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On the Table of Marks of a Direct Product of Finite Groups

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Contents

List of Tables vii

List of Figures ix

Acknowledgements xiii

Abstract xv

Chapter 1 Introduction 1

1.1 Introduction 2

1.2 Historical Overview 2

1.3 Notation 4

1.4 Outline of Thesis 5

Chapter 2 About the Table of Marks 7

2.1 Introduction 8

2.2 Fixed Points 8

2.3 The Mark Homomorphism 9

2.4 The Table of Marks 10

2.5 Current Methods 14

2.5.1 Pfeiffer's Method 15

2.5.2 Naughton's Method 16

2.6 Motivation 17
2.7 Where to begin? 18

Chapter 3 Bisets and double Burnside rings **19**
3.1 Introduction 20
3.2 Bisets 20
3.3 The double Burnside ring 22

Chapter 4 Sections **25**
4.1 Introduction 26
4.2 Sections 26
4.3 Normalizer and Centralizer 29
4.4 The Class Incidence Matrix of Sections 30
4.5 An Alternative Partial Order on Sections 35
4.6 Isomorphisms 39

Chapter 5 Subgroups of Direct Products **43**
5.1 Introduction 44
5.2 Goursat triples 44
5.3 Conjugacy Classes of Subgroups 53
5.4 Monomorphism Class Incidence Matrix 58

Chapter 6 The Bifree and Left-free Double Burnside Ring **61**
6.1 Introduction 62
6.2 The bifree double Burnside ring 62
6.3 The left-free double Burnside ring 65
6.4 The lattice of bifree subgroups of $G \times G$ 67
6.5 The bifree table of marks of $G \times G$ 68
6.6 The left-free Table of Marks 73

Chapter 7 The Table of Marks of a Direct Product **81**
7.1 Introduction 82
7.2 The Table of Marks of $G \times C_p$ 88
7.3 The Decomposition of the Table of Marks 89
7.4 The same K relation 90
7.5 The same P/K relation 93
7.6 The same P relation 96
7.7 Conclusion 98

Chapter 8 Application to double Burnside algebras **103**
8.1 Introduction 104
8.2 Cellular Algebras 104
8.3 The Natural Basis of the Double Burnside Ring 110
8.4 A Base Change 113
8.5 A Conjecture 130

List of Tables

2.1 The table of marks of A_5 10

2.2 The inverse of the table of marks of A_5 14

4.1 The class incidence matrix of sections of S_3 35

4.2 The class incidence matrix of sections of S_3 with respect to Proposition 4.5.3 38

5.1 Monomorphism class incidence matrix of S_3 59

6.1 The table of marks of A_4 70

6.2 The bifree table of marks of $A_4 \times A_4$ 70

6.3 The permuted bifree table of marks of $A_4 \times A_4$ 71

List of Figures

4.1 Depiction of Butterfly lemma 28
4.2 Depiction of Butterfly lemma when sections are linked 29
4.3 Decomposition of section incidence 31
4.4 Decomposition of section incidence with respect to Proposition 4.5.3 37

5.1 The position of a subgroup M in the subgroup lattice of $G_1 \times G_2$ 46
5.2 Decomposition of subgroup incidence with respect to Theorem 5.2.9 52

6.1 The bifree subgroup lattice of $S_4 \times S_4$ 67

Declaration

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Brendan Masterson

Dated: February 10, 2016

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Abstract

The table of marks was first introduced by William Burnside in his book “Theory of groups of finite order” in 1955 [9]. The table of marks counts the number of fixed points one subgroup has in the action of the cosets of another. In doing this it also encodes a lot of useful information about the subgroup lattice of a group G, including the index of each of G's subgroups in both G and their normalizers, containments and what cyclic subgroups G has [22].

Despite their usefulness they are extremely expensive to compute (the GAP table of marks library extends only as far as S_{13}). Thus one purpose of present research is find an efficient way to compute the table of marks of a direct product of finite group.

This is more difficult than one might expect. We consider a direct product of two finite groups $G_1 \times G_2$, using Goursat's lemma we hope to use knowledge of the table of marks of G_1 and G_2 to compute the table of marks of $G_1 \times G_2$. The methods developed in the present research also gives rise to a new base change matrix for the double Burnside algebra, $\mathbb{Q}B\left(G,G\right)$, which it will be conjectured gives a cellular basis for the algebra.

Chapter 1

Introduction

1.1 Introduction

This thesis is primarily concerned with the table of marks of the direct product of finite groups and its related structures such as bisets and double Burnside rings. In particular we want to take advantage of the biset structure associated with the subgroups of a direct product of finite groups.

To this end we develop a significant amount of theory for sections. A section of a group G is a pair of groups (P, K), where $K \trianglelefteq P \leq G$. We make some observations on the incidences between conjugacy classes of sections of a group G. This allows us to define a class incidence matrix for the sections of a finite group.

These results are further generalised to the conjugacy classes of subgroups of a direct product, hence simplifying the structure of the table of marks of a direct product of finite groups. We then also apply these results to produce a different base change matrix for the double Burnside ring of the symmetric group S_3 which gives a cellular basis for the algebra.

This introduction begins with a historical overview of the area, continues with a note on some of the notation used throughout the present work and finishes with an outline of the structure of the thesis.

1.2 Historical Overview

The Burnside ring was first introduced by William Burnside in his book "Theory of Groups of Finite Order" [9] as a way of describing how a finite group G can act on its G-sets. The table of marks encodes this information and presents it in the compact form of a lower triangular matrix with rows and columns indexed by the conjugacy classes of subgroups of G.

Definition 1.2.1. [23] Let G be a finite group and let $cl_G = \{H_1, \ldots, H_n\}$ be a complete set of representatives of the conjugacy classes of subgroups of G. For simplicity we will assume that $|H_i| \leq |H_j|$ for $i \leq j$.

- If X is any G-set. Then the function

$$\beta_X : cl_G \to \mathbb{Z},$$

$$H \mapsto |Fix_X(H)|$$

is called the *impression* of X.

- The *Table of marks* of G is the square matrix

$$M(G) = \left(\beta_{G/H_i}(H_j)\right)_{1 \le i,j \le n}.$$

This table includes useful information about the subgroup lattice of a group G, including the index of each subgroup in its normalizer, if G has any cyclic subgroups and gives a complete characterisation of the permutation representations of G. Despite its usefulness, it is very difficult to compute.

Many methods have been developed to more efficiently compute the table of marks of a finite group. These include computing the table of marks of G from the table of marks of maximal subgroups of G [28] and using the composition series of a solvable group G to compute the table of marks of G [24]. However despite these efforts the GAP table of marks library extends only to the symmetric group S_{13} [25]. It should be noted however that two non-isomorphic groups can have identical tables of marks [35].

The domain of the table of marks of G under the map $\beta_X(G)$, is known as the Burnside ring of G. This ring defined as follows:

Definition 1.2.2. The *Burnside ring* $B(G)$ of a finite group G is the Grothendieck group of the category of finite G-set with addition defined as disjoint union between formal differences of finite G-sets and multiplication defined as direct products between finite G-sets.

This ring is commutative [6] and the algebra $\mathbb{Q}B(G) := \mathbb{Q} \otimes_{\mathbb{Z}} B(G)$ is semisimple over $\mathbb{Q}$ [33] and generally well understood. A related object, which has been the centre of considerable research in the areas of group theory [34], fusion systems [30], representation theory [2, 7, 8] and algebraic topology [10], is the double Burnside ring. However despite all this research, much remains unknown about the structure of the double Burnside ring.

Definition 1.2.3. [6] Let G_1 and G_2 be finite groups. Then a (G_1, G_2)-*biset* X is both a left G_1-set and a right G_2-set, such that the G_1-action and G_2-action commute, i.e.,

$$\forall g_1 \in G_1, \forall x \in X, \forall g_2 \in G_2\colon\ (g_1 \cdot x) \cdot g_2 = g_1 \cdot (x \cdot g_2)\,.$$

Definition 1.2.4. The *double Burnside group* $B\,(G_1, G_2)$, where G_1 and G_2 are finite groups, is the Grothendieck group of the category of finite (G_1, G_2)-bisets.

The double Burnside group, denoted $B\,(G_1, G_2)$, is the biset analog of the classical Burnside group $B\,(G_1 \times G_2)$. However when $G_1 = G_2$ we can define a multiplication, under which the abelian group $B\,(G_1, G_2)$ would be closed, producing a ring structure on this group. In this thesis we aim to take advantage of the biset structure to construct the table of marks for a direct product.

1.3 Notation

Throughout this thesis we will discuss many different families of groups which we will denote as follows; the symmetric and alternating groups on n letters shall be denoted as S_n and A_n respectively, the cyclic and dihedral groups of order n will be denoted C_n and D_n respectively, the Klein four group will be denoted as V_4 and finally the quaternion group shall be denoted as Q_8. The set of all subgroups of group G will be denoted $\mathcal{S}_G$.

$\mathrm{Aut}(G)$, $\mathrm{Out}(G)$ and $\mathrm{Inn}(G)$ will denote the automorphism, outer automorphism and inner automorphism groups of a group G. Whereas $\mathrm{Aut}_G(H)$ and $\mathrm{Out}_G(H)$ denotes the group of automorphisms and outer automorphisms induced by G on a subgroup H.

1.4 Outline of Thesis

In Chapter 2 we review some basic results and properties of tables of marks. Current methods on how to compute the table of marks directly, via maximal subgroups and via cyclic extensions will also be reviewed. This will provide motivation for the present research.

Chapter 3 introduces bisets, Burnside rings and double Burnside rings. It will largely consist of recent results and attempts from researchers to understand the ring and algebra structure of the double Burnside ring and double Burnside algebras.

In Chapter 4 we discuss the many properties and uses of the sections of a group G. In this chapter we make some observations on the incidences of sections. This results in an alternative partial order on sections. We also consider the set of isomorphisms from sections of G into the isomorphism type of quotient group of the section. These results will be fundamental to the research carried out in the later chapters of this thesis.

In Chapter 5 we introduce Goursat's Lemma and use it to further elaborate on the structure of the subgroups of $G_1 \times G_2$ based on knowledge of the subgroup structure of G_1 and G_2, respectively. The chapter will finish with a theorem on a method to compute the conjugacy classes of subgroups of $G_1 \times G_2$ from the conjugacy classes of subgroups of G_1 and G_2, respectively.

Chapter 6 will look at the bifree and left-free double Burnside rings and make some interesting observations on the lattice of bifree subgroups of $G \times G$ and its related table of marks. We will observe that both the lattice of bifree subgroups and the bifree table of marks of $G \times G$ are very closely related to those of G. We will also produce formulas for the table of marks of the bifree and left-free subgroups.

Chapter 7 focuses on the classical table of marks of a direct product of two finite groups as well as some observations on a permuted table of marks. We develop a method to compute the table of marks of $G_1 \times G_2$ from the table of marks of G_1 and G_2 and their respective subgroups through using containment properties deduced from Goursat's lemma.

In Chapter 8 we will describe an application of our methods to double Burnside algebras. This will provide a base change for the double Burnside algebras which offers insight to the simple modules of the double Burnside algebra. This basis also suggests that the double Burnside algebra is a cellular algebra. We finish with a conjecture for future work in the area.

Chapter 2

About the Table of Marks

2.1 Introduction

In this chapter we will introduce the table of marks of a finite group G and describe its various properties in order to motivate why one would wish to compute the table of marks of a finite group. We will then describe existing methods which have been implemented to compute the table of marks, including Pfeiffer's method based on the maximal subgroups of a group G [28] and Naughton's method which computes the table of marks of G based on cyclic extensions of a subgroup $H \trianglelefteq G$ where H is of prime p index in G [24].

2.2 Fixed Points

Definition 2.2.1. Let G be a group and X be a finite non-empty set. The pair $(X, \cdot)$ is called a right G-set, and G is said to act on X if

$$\cdot : X \times G \to X$$

$$(x, g) \mapsto x \cdot g$$

satisfies $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ for all $g_1, g_2 \in G$, $x \in X$ and $x \cdot 1_G = x$ for all $x \in X$.

Let G be a finite group and X be a finite G-set. For any element $a \in G$ we define the set of *fixed points* of a on X as

$$Fix_X(a) := \{x \in X : x \cdot a = x\}.$$

Let A be a subset of G. Define the common set of fixed points of all $a \in A$ to be

$$Fix_X(A) := \{x \in X : x \cdot a = x \text{ for all } a \in A\} = \bigcap_{a \in A} Fix_X(a).$$

A G-set X is determined up to similarity by the fixed points of the subgroups of G as follows. If G acts on a set X, then by restriction so does every subgroup H of G. The number of fixed points of the subgroup H on a G-set X, $|Fix_X(H)|$, is called the *mark* of H on X. We now outline some basic properties of marks. Let X, Y be G-sets, $H \leq G$ and $g \in G$.

- $Fix_{X\cup Y}(H) = Fix_X(H) \cup Fix_Y(H)$. In particular $|Fix_{X\coprod Y} H| = |Fix_X(H)| + |Fix_Y(H)|$.
- $Fix_{X\times Y}(H) = Fix_X(H) \times Fix_Y(H)$. In particular $|Fix_{X\times Y}(H)| = |Fix_X(H)| \cdot |Fix_Y(H)|$.
- If X and Y are similar then $|Fix_X(H)| = |Fix_Y(H)|$.
- $Fix_X(H^g) = (Fix_X(H)) \cdot g$. In particular $|Fix_X(H)| = |Fix_X(H^g)|$.

2.3 The Mark Homomorphism

The transitive G-sets, up to similarity, form a $\mathbb{Z}$-basis for the Burnside ring. Let X be a finite G-set and let $H \leq G$. Then the *mark* $\beta_X(H)$ of H on X is defined as $\beta_X(H) = |Fix_X(H)|$. Hence we have $\beta_{X\coprod Y}(H) = \beta_X(H) + \beta_Y(H)$ and $\beta_{X\times Y}(H) = \beta_X(H) \cdot \beta_Y(H)$. It should be noted that marks are also constant on conjugacy classes of subgroups, i.e., $\beta_X(H) = \beta_X(H^g)$ for all $g \in G$.

We define β_X for each $[X] \in B(G)$ to be the n-tuple $\beta_X = (\beta_X(H_1), \ldots, \beta_X(H_n))$, where $\{H_1, \ldots, H_n\}$ is a complete set of conjugacy classes of subgroups of G, this gives rise to a ring homomorphism $\beta : B(G) \to \mathbb{Q}^n$ which is defined by:

$$\beta : X \mapsto \beta_X$$

This ring homomorphism is called the *mark homomorphism* of G. The codomain of the mark homomorphism, $\mathbb{Q}^n$, is known as the *ghost ring* of G. The image of the mark homomorphism can also be expressed in terms of subgroup containments between conjugacy classes of subgroups of G.

Theorem 2.3.1. [23] Let G be a finite group, H and K subgroups of G and $g \in G$. Then

$$\beta_{G/K}(H) = |N_G(K) : K| \cdot |\{K^g \geq H : g \in G\}|.$$

2.4 The Table of Marks

Definition 2.4.1. Let G be a finite group and $cl(G) = \{H_1, \ldots, H_n\}$ be a complete set of representatives of conjugacy classes of subgroups of G. The *table of marks* of a finite group is the square matrix

$$M(G) = \left(\beta_{G/H_i}(H_j)\right)_{1 \le i,j \le n}.$$

Here we will look at the example of the alternating group A_5. The group A_5 has nine conjugacy classes of subgroups, i.e., $[1]$, $[C_2]$, $[C_3]$, $[V_4]$, $[C_5]$, $[S_3]$, $[D_{10}]$, $[A_4]$ and $[A_5]$.

	1	C_2	C_3	V_4	C_5	S_3	D_{10}	A_4	A_5
$A_5/1$	60								
A_5/C_2	30	2							
A_5/C_3	20	.	2						
A_5/V_4	15	3	.	3					
A_5/C_5	12	.	.	.	2				
A_5/S_3	10	2	1	.	.	1			
A_5/D_{10}	6	2	.	.	1	.	1		
A_5/A_4	5	1	2	1	.	.	.	1	
A_5/A_5	1	1	1	1	1	1	1	1	1

Table 2.1: The table of marks of A_5

Definition 2.4.2. We define the *class incidence matrix* of a finite group G as the table of marks of G with each row divided by the index of the corresponding subgroup in its normalizer.

We will now present some properties of the table of marks in order to motivate why one would be interested in computing the table of marks.

Lemma 2.4.3. Let H, $K \leq G$. The following are true for the table of marks of a group G:

1. The first entry in each row of the table of marks, $M(G)$, is the index in G of the corresponding conjugacy class of subgroups
$$\beta_{G/K}(1) = |G : K|.$$

2. The entry on the diagonal gives the index of a representative of a conjugacy class of subgroups in its normalizer in G, i.e.
$$\beta_{G/K}(K) = |N_G(K) : K|.$$

3. The length of the conjugacy class $[K]$ of K is given by
$$|[K]| = |G : N_G(K)| = \frac{\beta_{G/K}(1)}{\beta_{G/K}(K)}.$$

4. The number of conjugates of K which contain H as a subgroup is
$$|\{K^g \geq H : g \in G\}| = \frac{\beta_{G/K}(H)}{\beta_{G/K}(K)}.$$

Proof. 1. Let X be a G-set. Clearly $\beta_X(1) = |X|$ and $|G/K| = |G : K|$, hence $\beta_{G/K}(1) = |G : K|$.

2. Let $K \leq G$, then $KgK = Kg$ if and only if $Kg \subseteq N_G(K)$, so we have $\beta_{G/K}(K) = |N_G(K) : K|$.

3. $\beta_{G/K}(1) = |G : K|$ and $\beta_{G/K}(K) = |N_G(K) : K|$.
$$\frac{\beta_{G/K}(1)}{\beta_{G/K}(K)} = \frac{|G : K|}{|N_G(K) : K|} = |G : N_G(K)| = |[K]|.$$

4. $\beta_{G/K}(H) = |N_G(K) : K| \cdot |\{K^g \geq H : g \in G\}|$ and $\beta_{G/K}(K) = |N_G(K) : K| \cdot |\{K^g \geq K : g \in G\}| = |N_G(K) : K|$. Thus
$$\frac{\beta_{G/K}(H)}{\beta_{G/K}(K)} = \frac{|N_G(K) : K| \cdot |\{K^g \geq H : g \in G\}|}{|N_G(K) : K|} = |\{K^g \geq H : g \in G\}|.$$

□

Remark 2.4.4. It should be noted that the notion of a class incidence matrix can be generalised to any partially ordered set which G acts upon.

Theorem 2.4.5. [23] Let H and K be subgroups of a finite group G. Then the number of conjugates of H contained in K is given by

$$\vartheta_G(K,H) = |\{H^g : g \in G, H^g \leq K\}| = \frac{\beta_{G/K}(H) \cdot \beta_{G/H}(1)}{\beta_{G/K}(1) \cdot \beta_{G/H}(H)}.$$

Corollary 2.4.6. [28] Let $K, H \leq G$. Then

$$\beta_{G/K}(H) = |G : K| \, \vartheta_G(K,H) / \vartheta_G(G,H).$$

The table of marks can also offer some interesting insights into the structure of the subgroups of a finite group G. For instance we have the following lemma on the marks of cyclic subgroups of a group G which through its proof makes explicit the connection between marks and permutation characters.

Let X be a G-set. The permutation character π_X of G on X is defined as $\pi_X(g) = |Fix_X(g)|$ for any element $g \in G$. This coincides with $\beta_X(\langle g \rangle)$ of the cyclic group generated by g on X. Therefore, the table of marks contains in the columns corresponding to the cyclic subgroups a complete list of transitive permutation characters $\pi_{G/H}$ of G corresponding to the transitive G-sets G/H.

The ghost ring $\mathbb{Q}^n$ has basis $e_i := (e_i(H_1), \ldots, e_i(H_n))$. Here e_i is the basis element corresponding to the conjugacy class of H_i, that is, $e_i(H_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function, and write

$$e_i = \sum_j e_{ij} \beta_{G/H_j}$$

with rational coefficients e_{ij}. The matrix $(e_{ij})_{1 \leq i,j \leq n}$ is then the inverse of the table of marks, $M(G)$.

Lemma 2.4.7. [22] Let $i, j \leq n$ and let e_{ij} and H_i be as above. Then

$$\sum_j e_{ij} = \begin{cases} \phi(|H_i|) / |N_G(H_i)|, & \text{if } H_i \text{ is cyclic,} \\ 0, & \text{otherwise,} \end{cases}$$

where ϕ denotes the Euler function.

Proof. Let e_i be defined as above. For any subgroup $H \leq G$ we set $e_i(H) = e_i(H_j)$ if H belongs to the j^{th} conjugacy class of subgroups of G. Then e_i defines a class function π_{e_1} on G via $\pi_{e_i}(g) = e_i(\langle g \rangle)$. Thus β_{G/H_j} this way yields the permutation character π_{G/H_j} and $\left\langle \pi_{G/H_j}, 1_G \right\rangle = 1$ and

$$\sum_j e_{ij} = \sum_j e_{ij} \left\langle \pi_{G/H_j}, 1_G \right\rangle$$

$$= \left\langle \sum_j e_{ij} \pi_{G/H_j}, 1_G \right\rangle = \langle \pi_{e_i}, 1_G \rangle,$$

where $\pi_{e_i} = 0$ and therefore $\langle \pi_{e_i}, 1_G \rangle = 0$ unless H_i is cyclic.

If H_i is cyclic then the result follows from the fact that $\phi(|H_i|)$ generators of H_i lie in

$$\phi(|H_i|) / |N_G(H_i) : C_G(H_i)|$$

different conjugacy classes of elements of G each of which contributes $1/|C_G(H_i)|$ to $\langle \pi_{e_i}, 1_G \rangle$. □

Example 2.4.8. To demonstrate this Lemma at work we will again look at the example of the alternating group A_5. The inverse of the table of marks of A_5 is below with the row sums in the right most column. The row sums corresponding to cyclic groups are the only row sums which are non-zero.

Remark 2.4.9. The sum of all the entries in $M^{-1}(G)$ is equal to one. This arises from the fact that the column sum of the inverse of the table of marks is 0 with the exception of the column which corresponds to G, as can be seen from considering the trivial action of G. We recall that the table of marks is a base change matrix between the Burnside ring, $B(G)$, and the ghost ring, $\mathbb{Q}^n$. Hence the image of $e_n \in \mathbb{Q}^n$ under $M(G)$ is the trivial transitive G-set, thus yielding the equation:

$$(0, 0, \ldots, 1) \cdot M(G) = (1, 1, \ldots, 1)$$

$A_5/1$	$\frac{1}{60}$									$\frac{1}{60}$
A_5/C_2	$\frac{-1}{4}$	$\frac{1}{2}$								$\frac{1}{4}$
A_5/C_3	$\frac{-1}{6}$	.	$\frac{1}{2}$							$\frac{1}{3}$
A_5/V_4	$\frac{1}{6}$	$\frac{-1}{2}$	.	$\frac{1}{3}$						0
A_5/C_5	$\frac{-1}{10}$	.	.	.	$\frac{1}{2}$					$\frac{2}{5}$
A_5/S_3	$\frac{1}{2}$	-1	$\frac{-1}{2}$	.	.	1				0
A_5/D_{10}	$\frac{1}{2}$	-1	.	.	$\frac{-1}{2}$	.	1			0
A_5/A_4	$\frac{1}{3}$	.	-1	$\frac{-1}{3}$	.	.	.	1		0
A_5/A_5	-1	2	1	.	.	-1	-1	-1	1	0
	1	C_2	C_3	V_4	C_5	S_3	D_{10}	A_4	A_5	Σ

Table 2.2: The inverse of the table of marks of A_5

$$\Rightarrow (0, 0, \ldots, 1) = (1, 1, \ldots, 1) \cdot M^{-1}(G).$$

From the above equation it is clear that the column sum of $M^{-1}(G)$ is 0 with the exception of the final column. Thus if one first sums along the columns of $M^{-1}(G)$ then along the rows it is clear that the sum of all the entries will come to 1.

Despite all the information which the table of marks provides on the subgroup lattice of a finite group G, it should however be noted that the table of marks does not determine a group up to isomorphism [35]. Examples exist of groups which are not isomorphic but have identical tables of marks.

2.5 Current Methods

The table of marks can be computed directly by explicitly computing the marks using the formula in Theorem 2.3.1. However, this requires complete knowledge of the subgroup lattice of a finite group G. The marks are calculated by counting inclusions between conjugacy classes of subgroups. This method works well if G is small and the subgroup lattice of G is known in advance. When this is not the case this kind of direct computation becomes impractical. There have been attempts made by Pfeiffer and Naughton to develop new methods to compute the table of marks. We will give a brief description of these methods here as they help motivate the present research.

2.5.1 Pfeiffer's Method

In his paper *The Subgroups of M_{24}, or How to compute the table of marks of a finite group* [28] Pfeiffer notes that large parts of the table of marks of G can be computed by examining marks from the table of marks of the maximal subgroups of G. In particular, if the table of marks of all the maximal subgroups of G are known one obtains by induction formulae all the rows of the table of marks of G with the exception of the row corresponding to the trivial representation of G. However this row consists entirely of 1's and can easily be added afterwards. This induction formula is given by the following theorem.

Theorem 2.5.1. [28] Let $L \leq H \leq G$ and $M \leq G$. Then the mark $\beta_{G/L}(M)$ is given by

$$\beta_{G/L}(M) = |N_G(M)| \sum_{M' \approx_G M} \frac{1}{|N_H(M')|} \beta_{G/L}(M')$$

where the sum ranges over all representatives M' of conjugacy classes of subgroups of H that are conjugate to M in G.

The difficulty is that the conjugacy classes of subgroups of G, which index the rows and columns of the table of marks, are not known in advance. Pfeiffers method utilises results such as the Sylow theorems to construct the conjugacy classes of subgroups G from those of its maximal subgroups which is known from their table of marks. The number of table of marks which can be computed using this method has since been exhausted.

2.5.2 Naughton's Method

Naughton's method computes the table of marks of a group G from the table of marks of a finite group H such that $H \trianglelefteq G$ is of prime index p [24]. This method begins with computing the conjugacy classes of subgroups of G from those of H. To achieve this Naughton identifies two types of conjugacy classes of subgroups of G, which he calls *blue* and *red* conjugacy classes of subgroups.

Blue conjugacy classes of subgroups of G are unions of H-conjugacy classes of subgroups. Red conjugacy classes of subgroups of G on the other hand correspond to certain conjugacy classes of subgroups of order p in normalizer quotients.

From there Naughton partitions the table of marks of G into three sections based on this observation. The top left section of the table represents fixed points of blue conjugacy classes of subgroups on other blue conjugacy classes of subgroups. He presents the following proposition as a formula for this portion of the table.

Proposition 2.5.2. [24] Suppose $L, M \leq H$. Let $g \in G/H$ and denote $L_i = L^{g^i}$, for $i = 0, 1, \ldots, p-1$. Then

$$\beta_{G/L}(M) = \sum_{i=0}^{p-1} \beta_{H/L_i}(M).$$

In particular if $[L]_G = [L]_H$ then $\beta_{G/L}(M) = p\beta_{H/L}(M)$.

The bottom left quarter of the table represents fixed points of blue subgroups on red subgroups. He denotes $\gamma(K) = K \cap H$ for $K \leq G$. The formula for this portion is given by the proposition:

Proposition 2.5.3. [24] Suppose that $K \leq G$ is a red subgroup with $\gamma(K) = L \leq H$. Then the coset spaces G/K and H/L are equivalent as H-sets. In particular,

$$\beta_{G/K}(M) = \beta_{H/L}(M)$$

for all subgroups $M \leq H$.

The remaining bottom right portion represents fixed points of red subgroups on other red subgroups. Naughton points out that marks in this particular portion often cannot be computed directly by one formula. He does however, give a number of different bounds on the marks in this portion which through repeated application should uniquely determine the marks. Although if this is not the case, one can compute the marks explicitly by counting incidences between the relevant conjugacy classes of subgroups.

It should be noted that not every group lends itself to this method of computation of its table of marks. In particular perfect groups are not suitable for this method as they do not contain a normal subgroup of prime index. However this method is extremely useful for computing the table of marks of a solvable group. We will consider this problem and propose a new method for computing the table of marks of $G \times C_p$, where p is prime in Chapter 7.

2.6 Motivation

The motivation for the present research stems from the study of the table of marks of a direct product. The aim is to compute the table of marks of $G_1 \times G_2$, where G_1 and G_2 are finite groups, from the table of marks of G_1 and G_2 and those of their subgroups.

The table of marks of a direct product of finite group is extremely difficult to compute when compared to the character table of the same group. This is because computing the table of marks involves enumerating fixed points over conjugacy classes of subgroups, whereas with the character table we work with the conjugacy classes of elements. In the case of a direct product of groups the conjugacy classes of elements are simply the direct product of the conjugacy classes of elements of the factor groups. This is not the case with the conjugacy classes of subgroups of a direct product of groups.

Another motivating element for this project is to develop methods which can be used to expand the GAP library of table of marks, which only extents as far as the symmetric group S_{13}. Many of the maximal subgroups of S_n have the form $S_{n-k} \times S_k$. Thus if we can understand the table of marks of $S_{n-k} \times S_k$ Pfeiffer's method can be applied to compute the table of marks of S_n.

2.7 Where to begin?

The first thing needed when attempting to compute the table of marks of any group is to have a complete description of the conjugacy classes of subgroups. Our task is to compute the table of marks of $G_1 \times G_2$ from the tables of marks of G_1 and G_2 and those of their subgroups. Thus, we must describe the conjugacy classes of subgroups of $G_1 \times G_2$ from the conjugacy classes of subgroups of G_1 and G_2 respectively. However to perform this task it would be advantageous to think of the subgroups of $G_1 \times G_2$ in terms of (G_1, G_2)-bisets. This is where we will begin in Chapter 3.

Chapter 3

Bisets and double Burnside rings

3.1 Introduction

Bisets have been central to a number of different research areas in both algebra and topology including: category theory [8, 32], representation theory [4] and fusion systems [12, 30] and they are vital to the study of double Burnside rings.

It will often be useful for us to think of the subgroups of a direct product in terms of bisets. This will ultimately lead to a decomposition of the table of marks in terms of these bisets. We begin this chapter with some useful definitions for bisets before we later investigate their relationship to double Burnside rings.

3.2 Bisets

Definition 3.2.1. [6] Let G_1 and G_2 be finite groups. Then a (G_1, G_2)-*biset* X is both a left G_1-set and a right G_2-set, such that the G_1-action and G_2-action commute, i.e.,

$$\forall g_1 \in G_1, \forall x \in X, \forall g_2 \in G_2,\ (g_1 \cdot x) \cdot g_2 = g_1 \cdot (x \cdot g_2)\,.$$

Note that since the actions of G_1 and G_2 on a (G_1, G_2)-biset, X, commute, then first quotienting out the action of G_1 then G_2 is the same as quotienting out by G_2 then G_1.

Definition 3.2.2. Let $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$, then we set

$$L * M := \{(g_1, g_3) \in G_1 \times G_3 |\ \exists\, g_2 \in G_2 : (g_1, g_2) \in L, (g_2, g_3) \in M\}\,.$$

Clearly $L * M$ is a subgroup of $G_1 \times G_3$.

For every (G_1, G_2)-biset X, we denote by X^{op} its *opposite biset*. This opposite biset is a (G_2, G_1)-biset. As sets X and X^{op} are equal, and the biset structure on X^{op} is given by $g_2 x^{op} g_1 := \left(g_1^{-1} x g_2^{-1}\right)$, where $g_1 \in G_1$, $g_2 \in G_2$, $x \in X$, and x^{op} is x viewed as an element in X^{op}. It is obvious that $X = (X^{op})^{op}$.

Likewise, for any subgroup $L \leq G_1 \times G_2$ we define the *opposite subgroup* $L^{op} \leq G_2 \times G_1$ as follows:

$$L^{op} := \{(g_2, g_1) \in G_2 \times G_1 : (g_1, g_2) \in L\}.$$

It is worth noting that for $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$, $(L * M)^{op} = M^{op} * L^{op}$. This also induces an isomorphism of (G_2, G_1)-bisets

$$(G_2 \times G_1)/L^{op} \to (G_1 \times G_2/L)^{op}, \quad (g_2, g_1) L^{op} \mapsto ((g_1, g_2) L)^{op}.$$

Definition 3.2.3. We define a map $\Pi : \mathcal{S}_{G_1 \times G_2} \times \mathcal{S}_{G_3 \times G_2} \to \mathcal{S}_{G_1 \times G_3}$ such that

$$\Pi : (M, L) \mapsto M * L^{op},$$

where $\mathcal{S}_{G_i \times G_j}$ denotes the set of subgroups of $G_i \times G_j$, $M \leq G_1 \times G_2$ and $L \leq G_3 \times G_2$.

Lemma 3.2.4. [6] Let G_1 and G_2 be groups.

1. If L is a subgroup of $G_1 \times G_2$, then the set $(G_1 \times G_2)/L$ is a *transitive* (G_1, G_2)-biset for the actions defined by

$$\forall g_2 \in G_2,\ \forall L(b, a) \in (G_1 \times G_2)/L,\ \forall g_1 \in G_1,\ g_1 \cdot L(b, a) \cdot g_2 = L\left(bg_1^{-1}, ag_2\right).$$

2. If X is a (G_1, G_2)-biset, choose a set $[G_1 \backslash X / G_2]$ of representatives of (G_1, G_2)-orbits on X. Then there is an isomorphism of (G_1, G_2)-bisets

$$X \cong \coprod_{x \in [G_1 \backslash X / G_2]} (G_1 \times G_2)/L_x,$$

where $L_x = (G_1, G_2)_x$ is the stabiliser of x in $G_1 \times G_2$, i.e. the subgroup of $G_1 \times G_2$ defined by

$$(G_1, G_2)_x = \{(g_1, g_2) \in G_1 \times G_2 | g_1 \cdot x = x \cdot g_2\}.$$

In particular, any transitive (G_1, G_2)-biset is isomorphic to $(G_1 \times G_2)/L$, for some subgroup L of $G_1 \times G_2$.

This covers most of the improtant information required to understand bisets. As such, we can begin to investigate their relationship to double Burnside rings.

3.3 The double Burnside ring

We recall that the Burnside ring $B(G)$ of a finite group G is the Grothendieck group of the category of finite G-set with addition defined by disjoint union between formal differences of finite G-sets and multiplication defined as direct products between finite G-sets.

The isomorphism types of transitive G-sets form the standard basis for the Burnside ring $B(G)$, in particular these transitive G-sets are equal to $[G/L]$, where $L \leq G$ and $[G/L]$ denotes the isomorphism type of the G-set G/L.

Definition 3.3.1. The *double Burnside group* $B(G_1, G_2)$, where G_1 and G_2 are finite groups, is the Grothendieck group of the category of finite (G_1, G_2)-bisets with addition induced by disjoint union of finite (G_1, G_2)-bisets.

The isomorphism types of transitive (G_1, G_2)-bisets form the standard basis for the double Burnside group $B(G_1, G_2)$, in particular these transitive (G_1, G_2)-bisets are isomorphic to $(G_1 \times G_2)/L$, where $L \leq G_1 \times G_2$. We can identify (G_1, G_2)-bisets with right $G_1 \times G_2$ -sets as follows:

$$x(g_1, g_2) := g_1^{-1} x g_2 \quad \text{and} \quad g_1 y g_2 := y\left(g_1^{-1}, g_2\right),$$

for every (G_1, G_2)-biset X, every right $(G_1 \times G_2)$-set Y, $g_1 \in G_1$, $g_2 \in G_2$, $x \in X$, $y \in Y$. This then allows us to associate the double Burnside group $B(G_1, G_2)$ with the classical Burnside group $B(G_1 \times G_2)$.

Suppose that X is a (G_1, G_2)-biset and Y is a (G_2, G_3)-biset. Then their cartensian product becomes a (G_1, G_3)-biset in the obvious way. Moreover, $X \times Y$ is also a left G_2-set via $g_2(x, y) := \left(x g_2^{-1}, g_2 y\right)$, for $g_2 \in G_2$, $x \in X$ and $y \in Y$. This G_2-action commutes with the $G_1 \times G_3$-action, and the set $X \times_{G_2} Y$ of G_2-orbits on $X \times Y$ inherits the (G_1, G_3)-biset structure. We call $X \times_{G_2} Y$ the *tensor product* of X and Y and denote the G_2-orbit of an element $(x, y) \in X \times Y$ by $x \times_{G_2} y \in X \times_{G_2} Y$.

The tensor product of bisets gives rise to a $\mathbb{Z}$-bilinear map

$$- \cdot_{G_2} - : B(G_1, G_2) \times B(G_2, G_3) \to B(G_1, G_3), \; ([X], [Y]) \mapsto [X \times_{G_2} Y],$$

where X is a (G_1, G_2)-biset and Y is a (G_2, G_3)-biset. The following Mackey-type formula shows how to express the tensor product of a standard basis element of $B(G_1, G_2)$ and a standard basis element of $B(G_2, G_3)$ as a sum of standard basis element of $B(G_1, G_3)$.

Proposition 3.3.2. [6] For $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$, one has

$$[G_1 \times G_2/L] \cdot_{G_2} [G_2 \times G_3/M] = \sum_{g_2 \in [\pi_2(L) \backslash G_2 / \pi_1(M)]} \left[G_1 \times G_3 / \left(L * {}^{(g_2,1)}M\right)\right] \in B(G_1, G_3),$$

where π_2 is the projection map $\pi_2 : G_1 \times G_2 \to G_2$, likewise π_1 is the projection map $\pi_1 : G_2 \times G_3 \to G_2$ and $[\pi_2(L) \setminus G_2/\pi_1(M)] \subseteq G_2$ denotes a set of double coset representatives.

If $G_1 = G_2 = G_3 = G$ then the above product of bisets will produce a ring structure on the double Burnside group $B(G, G)$. This defines the double Burnside ring $B(G, G)$.

In order to understand the conjugacy classes of subgroups of a direct product of finite groups $G_1 \times G_2$ in terms of (G_1, G_2)-bisets, we must first make a study of the sections of a group G. These will be the fundamental building blocks for much of the work which will be carried out in this research.

Chapter 4

Sections

4.1 Introduction

This chapter will be concerned with sections. Sections are fundamental to much of the work which will be carried out in later chapters. However there is not much research in the literature which specifically studies sections. As such in this chapter we will develop a significant amount of theory for sections.

Among the topics we will consider here are incidences between conjugacy classes of sections of a group G, partial orders on these sections, a class incidence matrix for the sections of G and sets of isomorphisms from sections of G into isomorphism types of subquotients of G. Each of these observations, as will be seen in later chapters, will have important implications on the subgroups of a direct product, its table of marks and even the structure of the related double Burnside ring.

4.2 Sections

Definition 4.2.1. Let G be a group. A *section* of G is a pair (P, K) where $K \trianglelefteq P \leq G$. We call P/K a *subquotient* of G. For a group U such that U is isomorphic to a subquotient of G we write $U \sqsubseteq G$. If $U \sqsubseteq G$ and $U \ncong G$ we write $U \sqsubset G$.

We will often refer to the *quotient* of a section, which we define as the quotient group P/K of a section (P, K). We may also refer to the isomorphism type of a section, by this we will mean the isomorphism type of the quotient of the section.

Let H be a subgroup of G. Then for every $g \in G$, we define $H^g := g^{-1}Hg$ and ${}^gH := gHg^{-1}$. For a section (P, K) of G, we set $(P, K)^g := (P^g, K^g)$ and ${}^g(P, K) := ({}^gP, {}^gK)$. Two sections in G are said to be conjugate in G if they are componentwise conjugate in G, i.e., two sections, (P, K) and (P', K') are conjugate in G if there exist $g \in G$ such that $P^g = P'$ and $K^g = K'$ and we write $(P, K) \approx_G (P', K')$. We write $[(P, K)]_G$ for the conjugacy class of a section (P, K) in G. We will also say that the size of a section is given by $|(P, K)| = |P/K|$.

Definition 4.2.2. Let (P', K') and (P, K) be sections of a finite group G. Then we have the partial order: (P', K') is contained in (P, K) if $P' \leq P$ and $K' \leq K$. We write $(P', K') \leq (P, K)$.

Proposition 4.2.3. The complete set of conjugacy classes of sections is given by the set

$$\mathscr{S}^G = \bigcup_{[K] \in \mathcal{S}_G^G} \{(P, K)^g : g \in N_G(K)/K, P \leq G, K \trianglelefteq P\}.$$

Proof. We denote by $\mathscr{S} = \{(P, K) : K \trianglelefteq P \leq G\}$ to be the set of all sections of G. First we note that the set $\mathscr{S}$ can be rewritten as the disjoint union of sets of sections which have all the same bottom group K, i.e.:

$$\mathscr{S} = \bigcup_{K \leq G} \{P \leq G : K \trianglelefteq P\}.$$

As is stated above, we consider two sections conjugate in G if and only if they are component-wise conjugate in G. Hence if we want to compute the conjugacy classes of sections, we must begin by matching the K's up to conjugacy. The stabiliser of K in G, $N_G(K)$, naturally acts on each of these sets. However since K acts trivially on each of the P's, we need only consider the action of $N_G(K)/K$. Thus we have

$$\mathscr{S}/\approx_G = \bigcup_{[K] \in \mathcal{S}_G^G} \{(P, K)^g : g \in N_G(K)/K, P \leq G, K \trianglelefteq P\}$$

We shall denote this set as $\mathscr{S}^G$. □

Through a similar argument $\mathscr{S}^G = \bigcup_{[P] \in \mathcal{S}_G^G} \{(P, K)^g : g \in N_G(P)/P, K \trianglelefteq P\}$ the set of of conjugacy classes of sections of G which have section isomorphism type $U \sqsubseteq G$, i.e.:

$$\mathscr{S}^G(U) := \left\{[(P, K)]_G \in \mathscr{S}^G : P/K \cong U\right\}.$$

Naturally, these sets will play an important role in computing the conjugacy classes of subgroups of $G_1 \times G_2$.

Definition 4.2.4. [6] We say two sections, (P_1, K_1) and (P_2, K_2) of G are linked if

$$P_2 \cap K_1 = P_1 \cap K_2,\ (P_2 \cap P_1)\, K_2 = P_2 \text{ and } (P_2 \cap P_1)\, K_1 = P_1;$$

and we write $(P_2, K_2) - (P_1, K_1)$.

Lemma 4.2.5 (Butterfly Lemma)**.** [20] Let (P_1, K_2) and (P_2, K_2) be two sections of a finite group G. Then there exists a canonical isomorphism

$$\rho\,(P_1, K_1; P_2, K_2) : P_2'/K_2' \to P_1'/K_1',$$

where $K_i \leq K_i' \trianglelefteq P_i' \leq P_i$ are defined as

$$P_1' := (P_1 \cap P_2)\, K_1,\ K_1' := (P_1 \cap K_2)\, K_1,\ P_2' := (P_1 \cap P_2)\, K_2 \text{ and } K_2' := (P_2 \cap K_1)\, K_2.$$

The below diagram illustrates this lemma.

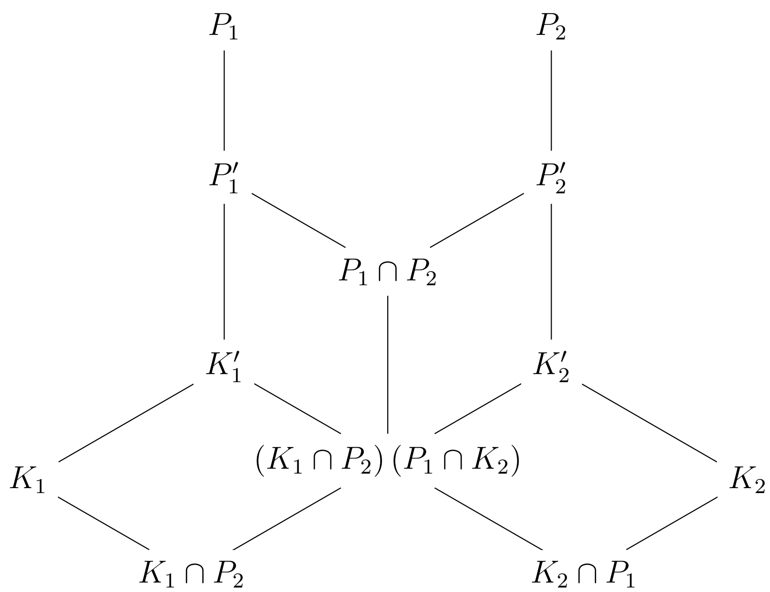


Figure 4.1: Depiction of Butterfly lemma

The sections (P_1, K_1) and (P_2, K_2) are linked if and only if this diagram reduces to

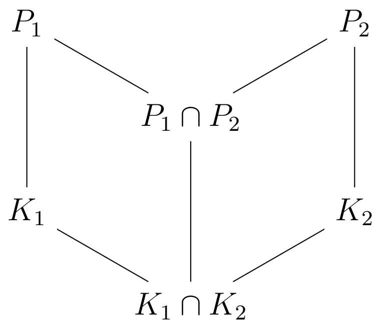


Figure 4.2: Depiction of Butterfly lemma when sections are linked

This induces an isomorphism $\rho\left(P_1, K_1; P_2, K_2\right) : P_2/K_2 \to P_1/K_1$ such that $pK_2 \mapsto pK_1$ for all $p \in P_1 \cap P_2$ between linked sections [6]. Thus linked sections have isomorphic section quotients.

4.3 Normalizer and Centralizer

Many of the properties which are defined for subgroups of G we can also define for sections of G. The concept of a normalizer and centralizer will be particularly useful in this study, as such they are defined here.

Definition 4.3.1. We write $N_G\left(P, K\right) := N_G\left(P\right) \cap N_G\left(K\right)$ for the *normalizer* of the section (P, K), that is all $g \in G$ such that $gPg^{-1} = P$ and $gKg^{-1} = K$.

Hence, we can define a group homomorphism $\gamma : N_G\left(P, K\right) \to \mathrm{Aut}\left(P/K\right)$ such that

$$n \mapsto \gamma\left(n\right) : pK \mapsto \left(pK\right)^n .$$

Definition 4.3.2. Let (P, K) be a section of G. The *centralizer* of a section $C_G\left(P, K\right)$ is the kernel of the homomorphism γ.

This definition gives rise to the induced group homomorphism

$$\overline{\gamma} : N_G\left(P, K\right)/C_G\left(P, K\right) \to \mathrm{Aut}\left(P/K\right),$$

$$n \cdot C_G(P, K) \mapsto \gamma(n) : pK \mapsto (pK)^n .$$

Clearly the elements of $C_G(P, K)$ commute with the elements of the group P/K since they are the elements of $N_G(P, K)$ which induce the trivial automorphism on P/K. Hence, $C_G(P, K) = C_{N_G(P,K)}(P/K)$.

Definition 4.3.3. Let (P, K) be a section of a finite group G. Then we define the *automizer* of the section (P, K) in G to be the section $(N_G(P, K), C_G(P, K))$ and we write $\mathscr{A}_G(P, K)$.

4.4 The Class Incidence Matrix of Sections

In Definition 4.2.2 we stated that we consider one section to be contained in another if it is componentwise contained in the other. We have also stated that we consider two sections of a finite group G to be conjugate in G if they are componentwise conjugate in G. Thus, we can count incidences between conjugacy classes of sections of G. This allows us to define an incidence matrix for the conjugacy classes of sections of G as follows:

Definition 4.4.1. Let $cl_{\mathscr{S}}(G) = \{(P_1, K_1), (P_2, K_2), \ldots, (P_n, K_n)\}$ be a complete set of representatives of the conjugacy classes of sections of a finite group G. The *class incidence matrix* of the sections of G is defined as

$$M_{\mathscr{S}}(G) = \left(\xi_{(P_i,K_i)}(P_j, K_j)\right)_{1\leq i,j\leq n},$$

where $\xi_{(P_i,K_i)}(P_j, K_j) = |\{(P_i, K_i)^g \geq (P_j, K_j) : g \in G\}|$.

Proposition 4.4.2. Let (P', K') and (P, K) be sections of a finite group G, such that $(P', K') \leq (P, K)$. Then there are uniquely determined sections of G, $(P, K) \geq (P_1, K_1) \geq (P_2, K_2) \geq (P', K')$, such that:

1. (P, K) and (P_1, K_1) have the same bottom group, i.e. $K = K_1$.

2. (P_2, K_2) and (P', K') have the same top group, i.e. $P_2 = P'$.

3. (P_1, K_1) and (P_2, K_2) have isomorphic quotients, and we have the isomorphism ψ : $P_2/K_2 \to P_1/K_1$, $\psi(pK_2) = pK_1$.

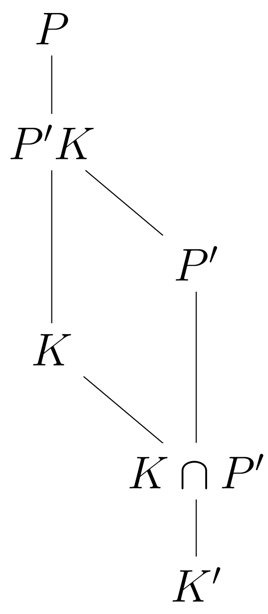


Figure 4.3: Decomposition of section incidence

Proof. Since $(P', K') \leq (P, K)$ we know $P' \leq P$ and $K' \leq K$ and by definition $K' \trianglelefteq P'$ and $K \trianglelefteq P$. By the second isomorphism theorem we have $K \cap P'$ is a normal subgroup of P', $P'K$ is a subgroup of P such that $K \trianglelefteq P'K$ and $(P'K)/K$ is isomorphic to $P'/(K \cap P')$. Hence, there are uniquely determined sections $(P', K \cap P') = (P_2, K_2)$ and $(P'K, K) = (P_1, K_1)$. □

From Proposition 4.4.2 it is clear to see that sections can be sub-sections of each other in one of three ways. If $(P', K \cap P') = (P'K, K) = (P, K)$, then (P', K') is contained in (P, K) with the property that they have the same P and we write $(P', K') \leq_P (P, K)$. If $(P', K') = (P', K \cap P')$ and $(P'K, K) = (P, K)$ then $(P', K') \leq (P, K)$ such that $P'/K' \cong P/K$ and we write $(P', K') \leq_{P/K} (P, K)$. Finally if $(P', K') = (P', K \cap P') =$

$(P'K, K)$ then $(P', K') \leq (P, K)$ such that $K' = K$ and we write $(P', K') \leq_K (P, K)$. Each of these relations defines a partial order on the sections of G. By Proposition 4.4.2 we have $\leq = \leq_P \circ \leq_{P/K} \circ \leq_K$, moreover by the uniqueness of the decomposition we have the following theorem.

Theorem 4.4.3. The class incidence matrix of sections $M_{\mathscr{S}}(G)$ with respect to the partial order $\leq$ is $M_{\mathscr{S}_K}(G) \cdot M_{\mathscr{S}_{P/K}}(G) \cdot M_{\mathscr{S}_P}(G)$, the class incidence matrices with respect to the same K, same P/K and same P relations respectively.

Proof. Let $M_{\mathscr{S}}(G)$ denote the class incidence matrix of sections with respect to the partial order $\leq$, and let $\mathscr{S}^G = \{[(P_1, K_1)], [(P_2, K_2)], \ldots, [(P_n, K_n)]\}$ denote the conjugacy classes of sections of G. Now we have that

$$(M_{\mathscr{S}}(G))_{i,j} = |\{(P, K) \in [(P_i, K_i)] : (P, K) \geq (P', K'), \text{for some } (P', K') \in [(P_j, K_j)]\}|.$$

We will let $M_{i,j}$ denote the set

$$\{(P, K) \in [(P_i, K_i)] : (P, K) \geq (P', K'), \text{for some } (P', K') \in [(P_j, K_j)]\}.$$

By Proposition 4.4.2 we have that there are sections $\left(\tilde{P}, \tilde{K}\right) \in [(P_l, K_l)]$ and $\left(\overline{P}, \overline{K}\right) \in [(P_k, K_k)]$, such that $(P', K') \leq_P \left(\overline{P}, \overline{K}\right) \leq_{P/K} \left(\tilde{P}, \tilde{K}\right) \leq_K (P, K)$. Thus for each $(P, K) \in [(P_i, K_i)]$ such that $(P', K') \leq (P, K)$ for a fixed $(P', K') \in [(P_j, K_j)]$ we can define the map $\psi : (P, K) \mapsto (l, k)$. Hence we have the set

$$M_{i,j}^{l,k} := \{(P, K) \in [(P_i, K_i)] : \psi((P, K)) = (l, k), (P, K) \geq (P', K')\},$$

for a fixed $(P', K') \in [(P_j, K_j)]$. Then clearly we have

$$M_{i,j} = \coprod M_{i,j}^{l,k}.$$

We also have that $\left|M_{i,j}^{l,k}\right| = \left|A_{i,l}\left(\tilde{P}, \tilde{K}\right)\right| \cdot \left|B_{l,k}\left(\overline{P}, \overline{K}\right)\right| \cdot |C_{k,j}(P', K')|$, where

- $A_{i,l}\left(\tilde{P}, \tilde{K}\right) = \left\{(P, K) \in [(P_i, K_i)] : (P, K) \geq_K \left(\tilde{P}, \tilde{K}\right)\right\}$,

- $B_{l,k}\left(\overline{P},\overline{K}\right)=\left\{\left(\tilde{P},\tilde{K}\right)\in[(P_l,K_l)]:\left(\tilde{P},\tilde{K}\right)\geq_{P/K}\left(\overline{P},\overline{K}\right)\right\}$,
- $C_{k,j}\left(P',K'\right)=\left\{\left(\overline{P},\overline{K}\right)\in[(P_k,K_k)]:\left(\overline{P},\overline{K}\right)\geq_{P}\left(P',K'\right)\right\}$.

Thus $\left(M_{\mathscr{S}}(G)\right)_{i,j} = |M_{i,j}| = \sum_{l,k}\left|A_{i,l}\left(\tilde{P},\tilde{K}\right)\right|\cdot\left|B_{l,k}\left(\overline{P},\overline{K}\right)\right|\cdot\left|C_{k,j}\left(P',K'\right)\right|$, since the sizes of the sets $A_{i,l}\left(\tilde{P},\tilde{K}\right)$, $B_{l,k}\left(\overline{P},\overline{K}\right)$ and $C_{k,j}\left(P',K'\right)$ are constant on conjugacy classes of sections. Therefore the class incidence matrix of sections of G with respect to the partial order $\leq$ is $M_{\mathscr{S}_K}(G)\cdot M_{\mathscr{S}_{P/K}}(G)\cdot M_{\mathscr{S}_P}(G)$, the class incidence matrices with respect to the same K, same P/K and same P relations respectively. □

Remark 4.4.4. Theorem 4.4.3 a general result which applies to arbitrary G-equivariant partial orders with a unique decomposition.

Theorem 4.4.5. Let (P,K') and (P,K) be sections of a finite group G such that $(P,K')\leq_P(P,K)$. Then

$$\xi_{(P,K)}\left(P,K'\right)=\left|\left\{K^g\geq K':gP\in N_G(P)/P\right\}\right|.$$

Proof. By definition $\xi_{(P,K)}(P,K')=|\{(P,K)^g\geq(P,K'):g\in G\}|$. In order for $(P,K)^g\geq(P,K')$, g must fix P. Therefore $g\in N_G(P)$. However, since $K\trianglelefteq P$ then if $g\in P$, $K^g=K$ so we need only consider the elements of the group $N_G(P)/P$. This gives us

$$\xi_{G/(P,K)}\left(P,K'\right)=\left|\left\{(P,K)^g\geq\left(P,K'\right):g\in G\right\}\right|$$

$$=\left|\left\{K^g\geq K':g\in N_G(P)/P\right\}\right|.$$

□

Theorem 4.4.6. Let (P',K) and (P,K) be sections of a finite group G such that $(P',K)\leq_K(P,K)$. Then

$$\xi_{(P,K)}\left(P',K\right)=\left|\left\{P^g\geq P':gK\in N_G(K)/K\right\}\right|.$$

Proof. By definition $\xi_{(P,K)}(P',K)=|\{(P,K)^g\geq(P',K):g\in G\}|$. In order for $(P,K)^g\geq(P',K)$, g must fix K. Therefore $g\in N_G(K)$. However, since $K\trianglelefteq P$ then if $g\in K$,

$P^g = P$ so we need only consider the elements of the group $N_G(K)/K$. This gives us

$$\begin{aligned}\xi_{(P,K)}(P',K) &= \left|\left\{(P,K)^g \geq (P',K) : g \in G\right\}\right| \\ &= \left|\left\{P^g \geq P' : g \in N_G(K)/K\right\}\right|.\end{aligned}$$

□

Remark 4.4.7. Let (P',K') and (P,K) be sections of a finite group G such that $(P',K') \leq_{P/K} (P,K)$. Then there is no simplification of the formula described in Definition 4.4.1.

Example 4.4.8. We will now construct the class incidence matrix of sections of the symmetric group S_3 using the formulas outlined in Theorem 4.4.6, Remark 4.4.7 and Theorem 4.4.5 respectively. Note that we index the rows and columns of these matrices first according to isomorphism type of the section quotient and then by isomorphism type of the larger group in the section. That is: $(1,1)$, (C_2,C_2), (C_3,C_3), (S_3,S_3), $(C_2,1)$, (S_3,C_3), $(C_3,1)$ and $(S_3,1)$.

$$M_{\mathscr{S}_K}(S_3) = \left(\begin{array}{cccc|cc|c|c}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1
\end{array}\right)$$

$$M_{\mathscr{S}_{P/K}}(S_3) = \left(\begin{array}{cccc|cc|c|c}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$M_{\mathscr{S}_P}(S_3) = \left(\begin{array}{cccc|cc|c|c}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

Now by Proposition 4.4.2 multiplying these matrices in this order gives us the class incidence matrix of the sections of S_3.

	$(1,1)$	(C_2,C_2)	(C_3,C_3)	(S_3,S_3)	$(C_2,1)$	(S_3,C_3)	$(C_3,1)$	$(S_3,1)$
$(1,1)$	1	·	·	·	·	·	·	·
(C_2,C_2)	3	1	·	·	1	·	·	·
(C_3,C_3)	1	·	1	·	·	·	1	·
(S_3,S_3)	1	1	1	1	1	1	1	1
$(C_2,1)$	3	·	·	·	1	·	·	·
(S_3,C_3)	1	·	1	·	1	1	1	1
$(C_3,1)$	1	·	·	·	·	·	1	·
$(S_3,1)$	1	·	·	·	1	·	1	1

Table 4.1: The class incidence matrix of sections of S_3

4.5 An Alternative Partial Order on Sections

It should be noted that the partial order which we have defined in Definition 4.2.2 is inconsistent with our idea of the size of a section, that is, that we can have two sections of a group G, (P', K') and (P, K) such that $(P', K') \leq (P, K)$ but $|P'/K'| \geq |P/K|$. For example $(S_3, 1)$ and (S_3, S_3) are both sections of the symmetric group S_3. Clearly $(S_3, 1) \leq (S_3, S_3)$ but $|S_3/1| \geq |S_3/S_3|$. Thus it would be advantageous to define a partial order which is consistent with our idea of size.

Proposition 4.5.1. Let (P', K') and (P, K) be sections of G. Define a relation $\leq'$ by $(P', K') \leq' (P, K)$ if $P' \leq P$ and $K \cap P' \leq K'$. Then $\leq'$ is a partial order on sections of G.

Proof. To show that $\leq'$ does indeed define a partial order on the sections of a finite group G we must show that it is reflexive, antisymmetric and transitive.

Reflexivity: Let (P, K) be a section of G. Clearly $(P, K) \leq' (P, K)$ since $P \leq P$ and $K \cap P = K \leq K$.

Antisymmetry: Let (P', K') and (P, K) be sections of G such that $(P', K') \leq' (P, K)$ and $(P, K) \leq' (P', K')$. Thus $P' \leq P$ and $P \leq P'$ hence $P = P'$. We also have $K \cap P' \leq K'$ and $K' \cap P \leq K$, however since we have shown $P = P'$, we have $K \cap P' = K \leq K'$ and $K' \cap P' = K' \leq K$, thus $K = K'$ so $(P', K') = (P, K)$.

Transitvity: Let (P'', K''), (P', K') and (P, K) be sections of G, such that $(P'', K'') \leq' (P', K')$ and $(P', K') \leq' (P, K)$. We wish to show $(P'', K'') \leq' (P, K)$. Clearly $P'' \leq P$ since $P'' \leq P'$ and $P' \leq P$. We know $K' \cap P'' \leq K''$ and $K \cap P' \leq K'$. Therefore $K \cap P'' \leq K'$ since $P'' \leq P'$ but then $K \cap P'' \leq K' \cap P''$ giving us $K \cap P'' \leq K''$. Hence $(P'', K'') \leq' (P, K)$.

Thus $\leq'$ defines a partial order on the sections of a finite group G. □

Remark 4.5.2. Let (P', K') and (P, K) be sections of a finite group G, such that $(P', K') \leq' (P, K)$. Then $|P'/K'| \leq |P/K|$ since $|P| / |K| \geq |P'K| / |K|$, $|P'K| / |K| = |P'| / |K \cap P'|$ by the second isomorphism theorem and finally $|P'| / |K \cap P'| \geq |P'| / |K'|$ by assumption. Thus $|P| / |K| \geq |P'| / |K'|$ as required.

Proposition 4.5.3. Let (P', K') and (P, K) be sections of a finite group G, such that $(P', K') \leq' (P, K)$. Then, there are uniquely determined sections of G, $(P, K) \geq' (P_1, K_1) \geq' (P_2, K_2) \geq' (P', K')$ such that

1. (P, K) and (P_1, K_1) have the same bottom group, i.e., $K = K_1$.

2. (P_2, K_2) and (P', K') have the same top group, i.e., $P_2 = P'$.

3. (P_1, K_1) and (P_2, K_2) have isomorphic quotient groups, i.e., $P_1/K_1 \cong P_2/K_2$.

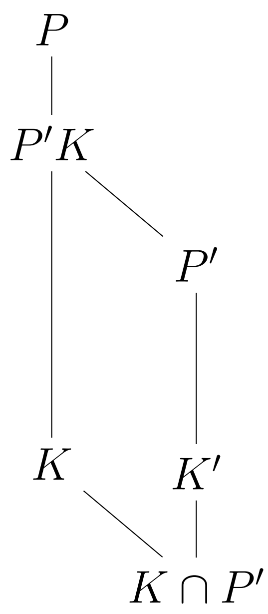


Figure 4.4: Decomposition of section incidence with respect to Proposition 4.5.3

Proof. Since $(P', K') \leq' (P, K)$ we know $P' \leq P$ and $K \cap P' \leq K'$ and by definition $K' \trianglelefteq P'$ and $K \trianglelefteq P$. Then by the second isomorphism theorem we have $K \cap P'$ is a normal subgroup of P', $P'K$ is a subgroup of P such that $K \trianglelefteq P'K$ and $(P'K)/K$ is isomorphic to $P'/(K \cap P')$. Hence, there are uniquely determined sections $(P', K \cap P') = (P_2, K_2)$ and $(P'K, K) = (P_1, K_1)$. □

Remark 4.5.4. Just as Proposition 4.4.2 gives a decomposition of the inclusions of sections with respect to the partial order defined in Definition 4.2.2 so does Proposition 4.5.3 with respect to the partial order defined in Definition 4.5.1. If $(P', K \cap P') = (P'K, K) = (P, K)$, then (P', K') is contained in (P, K) with the property that they have the same P and we write $(P', K') \geq_P (P, K)$. If $(P', K') = (P', K \cap P')$ and $(P'K, K) = (P, K)$ then $(P', K') \leq (P, K)$ such that $P'/K' \cong P/K$ and we write $(P', K') \leq_{P/K} (P, K)$. Finally if $(P', K') = (P', K \cap P') = (P'K, K)$ then $(P', K') \leq (P, K)$ such that $K' = K$ and we write $(P', K') \leq_K (P, K)$. Thus the partial order in Definition 4.5.1 has the effect of reversing the order on the same P relation $\leq_P$. This gives us the following proposition.

Proposition 4.5.5. The partial order $\leq'$ can be decomposed as $\leq' = \geq_P \circ \leq_{P/K} \circ \leq_K$. The class incidence matrix $M'_{\mathscr{S}}(G)$ is the product $M_{\mathscr{S}_K}(G) \cdot M_{\mathscr{S}_{P/K}}(G) \cdot M'_{\mathscr{S}_P}(G)$, the class incidence matrices with respect to the partial orders $\leq_K$, $\leq_{P/K}$ and $\geq_P$ relations respectively.

We can now compute an alternative class incidence matrix with respect to this new partial order by multiplying the matrices corresponding to the same K, same P/K and the reversed same P relations respectively. If we look again at the example of S_3 we obtain the following alternative class incidence matrix of sections by replacing the matrix for the $\leq_P$ relation with its transpose.

$S_3/(1,1)$	1	·	·	·	·	·	·	·
$S_3/(C_2,C_2)$	3	1	·	·	·	·	·	·
$S_3/(C_3,C_3)$	1	·	1	·	·	·	·	·
$S_3/(S_3,S_3)$	1	1	1	1	·	·	·	·
$S_3/(C_2,1)$	3	1	·	·	1	·	·	·
$S_3/(S_3,C_3)$	1	1	1	1	1	1	·	·
$S_3/(C_3,1)$	1	·	1	·	·	·	1	·
$S_3/(S_3,1)$	1	1	1	1	1	1	1	1
	$(1,1)$	(C_2,C_2)	(C_3,C_3)	(S_3,S_3)	$(C_2,1)$	(S_3,C_3)	$(C_3,1)$	$(S_3,1)$

Table 4.2: The class incidence matrix of sections of S_3 with respect to Proposition 4.5.3

This alternative class incidence matrix of sections has a number of advantages in comparison to the table defined with respect to the partial order in Definition 4.2.2. It has a number of things in common with the table of marks of a finite group G, for instance it is both lower triangular and its bottom row is entirely filled with 1's.

4.6 Isomorphisms

Definition 4.6.1. Let U be a subquotient of G, i.e., $U \sqsubseteq G$ then we consider the set of all isomorphisms from the sections of G onto U. This set will be denoted as follows:

$$\Lambda_{(G,U)} := \{((P,K),\theta) : (P,K) \text{ a section of } G, \theta : P/K \to U \text{ an isomorphism}\}.$$

The group G naturally acts on the domain of the isomorphisms in this set by conjugation. We will denote the G-orbit of $((P,K),\theta)$ by $[(P,K),\theta]_G$ and the set of G-orbits in $\Lambda_{(G,U)}$ as $\overline{\Lambda}_{(G,U)}$.

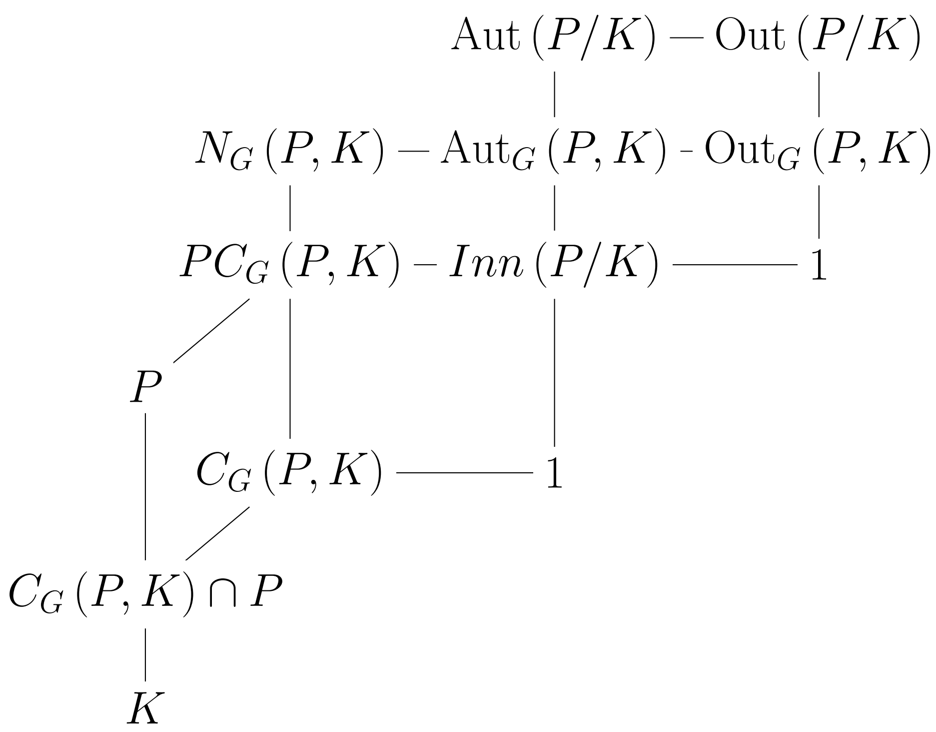
It is also important to note that $\mathrm{Aut}(U)$ acts on the set $\Lambda_{(G,U)}$ by composition of maps. Since $P/K \cong U$ we have that $\mathrm{Aut}(U) \cong \mathrm{Aut}(P/K)$. The G-action on the set $\Lambda_{(G,U)}$ commutes with the action of $\mathrm{Aut}(U)$ on this same set since composition of maps is associative. Thus $\Lambda_{(G,U)}$ is a $(G, \mathrm{Aut}(U))$-biset.

Lemma 4.6.2. The set $\Lambda_{(G,U)}/\mathrm{Aut}(U)$ is in bijection with the set of sections of G of isomorphism type U, which we have denoted by $\mathscr{S}(U)$.

Proof. The set $\Lambda_{(G,U)}$ is the set of all isomorphisms from the set $\mathscr{S}(U)$ to U. Hence quotienting out by the action of $\mathrm{Aut}(U)$ on $\Lambda_{(G,U)}$ produces a set which contains one isomorphism from each element of $\mathscr{S}(U)$ to U. Thus $\mathscr{S}(U)$ and $\Lambda_{(G,U)}$ are isomorphic as sets. □

Definition 4.6.3. Let (P,K) be a section of G and $\alpha : P/K \to U$ be an isomorphism. We denote by $\mathrm{Out}_G(P,K)$ the outer automorphisms of (P,K) induced by G and let $\mathscr{O}_\alpha$ be the $\mathrm{Out}(U)$ subgroup $(\mathrm{Out}_G(P,K))^\alpha$.

Lemma 4.6.4. Let $U \sqsubseteq G$. Recall that $\mathscr{S}^G(U)$ is the set of sections with isomorphism type U up to G conjugation and $\overline{\Lambda}_{(G,U)}$ denotes the set of G orbits of isomorphisms from sections of G into U. Then there is a bijective correspondence between the orbits of $\mathrm{Out}(U)$ on $\overline{\Lambda}_{(G,U)}$ and the set $\mathscr{S}^G(U)$.



Proof. We know from Lemma 4.6.2 that $\Lambda_{(G,U)}/\mathrm{Aut}(U)$ is in bijection with $\mathscr{S}(U)$ as sets, then clearly $\overline{\Lambda}_{(G,U)}/\mathrm{Aut}(U)$ is in bijection to $\mathscr{S}^G(U)$ as sets. However the inner automorphisms act trivially on the set $\overline{\Lambda}_{(G,U)}$, thus we only need to consider the action of $\mathrm{Out}(U)$ This gives us a bijective correspondence between the orbits of $\mathrm{Out}(U)$ on $\overline{\Lambda}_{(G,U)}$ and the set $\mathscr{S}^G(U)$. □

We denote by $\Lambda_{(G,U)}(P,K)$ the set of isomorphisms into U from a particular section (P,K) of G and $\overline{\Lambda}_{(G,U)}(P,K)$ is the set of G-orbits of $\Lambda_{(G,U)}(P,K)$.

Proposition 4.6.5. Let $[(P,K)]_G \in \mathscr{S}^G(U)$ and $\alpha : P/K \to U$ be an isomorphism. Then the orbit of $[(P,K),\alpha]_G$ in $\overline{\Lambda}_{(G,U)}(U)$ under the action of $\mathrm{Out}(U)$ is in bijection to the cosets $\mathrm{Out}(U)/\mathscr{O}_\alpha$.

Proof. Clearly $\mathscr{O}_\alpha$ is the stabilizer of $[(P,K),\alpha]_G$ in $\overline{\Lambda}_{(G,U)}(U)$, since the elements of $\mathscr{O}_\alpha$ are outer automorphisms of P/K induced by the conjugation action of G. Hence, the orbit of $[(P,K),\alpha]_G$ in $\overline{\Lambda}_{(G,U)}(U)$ is in bijection to the cosets $\mathrm{Out}(U)/\mathscr{O}_\alpha$. $\square$

Definition 4.6.6. Let $((P,K),\theta) \in \Lambda_{(G,U)}$. Then we define the *automizer* of θ as

$$\overline{\theta} : \overline{P}/\overline{K} \to \overline{U},$$

$$p\overline{K} \mapsto \theta^{-1} c_p \theta,$$

where $\left(\overline{P},\overline{K}\right) = \mathscr{A}_G(P,K)$, $p \in \overline{P}$ and $\overline{U} := (\mathrm{Aut}_G(P,K))^\theta \leq \mathrm{Aut}(U)$ and c_p is the conjugation map induced by p. We write $\mathscr{A}_G(\theta)$ for $\overline{\theta}$.

Naturally $\mathscr{A}_G(\theta)$ is a isomorphism from the automizer of the section (P,K) onto $\overline{U}$, that is $\mathscr{A}_G(\theta) \in \Lambda_{\left(G,\overline{U}\right)}(\mathscr{A}_G(P,K))$. These sets will be of great importance when computing the conjugacy classes of subgroups of $G_1 \times G_2$ as we will see in the next chapter and, later, the table of marks of $G_1 \times G_2$.

Chapter 5

Subgroups of Direct Products

5.1 Introduction

Direct products of groups are simply the cartesian product of two or more groups. Their conjugacy classes of element are cartesian products of the conjugacy classes of elements of the individual factor groups. However, their subgroups and conjugacy classes of subgroups are not so straightforward.

In this chapter we begin by asking the question: how do the conjugacy classes of subgroups of a direct product relate to the conjugacy classes of subgroups of the factor groups? Through the use of a result known as Goursat's Lemma we will be able to answer this question positively. Goursat's Lemma characterises subgroups of a direct product in terms of sections of the factor groups and an isomorphism between their section quotient. This, in later chapters, will lead to the study of the table of marks of a direct product and the related double Burnside ring.

5.2 Goursat triples

Sections can be very useful when working with subgroups of direct products. The next result is well known, and has first been introduced by Édouard Grousat in 1889 in his paper *Sur les substitutions orthogonales et les divisions régulières de l'espace* [18]. It characterises the subgroups of a direct product in terms of sections and an isomorphism between the quotient of the sections.

Goursat is perhaps best known for his work in complex analysis rather than group theory. He is particularly well known for his work on Cauchy's integral theorem, which is now also known as the Cauchy–Goursat theorem [17].

Theorem 5.2.1 (Goursat's Lemma, [1])**.** Let G_1 and G_2 be finite groups. Then there is a bijective correspondence between the subgroups M of $G_1 \times G_2$ and triples of the form $((P_1, K_1), (P_2, K_2), \theta)$, where (P_i, K_i) is a section of G_i, $i \in \{1, 2\}$ and $\theta : P_1/K_1 \xrightarrow{\cong}$

P_2/K_2 is an isomorphism. We will call these triples Goursat triples and write $M = ((P_1, K_1), (P_2, K_2), \theta)$, when the subgroup M is in bijection to the Goursat triple $((P_1, K_1), (P_2, K_2), \theta)$.

Proof. Let $M \leq G_1 \times G_2$ and $P_1 = \pi_1(M) = \{g_1 \in G_1 : (g_1, g_2) \in M \text{ for some } g_2 \in G_2\}$, i.e., P_1 is the image of the first projection of M into G_1. Likewise denote by $K_1 = \iota_1^{-1}(M) = \{g_1 \in G_1 : (g_1, 1_{G_2}) \in M\}$, that is, K_1 is the inverse image of the first inclusion of G_1 into $G_1 \times G_2$. The groups P_2 and K_2 are similarly defined.

We want to show that K_1 is a normal subgroup of P_1. For all $p \in P_1$ we have $pkp^{-1} \in K_1$ since $\iota_1 (pK_1p^{-1}) = \{(pkp^{-1}, 1_{G_2}) \in G_1 \times G_2 : pkp^{-1} \in M\} = \iota_1 (K_1)$. Thus $K_1 \trianglelefteq P_1$ and $K_2 \trianglelefteq P_2$. The projections π_i induce the isomorphisms $\overline{\pi}_i : M/ (K_1 \times K_2) \to P_i/K_i$. This gives rise to an isomorphism $\theta := \overline{\pi}_1\overline{\pi}_2^{-1} : P_1/K_1 \to P_2/K_2$ such that $\theta : p_1K_1 \mapsto p_2K_2$. Hence M defines a triple $((P_1, K_1), (P_2, K_2), \theta)$.

Conversely let $Q = ((P_1, K_1), (P_2, K_2), \theta)$ where (P_i, K_i) is a section of G_i and $\theta : P_1/K_1 \to P_2/K_2$ is an isomorphism. Then this triple defines a group

$$H = \{(g_1, g_2) \in P_1 \times P_2 : \theta (g_1K_1) = g_2K_2\} .$$

Clearly $\pi_i (H) = P_i$. We know that the K_i are normal subgroups of the P_i and $K_1 \subseteq Ker (\theta)$. Thus if $g_1 \in K_1$ then $(g_1, 1_{G_2}) \in H$. Likewise if $g_2 \in K_2$ then $(1_{G_2}, g_2) \in H$. Hence $H = M$. $\square$

The following result is elementary, however, it is necessary for much of the work carried out hereafter.

Lemma 5.2.2. Let $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$. Then $|M| = |P_1| \cdot |K_2| = |P_2| \cdot |K_1|$.

Proof. Since $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ we can write

$$M = \{(g_1, g_2) \in P_1 \times P_2 : \theta (g_1K_1) = g_2K_2\} .$$

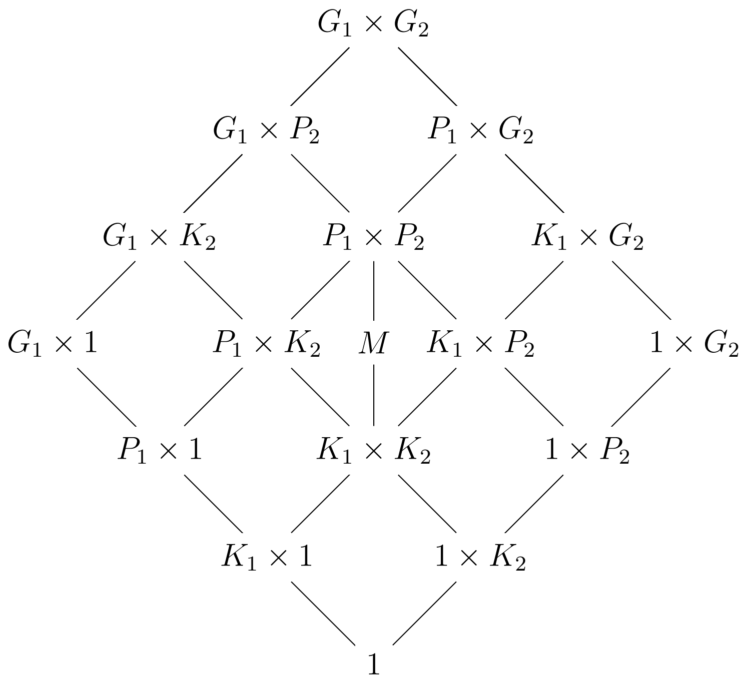


Figure 5.1: The position of a subgroup M in the subgroup lattice of $G_1 \times G_2$

Clearly $M \leq P_1 \times P_2$ hence $|M|$ divides $|P_1| \cdot |P_2|$. From the above alternative definition of M it follows that $|P_1 \times P_2 : M| = |P_i/K_i|$. Thus

$$\begin{aligned} |M| &= \frac{|P_1| \cdot |P_2|}{|P_2/K_2|} \\ &= |P_1| \cdot |K_2| \\ &= |P_2| \cdot |K_1| . \end{aligned}$$

□

Remark 5.2.3. Let $M \leq G_1 \times G_2$ and let $((P_1, K_1), (P_2, K_2), \theta)$ be the corresponding Goursat triple of M. Then $P_1 \times P_2$ is minimal among the subgroups of $G_1 \times G_2$ which are direct products which contain M, whereas $K_1 \times K_2$ is maximal among the subgroups of $G_1 \times G_2$ which direct products which are subgroups of M.

Theorem 5.2.4. Let $L = ((P_1', K_1'), (P_2', K_2'), \theta') \leq G_1 \times G_2$ and $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$. Then $L \leq M$ if and only if $P_i' \leq P_i$, $K_i' \leq K_i$ and $\theta(p_1' K_1) = \theta'(p_1' K_1') K_2$ for all $p_1' \in P_1'$.

Proof. ($\Rightarrow$) We know $L \leq P_1' \times P_2'$ and $M \leq P_1 \times P_2$ such that $P_1' \times P_2'$ is minimal among the direct products that contain L. Likewise $P_1 \times P_2$ is minimal among the direct products that contain M. Thus if $L \leq M$ we want to show $(P_1' \times P_2') \cap (P_1 \times P_2) = (P_1' \times P_2')$.

If $P_1' \times P_2' \nleq P_1 \times P_2$ then there exists $P_1'' \times P_2''$ such that $(P_1' \times P_2') \cap (P_1 \times P_2) = P_1'' \times P_2''$ where $L \leq P_1'' \times P_2''$ and $|P_1'' \times P_2''| < |P_1' \times P_2'|$. This is a contradiction since $P_1' \times P_2'$ is the minimal direct product which contains L. So $P_1' \times P_2' \leq P_1 \times P_2$.

We know $K_1' \times K_2'$ is a maximal among the direct products contained in L. Likewise $K_1 \times K_2$ is a maximal among the direct products contained in M. $K_1 \times K_2 \leq L$. Suppose $K_1' \times K_2' \nleq K_1 \times K_2$. Then $\langle K_1 \times K_2, K_1' \times K_2' \rangle = K_1'' \times K_2''$ such that $K_1'' \times K_2'' \leq M$ and $|K_1'' \times K_2''| > |K_1 \times K_2|$. This is a contradiction since $K_1 \times K_2$ is a maximal among the direct products contained in M. So $K_1' \times K_2' \leq K_1 \times K_2$.

We know now that $P_i' \leq P_i$, so for all $g_1 \in P_1'$, $g_2 \in P_2'$ we have $\theta(g_1 K_1) = g_2 K_2$. We also know $K_i' \leq K_i$ so $\theta(g_1 K_1') = g_2 K_2' = (g_2 K_2) K_2' = \theta'(g_1 K_1) K_2'$.

($\Leftarrow$) By definition $L \leq P_1' \times P_2'$, and assume $P_i' \leq P_i$. This implies $L \leq P_1 \times P_2$. $K_1' \times K_2'$ is a maximal among the direct products contained in L. Also $K_1' \times K_2' \leq K_1 \times K_2$. Thus $L \leq K_1 \times K_2$ since $K_1 \times K_2$ is a maximal among the direct products contained in M. Finally for $g_1 \in P_1'$, $\theta(g_1 K_1) = \theta'(g_1 K_1') K_2$. Hence $\theta(g_1 K_1') = \theta(g_1 K_1')$. The result follows that $L \leq M$. □

The multiplication defined in Definition 3.2.2 can be understood as multiplying two Goursat triples for subgroups, $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$, to produce a third Goursat triple $L * M \leq G_1 \times G_3$ via the following well known lemma.

Lemma 5.2.5. [2] Let $L = ((P_1, K_1), (P_2, K_2), \theta_1) \leq G_1 \times G_2$ and

$M = ((P_3, K_3), (P_4, K_4), \theta_2) \leq G_2 \times G_3$. Then

$$L * M = \left((P_1', K_1'), (P_4', K_4'), \overline{\theta}_1 \circ \rho(P_2, K_2; P_3, K_3) \circ \overline{\theta}_2\right),$$

where

- $K_2 \leq K_2' \trianglelefteq P_2' \leq P_2$, $K_3 \leq K_3' \trianglelefteq P_3' \leq P_3$ and the isomorphism $\rho(P_2, K_2; P_3, K_3)$ are determined using Lemma 4.2.5 applied to the sections (P_2, K_2) and (P_3, K_3) of G_2.
- $K_1 \leq K_1' \trianglelefteq P_1' \leq P_1$ and $K_4 \leq K_4' \trianglelefteq P_4' \leq P_4$ are determined by

$$P_1'/K_1 = \theta_1(P_2'/K_2), K_1'/K_1 = \theta_1(K_2'/K_2),$$

$$P_4'/K_4 = \theta_2^{-1}(P_3'/K_3), K_4'/K_4 = \theta_2^{-1}(K_3'/K_3);$$

- The isomorphisms $\overline{\theta}_1 : P_1'/K_1' \to P_2'/K_2'$ and $\overline{\theta}_2 : P_3'/K_3' \to P_4'/K_4'$ are induced by the isomorphisms θ_1 and θ_2.

Recall the from Definition 3.2.3 the map $\Pi : \mathcal{S}_{G_1\times G_2} \times \mathcal{S}_{G_3\times G_2} \to \mathcal{S}_{G_1\times G_3}$ defined by

$$\Pi : (M, L) \mapsto M * L^{op},$$

where $M \leq G_1 \times G_2$ and $L \leq G_3 \times G_2$.

Proposition 5.2.6. Let $M \leq G_1 \times G_2$. Then M can be written in the form $M = \Pi(M_1, M_2)$, for some $M_1 \leq G_1 \times U$ and $M_2 \leq G_2 \times U$.

Proof. Let $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$. Then clearly for $U \sqsubseteq G_i$ such that $P_i/K_i \cong U$ we can construct the isomorphisms $\theta_i : P_i/K_i \to U$ such that $\theta = \theta_1\theta_2^{-1}$. Then we have the groups $M_i = ((P_i, K_i), (U, 1), \theta_i) \leq G_i \times U$. Thus $M = \Pi(M_1, M_2)$. □

Proposition 5.2.7. Let L and $M \leq G_1 \times G_2$ such that $L = ((P_1', K_1), (P_2', K_2), \theta')$ and $M = ((P_1, K_1), (P_2, K_2), \theta)$, such that $(P_i', K_i) \leq_K (P_i, K_i)$. We define

- $L_i = ((P_i', K_i), (V, 1), \theta_i') \leq G_i \times V$, where $V \sqsubseteq G_i$ such that $P_i'/K_i \cong V$ and $L = \Pi(L_1, L_2)$,
- $M_i = ((P_i, K_i), (U, 1), \theta_i) \leq G_i \times U$, where $U \sqsubseteq G_i$ such that $P_i/K_i \cong U$ and $M = \Pi(M_1, M_2)$,
- the group homorphisms $\varphi_i : P_i'/K_i \to P_i/K_i$, $p_i'K_i \mapsto p_i'K_i$ and $\iota_i : V \to U$, $\iota_i := (\theta_i')^{-1} \varphi_i \theta_i$.

Then the homomorphims ι_i are injective, $L_i^{\iota_i} = ((P_i', K_i), (\iota_i(V), 1), \theta_i'\iota_i) \leq M_i$. Moreover $L \leq M$ if and only if where $\iota_1 = \iota_2$.

Proof. We begin by noting that clearly the maps φ_1 and φ_2 are clearly injective homomorphisms. Hence the maps θ_1 and θ_2 are also injective group homomorphisms since they are defined to be the composition of φ_i with two isomorphisms respectively.

Recall that $L_i^{\iota_i} = ((P_i', K_i), (\iota_i(V), 1), \theta_i\iota_i)$, clearly $\iota_i(V) \leq U$ and

$$\theta_i'\iota_i = \theta_i'(\theta_i')^{-1} \varphi_i\theta_i = \varphi_i\theta_i,$$

which gives us that $L_i^{\iota_i} \leq M_i$. If we assume that $\iota_1 = \iota_2$ then we have

$$(\theta_1')^{-1} \varphi_1\theta_1 = (\theta_2')^{-1} \varphi_2\theta_2$$

$$\varphi_1\theta_1 = \theta_1'(\theta_2')^{-1} \varphi_2\theta_2$$

$$\varphi_1\theta_1(\theta_2)^{-1} = \theta_1'(\theta_2')^{-1} \varphi_2$$

$$\varphi_1\theta = \theta'\varphi_2,$$

since $\theta_1(\theta_2)^{-1} = \theta$ and $\theta_1'(\theta_2')^{-1} = \theta'$ because $L = \Pi(L_1, L_2)$ and $M = \Pi(M_1, M_2)$ respectively. By assumption $(P_i', K_i) \leq_K (P_i, K_i)$. Thus $L \leq M$ if $\iota_1 = \iota_2$.

Assume $L \leq M$, we know that we can decompose $L = \Pi(L_i, L_2)$ and $M = \Pi(M_1, M_2)$, where $L_i = ((P_i', K_i), (V, 1), \theta_i')$ and $M_i = ((P_i, K_i), (U, 1), \theta_i)$ respectively. Again we define $\iota_i := (\theta_i')^{-1} \varphi_i\theta_i$. Since $L \leq M$ we have

$$\varphi_1\theta = \theta'\varphi_2$$

$$\varphi_1\theta_1\theta_2^{-1} = \theta_1'\left(\theta_2'\right)^{-1}\varphi_2$$

$$\varphi_1\theta_1 = \theta_1'\left(\theta_2'\right)^{-1}\varphi_2\theta_2$$

$$\left(\theta_1'\right)^{-1}\varphi_1\theta_1 = \left(\theta_2'\right)^{-1}\varphi_2\theta_2$$

thus $\iota_1 = \iota_2$ if $L \leq M$ as required.

□

Proposition 5.2.8. Let L and $M \leq G_1 \times G_2$ such that $L = \left(\left(P_1, K_1'\right), \left(P_2, K_2'\right), \theta'\right)$ and $M = \left(\left(P_1, K_1\right), \left(P_2, K_2\right), \theta\right)$, such that $\left(P_i, K_i'\right) \leq_P \left(P_i, K_i\right)$. We define

- $L_i = \left(\left(P_i, K_i'\right), \left(V, 1\right), \theta_i'\right) \leq G_i \times V$, where $V \sqsubseteq G_i$ such that $P_i'/K_i' \cong V$ and $L = \Pi\left(L_1, L_2\right)$,
- $M_i = \left(\left(P_i, K_i\right), \left(U, 1\right), \theta_i\right) \leq G_i \times U$,where $U \sqsubseteq G_i$ such that $P_i/K_i \cong U$ and $M = \Pi\left(M_1, M_2\right)$,
- the group homomorphisms $\varphi_i : P_i/K_i' \to P_i/K_i$, $p_iK_i' \mapsto p_iK_i$ and $\iota_i : V \to U$, $\iota_i := \left(\theta_i'\right)^{-1}\varphi_i\theta_i$.

Then the homomorphisms ι_i are surjective, $L_i \leq M_i^{\iota_i'^{-1}} = \left(\left(P_i, K_i\right), \left(V, Ker\left(\iota\right)\right), \theta_i\iota'^{-1}\right)$, where $\iota_i' : V/Ker(\iota) \to U$ is an isomorphism such that $\iota_i = \nu_i\iota_i'$ where $\nu_i : V \to V/Ker(\iota)$. Moreover $L \leq M$ if and only if $\iota_1 = \iota_2$.

Proof. Clearly the homomorphisms φ_i are surjective. Therefore the homomorphisms ι_i are surjective since they are defined to be the composition of of two isomorphisms and a surjection. By assumption $\left(P_i, K_i'\right) \leq_P \left(P_i, K_i\right)$ and obviously $\left(V, 1\right) \leq \left(V, Ker\left(\iota_i\right)\right)$. Recall that $\iota_i := \left(\theta_i'\right)^{-1}\varphi_i\theta_i$, then we have the following

$$\theta_i'\iota_i = \varphi_i\theta_i$$

$$\theta_i'\nu_i\iota_i' = \varphi_i\theta_i$$

$$\theta_i'\nu_i = \varphi_i\theta_i\left(\iota_i'\right)^{-1},$$

thus giving us $L_i \leq M_i^{\iota_i'^{-1}}$. If we assume that $\iota_1 = \iota_2$ then we have

$$(\theta_1')^{-1} \varphi_1 \theta_1 = (\theta_2')^{-1} \varphi_2 \theta_2$$

$$\varphi_1 \theta_1 = \theta_1' (\theta_2')^{-1} \varphi_2 \theta_2$$

$$\varphi_1 \theta_1 (\theta_2)^{-1} = \theta_1' (\theta_2')^{-1} \varphi_2$$

$$\varphi_1 \theta = \theta' \varphi_2,$$

since $\theta_1 (\theta_2)^{-1} = \theta$ and $\theta_1' (\theta_2')^{-1} = \theta'$ because $L = \Pi(L_1, L_2)$ and $M = \Pi(M_1, M_2)$ respectively. By assumption $(P_i', K_i) \leq_K (P_i, K_i)$. Thus $L \leq M$ if $\iota_1 = \iota_2$.

Assume $L \leq M$, we know that we can decompose $L = \Pi(L_i, L_2)$ and $M = \Pi(M_1, M_2)$, where $L_i = ((P_i', K_i), (V, 1), \theta_i')$ and $M_i = ((P_i, K_i), (U, 1), \theta_i)$ respectively. Again we define $\iota_i := (\theta_i')^{-1} \varphi_i \theta_i$. Since $L \leq M$ we have

$$\varphi_1 \theta = \theta' \varphi_2$$

$$\varphi_1 \theta_1 \theta_2^{-1} = \theta_1' (\theta_2')^{-1} \varphi_2$$

$$\varphi_1 \theta_1 = \theta_1' (\theta_2')^{-1} \varphi_2 \theta_2$$

$$(\theta_1')^{-1} \varphi_1 \theta_1 = (\theta_2')^{-1} \varphi_2 \theta_2$$

$$\iota_1 = \iota_2,$$

thus $\iota_1 = \iota_2$ if $L \leq M$ as required.

□

We will make an observation on the subgroup lattice of $G_1 \times G_2$ which will lead to a decomposition of the table of marks of $G_1 \times G_2$ in to three tables each of which can be understood in terms of the table of marks of G_1 or G_2 or those of their subgroups.

Theorem 5.2.9. Let $L \leq M \leq G_1 \times G_2$. Then, there are uniquely determined subgroups L' and M' of $G_1 \times G_2$ such that $L \leq L' \leq M' \leq M$, where L and L' have the same P_i groups in their Goursat triple, M and M' have the same K_i, and where L' and M' have linked isomorphic sections P_i/K_i.

Proof. The result is immediate from Theorem 5.2.4, the second isomorphism theorem and Proposition 4.4.2. Let $M = ((P_1, K_1), (P_2, K_2), \theta)$ and $L = ((P_1', K_1'), (P_2', K_2'), \theta') \leq G_1 \times G_2$, then we can see that we have the groups $L' = ((P_1', K_1 \cap P_1'), (P_2', K_2 \cap P_2'), \theta'')$ and $M' = \left((P_1' K_1, K_1), (P_2' K_2, K_2), \overline{\theta}\right)$ such that $M \geq_K M' \geq_{P/K} L' \geq_P L$. Where $\overline{\theta}$ is a restriction of θ to $P_1' K_1 / K_1$ and θ'' is a induced by $\overline{\theta}$ and the isomorphisms $\alpha_i : P_i' / (K_i \cap P_i') \to P_i' K_i / K_i$, $\alpha_i (p_i' (K_i \cap P_i')) = p_i' K_i$ for $p_i' \in P_i'$. □

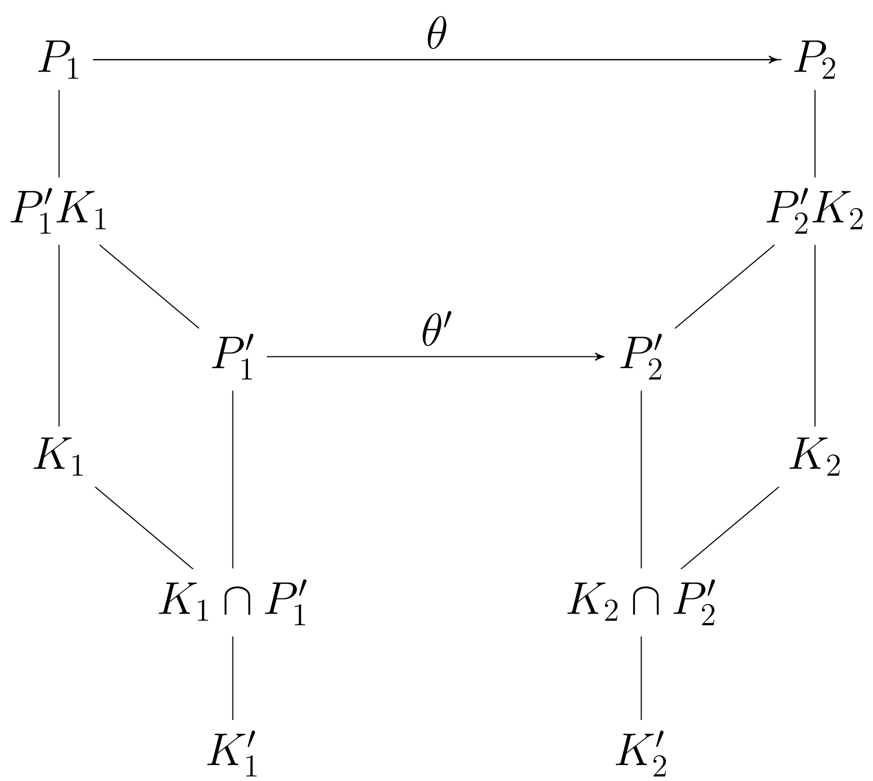


Figure 5.2: Decomposition of subgroup incidence with respect to Theorem 5.2.9

We write $L \leq_P M$ if $L' = M' = M$, $L \leq_K M$ if $L = L' = M'$, and $L \leq_{P/K} M$ if $L = L'$ and $M' = M$. Each of these relations is a partial order on the set of subgroups of $G_1 \times G_2$.

5.3 Conjugacy Classes of Subgroups

In this section we present a method for calculating the conjugacy classes of subgroups of the direct product $G_1 \times G_2$ by using only information about the conjugacy classes of subgroups of the factor groups and the structure of the outer automorphism groups of their subquotients. We begin with a useful remark.

Remark 5.3.1. Let $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$, where $P_i/K_i \cong U$ for some $U \sqsubseteq G_i$. Then, clearly θ can always be realised as a composition of maps $\theta_1\theta_2^{-1}$ for some $\theta_1 \in \Lambda_{(G_1,U)}(P_1, K_1)$ and $\theta_2 \in \Lambda_{(G_2,U)}(P_2, K_2)$.

Definition 5.3.2. Let $M = ((P, K), (U, 1), \theta) \leq G \times U$. Then we denote the *automizer* of M by

$$\mathscr{A}_G(M) = \left(\mathscr{A}_G(P, K), \left(\overline{U}, 1\right), \mathscr{A}_G(\theta)\right) \leq G \times \mathrm{Aut}(U),$$

where $\overline{U} = (\mathrm{Aut}_G(P, K))^{\theta}$ is the group of the G–induced automorphisms of U.

This is an important definition which through the Butterfly Lemma gives a rather nice description of the normalizer of a subgroup of $G_1 \times G_2$ in terms of Goursat triples.

Theorem 5.3.3. Let $M_i = ((P_i, K_i), (U, 1), \theta_i) \leq G_i \times U$ then the normalizer of $\Pi(M_1, M_2)$ in $G_1 \times G_2$ is

$$N_{G_1\times G_2}(\Pi(M_1, M_2)) = \Pi(\mathscr{A}_{G_1}(M_1), \mathscr{A}_{G_2}(M_2)).$$

Proof. Let $M_i := ((P_i, K_i), (U, 1), \theta_i) \leq G_i \times U$ and $M := \Pi(M_1, M_2)$. Hence we have that $M = \left((P_1, K_1), (P_2K_2), \theta_1\theta_2^{-1}\right)$. For M_i we have, by Definition 5.3.2

$$\mathscr{A}_{G_i}(M_i) = \left(\mathscr{A}_{G_i}(P_i, K_i), \left(\overline{U}_i, 1\right), \mathscr{A}_{\Lambda_{(G,U)}}(\theta_i)\right) \leq G_i \times \mathrm{Aut}(U).$$

By definition $P_i \trianglelefteq N_{G_i}(P_i, K_i)$ and $K_i \trianglelefteq C_{G_i}(P_i, K_i)$. Now following the construction described in Lemma 5.2.5 and Lemma 4.2.5 we will consider the group $\Pi(\mathscr{A}_{G_1}(M_1), \mathscr{A}_{G_2}(M_2))$. By Lemma 5.2.5 we have

$$\Pi\left(\mathscr{A}_{G_1}(M_1), \mathscr{A}_{G_2}(M_2)\right)$$
$$= \left(\left(\overline{P}_1, \overline{K}_1\right), \left(\overline{P}_2, \overline{K}_2\right), \overline{\mathscr{A}}_{\Lambda_{(G_1,U)}}(\theta_1)\, \rho\left(\overline{U}_1, 1; \overline{U}_2, 1\right) \overline{\mathscr{A}}_{\Lambda_{(G_2,U)}}(\theta_2)^{-1}\right).$$

We will begin by constructing the butterfly isomorphism in Lemma 4.2.5 $\rho\left(\overline{U}_1, 1; \overline{U}_2, 1\right)$. Let $P_1' = \left(\overline{U}_1 \cap \overline{U}_2\right) = P_2'$, $K_1' = \left(\overline{U}_1 \cap 1\right) = 1 = \left(\overline{U}_2 \cap 1\right) = K_2'$. Thus $\rho\left(\overline{U}_1, 1; \overline{U}_2, 1\right)$ is the identity map $\left(\overline{U}_1 \cap \overline{U}_2\right) \to \left(\overline{U}_1 \cap \overline{U}_2\right)$.

We have by Lemma 5.2.5 that $\overline{P}_1 / \overline{K}_1 = \theta_1^{-1}\left(\overline{U}_1 \cap \overline{U}_2\right)$ and $\overline{P}_2 / \overline{K}_2 = \theta_2^{-1}\left(\overline{U}_1 \cap \overline{U}_2\right)$. However $C_{G_i}(P_i, K_i) = Ker\left(\mathscr{A}_{\Lambda_{(G_i,U)}}(\theta_i)\right)$ hence $\overline{K}_i = C_{G_i}(P_i, K_i)$. Thus we have that $\overline{P}_1 = \theta_1^{-1}\left(\overline{U}_1 \cap \overline{U}_2\right) \overline{K}_1$, likewise $\overline{P}_2 = \theta_2^{-1}\left(\overline{U}_1 \cap \overline{U}_2\right) \overline{K}_2$. We now restrict the maps $\mathscr{A}_{\Lambda_{(G_i,U)}}(\theta_i)$ to the group $\overline{P}_i / C_{G_i}(P_i, K_i)$ to obtain the isomorphisms

$$\overline{\mathscr{A}}_{\Lambda_{(G_i,U)}}(\theta_i) : \overline{P}_i / \overline{K}_i \to \left(\overline{U}_1 \cap \overline{U}_2\right),$$

$$\overline{p}_i \overline{K}_i \mapsto \theta_i^{-1} c_{\overline{p}_i} \theta_i.$$

This defines the Goursat triple

$$\left(\left(\overline{P}_1, \overline{K}_1\right), \left(\overline{P}_2, \overline{K}_2\right), \overline{\mathscr{A}}_{\Lambda_{(G_1,U)}}(\theta_1)\, \rho\left(\overline{U}_1, 1; \overline{U}_2, 1\right) \overline{\mathscr{A}}_{\Lambda_{(G_2,U)}}(\theta_2)^{-1}\right).$$

For the moment, we will denote the subgroup defined by the above Goursat triple by $\overline{M}$. Now we must show that M is a normal subgroup of $\overline{M}$. Note that since $\overline{P}_i \leq N_{G_i}(P_i) \cap N_{G_i}(K_i)$ we have $K_i, P_i \trianglelefteq \overline{P}_i$. Hence for $\overline{p}_i \in \overline{P}_i$ we have $(P_i, K_i)^{\overline{p}_i} = (P_i, K_i)$. Now from the definition of the map $\overline{\mathscr{A}}_{\Lambda_{(G_1,U)}}(\theta_1)\, \rho\left(\overline{U}_1, 1; \overline{U}_2, 1\right) \overline{\mathscr{A}}_{\Lambda_{(G_2,U)}}(\theta_2)^{-1}$ that the following equality holds

$$\theta_1^{-1} c_{\overline{p}_1} \theta_1 = \theta_2^{-1} c_{\overline{p}_2} \theta_2$$

for all $(\overline{p}_1, \overline{p}_2) \in \overline{M}$. Clearly this reduces to $\theta_1 \theta_2^{-1} = c_{\overline{p}_1} \theta_1 \theta_2^{-1} c_{\overline{p}_2}^{-1}$. Thus we have

$$M^{(\overline{p}_1, \overline{p}_2)} = \left((P_1, K_1), (P_2, K_2), \theta_1 \theta_2^{-1}\right)^{(\overline{p}_1, \overline{p}_2)}$$
$$= \left((P_1, K_1)^{\overline{p}_1}, (P_2, K_2)^{\overline{p}_2}, c_{\overline{p}_1} \theta_1 \theta_2^{-1} c_{\overline{p}_2}^{-1}\right)$$

$$= \left((P_1, K_1), (P_2, K_2), \theta_1 \theta_2^{-1} \right) = M,$$

for all $(\overline{p}_1, \overline{p}_2) \in \overline{M}$, therefore $M \trianglelefteq \overline{M}$. By construction, it is also the largest possible subgroup of $G_1 \times G_2$ which contains M as a normal subgroup. Therefore $N_{G_1 \times G_2}(M) = \Pi\left(\mathscr{A}_{G_1}(M_1), \mathscr{A}_{G_2}(M_2)\right)$. □

Definition 5.3.4. Let $U \sqsubseteq G_i$ and $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ such that $P_i / K_i \cong U$. Note that $\theta = \theta_1 \theta_2^{-1}$ for some $\theta_i \in \Lambda_{(G_i, U)}$. We define the set of double cosets $\mathscr{D}_M := \mathscr{O}_{\theta_1} \setminus \mathrm{Out}(U) / \mathscr{O}_{\theta_2}$, where $\mathscr{O}_{\theta_i}$ are as defined in Definition 4.6.3.

Here is a bijective correspondence between the conjugacy classes of subgroups of $G_1 \times G_2$ and the set of double cosets $\mathscr{D}_M$, where $M \leq G_1 \times G_2$. Furthermore the set of double cosets $\mathscr{D}_M$ actually partitions the conjugacy classes of subgroups of $G_1 \times G_2$ to which M belongs.

The groups G_1, G_2 and $\mathrm{Aut}(U)$ each act on the set

$$\Lambda_{(G_1, U)} \times \Lambda_{(G_2, U)} = \left\{ P_1/K_1 \xrightarrow{\theta_1} U \xleftarrow{\theta_2} P_2/K_2 : \theta_i \in \Lambda_{(G_i, U)} \right\},$$

in such a way that these actions commute with one another since $\Lambda_{(G_i, U)}$ is a $(G_i, \mathrm{Aut}(U))$-biset and $\Lambda_{(G_1, U)} \times \Lambda_{(G_2, U)}$ is a (G_1, G_2)-biset.

Proposition 5.3.5. If we let $G_1 \times G_2$ act by conjugation on the set $\Lambda_{(G_1, U)} \times \Lambda_{(G_2, U)}$ we obtain the set $\overline{\Lambda}_{(G_1, U)} \times \overline{\Lambda}_{(G_2, U)}$.

Proof. The conjugation actions of G_1 and G_2 on $\Lambda_{(G_1, U)} \times \Lambda_{(G_2, U)}$ commute since composition of maps is associative. Thus by allowing G_1 and G_2 to act on this set from the left and right respectively we obtain the set

$$\left\{ [(P_1, K_1)]_{G_1} \xrightarrow{\theta_1} U \xleftarrow{\theta_2} [(P_2, K_2)]_{G_2} : \theta_i \in \overline{\Lambda}_{(G_i, U)} \right\},$$

which is clearly equal to the set $\overline{\Lambda}_{(G_1, U)} \times \overline{\Lambda}_{(G_2, U)}$. □

Proposition 5.3.6. Let $U \sqsubseteq G_i$. The set $\left(\Lambda_{(G_1, U)} \times \Lambda_{(G_2, U)}\right) / \mathrm{Aut}(U)$ is in bijection to the set of subgroups of $G_1 \times G_2$ whose Goursat triples have section isomorphism type U.

Proof. The group $\mathrm{Aut}(U)$ acts diagonally on the set $\Lambda_{(G_1,U)} \times \Lambda_{(G_2,U)}$ through composition of maps. Thus if we quotient out by the action of $\mathrm{Aut}(U)$ we obtain the set of isomorphisms from the set of sections of isomorphism type U in G_1 to the set of sections of isomorphism type U in G_2, through the composition of maps $\theta_1\theta_2^{-1}$ where $\theta_i \in \Lambda_{(G_i,U)}$. By Goursat's lemma each of these isomorphism defines a unique subgroup of $G_1 \times G_2$. □

If we now let G_1 and G_2 act on this set by conjugation on the left and right respectively this will clearly be isomorphic to the set of conjugacy classes of subgroups of $G_1 \times G_2$ whose Goursat triples have sections with isomorphism type U.

Theorem 5.3.7. There is a bijective correspondence between the set

$$\left(\overline{\Lambda}_{(G_1,U)} \times \overline{\Lambda}_{(G_2,U)}\right) / \mathrm{Out}(U)$$

and the set of conjugacy classes of subgroups of $G_1 \times G_2$ whose Goursat triples have section isomorphism type $U \sqsubseteq G_i$.

Proof. Recall that the actions of $G_1 \times G_2$ and $\mathrm{Aut}(U)$ on the set $\Lambda_{(G_1,U)} \times \Lambda_{(G_2,U)} = \left\{P_1/K_1 \stackrel{\theta_1}{\rightarrow} U \stackrel{\theta_2}{\leftarrow} P_2/K_2 : \theta_i \in \Lambda_{(G_i,U)}\right\}$ commute with one another. By Proposition 5.3.6 we know that the set $\left(\Lambda_{(G_1,U)} \times \Lambda_{(G_2,U)}\right) / \mathrm{Aut}(U)$ is in bijection to the set of subgroups of $G_1 \times G_2$. It should be noted that the group of inner automorphisms of U, $\mathrm{Inn}(U)$, acts trivially on this set, thus the acting group is in fact $\mathrm{Out}(U)$ not $\mathrm{Aut}(U)$. □

This result provides us with a method to compute the conjugacy classes of subgroups of $G_1 \times G_2$ from knowledge of the conjugacy classes of subgroups of G_1 and G_2 and the automorphism groups of their subquotients. This is a powerful result and will form the basis for much of the work which will be carried out throughout the rest of this thesis.

Corollary 5.3.8. The complete set of conjugacy classes of subgroups of $G_1 \times G_2$ is in bijection to the set

$$\coprod_{U \sqsubseteq G_i} \left(\overline{\Lambda}_{(G_1,U)} \times \overline{\Lambda}_{(G_2,U)}\right) / \mathrm{Out}(U).$$

Proof. The proof of this corollary is immediate from the fact that from Theorem 5.3.7 we know that the set $\left(\overline{\Lambda}_{(G_1,U)} \times \overline{\Lambda}_{(G_2,U)}\right)/\mathrm{Out}(U)$ is isomorphic to the set of conjugacy classes of subgroups of $G_1 \times G_2$ whose Goursat triples have sections with isomorphism type U. Hence by taking a disjoint union across all $U \sqsubseteq G_i$ we obtain a set which is isomorphic to the complete set of conjugacy classes of subgroups of $G_1 \times G_2$. □

Lemma 5.3.9. Let $U \sqsubseteq G_i$ and $L = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ such that $P_i/K_i \cong U$. Fix the isomortphisms $\theta_i : P_i/K_i \to U$ such that $\theta = \theta_1\theta_2^{-1}$. Then there is a bijective correspondence between the set $\mathscr{D}_L = \mathscr{O}_{\theta_1}\backslash\mathrm{Out}(U)/\mathscr{O}_{\theta_2}$ and the conjugacy classes of subgroups whose Goursat triples have sections conjugate to (P_1, K_1) and (P_2, K_2) in G_1 and G_2 respectively.

Proof. Recall from Proposition 4.6.5 that the conjugacy classes of sections with isomorphism type U is in bijection to the cosets $\mathrm{Out}(U)/\mathscr{O}_{\theta_i}$. Clearly, if two subgroups of $G_1 \times G_2$ are conjugate then their sections must also be conjugate. Thus we have that $\mathscr{O}_{\theta_1}\mathscr{O}_{\theta_2}$ stabilises $[L]_{G_1\times G_2}$. Hence the the conjugacy classes of subgroups whose Goursat triples have sections conjugate to (P_1, K_1) and (P_2, K_2) in G_1 and G_2 respectively is in bijection to $\mathscr{O}_{\theta_1}\backslash\mathrm{Out}(U)/\mathscr{O}_{\theta_2}$. □

Proposition 5.3.10. Let $L = ((P_1', K_1'), (P_2', K_2'), \theta') \leq G_1 \times G_2$ and $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$. Then $L \approx_{G_1\times G_2} M$ if only if the isomorphism $\theta = c_{g_1}^{-1}\theta' c_{g_2}$ for some $c_{g_i} \in \mathrm{Inn}(G_i)$.

Proof. Assume $L \cong_{G_1\times G_2} M$, then there exists $(g_1, g_2) \in G_1 \times G_2$ such that $L^{(g_1,g_2)} = M$. Hence $\left((P_1', K_1')^{g_1}, (P_2', K_2')^{g_2}, c_{g_1}^{-1}\theta' c_{g_2}\right) = ((P_1, K_1), (P_2, K_2), \theta)$, where the c_{g_i} are the conjugation maps induced by g_i. Thus $\theta = c_{g_1}^{-1}\theta' c_{g_2}$.

Suppose $\theta = c_{g_1}^{-1}\theta' c_{g_2}$ for some $c_{g_i} \in \mathrm{Inn}(G_i)$, there then exists $g_i \in G_i$ such that their induced conjugation maps, c_{g_i} give the equality $c_{g_1}\theta' c_{g_2} = \theta$. Clearly, if we conjugate the sections of L by the g_i we have $(P_i', K_i)^{g_i} = (P_i, K_i)$. Hence $((P_1', K_1')^{g_1}, (P_2', K_2')^{g_2}, c_{g_1}\theta' c_{g_2}) = ((P_1, K_1), (P_2, K_2), \theta)$. Finally this gives us $L^{(g_1,g_2)} = M$ completing the result. □

5.4 Monomorphism Class Incidence Matrix

In the previous chapter we defined the set of monomorphisms from sections of G to U where $U \sqsubseteq G$, $\Lambda_{(G,U)}$. By Goursat's lemma each θ in $\Lambda_{(G,U)}$ uniquely determines a subgroup of $G \times U$. That is for each $\theta : P/K \to U$ we can define the Goursat triple $((P, K), (U, 1), \theta)$ which we know by Theorem 5.2.1 defines a subgroup of $G \times U$. Thus we can count incidences between monomorphisms from sections of G into U.

Definition 5.4.1. Let $\overline{\Lambda}_{(G,U)}/\mathrm{Out}\,(U) = \{\theta_1, \ldots, \theta_n\}$. We define the *monomorphism class incidence matrix* of a finite group G to be the square matrix

$$M_\Lambda (G) = \bigoplus_{U \sqsubseteq G} (\xi_{\theta_i} (\theta_j))_{1 \leq i,j \leq n} ,$$

where $\xi_{\theta_i} (\theta_j) = \left| \left\{ ((P, K), (U, 1), \theta_i)^{(g,u)} \geq ((P', K'), (U, 1), \theta_j) : (g, u) \in G \times U \right\} \right|$.

Example 5.4.2. Here, we will look at the example of the symmetric group S_3. It is noteworthy that this monomorphism class incidence matrix is identical to the incidence matrix of the same P/K relation for completion of the class incidence matrix of sections in the previous chapter. However this, in general, is not the case, this only occurs because the outer automorphism groups for each of the sections is trivial in S_3.

In the next chapter we will begin to approach the central problem of the present research; computing the table of marks of a direct product of finite groups. This is quite an intricate problem; hence we will have to consider the table of marks of the bifree double Burnside ring and left-free double Burnside ring, subrings of the double Burnside ring. The bifree and left-free subgroups of $G_1 \times G_2$ have a more simple structure the arbitrary subgroups thus their respective table of marks should be easier to construct.

$(1,1) \xrightarrow{\theta_1} 1$	1	·	·	·	·	·	·	·
$(C_2,C_2) \xrightarrow{\theta_2} 1$	3	1	·	·	·	·	·	·
$(C_3,C_3) \xrightarrow{\theta_3} 1$	1	·	1	·	·	·	·	·
$(S_3,S_3) \xrightarrow{\theta_4} 1$	1	1	1	1	·	·	·	·
$(C_2,1) \xrightarrow{\theta_5} C_2$	·	·	·	·	1	·	·	·
$(S_3,C_3) \xrightarrow{\theta_6} C_2$	·	·	·	·	1	1	·	·
$(C_3,1) \xrightarrow{\theta_7} C_3$	·	·	·	·	·	·	1	·
$(S_3,1) \xrightarrow{\theta_8} S_3$	·	·	·	·	·	·	·	1
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8

Table 5.1: Monomorphism class incidence matrix of S_3

Chapter 6

The Bifree and Left-free Double Burnside Ring

6.1 Introduction

An excellent starting point for the study of the table of marks of the double Burnside ring is the study of two subrings of the full double Burnside ring with particularly nice properties, called the bifree double Burnside ring and the left-free double Burnside ring.

6.2 The bifree double Burnside ring

Definition 6.2.1. [3] Let X be a (G_1, G_2)-biset. X is a *bifree* biset if and only if $stab_{G_1 \times G_2}(x)$ is a twisted diagonal subgroup of $G_1 \times G_2$ for all $x \in X$.

Definition 6.2.2. A *bifree subgroup* of a direct product of groups is a group whose Goursat triple under Theorem 5.2 has the form $((P_1, 1_{G_1}), (P_2, 1_{G_2}), \theta) \leq G_1 \times G_2$.

Remark 6.2.3. The section quotient of a Goursat triple of a bifree subgroup of $G_1 \times G_2$ is naturally a subgroup of G_i since each section of a bifree subgroup has the form $(P, 1)$.

Proposition 6.2.4. If $M \leq G_1 \times G_2$ is bifree then all of its conjugates in $G_1 \times G_2$ are also bifree.

Proof. The proof of the above proposition clearly follows from Theorem 5.3.10. □

We consider these bifree bisets up to isomorphism, and denote the set of isomorphism classes by $B_+^{\triangledown}(G_1, G_2)$. $B_+^{\triangledown}(G_1, G_2)$ naturally has a semiring structure with disjoint union as addition and cartesian product as multiplication.

Definition 6.2.5. [7] The *bifree double Burnside group*, which we denote by $B^{\triangledown}(G_1, G_2)$, is the Grothendieck group of the semiring $B_+^{\triangledown}(G_1, G_2)$.

If $G_1 = G_2 = G$ then this group is closed under the $\cdot_G$ multiplication thus giving a ring structure.

In the paper *A ghost ring for the left-free double Burnside ring and an application to fusion systems* [3] Boltje and Danz note that the bifree double Burnside algebra, $\mathbb{Q}B(G,G)$, is semi-simple and that its ghost ring can be decomposed into a direct sum of rings, indexed by the isomorphism classes of subgroups of G. They describe the ghost ring as a direct product of endomorphism rings of permutation modules over outer automorphism groups of subgroups of G. This description of the ghost ring will provide much of the motivation for the present research, hence will provide details of the most relevant results here.

First we must begin with some notation; we denote by $\triangledown_{(G_1,G_2)}$ the set of bifree (G_1,G_2)-bisets, we will denote the ghost ring of the bifree double Burnside ring, $B^{\triangledown}(G,G)$, by $\widetilde{B}^{\triangledown}(G,G)$ and throughout this chapter R will be a commutative ring. We define $\Lambda^{\triangledown}_{(G,U)}$ to be set of isomorphisms from the set of free sections of G in to U and $\overline{\Lambda}^{\triangledown}_{(G,U)}$ to be the set of G-orbits of $\Lambda^{\triangledown}_{(G,U)}$. Denote by $[\lambda]$ the G-orbit of $\lambda\in\Lambda^{\triangledown}_{(G,U)}$.

Definition 6.2.6. [3] Let G_1 and G_2 be finite group, then for $U\sqsubseteq G_i$ we define the $\mathbb{Z}$-bilinear map

$$\sigma_{G_1,G_2,U}: B^{\triangledown}(G_1,G_2)\to \mathrm{Hom}_{\mathbb{Z}Out(U)}\left(\mathbb{Z}\overline{\Lambda}^{\triangledown}_{(G_1,U)},\mathbb{Z}\overline{\Lambda}^{\triangledown}_{(G_2,U)}\right),$$
$$[X]\mapsto\left([\mu]\mapsto\sum_{[\lambda]\in\overline{\Lambda}^{\triangledown}_{(G_1,U)}}\frac{\left|X^{\left(\left(\lambda^{-1}(U),1\right),\left(\mu^{-1}(U),1\right),\lambda\mu^{-1}\right)}\right|}{\left|C_{G_1}\left(\lambda^{-1}(U)\right)\right|}[\lambda]\right),$$

where X is a (G_1,G_2)-biset.

Since the map $\sigma_{G_1,G_2,U}([X])$ is defined on a $\mathbb{Z}$-basis, and since the definition does not depend on the choices of λ or μ in their classes, it is a well-defined group homomorphism. Moreover it can be verified that it respects $\mathbb{Z}\mathrm{Aut}(U)$-module structures. Collecting all these maps we obtain the map

$$\sigma_{G_1,G_2}: B^{\triangledown}(G_1,G_2)\to\bigoplus_{U\sqsubseteq G_i}\mathrm{Hom}_{\mathbb{Z}Out(U)}\left(\mathbb{Z}\overline{\Lambda}^{\triangledown}_{(G_1,U)},\mathbb{Z}\overline{\Lambda}^{\triangledown}_{(G_2,U)}\right).$$

The following theorem shows that the direct sum of homomorphism groups can serve as a ghost group and that σ_{G_1,G_2} can serve as a mark homomorphism translating the tensor product construction on bisets into componentwise composition of homomorphisms.

Theorem 6.2.7. [3] Let G, G_1, G_2 and G_3 be finite groups and let R be a commutative ring.

1. For $U \sqsubseteq G_i$, $a \in B^{\triangledown}(G_1, G_2)$ and $b \in B^{\triangledown}(G_2, G_3)$, one has
$$\sigma_{G_1,G_3,U}(a \cdot_{G_2} b) = \sigma_{G_1,G_2,U}(a) \circ \sigma_{G_2,G_3,U}(b).$$

2. The group homomorphism σ_{G_1,G_2} is injective with finite cokernel. If $|G_1 \times G_2|$ is invertible in R then the induced R-module homomorphism is an isomorphism.

3. The map
$$\sigma_{G,G} : B^{\triangledown}(G,G) \to \prod_{U \sqsubseteq G} \mathrm{End}^{\triangledown}_{\mathbb{Z}Out(U)}\left(\mathbb{Z}\overline{\Lambda}_{(G,U)}\right)$$
is an injective ring homomorphism with image of finite index, where the multiplication in the codomain is given by componentwise composition. If $|G|$ is invertible in R then the induced map
$$\sigma_{G,G} : RB^{\triangledown}(G,G) \to \prod_{U \sqsubseteq G} \mathrm{End}^{\triangledown}_{ROut(U)}\left(R\overline{\Lambda}_{(G,U)}\right)$$
is an R-algebra isomorphism.

They then consider the case where $G_1 = G_2 = G$ more closely. They begin by defining the set of group homomorphisms $\phi : U \to V$, where U, $V \leq G$, such that $\{(u, \phi(u)) : u \in U\}$ is a bifree subgroups of $G \times G$ by $\mathrm{Hom}_{\triangledown}(U, V)$ and note that automatically each $\phi \in \mathrm{Hom}_{\triangledown}(U, V)$ is injective since $\{(u, \phi(u)) : u \in U\}$ is a bifree subgroup. For all $U \leq G$ set $\mathrm{Aut}(U) := \mathrm{Hom}_{\triangledown}(U, U)$ and $\mathrm{Out}(U) := \mathrm{Aut}(U)/\mathrm{Inn}(U)$. Moreover they set $\overline{\mathrm{Hom}}_{\triangledown}(U, V) := \mathrm{Inn}(U)\backslash\mathrm{Hom}_{\triangledown}(U, V)$. Thus specializing to $V = G$, we obtain permutation $\mathbb{Z}\mathrm{Out}(U)$-module $\mathbb{Z}\overline{\mathrm{Hom}}_{\triangledown}(U, G)$ and its endomorphism ring
$$\mathrm{End}_{\mathbb{Z}\mathrm{Out}(U)}\left(\mathbb{Z}\overline{\mathrm{Hom}}_{\triangledown}(U, G)\right).$$

Then we can define the map
$$\widetilde{\sigma}_{G,U} : B^{\triangledown}(G,G) \to \mathrm{End}_{\mathbb{Z}\mathrm{Out}(U)}\left(\mathbb{Z}\overline{\mathrm{Hom}}_{\triangledown}(U,G)\right),$$

$$[X] \mapsto \left([\psi] \mapsto \sum_{[\phi] \in \overline{\mathrm{Hom}}_{\triangledown}(U,G)} \frac{\left| X^{\left((\phi(U),1),(\psi(U),1),\phi^{-1}\psi\right)} \right|}{\left| C_G\left(\phi\left(U\right)\right) \right|} \left[\phi^{-1}\right] \right).$$

Theorem 6.2.8. [3] Let G be a finite group and denote by $\overline{\mathcal{S}}_G$ a set of representative of isomorphism types of subgroups of G.

1. The collection of these maps, for $U \in_G$, gives an injective ring homomorphism with image of finite index.

$$\widetilde{\sigma}_G : B^{\triangledown}\left(G,G\right) \to \bigoplus_{U \in \overline{\mathcal{S}}_G} \mathrm{End}_{\mathbb{Z}\mathrm{Aut}(U)}\left(\mathbb{Z}\overline{\mathrm{Hom}}_{\triangledown}(U,G)\right)$$

2. If $|G|$ is a unit in R then the map

$$\widetilde{\sigma}_G : RB^{\triangledown}\left(G,G\right) \to \bigoplus_{U \in \overline{\mathcal{S}}_G} \mathrm{End}_{R\mathrm{Out}_{\triangledown}(U)}\left(R\overline{\mathrm{Hom}}_{\triangledown}(U,G)\right)$$

   is an isomorphism of R-algebras.

Since endomorphism rings of semisimple artinian modules are semisimple, the following corollary is immediate from Theorem 6.2.8 and Maschke's Theorem.

Corollary 6.2.9. [3] Let G be a finite group and let R be a field such that the numbers $|G|$ and $|\mathrm{Out}(U)|$ are invertible in R for all $U \in \overline{\mathcal{S}}_G$. Then the R-algebra $RB^{\triangledown}\left(G,G\right)$ is semisimple. The isomorphism classes of simple $RB^{\triangledown}\left(G,G\right)$-modules are in a bijective correspondence with pairs $(U,[V])$, where $U \in \overline{\mathcal{S}}_G$ and $[V]$ is the isomorphism class of a simple $R\mathrm{Out}\left(U\right)$-module V that occurs as a direct summand in the permutation module $R\overline{\mathrm{Hom}}_{\triangledown}\left(U,G\right)$.

6.3 The left-free double Burnside ring

Definition 6.3.1. Let X be a (G_1,G_2)-biset. Then we say that X is left-free if and only if $K_1\left(stab_{G_1 \times G_2}\left(x\right)\right) = 1$ for all $x \in X$.

Definition 6.3.2. Let G_1 and G_2 be finite groups. We define a *left-free* subgroup of $G_1 \times G_2$ to be a subgroup M of $G_1 \times G_2$ whose Goursat triple has the form $((P_1, 1), (P_2, K_2), \theta)$.

Definition 6.3.3. [7] The *left-free double Burnside group*, which we denote by $B^{\triangleleft}(G_1, G_2)$, is the Grothendieck group of the category of left-free finite (G_1, G_2)-bisets.

If $G_1 = G_2 = G$ then $B^{\triangleleft}(G, G)$ is closed under the multiplication, $\cdot_G$, given by Proposition 3.3.2 giving us a natural ring structure. In particular this gives us inclusions of unitary rings $B^{\triangledown}(G, G) \subseteq B^{\triangleleft}(G, G) \subseteq B(G, G)$. Unlike the bifree double Burnside ring, the left-free double Burnside ring is not semisimple [3]. The next result is again due to Boltje and Danz [3] and gives a nice decomposition of the left-free double Burnside ring in terms of the bifree double Burnside ring and the Jacobson radical of the left-free double Burnside ring.

Theorem 6.3.4. [3] Let G be a finite group and assume that R is a field such that $|G|$ and $|\mathrm{Out}(U)|$ are invertible in R, for every subgroup $U \leq G$. Moreover set $J := J(RB^{\triangleleft}(G, G))$, i.e., the Jacobson radical of the R-algebra $RB^{\triangleleft}(G, G)$. One has

$$RB^{\triangleleft}(G, G) = RB^{\triangledown}(G, G) \oplus J.$$

Corollary 6.3.5. [3] Let G be a finite group and R a field such that $|G|$ and $|\mathrm{Out}(U)|$ are units in R, for every subgroup $U \leq G$. Then the isomorphism classes of simple $RB^{\triangleleft}(G, G)$-modules and the isomorphism classes of simple $RB^{\triangledown}(G, G)$-modules are in natural bijective correspondence.

These results provide the first insight into how important the sets $\Lambda_{(G,U)}$ and $\overline{\Lambda}_{(G,U)}$ may be when attempting to study the structure of the ghost group of $B(G_1, G_2)$ and hence by extension its associated mark homomorphism and table of marks. Before we move on to study the table of marks of $B(G_1, G_2)$ we will now take a closer look at the subgroup lattice of $B^{\triangledown}(G, G)$ and its table of marks.

6.4 The lattice of bifree subgroups of $G \times G$

In this section we will be concerned with the case where $G_1 = G_2$. The subgroup lattice created by the bifree subgroups of $G \times G$ has some interesting properties. If, for example, we let $G = S_4$ we will have the following bifree lattice where each node corresponds to a conjugacy class of bifree subgroups:

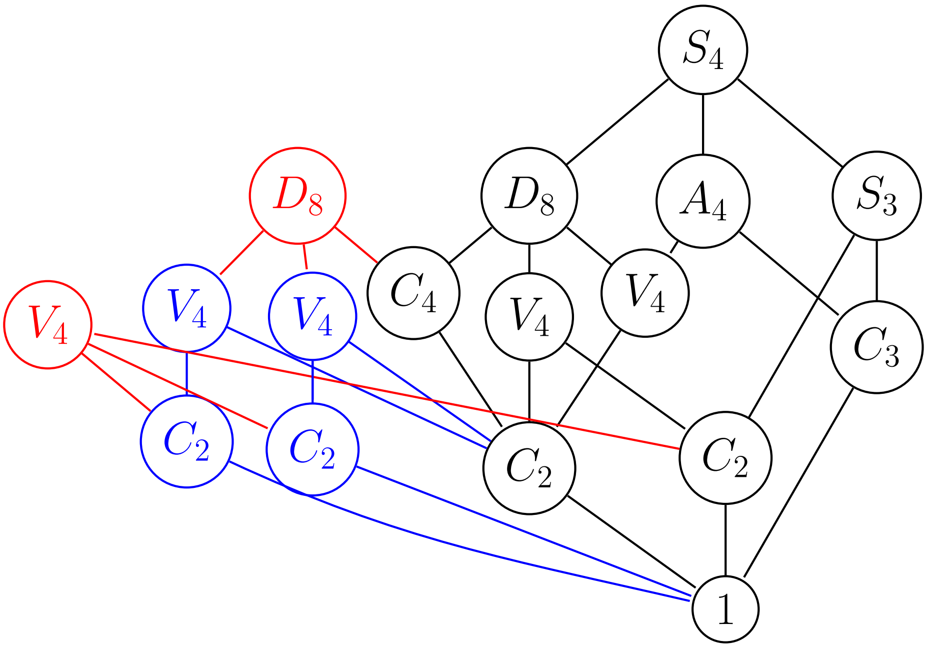


Figure 6.1: The bifree subgroup lattice of $S_4 \times S_4$

There are three kinds of subgroups in the bifree subgroup lattice of $G \times G$ depicted here by the three different colours:

1. The "black" subgroups have the form $M := ((P, 1), (P, 1), \theta)$, such that the isomorphism $\theta \in \mathrm{Inn}(G)$ where id is the identity isomorphism.

2. The "red" subgroups have the form $M := ((P, 1), (P, 1), \theta)$, such that the isomorphism $\theta \notin \mathrm{Inn}(G)$.

3. The "blue" subgroups have the form $M = ((P_1, 1), (P_2, 1), \theta)$ such that $P_1 \not\approx_G P_2$.

Lemma 6.4.1. The black subgroups form a sub-lattice of the lattice of bifree subgroups of $G \times G$ isomorphic to the subgroup lattice of G.

Proof. It is clear that $\Delta G = ((G, 1_G), (G, 1_G), \theta) \cong G$ will always exist as a bifree subgroup of $G \times G$. Without loss of generality we can chose $\Delta G = ((G, 1_G), (G, 1_G), id)$, where id is the identity automorphism in G, as a conjugacy classes representative. By Theorem 5.2.4 it is clear that all the subgroups of ΔG will also be bifree subgroups of $G \times G$. □

Corollary 6.4.2. A "black" subgroup cannot have a "red" or "blue" subgroup.

Proof. It is clear from Theorem 5.2.4 and from the fact that id can only restrict to an identity isomorphism that a black subgroup can only contain other black subgroups. □

However from the example above that a "red" subgroup can have a "blue" subgroup. By computation with larger examples an example can also be found where a "blue" subgroup can have a "red" subgroup. Note that "blue" and "red" subgroups will always contain the "black" subgroup 1.

6.5 The bifree table of marks of $G \times G$

The bifree table of marks is a sub-table of the table of marks obtained by restricting attention to conjugacy classes of bifree subgroups. Here we will propose a formula for obtaining the bifree table of $G \times G$ from the table of marks of G.

Theorem 6.5.1. Let $M = ((P, 1), (P, 1), id)$ and $L = ((P', 1), (P', 1), id) \leq G \times G$. Then

$$\beta_{G \times G/M}(L) = |C_G(P')| \cdot \beta_{G/P}(P').$$

Proof. We know from Theorem 6.4.1 that the subgroup lattice of G exist as a sub-lattice of the bifree subgroup lattice of $G \times G$. Therefore we can expect that the table of marks of G

will exist in the bifree table of marks of $G \times G$ in some form since the the lattice of G is preserved.

It can be seen from Theorem 2.4.5 that

$$\left|\{H_1^g \leq H_2 : g \in G\}\right| = \beta_{G/H_2}(H_1) \cdot |N_G(H_1)|^{-1} \cdot |H_2|$$

where H_1 and H_2 are subgroups of a finite group G.

By Goursat's Lemma we have $M = \{(m,m) : m \in P\}$ and $L = \{(l,l) : l \in P'\} \leq G \times G$, thus clearly $M \cong P$ and $L \cong P'$. This gives us that

$$\left|\left\{L^{(g_1,g_2)} \leq M : (g_1,g_2) \in G \times G\right\}\right| = \left|\{P'^g \leq P : g \in G\}\right|.$$

Then from Theorem 2.4.5 we have

$$|N_{G\times G}(L)|^{-1} \cdot |M| \cdot \beta_{G\times G/M}(L) = |N_G(P')|^{-1} \cdot |P| \cdot \beta_{G/P}(P'),$$

$$\beta_{G\times G/M}(L) = |N_G(P')|^{-1} \cdot |P| \cdot |N_{G\times G}(L)| \cdot |M|^{-1} \cdot \beta_{G/P}(P').$$

Clearly $|M| = |P|$ and it is also straightforward to check by Theorem 5.3.3 that $|N_{G\times G}(L)| = |N_G(P')| \cdot |C_G(P')|$.

$$\beta_{G\times G/M}(L) = |N_G(P')|^{-1} \cdot |P| \cdot |N_G(P')| \cdot |C_G(P')| \cdot |P|^{-1} \cdot \beta_{G/P}(P'),$$

$$\beta_{G\times G/M}(L) = |C_G(P')| \cdot \beta_{G/P}(P').$$

□

Example 6.5.2. We will examine the example of the table of marks of A_4 and the bifree table of marks of $A_4 \times A_4$.

Now we note the centralisers for each of the conjugacy classes of subgroups of A_4. $C_{A_4}(1) = A_4$, $C_{A_4}(C_2) = V_4$, $C_{A_4}(C_3) = C_3$, $C_{A_4}(V_4) = V_4$ and $C_{A_4}(A_4) = 1$. The bifree table of marks of $A_4 \times A_4$ is as follows:

To make the relationship between the bifree table of marks of $A_4 \times A_4$ and the table of marks of A_4 more explicit we will permute the rows and columns of Table 6.3.

	1	C_2	C_3	V_4	A_4
$A_4/1$	12				
A_4/C_2	6	2			
A_4/C_3	4	.	1		
A_4/V_4	3	3	.	3	
A_4/A_4	1	1	1	1	1

Table 6.1: The table of marks of A_4

	1	C_2	C_3	C_3'	V_4	V_4'	A_4	A_4'
$(A_4 \times A_4)/1$	144							
$(A_4 \times A_4)/C_2$	72	8						
$(A_4 \times A_4)/C_3$	48	.	3					
$(A_4 \times A_4)/C_3'$	48	.	.	3				
$(A_4 \times A_4)/V_4$	36	12	.	.	12			
$(A_4 \times A_4)/V_4'$	36	12	.	.	.	12		
$(A_4 \times A_4)/A_4$	12	4	.	3	4	.	1	
$(A_4 \times A_4)/A_4'$	12	4	3	.	.	4	.	1

Table 6.2: The bifree table of marks of $A_4 \times A_4$

It is clear to see that the top left-hand section of the above table is the table of marks of A_4 with each column scaled up by the size of the centraliser in A_4 of the corresponding conjugacy class of subgroups.

This result can be further generalised to all bifree subgroups of $G_1 \times G_2$.

Theorem 6.5.3. Let $M = ((P_1, 1), (P_2, 1), \theta)$ and $L = ((P_1', 1), (P_2', 1), \theta') \leq G_1 \times G_2$,

$(A_4 \times A_4)/1$	144							
$(A_4 \times A_4)/C_2$	72	8						
$(A_4 \times A_4)/C_3$	48	.	3					
$(A_4 \times A_4)/V_4$	36	12	.	12				
$(A_4 \times A_4)/A_4$	12	4	3	4	1			
$(A_4 \times A_4)/C_3'$	48	.	.	.	.	3		
$(A_4 \times A_4)/V_4'$	36	12	.	.	.	.	12	
$(A_4 \times A_4)/A_{4'}$	12	4	.	.	.	3	4	1
	1	C_2	C_3	V_4	A_4	C_3'	V_4'	A_4'

Table 6.3: The permuted bifree table of marks of $A_4 \times A_4$

such that $N_{G_1 \times G_2}(L) = \left(\left(\overline{P}_1, C_{G_1}(P_1')\right), \left(\overline{P}_2, C_{G_2}(P_2')\right), \overline{\theta}\right)$. Then

$$\beta_{G_1 \times G_2/M}(L) = \frac{\left|\overline{P}_2\right| \cdot \left|C_{G_1}(P_1')\right|}{|P_1|} \cdot \left|\left\{P_1'^{g_1} : g_i \in G_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|,$$

where $\varphi_{g_i} \colon P_i'^{g_i}/1 \to P_i/1$ is the canonical monomorphism induced by the inclusion of $P_i'^{g_i}$ into P_i.

Proof. From Theorem 5.2.4 we know that $L^{(g_1,g_2)} \leq M$ if and only if $P_i'^{g_i} \leq P_i$ and $c_{g_1}\theta = \theta' c_{g_2}$. By Theorem 5.2.7 we know that $M \geq L$ if and only if $(P_i', 1) \leq_K (P_i, 1)$ and $\iota_i = \iota_2$, such that $\iota_i = (\theta_i')^{-1} c_{g_i}\theta_i$, where $\theta = \theta_1\theta_2^{-1}$ and $\theta' = \theta_1'(\theta_2')^{-1}$. We begin by noting

$$\left|\left\{(P_1', 1)^{g_1} \leq (P_1, 1) : g_1 \in G_1\right\}\right|$$

$$= \left|\left\{P_1'^{g_1} \leq P_1 : g_1 \in G_1\right\}\right|.$$

There is a bijective correspondence between the set $\left\{L^{(g_1,g_2)} : g_i \in G_i, L^{(g_1,g_2)} \leq M\right\}$ and the set $\left\{P_1'^{g_1} : g_i \in G_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}$. Clearly there is a surjection which sends $L^{(g_1,g_2)}$ in the first set to $P_1'^{g_1} = \pi_1(L^{(g_1,g_2)})$. To see that this map is injective, suppose $P_1'^{g_1} = P_1'^{\tilde{g}_1}$ with $\tilde{g}_1 \in G_2$, $P_i'^{\tilde{g}_i} \leq P_i$, $c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2} = \varphi_{\tilde{g}_1}\theta$. Since $P_1'^{g_1} = P_1'^{\tilde{g}_1}$, we have $\varphi_{g_1} = \varphi_{\tilde{g}_1}$. Then

$$c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta = \varphi_{\tilde{g}_1}\theta = c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2},$$

so

$$P_2'^{g_2}/1 = \mathrm{Im}(c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2}) = \mathrm{Im}(c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2}) = P_2'^{\tilde{g}_2}/1,$$

so $P_2'^{g_2} = P_2'^{\tilde{g}_2}$, so $\varphi_{g_2} = \varphi_{\tilde{g}_2}$, so $c_{g_1}^{-1}\theta' c_{g_2} = c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}$ (because $\varphi_{g_2} = \varphi_{\tilde{g}_2}$ is a monomorphism), so $L^{(g_1,g_2)} = L^{(\tilde{g}_1,\tilde{g}_2)}$. Hence we have the equality

$$\left|\left\{L^{(g_1,g_2)} \leq M : (g_1,g_2) \in G_1 \times G_2\right\}\right|$$

$$= \left|\left\{P_1'^{g_1} : g_i \in G_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|.$$

Recall from Corollary 2.4.6 that

$$|N_{G_1\times G_2}(L)|^{-1} \cdot |M| \cdot \beta_{G_1\times G_2/M}(L) = \left|\left\{L^{(g_1,g_2)} \leq M : (g_1,g_2) \in G_1 \times G_2\right\}\right|.$$

Hence we have

$$|N_{G_1\times G_2}(L)|^{-1} \cdot |M| \cdot \beta_{G_1\times G_2/M} = \left|\left\{P_1'^{g_1} : g_i \in G_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|.$$

Note that $M \cong P_i$ and $N_{G_1\times G_2}(L) = \left(\left(\overline{P}_1, C_{G_1}(P_1')\right), \left(\overline{P}_2, C_{G_2}(P_2')\right), \overline{\theta}\right)$, thus we have the result

$$\beta_{G_1\times G_2/M}(L) = \frac{\left|\overline{P}_2\right| \cdot |C_{G_1}(P_1')|}{|P_1|} \cdot \left|\left\{P_1'^{g_1} : g_i \in G_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|.$$

□

Remark 6.5.4. If there exists $g_2 \in G_2$ for each $g_1 \in G_1$ such that $P_1'^{g_1} \leq P_1$ and $(\theta')^{-1} c_{g_1}\theta = c_{g_2}$ the formula in Theorem 6.5.3 reduces to:

$$\beta_{G_1\times G_2/M}(L) = \left|\overline{P}_2\right| \cdot |N_{G_1}(P_1')/C_{G_1}(P_1')|^{-1} \cdot \beta_{G_1/P_1}(P_1').$$

It should be noted that this matrix can be regarded as a same K class incidence matrix since the K_i in the case of bifree subgroups are fixed to be 1.

6.6 The left-free Table of Marks

Lemma 6.6.1. Let $M = ((P_1, 1), (P_2, K_2), \theta)$ and $L = ((P_1', 1), (P_2', K_2'), \theta')$ be left-free subgroups of $G_1 \times G_2$, such that $L \leq M$. Then there is a uniquely determined left-free subgroup L', such that $L \leq_{P/K} M' \leq_K M$.

Proof. From Theorem 5.2.9 we know that there exist uniquely determined subgroups, L' and M', such that $L \leq_P L' \leq_{P/K} M' \leq_K M$. Here we will show that if $L \leq_P L'$ then $L \leq_{P/K} L'$, and thus $L' = M'$.

If $L \leq_P L'$ then $L' = ((P_1', 1), (P_2', K_2 \cap P_2'), \theta'')$. Clearly then $P_2'/K_2' \cong P_2'/K_2 \cap P_2'$ thus $K_2 = K_2 \cap P_2'$ and $L = L'$. Now $M' = \left((P_1', 1), (P_2'K_2, K_2)\overline{\theta}\right)$ hence by Theorem 5.2.9 we have $L \leq_{P/K} M' \leq_K M$. □

Thus the left-free class incidence matrix of a direct product $G_1 \times G_2$ can be computed from the incidence matrices which correspond to these two relations.

Theorem 6.6.2. Let $M = ((P_1, 1), (P_2, K_2), \theta)$ and $L = ((P_1', 1), (P_2', K_2), \theta') \leq G_1 \times G_2$, such that $N_{G_1 \times G_2}(L) = \left(\left(\overline{P}_1, C_{G_1}(P_1')\right), \left(\overline{P}_2, C_{G_2}(P_2', K_2)\right), \overline{\theta}\right)$. Then $\beta_{G_1 \times G_2/M}(L)$

$$= \frac{\left|\overline{P}_2\right| \cdot |C_{G_1}(P_1')|}{|P_1| \cdot |K_2|} \cdot \left|\left\{P_1'^{g_1} : g_i \in G_i, g_2 \in N_{G_2}(K_2)/K_2, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|,$$

where $\varphi_{g_1} \colon P_1'^{g_1}/1 \to P_1/1$ and $\varphi_{g_2} \colon P'^{g_2}/K_2 \to P_2/K_2$ are the canonical monomorphisms induced by the inclusion of $P_i'^{g_i}$ into P_i.

Proof. From Theorem 5.2.4 we know that $L^{(g_1, g_2)} \leq M$ if and only if $P_i'^{g_i} \leq P_i$ and $c_{g_1}\theta = \theta' c_{g_2}$. By Theorem 5.2.7 we know that $M \geq L$ if and only if $(P_1', 1) \leq_K (P_1, 1)$, $(P_2', K_2) \leq_K (P_2, K_2)$ and $\iota_i = \iota_2$, such that $\iota_i = (\theta_i')^{-1} c_{g_i}\theta_i$, where $\theta = \theta_1\theta_2^{-1}$ and $\theta' = \theta_1'(\theta_2')^{-1}$. We begin by noting

$$\left|\left\{(P_1', 1)^{g_1} \leq (P_1, 1) : g_1 \in G_1\right\}\right|$$

$$= \left|\left\{P_1'^{g_1} \leq P_1 : g_1 \in G_1\right\}\right|$$

There is a bijective correspondence between the set $\left\{L^{(g_1,g_2)} : g_i \in G_i, L^{(g_1,g_2)} \leq M\right\}$ and the set $\left\{P_1'^{g_1} : g_1 \in G_1, g_2 \in N_{G_2}(K_2)/K_2, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}$. Clearly there is a surjection which sends $L^{(g_1,g_2)}$ in the first set to $P_1'^{g_1} = \pi_1(L^{(g_1,g_2)})$. To see that this map is injective, suppose $P_1'^{g_1} = P_1'^{\tilde{g}_1}$ with $\tilde{g}_1 \in G_2$, $\tilde{g}_2 \in N_{G_2}(K_2)/K_2$, $P_i'^{\tilde{g}_i} \leq P_i$, $c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2} = \varphi_{\tilde{g}_1}\theta$. Since $P_1'^{g_1} = P_1'^{\tilde{g}_1}$, we have $\varphi_{g_1} = \varphi_{\tilde{g}_1}$. Then

$$c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta = \varphi_{\tilde{g}_1}\theta = c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2},$$

so

$$P_2'^{g_2}/K_2 = \operatorname{Im}(c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2}) = \operatorname{Im}(c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2}) = P_2'^{\tilde{g}_2}/K_2,$$

so $P_2'^{g_2} = P_2'^{\tilde{g}_2}$, so $\varphi_{g_2} = \varphi_{\tilde{g}_2}$, so $c_{g_1}^{-1}\theta' c_{g_2} = c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}$ (because $\varphi_{g_2} = \varphi_{\tilde{g}_2}$ is a monomorphism), so $L^{(g_1,g_2)} = L^{(\tilde{g}_1,\tilde{g}_2)}$. We will denote the set

$$\left\{P_1'^{g_1} : g_1 \in G_1, g_2 \in N_{G_2}(K_2)/K_2, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}$$

by $\mathscr{K}$. Hence we have the equality

$$\left|\left\{L^{(g_1,g_2)} \leq M : (g_1,g_2) \in G_1 \times G_2\right\}\right|$$

$$= \left|\left\{P_1'^{g_1} : g_1 \in G_1, g_2 \in N_{G_2}(K_2)/K_2, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|.$$

Recall from Corollary 2.4.6 that

$$\left|N_{G_1\times G_2}(L)\right|^{-1} \cdot |M| \cdot \beta_{G_1\times G_2/M}(L) = \left|\left\{L^{(g_1,g_2)} \leq M : (g_1,g_2) \in G_1 \times G_2\right\}\right|.$$

Hence we have

$$\left|N_{G_1\times G_2}(L)\right|^{-1} \cdot |M| \cdot \beta_{G_1\times G_2/M} = |\mathscr{K}|.$$

Note that $|M| = |P_1| \cdot |K_2|$ and $N_{G_1\times G_2}(L) = \left(\left(\overline{P}_1, C_{G_1}(P_1')\right), \left(\overline{P}_2, C_{G_2}(P_2')\right)\overline{\theta}\right)$, thus we have the result $\beta_{G_1\times G_2/M}(L)$

$$= \frac{\left|\overline{P}_2\right| \cdot \left|C_{G_1}(P_1')\right|}{|P_1| \cdot |K_2|} \cdot \left|\left\{P_1'^{g_1} : g_1 \in G_1, g_2 \in N_{G_2}(K_2)/K_2, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}\right|.$$

□

Remark 6.6.3. If there exists $g_2K_2 \in N_{G_2}(K_2)/K_2$ for each $g_1 \in G_1$ of P'_1 such that $P_1'^{g_1} \leq P_1$ and $(\theta')^{-1} c_{g_1}\theta = c_{g_2}$ the formula in Theorem 6.6.2 reduces to:

$$\beta_{G_1\times G_2/M}(L) = \frac{|N_{G_1\times G_2}(L)|}{|N_{G_1}(P'_1)|} \cdot \frac{1}{|K_2|} \cdot \beta_{G_1/P_1}(P'_1).$$

Theorem 6.6.4. Let $M = ((P_1, 1), (P_2, K_2), \theta)$ and $L = ((P'_1, 1), (P'_2, K'_2), \theta')$ be subgroups of the direct product $G_1 \times G_2$ such that $(P'_2, K'_2) \leq_{P/K} (P_2, K_2)$. Then we have the equality

$$\left|\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in G_1 \times G_2\right\}\right| =$$

$$= \sum_{\mathscr{O}_2\delta_2\in\mathscr{O}_2\mathscr{O}_1} \left|\left\{M_2^{(g_2,\delta_2)} \geq L_2 : g_2 \in G_2\right\}\right|.$$

Proof. We fix the groups $L_1 := ((P'_1, 1), (U, 1), \theta'_1)$ and $L_2 := ((P'_2, K'_2), (U, 1), \theta'_2)$ such that $\Pi(L_1, L_2) = L$. Let $M_i \leq G_i \times U$ such that $M_1 = ((P_1, 1), (U, 1), \theta_1)$, $M_2 = ((P_2, K_2), (U, 1), \theta_2)$ and $\Pi(M_1, M_2) = M$. Without loss of generality we can assume $M \geq L$. Hence since $(P'_2, K'_2) \leq_{P/K} (P_2, K_2)$ and $(P'_1, 1) \leq_{P/K} (P_1, 1)$ we have the isomorphisms $\varphi_2 : P'_2/K'_2 \to P_2/K_2$ such that $p'_2K'_2 \mapsto p'_2K_2$ for $p'_2 \in P'_2$ and $\varphi_2 : P'_2/1 \to P_2/1$ such that $p'_21 \mapsto p'_21$ for $p'_1 \in P'_1$.

Clearly then $M^{(g_1,g_2)} \geq L$ if and only if $P'_i \leq P_i^{g_i}$, $K'_i \leq K_i^{g_i}$ and $\varphi_1 c_{g_1}^{-1}\theta c_{g_2} = \theta'\varphi_2$. We begin by noting:

$$\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in G_1 \times G_2\right\}$$

$$= \left\{\Pi(M_1, M_2)^{(g_1,g_2)} \geq \Pi(L_1, L_2) : (g_1, g_2) \in G_1 \times G_2\right\}.$$

When we decompose M into M_1 and M_2 we have $|\mathrm{Aut}(U)|$ many choices for θ_1 and θ_2. If we fix the isomorphisms $\theta_1 : P_1/1 \to U$ and $\theta_2 : P_2/K_2 \to U$ then we can obtain all possible isomorphisms from P_1/K_1 and P_2/K_2 to U by composing θ_i with $\delta_i \in \mathrm{Aut}(U)$. Then $M^{(g_1,g_2)} \geq L$ if and only if $(P'_i, K'_i) \leq_{P/K} (P_i, K_i)^{g_i}$ and the following equality is satisfied

$$(\theta'_1)^{-1} \varphi'_1 c_{g_1}^{-1}\theta_1\delta_1 = (\theta'_2)^{-1} \varphi'_2 c_{g_2}^{-1}\theta_2\delta_2 = \alpha \in \mathrm{Aut}(U),$$

where $\varphi_2' : P_2'/K_2' \to P_2^{g_2}/K_2^{g_2}$ such that $\varphi_2' : p_2'K_2' \mapsto p_2'K_2^{g_2}$ and where $\varphi_i' : P_i'/K_i' \to P_i^{g_i}/K_i^{g_i}$ such that $\varphi_i' : p_i'K_i' \mapsto p_i'K_i^{g_i}$.. However if allow this map α vary across all of Aut (U) we over count by a factor of $|\text{Aut}\,(U)|$, that is, if we allow $(\theta_i')^{-1}\varphi_i c_{g_i}^{-1}\theta_i\delta_i$ to vary across all of Aut (U) we will count each $M^{(g_1,g_2)} = \Pi\left(M^{(g_1,\delta_1)}, M^{(g_2,\delta_2)}\right)$ the size of Aut (U) many times. Hence we add the requirement that the isomorphisms $(\theta_i')^{-1}\varphi_i c_{g_i}^{-1}\theta_i\delta_i$ must equal a fixed element in Aut (U), the identity

$$(\theta_1')^{-1}\varphi_1 c_{g_1}^{-1}\theta_1\delta_1 = (\theta_2')^{-1}\varphi_2 c_{g_2}^{-1}\theta_2\delta_2 = 1 \in \text{Aut}\,(U)\,.$$

Obviously this then gives us the sets

$$\left\{M_i^{(g_i,\delta_i)} \geq L_i : g_i \in G_i, \delta_i \in \text{Aut}\,(U)\right\},$$

since if $(P_1', 1) \leq_{P/K} (P_1, 1)^{g_1}$ and $(P_2', K_2') \leq_{P/K} (P_2, K_2)^{g_2}$ and $\varphi_i c_{g_i}^{-1}\theta_i\delta_i = \theta_i'$, which is equivalent to the above requirement, then $M_i^{(g_i,\delta_i)} \geq L_i$. Now we need to find a condition on $\delta_1\delta_2^{-1}$ since it is required that $\varphi_1 c_{g_1}^{-1}\theta_1\delta_1\delta_2^{-1}(\theta_2)^{-1} = \theta_1'(\theta_2')^{-1}\varphi_2$ in order for $M^{(g_1,g_2)} \geq L$. We know that $(\theta_i')^{-1}\varphi_i c_{g_i}^{-1}\theta_i = \delta_i^{-1} \in \text{Aut}\,(U)$, thus

$$\delta_1\delta_2^{-1} = (\theta_1)^{-1} c_{g_1}\varphi_1^{-1}\theta_1(\theta_2')^{-1}\varphi_2 c_{g_2}^{-1}\theta_2.$$

Recall that $\theta_1'(\theta_2')^{-1} = \theta'$, this gives us

$$\delta_1\delta_2^{-1} = (\theta_1)^{-1} c_{g_1}\varphi_1^{-1}\theta'\varphi_2 c_{g_2}^{-1}\theta_2,$$

since, by assumption, $M \geq L$ we have that $\theta = \varphi_1^{-1}\theta'\varphi_2$ giving us

$$\delta_1\delta_2^{-1} = (\theta_1)^{-1} c_{g_1}\theta c_{g_2}^{-1}\theta_2,$$

$$= (\theta_1)^{-1} c_{g_1}\theta_1\theta_2^{-1} c_{g_2}^{-1}\theta_2.$$

Clearly then $\theta_i^{-1}c_{g_i}^{-1}\theta_i \in \mathscr{O}_i$, therefore $\delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2$.

Now we need to show that for all $M^{(g_1,g_2)} \geq L$ that there exist unique cosets $\mathscr{O}_i\delta_i \in$ Out $(U)/\delta_i$ with $\delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2$ such that $M^{(g_1,g_2)} = \Pi\left(M_1^{(g_1',\delta_1)}, M_2^{(g_2',\delta_2)}\right)$.

Clearly such cosets exist since we can always decompose $M^{(g_1,g_2)}$ as $\Pi\left(M_1^{(g_1,1)}, M_2^{(g_2,1)}\right)$ and $1 \in \mathscr{O}_1\mathscr{O}_2$.

It only remains to show that these cosets are unique. Suppose that

$$M^{(g_1,g_2)} = \Pi\left(M_1^{\left(g_1',\delta_1'\right)}, M_2^{\left(g_2',\delta_2'\right)}\right) = \Pi\left(M_1^{\left(g_1'',\delta_1''\right)}, M_2^{\left(g_2'',\delta_2''\right)}\right),$$

then we wan to show that $\mathscr{O}_i\delta_i' = \mathscr{O}_i\delta_i''$. By the above equality we have that $(P_i, K_i)^{g_i'} = (P_i, K_i)^{g_i''}$, thus $g_i' = \overline{g}_i \cdot g_i''$ where $\overline{g}_i \in N_{G_i}(P_i, K_i)/C_{G_i}(P_i, K_i)$. Recall that $\delta_i' = \theta_i^{-1} c_{g_i'} \varphi_i \theta_i'$ and $\delta_i'' = \theta_i^{-1} c_{g_i''} \varphi_i \theta_i'$. Then we have

$$\delta_i' = \theta_i^{-1} c_{g_i'} \varphi_i \theta_i' = \theta_i^{-1} c_{\overline{g}_i} c_{g_i''} \varphi_i \theta_i'$$

$$= \theta_i^{-1} c_{\overline{g}_i} \theta_i \theta_i^{-1} c_{g_i''} \varphi_i \theta_i'.$$

Clearly $\theta_i^{-1} c_{\overline{g}_i} \theta_i = \gamma_i \in \mathscr{O}_i$, this gives $\delta_i' = \gamma_i \delta_i''$ thus $\mathscr{O}_i\delta_i' = \mathscr{O}_i\delta_i''$. This gives us the following formula for the class incidence matrix for the subgroups of $G_1 \times G_2$ whose Goursat triples have isomorphic section quotients:

$$\left|\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in G_1 \times G_2\right\}\right|$$

$$= \sum_{\substack{\mathscr{O}_i\delta_i \in \mathrm{Out}(U)/\mathscr{O}_i \\ \delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2}} \left|\left\{\left(M_1^{(g_1,\delta_1)}, M_2^{(g_2,\delta_2)}\right) \geq (L_1, L_2) : (g_1, g_2) \in G_1 \times G_2\right\}\right|$$

$$= \sum_{\substack{\mathscr{O}_i\delta_i \in \mathrm{Out}(U)/\mathscr{O}_i \\ \delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2}} \left|\left\{M_1^{(g_1,\delta_1)} \geq L_1 : g_1 \in G_1\right\}\right| \cdot \left|\left\{M_2^{(g_2,\delta_2)} \geq L_2 : g_2 \in G_2\right\}\right|.$$

However since $P_1/1 \cong P_1'/1 \cong P_2/K_2 \cong P_2'/K_2'$ and by the assumption that $M \geq L$ we have that $P_1' = P_1$ if $M^{g_1,g_2} \geq L$ then clearly conjugation by g_1 acts trivially on P_1 and $M_1^{(g_1,\delta_1)} = L_1$. We can choose $\theta_1 = \theta_1$ thus $\delta_1 = \theta_1^{-1} c_{g_1} \theta_1 \in \mathscr{O}_1$

$$\sum_{\substack{\mathscr{O}_i\delta_i \in \mathrm{Out}(U)/\mathscr{O}_i \\ \delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2}} \left|\left\{M_1^{(g_1,\delta_1)} \geq L_1 : g_1 \in G_1\right\}\right| \cdot \left|\left\{M_2^{(g_2,\delta_2)} \geq L_2 : g_2 \in G_2\right\}\right|$$

$$= \sum_{\substack{\mathscr{O}_2\delta_2 \in \mathrm{Out}(U)/\mathscr{O}_2 \\ \delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2}} \left| \left\{ M_2^{(g_2,\delta_2)} \geq L_2 : g_2 \in G_2 \right\} \right|$$

$$= \sum_{\mathscr{O}_2\delta_2 \in \mathscr{O}_2\mathscr{O}_1} \left| \left\{ M_2^{(g_2,\delta_2)} \geq L_2 : g_2 \in G_2 \right\} \right| .$$

□

Example 6.6.5. Here we will consider the class incidence matrix of the left-free subgroups of $S_3 \times S_3$. Using the formula for the same K class incidence matrix we obtain the following matrix:

$$M_K^{\triangleleft}\left(S_3\right) = \begin{pmatrix} 36 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 18 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot \\ 12 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 6 & \cdot & \cdot & 2 & \cdot \\ 6 & \cdot & 2 & \cdot & 3 & \cdot & \cdot & 1 \end{pmatrix}$$

Now here is the same P/K class in incidence matrix:

$$M^{\triangleleft}_{P/K}(S_3) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Now if compute the matrix $\leq_K \cdot \leq_{P/K}$ we obtain the class incidence matrix of the left-free subgroups of $S_3 \times S_3$.

$$M^{\triangleleft}(S_3) = \begin{pmatrix} 36 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 18 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 18 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 12 & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot \\ 12 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 6 & 6 & \cdot & 6 & \cdot & 6 & \cdot & \cdot \\ 6 & \cdot & 2 & 6 & \cdot & \cdot & 2 & \cdot \\ 6 & \cdot & 2 & \cdot & 3 & \cdot & \cdot & 1 \end{pmatrix}$$

In the next chapter we will expand and generalise these results through the use the full Burnside ring to give a decomposition of the general table of marks of a direct product of finite groups.

Chapter 7

The Table of Marks of a Direct Product

7.1 Introduction

In this chapter we are interested in the table of marks of a direct product of groups. The table of marks simplifies working with the permutation representation of a group G, since it displays the number of fixed points of one subgroup on another.

However computing the table of marks is not an easy task. As such, it would be considered advantageous to develop a more efficient method for the computation of the table of marks than direct computation. Herein lies the motivation for the present research, where we aim to develop a method for computing the table of marks of a direct product $G_1 \times G_2$ which relies only on the table of marks G_1 and G_2 and knowledge of the outer automorphism group $\mathrm{Out}(U)$ where $U \sqsubseteq G_i$.

In order to motivate how one may be able to develop such a method we will begin with an example and show some similarities between the table of marks of a finite group G and the table of marks of $G \times G$.

Example 7.1.1. Here we take the example of S_3 and use it to examine the table of marks of $S_3 \times S_3$. The table of marks of S_3 is as follows:

$S_3/1$	6			
S_3/C_2	3	1		
S_3/C_3	2	.	2	
S_3/S_3	1	1	1	1
	1	C_2	C_3	S_3

Now the table of marks of $S_3 \times S_3$ is not the Kronecker product of the table of marks of S_3 with itself, i.e., the 16×16 matrix obtained by multiplying each entry of $M(S_3)$ with $M(S_3)$. Below is the table of marks of $S_3 \times S_3$:

1 :	36																					
2 :	18	6																				
3 :	18	.	6																			
4 :	18	.	.	2																		
5 :	12	.	.	.	12																	
6 :	12	.	.	.	.	12																
7 :	12	.	.	.	.	.	6															
8 :	9	3	3	1	.	.	.	1														
9 :	6	6	.	.	6	.	.	.	6													
10 :	6	.	6	.	.	6	.	.	.	6												
11 :	6	2	.	.	.	6	.	.	.	.	2											
12 :	6	.	2	.	6	.	.	.	.	.	.	2										
13 :	6	.	.	2	6	.	.	.	.	.	.	.	2									
14 :	6	.	.	2	.	6	.	.	.	.	.	.	.	2								
15 :	6	.	.	2	.	.	3	.	.	.	.	.	.	.	1							
16 :	4	.	.	.	4	4	4	.	.	.	.	.	.	.	.	4						
17 :	3	1	3	1	.	3	.	1	.	3	1	.	.	1	.	.	1					
18 :	3	3	1	1	3	.	.	1	3	.	.	1	1	.	.	.	.	1				
19 :	2	2	.	.	2	2	2	.	2	.	2	.	.	.	.	2	.	.	2			
20 :	2	.	2	.	2	2	2	.	.	2	.	2	.	.	.	2	.	.	.	2		
21 :	2	.	.	2	2	2	2	.	.	.	.	.	2	2	2	2	.	.	.	.	2	
22 :	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

We can now permute the rows and columns of the table of marks by isomorphic quotients of sections, i.e., permute the rows and columns such that $P_i/K_i \cong U \sqsubseteq S_3$. This gives us the following table:

36	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
18	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
12	.	12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	6	6	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
18	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
9	3	.	.	3	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
6	.	6	.	2	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	3	3	3	1	1	1	1	.	.	.	.	.	.	.	.	1	1	.	.	.	.
12	.	.	.	.	.	.	.	12	.	.	.	.	.	.	.	.	.	.	.	.	.
6	2	.	.	.	.	.	.	6	2	.	.	.	.	.	.	.	.	.	.	.	.
4	.	4	.	.	.	.	.	4	.	4	.	.	.	.	.	.	.	.	.	4	.
2	2	2	2	.	.	.	.	2	2	2	2	.	.	.	.	.	.	.	.	2	.
6	.	.	.	6	.	.	.	6	.	.	.	6	.	.	.	.	.	.	.	.	.
3	1	.	.	3	1	.	.	3	1	.	.	3	1	.	.	1	.	1	.	.	.
2	.	2	.	2	.	2	.	2	.	2	.	2	.	2	.	.	.	.	.	2	.
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.
6	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	.	.	.	.
6	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	2	.	2	.	.	.
2	.	2	.	.	.	.	.	2	.	2	.	.	.	.	.	2	2	2	2	2	2
12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6	.
6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	3	1

Notice now that the top left section, which corresponds to those subgroups whose sections have trivial quotient, is simply the Kronecker product of the table of marks of S_3 with itself. This is because those conjugacy classes of subgroups are direct products of pairs of conjugacy classes of subgroups of S_3. There are some other interesting properties of marks involving groups with Grousat triples with trivial section quotient which we will observe in Theorem 7.1.2, Lemma 7.1.3 and Lemma 7.1.4.

Theorem 7.1.2. Let G_1 and G_2 be finite groups and let $M = ((P_1, P_1), (P_2, P_2), id)$ and $L = ((P_1', P_1'), (P_2', P_2'), id) \leq G_1 \times G_2$. Then $\beta_{G_1 \times G_2/M}(L) = \beta_{G_1/P_1}(P_1') \cdot \beta_{G_2/P_2}(P_2')$.

Proof. Clearly $M \cong P_1 \times P_2$ and $L \cong P_1' \times P_2'$. Therefore we have $\beta_{G_1 \times G_2/M}(L) = \beta_{G_1 \times G_2/P_1 \times P_2}(P_1' \times P_2')$. Since G_1 and G_2 act independently on P_1 and P_1', and P_2 and P_2' respectively we obtain the equality

$$\beta_{G_1 \times G_2/P_1 \times P_2}(P_1' \times P_2') = \beta_{G_1/P_1}(P_1') \cdot \beta_{G_2/P_2}(P_2').$$

□

Lemma 7.1.3. Let G_1 and G_2 be finite groups. Let $L = ((K_1', K_1'), (K_2', K_2'), id) \leq G_1 \times G_2$ and $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ such that $P_i/K_i \cong U$, $U \sqsubseteq G_i$. Then

$$\beta_{G_1 \times G_2/M}(L) = \frac{1}{|P_1/K_1|} \beta_{G_1/K_1}(K_1') \cdot \beta_{G_2/K_2}(K_2').$$

Proof. Let $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ and $L = ((K_1', K_1'), (K_2', K_2'), id) \leq G_1 \times G_2$. From Theorem 2.4.5 we have that $\vartheta_{G_1 \times G_2}(M, L) = \left|\left\{L^{(g_1,g_2)} : (g_1, g_2) \in G_1 \times G_2, L^{(g_1,g_2)} \leq M\right\}\right|$ and from Corollary 2.4.6 that $\beta_{G_1 \times G_2/M}(L) = |G_1 \times G_2 : M| \frac{\vartheta_{G_1 \times G_2}(M,L)}{\vartheta_{G_1 \times G_2}(G_1 \times G_2, L)}$.

$$\begin{aligned}
\vartheta_{G_1 \times G_2}(M, L) &= \left|\left\{L^{(g_1,g_2)} : (g_1, g_2) \in G_1 \times G_2, L^{(g_1,g_2)} \leq M\right\}\right| \\
&= \left|\left\{((K_1', K_1'), (K_2', K_2'), id)^{(g_1,g_2)} : (g_1, g_2) \in G_1 \times G_2, L^{(g_1,g_2)} \leq M\right\}\right| \\
&= \left|\left\{K_1'^{g_1} : g_1 \in G, K_1'^{g_1} \leq K_1\right\}\right| \cdot \left|\left\{K_2'^{g_2} : g_2 \in G_2, K_2'^{g_2} \leq K_2\right\}\right|.
\end{aligned}$$

If $K_1'^{g_1} \leq K_1$ it is automatically a subgroup of P_1, similarly for $K_2'^{g_2}, K_2$ and P_2. Thus $\vartheta_{G_1 \times G_2}(M, L) = \vartheta_{G_1}(K_1, K_1') \cdot \vartheta_{G_2}(K_2, K_2')$. Similarly $\vartheta_{G_1 \times G_2}(G_1 \times G_2, L) = \vartheta_{G_1}(G_1, K_1') \cdot \vartheta_{G_2}(G_2, K_2')$.

Finally $\frac{|G_1 \times G_2|}{|M|} = \frac{|G_1| \cdot |G_2|}{|P_1| \cdot |K_2|}$, so $\frac{|P_1|}{|K_1|} \cdot \frac{|G_1| \cdot |G_2|}{|P_1| \cdot |K_2|} = \frac{|G_1||G_2|}{|K_1||K_2|}$. Putting these pieces together we have

$$\beta_{G_1 \times G_2/M}(L) = |G_1 \times G_2 : M| \, \frac{\vartheta_{G_1 \times G_2}(M, L)}{\vartheta_{G_1 \times G_2}(G_1, L)}$$

$$= \frac{1}{|P_1/K_1|} \frac{|G_1|\,|G_2|}{|K_1|\,|K_2|} \frac{\vartheta_{G_1}(K_1, K_1')\,\vartheta_{G_2}(K_2, K_2')}{\vartheta_{G_1}(G_1, K_1')\,\vartheta_{G_2}(G_2, K_2')}.$$

$$\Rightarrow \beta_{G_1 \times G_2/M}(L) = \frac{1}{|P_1/K_1|} \beta_{G_1/K_1}(K_1') \cdot \beta_{G_2/K_2}(K_2').$$

□

Lemma 7.1.4. Let G_1 and G_2 be finite groups such that $M = ((P_1, P_1), (P_2, P_2), id)$ and $L = ((P_1', K_1'), (P_2', K_2'), \theta') \leq G_1 \times G_2$. Then

$$\beta_{G_1 \times G_2/M}(L) = \beta_{G_1/P_1}(P_1') \cdot \beta_{G_2/P_2}(P_2').$$

Proof. Let $M = ((P_1, P_1), (P_2, P_2), id) \leq G_1 \times G_2$ and $L = ((P_1', K_1'), (P_2', K_2'), \theta') \leq G_1 \times G_2$. Then

$$\beta_{G_1 \times G_2/M}(L) = |N_{G_1 \times G_2}(M) : M| \left|\{M^{(g_1, g_2)} : (g_1, g_2) \in G_1 \times G_2, L \leq M^{(g_1, g_2)}\}\right|$$

$$= |N_{G_1 \times G_2}(M) : M| \left|\left\{((P_1, P_1), (P_2, P_2), id)^{(g_1, g_2)} : (g_1, g_2) \in G_1 \times G_2, L \leq M^{(g_1, g_2)}\right\}\right|.$$

$$= \frac{|N_{G_1 \times G_2}(M)|}{|M|} \left|\{P_1^{g_1} : g_1 \in G_1, P_1' \leq P_1^{g_1}\}\right| \left|\{P_2^{g_2} : g_2 \in G_2, P_2' \leq P_2^{g_2}\}\right|.$$

If $P_1' \leq P_1^{g_1}$ and $P_2' \leq P_2^{g_2}$ then $K_1' \leq P_1^{g_1}$ and $K_2' \leq P_2^{g_2}$. Also by Theorem 5.3.3 it is easy to check that $|N_{G_1 \times G_2}(M)| = |N_{G_1}(P_1)| \cdot |N_{G_2}(P_2)|$. Hence we have

$$\beta_{G_1 \times G_2/M}(L) = \beta_{G_1/P_1}(P_1') \cdot \beta_{G_2/P_2}(P_2').$$

□

It might be tempting at this point to assume that the diagonal block which correspond to the marks of subgroups of $G_1 \times G_2$ of section quotient U, say, on other subgroups of quotient U might just be a scalar multiple of the Kronecker product of rows and columns of the table of marks of $G_1 \times U$ whose conjugacy classes of subgroups have section quotient isomorphic to U and the rows and columns of the table of marks of $G_2 \times U$ whose conjugacy classes of subgroups also have section quotients isomorphic to U. This however is not the case. In the non trivial case other inclusions can be created which are not accounted for in the Kronecker product. We provide a counter example: first here are the rows and columns of the table of marks of $A_4 \times A_4$ whose conjugacy classes of subgroups have section quotient isomorphic to C_2;

$$\begin{array}{llll} 8 & & & \\ 8 & 4 & & \\ 8 & . & 4 & \\ 10 & 4 & 4 & 2 \end{array}$$

For comparison here is the Kronecker product of the rows and columns of the table of marks of $A_4 \times C_2$ whose section quotients are isomorphic to C_2 with themselves:

$$\begin{array}{llll} 16 & & & \\ 16 & 8 & & \\ 16 & . & 8 & \\ 16 & 8 & 8 & 4 \end{array}$$

We see from this comparison that the Kronecker product of the rows and columns specified in the table of marks of $A_4 \times C_2$ is almost exactly twice that of the corresponding rows and columns in the table of marks of $A_4 \times A_4$, which is what we might expect, however there are a couple of inclusions not accounted for by the Kronecker product. In the next section we will see that this construction with slight alteration can work in the case $G_1 \times G_2$ where G_2 is an abelian group.

7.2 The Table of Marks of $G \times C_p$

In this section we shall look at the special case of $G \times C_p$, where p is prime, and its table of marks. This is analogous to the problem considered by Naughton in his paper *Computing the table of marks of a cyclic extension* [24]. We will also give some generalisations to all cyclic groups and further to abelian groups. Since p is prime, and thus C_p has no proper subgroups, there are only three kinds of subgroups to consider. First there are those subgroups whose corresponding Goursat triple has the form $((P_1, P_1), (1, 1), \theta)$, these are the subgroups which Naughton labels as "*blue*" subgroups [24]. Second are those whose Goursat triple has the form $((P_1, P_1), (C_p, C_p), \theta)$. Finally, there are those subgroups whose Goursat triple have the form $((P_1, K_1), (C_p, 1), \theta)$. These second two types are what Naughton calls "*red*" subgroups [24].

As we saw from Theorem 7.1.2, when considering the table of marks of a direct product, rows and columns which correspond to conjugacy classes of subgroups with Goursat triple with trivial section form a sub-table which is simply the Kronecker product of the table of marks of the two factor groups.

Also by Lemma 7.1.3 for marks of subgroups with Grousat triples which have trivial sections on subgroups with Goursat triples with nontrivial section quotient is simply the Kronecker product of the rows from the table of marks of the factor groups corresponding to the possible K_i with each entry divided by the size of the section.

Whereas by Lemma 7.1.4 the marks of subgroups with Goursat triple which have nontrivial section on subgroups with Goursat triple which have trivial section is the Kronecker product of the columns from the table of marks of the factor groups corresponding to the possible P_i.

Thus the only part of the table of marks of $G \times C_p$ which is not already well understood, is that which is made up of marks of subgroups with Goursat triple with section of size p on other subgroups of this type. The following theorem provides some insights on these marks.

Theorem 7.2.1. Let $L = ((P_1', K'), (P_2', K_2'), \theta') \leq G \times C_p$ and $M = ((P, K), (C_p, 1), \theta) \leq G \times C_p$ such that L is a subgroup of some conjugate of M. Then the mark of L on M is a multiple of p.

Proof. Recall the following formula for marks from Theorem 2.3.1:

$$\beta_{G \times C_p/M}(L) = \left|N_{G \times C_p}(M) : M\right| \cdot \left|\left\{M^{(g,c)} \geq L : (g, c) \in G \times C_p\right\}\right|.$$

From Theorem 5.3.3 it can be checked that $N_{G \times C_p}(M) = \left(\left(\overline{P}, \overline{K}\right), (C_p, C_p), \overline{\theta}\right)$ where $\overline{K} = C_G(P, K) = \overline{P}$. This gives us the equality

$$\left|N_{G \times C_p}(M)\right| = \left|\overline{P}\right| \cdot |C_p| = \left|\overline{K}\right| \cdot |C_p|.$$

Hence we have that the mark $\beta_{G \times C_p/M}(L) = \frac{\left|\overline{P}\right| \cdot p}{|P|} \cdot \left|\left\{M^{(g,c)} \geq L : (g, c) \in G \times C_p\right\}\right|$, and $|P|$ divides evenly into $\left|\overline{P}\right|$ since $P \leq \overline{P}$. Thus the mark is multiple of p. ☐

Clearly the above result can be extended to all cyclic and abelian groups, since $C_G(G) = G$ when G is abelian, resulting in the following corollaries.

Corollary 7.2.2. Let $L = ((P_1', K_1'), (P_2', K_2'), \alpha') \leq G \times C_n$ and $M = ((P, K), (C_n, 1), \alpha) \leq G \times C_n$, $n \in \mathbb{Z}$, such that L is a subgroup some conjugate of M. Then the mark of L on M is a multiple of n.

Corollary 7.2.3. Let G_2 be a finite abelian group, $L = ((P_1', K_1'), (P_2', K_2'), \alpha') \leq G_1 \times G_2$, $M = ((P, K), (G_2, 1), \alpha) \leq G_1 \times G_2$, such that L is a subgroups of some conjugate of M. Then the number of fixed points M has under the action of L is a multiple of $|G_2|$.

7.3 The Decomposition of the Table of Marks

Recall from Theorem 5.2.9 that the subgroups of a direct product $G_1 \times G_2$ can be contained in each other as subgroups in one of three ways. This has consequences for the table of marks of a direct product and leads to the natural decomposition described in the following proposition.

This next theorem is central to all of the work presented in this thesis. Recall that the class incidence matrix of a finite group is the table of marks with each row divided by the corresponding diagonal entry. This theorem significantly simplifies the computation of the table of marks of a direct product of finite groups. It allows us to understand the class incidence matrix of a direct product of finite groups in terms of three other class incidence matrices which can be more easily computed from the smaller class incidence matrices of their subgroups.

Theorem 7.3.1. The class incidence matrix of $G_1 \times G_2$ is the product $M_K(G) \cdot M_{P/K}(G) \cdot M_P(G)$ where the class incidence matrices $M_K(G)$, $M_{P/K}(G)$ and $M_K(G)$ correspond to the partial orders Proposition 4.4.2.

Proof. The proof is similar to that of Theorem 4.4.3 in combination with Theorem 5.2.9. □

In the remainder of this chapter will present a formula for each of the relations described in Theorem 7.3.1. However we must make a number of definitions which will be useful in order to prove the theorems which follow.

7.4 The same K relation

In this section we will present some interesting properties of incidences between subgroups of $G_1 \times G_2$ whose Goursat triples have sections with the same K_i. In particular the main theorem in this section will provide us with a method to compute the incidences between subgroups which have the same K_i in the sections of their Goursat triples.

Proposition 7.4.1. Let $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ such that $P_i/K_i \cong U \sqsubseteq G_i$. Then all the subgroups of M which have the same K_i in the sections of their Goursat triples are in a bijective correspondence to subgroups of U.

Proof. Let $L = ((P_1', K_1), (P_2', K_2), \theta') \leq_K M$. Then by Theorem 5.2.7 that $\varphi_1\theta = \theta'\varphi_2$, where φ_i are injective group homomorphisms. Therefore θ' is the restriction of θ to P_1'/K_1. Thus we have that

$$L = \left((P_1', K_1), (\theta(P_1'/K_1)K_2, K_2), \theta\mid_{P_1'/K_1}\right),$$

where $\theta\mid_{P_1'/K_1}$ is the restriction of θ to P_1'/K_1. By the correspondence theorem we have that there is a bijective correspondence between the subgroups of P_1 and P_2 which contain K_1 and K_2 and the subgroups of P_1/K_1 and P_2/K_2 respectively. Thus there is a bijective correspondence between sections $(P_i', K_i) \leq_K (P_i, K_i)$ and subgroups of $U \cong P_i/K_i$.

We define the isomorphisms $\theta_i : P_i/K_i \to U$ such that $\theta = \theta_1\theta_2^{-1}$. Since the sections (P_i', K_i) are in bijective correspondence to the subgroups of U, the restriction of θ_i' to P_i'/K_i gives us a unique subgroup of U. Hence for each subgroup of U we can construct the unique Goursat triple

$$\left((P_1', K_1), (\theta(P_1'/K_1)K_2, K_2), \theta\mid_{P_1'/K_1}\right),$$

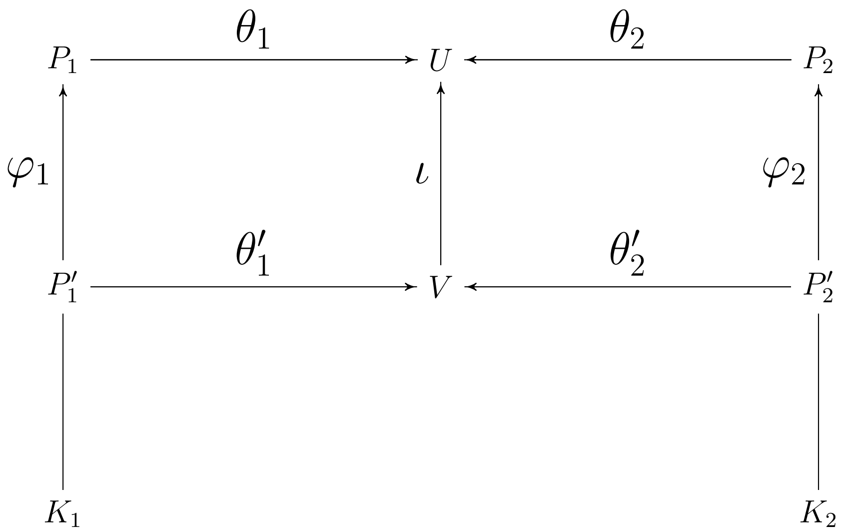
where $\theta\mid_{P_1'/K_1}$ is the restriction of θ to P_1'/K_1. Thus there is a bijective correspondence between the subgroups of M which have the same K_i in their Goursat triples and the subgroups of $U \cong P_i/K_i$. □

The next theorem provides us with a method to compute the marks of subgroups which have the same K_i in the sections of their Goursat triples from the bifree table of marks of $N_{G_1}(K_1) \times N_{G_2}(K_2)$.

Theorem 7.4.2. Let $M = ((P_1, K_1), (P_2, K_2), \theta)$ and $L = ((P_1', K_1), (P_2', K_2), \theta') \leq G_1 \times G_2$, such that $N_{G_1\times G_2}(L) = \left(\left(\overline{P}_1', C_{G_1}(P_1', K_1)\right), \left(\overline{P}_2', C_{G_2}(P_2', K_2)\right), \overline{\theta}\right)$. Then

$$\beta_{G_1\times G_2/M}(L) = \frac{\left|\overline{P}_2'\right| \cdot |C_{G_1}(P_1', K_1)|}{|P_1| \cdot |K_2|} \cdot |\mathscr{K}|,$$

where $\mathscr{K} = \left|\{P_1'^{g_1} : g_i \in N_{G_i}(K_i)/K_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\}\right|$, where the map $\varphi_{g_i} \colon P'^{g_i}/K_i \to P_i/K_i$ is the canonical monomorphism induced by the inclusion of $P_i'^{g_i}$ into P_i.



Proof. From Theorem 5.2.4 that $L^{(g_1,g_2)} \leq M$ if and only if $P_i'^{g_i} \leq P_i$ and $c_{g_1}\theta = \theta' c_{g_2}$. By Theorem 5.2.7 we know that $M \geq L$ if and only if $(P_i', K_i) \leq_K (P_i, K_i)$ and $\iota_i = \iota_2$, such that $\iota_i = (\theta_i')^{-1} c_{g_i}\varphi_{g_i}\theta_i$, where $\theta = \theta_1\theta_2^{-1}$ and $\theta' = \theta_1' (\theta_2')^{-1}$. We begin by noting

$$\begin{aligned}&\left|\left\{(P_1', K_1)^{g_1} \leq (P_1, K_1) : g_1 \in G_1\right\}\right| \\ &= \left|\left\{P_1'^{g_1} \leq P_1 : g_1 \in N_{G_1}(K_1)/K_1\right\}\right|.\end{aligned}$$

There is a bijective correspondence between the set $\left\{L^{(g_1,g_2)} : g_i \in G_i, L^{(g_1,g_2)} \leq M\right\}$ and the set $\left\{P_1'^{g_1} : g_i \in N_{G_i}(K_i)/K_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}$. Clearly there is a surjection which sends $L^{(g_1,g_2)}$ in the first set to $P_1'^{g_1} = \pi_1(L^{(g_1,g_2)})$. To see that this map is injective, suppose $P_1'^{g_1} = P_1'^{\tilde{g}_1}$ with $\tilde{g}_i \in N_{G_i}(K_i)/K_i$, $P_i'^{\tilde{g}_i} \leq P_i$, $c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2} = \varphi_{\tilde{g}_1}\theta$. Since $P_1'^{g_1} = P_1'^{\tilde{g}_1}$, we have $\varphi_{g_1} = \varphi_{\tilde{g}_1}$. Then

$$c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta = \varphi_{\tilde{g}_1}\theta = c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2},$$

so

$$P_2'^{g_2}/K_2 = \mathrm{Im}(c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2}) = \mathrm{Im}(c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}\varphi_{\tilde{g}_2}) = P_2'^{\tilde{g}_2}/K_2,$$

so $P_2'^{g_2} = P_2'^{\tilde{g}_2}$, so $\varphi_{g_2} = \varphi_{\tilde{g}_2}$, so $c_{g_1}^{-1}\theta' c_{g_2} = c_{\tilde{g}_1}^{-1}\theta' c_{\tilde{g}_2}$ (because $\varphi_{g_2} = \varphi_{\tilde{g}_2}$ is a monomorphism), so $L^{(g_1,g_2)} = L^{(\tilde{g}_1,\tilde{g}_2)}$. We will denote the set

$$\left\{P_1'^{g_1} : g_i \in N_{G_i}(K_i)/K_i, P_i'^{g_i} \leq P_i, c_{g_1}^{-1}\theta' c_{g_2}\varphi_{g_2} = \varphi_{g_1}\theta\right\}$$

by $\mathscr{K}$. Recall from Corollary 2.4.6 that

$$|N_{G_1\times G_2}(L)|^{-1}\cdot|M|\cdot\beta_{G_1\times G_2/M}(L)=\left|\left\{L^{(g_1,g_2)}\leq M:(g_1,g_2)\in G_1\times G_2\right\}\right|.$$

Hence we have

$$|N_{G_1\times G_2}(L)|^{-1}\cdot|M|\cdot\beta_{G_1\times G_2/M}=|\mathscr{K}|.$$

Note that $|M|=|P_1|\cdot|K_2|$ and $N_{G_1\times G_2}(L)=\left(\left(\overline{P}'_1,C_{G_1}(P'_1)\right),\left(\overline{P}'_2,C_{G_2}(P'_2)\right)\overline{\theta}\right)$, thus we have the result

$$\beta_{G_1\times G_2/M}(L)=\frac{\left|\overline{P}'_2\right|\cdot|C_{G_1}(P'_1)|}{|P_1|\cdot|K_2|}\cdot|\mathscr{K}|.$$

□

Remark 7.4.3. If there exists $g_2\in N_{G_2}(K_2)/K_2$ for each $g_1\in N_{G_1}(K_1)/K_1$ such that $P_1'^{g_1}\leq P_1$ and $(\theta')^{-1}c_{g_1}\theta=c_{g_2}$ the formula in Theorem 7.4.2 reduces to:

$$\beta_{G_1\times G_2/M}(L)=\frac{|N_{G_1\times G_2}(L)|}{|N_{G_1}(P'_1,K_1)|}\cdot\frac{1}{|K_2|}\cdot\beta_{\left(N_{G_1}(K_1)/K_1\right)/(P_1/K_1)}(P'_1/K_1).$$

7.5 The same P/K relation

Here we will present a method to compute incidences between subgroups of $G_1\times G_2$ whose Goursat triples have isomorphic section quotient. This will make a significant contribution to the computation of the table of marks of $G_1\times G_2$.

Theorem 7.5.1. Let $L=((P'_1,K'_1)(P'_2,K'_2),\theta')$ and $M=((P_1,K_1)(P_2,K_2),\theta)\leq G_1\times G_2$, where $P'_i/K'_i\cong P_i/K_i\cong U$ and $U\sqsubseteq G_i$. Then $\left|\left\{M^{(g_1,g_2)}\geq L:(g_1,g_2)\in G_1\times G_2\right\}\right|$

$$=\sum_{\substack{\mathscr{O}_i\delta_i\in\mathrm{Out}(U)/\mathscr{O}_i\\ \delta_1\delta_2^{-1}\in\mathscr{O}_1\mathscr{O}_2}}\left|\left\{M_1^{(g_1,\delta_1)}\geq L_1:g_1\in G_1\right\}\right|\cdot\left|\left\{M_2^{(g_2,\delta_2)}\geq L_2:g_2\in G_2\right\}\right|,$$

where $L=\Pi(L_1,L_2)$, $L_i=((P'_i,K'_i),(U,1),\theta'_i)\leq G_i\times U$, and $M=\Pi(M_1,M_2)$, $M_i=((P_i,K_i),(U,1),\theta_i)\leq G_i\times U$.

$$
\begin{array}{ccccc}
P_1/K_1 & \xrightarrow{\theta_1} & U & \xleftarrow{\theta_2} & P_2/K_2 \\
\uparrow \varphi_1 & & \uparrow 1 & & \uparrow \varphi_2 \\
P_1'/K_1' & \xrightarrow{\theta_1'} & U & \xleftarrow{\theta_2'} & P_2'/K_2'
\end{array}
$$

Proof. We fix $L_i := ((P_i', K_i'), (U, 1), \theta_i')$ such that $\Pi(L_1, L_2) = L$. Let $M_i \leq G_i \times U$ such that $M_i = ((P_i, K_i), (U, 1), \theta_i)$ and $\Pi(M_1, M_2) = M$. Without loss of generality we can assume $M \geq L$. Hence since $(P_i', K_i') \leq_{P/K} (P_i, K_i)$ we have the isomorphisms $\varphi_i : P_i'/K_i' \to P_i/K_i$ such that $p_i'K_i' \mapsto p_i'K_i$ for $p_i' \in P_i'$.

Clearly then $M^{(g_1,g_2)} \geq L$ if and only if $P_i' \leq P_i^{g_i}$, $K_i' \leq K_i^{g_i}$ and $\varphi_{g_1} c_{g_1}^{-1} \theta c_{g_2} = \theta' \varphi_{g_2}$, where $\varphi_{g_i} : P_i'/K_i' \to P_i^{g_i}/K_i^{g_i}$ such that $\varphi_{g_i} : p_i'K_i' \mapsto p_i'K_i^{g_i}$. We begin by noting:

$$\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in G_1 \times G_2\right\}$$

$$= \left\{\Pi(M_1, M_2)^{(g_1,g_2)} \geq \Pi(L_1, L_2) : (g_1, g_2) \in G_1 \times G_2\right\}.$$

When we decompose M into M_1 and M_2 we have $|\mathrm{Aut}(U)|$ many choices for θ_1 and θ_2. If we fix an isomorphism $\theta_i : P_i/K_i \to U$ then we can obtain all possible isomorphisms from P_i/K_i to U by composing θ_i with $\delta_i \in \mathrm{Aut}(U)$. Then $M^{(g_1,g_2)} \geq L$ if and only if $(P_i', K_i') \leq_{P/K} (P_i, K_i)^{g_i}$ and the following equality is satisfied

$$(\theta_1')^{-1} \varphi_{g_1} c_{g_1}^{-1} \theta_1 \delta_1 = (\theta_2')^{-1} \varphi_{g_2} c_{g_2}^{-1} \theta_2 \delta_2 = \alpha \in \mathrm{Aut}(U).$$

However if allow this map α vary across all of $\mathrm{Aut}(U)$ we over count by a factor of $|\mathrm{Aut}(U)|$, that is, if we allow $(\theta_i')^{-1} \varphi_{g_i} c_{g_i}^{-1} \theta_i \delta_i$ to vary across all the elements of $\mathrm{Aut}(U)$ we will count each $M^{(g_1,g_2)} = \Pi\left(M^{(g_1,\delta_1)}, M^{(g_2,\delta_2)}\right)$ the size of $\mathrm{Aut}(U)$ many times. Hence we add the requirement that the isomorphisms $(\theta_i')^{-1} \varphi_{g_i} c_{g_i}^{-1} \theta_i \delta_i$ must equal a fixed element in $\mathrm{Aut}(U)$, the identity

$$(\theta_1')^{-1} \varphi_{g_1} c_{g_1}^{-1} \theta_1 \delta_1 = (\theta_2')^{-1} \varphi_{g_2} c_{g_2}^{-1} \theta_2 \delta_2 = 1 \in \mathrm{Aut}(U).$$

Obviously this then gives us the sets

$$\left\{M_i^{(g_i,\delta_i)} \geq L_i : g_i \in G_i, \delta_i \in \mathrm{Aut}\,(U)\right\},$$

since if $(P_i', K_i') \leq_{P/K} (P_i, K_i)^{g_i}$ and $\varphi_{g_i} c_{g_i}^{-1} \theta_i \delta_i = \theta_i'$, which is equivalent to the above requirement, then $M_i^{(g_i,\delta_i)} \geq L_i$. Now we need to find a condition on $\delta_1 \delta_2^{-1}$ since it is required that $\varphi_{g_1} c_{g_1}^{-1} \theta_1 \delta_1 \delta_2^{-1} (\theta_2)^{-1} = \theta_1' (\theta_2')^{-1} \varphi_{g_2}$ in order for $M^{(g_1,g_2)} \geq L$. We know that $(\theta_i')^{-1} \varphi_{g_i} c_{g_i}^{-1} \theta_i = \delta_i^{-1} \in \mathrm{Aut}\,(U)$, thus

$$\delta_1 \delta_2^{-1} = (\theta_1)^{-1} c_{g_1} \varphi_{g_1}^{-1} \theta_1' (\theta_2')^{-1} \varphi_{g_2} c_{g_2}^{-1} \theta_2.$$

Recall that $\theta_1' (\theta_2')^{-1} = \theta'$, this gives us

$$\delta_1 \delta_2^{-1} = (\theta_1)^{-1} c_{g_1} \varphi_{g_1}^{-1} \theta' \varphi_{g_2} c_{g_2}^{-1} \theta_2,$$

if $M^{(g_1,g_2)} \geq L$ then $c_{g_1} \varphi_{g_1}^{-1} \theta' \varphi_{g_2} c_{g_2}^{-1} = \theta$ this gives us

$$\delta_1 \delta_2^{-1} = (\theta_1)^{-1} c_{g_1} \varphi_{g_1}^{-1} \theta' \varphi_{g_2} c_{g_2}^{-1} \theta_2 = (\theta_1)^{-1} \theta \theta_2 = 1.$$

Clearly then $\delta_1 = \delta_2$ therefore $\delta_1 \delta_2^{-1} \in \mathscr{O}_1 \mathscr{O}_2$.

Now we need to show that for all $M^{(g_1,g_2)} \geq L$ that there exist unique cosets $\mathscr{O}_i \delta_i \in \mathrm{Out}\,(U)/\mathscr{O}_i$ with $\delta_1 \delta_2^{-1} \in \mathscr{O}_1 \mathscr{O}_2$ such that $M^{(g_1,g_2)} = \Pi\left(M_1^{(g_1',\delta_1)}, M_2^{(g_2',\delta_2)}\right)$ for some $g_i' \in G_i$ such that $M_i^{(g_i,\delta_i)} \geq L_i$.

Clearly such cosets exist since we can always decompose $M^{(g_1,g_2)}$ as $\Pi\left(M_1^{(g_1,1)}, M_2^{(g_2,1)}\right)$ and $1 \in \mathscr{O}_1 \mathscr{O}_2$.

It only remains to show that these cosets are unique. Suppose that

$$M^{(g_1,g_2)} = \Pi\left(M_1^{(g_1',\delta_1')}, M_2^{(g_2',\delta_2')}\right) = \Pi\left(M_1^{(g_1'',\delta_1'')}, M_2^{(g_2'',\delta_2'')}\right),$$

then we want to show that $\mathscr{O}_i \delta_i' = \mathscr{O}_i \delta_i''$. By the above equality we have that $(P_i, K_i)^{g_i'} = (P_i, K_i)^{g_i''}$, thus $g_i' = \overline{g}_i \cdot g_i''$ where $\overline{g}_i \in N_{G_i}(P_i, K_i)/C_{G_i}(P_i, K_i)$. Recall that $\delta_i' = \theta_i^{-1} c_{g_i'} \varphi_{g_i}^{-1} \theta_i'$ and $\delta_i'' = \theta_i^{-1} c_{g_i''} \varphi_{g_i}^{-1} \theta_i'$. Then we have

$$\delta_i' = \theta_i^{-1} c_{g_i'} \varphi_{g_i}^{-1} \theta_i' = \theta_i^{-1} c_{\overline{g}_i} c_{g_i''} \varphi_{g_i}^{-1} \theta_i'$$

$$= \theta_i^{-1} c_{\overline{g}_i} \theta_i \theta_i^{-1} c_{g_i''} \varphi_{g_i} \theta_i'.$$

Clearly $\theta_i^{-1} c_{\overline{g}_i} \theta_i = \gamma_i \in \mathscr{O}_i$, this gives $\delta_i' = \gamma_i \delta_i''$ thus $\mathscr{O}_i \delta_i' = \mathscr{O}_i \delta_i''$. This gives us the following formula for the class incidence matrix for the subgroups of $G_1 \times G_2$ whose Goursat triples have isomorphic section quotients:

$$\left|\left\{M^{(g_1,g_2)} \geq L : (g_1,g_2) \in G_1 \times G_2\right\}\right|$$

$$= \sum_{\substack{\mathscr{O}_i\delta_i \in \mathrm{Out}(U)/\mathscr{O}_i \\ \delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2}} \left|\left\{\left(M_1^{(g_1,\delta_1)}, M_2^{(g_2,\delta_2)}\right) \geq (L_1, L_2) : (g_1,g_2) \in G_1 \times G_2\right\}\right|$$

$$= \sum_{\substack{\mathscr{O}_i\delta_i \in \mathrm{Out}(U)/\mathscr{O}_i \\ \delta_1\delta_2^{-1} \in \mathscr{O}_1\mathscr{O}_2}} \left|\left\{M_1^{(g_1,\delta_1)} \geq L_1 : g_1 \in G_1\right\}\right| \cdot \left|\left\{M_2^{(g_2,\delta_2)} \geq L_2 : g_2 \in G_2\right\}\right|.$$

□

7.6 The same P relation

Here we present a method to calculate incidences between groups which have the same P_i in their Goursat triples. This is the final component needed to compute the table of marks of $G_1 \times G_2$.

Theorem 7.6.1. Let $L = ((P_1, K_1'), (P_2, K_2'), \theta')$ and $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$. Then $\left|\left\{M^{(g_1,g_2)} \geq L : (g_1,g_2) \in G_1 \times G_2\right\}\right|$

$$= \left|\{K_1^{g_1} : g_i \in N_{G_i}(P_i)/P_i, K_i^{g_i} \geq K_i', \varphi_{g_1} c_{g_1}^{-1} \theta c_{g_2} = \theta' \varphi_{g_2}\}\right|,$$

where $\varphi_{g_i} \colon P_1/K_i' \to P_i/K_i^{g_i}$ is the canonical epimorphism induced by the inclusion of K_i' into $K_i^{g_i}$.

$$
\begin{array}{ccccc}
P_1/K_1 & \xrightarrow{\theta_1} & U & \xleftarrow{\theta_2} & P_2/K_2 \\
\uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \varphi_2} \\
P_1/K_1' & \xrightarrow{\theta_1'} & V & \xleftarrow{\theta_2'} & P_2/K_2'
\end{array}
$$

Proof. We know from Theorem 5.2.4 that $M^{(g_1,g_2)} \geq L$ if and only if $P_i^{g_i} \geq P_i$, $K_i^{g_i} \geq K_i'$ and $c_{g_1}^{-1}\theta c_{g_2} = \theta'$. Note that since g_i must fix the P_i, $g_i \in N_{G_i}(P_i)$. However $K_i \lhd P_i$, hence if $g_i \in P_i$ then g_i it fixes the set K_i. Thus we only need to consider the case where $g_i \in N_{G_i}(P_i)/P_i$. This gives us the following equality of set:

$$
\begin{aligned}
&\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in G_1 \times G_2\right\} \\
= &\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in N_{G_1}(P_1)/P_1 \times N_{G_2}(P_2)/P_2\right\}.
\end{aligned}
$$

There is a bijective correspondence between the set $\{M^{(g_1,g_2)} : g_i \in G_i, M^{(g_1,g_2)} \geq L\}$ and the set $\{K_1^{g_1} : g_i \in N_{G_i}(P_i), K_i^{g_i} \geq K_i', \varphi_{g_1} c_{g_1}^{-1}\theta c_{g_2} = \theta'\varphi_{g_2}\}$. Clearly there is a surjection which sends $M^{(g_1,g_2)}$ in the first set to $K_1^{g_1} = k_1(M^{(g_1,g_2)})$ in the second set. To see that this map is injective, suppose $K^{g_1} = K^{\tilde{g}_1}$ with $\tilde{g}_i \in N_{G_i}(P_i)$, $K_i^{\tilde{g}_i} \geq K_i'$, $\varphi_{\tilde{g}_1} c_{\tilde{g}_1}^{-1}\theta c_{\tilde{g}_2} = \theta'\varphi_{\tilde{g}_2}$. Since $K^{g_1} = K^{\tilde{g}_1}$, we have $\varphi_{g_1} = \varphi_{\tilde{g}_1}$. Then

$$\varphi_{g_2} c_{g_2}^{-1}\theta^{-1} c_{g_1} = \theta'^{-1}\varphi_{g_1} = \theta'^{-1}\varphi_{\tilde{g}_1} = \varphi_{\tilde{g}_2} c_{\tilde{g}_2}^{-1}\theta^{-1} c_{\tilde{g}_1},$$

so

$$K_2^{g_2}/K_2' = \mathrm{Ker}(\varphi_{g_2} c_{g_2}^{-1}\theta^{-1} c_{g_1}) = \mathrm{Ker}(\varphi_{\tilde{g}_2} c_{\tilde{g}_2}^{-1}\theta^{-1} c_{\tilde{g}_1}) = K_1^{\tilde{g}_2}/K_2',$$

so $K_2^{g_2} = K_2^{\tilde{g}_2}$, so $\varphi_{g_2} = \varphi_{\tilde{g}_2}$, so $c_{g_2}^{-1}\theta^{-1} c_{g_1} = c_{\tilde{g}_2}^{-1}\theta^{-1} c_{\tilde{g}_1}$ (because $\varphi_{g_2} = \varphi_{\tilde{g}_2}$ is an epimorphism), so $L^{(g_1,g_2)} = L^{(\tilde{g}_1,\tilde{g}_2)}$. Hence we have

$$
\begin{aligned}
&\left|\left\{M^{(g_1,g_2)} \geq L : (g_1, g_2) \in G_1 \times G_2\right\}\right| \\
= &\left|\{K_1^{g_1} : g_i \in N_{G_i}(P_i)/P_i, K_i^{g_i} \geq K_i', \varphi_{g_1} c_{g_1}^{-1}\theta c_{g_2} = \theta'\varphi_{g_2}\}\right|,
\end{aligned}
$$

□

7.7 Conclusion

The combination of Theorem 7.4.2, Theorem 7.5.1 and Theorem 7.6.1 provide us with a method for computing the table of marks of $G_1 \times G_2$ from the table of marks of G_1 and G_2 and those of their subgroups by Theorem 7.3.1. This greatly simplifies the computation for the table of marks of a direct product of finite groups. Here we will present an example of these computations on the symmetric group S_3.

Example 7.7.1. The same K, same P/K and same P matrices are displayed below and if they are multiplied in the order they are presented in then this will yield a permuted table of marks for $S_3 \times S_3$.

$$M_K = \left(\begin{array}{cccccccccccccccc|cccc|c|c}
36 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 3 & 1
\end{array}\right)$$

$$M_{P/K} = \left(\begin{array}{cccccccccccccccc|cccc|c|c}
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9 & 3 & \cdot & \cdot & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & 3 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}\right)$$

$$M_P = \left(\begin{array}{cccccccccccccccc|cccc|c|c}
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

It should be noted that the matrices corresponding to the $\leq_K$, $\leq_{P/K}$ and $\leq_P$ will always be lower triangular, block diagonal and upper triangular respectively. By multiplying the resulting matrices in accordance with Theorem 7.3.1 we get the following 22×22 matrix, which is a permuted version of the table of marks for $S_3 \times S_3$.

$$M\left(S_3\right) = \left(\begin{array}{cccccccccccccccc|cccc|c|c}
36 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
18 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
12 & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & 6 & 6 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
18 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
9 & 3 & \cdot & \cdot & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & \cdot & 6 & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
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6 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot \\
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2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
18 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
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2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot \\
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6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 3 & 1
\end{array}\right)$$

Chapter 8

Application to double Burnside algebras

8.1 Introduction

In this chapter we use methods developed in the previous chapter to produce a base change matrix for the double Burnside algebra of the symmetric group on three letters, $\mathbb{Q}B(S_3, S_3)$. This basis new basis will be conjectured to be a cellular basis for the double Burnside algebra, meaning the double Burnside algebra is a cellular algebra as defined in [19]. This will offer new insights into the simple modules and radical of $\mathbb{Q}B(S_3, S_3)$ which have proven difficult to compute for researchers [8, 2]. This conjecture will form the basis for future work. We begin with a detailed introduction into cellular algebras.

8.2 Cellular Algebras

Cellular algebras are a family of algebras first introduced by Graham and Lehrer in their paper *Cellular Algebras* [19]. Since being first defined many families of algebras have been shown to be cellular, including: Iwahori–Hecke algebras [16], the Brauer centraliser algebras, the Temperly–Lieb algebras and the Ariki–Koike algebras [19]. In this section we will introduce some background from Graham and Lehrer's paper. Throughout this section R will be a commutative ring with identity.

Definition 8.2.1. [19] A *cellular algebra* over R is an associative (unital) algebra A, together with *cell datum* (Λ, M, C, i) where

1. Λ is a partially ordered set (poset) and for each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of "tableaux of type λ") such that $C : \coprod_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \to A$ is an injective map with image an R-basis of A.

2. If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, write $C(S, T) = C^{\lambda}_{S,T} \in A$. Then i is an R-linear anti-involution of A such that $i\left(C^{\lambda}_{S,T}\right) = C^{\lambda}_{T,S}$.

3. If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for any element $a \in A$ we have

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S)\, C^{\lambda}_{S',T} \,(\text{mod}\, A(< \lambda))$$

where $r_a(S', S) \in R$ is independent of T and where $A(< \lambda)$is the R-submodule of A generated by $\left\{C^{\mu}_{S'',T''} : \mu < \lambda; S'', T'' \in M(\mu)\right\}$.

The above definition can be difficult to work with and use in order to prove that a particular algebra is a cellular algebra because it would involve picking a suitable basis in order to prove that the basis is cellular. To this end, we will use another definition introduced by Changchang Xi [11] which does not rely on a choice of basis.

Definition 8.2.2. [11] Let A be an R-algebra. Assume that there is an involution i on A. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $W \subset J$ such that W has finite R-dimension and that there is an isomorphism of A-bimodules $\alpha : J \simeq W \otimes_R i(W)$ (where $i(W) \subset J$ is the i-image of W) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & W \otimes_R i(W) \\ {\scriptstyle i}\downarrow & & \downarrow {\scriptstyle x \otimes y \mapsto i(y) \otimes i(x)} \\ J & \xrightarrow[\alpha]{} & W \otimes_R i(W) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J_1 \oplus J_2 \oplus \ldots \oplus J_n$ (for some n) with $i(J_j) = J_j$ for each j and such that setting $J'_j = \oplus_{l=1}^{j} J_l$ gives a chain of two-sided ideals of $A : 0 = J'_0 \subset J'_1 \subset J'_2 \subset \ldots \subset J'_n = A$ (each of them fixed by i) and for each j $(j = 1, \ldots, n)$ the quotient $J_j = J'_j / J'_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J'_{j-1}.

We will now outline some known results and properties of cellular algebras since they will be useful to recall and it will provide context for the work carried out in this chapter.

Lemma 8.2.3. [19] Suppose Φ is an ideal of Λ (i.e. $\phi \in \Phi, \lambda \in \Lambda, \lambda \leq \phi \Rightarrow \lambda \in \Phi$). Then $A(\Phi) = \left\langle C^{\lambda}_{S,T} | \lambda \in \Phi \right\rangle_R$ is a two-sided ideal of A.

Definition 8.2.4. [19] If $\Phi' \subseteq \Phi$ are two ideals of Λ, we define $\mathcal{Q}(\Phi \setminus \Phi')$ as the (A, A) bimodule $A(\Phi)/A(\Phi')$.

Lemma 8.2.5. [19] Let $\lambda \in \Lambda$ and $a \in A$. Then for any elements S_1, S_2, T_1 and $T_2 \in M(\lambda)$, we have

$$C^{\lambda}_{S_1,T_1} a C^{\lambda}_{S_2,T_2} \equiv \phi_a(T_1, T_2)\, C^{\lambda}_{S_1,T_2} \ (\text{mod } A(<\lambda)),$$

where $\phi_a(T_1, T_2) \in R$ depends only on a, T_1 and S_1 (i.e., is independent of T_2 and S_2).

Remark 8.2.6. [19] Clearly $\mathcal{Q}(\Phi \backslash \Phi')$ depends only on the set $\Phi \backslash \Phi'$, and not on Φ and Φ'. This accounts for the notation.

Observe that we have an obvious R-module monomorphism: $\mathcal{Q}(\Phi \backslash \Phi') \to A$, whose image is $A(\Phi \backslash \Phi')$ in the notation above. We shall making particular use of $A(\{\lambda\})$ and $\mathcal{Q}(\{\lambda\})$ where $\lambda \in \Lambda$ [19].

Definition 8.2.7. [19] For each $\lambda \in \Lambda$ define the (left) A-module $W(\lambda)$ as follows: $W(\lambda)$ is a free R-module with basis $\{C_S | S \in M(\lambda)\}$ and A-action defined by

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S)\, C_{S'} \quad (a \in A, S \in M(\lambda))$$

where $r_a(S', S)$ is the element of R defined in Definition 8.2.1 3. It is called the *cell representation* of A corresponding to $\lambda \in \Lambda$.

Lemma 8.2.8. [19]

1. There is a natural isomorphism of R-modules $C^{\lambda} : W(\lambda) \otimes_R i(W(\lambda)) \to A(\{\lambda\})$, defined by $(C_S, C_T) \to C^{\lambda}_{S,T}$ $(S, T \in M(\lambda))$. If $A(\{\lambda\})$ is identified with the (A, A) bimodule $\mathcal{Q}(\{\lambda\})$ then C^{λ} becomes an isomorphism of (A, A) bimodules.

2. $A = \bigoplus_{\lambda \in \Lambda} A(\{\lambda\})$ (as R-modules).

3. If $a \in A(\{\lambda\})$ and $S, T \in M(\mu)$, $(\lambda, \mu \in \Lambda)$ then $r_a(S,T) = 0$ unless $\lambda \geq \mu$.

It is in fact the above lemma which motivates the basis–free definition of a cellular algebra described in Definition 8.2.2 [11].

Definition 8.2.9. [19] For $\lambda \in \Lambda$, define $\phi_\lambda : W(\lambda) \times W(\lambda) \to R$ by $\phi_\lambda(C_S, C_T) = \phi_1(S,T)$, $S, T \in M(\lambda)$, extended bilinearly.

Proposition 8.2.10. [19] Keep the notation above and let $\lambda \in \Lambda$. Then

1. The form ϕ_λ is symmetric; i.e., for $x, y \in W(\lambda)$, one has$\phi_\lambda(x,y) = \phi_\lambda(y,x)$.

2. For $x, y \in W(\lambda)$ and $a \in A$, we have

$$\phi_\lambda(i(a)x, y) = \phi_\lambda(x, ay).$$

3. For $x, y, z \in W(\lambda)$ we have

$$C^\lambda(x \otimes y)\, z = \phi_\lambda(y, z)\, x.$$

Definition 8.2.11. [19] Let V be any left A-module, with Φ, Φ' as above. Then $V(\Phi \setminus \Phi')$ is defined as the A-module $\mathcal{Q}(\Phi \setminus \Phi') \otimes_A V$. In particular, if Φ' is empty, we have

$$V(\Phi) = \mathcal{Q}(\Phi) \otimes_A V \cong A(\Phi) \otimes_A V.$$

Lemma 8.2.12. [19] Let Φ be an ideal of Λ.

1. If V is any projective A-module, there is a natural isomorphism: $V(\Phi) \to A(\lambda)V$ defined by $a \otimes v \mapsto av$ $(a \in A(\Phi), v \in V)$.

2. If e is an idempotent of A, then

$$A(\Phi)Ae = A(\Phi)e = A(\Phi) \cap Ae.$$

Lemma 8.2.13. [19]

1. If, in Definition 8.2.11 and Lemma 8.2.12, V is projective (as A-module), then for any two ideals $\Phi \subseteq \Phi'$ of Λ, we have an exact sequence

$$0 \to V(\Phi) \to V(\Phi') \to V\left(\Phi' \setminus \Phi\right) \to 0.$$

2. For any finitely generated projective A-module V, there is a filtration $0 = V_0 \leq V_1 \leq V_2 \ldots \leq V_d = V$ of V by projective modules V_i, such that $V_i/V_{i-1} \cong V(\{\lambda\})$ for some $\lambda \in \Lambda$.

Lemma 8.2.14. [19] Let V be any A-module and let $\lambda \in \Lambda$. Define the R-module V^λ by $V^\lambda := i\left(W\left(\lambda\right)\right) \otimes_A V$.

1. In the notation of Definition 8.2.11, $V\left(\{\lambda\}\right) \cong W\left(\lambda\right) \otimes_R V^\lambda$.

2. If $\phi_\lambda \neq 0$ and R is an integral domain, then $Hom_A\left(V\left(\{\lambda\}\right), W\left(\lambda\right)\right) \cong Hom_R\left(V^\lambda, R\right)$. (as R-modules).

Henceforth we will assume hat R is a field and that all modules are finite dimensional over R.

Definition 8.2.15. [19] Let (Λ, M, C, i) be a cell datum. For $\lambda \in \Lambda$, define the radical

$$rad\left(\lambda\right) := \left\{x \in W\left(\lambda\right) | \phi_\lambda(x, y) = 0 \text{ for all } y \in W(\lambda)\right\}.$$

Proposition 8.2.16. [19] Let $\lambda \in \Lambda$ as above. Then

1. $rad\left(\lambda\right)$ is an A-submodule of $W\left(\lambda\right)$.

2. If $\phi_\lambda \neq 0$, the quotient $W\left(\lambda\right)/rad\left(\lambda\right)$ is absolutely irreducible.

3. If $\phi_\lambda \neq 0$, $rad(\lambda)$ is the radical of the A-module $W\left(\lambda\right)$ (i.e., the minimal submodule with semisimple quotient).

Definition 8.2.17. [19] Denote the (absolutely irreducible) A-module $W(\lambda)/rad(\lambda)$ $(\lambda \in \Lambda, \phi_\lambda \neq 0)$ by L_λ.

Theorem 8.2.18. [19] Let R be a field and let (Λ, M, C, i) be a cell datum for the R-algebra A. For each $\lambda \in \Lambda$, define the (left) A-module $W(\lambda)$ and bilinear form ϕ_λ on $W(\lambda)$. Let $\Lambda_0 = \{\lambda \in \Lambda | \phi_\lambda \neq 0\}$.

1. The set $\{L_\lambda | \lambda \in \Lambda_0\}$ is a complete set of (representatives of equivalence classes of) absolutely irreducible A-modules.

2. If V_λ is the principal indecomposable A-module with head isomorphic to L_λ then $V_\lambda \cong V_\lambda(\leq \lambda)$ in the notation of definition 8.2.11 $(\lambda \in \Lambda_0)$.

As a consequence of the above theorem, each A-module $W(\lambda)$ $(\lambda \in \Lambda)$ has a composition series with quotient isomorphic to L_μ (some $\mu \in \Lambda_0$). Since the Jordan-Hölder theorem applies here, we may speak of the multiplicity of L_μ in $W(\lambda)$.

Definition 8.2.19. [19] For $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, write $d_{\lambda\mu}$ for the multiplicity of L_μ in $W(\lambda)$. The matrix $(d_{\lambda\mu})_{\lambda \in \Lambda, \mu \in \Lambda_0}$ will be denoted D; it is called the *decomposition matrix* of A.

Proposition 8.2.20. [19] The matrix D is upper triangular, i.e., $d_{\lambda\mu} = 0$ unless $\lambda \leq \mu$ and $d_{\lambda\lambda} = 1$.

Theorem 8.2.21. [19] Let V_λ be the projective indecomposable A-module corresponding to $\lambda \in \Lambda_0$. Then

1. $V_\lambda \cong Ae \cong i(Ae)$ for some (primitive) idempotent e of A such that $eA(\{\lambda\}) \neq 0$.

2. If $\lambda \geq \mu$, then $dim_R (V_\lambda)^\mu = d_{\mu\lambda}$ $(\lambda \in \Lambda_0, \mu \in \Lambda)$, where $V^\mu = i(W(\mu)) \otimes_A V$ for any A-module V.

3. If $c_{\lambda\mu}$ is the multiplicity of L_μ in V_λ $(\lambda, \mu \in \Lambda_0)$ then writing $C = (c_{\lambda\mu})_{\lambda,\mu \in \Lambda_0}$ we have $C = D^t D$.

Theorem 8.2.22. [19] Let A be an R-algebra (R a field) with cell atum (Λ, M, C, i). Then the following are equivalent.

1. The algebra A is semi–sipmple.

2. The nonzero cell representations $W(\lambda)$ are irreducible and pairwise inequivalent.

3. The form ϕ_λ is nondegenerate (i.e., $rad(\lambda) = 0$) for each $\lambda \in \Lambda$.

We finish this section with the following remark.

Remark 8.2.23. [19] Let A be a cellular algebra and R a field. The irreducible modules of A are parametrised by

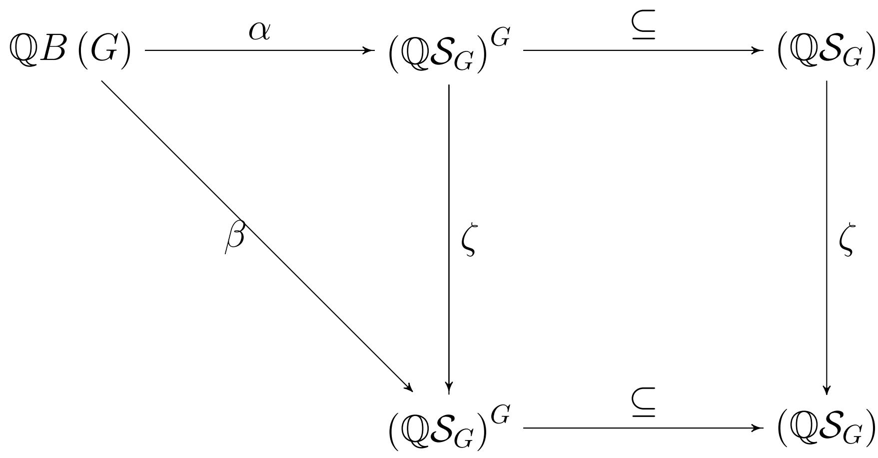
$$\Lambda_0 := \{\lambda \in \Lambda | \phi_\lambda \neq 0\},$$

and their dimensions are given by

$$dim_R (L_\lambda) = |M(\lambda)| - dim_R (rad(\lambda)).$$

8.3 The Natural Basis of the Double Burnside Ring

Recall that the Burnside ring $B(G)$ is the Grothendieck ring of the category of transitive G-sets. We can define the following maps $\alpha : B(G) \to \mathbb{Q}\mathcal{S}_G$, $[X] \mapsto \sum_{x \in X} stab_G(x)$ and $\zeta : \mathbb{Q}\mathcal{S}_G \to \mathbb{Q}\mathcal{S}_G$, $U \mapsto \sum_{U' \leq U} U'$, such that $\beta = \zeta \circ \alpha$, where $\mathcal{S}$ is the set of subgroups of G and β is the mark homomorphism [2]. In particular, we have the following commutative diagram



Recall from Definition 3.3.1 that the double Burnside ring $B(G,G)$ is that Grothendieck ring of the category of transitive (G,G)-bisets. The multiplication on $B(G,G)$ induces a unique algebra structure on the fixed point set that turns the mark homomorphism, $\beta_{G,G}$ into a $\mathbb{Q}$-algebra isomorphism. In their paper *A ghost algebra of the double Burnside algebra in characteristic zero* [2], Boltje and Danz wish to investigate the multiplication on $(\mathbb{Q}\mathcal{S}_{G\times G})^{G\times G}$, where $\mathcal{S}_{G\times G}$ denotes the set of all subgroups of $G\times G$, and whether there is a 'natural' $\mathbb{Q}$-algebra structure on the bigger vector space $\mathbb{Q}\mathcal{S}_{G\times G}$ extending the multiplication on the fixed points.

Boltje and Danz [2] use the above α and ζ maps for the ordinary Burnside ring $B(G)$ to construct similar maps, $\alpha_{G,G} : \mathbb{Q}B(G,G) \to \mathbb{Q}\mathcal{S}_{G\times G}$, $[X] \mapsto \sum_{x\in X} stab_{G\times G}(x)$ and $\zeta_{G,G} : \mathbb{Q}\mathcal{S}_{G\times G} \overset{\cong}{\to} \mathbb{Q}\mathcal{S}_{G\times G}$, $L \mapsto \sum_{L'\leq L} L'$, for the the double Burnside ring $B(G,G)$.

They state that the key observation leading to answers to the above questions is that the additive map $\alpha_{G,G}$ is almost multiplicative when $\mathbb{Q}\mathcal{S}_{G\times G}$ is viewed as a monoid algebra over the monoid $(\mathcal{S}_{G\times G}, *)$. This map is multiplicative if one introduces a 2-cocycle on the monoid $\mathcal{S}_{G,G}$ and uses the twisted monoid algebra structure on $\mathbb{Q}\mathcal{S}_{G\times G}$, which we denote by A. Thus, the double Burnside ring and its representation theory can be viewed as a result of idempotent condensation applied to a twisted monoid algebra. The $\mathbb{Q}$-linear isomorphism $\zeta_{G,G}$ transports the twisted monoid algebra structure A on $\mathbb{Q}\mathcal{S}_{G\times G}$ to another algebra structure, which we denote be $\tilde{A}$ on the same $\mathbb{Q}$-vector space. It turns out that the composition $\zeta_{G,G} \circ \alpha_{G,G}$ is equal to the mark homomorphism $\beta_{G,G}$ and we obtain a commutative diagram of algebra isomorphisms, with $\tilde{e} := \zeta_{G,G}(e)$, where e is an idempotent in A.

$$\begin{array}{ccccc} \mathbb{Q}B(G,G) & \xrightarrow{\alpha_{G,G}} & (\mathbb{Q}\mathcal{S}_{G\times G})^{G\times G} = eAe & \xrightarrow{\subseteq} & (\mathbb{Q}\mathcal{S}_{G\times G}) = A \\ & \searrow^{\beta_{G,G}} & \downarrow \zeta_{G,G} & & \downarrow \zeta_{G,G} \\ & & (\mathbb{Q}\mathcal{S}_{G\times G})^{G\times G} = \tilde{e}A\tilde{e} & \xrightarrow{\subseteq} & (\mathbb{Q}\mathcal{S}_{G\times G})\,\tilde{A} \end{array}$$

Definition 8.3.1. [2] Let G_1, G_2 and G_3 be finite groups and let R be a commutative ring such that $|G_2|$ is invertible in R. For $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G_1 \times G_2$ and $L = ((P_1', K_1'), (P_2', K_2'), \theta') \leq G_2 \times G_3$ we set

$$\kappa(M, L) = \frac{|K_2 \cap K_1'|}{|G_2|} \in R$$

and we define the R-bilinear map

$$- *_{G_2}^{\kappa} - : R\mathcal{S}_{G_1\times G_2} \times R\mathcal{S}_{G_2\times G_3} \to R\mathcal{S}_{G_1\times G_3}$$

by

$$M *_{G_2}^{\kappa} L = \kappa(M, L) \cdot (M * L).$$

Recall that the tensor product of bisets gives rise to the to the following $\mathbb{Z}$-bilinear map

$$- \cdot_{G_2} - : B(G_1, G_2) \times B(G_2, G_3) \to B(G_1, G_3),\ ([X], [Y]) \mapsto [X \times_{G_2} Y],$$

where X is a (G_1, G_2)-biset and Y is a (G_2, G_3)-biset.

Proposition 8.3.2. [2] Let G_1, G_2 and G_3 be finite groups, and let R be a commutative ring such that $|G_2|$ is invertible in R. Moreover let $a \in RB(G_1, G_2)$ and $b \in RB(G_2, G_3)$. Then

$$\alpha_{G_1,G_3}(a \cdot_{G_2} b) = \alpha_{G_1,G_2}(a) *^{\kappa}_{G_2} \alpha_{G_2,G_3}(b).$$

Using the isomorphism $\zeta_{G,G}$ Boltje and Danz [2] are able to improve on results by Bouc [6] on the parametrisation of the simple $\mathbb{Q}B(G,G)$-modules. Recall that the simple $\mathbb{Q}B(G,G)$-modules are parametrised by (U,V), where $U \sqsubseteq G$ and V is a simple $\mathbb{Q}Out(U)$-module. Boltje and Danz give a necessary condition for V to occur in this parametrisation. However this procedure is extremely difficult to work with.

In the next section we provide an alternative $\zeta_{G,G}$ map for $\mathbb{Q}B(S_3, S_3)$ based on the observations we made in Theorem 7.3.1 and Proposition 4.5.1 on subgroup incidences and alternative partial orders which can be defined on sections. Using these observations we will show that $\mathbb{Q}B(S_3, S_3)$ is cellular algebra and thus simplify the computation of the simple modules therein.

8.4 A Base Change

The table of marks is a base change matrix between the Burnside ring of of a finite group and its ghost ring. In the previous chapter we developed methods to construct the table of marks of a direct product. In this section we will use the example of the double Burnside ring of the symmetric group S_3, $B(S_3, S_3)$, to illustrate a base change for the double Burnside ring which, it will be conjectured, will produce a cellular basis for the double Burnside ring.

In order to perform this base change we will adopt the partial order we defined in Proposition 4.5.1. We will now count subgroup incidences in the same P matrix in terms of the size of the set

$$\left|\left\{L^{(g_1,g_2)} \leq M : (g_1, g_2) \in G_1 \times G_2\right\}\right|,$$

that is, we are are reversing the count in this transposed same P relation so instead of counting the number of conjugates of M which contain a fixed L as we did in Theorem 7.6.1, we are now counting the number of conjugates of L which are contained in a fixed M with the same P_i in their Goursat triples. We obtain the following class incidence matrix for this transposed same P relation

$$M'_P = \left(\begin{array}{cccccccccccccccc|cccc|c|c}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & 6 & \cdot & 1
\end{array}\right)$$

We obtain the following base change matrix for the double Burnside ring $B(S_3, S_3)$ by first post multiplying the table of marks of $S_3 \times S_3$ with the inverse of the old same P relation then post multiply again with the above matrix for the new same P relation.

$$M'(S_3) = \left(\begin{array}{cccccccccccccccc|cccc|c|c}
36 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
18 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
12 & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & 6 & 6 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
18 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
9 & 3 & \cdot & \cdot & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & \cdot & 6 & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & 2 & 2 & 2 & \cdot & \cdot & \cdot & \cdot & 2 & 2 & 2 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & 1 & \cdot & \cdot & 3 & 1 & \cdot & \cdot & 3 & 1 & \cdot & \cdot & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
18 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & \cdot & 6 & \cdot & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 2 & \cdot & \cdot & \cdot & \cdot \\
6 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & 2 & \cdot & 2 & \cdot & \cdot & \cdot \\
2 & \cdot & 2 & \cdot & \cdot & 2 & \cdot & 2 & 2 & \cdot & 2 & \cdot & \cdot & 2 & \cdot & 2 & 2 & 2 & 2 & 2 & \cdot & \cdot \\
\hline
12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot \\
\hline
6 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & 6 & 2 & \cdot & \cdot & 6 & 3 & 1
\end{array}\right)$$

We will begin by creating a table for each $U \sqsubseteq S_3$ with rows and columns indexed by the elements of $\overline{\Lambda}_{(S_3,U)}$, the entries of each of these tables are the basis elements of $B(S_3, S_3)$ of type U (i.e. the transitive bisets which correspond to conjugacy classes of subgroups of $S_3 \times S_3$ whose Goursat triples have section isomorphism type U).

$1/1$	1	2	7	9
$C2/C2$	3	8	11	17
$C3/C3$	5	12	16	19
$S3/S3$	14	18	20	22

$C2/1$	4	13
$S3/C3$	15	21

$C3/1$	6

$S3/1$ $\boxed{10}$

For a basis of the double Burnside algebra $\mathbb{Q}B\left(S_3, S_3\right)$ we use the matrix representation of the conjugacy classes of subgroups of $S_3 \times S_3$ with respect to the multiplication defined in Definition 8.3.1 by Boltje and Danz in [2].

In the previous chapter we developed a method of calculating the table of marks of a direct product of finite groups from two class incidence matrices with respect to the partial orders $\leq_P$ and $\leq_{P/K}$ and a third matrix which acts as a table of marks for the partial order $\leq_K$, i.e. $M\left(G\right) = M_K\left(G\right) M_{P/K}\left(G\right) M_P\left(G\right)$. Here we use a different base change matrix which we obtain by reversing the count of incidences for the class incidence matrix with respect to the partial order $\leq_P$. This gives us the base change matrix $M'\left(G\right) = M_K\left(G\right) M_{P/K}\left(G\right) M'_P\left(G\right)$, where $M'_P\left(G\right)$ is the class incidence matrix with respect to the partial order $\geq_P$.

Let $\{c_1, \ldots, c_{22}\}$ be the complete set of basis elements. Below we give a compact representation of these 22×22 matrices under this base change with respect to right multiplication.

$$c_1 := \frac{1}{6} \cdot \begin{pmatrix} 1 & 3 & 5 & 14 \\ 1 & 3 & 5 & 14 \end{pmatrix}$$

$$c_2 := \frac{1}{6} \cdot \begin{pmatrix} 1 & 3 & 5 & 14 \\ 2 & 8 & 12 & 18 \end{pmatrix}$$

$$c_3 := \frac{1}{2} \cdot \begin{pmatrix} 2 & 8 & 12 & 18 \\ 1 & 3 & 5 & 14 \end{pmatrix}$$

$$c_4 := \frac{1}{2} \cdot \begin{pmatrix} 4 & 15 \\ 4 & 15 \end{pmatrix}$$

$$c_5 := \frac{1}{3} \cdot \begin{pmatrix} 6 & 7 & 11 & 16 & 20 \\ -5 & 1 & 3 & 5 & 14 \end{pmatrix}$$

$$c_6 := \frac{1}{3}\begin{pmatrix} 6 & 7 & 11 & 16 & 20 \\ 6+2\cdot 16 & -7 & -11 & -16 & -20 \end{pmatrix}$$

$$c_7 := \frac{1}{6}\cdot\begin{pmatrix} 1 & 3 & 5 & 14 \\ 7 & 11 & 16 & 20 \end{pmatrix}$$

$$c_8 := \frac{1}{2}\cdot\begin{pmatrix} 2 & 8 & 12 & 18 \\ 2 & 8 & 12 & 18 \end{pmatrix}$$

$$c_9 := \frac{1}{6}\cdot\begin{pmatrix} 1 & 3 & 5 & 14 \\ 9 & 17 & 19 & 22 \end{pmatrix}$$

$$c_{10} := \begin{pmatrix} 9 & 10 & 13 & 17 & 19 & 21 & 22 \\ 9 & 10-6\cdot 21-6\cdot 22 & 13 & 17 & 19 & 21 & 22 \end{pmatrix}$$

$$c_{11} := \frac{1}{2}\cdot\begin{pmatrix} 2 & 8 & 12 & 18 \\ 7 & 11 & 16 & 20 \end{pmatrix}$$

$$c_{12} := \frac{1}{3}\cdot\begin{pmatrix} 6 & 7 & 11 & 16 & 20 \\ -12 & 2 & 8 & 12 & 18 \end{pmatrix}$$

$$c_{13} := \frac{1}{2}\cdot\begin{pmatrix} 4 & 15 \\ 13 & 21 \end{pmatrix}$$

$$c_{14} := \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

$$c_{15} := \begin{pmatrix} 10 \\ 15 \end{pmatrix}$$

$$c_{16} := \frac{1}{3}\cdot\begin{pmatrix} 6 & 7 & 11 & 16 & 20 \\ -16 & 7 & 11 & 16 & 20 \end{pmatrix}$$

$$c_{17} := \frac{1}{2} \cdot \begin{pmatrix} 2 & 8 & 12 & 18 \\ 9 & 17 & 19 & 22 \end{pmatrix}$$

$$c_{18} := \begin{pmatrix} 10 \\ 18 \end{pmatrix}$$

$$c_{19} := \frac{1}{3} \cdot \begin{pmatrix} 6 & 7 & 11 & 16 & 20 \\ -19 & 9 & 17 & 19 & 22 \end{pmatrix}$$

$$c_{20} := \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

$$c_{21} := \begin{pmatrix} 10 \\ 21 \end{pmatrix}$$

$$c_{22} := \begin{pmatrix} 10 \\ 22 \end{pmatrix}$$

We know that each basis element of $B(S_3, S_3)$ can be realised as a pair of isomorphisms, that is for a subgroup $M = ((P_1, K_1), (P_2, K_2), \theta) \leq S_3 \times S_3$ such that $P_i/K_i \cong U \sqsubseteq S_3$, M can be realised as the pair $(((P_1, K_1), \theta_1), ((P_2, K_2), \theta_2))$ where $((P_i, K_i), \theta_i) \in \Lambda_{(S_3,U)}(P_i, K_i)$ such that we can define the groups $M_i = ((P_i, K_i), (U, 1), \theta_i) \leq S_3 \times U$ and $M = \Pi(M_1, M_2)$.

Recall for S_3 we have eight conjugacy classes of monomorphisms and

$$\overline{\Lambda}_{S_3} = \coprod_{U \sqsubseteq S_3} \overline{\Lambda}_{(S_3,U)} = \left\{ [(P_1, K_1), \theta_1]_{S_3}, \ldots, [(P_8, K_8), \theta_8]_{S_3} \right\}$$

is a complete set of representatives of the conjugacy classes of monomorphisms of S_3. Then each basis element is indexed by a pair $\left([(P_i, K_i), \theta_i]_{S_3}, [(P_j, K_j), \theta_j]_{S_3}\right)$, such that $[(P_i, K_i), \theta_i]_{S_3}, [(P_j, K_j), \theta_j]_{S_3} \in \overline{\Lambda}_{S_3}$. Consider the projection

$$\pi\left([(P_i, K_i), \theta_i]_{S_3}, [(P_j, K_j), \theta_j]_{S_3}\right) = [(P_j, K_j), \theta_j]_{S_3}.$$

The matrix representation of the basis elements of $B(S_3, S_3)$, prior to the above base change, has the following property: If we add, for each $\left([(P_j, K_j)]_{S_3}, \theta_j\right)$ the columns of each of these matrices that correspond to basis vectors with the second component of their pair equal to $\left([(P_j, K_j)]_{S_3}, \theta_j\right)$, we obtain a sets of 22×8 matrices, where all rows corresponding to a fixed second component are identical. This property is a consequence of the multiplication in the double Burnside ring $B(G, G)$. Recall that this multiplication is given by

$$[G \times G/L] \cdot_G [G \times G/M] = \sum_{g \in \left[P_2' \backslash G/P_1\right]} \left[G \times G/\left(L * {}^{(g,1)}M\right)\right] \in B(G, G),$$

where $L = ((P_1', K_1'), (P_2', K_2'), \theta')$ and $M = ((P_1, K_1), (P_2, K_2), \theta) \leq G \times G$. The entries in these matrix representations are determined by the above formula. In other word, by resizing the second section in L and the first section in M conjugated by $g \in [P_2' \setminus G/P_1]$ as described in the Butterfly Lemma in order for them to have isomorphic section quotient. This is then communicated to the first section of L and the second section of M via the isomorphisms given in the Butterfly Lemma.

Hence the rows of the portion of each matrix representation indexed by the columns of matrices which represent conjugacy classes of subgroups M which have second section conjugate to (P_j, K_j) and rows which represent the conjugacy classes of subgroups L that have first section conjugate to (P_i, K_i) are permuted copies of each other. Clearly then the column sum for each of these rows will be identical. Thus we can select one row for each class of conjugacy classes of monomorphisms of S_3, to get a set of 8×8 matrices. Clearly the map $\phi : \mathbb{Q}^{22\times 22} \to \mathbb{Q}^{8\times 8}$ described by this procedure is a $\mathbb{Q}$-algebra homomorphism.

Remarkably when this algebra homomorphism is applied after basis change we obtain a one to one correspondence, as will be seen, between these new basis elements and matrices in $\mathbb{Q}^{8\times 8}$. Hence ϕ becomes an injective algebra homomorphism.

$$c_1 = \left(\begin{array}{cccc|cc|c|c}
1/6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_2 = \left(\begin{array}{cccc|cc|c|c}
\cdot & 1/6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_3 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_4 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_5 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
-1/3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_6 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1/3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & 2/3 & \cdot & \cdot & \cdot & 1/3 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_7 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & 1/6 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_8 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_9 = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & 1/6 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{10} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & -6 & \cdot & -6 & \cdot & 1
\end{array}\right)$$

$$c_{11} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{12} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1/3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & -1/3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{13} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{14} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{15} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{16} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1/3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & -1/3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{17} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$c_{18} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$
c_{19} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1/3 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & -1/3 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

$$
c_{20} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

$$
c_{21} = \left(\begin{array}{cccc|cc|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{array}\right)
$$

$$c_{22} = \left(\begin{array}{cccc|cc|c|c} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array} \right)$$

Notice that the above matrices suggests two further steps in the base change:

$$c_6 \to c_6 + (-1) \cdot c_{16},$$

$$c_{10} \to c_{10} + (-6) \cdot c_{21} + (-6) \cdot c_{22}.$$

After applying these final two steps we obtain the following representation of the basis of $\mathbb{Q}B\,(S_3, S_3)$. Here each 8×8 basis matrix can be realised as a matrix whose only non-zero entries are the scalars accompanying the basis elements in the positions given in the table:

$1/1 \to 1$	$\left(\frac{1}{6}\right) 1$	$\left(\frac{1}{6}\right) 2$	$\left(\frac{1}{6}\right) 7$	$\left(\frac{1}{6}\right) 9$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_2/C_2 \to 1$	$\left(\frac{1}{2}\right) 3$	$\left(\frac{1}{2}\right) 8$	$\left(\frac{1}{2}\right) 11$	$\left(\frac{1}{2}\right) 17$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_3/C_3 \to 1$	$\left(\frac{1}{3}\right) 5$	$\left(\frac{1}{3}\right) 12$	$\left(\frac{1}{3}\right) 16$	$\left(\frac{1}{3}\right) 19$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$S_3/S_3 \to 1$	$\cdot$	$\cdot$	$\cdot$	$(1)\, 10$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_2/1 \to C_2$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\left(\frac{1}{2}\right) 4$	$\left(\frac{1}{2}\right) 13$	$\cdot$	$\cdot$
$S_3/C_3 \to C_2$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$(1)\, 10$	$\cdot$	$\cdot$
$C_3/1 \to C_3$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\left(\frac{1}{3}\right) 6$	$\cdot$
$S_3/1 \to S_3$	$(1)\, 14$	$(1)\, 18$	$(1)\, 20$	$(1)\, 22$	$(1)\, 15$	$(1)\, 21$	$\cdot$	$(1)\, 10$

Theorem 8.4.1. The double Burnside algebra $\mathbb{Q}B(S_3, S_3)$ is isomorphic to the cellular sub-algebra of $\mathbb{Q}^{8\times 8}$

$$\overline{\mathbb{Q}}^{8\times 8} = \left\{ \left(\begin{array}{cccc|cc|c|c} \frac{1}{6}x_1 & \frac{1}{6}x_2 & \frac{1}{6}x_7 & \frac{1}{6}x_9 & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2}x_3 & \frac{1}{2}x_8 & \frac{1}{2}x_{11} & \frac{1}{2}x_{17} & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{3}x_5 & \frac{1}{3}x_{12} & \frac{1}{3}x_{16} & \frac{1}{3}x_{19} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_{10} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \frac{1}{2}x_4 & \frac{1}{2}x_{13} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x_{10} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3}x_6 & \cdot \\ \hline x_{14} & x_{18} & x_{20} & x_{22} & x_{15} & x_{21} & \cdot & x_{10} \end{array} \right) : \begin{array}{l} x_i \in \mathbb{Q}, \\ i \in \{1,\ldots,22\} \end{array} \right\}$$

Proof. Let $cl(S_3 \times S_3) := \{H_1, \ldots, H_{22}\}$ be a complete set conjugacy class representatives of subgroups of $S_3 \times S_3$. Recall that the ghost algebra $\mathbb{Q}^{22}$ of $\mathbb{Q}B(S_3, S_3)$ has basis $\{e_i = (e_i(H_1), \ldots, e_i(H_{22})) : i \in \{1, \ldots, 22\}\}$, where e_i is the basis element corresponding to the conjugacy class of H_i, that is $e_i(H_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Let $\overline{\mathbb{Q}}^{8\times 8}$ be the cellular sub-algebra of $\mathbb{Q}^{8\times 8}$ equal to the above set. Thus are able to construct the $\mathbb{Q}$-algebra isomorphism

$$\gamma : \mathbb{Q}^{22} \to \overline{\mathbb{Q}}^{8\times 8},$$

$$\gamma : e_i \mapsto c_i.$$

Which by extension through the mark homomorphism gives us that $\mathbb{Q}B(S_3, S_3) \cong \overline{\mathbb{Q}}^{8\times 8}$ as $\mathbb{Q}$-algebras. Hence $\mathbb{Q}B(S_3, S_3)$ is cellular. □

Remark 8.4.2. Note that our involution on $\mathbb{Q}B(S_3, S_3)$ is the opposite operation, ^{op},on bisets. Using the notation from [11] we let J_U be the two-sided ideal with basis the conjugacy classes of subgroups of $S_3 \times S_3$ whose Goursat triples have section quotient $U \sqsubseteq S_3$ under the base change described above. This is cell ideal since there exists the left-sided ideal W_U which is the ideal generated by the subgroups of $S_3 \times U$ whose Goursat triples have isomorphisms $\theta \in \overline{\Lambda}_{(S_3,U)}$, this gives us $W_U \subseteq J_U$ and we have $J_U \cong W_U \otimes_{\mathbb{Q}} W_U^{op}$.

This gives the following

$$J_1 = \langle c_1, c_3, c_5, c_{14}, c_2, c_8, c_{12}, c_{18}, c_7, c_{11}, c_{16}, c_{20}, c_9, c_{17}, c_{19}, c_{22} \rangle,$$

$$W_1 = \langle c_1, c_3, c_5, c_{14} \rangle \,, \quad W_1^{op} = \langle c_1, c_2, c_7, c_9 \rangle \,,$$

$$J_{C_2} = \langle c_4, c_{15}, c_{13}, c_{21} \rangle \,,$$

$$W_{C_2} = \langle c_4, c_{15} \rangle \,, \quad W_{C_2}^{op} = \langle c_4, c_{13} \rangle \,,$$

$$J_{C_3} = \langle c_6 \rangle = W_{C_3} = W_{C_3}^{op},$$

$$J_{S_3} = \langle c_{10} \rangle = W_{S_3} = W_{S_3}^{op}.$$

Clearly we can write $\mathbb{Q}B\left(S_3, S_3\right) = J_1 \oplus J_{C_2} \oplus J_{C_3} \oplus J_{S_3}$ and $J_U^{op} = J_U$ for all $U \sqsubseteq S_3$. Now we define

$$J_U' := \bigoplus_{\substack{V \sqsubseteq S_3 \\ |V| \leq |U|}} J_V.$$

This gives us a chain of two-sided ideals $0 = J_0' \subset J_1' \subset J_{C_2}' \subset J_{C_3}' \subset J_{S_3}' = \mathbb{Q}B\left(S_3, S_3\right)$. Let $J_2' = J_{C_2}'$, $J_3' = J_{C_3}'$ and $J_4 = J_{S_3}'$. Then clearly $J_j = J_j'/J_{j-1}'$ remains a cell ideal of the quotient $\mathbb{Q}B\left(S_3, S_3\right)/J_{j-1}'$. Thus $\mathbb{Q}B\left(S_3, S_3\right)$ is a cellular algebra.

It can be seen from the table above that under this base change the basis elements

$$\{c_{14}, c_{18}, c_{20}, c_{22}, c_9, c_{17}, c_{19}, c_{15}, c_{21}\}$$

form a basis of the radical of the algebra $\mathbb{Q}B\left(S_3, S_3\right)$; moreover we have the following simple modules, where the pair (U, V) is given by $U \sqsubseteq S_3$ and V is an isomorphism type of a simple $\mathbb{Q}\mathrm{Out}(U)$-module.

$$(1, 1) = \langle c_1, c_3, c_5 \rangle \,,$$

$$(C_2, 1) = \langle c_4 \rangle \,,$$

$$(C_3, 1) = \langle c_6 \rangle \,,$$

$$(S_3, 1) = \langle c_{10} \rangle \,.$$

This is consistent with a result of Bouc, Stancu and Thévenaz [7] which states that the dimension of $(1, 1)$ is equal to the number of conjugacy classes of cyclic subgroups of a group G.

8.5 A Conjecture

The above procedure is not unique to the example of $S_3 \times S_3$ but can be reproduced in an unsystematic way in many other examples. However we have yet to prove the cellularity of the double Burnside algebra in general because we as of yet do not fully understand how the action of $\mathrm{Out}(U)$ effects this base change for the double Burnside algebra. Thus we will make the conjecture.

Conjecture 8.5.1. Let G be a finite group and R be a commutative ring with characteristic zero, such that $|G|$ is invertible in R. Then the algebra $RB(G, G)$ is a cellular algebra.

Future work will attempt to prove this conjecture and use it to study the simple modules and radical of the double Burnside algebra both in characteristic zero and positive characteristic.

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