}(e) \frac{3}{16} \frac{\mu e^2 c \omega_B}{\pi^3} \frac{x^2 \gamma^3 \theta^3}{b} K^2 \left( \frac{x}{2}\gamma \right) K^2 \left( \frac{x}{2}\gamma \right) \tag{2.48}
\]
\[
\rho_{x22} = \frac{3}{16} \frac{\mu e^2 c \omega_B}{\pi^3} \frac{x^2 \gamma^4 \theta^2}{b} K^2 \left( \frac{x}{2}\gamma \right) \tag{2.49}
\]

### 2.3.6 Power Law Polarisation-emission Tensor

When there is a power law of particles, the polarisation-emission tensor for that population of particles is

\[
n_x(n) = 2\pi \int_0^\infty N(E) \int \phi(\alpha) b P_x(n) d\Omega(n) dE \tag{2.50}
\]

In order to solve this it is possible to represent \( \alpha \) as \( \alpha' + \theta \), where \( \alpha' \) is the viewing angle. Then the solid angle is represented as \( d\Omega(n) = 2\pi \sin \alpha d\theta \). \( \alpha = \alpha' + \theta \) can now be substituted into Equation 2.50. To third order, the particle pitch angle distribution can be written as

\[
\phi(\alpha' + \theta) \sin (\alpha' + \theta) = f(\alpha') + k(\alpha')\theta + h(\alpha')\theta^2 \tag{2.51}
\]
where

\[ f(\alpha') = \phi(\alpha') \sin \alpha' \]  
\[ k(\alpha') = \phi'(\alpha') \sin \alpha' + \phi(\alpha') \cos \alpha' \]  
\[ h(\alpha') = \phi''(\alpha') \sin \alpha' + 2\phi'(\alpha') \cos \alpha' - \phi(\alpha') \sin \alpha' \]  

with \( \phi(\alpha) \) as the particle pitch angle distribution and the primes representing the derivative with respect to \( \theta \). Furthermore, \( \theta \) is a very small argument (from physical arguments it can be shown that \( \theta \) is of the order of \( 1/\gamma \), and \( \gamma \) is large for synchrotron radiation). Therefore the error involved in substituting \( \alpha \) for \( \alpha' \) is very small. This gives the polarisation-emission tensor as

\[ n_{x1} = \frac{1}{4\sqrt{2}} \mu_e^2 c f B_0 \left( \frac{3}{2} \right)^{\frac{7}{2}} \sin \alpha \frac{p+1}{2} f^{\frac{1}{2}} \left[ \phi(\alpha) \left( J_{\frac{p+1}{2}} - L_{\frac{p+1}{2}} \right) + \frac{3h(\alpha)}{2} \left( f_{B_0} f \right) \left( 3Q_{\frac{p+1}{2}} - 2L_{\frac{p+1}{2}} - J_{\frac{p+1}{2}} \right) \right] \]  
\[ n_{x2} = \text{sign}(e) \frac{1}{4\sqrt{2}} \mu_e^2 c k(\alpha) \left( \frac{3}{2} \right)^{\frac{7}{2}} f B_0 f^{\frac{1}{2}} \sin \alpha \frac{p+1}{2} \left[ \phi(\alpha) \left( J_{\frac{p+1}{2}} - L_{\frac{p+1}{2}} \right) + \frac{9h(\alpha)}{8} \left( f_{B_0} f \right) \left( Q_{\frac{p+1}{2}} - J_{\frac{p+1}{2}} \right) \right] \]  

where

\[ Q_n = \int_0^\infty x^{n-1} K_{\frac{1}{2}} (x) \, dx \]  

and the other parameters are as before. This will be referred to as the de Búrca and Shearer (DS) formulation for the remainder of this work.

### 2.4 DS Formulation Predictions

The DS formulation predicts that the circular polarisation of the emission will not exceed one hundred percent. By this metric it is successful. It also decays to the WL formulation in low magnetic fields, which is a good property of any theory attempting
Figure 2.3: The circular polarisation for a power law distribution of particles with a power law index of 1.42, at a frequency of $5.212 \times 10^{14}$ Hz. Here WL stands for Westfold and Legg, the original emission theory, and theta stands for the particle pitch angle (in degrees). The PAD used was the Inversely Linear PAD. Here This Work stands for the DS model. Figure reproduced from [25].

To expand a previous theory to unexplored areas, it makes predictions on how the polarisation of the emission will change - in particular, that in the intermediate range of magnetic fields ($10^4-10^8$ Gauss) the linear polarisation will switch (smoothly) from a low-magnetic field value to a high magnetic field value. At the extremes of the magnetic fields the amount of linear polarisation will be fixed - again this is in agreement with the WL model.
2.4.1 Range of Validity

Before discussing the results that are obtained by the DS model, it is useful to discuss the range of validity over which the model can be trusted. Like the WLG model, the DS model has a number of assumptions associated with it, including

- uniform magnetic fields,
- relativistic electrons,
- low plasma density,
- no self-absorption,
- reference origin close to the position of the particle during time of observation,
- relativistic particle motion and
- magnetic field less than critical magnetic field ($B << B_{\text{crit}}$)

The maximum amount of energy that the particle can emit (the maximum energy contained by the particle) is $\gamma mc^2$. This then leads to the condition that $h(\alpha)fB_0/f << \gamma mc^2$. Substituting in for $fB_0$ and using order of magnitude estimates, this condition reduces to the requirement that $B << B_{\text{crit}}/\gamma$, where $B_{\text{crit}}$ is of the order of $10^{20}$ Gauss. This leads to the requirement that the magnetic field be less than $10^{13}$ Gauss - excluding the quantum regime.

2.4.2 Circular Polarisation in the DS Model

The DS model predicts that the circular polarisation that is seen will be at a maximum at approximately $10^6$ Gauss. Below this value the DS model qualitatively agrees with the predictions of the WL model i.e. it increases exponentially (or linearly on a log scale) as in Figure 2.1. As it reaches approximately $10^6$ Gauss the circular polarisation reaches a maximum before beginning to drop (Figure 2.3). The maximum that was reached is dependant on the frequency, the pitch angle, the pitch angle distribution and the magnetic field - with initial results showing that as the pitch angle decreases the magnetic field at which the circular polarisation peaks also decreases.
Figure 2.4: The linear polarisation change with regard to the magnetic field for a particle power law index of 1.42 and frequency \(5.212 \times 10^{14}\) Hz and theta for the particle pitch angle (in degrees). As can be seen, the linear polarisation is steady at low magnetic fields and at high magnetic fields, with the linear polarisation changing smoothly between the two values in the intermediate range of magnetic field values. Here This Work stands for the DS model. The PAD used here was the Inverse Linear PAD. Figure reproduced from [25].

Once the maximum has been reached the circular polarisation begins to decrease again, until at high magnetic fields (\(\sim 10^{12}\) G) the amount of circular polarisation is again negligible.

2.4.3 Linear Polarisation in the DS Model

The amount of linear polarisation in the WL model is fixed - it is not dependant on the magnetic field, but rather on the particle power law index. The DS model predicts three
ranges for the linear polarisation. In low ($< 10^4$ G) and high ($> 10^{10}$ G) magnetic fields, the amount of linear polarisation is fixed and is primarily dependant on the particle power law index. In intermediate magnetic fields, the amount of linear polarisation varies smoothly from the low magnetic field polarisation to the high magnetic field polarisation (this is due to the expansion of the particle PAD to the next order in the DS model giving two components in the formulæ used to calculate the Stokes parameters. In the Westfold and Legg model there is only one component to the formulæ used to calculate the Stokes parameters.) (Figure 2.4). While the linear polarisation in Figure 2.4 increases as the magnetic field increases, this is not a necessary property. The amount of linear polarisation is very dependent on the particle power law index. This could offer an independent check on the power law indices of regions suspected of emitting synchrotron radiation, provided those regions have appreciable magnetic fields. In general, the power law index of the electrons will be the dominating factor for the amount of linear polarisation that is seen.
Chapter 3

Pulsar Optical Reverse Engineering Code

3.1 Introduction

The Pulsar Optical Reverse Engineering Code (POREC) was developed in NUI Galway in an attempt to constrain high energy emission theories for pulsars [63].

There are four main sections to POREC. These are:

1. The underlying physical model of both the emission process and the volume surrounding the pulsar.

2. The computational component which calculates the simulated light curves that would be seen from Earth.

3. The statistical component which selects the best fit between the simulated light curves and the observational data.

4. A reverse engineering component which allows the user to find out where in the magnetosphere the emission occurs.

In theory, once the emission location is known, it is possible to constrain optical and high energy emission theories.
Pulsars are complicated objects, and have a number of unknown parameters associated with them (e.g. inclination angle, viewing angle, pitch angle distribution, pitch angle distribution cut-off limit). POREC does not assume, a priori, any restriction on the unknown parameters. Instead, POREC treats these unknown parameters as a dimension in the search space to be covered. By searching over the entire parameter space, and comparing the resulting simulations to the observations, it is possible to constrain these unknown parameters. POREC is therefore attempting to constrain the various emission theories in two dimensions at the same time. Firstly, it constrains the location of the emission in the magnetosphere, thereby constraining those emission theories whose primary difference is the emission location. Secondly, POREC is also attempting to constrain the unknown pulsar parameters. Due to the complicated geometry surrounding a pulsar, the same light curves can result from many different theories with different geometrical parameters. Limiting the physical geometries allowed puts constraints on the emission theories.

As there is only one emission process in the optical domain (incoherent synchrotron radiation) this was the spectral range chosen to be simulated by POREC. In general, optical and high energy emission theories which predict emission from within the light cylinder have overlapping vacuum gaps (see Chapter 1). Therefore constraining the location of the optical emission leads to a constraint on the high energy emission theories. With only one emission process in the optical domain, there are less unknowns to consider (such as the ratio of the curvature emission to the inverse Compton emission, etc.) which reduces the parameter space to be considered.

While POREC simulates pulsar emission similar to that observed, there are still a number of problems associated with it. Firstly and most significantly, the synchrotron radiation formulæ that were used in the initial runs are incorrect in high magnetic fields (see Chapter 2). Secondly, the simulated intensity profiles were extremely broad, much broader than the intensity profiles that are actually observed. Thirdly, while a number of different pitch angle distribution functions were simulated, there is no guarantee that the physical distribution was among them. Finally, individual best fit parameters did not always agree with one another.

Over the course of this PhD, POREC has been modified and developed in a number of different ways. These include modifications to reduce the number of runs of POREC
required, modifications to change the incoherent synchrotron radiation formulæ used, and a new particle pitch angle distribution function which has been developed.

Keeping the limitations of POREC in mind, the core design will now be covered. The initial design will be laid out, with the changes that have been made over the course of this PhD clearly marked.

3.2 Design

In general, the design differences between POREC and POREC2.0 are only related to the change in the emission mechanism. For this reason, POREC is used here to refer to the code. Unless otherwise mentioned it should be assumed that POREC2.0 has the same design components as POREC itself.

3.2.1 Physical Model

One of the primary advantages to working in the optical domain is the relatively simple physical model that can be used. While there is still no consensus on where in the magnetosphere the emission occurs, there is a general consensus that the emission in the optical regime is due to incoherent synchrotron radiation (see Chapter 1). The primary assumption behind POREC is that the emission from isolated neutron stars in the optical domain is from incoherent synchrotron radiation from a power law spectrum of particles.

For a power law distribution of particles $N(\gamma) = \gamma^{-p}$ the formulæ for incoherent synchrotron radiation were first calculated in relation to Jupiter, which has a magnetic field of approximately one Gauss [34][49][95]. Full details of this formulation are covered in Chapter 2, it is important to note that this model fails in high magnetic fields. A new model was developed in order to overcome this problem. While POREC originally used the WLG model, this has been replaced in POREC2.0 with the DS model (Chapter 2).

The second assumption behind POREC is that the emission can occur from anywhere in the open magnetosphere surrounding a pulsar. While this is extremely unlikely to be correct, allowing the emission to occur from anywhere in the magnetosphere is an excellent method to constrain the various emission theories. For example, if POREC
Figure 3.1: The various stages of the design of POREC. By splitting the design into separate areas, each part of the design can be worked on independently of the whole code. This modular approach means that individual components (e.g. radiation function) are capable of being changed without changing the structure of POREC itself. This figure is reproduced from [63].
POREC predicts that there is no emission seen from the Earth for a particular inclination angle and viewing angle, then it can be concluded that none of the known pulsars are at that particular combination of inclination and viewing angle, therefore a constraint on the emission theories has been found. Of course, there is currently only one pulsar where the polarisation properties in the optical domain are known, reducing the effectiveness of POREC in distinguishing between different emission theories.

The magnetic field surrounding the pulsar will be covered briefly below. While a new model for the magnetic field surrounding a pulsar was published in 2005 [62] this model has not yet been incorporated into POREC. Future work would include incorporating a more realistic magnetic field surrounding the pulsar.

For a rotating dipole, the retarded, or Liénard-Wiechart, potentials are used to generate the magnetic and electric fields. The retarded potentials are given by

\[
\phi(r, t) = \int \frac{\rho(r', t')}{|r - r'|} d^3 r' \tag{3.1}
\]
\[
A(r, t) = \frac{1}{c} \int \frac{j(r', t')}{|r - r'|} d^3 r' \tag{3.2}
\]

where \( \rho(r', t') \) is the charge density, \( j(r', t') \) is the current density and the prime refer to the that variable at the retarded time, where

\[
t' = t - \frac{|r - r'|}{c} \tag{3.3}
\]

The calculation of a magnetic field for a rotating magnetic dipole (of moment \( |m| \)) can then be found by obtaining the field of an electric dipole (of moment \( |p| \)) and using the fact that the Maxwell’s equations are symmetric with respect to the substitutions

\[
E_e \rightarrow -H_m \tag{3.4}
\]
\[
H_e \rightarrow -E_m \tag{3.5}
\]
\[
|p| \rightarrow |m| \tag{3.6}
\]
Briefly, this leads to a magnetic field of

\[
B_m = \frac{3\hat{r} \cdot (\hat{r} \cdot m(t_r)) - m(t_r)}{r^3} - \frac{3\hat{r} \cdot (\hat{r} \cdot \dot{m}(t_r)) - \dot{m}(t_r)}{cr^2} + \frac{\dot{r} \cdot (\hat{r} \cdot \ddot{m}(t_r)) - \ddot{m}(t_r)}{c^2r} \tag{3.7}
\]

\[
E_m = \left(\frac{\dot{m}(t_r)}{cr^2} + \frac{\ddot{m}(t_r)}{c^2r}\right) \times \hat{r}, \quad t_r = t_{ret} \tag{3.8}
\]

which is the magnetic field structure that is in use in POREC [63] (where $ab$ stands for the dot product). Those interested in the derivation, together with the effects of rotation, of this magnetic field are referred to [63].

Once the magnetic field, emission location and process is known, the number of particles emitting at any particular location needs to be set. POREC, rather arbitrarily, sets the number of particles emitting from any location to be the Goldreich-Julian number of particles. While anywhere in the magnetosphere which has the Goldreich-Julian number of particles has enough ions to satisfy Ohm’s Law, and hence stop a parallel electric field from developing, it is a sufficient first approximation to the number of particles that emit. As we are generally interested in the ratio of the emission, and in particular in the percentage polarisation, this does not majorly effect the overall accuracy.

Another unknown is the particles’ pitch angle distribution (PAD). Physically, there are no a priori restrictions on the distribution of the particles in the magnetosphere. There are also no restrictions to the PAD from observations. In principle, the PAD can be of any form - patchy, symmetric, anti-symmetric (about the magnetic field), cone-shaped, etc. In practice the more complex PAD are harder to model, and so the PAD is limited to the simpler models. By treating the PAD as a parameter in the search space to be iterated over, it is hoped that the PAD can also be constrained.

The number of different possible PAD functions is too great for POREC to comprehensively cover them all. Originally, POREC was restricted to four PAD functions, which are (Figure 3.2):

1. Isotropic: the same number of particles at all pitch angles.


3. Circular: Similar to Isotropic but with a drop off at higher angles.
Chapter 3. *POREC*

![Graphs of different PAD functions](image)

**Figure 3.2:** The four original pitch angle distribution functions programmed into *POREC*. Here the cut-off point for each function was 20 degrees (for the Gaussian function this results in particles spread out to 1.5 times the cut-off).

4. Linear: Linearly decreasing number of particles as the pitch angle increases.

These were assumed to be valid up to a maximum pitch angle (or a cut-off angle) \( \text{PAD}_{co} \), after which there were no emitting particles.

When the DS model for emission is used, all of the different PAD functions covered so far predict no emission from the high magnetic fields (Chapter 5). A new PAD was developed where the number of particles increases linearly as the angle increases. Physically this system would arise if the particles at smaller angles emitted a higher proportion of their energy per unit time than particles at higher angles. Having a PAD in this shape is one of the requirements of the DS model, and gives quite different results than the original results from *POREC* (Figure 3.3). When the DS model is used outside of its range of validity, it predicts zero or negative emission!

As mentioned above, in addition to the PAD functions themselves, there was also \( \text{PAD}_{co} \). Above the \( \text{PAD}_{co} \) it was assumed that there were no longer any particles
Figure 3.3: The pitch angle distribution function used with the DS model. This distribution predicts non-zero emission in high magnetic fields.

which were capable of emitting. In the simulations performed for the results presented in this work (Chapter 4) the PAD$_{co}$ angle was varied between 1 degree and 20 degrees. In general, it was found that the PAD function and the PAD$_{co}$ had a smaller effect on the emission than the viewing angle and inclination angle.

Pulsars are not stationary sources, and at high altitudes the emitting location will be travelling at a significant portion of the speed of light (it is assumed that the source is moving with the magnetic field, which is in turn co-rotating with the pulsar). This introduces frame-dragging effects into the emission seen on Earth. POREC has a component which takes this frame-dragging into account. As this was not a primary part of the work described here, this component was never changed. For full details on how the frame-dragging is taken into account the interested reader is invited to peruse [63] and [56].
3.2.2 Computational Design

Once the basic design of POREC had been finalised, it was written in FORTRAN using a Message Passing Interface (MPI) for parallel processing. As each point in the open magnetosphere emits independently of the other points, POREC is an example of an embarrassingly parallel problem. This means that the speed up that can be obtained by increasing the number of processors is theoretically linear.

POREC has two main computational components. Firstly, it calculates whether a point is in the open magnetosphere, the closed magnetosphere, or outside of the magnetosphere entirely. Secondly, for each point in the open magnetosphere, POREC calculates the emission that would be seen on Earth for each viewing angle and phase bin. The total emission is then obtained and the simulated light-curves are outputted (together with the reverse engineering information if need be).

POREC calculates emission from discrete points in the magnetosphere. The volume surrounding the magnetosphere is computationally represented in spherical coordinates, where the parameters that are used to generate the number of radial, poloidal and toroidal points to be used are set on compile time. For the toroidal and poloidal points, the points are populated in discrete angles between 0 and $\pi$, and between 0 and $2\pi$, respectively. The number of radial points is a function of the inner radius of the pulsar (initially assumed to be 10 km) and the outer radius of the area of interest (initially assumed to be 6 times the light cylinder radius of the pulsar). Both the inner radius and outer radius to be used can be set on runtime.

Once the inner and outer radii have been set, the $x$, $y$, and $z$ coordinates of all of the points are found. Each point is checked to see if it lies within the open magnetosphere, which is calculated numerically.

There are two steps to check whether or not a point is in the open magnetosphere.

The $x$ and $y$ coordinates are checked to ensure that the $x$-$y$ distance is not greater than the light-cylinder radius (i.e. the point is checked to see if it is in the magnetosphere or outside the light cylinder). All points outside the magnetosphere can instantly be discarded as there is no emission from those points.
For the remainder of the points, POREC then checks to see if it is in the open or closed magnetosphere. To do this, the field line is followed in both directions until it either leaves the magnetosphere or it once again joins with the pulsar. To follow the field lines, the magnetic field strength and direction at the point of interest are found. The code then takes a small step in the direction of the magnetic field. The magnetic field strength and direction are once again found at this new point, and another step is taken. This is then repeated until the new point found is within the pulsar radius (i.e. the original point is in the closed magnetosphere) or outside the light cylinder radius (i.e. the original point is in the open magnetosphere). If the field line ends at the pulsar, then the direction is reversed, and the entire process starts again from the original point. If the field line going in both directions ends in the pulsar then the point is in the closed magnetosphere. If the field line goes outside the light cylinder at any point then the point is in the open magnetosphere.

Once the points in the open magnetosphere are known, POREC calculates the emission from each point. At this point, there are two modes that can be used in the code. The first simply adds together the emission from each point and outputs the total emission,
discarding the emission seen from each point in the magnetosphere. This makes the code faster to run and takes up less hard drive space. When the results are compared to the observations it is only the total emission that is of interest. In the second mode, the emission from each point in the magnetosphere is recorded and stored. This mode then allows the reverse engineering to be done, as you can recover exactly where in the magnetosphere the emission occurs.

As mentioned, POREC is written using a MPI interface. In order to calculate whether a point is in the open magnetosphere each processor takes a portion of the total points to be checked, without performing any load balancing. Once the status of the emission points is known, the remainder of the code is parallelised using a master-slave paradigm. The master-slave paradigm sets the first processor to be the master processor, which then gives out jobs to each of the other processors (slaves). The master processor keeps track of which points have been calculated, but doesn’t do any of the calculations itself.

The master-slave paradigm has some advantages - it is easier to understand, implement and maintain. It also makes load balancing easier to do. The code is balanced by setting the number of points that are sent to each individual slave to ensure that the master is always sending out points and the slaves are always working. POREC initially had a load balanced such that if the total number of points being calculated was less than thirty five thousand, or if the total number of points divided by the number of slave processes was less than a hundred, the number of points sent to each slave was ten. Otherwise, the number of points sent to each slave was a hundred. This works well for a small number of processors as the slave processors work through the points sent relatively quickly, while the master, who has only to send out a small number of instructions (equal to the number of slaves) is able to keep up with the slaves. However, as more processors are used, this load balancing becomes worse, with the slave waiting for the master to send out the points for longer, while the master has a backlog of instructions to send out.

In order to solve this, the number of points sent to each slave was modified by the number of processors. The more processors there were, the more points were sent out at each instruction. The idea of rewriting the code was briefly explored in order to remove the master node, in an attempt to make load balancing easier. However, it was ultimately decided that attempting to load balance without using a master-slave
paradigm would be extremely difficult and require substantial modifications to the code. The projected benefits would justify the effort that would be needed.

As mentioned previously, there are a number of unknowns involved in pulsar studies. Computationally, there are four different dimensions to the parameter space to be covered. The dimensions are the PAD, the $\text{PAD}_{\text{co}}$, the inclination angle ($\chi$) and the viewing angle ($\eta$). Each will now be considered in turn.

Firstly, POREC originally had four PAD functions, which were

\[
\begin{align*}
\text{Isotropic} & \rightarrow \phi(\theta) = 1 \\
\text{Circular} & \rightarrow \phi(\theta) = \text{abs}(\cos(\theta)) \\
\text{Linear} & \rightarrow \phi(\theta) = 1 - \frac{\sin \theta}{\sin \theta_{\text{max}}} \\
\text{Gaussian} & \rightarrow \phi(\theta) = \exp\left(-\frac{2\theta^2}{\theta_{\text{max}}^2}\right)
\end{align*}
\]

and another PAD function was added for use with POREC2.0 (Chapter 2).

\[
\text{Inverse} \rightarrow \phi(\theta) = \frac{\sin(\theta)}{\sin \theta_{\text{max}}}
\]

Due to the fact that none of the original 4 PAD functions give emission in high magnetic fields (Section 5.1.3), only the Inverse PAD will be considered in POREC2.0. Note that this is not optimal, as the inverse PAD is physically unrealistic. Computationally, the PAD that is to be used is a parameter that is passed into the program. This parameter is set on run-time.

Secondly, there is the $\text{PAD}_{\text{co}}$. The different cut-off limits used here were 1, 2, 5, 10 and 20 degrees, leaving a total of 6 different steps for that dimension of the parameter search space. Once again, this parameter was set at the run-time of the code, as opposed to the compile time.

Thirdly, there is the viewing angle to be considered. POREC was set up such that the emission is calculated for all viewing angles and all phases for a particular inclination angle every time the code is run. To do so, POREC steps through the viewing angles one degree at a time. This parameter is set up on compilation. When POREC calculates the emission at a particular point it calculates the emission that would be seen from
this point for each different viewing angle, and stores this in an array. Once all of the points have been calculated, it adds together all of the emission from each different viewing angle separately, and outputs the emission for each viewing angle to a single file.

Finally, the last parameter of the search space to be considered is the inclination angle. When POREC was first written it was run on the Hamilton supercomputer, where it would take approximately twenty four hours to run for each inclination angle at a resolution of $200^3$ points. By 2011 it was being run on the Stoney and Magma supercomputers in NUI Galway, where a single inclination angle at the same resolution would take approximately 2 hours to run. It is currently being run on the Fionn supercomputer, where a similar job takes approximately 7 minutes. Originally, a new job was created every time that a different inclination angle needed to be checked. This was due to time constraints - for 36 inclination angles, a single job would take over...
a month to complete on Hamilton! One of the first changes made when this project was started was to rewrite the inputs of POREC such that it ran a number of different inclination angles in a single job. The resolution chosen here was 5 degrees, such that it ran at 0 degrees (an aligned rotator), 5 degrees, etc. up to 180 degrees. This resolution was set on compilation.

The only other consideration is the phase of the emission, here set between 0 and 1. POREC calculates the emission for each phase at the same time as it calculates the emission for each viewing angle, and the number of phase steps to be taken is set on compile time. Initially, POREC was compared against the observations made by Smith et al. in 1988 [85], which had 200 phase bins. However, in 2003 new observations were made of the optical emission from the Crab pulsar, which were published in 2009 [84], and as such the newer simulations have 500 phase bins per phase, in agreement with these new observations.

In conclusion, the code works as follows: firstly, the pulsar parameters are set by reading them from an input file (e.g. for the Crab pulsar the pulsar name of ‘Crab’ would be entered). Depending on the pulsar used, other parameters are set which are stored internally in the code (e.g. period, period derivative). Once the pulsar parameters are set, they are sent out to each of the processors so each processor has a local copy. The inclination angle is set to 0 degrees and the points are farmed out in equal sized chunks to all processors to see whether they are in the open magnetosphere. Once the position of all the points has been found, the master processor sends out points to the slave processes, which then calculate the emission and polarisation as seen on Earth. Once these properties have been found for each point, the total emission and polarisation at a particular viewing angle and phase are found by summing the emission/polarisation from each point at that viewing angle and phase. This is then recorded. If an inverse mapping approach is being taken, then position and intensity (and for POREC2.0, the polarisation) of each point is also stored. The inclination angle is then increased by the amount set on compile time, and the entire process begins anew.

This leaves an enormous number of different light-curves and polarisation curves to be compared. For a single pulsar, there are $36 \times 180 \times 5 \times 5 = 162000$ different light curves produced. Each of these light curves needs to be compared against the observations.
3.2.3 Statistical Component

The statistical analysis of the results from POREC is important in cutting down the light curves that are produced. The vast majority of the light curves that are calculated will not agree with the observations at all, and being able to eliminate these light curves from consideration is an important step in the reverse engineering process.

As such, the first part of the project was to develop a statistical method to eliminate light curves. This was done using a $\chi^2$-squared function. In order to do a $\chi^2$-squared test against the observations, the maximum emission was set to be at phase 0. The phase of the observations is arbitrary, and so it is common practice to put the beginning of the phase at the peak of the pulse profile. Next, the data is normalised such that the peak of the emission has a total intensity of one, and the rest of the emission is scaled to this peak. The same is done to the observational data. The difference between the observed emission and the simulated emission is then used for the Chi-squared test.

$$\chi^2 = \frac{(\text{Simulated emission} - \text{Observed Emission})^2}{\text{Observed Emission}}$$  \hspace{1cm} (3.14)

In practice it was found that using a chi-squared function such as this did not give the best matches to the observations. Matches were manually viewed in order to obtain the best fit to observations.

3.2.4 Reverse Engineering

The goal behind POREC is to constrain the emission location for isolated neutron stars. To do so, it is necessary to see where the simulated emission occurs in the open magnetosphere. As such, POREC includes a flag that, when set, will record where in the open magnetosphere the emission occurred. This gives two things for theorists to use. Firstly, the position of the emission can give a lot of information. Knowing that the majority of the emission that POREC sees is from the volume of the magnetosphere corresponding to a particular model would be evidence in favour of that model of emission. This will be further discussed in Chapter 4. Secondly, it gives the relative intensities at different portions of the magnetosphere. This can also be used to constrain the emission theories. Visualisations of the different emission volumes
can possibly allow theorists to understand the different effects that can occur in the magnetosphere more easily.
Chapter 4

Results from POREC and POREC2.0

4.1 Introduction

This chapter has two main components. Firstly, it will cover the results from POREC, and secondly it will cover the results from POREC2.0. POREC’s results were originally published in [56]. This author’s involvement in that publication consisted of the comparison of the intensity to the observed emission from the Smith observations [85]. Unbeknownst to the authors of [56] at the time, there was a serious problem with the underlying mathematical theory [25] (Chapter 2). However, the initial results from POREC will still be detailed, and then compared against the new results from POREC2.0.

As of 2011, there are 12 known optical pulsars [57]. Of these, the Crab pulsar is the brightest, with a magnitude of 16.6 [57], with the next brightest rotation powered pulsar having a magnitude of 22 [57]. While there have been some interesting observations that could be new optical counterparts [59], these new observations are of stars which are also much fainter than the Crab. As a consequence, the Crab pulsar is by far the most studied optical pulsar. To date, it is the only pulsar that has time-resolved optical polarisation measurements. Given that one of the reasons to undertake the simulations was to use the polarisation simulations to constrain the emission theories, the results presented here will be concentrated on the Crab pulsar.
The Crab pulse profile has several distinctive features which any simulation has to be able to reproduce. These include:

1. **Double Peaks:** The Crab pulsar has two peaks, with the second peak located approximately 0.4 in phase after the main peak.

2. **Relative Intensity:** The secondary peak intensity ratio is 0.3 times the intensity ratio of the main peak. Any simulation should be capable of reproducing this ratio.

3. **Bridge emission:** The Crab is seen to have bridge emission. Simulations should be able to reproduce this bridge emission in addition to the double peak structure.

4. **Linear Polarisation:** The linear polarisation of the Crab pulsar has a distinct structure over the course of one rotation, as can be seen in both [85] and [84]. Any simulation should be able to reproduce this.

In addition to these observational constraints, there is also the polarisation angle and the polarisation degree which can be used to distinguish between various simulations.

### 4.2 POREC Results using WLG Model

Initial results for POREC using the WLG model can be found in [56]. A subsection of that paper will be covered here, detailing this author’s contributions. The main contributions to this paper by this author lay in the selection of the best-fitting models to the observations. Intensity checks found that there were a number of best-fitting simulations, dependant on the inclination angle and viewing angle. The amount of overlap in these solutions was reduced by including the linear polarisation results.

The simulated Stokes parameters were found to be dependant on the inclination angle, the viewing angle, the particle pitch angle distribution and the particle pitch angle distribution cut-off ($\alpha$, $\chi$, PAD, PAD$_{\alpha}$), with the maximum effects being from the inclination angle and the viewing angle. For the initial simulations, the phase was split into two hundred bins, and was normalised to between 0 and 1. This was then compared to optical observations which had 200 phase bins over the course of a single rotation [85].
Chapter 4. POREC Results

While these results were based on a faulty underlying premise, it is still interesting to compare how changing the emission formulæ changed the results. Pulsars have dependencies not only on the emission formulæ but also on geographical considerations - which were unchanged between the different versions of POREC. By studying where and how the simulations changed with the introduction of the DS model, it is possible to judge whether effects are coming from geographical considerations, as opposed to being due to the emission itself. Effects that occur due to geographical considerations would be expected to be present at all wavelengths.

4.2.1 PAD and PAD\textsubscript{co}

Initial results found that there were similar trends for the variation in the Stokes parameters for different PAD and PAD\textsubscript{co} \cite{56}. Furthermore, it was found that varying the PAD\textsubscript{co} had a greater effect on the results than the PAD did. In general, the larger the PAD\textsubscript{co} value the broader the resulting pulse peak. Varying the PAD and the PAD\textsubscript{co} had no effect on the phase difference between the peaks (where there was a double peaked structure to be seen). Given this behaviour, the analysis was mainly confined to the isotropic PAD, with the expectation that other PAD would have similar results. The PAD\textsubscript{co} chosen for the analysis was 20°.

4.2.2 Comparison to Observations

The initial simulations from POREC found that in order to get a double peaked structure from the Crab (which is observed to have a double peak structure at all wavelengths) the inclination angle had to be high ($\gtrsim 50^\circ$). At smaller inclination angles the emission had a single peak structure for all viewing angles. This was the first restriction to the parameter space.

The relative intensity of the two peaks was controlled by a combination of the inclination angle and the viewing angle. A double peak with a relative intensity of $\sim 0.3$ is only possible for a small range of viewing angles ($\sim 2.5^\circ$) around a central viewing angle $\chi_0$, where $\chi_0$ was a function of $\alpha$ \cite{56}. It was found that $\chi_0$ decreased steadily as the inclination angle increased. This leads to a second restriction, as only for $\alpha \gtrsim 50^\circ$ and
for viewing angles $\Delta \chi \sim 2.5^\circ$ about $\chi_0 = \chi_0(\alpha)$ were the simulations able to reproduce the observed ratio between the main and secondary peaks of the Crab pulsar.

Bridge emission was also a function of the inclination angle, with a high bridge emission seen at low inclination angles. As the inclination angle increased the amount of bridge emission decreased, with the emission disappearing for orthogonal rotators. For the inclination angle and viewing angle that best fit the intensity and polarisation measurements there was no significant bridge emission.

POREC was unable to reproduce the peak phase separation. This could be due to allowing emission from anywhere in the open magnetosphere, as the inherent symmetry between the primary pole and the secondary pole kept the two peaks offset by 0.5 in phase [56].

### 4.2.3 Inverse Mapping

Once the best fit was obtained, a reverse engineering approach was used to find where in the magnetosphere the emission occurred.

Simulations showed that the double peak structure observed was due to emission being seen from both poles. At lower inclination angles there is only a single pole of emission visible, leading to a broad single peak. At higher inclination angles emission from both poles is seen. As the inclination angle was increased the ratio of emission seen from the secondary pole increases (where secondary pole here refers to the pole which contributes less to the emission seen on Earth over the entire phase).

The dominant emission came from close to the neutron star surface ($\sim 0.2R_{\text{lc}}$). Emission was also seen far out into the magnetosphere, however this emission was much weaker than the emission from close to the neutron star. This is unfortunate, as it means that the dominant emission in previously run scenarios came from the areas with the highest magnetic fields. As such these simulations should not be trusted.

### 4.3 POREC2.0: The DS Model

In POREC2.0, the synchrotron emission formulæ were updated from the WLG model to the DS model. While there are still some problems with the DS model of incoherent
Chapter 4. *POREC Results*

Figure 4.1: Inverse mapping results of the intensity simulations from POREC. Here the emission seen on the right is from the primary peak, while the emission on the left comes from the secondary peak. As can be seen the emission is coming from opposite poles. The blue circles represent the light cylinder, while the green lines give selected LOFL. Figure taken from [56]. Note that the colour here is intensity in units of ergs/cm$^2$s, however as the intensity can be changed by changing the number of particles the absolute intensity is relatively arbitrary. It is only the relative intensity that is of interest. The banding structure seen here is due to the numerics used to calculate the emission points.

synchrotron radiation (Section 5.1), it does solve the issue of the circular polarisation going above 100% in high magnetic fields. The results from POREC2.0 are interesting and could provide insights into the various different theories surrounding pulsar emission. As the main difference between pulsar theories which have emission located in the magnetosphere is in the emission location, POREC2.0’s inverse mapping offers the opportunity to radically constrain emission theories through constraining the various polarisation properties of the emission seen.

Further discussion of POREC2.0’s limitations will be made in Chapter 5 and Chapter 6.

POREC2.0 can compare the results from its simulations to a number of different physical parameters. Currently, the simulations will be compared to the observed intensity and the linear polarisations (Q and U Stokes parameters). POREC2.0 also predicts the circular polarisation, but to date there have been no observations of pulsar circular polarisation. As such the circular polarisation simulations stand as a prediction by the model, to be compared against future observations. While the linear polarisation measurements show that POREC2.0 does not predict the observations, seeing a flip in the sign of the circular polarisation would be an argument in favour of a two-pole emitter model of pulsar emission (Section 4.3.3).
4.3.1 Intensity

In general, POREC2.0 finds a better match for the intensity profile than POREC. Despite this there are still some areas of concern with POREC2.0.

Like POREC, POREC2.0 found that the PAD$_{co}$ did not have a large effect on the simulated observations. As the PAD$_{co}$ was decreased, the range of inclination and viewing angles for which an appropriate secondary peak was seen decreased. The central value of the viewing angle at which an appropriate secondary peak was seen
POREC2.0 found, in agreement with POREC, that the double peaked structure of the crab could be reproduced (within the limits already mentioned) for a small range of values of the viewing angle $\chi$ around a critical value of the viewing angle $\chi_0$, provided that the inclination angle was between approximately 50° and 140°. As the inclination angle increased, the viewing angle decreased from approximately 90° at an inclination angle of 50° to approximately 25° at an inclination angle of 140°.

As can be seen from Figure 4.2 and Figure 4.4, the width of the pulse peaks increases as the inclination angle increases. Lower inclination angles give a better fit to the intensity observations in terms of the width of the pulse profile. However, inclination angles which have narrower pulse profiles do not produce significant off-peak or bridge emission.

As can also be seen in Figure 4.2, the simulated bridge emission is not smooth at lower inclination angles.
Chapter 4. POREC Results

Figure 4.4: For higher inclination angles, a lower viewing angle was required in order to see the double peaked structure of the Crab pulsar. At inclination angles higher than 140° the secondary peak was always greater than 0.3 times the primary peak. Here the inclination angle is 140°.

The intensities predicted by POREC2.0 are a much better fit to the observations than those seen by POREC. Despite this there are a number of issues with POREC2.0. There are two main problems to be considered. Firstly, the secondary peak is exactly 0.5 out of phase with the primary peak, regardless of different parameters used. This is in disagreement with the observations of the Crab pulsar, where the secondary peak is approximately 0.4 out of phase with the primary peak. This could be due to the inherent symmetry in allowing emission to occur from anywhere in the open magnetosphere.

Secondly, there is the matter of the off-peak emission. For narrow peaks such as that seen in the Crab pulsar, POREC2.0 predicts essentially no off-peak emission. Again,
this is not in agreement with the observations. One explanation for this could be that the off-peak emission is from a thermal source, or is emission seen from the synchrotron knot beside the pulsar, rather than intrinsic synchrotron emission from the Crab itself.

### 4.3.2 Linear Polarisation Comparisons

No combination of inclination angle/viewing angle were found which matched the observations. This suggests that either the PAD should be changed, or that the emission locations should be constrained. This will be further discussed in Section 4.3.4.

In general, the degree of linear polarisation predicted by POREC2.0 was much higher than that observed, with the simulated %Q and %U frequently going above 20%. The maximum observed Q and U is of the order of 6%.

### 4.3.3 Circular Polarisation

One thing to note from POREC2.0 is that any intensity profile with a double peak has a circular polarisation which flips sign over the course of the phase. This is due to the emission for the secondary peak coming from the secondary pole of the pulsar. POREC2.0 also predicts a distinctive circular polarisation (Figure 4.6).

### 4.3.4 Reverse Engineering

Note that the images chosen in this section are for display purposes only - the inclination and viewing angles chosen will not always reproduce the emission seen from the Crab. The images are chosen to give a general idea of the overall behaviour of the different components over all inclination and viewing angles.

While no combination of the various inclination angles, viewing angles, PAD and PAD$_{co}$ agree with the observations, it is still instructive to look at the emission locations given by POREC2.0. In particular, there are two main types of emission seen. The first, which will be called the slot gap component in this work, is the emission that is seen from around the LOFL. Secondly, there is emission from far out in the magnetosphere (typically greater than 200 kilometers in the $z$ direction). Here this will be called the outer gap component. The border between these two components is relatively
Figure 4.5: Q vs U plots predicted by POREC2.0 do not agree with observations. Here an example of one of the predicted QU plots is shown. No combination of inclination angle/viewing angle was found which predicted the correct shape to the QU plot. In general, the amount of linear polarisation predicted was too high. Here an inclination angle of 120° was used. The black dots on this diagram are the observation of the linear polarisation from the Crab pulsar [84].

fluid, with the outer gap component merging smoothly into the slot-gap component over the course of the pulsar phase, and vice versa. The relative intensity of the slot gap component is much higher than that of the outer gap component, and the two components have different levels of circular and linear polarisation. Crucially, the circular polarisation for each component is a different sign, suggesting that measuring the circular polarisation seen by the Crab could offer insights into where the emission occurs.

In addition, the inclination angle and viewing angle has a stronger effect on the slot gap
Figure 4.6: For any inclination angle and viewing angle where a secondary peak is seen, there is a flip in the sign of the circular polarisation as the secondary peak emission becomes dominant. Here an inclination angle of 90° was used.

component than it does on the outer gap component. The slot gap component is not visible at all inclination angles and viewing angles, however the outer gap component is always visible.

Pulsar phase has, in general, a much stronger influence on the slot gap component than on the outer gap component. Even at those inclination and viewing angles at which the slot gap component is visible, there are phase periods where the emission will not be visible. In contrast, while the emission from close to the magnetosphere can disappear for the outer gap component, the emission from far out in the magnetosphere is in general always visible. However, this emission is orders of magnitude weaker than the emission from close to the pulsar. Thus the weak outer gap component of the emission could be an explanation for the off-peak or bridge emission that is seen.
Figure 4.7: Here, the emission can be broadly split into two main components. There is the component from close to the LOFL, which is brighter, and a component from far out in the magnetosphere. These two components have markedly different linear polarisations. Here the inclination angle is 70° with the viewing angle of 94°, at a phase of 0.15 [14]. Here the intensity given is in units of ergs/s. This is relatively arbitrary, as the total intensity can be manipulated by changing the number of emitting particles.

4.3.4.1 Slot Gap Polarisation Properties

The polarisation of emission from the slot-gap category is quantitatively different to that from the outer gap category. The linear polarisation of the emission from the area surrounding the LOFL falls into bands, where these bands are at approximately constant radial distances around the pulsar. These bands exhibit a range of different polarisation going from approximately 60% positively polarised to approximately 60% negatively polarised (Figure 4.9). Both the U and Q polarisation is in these bands, however the band of U polarisation and Q polarisation do not in general overlap.

The circular polarisation of the emission from the slot-gap model is in general a different sign than the emission from the outer gap component.
Chapter 4. POREC Results

4.3.4.2 Outer Gap Category Linear Polarisation

Emission from the outer gap is polarised differently to that from the slot-gap category. The outer gap emission stretches from relatively close to the neutron star’s surface to far out in the magnetosphere. While the polarisation is still banded, now the bands are in the \( z \) direction, as opposed to being at a constant radial distance from the neutron star (Figure 4.10).

As with the slot-gap component, the linear polarisation of these bands change from being approximately 60% positively polarised to 60% negatively polarised. Again the Q band and U band do not in general overlap.

Figure 4.8: Here, the two different components share markedly different circular polarisation properties. Here an inclination angle of 70° and a viewing angle of 94° is used, at a phase of 0.5 [14].
Figure 4.9: Here the linear polarisation of the slot gap component and the outer gap component are markedly different, though both show a banded structure. Here the inclination angle is 70°, the viewing angle is 94° and the phase is 0.5 [14].

Figure 4.10: As can be seen here, the linear polarisation values are in bands stretching from near the pulsar to far out in the magnetosphere. Here an inclination angle of 70° and a viewing angle of 10° is used, at an initial phase of 0 [14].
Chapter 4. **POREC Results**

![Figure 4.11: The amount of linear polarisation at any particular point does not remain constant over the complete phase. Instead, the bands of polarised emission move around. Here an inclination angle of 70° and a viewing angle of 10° were chosen to illustrate this. In this particular case the second half of the phase is approximately the same as the first half, but this is not the case in general [14].](image)

These bands travel around as the phase changes. The linear polarisation of any point will change over as the phase changes, but the band of linear polarisation will remain (Figure 4.11, Figure 4.12).

The intensity of the emission decreases with radial distance from the neutron star, regardless of whether the emission is coming from the outer gap or the slot-gap component.
Figure 4.12: The percentage values for U over the course of a single phase. As can be seen the banded structures move around over the course of a phase. Here an inclination angle of 70° and a viewing angle of 10° is used [14].
Chapter 5

Discussion

The Westfold, Legg and Gleeson (WLG) model of incoherent synchrotron radiation fails in the optical domain in high magnetic fields. WLG’s formulation was before the discovery of pulsars and the potential of magnetic fields greater than $10^6$ G. The original intent in developing the de Burca & Shearer model was to address this problem, and then to apply the new model developed to pulsars, which are dominated, at least close to the pulsar, by extremely high magnetic fields at which the WLG model fails. By this metric, it has been a success, with the Pulsar Optical Reverse Engineering Code having been updated to include the DS model. Specifically we show that the peak circular polarisation is about 20% at $10^6$ G as distinct from a circular polarisation which goes up linearly with magnetic field under the WLG model and consequently becomes greater than 100%. We also show that the linear polarisation goes through a small, but rapid change, at the same magnetic field. This has allowed us to examine

5.1 Problems with the DS Formulation

There are a number of issues with the DS formulation for synchrotron emission. Firstly, this formulation puts unrealistic restraints on the particle PAD. Secondly, the expansion of the PAD to an odd order ($\theta^3, \theta^5$, etc.) will likely once again cause the circular polarised component to go over one hundred percent. And finally, as with the WL model, the energy range of the particles used in the DS model goes between 0 and $\infty$, which is clearly an unrealistic scenario. Each of these problems will now be discussed.
5.1.1 Integrating with Respect to Infinity

Westfold, Legg and Gleeson chose to correct the integration between 0 and \( \infty \) using correction factors - a set of equations that are applied to the underlying model to correct for the changes due to the finite integration limits. However, with the DS formulation, there are two different functions which are integrated between 0 and \( \infty \). This makes these correction factors much more complicated. However, it is still possible to follow the same formulation as the WLG model in two different ways.

Firstly, it is possible to apply the correction factors to the integrations as the separate Stokes parameters are calculated. In this method a different correction factor would be obtained for each of the different integrations \( J_n, L_n, R_n \) and \( Q_n \). If these correction factors are called \( C_i(x_1, x_2) \), where \( i \) goes between 1 and 4 (with \( x_1 \) and \( x_2 \) as before (Chapter 2)) then the polarisation tensor would become

\[
n_{x_{11}} = \frac{1}{4\sqrt{2}} \mu_e^2 c f B_0 \left( \frac{3}{2} \right)^{\frac{p}{2}} \sin \theta^{{\frac{p+1}{2}}} f^{1-\frac{p}{2}} \left[ \phi(\alpha) \left( J_{\frac{p+1}{2}} C^1(x_1, x_2) + L_{\frac{p+1}{2}} C^2(x_1, x_2) \right) \right] \\
+ \frac{3h(\alpha)}{2} \left( \frac{f B_0}{f} \right) \left( 3Q_{\frac{p+3}{2}} C^4(x_1, x_2) - 2L_{\frac{p+3}{2}} C^2(x_1, x_2) - J_{\frac{p+3}{2}} C^1(x_1, x_2) \right)
\]

with analogous expressions for \( n_{x_{12}} \) and \( n_{x_{22}} \). This is not an elegant solution, and does not allow the emission–polarisation tensor and the correction factors to be calculated separately, which is the main benefit of using correction factors. As such, this method is not recommended.

The second method that could be used is to devise a single correction function for each Stokes parameter that incorporates the corrections due to all of the integrations over \( E \), rather than splitting the correction for each integration into its own correction function. While this does have the benefit of allowing the emission and the correction functions to be calculated separately, the correction functions would be extremely complicated. Each of the correction functions would need to take into account not only the differences arising from the integration between 0 and \( \infty \) but also the differences from the extra factors of the frequency, magnetic field etc. arising in the emission–polarisation tensor.

While this author has not attempted to calculate these correction functions to date, it
seems likely that finding them would require a significant time investment. As such, this is not the method that would be advised.

In fact the method that this author would recommend would be to take the integrations between $E_1$ and $E_2$ into account explicitly as the emission-polarisation tensor is calculated. The integration of Bessel functions leads to solutions in terms of a hypergeometric function, which is defined as

$$\text{Hypergeom}([a_1, a_2, a_3, \ldots, a_n], [b_1, b_2, b_3, \ldots, b_m], d) = \sum_{k=0}^{\infty} \frac{d^k}{k!} \prod_{i=1}^{n} \text{Pochhammer}(a_i, k) \prod_{i=1}^{m} \text{Pochhammer}(b_i, k)$$  \hspace{1cm} (5.2)

where

$$\text{Pochhammer}(a, b) = \frac{\Gamma(a + b)}{\Gamma(a)}$$ \hspace{1cm} (5.3)

and

$$\Gamma(n) = \int_{0}^{\infty} x^{n-1} e^{-x} dx$$ \hspace{1cm} (5.4)

Using this it is possible to calculate the integration of the Bessel functions between $E_1$ and $E_2$ explicitly. Note that these expressions have been calculated for $J_n$, $L_n$ and $R_n$ in [34].

5.1.2 Expanding the Particle PAD Function to the Next Order

Expanding the particle PAD to the third order in $\theta$ without expanding the velocity beyond the first order means that the results can not be trusted - as the errors that occur due to the expansion of the velocity will dominate. However, to illustrate a flaw in the DS theory, the effects of expanding the PAD to the third order in $\theta$ will now be considered.

Ultimately, the reason for the problems in the expansion of the PAD comes from the integration over $\theta$ - in particular from the assumption that the integration goes between 0 and $\infty$. When it is assumed that the integration goes between 0 and $\infty$ it can be shown that the integration of odd functions disappear. This means that the difference
in going from the expansion of the PAD from a even order in \( \theta \) to an odd order in \( \theta \) will only effect \( n_{x12} \) - changing the circular polarisation only.

In order to illustrate this it is important to trace how the various variables are substituted over the course of the integrations. In particular, the variable of interest is \( \theta \), and the relationship that this has to the energy, and consequently to the magnetic field and the dimensionless quantity \( x \).

Expanding the particle pitch angle distribution to the next order gives a representation of \( \phi(\theta + \alpha') \) as follows

\[
\phi(\alpha' + \theta) \sin(\alpha' + \theta) = f'(\alpha') + g'(\alpha')\theta + h'(\alpha')\theta^2 + i'(\alpha')\theta^3 \tag{5.5}
\]

where \( f'(\alpha') \), \( g'(\alpha') \), \( h'(\alpha') \) and \( i'(\alpha') \) are all some function of \( \alpha' \). The important part to note is the various powers of \( \theta \).

When solving the integral over \( \theta \), there are a number of different substitutions. The first substitution of interest here is

\[
\theta = \frac{1}{\gamma} R \sin(\phi) \tag{5.6}
\]

Due to this substitution, each power of \( \theta \) corresponds to an inverse power of \( \gamma \). \( \gamma \) is then related to \( x \), the parameter that is actually used to solve these integrals, through the second substitution

\[
\gamma = \left( \frac{A'}{x} \right)^\frac{1}{2}
\]

where

\[
A' = \frac{2}{3} \frac{f}{f_{B_0} \sin \alpha} \tag{5.7}
\]

which leads, in conclusion, that the power of the magnetic field in the formulae for the emission-polarisation tensor is related to the power of \( \theta \) in the integrals, as well as being related to the other values of \( x \) which arise over the course of the integrals. These other values of \( x \) are the same for all of the different components of the emission-polarisation tensor. The article of interest here is the circular polarisation (as this becomes unrealistic when the PAD is expanded to the first order in \( \theta \)). As such, it is only the relative power (between \( I \) and \( V \)) that is of interest. The relationship between
the magnetic field and the circular polarisation is set by the various powers of \( \theta \) in the integrals in \( I \) and \( V \).

When \( \theta \) is expanded to an odd power, there is an extra power of \( \theta \) in \( V \) which leads to a higher power of the magnetic field in \( V \) than there is in \( I \). As such, when the ratio is taken (and it is assumed that the highest powers of \( B \) dominate - i.e. this only concerns emission from high magnetic fields) the ratio for the circular polarisation will be

\[
\frac{V}{I} \propto B^{\frac{1}{2}}
\]

which at high magnetic fields will lead to the circular polarisation being greater than 100%.

If the PAD is expanded to even powers of \( \theta \), then there will be an extra factor of \( B \) in \( I \) which does not occur in \( V \). This will in turn lead to a circular polarisation of (again assuming that the largest powers of \( B \) dominate)

\[
\frac{V}{I} \propto B^{-\frac{1}{2}}
\]

and the circular polarisation will disappear in high magnetic fields. Obviously this is non-ideal.

Now it is important to note that in order to prove that this is valid in the magnetic field ranges of interest (10^6-10^{12} G) it is necessary that the mathematics actually be carried out, which would be extremely challenging. Each expansion of anything in \( \theta \) leads to exponential growth in the number of integrations over \( \theta \) to be carried out. However, it could be possible with the aid of computer programs.

It is possible that rather than attempting to expand the particle PAD the integration range of \( \theta \) should be looked at. The assumption behind expanding the integration to being over infinity is as follows: From physical arguments it is possible to show that the particle emission will only be seen for a small angle proportional to \( 1/\gamma \), where \( \gamma \) is the Lorentz factor. For synchrotron radiation, the basic assumption is that the particle is travelling at close to the speed of light - i.e. that the \( \gamma \) factor is high (frequently assumed to be of the order of 10^7 for pulsars). When the total emission seen over the course of a single rotation is taken, the integration over \( \theta \) should go between 0 and \( 2\pi \). In comparison to these limits, the range at which the integration is non-zero is very
small, and as such the integration can be taken between 0 and ∞ without much error occurring.

That said, the fact that the expansion of the PAD for a power law of particles causes the circular polarisation to be extremely large ($\gtrsim 10^6$) for odd expansion and very small ($< 10^{-2}$) for even expansions in high magnetic fields gives a reason to question this assumption.

In addition, it is found that in high magnetic fields, expanding to the next order of $\theta$ has a large effect on the simulated emission seen. This seems physically unlikely. $\theta$ is extremely small, and so it should be assumed that higher orders of $\theta$ will have smaller effects on the results than lower orders of $\theta$. This is not the behaviour seen in the DS model.

### 5.1.3 Restraints on the Particle PAD Function

The DS formulation does stop the circular polarisation going over 100% in high magnetic fields, but it does so at a high cost. The necessity of having $h(\alpha)$ be positive is a strong constraint on the different particle PAD that can be used. Furthermore there is no physical reason for this restriction, it falls directly from the mathematics used. This could be a sign that it is a mathematical construct rather than a physical barrier - in much the same way that there are mathematical singularities at the Schwartzchild radius of a black hole which disappear when the right coordinate system is used.

There are four particle PAD that are used in POREC)(Section 3). These are the Isotropic PAD, the Circular PAD, the Linear PAD, and the Guassian PAD. Each of these fails to meet the requirements that $h(\alpha)$ be positive. Each will now be examined turn.

The isotropic PAD is simply

\[
\phi(\theta) = \begin{cases} 
1 & \theta_{\min} < \theta < \theta_{\max} \\
0 & \theta_{\min} > \theta > \theta_{\max} 
\end{cases} 
\]  

(5.9)
which leads to a first derivative of 0 and a second derivative of 0. This in turn leads to

\[ h(\alpha) = 0 + 0 - 1 \sin(\theta) \]  

(5.10)

i.e. \( h(\alpha) \) is greater than zero only when \( \sin(\theta) \) is less than than zero. This corresponds to \( \theta \) greater than 180°, which is much too large for the purposes of calculating synchrotron radiation (remember that \( \theta \) is going to be of the order of \( 1/\gamma \)).

The next PAD to look at is the linear PAD, which is defined as

\[ \phi(\theta) = 1 - \frac{\sin \theta}{\sin \theta_{\max}} \]  

(5.11)

This has first and second derivatives

\[ \phi'(\theta) = -\frac{\cos \theta}{\sin \theta_{\max}} \]  

(5.12)

\[ \phi''(\theta) = -\frac{\sin \theta}{\sin ^2 \theta_{\max}} \]  

(5.13)

which in turn leaves

\[ h(\alpha) = \frac{\sin ^2 \theta}{\sin \theta_{\max}} - 2 \frac{\cos ^2 \theta}{\sin \theta_{\max}} - \sin \theta + \frac{\sin ^2 \theta}{\sin \theta_{\max}} \]  

(5.14)

\[ = -2 \frac{\cos(2\theta)}{\sin \theta_{\max}} - \sin \theta \]  

(5.15)

which is clearly less than 0 in the range of interest. So this PAD also fails.

Thirdly, there is the circular PAD, which is

\[ \phi(\theta) = \text{abs}(\cos(\theta)) \]  

(5.16)

This has first and second derivative

\[ \phi'(\theta) = -\frac{\sin \theta \cos \theta}{|\cos \theta|} \]  

(5.17)

\[ \phi''(\theta) = -\frac{\cos ^4 \theta}{|\cos ^3 \theta|} \]  

(5.18)

which leaves \( h(\alpha) \) as clearly negative. As such, this PAD does not satisfy the requirements.
The fourth PAD is a Gaussian, which has a general formula of

$$\phi(\theta) = a \exp\left(-\frac{(\theta - b)^2}{2c^2}\right)$$ (5.19)

where $a$ controls the height, $b$ controls the position of the centre of the Gaussian, and $c$ controls the width of the Gaussian. The first and second derivatives are then

$$\phi'(\theta) = \left(\frac{b - \theta}{c^2}\right) \phi(\theta)$$ (5.20)

$$\phi''(\theta) = \frac{d}{d\theta}\left(\frac{b - \theta}{c^2}\phi(\theta)\right) = \left(\frac{(b - \theta)^2}{c^4} - \frac{1}{c^2}\right) \phi(\theta)$$ (5.21)

which in turn gives

$$h(\alpha) = \frac{1}{c^4} \left(\left[(b - \theta)^2 - c^2\right] \sin \theta - 2c^2(b - \theta) \cos \theta - c^2 \sin \theta\right) \phi(\theta)$$ (5.22)

Now consider that the PAD is positive (as a negative PAD makes no physical sense), and $c^4$ is also positive. As such you can multiply by $c^4/\phi(\theta)$ to get a new function which has to be greater than zero, $h'(\alpha)$. This function is

$$h'(\alpha) = \left[(b - \theta)^2 - 2c^2\right] \sin \theta - 2c^2(b - \theta) \cos \theta$$ (5.23)

Now consider the midpoint, at which $\theta = b$. At this point, $h'(\alpha)$ is negative provided that $\theta < \pi$. There will also be a small region surrounding the midpoint where $h(\alpha)$ will also be negative. As such, the Gaussian distribution is not one that can be used.

Finally, there is a PAD that does not fail the requirements. This is the inverse PAD, which is defined as

$$\phi(\theta) = \frac{\sin(\theta)}{\sin \theta_{\text{max}}}$$ (5.24)

which has a first and second derivative of

$$\phi'(\theta) = \frac{\cos \theta}{\sin \theta_{\text{max}}}$$ (5.25)

$$\phi''(\theta) = -\frac{\sin \theta}{\sin \theta_{\text{max}}}$$ (5.26)
which leaves

\[ h(\alpha) = \frac{2}{\sin \theta_{\text{max}}} \left( \cos^2 \theta - \sin^2 \theta \right) \]  \hspace{1cm} (5.27)

which is positive provided that \( \theta \) is less than 45°. There is some concern that the primary reason for choosing this PAD was due to mathematical constraints, rather than a physical reason.

With that in mind, now consider the results of POREC2.0 on the different emission theories.

5.2 POREC2.0 Results and the Application to Current Pulsar Theories

Unfortunately, the results from POREC2.0 are not enough to distinguish between the different pulsar emission theories at the moment. There are several aspects of the results that can be pointed to as evidence for one theory or another.

There are currently no theories in the literature that can explain all of the observations from the Crab pulsar [84], although there are some theories that can explain certain aspects [89][96].

5.2.1 Explaining the Polarisation Results?

POREC2.0 was unable to reproduce the distinctive polarisation curve that is seen from the Crab pulsar. Despite this, POREC2.0 offers clues as to how the polarisation results occur.

Consider the Q component. In the observations, this component is essentially non-existent except for the two pulse peaks. During the primary peak the Q component initially decreases to approximately minus 6%. When the pulse peak reaches approximately its FWHM the Q component switches sign and decreases in magnitude to less than 2% (Figure 5.1) as the pulse peak is reached. The polarisation then decreases again to the off peak value as the peaked emission disappears. Now consider the secondary pulse. Here the polarisation does not change sign. The Q component peaks
Figure 5.1: Here the behaviour of the linear polarisation modes are compared against phase. As can be seen, for the majority of the pulse there is no linear polarisation to be seen. During the primary peak, the U component increases until the peak reaches approximately its FWHM. At that point the polarisation switches sign and increases in magnitude. The Q component, on the other hand, behaves in the opposite manner, initially being large and negative, and becoming smaller and switching sign in the middle of the pulse peak. In the inter-pulse, the Q component peaks before the pulse peak, while the U component peaks after the pulse peak. Here the observations used were taken by [84].

approximately half-way through the pulse. There is a faint sign of the polarisation switching sign, but before this happens the pulse ends and the polarisation returns to the off-peak polarisation. When the U component is considered, the same approximate behaviour is seen. Here the U component starts small and positive, peaking slightly after the Q component in phase. Again, it switches sign, with the negative U peaking at the pulse peak. It then decreases to approximately its off-peak value as the peak emission ceases. In the secondary peak, the U component peaks as the pulse does, and then returns to the off-pulse percentage as the pulse emission ceases to be seen.

This is naturally explained by the banded structures that are seen for the polarisation in POREC2.0. Here firstly the pulse has a band of highly negatively Q polarised emission. As it crosses into a band of highly positive Q polarised emission the polarisation switches sign. The magnitude of the positive polarisation being less than that of the negative can be explained either through the negative band being still visible or the band of positively polarised Q emission being less polarised than that of the negatively polarised emission. The same can be seen in the secondary peak, only now the full peak is no longer visible and as such the polarisation does not switch sign, as the positively polarised emission
sites are never visible. In a similar argument, the Q emission starts with a highly positively polarised component and moves to a negatively polarised component.

It should be noted here that some of this polarisation occurs due to the rotation of the pulsar, rather than due to any intrinsic effects from the synchrotron radiation itself. As such it is expected that the same polarisation properties will be visible for higher energy emission as well. Theoretically the same polarisation properties should be visible for the radio emission, but studying that is beyond the scope of this work.

5.2.2 POREC2.0 and the Slot Gap Model

POREC2.0 predicted that for a range of different viewing and inclination angles, the slot gap component of the emission naturally occurs without any a priori assumptions. As the emission from the slot gap component is in general stronger than the emission from the outer gap component (due to a weaker magnetic field for the outer gap component) the slot gap component of the emission dominated when it was present.

POREC2.0 found that when emission arose from the area surrounding the LOFL, the emission was not consistently visible over the course of the entire pulse. In fact, there were significant parts of the pulse where the emission was not visible at all. As such, POREC2.0’s slot gap component was unable to explain the off-pulse emission.

It should be noted that the width of the emission seen in POREC2.0 is much bigger than that predicted by the slot gap model. The width of the slot gap model follows the formula

\[ \Lambda = PB_{0,12}^{-\frac{1}{2}} \]  

(5.28)

where \( P \) is the pulsar period and \( B_{0,12} \) is the surface magnetic field in units of \( 10^{12} \) G. The resolution of the emission from POREC and POREC2.0 is of the order of 8000 kilometers/250 - orders of magnitude larger than that predicted by the slot gap model.

5.2.3 POREC2.0 and the Outer Gap Model

POREC2.0 also showed that emission naturally arose for an outer gap component. This emission is qualitatively different to that from the slot gap component in terms of
the polarisation. The emission from the outer gap component was visible over the full course of the pulsar rotation, albeit with much different intensities. As such, the outer gap component is plausibly able to reproduce the off-peak emission. The emission was in general weak, with POREC2.0 predicting much stronger emission from close to the pulsar (in agreement with POREC [56]).

In conclusion, POREC2.0 does not offer any evidence for one model over the other.

5.2.4 Pulsars as Two-pole Emitters?

POREC2.0 strongly supports the two-pole model of pulsar emission. In this model, the secondary peak of the pulse profile is due to the emission from the secondary pole being visible, as opposed to the emission being due to a cone-like structure. However, due to POREC2.0 allowing emission from all points in the magnetosphere it is unlikely that a cone structure could occur. Therefore this can not be used as an argument against models with a cone shaped emission structure.

One method to distinguish between a two-pole emitter and a cone-like structure would be to measure the circular polarisation. For any cone-like structure the circular polarisation would be polarised in the same direction - in other words, it would be expected that the circular polarisation would have the same sign over the course of the entire pulse. For a two-pole emitter this is no longer the case, with the secondary pole expected to have circular polarisation in the opposite direction to the first pole (this being due to the rotation of the pulsar). Measurement of circular polarisation of opposite signs would therefore be strong evidence that the emission is coming from separate poles.

For all of the viewing angles and inclination angles at which POREC2.0 found intensity profiles which matched the observations, the circular polarisation changed sign.

5.3 Computational Artefacts?

One theory as to the reason for the banding structure that is evident in the linear polarisation is that it is due to a computational artefact. This is tested in two different ways.
Chapter 5. Discussion

Figure 5.2: A comparison between runs where the distance between the points was changed. In the top image on the left, the full resolution of 800,000 points was used. In the image in the centre the \( \phi \) distance between the points was doubled, whereas in the image on the bottom the \( \theta \) distance between the points was doubled. As can be seen changing the resolution of the points used had little effect on the banding structure that is seen [14]. Here the left row shows the \( \%Q \) while the right row shows the \( \%U \) seen.
Chapter 5. *Discussion*

**Figure 5.3:** On the left is the simulated emission seen. On the right is the simulated random values sent through the computational pipeline. As can be seen, there is still a small amount of patterning to be seen on the random emission, however, it is nowhere near the patterns that are seen in the actual simulated emission. Here the inclination angle is 70°, the viewing angle is 86°. The phase is at 0.14 [14].

Firstly, it is tested to see whether or not this is an effect of the resolution. If this was due to the resolution used it is expected that changing the resolution will change the banded structures that are seen - either making them bigger or smaller. When the distance between the emission points is doubled, the size of the bands would correspondingly double. POREC2.0 defines distances between points in terms of three spherical parameters, $r$, $\theta$, and $\phi$. In order to test to see if this banding was due to a computational artefact, the code was run twice more, firstly doubling the $\theta$ distances between the points, and secondly doubling the $\phi$ distance between the points. The $r$ distance was not checked as the banding structure is clearly not related to the radial
distance of the points. This did not substantially change the results (Figure 5.2).

Another way to check for computational artefacts is to send random numbers through the computational pipeline. In order to obtain the emission there are a number of steps taken:

1. The polarisation-tensor is calculated in the frame co-rotating with the pulsar.
2. The frame-transformations are applied.
3. Phase corrections are applied so that the emission as seen on Earth is being calculated.

In order to test where the pattern from emission occurred, random values were substituted into the pipeline instead of the originally calculated polarisation tensor. The resultant emission did show some patterns (Figure 5.3), in particular, the zero polarisation values appear to be set by some process other than the calculation of the emission (this is likely to be due to the frame-transformations). Despite this, it can be clearly seen that the patterned behaviour previously simulated has disappeared, suggesting that this is not due to a computational artefact.

When the code is tested without any frame transformations the amount of linear Q polarisation is steady at approximately 60%, while the U polarisation is obviously at 0%. This suggests that the banding effect is partially due to the frame transformations and partially due to the underlying emission.

### 5.4 Relativistic Rotating Vector Model

It is interesting to look at the relativistic rotating vector model (RRVM) and compare it to POREC2.0. POREC2.0 does not use the RRVM, but the results share a certain symmetry with the model. In the RRVM the relativistic rotation is essentially responsible for the observed polarisation of the Crab pulsar [28]. The results from POREC2.0 also suggest that the polarisation is due to the frame-transformation effects - in other words from the relativistic motion of the source.
Chapter 5. Discussion

While the RRVM and POREC2.0 do agree on the cause of the polarisation, the RRVM is mainly concerned with the polarisation angle - something which was not looked at in POREC2.0 (as it was clear that the polarisation results did not match the observations).

5.5 Conclusions

POREC2.0 offers evidence for pulsars being two-pole emitters. POREC2.0 is currently unable to offer evidence for or against the slot gap or the outer gap model, the two most widely held models for pulsar emission. POREC2.0 does offer evidence that it is possible to naturally obtain the double peaked structure of the Crab by emission from within the magnetosphere, however this was already widely known.

POREC2.0 was originally designed with the idea of using a reverse engineering tactic to limit the emission areas possible, and thereby to limit the models of pulsar emission. Unfortunately due to the large amount of unknown physical quantities (i.e. PAD, PAD$_{co}$, inclination angle, viewing angle), POREC2.0 was unable to constrain any emission models. The same output can be seen from emission from different areas of the magnetosphere for different inclination angles and viewing angles (in mathematical terms, the emission seen is degenerate), which means that a reverse engineering approach will not be able to limit the emission locations until the unknown physical parameters have been discovered.

There is still some concern over the effect that the unrealistic PAD had upon the results seen. However, this author believes that the main results that are being considered from POREC (namely the banding structure of the polarisation in the pulsar magnetosphere) would not be effected by the unrealistic PAD used.
Chapter 6

Future Work

6.1 POREC2.0

The original idea behind POREC and POREC2.0 was a simple one. As previously mentioned, POREC2.0 attempts to constrain the emission locations in the pulsar magnetosphere. Once a constraint on the emission location in the pulsar magnetosphere is found, this can be used to constrain the emission theories themselves, which in general have very different emission locations.

It was originally decided to simulate emission in the optical domain because there is only one emission process in the optical regime, incoherent synchrotron radiation (see Chapter 1). This significantly simplifies simulations, but at the cost of having comparatively few observations to compare simulations against. While there are a large number of radio and $\gamma$-ray pulsars known, to date there are only 12 known optical pulsars [57]. Of those, only one pulsar, the Crab pulsar, has had any polarisation observations [84][85], and even for the Crab pulsar there have been no circular polarisation observations to date.

POREC2.0 attempts to calculate the emission with as few assumptions as possible. Rather than assume a particular value for an unknown, POREC2.0 instead makes that unknown a dimension in the parameter search space. This lead to a large search space to be considered.
It was found that the emission seen is degenerate - it is possible to get the same output from different emission areas, provided the other unknown quantities (e.g. inclination angle) are also changed. As such, POREC2.0 is a failure in its stated goal, it cannot distinguish between different emission locations without better knowledge of the underlying physical parameters of the pulsar being simulated.

However, it is still tempting to try to distinguish between the different emission theories using a reverse engineering approach. To date, no theory is fully able to explain all of the emission seen from pulsars, and any tool to distinguish between the different theories should be fully examined before being discarded as unhelpful.

With this in mind, there are a number of different areas in which POREC2.0 can be helpful. In Section 5.2.1 it was shown that the polarisation of the emission seen followed certain bands (partially due to the frame-transformation effects). It is theoretically possible to use these bands, together with the polarisation observations, to restrict the location of the pulsar emission.

Some of the ways in which POREC2.0 could be improved will now be detailed.

### 6.1.1 Emission location

POREC2.0 currently allows emission to occur anywhere in the open magnetosphere, a clearly non-physical arrangement. Modifying POREC2.0 so that it would be possible to constrain the emission location itself could lead to better constraints on the emission theories. Currently, broad constraints are possible (e.g. to only allow emission from within a certain radial distance of the pulsar, etc), however these lack the precision needed to test current emission theories.

It seems likely that allowing emission to occur from anywhere in the open magnetosphere is the reason for POREC2.0’s inability to obtain the phase separation between the peaks that is observed. Once the symmetry of the emission locations is constrained, the phase difference between the peaks change so the peaks are no longer exactly 0.5 out of phase with one another.

Rewriting POREC2.0 in order to be able to constrain the emission locations is a big task - it would require a complete rewrite of how POREC2.0 designates the emission
locations. As a consequence of this, it would also require a rewrite of how POREC2.0 checks an emission location to see whether or not emission is allowed. Overall, this would require a rewrite of approximately half of the total code. However, this author believes that without such a rewrite POREC2.0 has essentially reached its limitations.

POREC2.0 did show that there were different linear and circular polarisation profiles to the emission (Section 5.2.1), dependent on where in the pulsar magnetosphere the emission occurred. As such it is theoretically possible to constrain the emission locations using this polarisation data. However, to do so there must be a clear method to judge whether a particular emission location should be included. This would be difficult to do for each emission point without adding considerable computational overhead.

6.1.2 Magnetic Field

The magnetic field that is currently in use in POREC2.0 is the Deutsch field, which may not be accurate at large distances from the pulsar [62]. Implementing the new model for the magnetic field (as detailed in [62]) should improve the accuracy of the results from far out in the magnetosphere.

6.1.3 Different Emission Components

POREC2.0 is in general written in sections, which is better in terms of maintainability in the long term. It also allows essentially the same code to be used to test a number of different emission regimes. Currently, POREC2.0 only calculates emission from incoherent synchrotron radiation, and as such is limited to the optical regime.

The majority of the code in POREC2.0 is not used in calculating the emission itself, rather it is concerned with setting up the environment surrounding the pulsar. As such, this code can be used regardless of which emission regime is of interest.

Currently, the emission function and the frame-transformation effects are in a single function. This is problematic for two reasons.

Firstly, it makes debugging the code unnecessarily difficult. Whenever this function throws an error there are two different things that must be checked. Refactoring this into two separate function would make the maintainability of the code much easier.
Secondly, once this is in two separate functions it becomes easy to test POREC2.0 with other emission mechanisms. Once the calculation of the emission in the co-rotating frame is separated out from the frame-transformation effects, it becomes much easier to slide in other functions which will calculate the emission at a particular frequency range due to other effects - e.g. inverse Compton scattering at higher energies.

Pulsars emit radiation over the entire spectrum. It would be helpful to have a single code which could simulate the emission over the entire spectrum at the same time. Given the design of POREC2.0, each emission regime could be separately programmed. Implementing each regime would be a much quicker task than redesigning the emission location code, and as such should be high on the list of priorities, should work on POREC2.0 continue.

6.1.4 Output

The output from POREC, and from POREC2.0, could be significantly improved. There is no designated pipeline to take the raw simulations and convert it into usable files.

Currently, output from a particular inclination angle in POREC2.0 is sent into a number of different ASCII files, where the number of files is equal to the number of cores that POREC2.0 is run on minus 1. So if POREC is run on 2 nodes on Fionn (48 cores) there are 47 different output files holding the reverse engineering information. There is no structure saying where the output for a particular viewing angle or phase will go. This leads to problems post-processing.

Currently, to obtain usable (vtk) files from the output of POREC, these steps are taken:

1. (Optional) Use \textit{cat} or \textit{sed} (or the tool of your choice) to combine all of the different output files into a single file holding all the data for all viewing angles and phase bins for a particular inclination angle.

2. Use \textit{awk} to split the combined file (or original output) into individual viewing angles files.

3. Set up a place to store the vtk files.

4. Use a script of your choice to take in the VAfiles and output vtk files (Here the programming language R was used).
While there is currently a bashscript that follows this pipeline, it is extremely slow (this is mainly due to using R to create the vtk files). Output from POREC2.0 for a particular inclination angle is frequently of the order of 100 GB, and the bashscript can take over three days to run. This is clearly non-ideal.

By refactoring the output of POREC2.0 (e.g. to use netCDF) this pipeline could be completely avoided - making the analysis of the results from POREC2.0 much much faster.

### 6.1.5 Statistical Component

POREC2.0 was originally designed to use a statistical approach to find the best-fit light curves. This was, in the main, not a success. When a Chi-squared function was used to find the best-fitting light curve, the results were not ideal. The light curves which best-fit the model often had significant problems. As an example, the best Chi-squared fit for the intensity gave a single peak as the best-fit, as this matched the peak of the observed emission extremely well. However, for the simulation to match observations of the Crab a secondary peak must clearly exist. Therefore the Chi-squared function was not outputting the best fit of the observations.

As such, much of the analysis carried out in this thesis was though manually finding the best fitting light curves. If POREC2.0 is to be used for running large numbers of simulations, this is infeasible. Writing a pipeline which can consistently find the best fit from POREC2.0 would be a huge help in simplifying analysis. It would be extremely difficult to write, and currently the author has no suggestions as to how you should computationally choose the best fitting pulse profile from the simulations.

Future observations of optical pulsars will include the intensity, the linear polarisation and the circular polarisation. This will decrease the degrees of freedom that POREC currently has, and make statistical analysis easier.
6.2 Synchrotron Formulae

6.2.1 PAD Problems

As mentioned in Chapter 5, the synchrotron radiation formulae lead to an unrealistic PAD function being required to obtain non-zero emission from high magnetic fields. This presents a large problem for future simulations, and rectifying this problem should be of the highest priority.

It seems likely to the author that this is a problem due to one of the substitutions or the assumptions made during the derivation of the mathematical model, as opposed to being a physical limitation. That said, the substitution required to prevent this result has not yet been found by this author.

Currently expanding the PAD out to third order has a large effect on the circular polarisation that is calculated. However, in this case $\theta$ is small, and therefore $\theta^3$ is extremely small. The expansion of $\theta$ out to the next order should have a correspondingly small effect on the results. Here it is found that expanding $\theta$ has a large effect on the results in high magnetic fields.

Future work would involve investigating the derivation in an attempt to remove the constraint on the PAD.

6.2.2 Realistic Energy Limits

Currently the DS model assumes that the energy of the particles goes between 0 and $\infty$, a clearly unrealistic assumption! In actuality, the energy will go between a lower limit $E_1$ and an upper limit $E_2$. Furthermore, emission will not be seen below the fundamental frequency. Correcting for these flaws would improve the simulations.

Gleeson, Legg and Westfold corrected for these through using a set of correction functions. The original WL predictions were then multiplied by these correction factors to take the finite integration limits into account. The WL model only had a single integration (Equation 2.6) to correct for each Stokes parameter. This is no longer the case in the DS model.
Chapter 6. *Future Work*

There are a number of different methods that can be used to take the finite integration limits into account. Full details of these are given in Section 5.1. Correcting for the finite integration limits would clearly improve POREC2.0’s simulated emission profiles.

6.3 Conclusions

Using a reverse engineering approach to constrain pulsar emission theories is a viable strategy. The current implementation of this approach, POREC2.0, is flawed, but the underlying strategy is still sound. With the results seen from POREC2.0 (namely the banding effect of the linear polarisation, together with the fact that the circular polarisation results) this inverse mapping approach become even more useful, as now there is a way to actually constrain where the emission is occurring due to the polarisation measurements. While there is still a lot of work to do, inverse mapping approaches such as POREC2.0 offer an excellent way to move forward in choosing between emission theories.
Appendix A

Airy Functions

A.1 Airy Functions

It is possible to define the Airy function as in integral in such a way that

\[ Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[ i (zt + \frac{1}{3}t^3) \right] dt \quad (A.1) \]

and using the property

\[ \frac{d}{dz} \int f(x, z) dx = \int \frac{df(z, x)}{dz} dx \quad (A.2) \]

we can write

\[ Ai'(z) = \frac{d}{dz} Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[ i \left( zt + \frac{1}{3}t^3 \right) \right] dt \quad (A.3) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{dz} \left( \exp \left[ i \left( zt + \frac{1}{3}t^3 \right) \right] \right) dt \quad (A.4) \]

\[ = \frac{i}{2\pi} \int_{-\infty}^{\infty} t \exp \left[ i \left( zt + \frac{1}{3}t^3 \right) \right] dt \quad (A.5) \]
Appendix A. Airy Functions

\[
Ai''(z) = \frac{d}{dz} Ai'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{dz} \left( t \exp \left[ i \left( zt + \frac{1}{3} t^3 \right) \right] \right) dt
\]  

(A.6)

\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} t^2 \exp \left[ i \left( zt + \frac{1}{3} t^3 \right) \right] dt
\]  

(A.7)

\[
Ai'''(z) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} t^3 \exp \left[ i \left( zt + \frac{1}{3} t^3 \right) \right] dt
\]  

(A.8)

and as the Airy function has the property

\[
Ai''(z) = zAi(z)
\]  

(A.9)

and we can use that to say that

\[
Ai'''(z) = \frac{d}{dz} Ai''(z) = \frac{d}{dz} zAi(z) = Ai(z) + zAi'(z)
\]  

(A.10)

It is also possible to convert the Airy function to Bessel functions. The Airy function is related to the modified Bessel function of the second kind as follows [2]

\[
Ai(z) = \frac{1}{\pi} \sqrt{\frac{z}{3}} K_{\frac{1}{3}} \left( \left( \frac{2}{3} z \right)^{\frac{3}{2}} \right)
\]  

(A.11)

\[
Ai'(z) = -\frac{z}{\pi \sqrt{3}} \frac{1}{K_{\frac{2}{3}}} \left( \left( \frac{2}{3} z \right)^{\frac{3}{2}} \right)
\]  

(A.12)

A.1.1 Converting integrals to Airy functions

There are four separate integrals to consider

A.1.1.1 \[ \int_{-\infty}^{\infty} y^3 \exp \left[ i \frac{2}{3} \eta \left( y + \frac{1}{3} y^3 \right) \right] dy \]

Here replace \( y \) by \( t = (\frac{2}{3} \eta)^{\frac{1}{3}} y \) in

\[ \int_{-\infty}^{\infty} y^3 \exp \left[ i \frac{3}{2} \eta \left( y + \frac{1}{3} y^3 \right) \right] dy \]  

(A.13)
Appendix A. Airy Functions

to get

\[
\left(\frac{3\eta}{2}\right)^{-\frac{4}{3}} \int_{-\infty}^{\infty} t^3 \exp \left[ i \left(\frac{3}{2} \eta \right)^{\frac{2}{3}} t + \frac{1}{3} t^3 \right] \, dt = \left(\frac{3\eta}{2}\right)^{-\frac{4}{3}} 2\pi i Ai''' \left(\left(\frac{3}{2} \eta \right)^{\frac{2}{3}} \right) \tag{A.14}
\]

and using the properties of Airy Functions this becomes

\[
\left(\frac{3}{2}\right)^{-\frac{4}{3}} (-2\pi i) \left[ \left(\frac{3}{2} \eta \right)^{\frac{2}{3}} Ai' \left(\left(\frac{2}{3} \eta \right)^{\frac{2}{3}} \right) + Ai \left(\left(\frac{2}{3} \eta \right)^{\frac{2}{3}} \right) \right] \tag{A.15}
\]

which in turn is equal to

\[
i \left[ -\frac{2}{\sqrt{3}} K_{\frac{2}{3}}(\eta) + \frac{4}{3\sqrt{3} \eta} K_{\frac{1}{3}}(\eta) \right] \tag{A.16}
\]

A.1.1.2 \( \int_{-\infty}^{\infty} y^2 \exp \left[ i \left(\frac{3}{2} \eta \right)^{\frac{2}{3}} y + \frac{1}{3} y^3 \right] \, dy \)

Again substitute \( y \) by \( t = \left(\frac{3}{2} \eta \right)^{\frac{1}{3}} y \) in

\[
\int_{-\infty}^{\infty} y^2 \exp \left[ i \frac{3}{2} \eta \left( y + \frac{1}{3} y^3 \right) \right] \, dy \tag{A.17}
\]

to get

\[
\left(\frac{3}{2} \eta \right)^{-1} \int_{-\infty}^{\infty} t^2 \exp \left[ i \left(\frac{3}{2} \eta \right)^{\frac{2}{3}} t + \frac{1}{3} t^3 \right] \, dt \tag{A.18}
\]

and by multiplying above and below by \(-2\pi\) this then becomes

\[
-2\pi \left(\frac{3}{2} \eta \right)^{-1} Ai'' \left(\left(\frac{3}{2} \eta \right)^{\frac{2}{3}} \right) = -2\pi \left(\frac{3}{2} \eta \right)^{-\frac{1}{3}} Ai \left(\left(\frac{3}{2} \eta \right)^{\frac{2}{3}} \right) = -\frac{2}{\sqrt{3}} K_{\frac{1}{3}}(\eta) \tag{A.19}
\]

A.1.1.3 \( \int_{-\infty}^{\infty} y \exp \left[ i \frac{2}{3} \eta \left( y + \frac{1}{3} y^3 \right) \right] \, dy \)

This is equal to \( i \frac{2}{\sqrt{3}} K_{\frac{2}{3}}(\eta) \) [50].

113
Appendix A. Airy Functions

A.1.1.4 \[ \int_{-\infty}^{\infty} \exp \left( i \frac{3}{2} \eta \left( y + \frac{1}{3} y^3 \right) \right) dy \]

This is equal to \( \frac{2}{\sqrt{3}} K_{1/3} (\eta) \) [50].
Appendix B

General Theta Integrals

B.1 General Theta Integrals

\[ \int_{-\infty}^{\infty} \theta_{\gamma} e^{i \theta^2 b K_{\frac{2}{3}} (\frac{x}{2} \theta_{3})} d\theta \]

To begin with, the Bessel functions are written out explicitly to get

\[ \int_{-\infty}^{\infty} \theta_{\gamma} e^{i \theta^2 b K_{\frac{2}{3}} (\frac{x}{2} \theta_{3})} d\theta = \int_{-\infty}^{\infty} \theta_{\gamma} e^{i \theta^2 b \sqrt{3} \theta_{\gamma}} \exp \left[ \frac{i 3 x}{4} \left( \theta_{\gamma}^2 u + \frac{1}{3} u^3 \right) \right] \]

\[ \frac{\sqrt{3}}{2 \theta_{\gamma}} \exp \left[ \frac{i 3 x}{4} \left( \theta_{\gamma}^2 v' + \frac{1}{3} v'^3 \right) \right] dv' du d\theta \]  

(B.1)

and simplifying and substituting \( v = -v' \) gives

\[ \int_{-\infty}^{\infty} \theta_{\gamma} e^{i \theta^2 b K_{\frac{2}{3}} (\frac{x}{2} \theta_{3})} d\theta = \frac{3}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_{\gamma}^{-2} e^{2 \theta^2 b} \times \]

\[ \exp \left[ \frac{3 x}{4} \left( \theta_{\gamma}^2 (u - v) + \frac{1}{3} (u^3 - v^3) \right) \right] du dv d\theta \]  

(B.2)

Now making the substitutions

\[ y = \frac{u - v}{2} \quad z = \frac{u + v}{2} \quad dudv = 2dydz \]  

(B.3)
Appendix B. General Theta Integrals

gives

\[
\int_{-\infty}^{\infty} \theta^a \theta^b K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{3}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta^{-2} \theta^{2b} \times \\
\exp \left[ \frac{3x}{2} \left( (\theta^2 + z^2) y + \frac{1}{3} y^3 \right) \right] \, du \, dv \, d\theta
\]  
(B.4)

and making a second substitution, where \( \phi \) goes between 0 and \( \pi \), and \( R \) between 0 and infinity

\[
\theta = \frac{1}{\gamma} R \sin(\phi) \\
z = R \cos(\phi) \\
dz \, d\theta = \frac{1}{\gamma} R \, dR \, d\phi
\]  
(B.5)

gives

\[
\int_{-\infty}^{\infty} \theta^a \theta^b K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta = \sqrt{3} \int_{0}^{2\pi} \int_{0}^{\infty} (1 + R^2 \sin^2(\phi))^{a-2} \left( \frac{R \sin(\phi)}{\gamma} \right)^{2b} (1 + R^2)^{\frac{1}{2}} \times \\
K_{\frac{1}{3}} \left( x \left( 1 + R^2 \right)^{\frac{1}{2}} \right) \frac{1}{\gamma} R \, dR \, d\phi
\]  
(B.6)

B.1.2 \[ \int_{-\infty}^{\infty} \theta^a \theta^b K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{z}{2} \theta^3 \right) d\theta \]

In the same manner

\[
\int_{-\infty}^{\infty} \theta^a \theta^b K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta = \sqrt{3} \int_{0}^{2\pi} \int_{0}^{\infty} (1 + R^2 \sin^2(\phi))^{a-3} \left( \frac{R^2 \sin^2(\phi)}{\gamma^2} \right)^{b} \times \\
(1 + R^2)^{\frac{1}{2}} K_{\frac{2}{3}} \left( x \left( 1 + R^2 \right)^{\frac{1}{2}} \right) \frac{1}{\gamma} R \, (1 + R^2)^{\frac{1}{2}} \, dR \, d\phi
\]  
(B.7)

where the other terms are functions of \( \cos(n\phi) \) and hence disappear when integrated with respect to \( \phi \) (where \( n \) is an integer).
Appendix B. General Theta Integrals

B.1.3 \[ \int_{-\infty}^{\infty} \theta_{\gamma}^{a+b} K_{\frac{2}{3}}^{2} \left( \frac{x}{2} \theta_{\gamma}^{3} \right) d\theta \]

Finally, with the same substitutions

\[ \int_{-\infty}^{\infty} \theta_{\gamma}^{a+b} K_{\frac{2}{3}}^{2} \left( \frac{x}{2} \theta_{\gamma}^{3} \right) d\theta = \frac{3}{2^{\gamma+1}} \int_{-\infty}^{\infty} \int_{0}^{2\pi} (1 + R^{2} \sin^{2}(\phi))^{a-4} (R^{2} \sin^{2}(\phi))^{b} \times \]

\[ (R^{2} \cos^{2}(\phi) - y^{2}) \exp \left[ \frac{3x}{2} \left( 1 + R^{2} y + \frac{1}{3} y^{3} \right) \right] dydRd\phi \]

\[ = \sqrt{3} \int_{0}^{2\pi} \int_{0}^{\infty} (1 + R^{2} \sin(\phi))^{a-4} \left( \frac{R^{2} \sin^{2}(\phi)}{\gamma^{2}} \right)^{b} \left[ \frac{3R^{2}}{2} + 1 \right] \times \]

\[ K_{\frac{1}{3}} \left( x(1 + R^{2})\frac{2}{3} \right) \frac{1}{\gamma R(1 + R^{2})^{\frac{1}{4}}} dRd\phi \]

where again there are other terms that will be a function of \( \cos(n\phi) \) and will disappear upon integration.
Appendix C

Theta Integrals

C.1 Theta Integrations

C.2 Integrating over $\phi$

When integrating over $\phi$, firstly convert the terms of $\sin^n \phi$ to $K + f(\cos(m\phi))$, $K$ is a real number, and $n$ and $m$ are integers. Upon integrating all of the terms that are a function of $\cos(m\phi)$ will become zero, and $K$ will become $2K\pi$.

C.3

\[ \int_{-\infty}^{\infty} \theta^2 \gamma^2 K_1 \left( \frac{x}{2} \theta^3 \right) d\theta \]

By B.6

\[ \int_{-\infty}^{\infty} \theta^2 \gamma^2 K_1 \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \left( \frac{R \sin \phi}{\gamma} \right)^2 d\phi \times \]

\[ K_1 \left( x (1 + R^2)^{\frac{3}{2}} \right) (1 + R^2)^{\frac{1}{2}} RdR \]

\[ = \frac{\sqrt{3}}{\gamma^3} \int_{0}^{2\pi} \frac{R^2}{2} K_1 \left( x (1 + R^2)^{\frac{3}{2}} \right) d\phi (1 + R^2)^{\frac{1}{2}} RdR \] (C.2)

+ cosine terms
and integrating with respect to $\phi$ gives

$$\int_{-\infty}^\infty \theta^2 \frac{\theta^2 K^2_{\frac{3}{2}}}{\gamma^3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi \sqrt{\gamma}}{\gamma^3} \int_{0}^{\infty} R^2 K_{\frac{1}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \left( 1 + R^2 \right)^{\frac{1}{2}} R dR \quad (C.3)$$

substituting

$$\mu = x \left( 1 + R^2 \right)^{\frac{3}{2}} \quad (C.4)$$

gives

$$\int_{-\infty}^\infty \theta^2 \frac{\theta^2 K^2_{\frac{3}{2}}}{\gamma^3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{\gamma^3 \sqrt{\gamma}} \int_{0}^{\infty} \left( \frac{\mu}{x} \right)^{\frac{3}{2}} \left( 1 - 1 \right) K_{\frac{1}{3}}(\mu) d\mu \quad (C.5)$$

**C.4**

$$\int_{-\infty}^\infty \theta^4 \frac{\theta^2 K^2_{\frac{3}{2}}}{\gamma^3} \left( \frac{5 \theta^3}{2} \right) d\theta$$

By B.6

$$\int_{-\infty}^\infty \theta^4 \frac{\theta^2 K^2_{\frac{3}{2}}}{\gamma^3} \left( \frac{x}{2} \theta^3 \right) d\theta = \sqrt{\frac{3}{\gamma}} \int_{0}^{2\pi} \left( 1 + R^2 \sin^2 \phi \right) \left( \frac{R \sin \phi}{\gamma} \right)^2 d\phi \times$$

$$K_{\frac{1}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \left( 1 + R^2 \right)^{\frac{1}{2}} R dR \quad (C.6)$$

$$= \sqrt{\frac{3}{8 \gamma^3}} \int_{0}^{2\pi} \left( \frac{5 R^6}{16} + 2 \frac{3 R^4}{8} + \frac{R^2}{2} \right) K_{\frac{1}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \times$$

$$\left( 1 + R^2 \right)^{\frac{1}{2}} R dR + \text{cosine terms} \quad (C.7)$$

and integrating with respect to $\phi$ and substituting in C.4 gives

$$\int_{-\infty}^\infty \theta^2 \frac{\theta^2 K^2_{\frac{3}{2}}}{\gamma^3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{8 \gamma^3 x \sqrt{\gamma}} \int_{x}^{\infty} \left( 5 \left( \frac{\mu}{x} \right)^{\frac{6}{2}} - 3 \left( \frac{\mu}{x} \right)^{\frac{4}{2}} - \left( \frac{\mu}{x} \right)^{\frac{2}{2}} - 1 \right) K_{\frac{1}{3}}(\mu) d\mu \quad (C.8)$$
Appendix C. Theta Integrals

C.5 \[ \int_{-\infty}^{\infty} \theta^2 \theta^4 K_\frac{2}{3}^2 \left( \frac{x}{2} \theta^3 \right) d\theta \]

By B.6

\[ \int_{-\infty}^{\infty} \theta^2 \theta^4 K_\frac{2}{3}^2 \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{\infty} \int_{0}^{\frac{2\pi}{4}} \left( \frac{R \sin \phi}{\gamma} \right)^4 d\phi \times \]

\[ K_\frac{2}{3} \left( x \left( 1 + R^2 \right)^\frac{3}{2} \right) \left( 1 + R^2 \right)^\frac{1}{2} R dR \] (C.9)

\[ = \frac{\sqrt{3}}{\gamma^3} \int_{0}^{\infty} \int_{0}^{\frac{2\pi}{8}} \frac{3R^4}{d\phi} \times \]

\[ K_\frac{2}{3} \left( x \left( 1 + R^2 \right)^\frac{3}{2} \right) \left( 1 + R^2 \right)^\frac{1}{2} R dR + \cosine \ terms \] (C.10)

and substituting C.4 and integrating with respect to \( \phi \) gives

\[ \int_{-\infty}^{\infty} \theta^2 \theta^4 K_\frac{2}{3}^2 \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{4\gamma^3 x \sqrt{3}} \int_{x}^{\infty} \left( 3 \left( \frac{\mu}{x} \right)^\frac{4}{3} - 6 \left( \frac{\mu}{x} \right)^2 + 3 \right) K_\frac{1}{3}(\mu) d\mu \] (C.11)

C.6 \[ \int_{-\infty}^{\infty} \theta^6 \theta^2 K_\frac{2}{3}^2 \left( \frac{x}{2} \theta^3 \right) d\theta \]

By B.6

\[ \int_{-\infty}^{\infty} \theta^6 \theta^2 K_\frac{2}{3}^2 \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{\infty} \int_{0}^{\frac{2\pi}{4}} \left( 1 + R^2 \sin^2 \phi \right)^4 \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right) d\phi \times \]

\[ K_\frac{2}{3} \left( x \left( 1 + R^2 \right)^\frac{3}{2} \right) \left( 1 + R^2 \right)^\frac{1}{2} R dR \] (C.12)

and integrating with respect to \( \phi \) and substituting C.4 gives

\[ \int_{-\infty}^{\infty} \theta^6 \theta^2 K_\frac{2}{3}^2 \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{128\gamma^3 x \sqrt{3}} \int_{x}^{\infty} \left[ 63 \left( \frac{\mu}{x} \right)^{10/3} - 35 \left( \frac{\mu}{x} \right)^{8/3} - 10 \left( \frac{\mu}{x} \right)^{6/3} ight. \]

\[ - 6 \left( \frac{\mu}{x} \right)^{4/3} - 5 \left( \frac{\mu}{x} \right)^{2/3} - 7 \left] K_\frac{1}{3}(\mu) d\mu \right. \] (C.13)
C.7  \[ \int_{-\infty}^{\infty} \theta^4 \theta^4 K_\frac{2}{3} \left( \frac{x}{2} \theta^3 \right) d\theta \]

By B.6

\[ \int_{-\infty}^{\infty} \theta^4 \theta^4 K_\frac{2}{3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \left( R^2 \sin^2 \phi \right)^2 \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right)^2 d\phi \times K_\frac{2}{3} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) (1 + R^2)^{\frac{1}{2}} RdR \quad (C.14) \]

and integrating with respect to \( \phi \) and substituting C.4 gives

\[ \int_{-\infty}^{\infty} \theta^4 \theta^4 K_\frac{2}{3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{64\gamma x \sqrt{3}} \int_{x}^{\infty} \left[ 35 \left( \frac{\mu}{x} \right)^{\frac{3}{2}} - 60 \left( \frac{\mu}{x} \right)^{\frac{5}{2}} + 18 \left( \frac{\mu}{x} \right)^{\frac{4}{2}} + 4 \left( \frac{\mu}{x} \right)^{\frac{6}{2}} + 3 \right] K_1 \left( \mu \right) d\mu \quad (C.15) \]

C.8  \[ \int_{-\infty}^{\infty} \theta^6 \theta^4 K_\frac{2}{3} \left( \frac{x}{2} \theta^3 \right) d\theta \]

By B.6

\[ \int_{-\infty}^{\infty} \theta^6 \theta^4 K_\frac{2}{3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \left( R^2 \sin^2 \phi \right)^2 \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right)^3 d\phi \times K_\frac{2}{3} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) (1 + R^2)^{\frac{1}{2}} RdR \quad (C.16) \]

and substituting C.4 and integrating with respect to \( \phi \) gives

\[ \int_{-\infty}^{\infty} \theta^6 \theta^4 K_\frac{2}{3} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{8\gamma x \sqrt{3}} \int_{x}^{\infty} \left[ 5 \left( \frac{\mu}{x} \right)^{\frac{3}{2}} - 15 \left( \frac{\mu}{x} \right)^{\frac{5}{2}} + 15 \left( \frac{\mu}{x} \right)^{\frac{4}{2}} - 5 \right] K_1 \left( \mu \right) d\mu \quad (C.17) \]
C.9 \[ \int_{-\infty}^{\infty} \frac{\theta^2}{\gamma} K^2_{\frac{1}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta \]

By B.6

\[
\int_{-\infty}^{\infty} \frac{\theta^2}{\gamma} K^2_{\frac{1}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} d\phi K_{\frac{1}{4}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \left( 1 + R^2 \right)^{\frac{1}{2}} RdR \quad (C.18)
\]

and integrating and substituting C.4 gives

\[
\int_{-\infty}^{\infty} \frac{\theta^2}{\gamma} K^2_{\frac{1}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta = \frac{2\pi}{\gamma x \sqrt{3}} \int_{x}^{\infty} K_{\frac{1}{3}} (\mu) d\mu \quad (C.19)
\]

C.10 \[ \int_{-\infty}^{\infty} \frac{\theta^4}{\gamma} K^2_{\frac{2}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta \]

By B.9

\[
\int_{-\infty}^{\infty} \frac{\theta^4}{\gamma} K^2_{\frac{2}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} d\phi \times K_{\frac{1}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \left( 1 + R^2 \right)^{\frac{1}{2}} RdR \quad (C.20)
\]

which upon integration by \( \phi \) and substitution of C.4 gives

\[
\int_{-\infty}^{\infty} \frac{\theta^4}{\gamma} K^2_{\frac{2}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta = \frac{\pi}{\gamma x \sqrt{3}} \int_{x}^{\infty} \left( 3 \left( \frac{\mu}{x} \right)^{\frac{3}{2}} - 1 \right) K_{\frac{1}{3}} (\mu) d\mu \quad (C.21)
\]

C.11 \[ \int_{-\infty}^{\infty} \frac{\theta^6}{\gamma} K^2_{\frac{2}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta \]

By B.9

\[
\int_{-\infty}^{\infty} \frac{\theta^4}{\gamma} K^2_{\frac{2}{3}} \left( \frac{x}{2} \theta^3_{\gamma} \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} d\phi \times \int_{0}^{(1 + R^2 \sin^2 \phi)^{\frac{3}{2}}} \left( \frac{3}{2} R^2 + 1 \right) d\phi \times K_{\frac{1}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \left( 1 + R^2 \right)^{\frac{1}{2}} RdR \quad (C.22)
\]
which upon integration by $\phi$ and substitution of C.4 gives
\[
\int_{-\infty}^{\infty} \theta^4 K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{16 \gamma x \sqrt{3}} \int_{x}^{\infty} \left[ 15 \left( \frac{\mu}{x} \right)^{\frac{4}{3}} + 3 \left( \frac{\mu}{x} \right)^{\frac{2}{3}} \right]
\]
\[+ 5 \left( \frac{\mu}{x} \right)^{\frac{2}{3}} + 9 \right] K_{\frac{4}{3}} (\mu) d\mu \quad (C.23)
\]

C.12 \quad \int_{-\infty}^{\infty} \theta^8 K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta

By B.9
\[
\int_{-\infty}^{\infty} \theta^4 K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \int_{0}^{R} (1 + R^2 \sin^2 \phi)^{\frac{4}{3}} \left( \frac{3}{2} R^2 + 1 \right) d\phi \times
\]
\[K_{\frac{4}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) \left( 1 + R^2 \right)^{\frac{1}{2}} RdR \quad (C.24)
\]

which upon integration by $\phi$ and substitution of C.4 gives
\[
\int_{-\infty}^{\infty} \theta^4 K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) d\theta = \frac{\pi}{256 \gamma x \sqrt{3}} \int_{x}^{\infty} \left[ 189 \left( \frac{\mu}{x} \right)^{\frac{10}{3}} + 35 \left( \frac{\mu}{x} \right)^{\frac{8}{3}} + 50 \left( \frac{\mu}{x} \right)^{\frac{6}{3}} \right]
\]
\[+ 54 \left( \frac{\mu}{x} \right)^{\frac{4}{3}} + 65 \left( \frac{\mu}{x} \right)^{\frac{2}{3}} + 119 \right] K_{\frac{4}{3}} (\mu) d\mu \quad (C.25)
\]

C.13 \quad \int_{-\infty}^{\infty} \theta^7 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right)

By B.7
\[
\int_{-\infty}^{\infty} \theta^7 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \int_{0}^{R} (1 + R^2 \sin^2 \phi)^{\frac{4}{3}} \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right) d\phi \times
\]
\[\left( 1 + R^2 \right) K_{\frac{4}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) RdR \quad (C.26)
\]
and integrating with respect to $\phi$ and substituting with C.4 gives

\[
\int_{-\infty}^{\infty} \theta^7 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\pi}{128 \gamma x \sqrt{3}} \int_{x}^{\infty} \left( 63 \left( \frac{\mu}{x} \right)^{\frac{11}{3}} - 35 \left( \frac{\mu}{x} \right)^{\frac{9}{3}} - 10 \left( \frac{\mu}{x} \right)^{\frac{7}{3}} \right. \\
\left. \right. - 6 \left( \frac{\mu}{x} \right)^{\frac{5}{3}} - 5 \left( \frac{\mu}{x} \right)^{\frac{3}{3}} - 7 \left( \frac{\mu}{x} \right)^{\frac{1}{3}} \right) K_{\frac{2}{3}} (\mu) d\mu
\]

(C.27)

C.14 \[
\int_{-\infty}^{\infty} \theta^5 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right)
\]

By B.7

\[
\int_{-\infty}^{\infty} \theta^5 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \int_{0}^{\infty} \left( 1 + R^2 \sin^2 \phi \right)^2 \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right) d\phi \times \\
(1 + R^2) K_{\frac{2}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) R dR
\]

(C.28)

and integrating with respect to $\phi$ and substituting with C.4 gives

\[
\int_{-\infty}^{\infty} \theta^5 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\pi}{8 \gamma x \sqrt{3}} \int_{x}^{\infty} \left( 5 \left( \frac{\mu}{x} \right)^{\frac{7}{3}} - 3 \left( \frac{\mu}{x} \right)^{\frac{5}{3}} - \right. \\
\left. \left( \frac{\mu}{x} \right)^{\frac{3}{3}} - \left( \frac{\mu}{x} \right)^{\frac{1}{3}} \right) K_{\frac{2}{3}} (\mu) d\mu
\]

(C.29)

C.15 \[
\int_{-\infty}^{\infty} \theta^3 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right)
\]

By B.7

\[
\int_{-\infty}^{\infty} \theta^3 \theta^2 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \int_{0}^{\infty} \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right) d\phi \times \\
(1 + R^2) K_{\frac{2}{3}} \left( x \left( 1 + R^2 \right)^{\frac{3}{2}} \right) R dR
\]

(C.30)
and integrating with respect to $\phi$ and substituting with C.4 gives

$$
\int_{-\infty}^{\infty} \theta^3 \theta^3 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\pi}{\gamma x \sqrt{3}} \int_{x}^{\infty} \left( \left( \frac{\mu}{x} \right)^{\frac{2}{3}} - \left( \frac{\mu}{x} \right)^{\frac{1}{3}} \right) K_{\frac{2}{3}}(\mu) d\mu \quad (C.31)
$$

**C.16**

$$
\int_{-\infty}^{\infty} \theta^5 \theta^4 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right)
$$

By B.7

$$
\int_{-\infty}^{\infty} \theta^5 \theta^4 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \int_{0}^{(1 + R^2 \sin^2 \phi)^2} \left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right)^2 d\phi \times (1 + R^2) K_{\frac{2}{3}} \left( x \left( 1 + R^2 \right)^{\frac{2}{3}} \right) RdR \quad (C.32)
$$

and integrating with respect to $\phi$ and substituting with C.4 gives

$$
\int_{-\infty}^{\infty} \theta^5 \theta^4 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\pi}{64\gamma x \sqrt{3}} \int_{x}^{\infty} \left[ \frac{35}{\gamma} \left( \frac{\mu}{x} \right)^{\frac{5}{3}} - 60 \left( \frac{\mu}{x} \right)^{\frac{2}{3}} + 98 \left( \frac{\mu}{x} \right)^{\frac{1}{3}} - 156 \left( \frac{\mu}{x} \right)^{\frac{2}{3}} + 83 \left( \frac{\mu}{x} \right)^{\frac{1}{3}} \right] K_{\frac{2}{3}}(\mu) d\mu \quad (C.33)
$$

**C.17**

$$
\int_{-\infty}^{\infty} \theta^5 \theta^4 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right)
$$

By B.7

$$
\int_{-\infty}^{\infty} \theta^5 \theta^4 K_{\frac{1}{3}} \left( \frac{x}{2} \theta^3 \right) K_{\frac{2}{3}} \left( \frac{x}{2} \theta^3 \right) = \frac{\sqrt{3}}{\gamma} \int_{0}^{2\pi} \int_{0}^{\left( \frac{R^2 \sin^2 \phi}{\gamma^2} \right)^2} d\phi \times (1 + R^2) K_{\frac{2}{3}} \left( x \left( 1 + R^2 \right)^{\frac{2}{3}} \right) RdR \quad (C.34)
$$
and substituting C.4 and integrating with respect to $\phi$ gives

$$
\int_{-\infty}^{\infty} \theta^5 \theta^2 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) = \frac{\pi}{4 \gamma x \sqrt{3}} \int_{x}^{\infty} \left[ 3 \left(\frac{\mu}{x}\right)^{\frac{3}{2}} - 6 \left(\frac{\mu}{x}\right)^{\frac{1}{2}} + 3 \left(\frac{\mu}{x}\right)^{\frac{1}{2}} \right] K_{\frac{2}{3}}(\mu) d\mu \quad (C.35)
$$

\begin{align*}
\text{C.18} & \quad \int_{-\infty}^{\infty} \theta^5 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) \\
\text{By B.7} & \quad \int_{-\infty}^{\infty} \theta^5 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) = \sqrt{3} \int_{0}^{2\pi} (1 + R^2 \sin^2 \phi)^{\frac{3}{2}} d\phi \times \\
& \quad \left(1 + R^2\right) K_{\frac{2}{3}} \left(x (1 + R^2)^{\frac{3}{2}}\right) R dR \quad (C.36)
\end{align*}

and substituting C.4 and integrating with respect to $\phi$ gives

$$
\int_{-\infty}^{\infty} \theta^2 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) = \frac{\pi}{4 \gamma x \sqrt{3}} \int_{x}^{\infty} \left[ 3 \left(\frac{\mu}{x}\right)^{\frac{3}{2}} + 2 \left(\frac{\mu}{x}\right)^{\frac{3}{2}} + 3 \left(\frac{\mu}{x}\right)^{\frac{1}{2}} \right] K_{\frac{2}{3}}(\mu) d\mu \quad (C.37)
$$

\begin{align*}
\text{C.19} & \quad \int_{-\infty}^{\infty} \theta^3 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) \\
\text{By B.7} & \quad \int_{-\infty}^{\infty} \theta^3 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) = \sqrt{3} \int_{0}^{2\pi} (1 + R^2) K_{\frac{2}{3}} \left(x (1 + R^2)^{\frac{3}{2}}\right) d\phi R dR \quad (C.38)
\end{align*}

and substituting C.4 and integrating with respect to $\phi$ gives

$$
\int_{-\infty}^{\infty} \theta^3 K_{\frac{1}{3}} \left(\frac{x}{2} \theta^2\right) K_{\frac{2}{3}} \left(\frac{x}{2} \theta^2\right) = \frac{2 \pi}{\gamma x \sqrt{3}} \int_{x}^{\infty} \left(\frac{\mu}{x}\right)^{\frac{3}{2}} K_{\frac{2}{3}}(\mu) d\mu \quad (C.39)
$$
Appendix D

Integral Conversions

D.1 \( \int_x^\infty K_{\frac{1}{3}}(\mu) \, d\mu \)

We have that

\[
\int_x^\infty K_{\frac{1}{3}}(\mu) \, d\mu = \int_x^\infty K_{\frac{5}{3}}(\mu) \, d\mu - \frac{4}{3} \int_x^\infty \mu^{-1} K_{\frac{2}{3}}(\mu) \, d\mu \quad (D.1)
\]

and integration by parts gives

\[
u = \mu^{-\frac{2}{3}} K_{\frac{2}{3}}
\]

\[
\begin{align*}
\frac{du}{d\mu} &= \frac{d}{d\mu} \left( \mu^{-\frac{2}{3}} K_{\frac{2}{3}}(\mu) \right) \\
&= -\mu^{-\frac{2}{3}} K_{\frac{2}{3}}(\mu) \\
&= -\mu^{-\frac{2}{3}} K_{\frac{2}{3}}(\mu) \\
\end{align*}
\]

\[
u = \mu^{-\frac{2}{3}}
\]

\[
\begin{align*}
v &= \frac{3}{2} \mu^{\frac{2}{3}} \\
\int_x^\infty K_{\frac{1}{3}}(\mu) \, d\mu &= F(x) - 2K_{\frac{2}{3}}(\mu) - 2F(x) \quad (D.4)
\int_x^\infty K_{\frac{1}{3}}(\mu) \, d\mu &= F(x) + 2K_{\frac{2}{3}}(x) - 2F(x) = 2K_{\frac{2}{3}}(x) - F(x) \quad (D.5)
\end{align*}
\]
Appendix D. Integral Conversions

D.2 \[ \int_{x}^{\infty} \left( \frac{x}{\mu} \right)^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu \]

\[ \int_{x}^{\infty} \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = \int_{x}^{\infty} \mu^{-\frac{2}{3}} \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu \]

\[ = - \int_{x}^{\infty} \mu^{-\frac{2}{3}} \frac{d}{d\mu} \left( \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) \right) d\mu \] (D.6)

and integration by parts gives

\[ \int_{x}^{\infty} \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = -\mu^{\frac{2}{3}} K_{\frac{4}{3}} + \frac{2}{3} \mu^{-\frac{1}{3}} K_{\frac{2}{3}}(\mu) \]

\[ = -\mu^{\frac{2}{3}} K_{\frac{3}{3}}(\mu) - \frac{2}{3} \mu^{-\frac{1}{3}} K_{\frac{3}{3}}(\mu) + \frac{2}{3} \mu^{-\frac{1}{3}} K_{\frac{3}{3}}(\mu) \] (D.7)

\[ = x^{\frac{2}{3}} K_{\frac{3}{3}}(x) \] (D.8)

which gives

\[ \int_{x}^{\infty} \left( \frac{x}{\mu} \right)^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = K_{\frac{2}{3}}(\mu) \] (D.9)

D.3 \[ \int_{x}^{\infty} \left( \frac{x}{\mu} \right)^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) \]

we have

\[ \int_{0}^{\infty} \left( \frac{x}{\mu} \right)^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) = \frac{1}{x^{\frac{4}{3}}} \int_{x}^{\infty} \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) d\mu = -\frac{1}{x^{\frac{4}{3}}} \int_{x}^{\infty} \frac{d}{d\mu} \left( \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) \right) d\mu \]

\[ = -\frac{1}{x^{\frac{4}{3}}} \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} = K_{\frac{4}{3}}(x) \] (D.10)

If required we can convert this into

\[ K_{\frac{4}{3}}(x) = K_{\frac{2}{3}}(x) + \frac{2}{3x} K_{\frac{1}{3}}(x) \] (D.11)
D.4 \[ \int \frac{(\mu)^{\frac{6}{3}}}{x^{\frac{1}{3}}} K_{\frac{1}{3}}(\mu) \]

Firstly we solve
\[ \int_{x}^{\infty} \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = \int_{x}^{\infty} \mu^{\frac{2}{3}} \mu^{\frac{4}{3}} K_{\frac{4}{3}}(\mu) d\mu = -\int_{x}^{\infty} \mu^{\frac{2}{3}} \frac{d}{d\mu} \left( \mu^{\frac{4}{3}} K_{\frac{4}{3}}(\mu) \right) d\mu \quad \text{(D.16)} \]

and using integration by parts we have
\[ u = \mu^{\frac{2}{3}} \quad \quad \quad \quad du = \frac{2}{3} \mu^{-\frac{1}{3}} \quad \text{(D.17)} \]
\[ dv = \frac{d}{d\mu} \left( \mu^{\frac{4}{3}} K_{\frac{4}{3}}(\mu) \right) \quad \quad \quad \quad v = \mu^{\frac{4}{3}} K_{\frac{4}{3}}(\mu) \quad \text{(D.18)} \]

to get
\[ \int_{0}^{\infty} \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = -\mu^{2} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} + \frac{2}{3} \int_{x}^{\infty} K_{\frac{4}{3}}(\mu) d\mu \quad \text{(D.19)} \]
\[ = -\mu^{2} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} + \frac{2}{3} \int_{x}^{\infty} K_{\frac{4}{3}}(\mu) d\mu + \frac{4}{9} \int_{x}^{\infty} K_{\frac{2}{3}}(\mu) d\mu \quad \text{(D.20)} \]

But as we can substitute for both of these integrations, this gives
\[ \int_{0}^{\infty} \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = -\mu^{2} K_{\frac{2}{3}}(\mu) + \frac{2}{3} \left[ -\mu K_{\frac{2}{3}}(\mu) - \frac{2}{3} F(x) \right] + \frac{4}{9} \left[ -2 K_{\frac{2}{3}}(\mu) - F(x) \right] \quad \text{(D.21)} \]
\[ = -\mu^{2} K_{\frac{2}{3}}(\mu) - \frac{2}{3} \mu K_{\frac{1}{3}}(\mu) - \frac{16}{9} K_{\frac{2}{3}}(\mu) - \frac{8}{9} F(x) \quad \text{(D.22)} \]

which finally gives
\[ \int_{x}^{\infty} \left( \frac{\mu^{\frac{6}{3}}}{x^{\frac{1}{3}}} K_{\frac{1}{3}}(\mu) \right) d\mu = K_{\frac{2}{3}}(x) + \frac{2}{3x} K_{\frac{4}{3}}(x) + \frac{16}{9x^{2}} K_{\frac{2}{3}}(x) - \frac{8}{9x^{2}} F(x) \quad \text{(D.23)} \]
Appendix D. Integral Conversions

D.5  \[ \int_x^\infty \left( \frac{\mu}{x} \right)^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) d\mu \]

As before, we take out the dependence on \( x \) to get

\[ \int_x^\infty \mu^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) d\mu = \int_x^\infty \mu^{\frac{4}{3}} \mu^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) d\mu = \int_x^\infty \mu^{\frac{4}{3}} \frac{d}{d\mu} \left( \mu^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) \right) d\mu \quad (D.24) \]

Now consider

\[ \int_x^\infty \mu^{\frac{4}{3}} K_{\frac{1}{3}}(\mu) d\mu = \int_x^\infty \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu \quad (D.25) \]

\[ = - \left[ \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) \right]_x^\infty - 2 \int_x^\infty \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu \quad (D.26) \]

and substituting for \( \int_x^\infty \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu \) gives

\[ \int_x^\infty \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = -\mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) |_x^\infty - 2\mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) |_x^\infty \quad (D.27) \]

and finally substituting this back gives

\[ \int_x^\infty \mu^{\frac{2}{3}} K_{\frac{1}{3}}(\mu) d\mu = K_{\frac{1}{3}}(x) - \frac{4}{3x} K_{\frac{1}{3}}(x) - \frac{8}{3x^2} K_{\frac{2}{3}}(x) \quad (D.28) \]
Appendix D. Integral Conversions

D.6 \( \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{10} K_{\frac{1}{3}}(\mu) d\mu \)

Writing

\[
\int_{x}^{\infty} \mu^{10} K_{\frac{1}{3}}(\mu) d\mu = \int_{x}^{\infty} \mu^{2} \mu^{4} K_{\frac{1}{3}}(\mu) d\mu = -\int_{x}^{\infty} \mu^{2} \frac{d}{d\mu} \left( \mu^{4} K_{\frac{1}{3}}(\mu) \right) d\mu \tag{D.29}
\]

\[
= -\mu^{10} K_{\frac{1}{3}}(\mu)|_{x}^{\infty} + \int_{x}^{\infty} \mu^{7} K_{\frac{1}{3}}(\mu) d\mu \tag{D.30}
\]

\[
= -\mu^{10} K_{\frac{1}{3}}(\mu)|_{x}^{\infty} - 2 \int_{x}^{\infty} \frac{d}{d\mu} \left( \mu^{7} K_{\frac{1}{3}}(\mu) \right) d\mu \tag{D.31}
\]

\[
= x^{10} K_{\frac{1}{3}}(x) + 2x^{7} K_{\frac{1}{3}}(x) \tag{D.32}
\]

and this in turn gives

\[
\int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{10} K_{\frac{1}{3}}(\mu) d\mu = K_{\frac{1}{3}}(x) + \frac{2}{x} K_{\frac{1}{3}}(x) \tag{D.33}
\]

\[
= K_{\frac{1}{3}}(x) + \frac{8}{3x} K_{\frac{1}{3}}(x) + \frac{16}{3x^{2}} K_{\frac{1}{3}}(x) + \frac{32}{9x^{3}} K_{\frac{1}{3}}(x) \tag{D.34}
\]

D.7 \( \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{\frac{1}{3}} K_{\frac{2}{3}}(\mu) d\mu \)

As before, we write

\[
\int_{x}^{\infty} \mu^{\frac{1}{3}} K_{\frac{2}{3}}(\mu) d\mu = \int_{x}^{\infty} \mu^{-\frac{1}{3}} \left[ \mu^{\frac{2}{3}} K_{\frac{2}{3}}(\mu) \right] d\mu \tag{D.35}
\]

\[
\tag{D.36}
\]

and using integration by parts this becomes

\[
\int_{x}^{\infty} \mu^{\frac{1}{3}} K_{\frac{2}{3}}(\mu) d\mu = \frac{3}{2} \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} - \frac{3}{2} \int_{0}^{\infty} \frac{d}{d\mu} \left( \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) \right) d\mu \tag{D.37}
\]

\[
\tag{D.38}
\]

\[
= \frac{3}{2} \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} - \frac{3}{2} \mu^{\frac{4}{3}} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} \tag{D.38}
\]

\[
= \frac{3}{2} x^{\frac{4}{3}} K_{\frac{2}{3}}(x) + \frac{3}{2} x^{\frac{4}{3}} \left[ K_{\frac{2}{3}}(x) + \frac{2}{3x} K_{\frac{1}{3}}(x) \right] \tag{D.39}
\]

133
which gives that
\[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{\frac{1}{3}} K_{\frac{2}{3}}(\mu) d\mu = K_{\frac{1}{3}}(x) \] (D.40)

**D.8** \[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right) K_{\frac{2}{3}}(\mu) d\mu \]

We have that
\[ \int_{x}^{\infty} \mu K_{\frac{2}{3}}(\mu) d\mu = \int_{x}^{\infty} \mu^{-\frac{2}{3}} \left[ \mu^{\frac{5}{3}} K_{\frac{2}{3}}(\mu) \right] d\mu \] (D.41)

\[ = - \int_{x}^{\infty} \mu^{-\frac{2}{3}} \frac{d}{d\mu} \left( \mu^{\frac{5}{3}} K_{\frac{2}{3}}(\mu) \right) d\mu \] (D.42)

and we now use integration by parts to get
\[ \int_{x}^{\infty} \mu K_{\frac{2}{3}}(\mu) d\mu = -\mu K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} - \frac{2}{3} \int_{x}^{\infty} K_{\frac{2}{3}}(\mu) d\mu \] (D.43)

which gives
\[ \int_{x}^{\infty} \frac{\mu}{x} K_{\frac{2}{3}}(\mu) d\mu = K_{\frac{2}{3}}(x) - \frac{2}{3x} F(x) \] (D.44)

**D.9** \[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{\frac{5}{3}} K_{\frac{2}{3}}(\mu) d\mu \]

Note that
\[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{\frac{5}{3}} K_{\frac{2}{3}}(\mu) d\mu = -\frac{1}{x^{\frac{2}{3}}} \int_{x}^{\infty} \frac{d}{d\mu} \left( \mu^{\frac{5}{3}} K_{\frac{2}{3}}(\mu) \right) d\mu = -\left( \frac{\mu}{x} \right)^{\frac{5}{3}} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} = K_{\frac{2}{3}}(x) \] (D.45)

\[ = K_{\frac{1}{3}}(x) + \frac{4}{3x} K_{\frac{2}{3}}(x) \] (D.46)
\[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{7/3} K_{\frac{2}{3}}(\mu) d\mu \]

\[ \int_{x}^{\infty} \mu^{7/3} K_{\frac{2}{3}}(\mu) d\mu = -\int_{x}^{\infty} \frac{\mu^{2}}{d\mu} \left( \mu^{5/3} K_{\frac{2}{3}}(\mu) \right) d\mu \quad (D.47) \]

\[ = - \left[ \mu^{5/3} K_{\frac{2}{3}}(\mu) \bigg|_{x}^{\infty} - \frac{2}{3} \int_{x}^{\infty} \mu^{5/3} K_{\frac{2}{3}}(\mu) d\mu \right] \quad (D.48) \]

And considering the second term

\[ \int_{x}^{\infty} \mu^{4/3} K_{\frac{2}{3}}(\mu) d\mu = \int_{x}^{\infty} \mu^{4/3} K_{\frac{1}{3}}(\mu) d\mu + \frac{4}{3} \int_{x}^{\infty} \mu^{1/3} K_{\frac{2}{3}}(\mu) d\mu \quad (D.49) \]

\[ = -\mu^{4/3} K_{\frac{1}{3}}(\mu) - \frac{4}{3} \mu^{1/3} K_{\frac{2}{3}}(\mu) \quad (D.50) \]

and substituting this in gives

\[ \int_{x}^{\infty} \mu^{7/3} K_{\frac{2}{3}}(\mu) d\mu = -\mu^{7/3} K_{\frac{2}{3}}(\mu) - \frac{2}{3} \mu^{4/3} K_{\frac{1}{3}}(\mu) - \frac{8}{9} \mu^{1/3} K_{\frac{2}{3}}(\mu) \quad (D.52) \]

and dividing this by \( x^{7/3} \) gives

\[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{7/3} K_{\frac{2}{3}}(\mu) d\mu = K_{\frac{1}{3}}(x) + \frac{2}{3x} K_{\frac{1}{3}}(x) + \frac{8}{9x^2} K_{\frac{1}{3}}(x) \quad (D.53) \]
Appendix D. Integral Conversions

D.11 \[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{3} K_{\frac{2}{3}}(\mu) d\mu \]

\[ \int_{x}^{\infty} \mu^{3} K_{\frac{2}{3}}(\mu) d\mu = - \int_{x}^{\infty} \mu^{\frac{4}{3}} \frac{d}{d\mu} \left( \mu^{\frac{2}{3}} K_{\frac{2}{3}}(\mu) \right) d\mu \]  \hspace{1cm} (D.54)

\[ = - \mu^{3} K_{\frac{2}{3}}(\mu) + \frac{4}{3} \int_{x}^{\infty} \mu^{2} K_{\frac{2}{3}}(\mu) d\mu \]  \hspace{1cm} (D.55)

\[ = - \mu^{3} K_{\frac{2}{3}}(\mu) + \frac{4}{3} \int_{x}^{\infty} \mu^{2} K_{\frac{2}{3}}(\mu) d\mu + \frac{16}{9} \int_{x}^{\infty} \mu K_{\frac{2}{3}}(\mu) d\mu \]  \hspace{1cm} (D.56)

and substituting from before gives

\[ \int_{x}^{\infty} \mu^{3} K_{\frac{2}{3}}(\mu) d\mu = - \mu^{3} K_{\frac{2}{3}}(\mu) - \frac{8}{3} \mu^{2} K_{\frac{2}{3}}(\mu) - \frac{32}{9} \mu K_{\frac{2}{3}}(\mu) - \frac{128}{27} K_{\frac{2}{3}}(\mu) - \frac{64}{27} F(x) \]  \hspace{1cm} (D.57)

and this finally gives

\[ \int_{x}^{\infty} \left( \frac{\mu}{x} \right)^{3} K_{\frac{2}{3}}(\mu) d\mu = K_{\frac{4}{3}}(x) + \frac{8}{3x} K_{\frac{2}{3}}(x) \]

\[ + \frac{32}{9x^{2}} K_{\frac{4}{3}}(x) + \frac{128}{27x^{3}} K_{\frac{2}{3}}(x) - \frac{64}{27x^{3}} F(x) \]  \hspace{1cm} (D.58)

D.12 \[ K_{n,\mu} = \int_{0}^{\infty} x^{n} K_{\mu}(x) dx = \frac{1}{n-\mu+1} \int_{0}^{\infty} x^{n+1} K_{\mu-1}(x) dx \]

\[ K_{n,\mu} = \int_{0}^{\infty} x^{n} K_{\mu}(x) dx = \int_{0}^{\infty} x^{n-\mu} x^{\mu} K_{\mu}(x) dx \]  \hspace{1cm} (D.59)

So setting

\[ dv = x^{n-\mu} dx \quad v = \frac{x^{n-\mu+1}}{n-\mu+1} \quad u = x^{\mu} K_{\mu}(x) \quad du = -x^{\mu} K_{\mu-1}(x) \]  \hspace{1cm} (D.60)
\[ K_{n,\mu} = \int_{0}^{\infty} x^n K_{\mu}(x) \, dx = \left[ \frac{x^{n+1}}{n - \mu + 1} K_{\mu}(x) \right]_{0}^{\infty} + \int_{0}^{\infty} \frac{x^{n-\mu+1}}{n - \mu + 1} x^{\mu} K_{\mu+1}(x) \, dx \] (D.61)

\[ = \frac{1}{n - \mu + 1} \int_{0}^{\infty} x^{n+1} K_{\mu-1} \, dx \] (D.62)
Appendix E

Bessel Function Relations

There are a number of Bessel function relationships that were extremely important. The relationships given here have been used extensively in deriving the DS model.

\[ K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x} K_n(x) \quad (E.1) \]

\[ K'_n(x) = \frac{1}{2} [K_{n-1}(x) + K_{n+1}(x)] \quad (E.2) \]

\[ K'_n(x) = - K_{n-1}(x) - \frac{n}{x} K_n(x) \quad (E.3) \]

\[ K'_n(x) = \frac{n}{x} K_n(x) - K_{n+1}(x) \quad (E.4) \]

\[ \frac{d}{dx} (x^n K_n(x)) = -x^n K_{n-1}(x) \quad (E.5) \]

\[ \frac{d}{dx} (x^{-n} K_n(x)) = -x^{-n} K_{n+1}(x) \quad (E.6) \]

\[ K_n(x) = K_{-n}(x) \quad (E.7) \]
Bibliography


Bibliography


