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Theoretical and Numerical Analysis of Rigid-body Impacts with Friction

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics

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Declaration

I declare that the work contained in this thesis is all my own and that I have not obtained a degree at NUIG or elsewhere on the basis of this work,

Shane Burns.
Summary

This thesis gives a flavour of the area of rigid body impacts with friction, an area which has far reaching applications in engineering, sports science and everyday life. The focus of this work will be on the two main streams of this field, theoretical and numerical. This thesis will present an overview of the general subject of rigid-body impact, including discussion and analysis of the validity of one’s choice of impact law and the numerical techniques required for the simulation of rigid-body impacting systems.

Two impact laws will be introduced in Chapter 3 and a direct comparison will be made in order to examine the varying dynamics that can be achieved using both a basic and a complex impact law and to explore some of the problems that can occur with a more basic formulation. It will be demonstrated that for certain regions in parameter space the two formulations are equivalent, however, for many other regions the two formulations can vary greatly.

A hybrid event-driven numerical scheme is one in which smooth dynamics are described by differential equations, which can be solved numerically using standard techniques, and non smooth events which are described by maps. In Chapter 5, a hybrid event-driven numerical scheme for the implementation of the Energetic impact law described in Chapter 3 is presented. Moreover, the framework necessary for the long term simulation of mechanical systems with impacts and chatter is derived.

This thesis also gives an overview of the phenomena known as the Painlevé Paradox in Chapter 6 and presents a numerical experiment to show the occurrence of the paradox for a mechanical system.
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A special thanks to my family for their encouragement and support. I would also like to thank all of my friends and fellow PhD students who supported me in writing, and incented me to strive towards my goal.
Chapter 1

Introduction

This thesis presents a collection of various problems concerning rigid-body impacts with friction and gives a possible resolution to these problems. The focus of this thesis is twofold, theoretical and numerical. The work in this thesis makes a contribution to both these branches of rigid body impact theory. This Chapter begins with a historical overview of some of the early contributors in the field of impact mechanics and in doing so sets the scene for the more modern developments in the field. The current state of impact mechanics is then described, including the necessity of the modern techniques. This Section is then followed by an introduction to non-smooth systems, an area fundamentally linked to impact mechanics. We conclude this Chapter with the specific contributions of this thesis together with an outline of the remainder of this thesis.

1.1 History of Impact Mechanics

In this Section we present a historical overview of some of the early contributions made to the field of impact mechanics. The purpose of this Section is to give a flavour of some of the early contributions made and to pave the way for the modern contributions to the field.

*Leonardo Da Vinci (1452-1519), Italy* The early developments in the theory of impacts and friction came during the Renaissance period, beginning with the contributions from the engineer and scientist Leonardo Da Vinci. Da Vinci was investigating the phenomenon of friction between two bodies in contact. His results could be summarized as two fundamental laws of friction:
1. The frictional force is proportional to the applied load (normal force).

2. The frictional force is independent of the contact surface area.

Da Vinci’s first law is now attributed to the French physicist Guillaume Amontons. As was common in scientific discovery, Da Vinci’s findings were lost and rediscovered 200 years later by Amontons.

---

Galileo Galilei (1564-1642), Italy. Working at the University of Pisa Galileo was performing experiments to determine the relationship between distance, velocity and time of falling objects. During the experiments Galileo began thinking about the forcing involved at impact. Upon many observations he concluded that “the impetus of collision depends on the relative velocity”.

Galileo was the first to note that the impetus of collision is directly proportional to the normal component of relative velocity [2].

---

René Descartes (1596 - 1650), France. Descartes published a book *Dioptrics* in 1637 in which he used the analogy of light reflection to the rebound of a ball. He proposed seven rules for impact of elastic bodies, most of which were entirely wrong [3]. This did however stimulate further exploration into impact mechanics by other scientists.

---

John Wallis (1616-1703), England. Wallis wrote a paper to the Royal society in which he dealt with impacts of hard bodies. Wallis concluded that "If a body in motion collides with a body at rest, and the latter is such that it is not moving nor prevented from moving by an external cause, after the impact the two bodies will go together with a velocity given by"

\[ \frac{W_1 v_1 + W_2 v_2}{W_1 + W_2}, \]

where \( W_1, W_2 \) are the weights of the objects and \( v_1, v_2 \) are the initial velocities of the objects [4].

---

Christopher Wren (1632-1723), England. The most significant contribution Wren made to Impact Mechanics was the experiment he performed involving impacting pendula. Wren released pendula of equal length but different weight from different heights. The heights were fixed such that the pendula would rebound to the same
proportion of its initial release height. He found that if two elastic bodies collide
with velocities inversely proportional to the weight of the other body, then velocity
reversal will occur [3].

*Christian Huyghens (1629 - 1695), Holland.* Huyghens was the first to distinguish
between momentum and kinetic energy [5], but was not able though to determine
the relationship between them. Huyghens used two concepts:

- He gives a translational velocity to the entire system such that the centre of
gravity of the system is stationary both before and after collision.
- During a collision each body has a speed of approach that is inversely propor-
tional to its weight.

He also determined an expression to calculate the moment of inertia of a solid.

*Edmé Mariotte (1620-1684), France.* Mariotte focused on the elastic collision of
spheres. He divided the velocity into components normal and tangential to the com-
mon tangent plane. He considered spheres of varying and equal masses. Mariotte’s
analysis involved the conservation of *quantity of motion* [4].

*Isaac Newton (1642-1727), England.* Newton’s third law states ”To every action
there is an equal and opposed reaction”

Newton decided to show experimentally that this law held for collisions. He
conducted experiments involving impacting spheres of different material and he con-
cluded that: *If the rule is to be tried in bodies not perfectly hard we are only to
diminish the reflection in such a certain proportion as the quantity of the elastic
force requires*

This lead to the newtonian coefficient of restitution, which relates the relative
post impact normal velocity of the two bodies to the pre impact relative normal
velocity of the two bodies [2].

*Leonhard Euler (1707-1783), Switzerland.* Leonard Euler theorised that the frictional
force developed as a result of interlocking triangular irregularities at the surfaces of
the bodies in contact. Further, he stated that the coefficient of friction was equal to
the gradient of these irregularities. A variation of this model which considers friction
on the atomic level, the *Tomlinson Model*, is still used today [6].
Euler was the first person to model impact using the construct of an infinitesimal particle located at contact point of the colliding bodies. He noted that the contact force which results from compressing the particle acts only in the direction normal to the common tangent plane. This idea helped develop the concept of energy stored during a compression phase and energy released during a restitution phase. He then began to consider non collinear impacts. From this he realised that the changes in relative velocity would now be due to translational and rotational components. Leonhard Euler was the first to distinguish between static and kinetic friction and was also the first to use $\mu$ as the symbol for the coefficient of friction.

Euler also presented the first analytic solution to a contact problem with friction. He considered the problem of a rope or belt wrapped around a rough cylindrical surface.

Guillaume Amontons (1663 -1705), France. In 1699, Amontons published his 3 laws of friction.

1. The force of friction is directly proportional to the applied load.
2. The force of friction is independent of the apparent area of contact.
3. Kinetic friction is independent of the sliding velocity

Unfortunately, he could not validate these laws at the time.

Charles Augustin Coulomb (1736 -1806), France. The law of dry friction, which relates the tangential force generated during a collision $F_T$ to the normal force $F_N$, is named after the French engineer Coulomb ($F_T = \pm \mu F_N$). He confirmed Amontons’ results and established that sliding friction is independent of sliding speed in a first order approximation. Coulomb performed various experiments involving material surface composition, lubrication, sliding speed, humidity, temperature etc and published his results in his book Theory of Simple Machines [3]. His experimentation also lead to variations to the then standard law of friction. He found that the longer a body remains in stationary contact with a surface, the greater the static force will be [2].

Osborne Reynolds (1842 - 1912), Ireland. Osborne examined what was happening in the contact area during rolling contact. He determined that on a driven wheel
there will always be points where there will be no slip and areas where there will be slip. Osborne also accounted for the energy loss during rolling because of sliding contact.

Heinrich Hertz (1857-1894), Germany. Hertz is attributed as being the founder of classical contact mechanics. He solved the problem of contact between two elastic bodies with curved surfaces. He considered the nature of the localized deformation at contact and the distribution of pressure between the two contacting surfaces. His method involved assigning a shape to the contact surface satisfying certain boundary conditions [4].

David Tabor (1913 - 2005), England. Tabor, along with his colleague Frank Bowden, examined the roughness of the contact surfaces during contact. Because of this roughness, the contact area between bodies is much smaller than previously thought. They defined a coefficient of friction as the ratio of the critical shear stress to the hardness of the materials in question [4].

1.2 Impact Mechanics

Impact mechanics is concerned with the reaction forces and impulses that develop as a result of two or more bodies colliding and the governing dynamics which ensue [4,7,8]. The main objective of impact mechanics is to develop and understand methods for calculating the changes in rigid body motion due to an impact, specifically velocity changes [9]. Impacts are very prevalent in every day life, for example, in using a keyboard, hammering a nail and the action of walking. The understanding of the mechanics of impacts has far reaching applications in areas such as mechanical engineering, civil engineering, vehicle collision analysis and sport science [10].

Collisions or impacts in mechanical systems are very common and in many mechanical engineering applications often cause undesired wear and noise and can thus be very problematic and expensive. Two recent experimental examples of systems with such issues include an engine cam follower [10] and a magnetic bearing system [11]. During impact, energy is dissipated through motion in both the normal and tangential direction (friction) and the underlying mechanisms by which this happens has wide reaching effects on both the short-term and long-term dynamics.
of that system. For instance moving parts in machines are often subject to unwanted impacts and the understanding of impact mechanics can help in resolving such problems [11]. In civil engineering the classical problem of a rocking block in the presence of friction has been studied extensively as a tool in understanding the dynamics caused by earthquakes [12, 13]. Similarly, impact mechanics can be an aid in, for instance, the design of golf clubs and baseball bats, where impacts are prevalent [4].

Rigid body mechanics, the preliminaries of which will be discussed in Chapter 2, is very accurate in predicting the motion of solid objects under the application of known forces such as gravity. For most scientific and engineering purposes, the laws of linear and angular momentum can be considered exact. The laws which govern how a body will deform in response to an applied force however are not exact, and when used together with rigid body theory can lead to inaccurate predictions. For this reason most analysis of impact mechanics has been restricted to collisions involving only simple geometric structures as the mathematics becomes increasingly complex for more complicated structures [14]. If we wanted to exactly model the collision process we would need more exact laws to describe friction and non-linear deformations. We would also require details on the micro scale of the contacting surfaces of the colliding bodies together with exact geometric and inertia measurements of the bodies, assuming also that they do not vary in time. Even if all of these quantities were known, we would then be limited by the computational cost in performing such calculations.

For the above reasons, it is necessary to consider alternative simpler approaches, which may compromise on the accuracy, but still have useful practical applications. Rigid body impact theory is the study and analysis of these approaches.

In the late 19th century, Routh [15] presented an incremental rigid body impact model for calculating the outcomes of rigid body collisions with friction. Routh’s approach assumed point contact and that there is no compliance in the tangential direction. In more recent times, Routh’s approach has been revisited by Keller [16], who performed some analysis using Routh’s formulation, Wang and Mason [17], who consider the 2D version of Routh’s model, Ivanov [18], who proved nonnegative energy dissipation for the 3D case and Batlle [19], who considered rough balanced collisions for the 3D case. These models are discussed in greater detail in Chapter 2.

An alternative to the incremental modelling approach is to use an algebraic approach. Brach [9, 20, 21] considers an impact model in which all of the fundamental
CHAPTER 1. INTRODUCTION

underlying equations are linear and algebraic. Smith [22] describes an impact model in which the fundamental equations are nonlinear and depend on two parameters only. Chatterjee [23, 24] presents an algebraic impact model that allows for 3D collisions. More detail will be given on these approaches in Chapter 2 and Chapter 3.

Nordmark [1, 25] developed an impact model using a new energetic coefficient of restitution described by Stronge [4]. A set of mappings are defined and the corresponding map for a given impact depends on what region of parameter space the system is in at impact. Further details on this approach will be given in Chapter 3.

1.3 Nonsmooth Systems

Nonsmooth systems are systems in which a sudden change in state variables or vector fields occur. One of the most common examples of a nonsmooth system in everyday life is that of an impact or collision between two objects. Some examples include turbine-blade dampers, friction-clutch vibrations, landing-gear dynamics [26], passive walkers [27, 28] and braille printers [29].

The use of nonsmooth system theory to predict and understand the kinematics of colliding rigid bodies in the presence of impacts and friction is a useful commodity in engineering in particular and research of such systems in general [7, 11, 26, 30–32]. It is well known that nonsmooth systems can exhibit complex behaviour that cannot be found in smooth systems. The class of systems with combinations of ODEs and maps, that we use here for mechanical systems with impact and/or friction, is often termed as piecewise-smooth (PWS) systems. In recent years the interest of discontinuity-induced bifurcations (DIBs) found in PWS systems have increased dramatically, and as mentioned above the main driver of the analysis of DIBs have been the hybrid system approach, where local behaviour can numerically be pinpointed with high accuracy [33–37]. An example of a DIB, and something that will be seen in this thesis, is a grazing bifurcation. A grazing bifurcation occurs when, under parameter variation, a trajectory of a periodic orbit makes tangential contact with the discontinuity surface, resulting in a change in the system dynamics [35, 36]. A discontinuity surface is a surface in state space where the vector field is no longer physically defined. For example, a discontinuity surface of an impacting mechanical system would represent the surface tangential to the point of contact. Once this
surface has been reached, it no longer makes physical sense to use the non contacting-vector fields for the system. In particular DIBs in impacting systems without friction have been studied extensively and some classification methodologies have been developed in [38–40], but also impacting systems with friction have been studied from a DIB point of view [1, 41, 42].

Another specific feature of impacting systems with non-compliant impacts is chatter, which is the phenomenon whereby a system goes through an infinite number of impacts in a finite time period. Previous works on chatter have considered both frictionless systems [34, 37, 43] and systems with impacts and friction [1, 25, 41]. An interesting example of an engineering-based frictionless system is analysed in [44], where the problem of gear rattle (chatter) in Roots blower vacuum pumps is considered, and where the rattle is induced from the gear teeth losing and regaining contact. Similarly, in cam-follower systems for certain conditions the follower detaches from the cam, resulting in a series of unwanted impacts or chatter [10, 33].

It is worth noting that the term chattering is also used in control theory to refer to a large number of switches, and also in mechanical engineering to refer to a large number of sequential impacts.

1.4 Contribution of this Thesis

This thesis mainly makes two fundamental contributions in the area of rigid body impact with friction.

1. It presents an overview of the general subject of rigid-body impact, including discussion and analysis of the validity of one’s choice of impact law. Further to this, it develops a novel extension to a classical impact law formulation.

2. It presents a hybrid-event driven numerical scheme for the implementation of an Energetic impact law. Moreover, it provides a framework necessary for the long term simulation of mechanical systems with impact and chatter.

In addition, this thesis also

3. gives an overview of the phenomena known as the Painlevé Paradox and presents a numerical experiment to show the occurrence of the paradox for a mechanical system.
In terms of the current research landscape of impacting systems with friction, the work in this thesis is mainly application based. We attempt to build a platform for engineers and scientists interested in analysing systems with impact and friction. Our focus is to illustrate how too simple an impact law formulation can lead to inaccurate post impact velocity mappings and to show that in certain cases a more advanced impact law is required. We build on the state of the art in terms of impact law formulations and show how it can be implemented in a numerical simulation scheme. These techniques provide the platform for researchers to analyse rigid body impacting systems using an advanced impact law. In terms of the Nonsmooth aspect of this work, we provide a simple first step framework capable of considering long term dynamics for a rigid body impacting system with friction. This framework also provides a first step in developing methods for bifurcation and stability analysis of systems with impacts and friction, aiding in the current state of the art in nonsmooth dynamical systems analysis.

1.5 Outline of this Thesis

In this Section we give a brief outline of the organisation of this thesis.

Chapter 2 presents some preliminaries regarding rigid body dynamics with impact and friction. In Section 2.1, the concept of equations of motion are defined and discussed from both a Newtonian and Lagrangian viewpoint. The relationship between force, impulse and the conservation principles are examined in Section 2.2. In Sections 2.3 and 2.4 the concepts behind the physics of colliding bodies is discussed together with how energy removal is modelled mathematically. The Chapter concludes in Section 2.5 with an overview of some commonly used impact laws.

Chapter 3 presents two impact law formulations, The Brach impact law and the Energetic impact law, using the preliminaries discussed in Chapter 2. An extension to the Brach impact law is derived using an energetic analysis and a general framework for a two body collision using the Energetic impact law is presented. The two impact laws are compared and a detailed discussion on the applicability of the Brach impact law is presented. The work in this Chapter has been published in [45].

Chapter 4 contains some preliminaries regarding numerical methods for the simulation of impacting systems. An overview of the different numerical methods is given in Section 4.1, together with a discussion on the advantages of using one method over
another. In Section 4.2, the specific numerical techniques are described and illustrated by means of an example system. The Chapter finishes with a discussion in Section 4.3 regarding bifurcation diagram computations.

Chapter 5 builds on the ideas presented in Chapter 4 and develops a general framework for the long term simulation of impacting rigid bodies with friction and with relatively few contact points. A hybrid event driven scheme is derived to allow for the simulation of a slender rod impacting a periodically oscillating surface, thus making it possible to simulate through chatter sequences. The techniques and results presented in this Chapter has been published in [46].

Chapter 6 presents a discussion and analysis on the Painlevé paradox. The classical Painlevé formulation for a slender rod in contact with a non-compliant plane is considered, and its relevance to the Brach impact law is discussed. A second model system, a double pendulum in contact with a sliding rough belt, is examined and the conditions under which the Painlevé paradox can occur are derived and simulated computationally. This work has been submitted for publication.

Chapter 7 contains the concluding remarks and discussion on the thesis and gives an outlook into the possible directions and areas worth pursuing for future work.
Chapter 2

Dynamics of Rigid Bodies with Impact and Friction

In this Chapter we will present the preliminaries necessary for the analysis of rigid body dynamics of mechanical systems subject to impacts and friction. Dynamics is one of the most important branches of mechanics as it is the study of the motions of bodies, which may interact, and the use of postulated force, energy and momentum laws to describe these motions. For every system analysed in this thesis the first step in the analysis is to model the dynamics mathematically. We will discuss how to derive equations of motion which describe the motion of a system, and we will also examine force-impulse formulations and conservation properties of these systems. These concepts will then be extended to describe the physics behind a collision between two or more bodies. Further, we will introduce the concept of an impact law and give a review of some of the different impact laws used in both a scientific research setting and for engineering applications.

2.1 Equations of Motion

In this Section we will present two classical formulations for deriving equations of motion of rigid-body systems. Equations of motion are fundamental differential equations, based on force or energy principles, which describe the principle motion of a system, typically the centre of mass of a system [47]. The centre of mass of a body may be viewed as a point on a body where it can be assumed that all of the mass is located [48]. For a given force, at a specified instance in time, the equations
of motion describes the corresponding acceleration of the system [26]. Firstly, in Section 2.1.1 we present a Newtonian formulation, which is used in Chapters 3 and 5. Secondly, in Section 2.1.2 we present a Lagrangian formulation, which is used in Chapter 6. Both formulations will give equivalent equations of motion, although some algebraic manipulation may be required to show this. We begin with some fundamental definitions of principles and terms used throughout this thesis.

**Degrees of freedom.** The number of degrees of freedom of a mechanical system can be defined as the number of coordinates required to specify the configuration of the system minus the number of independent constraint equations. For our purposes, when we refer to degrees of freedom we mean geometrical degrees of freedom, as described above. Often, when degrees of freedom are referred to, the meaning is dynamical degrees of freedom, which are defined the same as geometrical degrees of freedom, but in addition they also include velocities in the number of degrees of freedom [26].

**Generalised Coordinates.** There are an infinite number of choices of coordinate systems to describe the configuration of a given system. The only requirement is that there are at least as many coordinates as there are degrees of freedom, otherwise it would be impossible to fully describe the system. In general, however, the number of system coordinates is equal to the number of degrees of freedom plus the number of independent constraint equations. It is possible to transform between different coordinate systems, for example by transforming from cartesian to spherical polar coordinates. This is known as a coordinate transformation and the new set of coordinates equally describes the mechanical system. For certain systems it is advantageous and often more intuitive to use one set of coordinates over the other. We can now define generalised coordinates as any set of coordinates which describe the configuration of a mechanical system [26].

**Constraints.** The possible motions of a system are dependent on geometrical restrictions or constraints [48]. For example, a simple pendulum is constrained to move in a plane due to the connection or pivot point restraining the motion. In general, for a system that can be fully described by \( n \) generalised coordinates \( q_1, q_2, \ldots, q_n \), the corresponding constraint equations are of the form

\[
\phi_k (q_1, q_2, \ldots, q_n) = 0, \quad k = 1, 2, \ldots, m. \tag{2.1}
\]
Constraints that are of the form (2.1) are known as holonomic constraints and are the only form which will be considered for the work in this thesis. Constraints which depend explicitly on time are known as rheonomic and ones which do not depend on time are known as scleronomic [26].

These definitions will be used in what follows to describe the Newtonian and Lagrangian formulations of equations of motion.

2.1.1 Newtonian Mechanics

The Newtonian formulation is based inherently on forces and accelerations acting on a system of particles or on one or multiple bodies. It is necessary therefore to first make the distinction between the forces which can act on a system and then explain how the different forces affect the acceleration of their centres of mass. Internal forces are interaction forces which occur between a system of particles or between interacting rigid bodies. External forces are forces which arise from sources external to the system. As an example, consider a baseball. The forces holding the particles of the baseball together are the internal forces whereas if the baseball is struck by a bat, the force exerted by the bat on the ball is an external force. In this system the force due to gravity is also an external force. It is necessary to make this distinction when deriving equations of motion. In what follows, we will define the Newtonian formulation for the equations of motion of a rigid body system. To this purpose, we will begin by deriving the Newtonian formulation for the classical example of a system of particles and then extend the theory for the case of a rigid body. In doing so we will show, by example, that as a system of rigidly connected particles grows large, it can be viewed as a rigid body.

Dynamics of a System of Particles

We consider a system of $n$ particles in three dimensional space (see Figure 2.1 in which four of the particles are shown). Let $m_i$ denote the mass of the $i$th particle and let $q_i$ be the position vector of the $i$th particle relative to the origin $O$ of the inertial frame. At this point we will define what is meant by the velocity and acceleration of a particle. We define the velocity vector $\dot{q}_i$ as the time derivative of the position vector and we define the acceleration vector $\ddot{q}_i$ as the time derivative of the velocity vector $\dot{q}_i$. 

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Figure 2.1: The internal $f_{ij}$ and external forces $F_k$ acting on a system of particles with masses $m_i$, four of which are depicted in the figure. It is clear that the internal forces $f_{ij}$ act in equal and opposite pairs.

If we denote by $F_i$ the external force acting on the $i$th particle then from Newton’s second law we have

$$m_i \ddot{q}_i = F_i + \sum_{j=1}^{n} f_{ij},$$

where $f_{ij}$ are the internal force vectors and represent the force exerted on particle $i$ due to particle $j$. If we now sum over all the particles in the system we get

$$\sum_{i=1}^{n} m_i \ddot{q}_i = \sum_{i=1}^{n} F_i + \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij},$$

but from Newton’s third law [49], which states that the internal forces will always occur in equal and opposite pairs, we know that the interaction forces between any two particles are equal and opposite so that

$$f_{ij} = -f_{ji} \implies \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} = 0.$$

From this we then arrive at the expression for Newton’s Second Law of motion for a system of $n$ particles

$$\sum_{i=1}^{n} m_i \ddot{q}_i = \sum_{i=1}^{n} F_i,$$

(2.2)
Defining the total external force as \( F = \sum_{i=1}^{n} F_i \), the total system mass as \( m = \sum_{i=1}^{n} m_i \), then by definition we have that the centre of mass location of the mass system \( q_G \) is given by
\[
q_G = \frac{1}{m} \sum_{i=1}^{n} m_i q_i.
\]
This allows us to rewrite (2.2) as
\[
m\ddot{q}_G = F, \tag{2.3}
\]
which is the general expression for Newton’s Second Law of motion.

Consider again the system of particles in Figure 2.1. We define another fundamental quantity, the \textit{linear momentum} of a particle, as
\[
\rho_i = m_i \dot{q}_i, \tag{2.4}
\]
and the total \textit{linear momentum} of the system is found by summing the linear momentum of all the particles in the system so that
\[
\rho = \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} m_i \dot{q}_i. \tag{2.5}
\]
We define the \textit{angular momentum} \( H \) of a particle as
\[
H_i = q_i \times m_i \dot{q}_i, \tag{2.6}
\]
where \( \times \) is the \textit{vector product}. The vector product of two vectors \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) is defined as the vector such that
\[
a \times b = (c_1, c_2, c_3),
\]
where
\[
c_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad c_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad c_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.
\]
The total angular momentum \( H \) of the system is then found by summing the angular
CHAPTER 2. RIGID-BODY DYNAMICS

momentum of all the particles in the system to give

\[ H = \sum_{i=1}^{n} H_i = \sum_{i=1}^{n} q_i \times m_i \dot{q}_i. \]  \hspace{1cm} (2.7)

One can also calculate a moment \( M \) about the origin \( O \) of the system (see Figure 2.1) due to the external forces acting on the system where

\[ M = \sum_{i=1}^{n} q_i \times F_i. \]

Further, it can be shown that the rate of change of the angular momentum about a fixed point of the system is equal to the total moment about \( O \) of the external forces acting on the system [50], such that

\[ M = \dot{H}. \]  \hspace{1cm} (2.8)

Equation (2.8) is also known as Euler’s Second Law and is a useful tool for describing the motion of rigid bodies. This law describes how the change of angular momentum of a rigid body is determined by the moment of forces and couples acting on the body [50].

So far in this Section we have described the Newtonian formulation and from this arrived at two second order differential equations for calculating equations of motion of a system of particles, namely Newton’s second law given by (2.3) and Euler’s second law given by (2.8). We will now extend this formulation for the case of a rigid body.

Extension to a Rigid body

As a motivating example, consider a system of four unconstrained particles. Each particle requires at least three coordinates to fully describe the position and thus the full system has twelve degrees of freedom. Now suppose that the four particles are placed at the corners of a square whose sides are formed by four rigid rods, and whose diagonals are joined by two rigid rods, but the particles are otherwise unconstrained and free to move. The rigid connections correspond to six independent constraint equations meaning that the system has \( 12 - 6 = 6 \) degrees of freedom left. If we were to add a fifth particle to the system and thus introducing three more generalised
coordinates then we would require three more rigid connections to rigidly attach the particle to the system. This example shows that the number of degrees of freedom remains as $15 - 9 = 6$. In the limiting case, as the number of particles becomes large, the system of particles can be viewed as a *rigid body* and further, in general, a rigid body has six degrees of freedom. The additional degrees of freedom associated with a rigid body are a result of rotational degrees of freedom. The body is free to rotate about each of its axis, giving three additional degrees of freedom. The main consequence of this is that we can use (2.3) and (2.8) to describe the motion of rigid-bodies.

### 2.1.2 Lagrangian Mechanics

The application of Newton’s laws of motion becomes increasingly difficult as the complexity of the system increases. The main reason for this is that the forces become difficult to identify in the required vectorial form and also that a high level of insight for each given system is required [50]. An alternative approach to this is to use a Lagrangian formulation, which we will present here. This approach is particularly useful for systems with multiple degrees of freedom. The Lagrangian formulation provides a powerful means of obtaining the equations of motion in terms of the generalised coordinates of the system illustrated in Figure 2.1. The equations of motion are derived based on an energetic calculation, expressed in terms of the generalised coordinates. This approach requires the calculation of velocities and displacements of the system rather than forces and accelerations as in the Newtonian approach [51].

**Lagrange’s Equations**

The *kinetic energy* $T$ of a system of $n$ particles with position vectors $q_i$ and masses $m_i$ can be defined as

\[
T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i \cdot \dot{q}_i, \tag{2.9}
\]

where $\cdot$ is the *scalar product*. The scalar product of two vectors $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ is defined as

\[
a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + ... + a_n b_n.
\]
The purpose of defining $T$ in this manner is to calculate the energy associated with each degree of freedom of the system. By extension, for a rigid body, we can write

$$
T = \frac{1}{2} M \dot{q} \cdot \dot{q},
$$

where $M \in \mathbb{R}^{n \times n}$ is the mass matrix of the system. The mass matrix of a system is a symmetric matrix that relates the generalised velocity components $\dot{q}_i$ to the system kinetic energy. For the analysis and techniques used throughout this thesis (2.9) and (2.10) are the only definition we require for kinetic energy.

The second form of energy that needs to be considered here is the potential energy $V$. This is the energy that an object has, due to its position in space, under the action of some force field. Some examples include the gravitational potential energy of an object that depends on the vertical position and mass of the object, the electrostatic potential energy of a charged particle in an electric field and the elastic potential energy of a compressed spring [47]. For our purposes, the potential energy of a system is a function that satisfies

$$
Q_i = -\nabla V = -\frac{\partial V}{\partial q_i},
$$

where $Q_i$ are the generalised forces which can be derived from potential functions. In other words the force $Q_i$ must be a conservative force, which is a force that is not dissipative in nature and any mechanical process acting under the influence of this force is reversible. Moreover, a conservative force is a force in which the work done $W$ by the force $Q_i$ between any two points is independent of the path taken [50].

The Lagrangian $L$ of a system can be defined as

$$
L = T - V
$$

and Lagrange’s equations of motion, which are a system of second order differential equations, are given by

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \ddot{Q}_i,
$$

where $\ddot{Q}_i$ are the generalised forces that can not be derived from a potential function. Examples of such forces are friction and time-dependent forces. A system with $n$ degrees of freedom can be described, in general, by $n$ second order differential
equations. The right hand side of (2.13) is zero when all of the generalised forces of the system are conservative. Once the Lagrangian has been calculated for the system, obtaining the equations of motion only involves the differentiation of certain system quantities, as seen in Section 2.1.3. The focus on energies rather than forces and accelerations means that the primary analysis is on scalar quantities which is inherently more simple than the Newtonian formulation [47].

2.1.3 Example

To illustrate the two techniques described in Sections 2.1.1 and 2.1.2, we will derive the equations of motion for a model system using both a Newtonian and a Lagrangian formulation. Consider a pendulum which is subject to the force of gravity and that can impact with a smooth surface as illustrated in Figure 2.2. Assume that the mass of the pendulum is located at a point mass at the end of the string with mass \( m \) and that the string connecting the mass to the pivot point is massless and inextensible. This system can be described in terms of the generalised coordinate \( \theta \).

![Figure 2.2: A pendulum in space subject to the force of gravity which can impact with a smooth surface.](image)


NEWTONIAN FORMULATION

The rotational dynamics of this system could be described by using (2.8), however, we will instead consider components acting in the horizontal direction $n_1$ and vertical direction $n_2$ and then use Newton’s Second law given by (2.3).

The position of the mass in the $n_1$-$n_2$ frame is given by

$$q := (q_1, q_2)^T = (l \sin(\theta), -l \cos(\theta))^T,$$

(2.14)

and the velocity is given by the time derivative of the position

$$\dot{q} := (\dot{q}_1, \dot{q}_2)^T = \left( l \cos(\theta) \dot{\theta}, l \sin(\theta) \dot{\theta} \right)^T,$$

(2.15)

and further the acceleration is given by the time derivative of the velocity

$$\ddot{q} := (\ddot{q}_1, \ddot{q}_2)^T = \left( l \cos(\theta) \ddot{\theta} - l \sin(\theta) \dot{\theta}^2, l \sin(\theta) \ddot{\theta} + l \cos(\theta) \dot{\theta}^2 \right)^T.$$

(2.16)

Considering the total force forces $F$ acting on the system in Figure 2.2 yields

$$F = (\lambda - T \sin(\theta), -mg + T \cos(\theta))^T.$$

(2.17)

Further, using Newton’s second law

$$F = m \ddot{q}_G = m \ddot{q},$$

(2.18)

we can write

$$(\lambda - T \sin(\theta), -mg + T \cos(\theta))^T = m \left( l \cos(\theta) \ddot{\theta} - l \sin(\theta) \dot{\theta}^2, l \sin(\theta) \ddot{\theta} + l \cos(\theta) \dot{\theta}^2 \right)^T,$$

(2.19)

which gives the following two equations

$$\lambda - T \sin(\theta) = m \left( l \cos(\theta) \ddot{\theta} - l \sin(\theta) \dot{\theta}^2 \right)$$

(2.20)

and

$$-mg + T \cos(\theta) = m \left( l \sin(\theta) \ddot{\theta} + l \cos(\theta) \dot{\theta}^2 \right).$$

(2.21)
Multiplying (2.20) by \( \cos(\theta) \), multiplying (2.21) by \( \sin(\theta) \) and summing the two gives

\[
\ddot{\theta} = -\frac{g}{l} \sin(\theta) - \frac{\lambda}{ml} \cos(\theta),
\]

(2.22)

the equations of motion of the system.

**Lagrangian Formulation**

For the Lagrangian formulation it is necessary to first construct the Lagrangian. The kinetic energy \( T \) of the particle can be calculated using (2.10) together with (2.15)

\[
T = \frac{1}{2} M \dot{q} \cdot \dot{q} \Rightarrow
\]

(2.23)

\[
T = \frac{1}{2} m \left( l \cos(\theta) \dot{\theta}, l \sin(\theta) \dot{\theta} \right) \cdot \left( l \cos(\theta) \dot{\theta}, L \sin(\theta) \dot{\theta} \right)^T \Rightarrow
\]

(2.24)

\[
= \frac{1}{2} m l^2 \dot{\theta}^2
\]

(2.25)

Next, we calculate the potential energy of the system using (2.11) such that

\[
Q_i = -\nabla V = -\frac{\partial V}{\partial q_i},
\]

where

\[
Q_i = (0, -mg)^T
\]

(2.26)

\[
\Rightarrow V = -mgq_2
\]

(2.27)

\[
\Rightarrow V = mgl \cos(\theta)
\]

(2.28)

the potential energy of the particle.

When the mass is in contact with the surface there is a normal force \( \lambda \) which is can not be considered as a conservative force and thus we can not derive a potential function from it. We therefore define

\[
\hat{Q}_i := l \cos(\theta) \lambda.
\]

(2.29)

The Lagrangian of the particle can then be determined using (2.12) to give

\[
L = \frac{1}{2} ml^2 \dot{\theta}^2 - mg \cos(\theta)
\]

(2.30)
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and finally, the equations of motion of the system can be calculated using (2.13) to give

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \dot{Q}_i \Rightarrow (2.31)
\]

\[
\ddot{\theta} = -\frac{g}{l} \sin(\theta) - \frac{\lambda}{ml} \cos(\theta),
\]

which is consistent with the equations of motion obtained using the Newtonian formulation in Section 2.1.1.

### 2.2 Force, Impulse and Conservation Principles

In this Section we describe the relationship between force and impulse and also present a statement of two fundamental conservation principles used extensively in the analysis of rigid-body impacts. The impact-law formulations that will be introduced in Chapter 3 are fundamentally based on the force-impulse relationship and conservation principles presented in this Section.

#### 2.2.1 The relationship Between Force and Impulse

Consider (2.3) presented in Section 2.1.1, which relates the centre of mass acceleration of a system to the total external force, i.e.

\[
m\ddot{q}_G = F. \tag{2.33}
\]

If we integrate (2.33) with respect to time over the interval \([t_0, t_1]\) and if we assume that \(F\) is constant throughout the time interval we get

\[
\int_{t_0}^{t_1} F\,dt = m(v_{G1} - v_{G0}) = \rho_1 - \rho_0 := P, \tag{2.34}
\]

which states that the change in linear momentum of a rigid body over a time interval is equal to the total impulse of the external forces acting over the same time interval [48], where \(v_{G1} = \dot{r}_G(t_1)\) is the centre of mass velocity at time \(t_1\) and \(v_{G0} = \dot{r}_G(t_0)\) is the centre of mass velocity at time \(t_0\) and where \(\rho_1 = mv_{G1}\) and \(\rho_0 = mv_{G0}\). The time integral of the force \(F\) is known as the linear impulse and we denote this by \(P\). Further, we define the linear momentum \(\rho\) of the system as \(\rho = mv_G\).
For rigid-body impacts, which will be discussed in detail in Chapter 3, the impulse and force generated as a result of a collision can be related using (2.34), provided the contact duration is sufficiently small. This allows us to use impulse formulations as opposed to force formulations which, for computational purposes, are easier to implement. Force formulations require contact durations to be specified, whereas impulse formulations, which naturally occur over an infinitesimal time period, do not.

### 2.2.2 Conservation of Linear Momentum

The impulse $P$ given by (2.34) has components in one, two or three dimensions, depending on the system in question. If any of these components are zero, for a given instance in time, then linear momentum is conserved in that direction. Furthermore, if there are no external forces acting on the system, regardless of the nature of the internal forces, then the total linear momentum will be a constant. This is the principle of conservation of linear momentum [47, 50]. It is important to mention the significance of the internal forces. Internal forces are allowed to occur because Newton’s third law guarantees that these forces occur in equal and opposite pairs and hence they cancel out exactly when we sum the total force on the system. Thus the presence of internal forces in the system does not invalidate the principle of conservation of linear momentum.

### 2.2.3 Conservation of Angular Momentum

If the moment of the external forces about the origin $O$ or about another fixed point of the system is zero at a given instance in time, then the angular momentum about the corresponding point is conserved. This is the principle of conservation of angular momentum. It is possible for the angular momentum to be conserved about one axis but not another [47, 50]. The law of conservation of angular momentum is one of the most fundamental of the conservation laws and is of particular importance for the analysis of rigid bodies with rotational degrees of freedom. In essence, this law states that a rigid rotating body, for example, continues to spin at a constant rate and with the same orientation unless acted upon by an external torque.

We have now defined all of the preliminaries necessary to describe the dynamics of systems of particles and rigid bodies under the action of forces. Further, the rela-
tionship between force and impulse has been derived, together with the conservation of linear and angular momentum principles. In what follows, we will use and extend these concepts to mathematically model the change in dynamics of particles and rigid bodies as a result of an impact.

2.3 Physics of Colliding Bodies

The term *impact* describes an interaction, of short contact duration, between two bodies that typically involves large accelerations, small changes in position and orientation, and finite changes in velocities [24]. During a two or multi-body collision, contact forces are generated between the bodies. These forces act over a very short time period, generating stress waves which propagate through the bodies in a direction away from the contact surface [14]. This process acts to deform the contact surface into a mutual configuration. In an ideal situation this configuration could be used to predict the full kinematics of the rigid bodies throughout the impact phase. It has however proven extremely difficult to calculate the geometry of this configuration whilst also satisfying equations of motion which give forces to each body consistent with Newton’s third law of motion. An alternative to this approach is to define different classes of impacts and apply suitable *impact laws* accordingly [4].

Assume that the configuration of the bodies at the beginning of the impact phase is known, along with their mass matrices and moments of inertia. Then, provided the centre of mass velocities and angular velocities at impact are known, an impact law can be used to predict the velocity components of the bodies after the collision. Moreover, an impact law is a physical law which is based on theory, observation and experimentation that is used to model the physics of a collision between bodies. We define an *impact mapping* as the individual components of the impact law which map the relative velocities before a collision to the relative velocities after a collision, using the masses, inertias and pre impact velocity components of the system or body.

2.4 Modelling Energy Removal

Impacts are dissipative in their nature. The process of two bodies colliding removes a certain portion of the system’s initial energy. Some of this energy is used to deform the colliding bodies into a mutual configuration and some is transmitted through
Figure 2.3: Schematic illustrating a collision between two rigid bodies with contact point \( C \). The contact forces \( \lambda_1 \) and \( \lambda_2 \) are indicated as well as the generalised coordinates \( q_1 \) and \( q_2 \) of \( B \) and the generalised coordinates \( q'_1 \) and \( q'_2 \) of \( B' \). The subscript 1 denotes the direction tangent to the contact plane and the subscript 2 denotes the direction normal to the tangent plane. The coordinates of the contact point are \( q_{1C} \) and \( q_{2C} \) respectively.

heat, sound vibrations and other mechanical mechanisms [9]. The exact mechanisms by which energy is transferred during an impact involve extremely complex processes and leads to great difficulties in the mathematical analysis of impact problems. To date, complete solutions have only been obtained for impacting systems with very simple geometries, and thus no complete theory for the energy transfer process during an impact has been developed [14]. It is therefore necessary to employ more approximated methods to model the energy removal process. The techniques used to mathematically model this vary vastly depending on the nature of the impact in question. Some approaches model the impact process using a series of springs and mechanical dampers, as will be discussed in Section 2.5, while others use experimentally measurable coefficients to describe the energy removal process.

Elastic collisions are those in which there is no energy loss as a result of the collision. Inelastic or plastic collisions are ones in which all of the energy is lost as a result of the collision. For the class of impacts considered in this thesis, we will analyse systems which lie in between these two extremes.

In what follows, we will use the two body collision shown in Figure 2.3 to illustrate
some of the concepts and techniques used for mathematically modelling the collision process. Consider the collision between the body $B$, with centre of mass coordinates $(q_1, q_2)$, and the body $B'$, with centre of mass coordinates $(q'_1, q'_2)$, which occurs at an isolated contact point $C$. We denote by $(\dot{q}_1, \dot{q}_2)$ and $(\dot{q}'_1, \dot{q}'_2)$, the centre of mass velocities of $B$ and $B'$, respectively, and we denote by $(\dot{q}_{1C}, \dot{q}_{2C})$ and $(\dot{q}'_{1C}, \dot{q}'_{2C})$, the contact point velocities of $B$ and $B'$, respectively. We define the relative velocity between the centres of mass of $B$ and $B'$ as

$$(\check{\dot{q}}_1, \check{\dot{q}}_2) = (\dot{q}_1 - \dot{q}'_1, \dot{q}_2 - \dot{q}'_2),$$

and we define the relative velocity between the contact point of $B$ and $B'$ as

$$(\check{\dot{q}}_{1C}, \check{\dot{q}}_{2C}) = (\dot{q}_{1C} - \dot{q}'_{1C}, \dot{q}_{2C} - \dot{q}'_{2C}).$$

Tangential forces $\pm \lambda_1$ and normal forces $\pm \lambda_2$ are generated at contact, which, for a sufficiently short contact duration and for a large force, may be viewed as the impulses $\pm P_1$ and $\pm P_2$, respectively. $\lambda_1$, $\lambda_2$, $P_1$, $P_2$ act on body $B$ and $-\lambda_1$, $-\lambda_2$, $-P_1$, $-P_2$ act on body $B'$. In what follows we will discuss the various coefficients used for modelling energy removal, as this approach is used exclusively throughout this thesis. The impact law formulations we focus on involve a combination of the coefficients discussed in the next section together with a work or momentum principle.

### 2.4.1 Coefficient of Restitution

For collisions between rigid bodies, removal of energy in the direction normal to the contact plane is modelled using a coefficient of restitution. Coefficients of restitution have mainly been defined in two ways. Firstly, in terms of a relation between the normal impulse applied during the restitution phase to that applied during compression. Secondly, in terms of a relation between the normal component of the relative velocity before and after the collision [52]. Using a coefficient approach the impact is considered in two segments, the segment immediately before impact, denoted by $-$ and the segment immediately after impact, denoted by $+$. There are three commonly used coefficients of restitution, namely, Newtonian, Poisson and Energetic, a brief description of which we will present here.
1. The *Newtonian restitution law*

\[ e = -\frac{\ddot{q}_2^+}{\dot{q}_2^+}, \]

(2.35)

which models the removal of energy in the normal direction using a kinematic coefficient of restitution \( e \), with \( e \) taking values between 0 and 1. This model defines the coefficient \( e \) as the negative ratio of the relative normal velocity at the contact point before the collision to the relative normal velocity at the contact point after the collision [4]. The negative sign comes from the typical physical situation that the normal relative velocity is reversed as a result of a collision. A negative value of \( e \) would, for example, imply penetration through a barrier, a class of impact not considered in this thesis. A value of \( e \) greater than 1 would imply an increase in the normal component of the relative velocity as a result of the collision. Such phenomena can be likened to the bounce of a superball, in which, with each impact the direction of spin is reversed and the normal component of the relative velocity can increase as a result of the collision [8].

2. The *Poisson restitution law*

\[ e_p = \frac{P_{2_{\text{exp}}}}{P_{2_{\text{comp}}}}, \]

(2.36)

which states that the ratio of the normal impulse during the compression phase \( P_{2_{\text{comp}}} \) to that in the expansion phase \( P_{2_{\text{exp}}} \) is a constant denoted by \( e_p \). This model assumes that, in compression, the contact surfaces act as if they were inelastic so that at the end of this phase the impulse \( P_{2_{\text{comp}}} \) is such as to bring the relative normal velocity to zero and, in restitution, the impulse \( P_{2_{\text{exp}}} \) is some scalar \( e_p \) times the impulse during compression [48].

3. The *Energetic restitution law*

\[ e_r^2 = \frac{W(P_{2_i}) - W(P_{2_{\text{comp}}})}{W(P_{2_{\text{comp}}})}, \]

states that the square of the coefficient of restitution is the negative ratio of the elastic strain energy released during restitution to the internal energy of deformation absorbed during compression [4]. The individual work components
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$W$ are defined as follows

$$W(P_{2f}) = \int_0^{t(P_{2f})} \lambda_2 \dot{q}_2 dt', \quad W(P_{2\text{comp}}) = \int_0^{t(P_{2\text{comp}})} \lambda_2 \dot{q}_2 dt',$$

where $P_2$ is the normal component of the impulse, $\dot{q}_2(P_2)$ is the relative normal velocity, $\lambda_2$ the normal component of the contact force, $P_{2f}$ is the terminal impulse achieved at separation and where $P_{2\text{comp}}$ is the final impulse achieved at the end of the compression phase.

The work released to the system during the restitution phase is defined as $W(P_{2f}) - W(P_{2\text{comp}})$ and the work done during the compression phase is defined as $W(P_{2\text{comp}})$. For a significantly short contact duration the force can be related to the differential of impulse, $dP_2 = \lambda_2 dt'$ so that

$$e_2^2 = -\frac{\int_{P_{2\text{comp}}}^{P_{2f}} \dot{q}_2(P_2) dP_2}{\int_0^{P_{2\text{comp}}} \dot{q}_2(P_2) dP_2}. \quad (2.37)$$

Stronge [4] views the impact phase as being composed of a compression phase followed by a restitution phase. During compression, kinetic energy is stored as internal deformation energy until the normal relative contact point velocity is brought to zero. At this point the restitution phase begins and the stored energy is released.

Typically a Newtonian coefficient or a Poisson coefficient is used to model this energy removal in the normal direction [4], and in many formulations both give an equivalent impact mapping [8,9,24]. However, for situations where the direction of the relative tangential velocity at the contact point can vary throughout the impact phase both the Newtonian and Kinematic coefficients may violate energy conservation [17]. It is then necessary to use the energetic coefficient of restitution. A comparison of the dynamics one can achieve using the three coefficients is presented in [52], together with a discussion detailing for what conditions the coefficients are equivalent.

2.4.2 Amontons-Coulomb Friction Law

For the analysis carried out in this thesis, we assume that any tangential force $\lambda_1$ generation (see Figure 2.3), as a result of an impact, is due exclusively to roughness of
the contact surfaces, known as friction. Further, we assume that the frictional force can be modelled using the Amontons-Coulomb friction law [53]. The Amontons-Coulomb friction law gives a relation between the friction force $\lambda_1$, generated as a result of bodies being in contact, and the normal force $\lambda_2$ in terms of a dimensionless parameter $\mu$ such that

$$
\lambda_1 = \begin{cases} 
\mu \lambda_2, & \dot{\tilde{q}}_{1C} < 0 \\
[-\mu \lambda_2, \mu \lambda_2], & \dot{\tilde{q}}_{1C} = 0 \\
-\mu \lambda_2, & \dot{\tilde{q}}_{1C} > 0
\end{cases} 
$$

where as before, $\dot{\tilde{q}}_{1C}$ is the relative tangential velocity at the contact point, for some non-negative constant $\mu$ representing a coefficient of friction.

Figure 2.4: Schematic illustrating the tangential forcing $\lambda_1$ as a function of initial relative tangential velocity at the contact point $\dot{\tilde{q}}_{1C}$.

The sign assigned to the tangential force $\lambda_1$ is positive (+) when the relative tangential contact point velocity between the two bodies is negative and it is negative (−) when the relative tangential contact point velocity is positive. In other words, the frictional force acts in a direction opposite to the relative tangential contact point velocity. For the case of zero relative tangential velocity $\lambda_1$ can take any value in $[-\mu \lambda_2, \mu \lambda_2]$ as depicted in Figure 2.4. This law assumes that the frictional force (the tangential force $\lambda_1$) is proportional to the normal force $\lambda_2$ and is independent of any sliding speed [53].
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Using (2.3), and considering the case \( \dot{q}_{1C} < 0 \), we can rewrite (2.38) as

\[
M \frac{d\dot{q}_1}{dt} dt = \mu M \frac{d\dot{q}_2}{dt} dt,
\]

(2.39)

where \( M \) is the mass matrix of the system of bodies. Further, for a sufficiently small contact duration the coefficient \( \mu \) can also be defined as the ratio of impulses as opposed to the ratio of forces. Letting \([t_0, t_1]\) be the time interval over which the contact forces act we have

\[
\int_{t_0}^{t_1} \frac{d\dot{q}_1}{dt} dt = \mu \int_{t_0}^{t_1} \frac{d\dot{q}_2}{dt} dt,
\]

(2.40)

and letting \( t_1 - t_0 \to 0 \) leads to

\[
\mu = \frac{\dot{q}_1^- - \dot{q}_1^+}{\dot{q}_2^- - \dot{q}_2^+},
\]

(2.41)

where \( \dot{q}_1^- \) and \( \dot{q}_2^- \) are the velocity components of the centre of mass just before the impact and \( \dot{q}_1^+ \) and \( \dot{q}_2^+ \) are the velocity components just after the impact. In this setting (and in what follows) the parameter \( \mu \) is referred to as the impulse ratio. Note that the case \( \dot{q}_{1C} > 0 \) can be equally considered by assigning a negative value to the impulse ratio in (2.41). We define the impulse ratio as the ratio of the tangential impulse to the normal impulse generated due to a collision between two or more bodies. The mechanism behind the tangential impulse generation is highly dependent on the nature of the collision in question. During vehicle collisions, for example, the impulse ratio can be viewed as a physical constant representing the results of shearing, tearing and deformation of the colliding bodies [9]. This idea was considered previously where the relationship between force and impulse was analysed. We note that there are variations of (2.38) which assume \( \lambda_1 \) changes as a function of the relative tangential velocity at the contact point \( \dot{q}_{1C} \) [54]. These formulations have not been considered however for the models considered in this thesis.
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2.5 A Review of Some Commonly Used Impact Laws

There are two main categories of impact: compliant impacts and non compliant impacts. Within each category there are some impact laws that include friction and some that do not. The material properties of a body, the duration of the impact phase, the configuration of the bodies at impact and the incident relative velocity at contact are the main factors in deciding which impact law to use [24].

Compliant impacts allow for deformation of the contact regions and consider visco-elastic effects throughout the impact phase. Collisions between nonmetallic bodies are typical examples of compliant impacts. In this situation time dependent forcing needs to be considered as impulse cannot be used as the independent variable in the analysis of the collision.

The Maxwell model [4, 14], for example, uses a linear spring and a damper in series configuration to capture the behaviour of the impact phase. In this model the

\[ \begin{align*}
\dot{q}_2 & \quad (x + x_0) \\
H & \quad c \\
\quad k & \quad H' \\
\quad y + y_0
\end{align*} \]

Figure 2.5: Collinear impact between two bodies \( H \) and \( H' \) separated by a Maxwell linear viscoelastic element with spring stiffness \( k \) and damping force constant \( c \). We denote the relative displacement of the two bodies by \( y \) and the portion of this which is due to the compression of the dashpot as \( x \). The initial relative velocity between the two bodies at impact is denoted by \( \dot{q}_2 \).

normal force increases continuously during compression while also storing energy, and during restitution this energy is released to the system. This model thus allows for coefficients of restitution with values between 0 and 1. The normal forcing, which is the same in the spring and damper, arises due to the relative displacement of the two bodies. This component type modelling was not considered for the work in this
thesis, but to give a flavour of the solutions it gives we will present an example using the Maxwell model, describing the eccentric collision between two bodies.

Following the analysis of this model presented in [4] we consider the schematic in Figure 2.5. We let the spring have a spring constant $k$ and rest length $y_0$, while the damper has a damping force constant $c$ with a rest length $x_0$. The combined relative displacement of the bodies $y$, generates the normal force $\lambda$ between the two bodies. This combined relative displacement $y$ has a portion which is due exclusively to the compression of the damper, which we denote by $x$.

$$\lambda = -k \left( y - x \right) . \tag{2.42}$$

The same force acts in the spring and the dashpot, which allows us to write

$$\lambda = -k \left( y - x \right) = -c \dot{x}. \tag{2.43}$$

Further if $H$ has mass $M$ and $H'$ has mass $M'$ then we can define the effective mass $m$ of the system such that $m^{-1} = M^{-1} + M'^{-1}$, and then the equations of motion are given by

$$m \ddot{y} = -k \left( y - x \right) \tag{2.44}$$

Differentiating (2.43) with respect to time gives

$$-k \left( \ddot{y} - \dot{x} \right) = -c \ddot{x} \tag{2.45}$$

Adding (2.44) and (2.45) gives

$$\ddot{y} - \ddot{x} = -k \left( \frac{1}{m} \left( y - x \right) - \frac{k}{c} \left( \dot{y} - \dot{x} \right) \right) . \tag{2.46}$$

Further, if we let $w = y - x$ then we arrive at the following second order differential equation for an oscillator

$$\ddot{w} + \omega_0^2 w + 2\zeta \omega_0 \dot{w} = 0, \tag{2.47}$$

where $\omega_0^2 = \frac{k}{m}$ and where $\zeta = \frac{m \alpha}{2c}$. If we consider this model describing a collision between two bodies with initial normal relative velocity $\dot{q}_2$ and initial relative position equal to zero, the initial conditions can be written as follows

$$x(0) = y(0) = w(0) = 0 \tag{2.48}$$
\begin{align*}
\dot{y}(0) = \dot{z}(0) = \dot{q}_2
\end{align*}

we arrive at the following solution
\begin{align*}
w(t) = e^{-\zeta \omega_0 t} \left( C_1 \cos \left( \omega_0 \sqrt{1 - \zeta^2} t \right) + C_2 \sin \left( \omega_0 \sqrt{1 - \zeta^2} t \right) \right)
\end{align*}

and applying the initial conditions given by (2.48) and (2.49) gives
\begin{align*}
C_1 = 0, \quad C_2 = \frac{\dot{q}_2}{\sqrt{1 - \zeta^2}}.
\end{align*}

Further, using (2.43) we arrive at the following expression for the forcing that occurs during the compression period of the impact phase
\begin{align*}
\lambda = k e^{-\zeta \omega_0 t} \frac{\dot{q}_2}{\sqrt{1 - \zeta^2}} \sin \left( \omega_0 \sqrt{1 - \zeta^2} t \right).
\end{align*}

An intuitive way to analyse this solution is to consider a non-dimensionalised form of (2.51). This non-dimensionalisation or rescaling allows us to extract characteristic properties of the system as we do not need to concern ourselves with units. We let
\begin{align*}
\lambda_* = \frac{\lambda}{m \omega_0 q} \quad \text{and} \quad t_* = \omega_0 t
\end{align*}

and under these substitutions (2.51) can then be written as
\begin{align*}
\lambda_* = \sqrt{\frac{k}{m} e^{-\zeta t_*}} \frac{1}{\sqrt{1 - \zeta^2}} \sin \left( \sqrt{1 - \zeta^2} t_* \right).
\end{align*}

In Figure 2.6 we plot $\lambda_*$ as a function of $t_*$ to analyse the properties of this impact model of the compression phase. The solution for $c = \infty$ corresponds to a perfectly elastic collision, the solution for $c = m \omega_0$ represents a collision with a large amount of damping and the solution for $c = \frac{1}{2} m \omega_0$ represents the lower bound of physicality for this model as it calculates a value of zero for $\lambda_*$. It is clear from Figure 2.6 that the Maxwell model predicts an asymmetrical force which increases with the ratio of damping $\gamma$. This model is useful for predicting how the normal force develops throughout the compression phase of an impact.

In practice it has been found that most materials do not behave as predicted by the Maxwell model. For more standard materials more elaborate spring-damper models are required. The simple addition of a second spring in parallel with the Maxwell setup gives a useful model known as the Standard Linear Solid (SLS). The effect of the parallel spring is that once the applied stress has been removed,
the system will have instantaneous elastic recovery, followed by a slowed recovery towards zero strain [4,14]. These models have been further extended in more recent years by Stoianovici and Hurmuzlu [55] and Chatterjee [24] to include forcing that depends on the relative velocity and the displacement. This is achieved using a linear spring-nonlinear damper approach [4].

Another technique for modelling compliant impacts, and the first to incorporate a theory of local indentation, is the *Hertz law* (Section 1.1) [53,54]. Hertz viewed the contact between two bodies as an analogous problem in electrostatics. A potential-like solution for the contact problem of two compliant bodies was constructed which described the stresses and deformations in a region around the contact point in terms of the elastic and geometrical properties of the bodies [14]. This approach assumes that deformations will only occur within a certain area around the contact region and outside this region they can be considered negligible. Hertzian theory predicts the shape of this area of contact and how it increases in size with the applied load.

An improvement to Hertzian contact theory was presented in 1971 by Johnson, Kendall and Roberts (JKR-Theory) [54,56]. They extended the original Hertzian contact theory theory by solving for the case of adhesive contact between elastic bodies. This problem plays a key role in many technological industries, in particular for the semi-conductor device industry where particle contamination is a limiting

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**Figure 2.6:** Evolution of the non-dimensionalised force $\lambda_*$ that occurs during the compression phase of the impact as a function of the non-dimensionalised time $t_*$ for $m = 1$ and $k = 1$. 

$$c = \infty$$

$$c = m\omega_0$$

$$c = \frac{1}{2}m\omega_0$$
factor in production. JKR-Theory uses a balance between the elastic energy stored during compression and the loss in surface energy. An alternative adhesion theory was published in 1975 by Derajaguin, Müller and Toporov (DMT-Theo-ry) [57]. It was agreed that both theories gave consistent results, however, JKR-Theory is more widely used.

There are of course other compliant impact models. Although interesting this class of impact is not the focus of this thesis and will not be discussed in any further detail.

Non-compliant impacts, the emphasis of this thesis, are ones in which certain rigidity constraints are satisfied. The assumptions of rigid body impact theory can be summarised as follows:

**No compliance of the contact regions.** Although the contact area is small in comparison with the cross-sectional dimensions of the impacting bodies, a relatively large contact force is generated. This force quickly changes the normal component of the relative velocity and therefore minimises any interpenetration of the bodies [4].

**Point Contact.** It is necessary to have a well defined tangent plane at the area of contact, particularly when dealing with friction. The normal and tangential direction need to be clearly isolated to apply the impulses correctly [24]. This assumption is valid provided the dimensions of the contacting region are small in comparison with the other dimensions, as for a relatively large object, a sufficiently small surface may be approximated as a point.

**No Moment impulse.** The assumption of point contact allows any moment impulse that would be generated due to a surface contact to be neglected. This assumption is valid only if the contact area is sufficiently small [58].

**Infinitesimal contact duration.** It is evident from observation and experimentation that the contact duration for rigid bodies is extremely small. The contact duration, for example, of a hockey stick striking a stationary ball is approximately 2 ms [4]. The infinitesimal contact duration leads naturally to an impulse analysis as opposed to a force analysis. It also simplifies the computationally implementation as a time interval over which a force acts does not need to be defined [4].
No change in generalised coordinates throughout the impact phase. A consequence of zero compliance of the contact regions and the infinitesimal contact duration means that there is insufficient time for the bodies to be displaced by any significant amount. Again this is advantageous from a computational point of view [16].

Newton’s Third Law of motion. It is assumed that equal and opposite forces and impulses are applied to colliding bodies throughout the impact phase [14].

Finite active forces can be neglected. The finite forces (for example gravity) do no work during the collision and can therefore be considered negligible. If we assume no change in generalised coordinates throughout the impact phase, then the finite forces do no work as there is no displacement in the line of action of the force. The contact forces are large in comparison to any other interaction force [24].

A real world example of a non-compliant impact would be a steel on steel collision [4]. Analysis of non compliant impacts is particularly important in the area of mechanical engineering. Many machines with moving elements are subject to unwanted impacts which in certain instances can lead to wear and possible shutdown [11].

One of the simplest and most commonly used impact laws is Whittaker’s Model [59] which uses the Newtonian restitution law given by (2.35) together with an impulse argument to define the impact law. Whittaker defines the relative tangential contact point velocity $\dot{q}_{1C}^+$ after collision as

$$
\dot{q}_{1C}^+ \begin{cases} 
0, & P_1 \leq \mu P_2, \\
\frac{P_1}{|P_1|}, & P_1 = -\mu P_2 \frac{P_2}{|P_2|}.
\end{cases}
$$

In the second instance where $P_1 = -\mu P_2 \frac{P_2}{|P_2|}$, the relative tangential contact point velocity after collision $\dot{q}_{1C}^+$ is calculated using (2.35) together with (2.41). Such a simple model however naturally has its disadvantages. It has been shown that for a broad range of parameter values conservation of energy is violated [24].

Smith [22] presented an energy conserving alternative to Whittaker’s model. Again (2.35) is used to model the energy dissipation in the normal direction, however, $P_1$ is now defined as a weighted average of the relative tangential velocity before
and after impact as
\[
P_1 = -\mu P_2 \frac{|\dot{q}_{1C}^-| |\dot{q}_{1C}^- + |\dot{q}_{1C}^+| |\dot{q}_{1C}^+|}{|\dot{q}_{1C}^-|^2 + |\dot{q}_{1C}^+|^2}.
\]
Smith states that if the relative tangential velocity changes direction during impact, then it is reasonable to assume that \(P_1\) would be some average of the tangential velocity before and after collision. This leads to the following system of equations
\[
M \left( \begin{array}{c} -\frac{1}{1+e} \dot{q}_{2C}^- \\ \dot{\hat{q}}_{1C}^- - \dot{\hat{q}}_{1C}^+ \end{array} \right) = P_2 \left( \begin{array}{c} \frac{1}{|\dot{q}_{1C}^-|^2 + |\dot{q}_{1C}^+|^2} \\ -\mu |\dot{q}_{1C}^-| |\dot{q}_{1C}^- + |\dot{q}_{1C}^+| |\dot{q}_{1C}^+| \end{array} \right),
\]
where \(M\) is the mass matrix of the system. For a particle collision \(M\) is a scalar and for a planar collision, \(M\) is a \(2 \times 2\) matrix. The system of equations given by (2.53), however, is non-linear and can be solved numerically using iterative techniques [24].

Routh’s model uses Poisson’s definition of the coefficient of restitution (2.36) together with the Amontons-Coulomb friction law (3.30) to define the tangential impulse [15]. The method employs a graphical technique to predict the total impulse for a given impact. This involves constructing an impulse space diagram in a tangential and normal coordinate system. The normal component of the impulse increases until the restitution law indicates the impact process has finished. The tangential component of the impulse increases according to the Amontons-Coulomb law and the method guarantees to not violate energy conservation [17]. One of the observed short comings with this impact model is that it assumes zero tangential compliance of the contact region and thus cannot predict for example relative tangential velocity reversal for the case of impacting spheres [24].

Glocker and Pfeiffer [60] present an impact law for 2D planar collisions that uses Poisson’s definition of the coefficient of restitution (2.36), but with a slight extension. This approach involves considering the impulse as two distinct segments of compression \(P_{\text{comp}}\) and expansion \(P_{\text{exp}}\). The compression phase is defined such that the two conditions
\[
P_{\text{comp}} = M (\dot{q}_{\text{comp}} - \dot{q}^-), \quad \text{and} \quad \dot{q}_{\text{comp}} = 0,
\]
must hold, where \(M\) is the mass matrix of the system and \(\dot{q}_{1\text{comp}}\) is defined as
\[
\dot{q}_{1\text{comp}} \begin{cases} 
0 & |P_{\text{comp}}| \leq \mu P_{2\text{comp}}, \\
\neq 0 & -\mu P_{2\text{comp}} |\dot{q}_{1\text{comp}}|.
\end{cases}
\]
The statement that $\dot{q}_{2_{\text{comp}}}=0$ at the end of the expansion phase comes from the argument that during compression the surface acts as if it was inelastic and so would naturally bring the normal velocity to zero [48]. The expansion phase is defined first using Poisson’s restitution law (2.36)

$$P_{2_{\text{exp}}} = e_p P_{1_{\text{comp}}},$$

where $e_p$ is the Poisson coefficient of normal restitution, and second by the conditions depending on $P_{1_{\text{exp}}}$

$$\begin{align*}
\dot{q}_{1_{\text{exp}}} & = 0 & |P_{1_{\text{exp}}}^*| & \leq \mu^* P_{2_{\text{exp}}}^*, \\
\dot{q}_{1_{\text{exp}}} & \neq 0 & P_{1_{\text{exp}}}^* \text{sign}(P_{1_{\text{comp}}}) & - \mu^* P_{2_{\text{exp}}} \frac{\dot{q}_{1_{\text{exp}}}}{|\dot{q}_{1_{\text{exp}}}|},
\end{align*}$$

where $\mu^* = (1 - \alpha) \mu$, for an arbitrary scalar $\alpha \in [0, 1]$, and where $P_{1_{\text{exp}}}^* = \alpha \mu P_{2_{\text{exp}}}$. The quantity $\alpha$ is then chosen in terms of two new collision parameters, $e_t$ and $\nu$ which model tangential restitution. The method for determining $\alpha$ involves solving a set of complementarity conditions and it can be shown that

$$\alpha = \frac{P_{1_{\text{exp}}}^*}{\mu P_{2_{\text{exp}}}} = \frac{\nu + e_t}{2}.$$ 

The quantity $P_{1_{\text{exp}}}^*$ represents the idea of stored impulse in the system and is used to describe the interaction of the tangential impulse during the expansion phase.

Chatterjee [23,24] presents an algebraic impact law that extends to 3D collisions. The inherent difficulty when considering this class of impact is the correct treatment of the impulse. For the 3D case it is necessary to assign a direction to the tangential impulse as there will be two independent tangential directions. A mistake in this choice could result in energy gains when the normal impulse is calculated. Chatterjee’s approach involves describing the impulse as a weighted sum of an inelastic collision without friction and an inelastic collision with sticking motion after impact. This model uses two restitution parameters, a normal coefficient, $e_2$, which can be either the Newtonian $e$ or Poisson coefficient $e_p$ of restitution, together with a tangential coefficient $e_1 \in [-1, 1]$ which is a measure of the degree of tangential velocity reversal as a result of the collision. The standard Amontons-Coulomb friction law (2.38) is used and the coefficient $\mu$ is selected such as to ensure a physical frictional impulse is generated at impact.
Chapter 3

Comparison of Two Impact Laws

Impact laws vary vastly in complexity, from the basic to the advanced. How to choose one formulation over another is an important question for scientists and engineers analysing impacts in a research setting. The wrong choice of impact law may lead to energy gains in the system and unphysical post impact kinematic predictions. The aim of this Chapter is to explore the varying dynamics that can be achieved using both a basic and a complex impact law and to highlight some of the problems that can occur with a too simplistic formulation. Both impact laws are for planar rigid body collisions with friction. For this purpose we use the extended Brach impact law [45] as the simple impact law and we use the Energetic impact law [1,25] for the advanced one. For the Brach impact law [21,30,61] we will derive a novel extension, using energetic considerations, to allow for long term simulation of impacting mechanical systems. This impact law had previously only been used for specific engineering applications. The form of the Brach impact law is the most commonly used impact law formulation for most engineering applications. For the Energetic impact law we derive a general framework for using this impact law for a two body collision, including translations between reference frames. The Energetic impact law represents the most advanced impact law used for planar rigid bodies with friction. We will compare the dynamics achieved using both impact laws and show that for certain regions of parameter space the Brach impact law will give unphysical post impact velocities and it is therefore necessary to use the Energetic impact law.
CHAPTER 3. COMPARISON OF TWO IMPACT LAWS

3.1 The Brach Impact Law

In this Section we will describe the Brach impact law [9, 21], and further we will present an extension to the Brach impact law to allow for the simulation of multiple impacts in a rigid body mechanism [45]. For this impact law, we will consider the impulse ratio representing a coefficient of friction. Moreover, that any tangential force generation is exclusively due to Amontons-Coulomb friction. In what follows, we will derive this impact law for the case of a planar surface collision.

3.1.1 Planar Rigid-body Impacts with Friction

Consider a single planar rigid body in contact with a noncompliant plane at a point C (see Figure 3.1). We let $\theta \in (0, \pi)$ be the angle of rotation of the body relative to the plane we define $L$ as the distance from $G$ to the contact point $C$. Further, let $q_1$ denote the tangential and let $q_2$ denote the normal position of the centre of mass $G$ of the rigid body relative to the initial configuration such that the generalised coordinates $q$ can be written as

$$ q = (q_1, q_2, \theta)^T, $$

Figure 3.1: A planar rigid body with centre of mass $G$ with generalised coordinates $(q_1, q_2)$ in contact with a non-compliant plane at point $C$. We let $x$ and $y$ denote the distance between the contact point $C$ and the centre of mass $G$ in the tangential and normal direction respectively.
then the position $c$ of the contact point $C$ is given by
\[ c := (c_1, c_2)^T = (q_1 - L \cos(\theta), q_2 - L \sin(\theta))^T = (q_1 - x, q_2 - y)^T, \] (3.1)

where
\[ x = L \cos(\theta), \quad y = L \sin(\theta). \]

In order to relate how the contact force $\lambda$ will affect the centre of mass acceleration we need to consider
\[ \frac{\partial c}{\partial q} = \begin{pmatrix} 1 & 0 & L \sin(\theta) \\ 0 & 1 & -L \cos(\theta) \end{pmatrix}. \]

Further if we let
\[ \dot{q} = (\dot{q}_1, \dot{q}_2, \dot{\theta})^T, \quad \ddot{q} = (\ddot{q}_1, \ddot{q}_2, \ddot{\theta})^T \]

be the velocity and acceleration of the centre of mass respectively, then the equations of motion at the moment of impact can be determined by using (2.3) to give
\[ M \ddot{q} = F + \left( \frac{\partial c}{\partial q} \right)^T \lambda, \] (3.2)

where $M$ is the mass matrix of the body and $F$ is the vector of external forces, such that
\[ M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ T \end{pmatrix}, \] (3.3)

where $m$ is the mass, $I$ is the moment of inertia, $F_1$ and $F_2$ are respectively the external tangential and normal forces and $T$ is the external torque. Also, we let
\[ \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \]

where $\lambda_1$ and $\lambda_2$ are respectively the tangential and normal forces generated at impact.

The impact phase is defined as the compression and restitution segments of the impact and will be considered here to be instantaneous. We also assume that the contact area is small in comparison to the size of the body and that there is no change in generalised coordinates throughout the impact phase. Further, for future
reference, we let \( V_T \) and \( V_N \) be the tangential and normal components, respectively, of the linear velocity \( V \) of \( G \), respectively, and \( \dot{\theta} \) the angular velocity about \( G \). Similarly, we let \( V_{TC} \) and \( V_{NC} \) be the tangential and normal components of the linear velocity \( V_C \) at \( C \). This gives that

\[
V := \begin{pmatrix} V_T \\ V_N \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{\theta} \end{pmatrix}, \quad (3.4)
\]

\[
V_C := \begin{pmatrix} V_{TC} \\ V_{NC} \end{pmatrix} = \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 + L \sin(\theta)\dot{\theta} \\ \dot{q}_2 - L \cos(\theta)\dot{\theta} \end{pmatrix} = \begin{pmatrix} V_T + y\dot{\theta} \\ V_N - x\dot{\theta} \end{pmatrix}. \quad (3.5)
\]

The analysis in the next Section will be based on the equations and notation introduced above.

### 3.1.2 Derivation of the Brach Impact Law

In this Section we will follow [9] to derive the Brach impact law for a planar rigid body using the Amontons-Coulomb friction law and the Newtonian restitution law.

First, the Amontons-Coulomb friction law, as described in Section 2.4.2, relates the tangential and normal components \( \lambda_1 \) and \( \lambda_2 \), respectively, of the reaction force occurring at impact through a coefficient of friction \( \mu_f \) such that

\[
\lambda_1 = \pm \mu_f \lambda_2. \quad (3.6)
\]

In what follows we will denote the sign by \( s \) so that \( \lambda_1 = s\mu_f \lambda_2 \), where \( s = \pm 1 \).

The conditions for which the impulse ratio is equivalent to the conventional friction coefficient will be discussed in Section 3.1.2.

Second, Newton’s restitution law given by (2.35) is used. It is necessary to define the contact point velocity before the impact \( V_{NC}^- \) in order to use this law. From (3.5) we have that

\[
V_{NC}^- = V_N^- - x\dot{\theta}^-,
\]

which then allows us to write

\[
V_{NC}^+ = V_N^+ + \dot{\theta}^+ x = -e \left( V_N^- - x\dot{\theta}^- \right) = -e V_{NC}^-,
\]

where \( \dot{\theta}^- \) and \( \dot{\theta}^+ \) are the angular velocities just before and after the impact, re-
spectively. The parameter $e$ is the restitution coefficient, where the value assigned to $e$ depends on the material properties of the colliding objects. This assumption accounts for the visco-elastic effects which lead to the removal of energy. Note, in the examples considered in this Chapter, and in the rest of this thesis, $e$ will be given a value between 0 and 1, however, it will be shown here that energy will not be introduced into the system for all $e \in [-1, 1]$. Notice that a negative $e$ implies further penetration through the plane, which will not be considered here.

Third, we consider the angular momentum of the rigid body about the point of contact $C$. We neglect the moment impulse since we assume a contact point that is fixed throughout the impact phase. For a surface contact, a moment impulse would be generated between the rigid body and the plane and this would need to be considered for the correct analysis of the impact phase [58]. Using conservation of angular momentum we get

$$(-c \times mV^+) - (-c \times mV^-) = mk_c^2 \left(\dot{\theta}^+ - \dot{\theta}^-\right)$$  \hspace{1cm} (3.8)

which gives

$$-k_c^2\dot{\theta}^+ + yV_T^+ - xV_N^+ = -k_c^2\dot{\theta}^- + yV_T^- - xV_N^-, \hspace{1cm} (3.9)$$

where $k_c$ is the radius of gyration of the rigid body. The radius of gyration is essentially a measure of the rotational inertia for the centre of mass of the rigid body about the axis normal to the plane [4].

Using (3.7), (3.9) and the impulse ratio given by (2.41) ($\mu = \frac{\dot{q}_2 - \dot{q}_1}{\dot{q}_2 + \dot{q}_1}$) we get the impact law

$$V_N^+ = V_N^- - k_c^2 \frac{(1 + e) \left(V_N^- - x\dot{\theta}^-\right)}{k_c^2 + x(x - s\mu y)}, \hspace{1cm} (3.10)$$

$$V_T^+ = V_T^- - s\mu k_c^2 \frac{(1 + e) \left(V_N^- - x\dot{\theta}^-\right)}{k_c^2 + x(x - s\mu y)}, \hspace{1cm} (3.11)$$

$$\dot{\theta}^+ = \dot{\theta}^- + \frac{(x + s\mu y)(1 + e) \left(V_N^- - x\dot{\theta}^-\right)}{k_c^2 + x(x - s\mu y)}. \hspace{1cm} (3.12)$$

We note that from the definition of $y$ (see Figure 3.1) it is clear that $y$ can only take positive values, however, $x$ can be both positive and negative depending on whether the centre of mass of the rigid body is leading or trailing the contact point at impact.
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We also notice that the impact mapping has a singularity when

\[ \mu = \mu_s := \frac{k_r^2 + x^2}{s xy}. \]  

(3.13)

The significance of \( \mu_s \) will be discussed in what follows and will also be examined in Chapter 6 for one of the model examples presented there.

Bounds on the Impulse Ratio

The impact law given by the mappings (3.10)-(3.12) is relatively easy to implement in a numerical simulation scheme, but as it stands it is well known to give inconsistent results [45] where certain choices of the impulse ratio \( \mu \) lead to unphysical results by introducing energy into the systems. This motivates the need to investigate under what conditions \( \mu \) can be viewed as the conventional coefficient of friction \( \mu_f \) and what the choices are in circumstances where it is no longer valid. If it is necessary to impose a value for \( \mu \) different to \( \mu_f \), then in such an instance, the impact law may not correctly model the impact in question. Here we propose to impose three bounds on \( \mu \) depending on kinematic and energetic restrictions. The analysis follows and extends the approach taken in [9].

The first bound that will be considered is the critical value of \( \mu \) that brings the relative tangential velocity of the contact point to zero at impact (stick). At this point \( \mu \) is large enough to cause a stick motion and any further increase in \( \mu \) would lead to a non-physical velocity mapping (such as velocity reversals) and possibly introduce energy into the system. From (3.5) and enforcing stick after impact, i.e. assuming

\[ V_T^+ + \dot{\theta}^+ y = 0, \]

the bound on \( \mu = \mu_C \) is given by

\[ \mu_C := s \frac{(1 + e) xyV_{NC} - (k_r^2 + x^2) V_T}{(1 + e) (k_r^2 + y^2) V_{NC} + xyV_T}. \]  

(3.14)

This gives the first bound on \( \mu \), i.e. depending on the direction of the tangential contact point velocity prior to impact, the value of the impulse ratio \( \mu \) can not exceed \( \mu_C \).

The second and third bounds on \( \mu \) to be considered ensure that no energy is added to the system at impact through the impact mapping. An intuitive way to
examine the energy in the system is to consider how the kinetic energy loss varies as a function of the impulse ratio $\mu$. Values of $\mu$ for which the kinetic energy loss is positive will show the $\mu$ values that will lead to a physical realistic impact mapping and the regions in which the kinetic energy loss is negative will show the values of $\mu$ that lead to a non-physical mapping. An expression for the kinetic energy loss at impact $T_L(\mu)$ is found by using (3.10)-(3.12) together with the definition of kinetic energy

$$T_L(\mu) = \frac{m}{2} \left( (V_N^-)^2 - (V_N^+)^2 + (V_T^-)^2 - (V_T^+)^2 \right) + \frac{I}{2} \left( (\dot{\theta}^-)^2 - (\dot{\theta}^+)^2 \right). \tag{3.15}$$

Solving $T_L(\mu) = 0$ for $\mu$ gives two solutions (since $T_L(\mu)$ is a ratio of quadratics in $\mu$), $\mu = \mu_{E_1}$ and $\mu = \mu_{E_2}$ (where $\mu_{E_1}$ is the larger of the two roots), that defines an interval on $\mu$ for which which $T_L < 0$ and where energy will be introduced into the system. The bounds on this interval are the following

$$\mu_{E_1}, \mu_{E_2} = \frac{s V_T^- (k_r^2 + x^2) + eyV_N^-}{(e + 1) (k_r^2 + y^2) V_N^- + 2xyV_T^-} \sqrt{\frac{(1 - e^2) k_r^2 (x^2 + y^2 + k_r^2) (V_N^-)^2 + ((x^2 + k_r^2) V_T^- + xyV_N^-)^2}{(e + 1) (k_r^2 + y^2) V_N^- + 2xyV_T^-}}. \tag{3.16}$$

Note, for the special case where $e = 1$, (3.16) reduces to

$$\mu_{E_1}, \mu_{E_2} = \frac{s V_T^- (k_r^2 + x^2) + eyV_N^-}{2 (k_r^2 + y^2) V_N^- + 2xyV_T^-} \pm \frac{s \sqrt{((x^2 + k_r^2) V_T^- + xyV_N^-)^2}}{2 (k_r^2 + y^2) V_N^- + 2xyV_T^-}, \tag{3.17}$$

to give

$$\mu_{E_1} = \frac{s V_T^- (k_r^2 + x^2) + eyV_N^-}{(k_r^2 + y^2) V_N^- + 2xyV_T^-}, \quad \mu_{E_2} = 0 \tag{3.18}$$

which shows that one of the bounds, $\mu_{E_2}$ by convention, will always be 0.

In order to decide which bound to select for a given impact, another critical value for the impulse ratio $\mu$ has to be considered. Consider the value $\mu = \mu_M$ at which the energy loss in the system is a maximum. This value can be found by solving
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\[ \frac{\partial T_L(\mu)}{\partial \mu} = 0 \text{ for } \mu, \text{ which yields} \]

\[ \mu_M = 2s \frac{(x^2 + k_r^2) \left( V_T^- (x^2 + k_r^2) + k_r^2 y \dot{\theta}^- + x y V_N^- \right)}{(e + 1) \left( x^2 + k_r^2 \right) k_r^2 V_{NC}^- + (x^2 + k_r^2) \left( V_N^- y^2 + x y V_T^- \right) + e \left( V_N^- y^2 k_r^2 - y^2 k_r^2 x \dot{\theta}^- \right)}. \]  

(3.19)

The expression for \( \mu_M \) is important when determining the sign assigned to the bound used to select the impulse ratio \( \mu \). This will ensure that the tangential impulse is physical for a given impact.

It is interesting to note that

\[ \mu_C|_{e=0} = \mu_E_1|_{e=1} = \mu_M|_{e=0} = s \frac{(k_r^2 + x^2) V_T^- + x y V_N^-}{(k_r^2 + y^2) V_{NC}^- + x y V_T^-}. \]  

(3.20)

for the non-zero solution in (3.16), which is consistent with Brach’s observation in [9].

The expression for \( \mu_E \) in (3.16) also contains a first order singularity at a value

\[ \frac{V_T^-}{V_{NC}^-} = \frac{(e + 1) (k_r^2 + y^2)}{2x y}. \]

The significance of this is that as this ratio is approached \( \mu_E \) becomes infinite meaning \( T_L \) will never cross through the \( x^- \) axis and thus there will be regions where \( T_L \) is always positive and regions where it is always negative.

**Extending the Brach Impact Law**

Analysis of the impact law given by (3.10)-(3.12) together with the bounds (3.14), (3.16) and (3.19) will lead to a large number of possible cases. All these cases were not analysed or discussed by Brach [9,20,21,30,58,61], where the impact law was originally introduced and analysed, and in particular a full exploration of the parameter space has not been considered. Previous work has only analysed a small range of parameters, which due to the specific application they were designed for, avoid most of the difficulties inherent with this impact law. Very little has been mentioned in the literature about the significance of \( \mu_E_1 \) and \( \mu_E_2 \). In order to make the impact law consistent in each case we will here provide an extension to the original Brach formulation that will be consistent and not lead to impacts that add energy into the system and will enable the simulation of long-term dynamics [45]. First,
examination of (3.16) shows that $\mu_{E_1}$ and $\mu_{E_2}$ will always be real provided $e \in [-1, 1]$. This means that theoretically both values are allowed as no energy is added to the system for these $\mu$ values. However, only one of them can have physical significance for a given impact. We notice that as $\mu \to \pm \infty$, $T_L(\mu)$ (see (3.15)) is bounded and the sign depends on the system and impact parameters. The significance of this is that there may be $\mu$ intervals of infinite length where energy is never introduced (see Figure 3.4(a)) into the system or $\mu$ intervals of infinite length where energy is always introduced (see Figure 3.5(a)). By considering the relationships between $\mu_{E_1}$, $\mu_{E_2}$, $\mu_M$ and the singularity at $\mu_s$ (see (3.13)) we can use the energy curve $T_L(\mu)$ to develop a comprehensive scheme for selecting the $\mu$ value for a given impact. We propose the following alteration to Brach’s scheme described in [21] for selecting a $\mu$ value. We let

$$\mu = \text{sgn}(\mu_M) \min(\mu_f, |\mu_C|, |\mu_E|),$$

(3.21)

where

$$\mu_E = \begin{cases} 
\mu_{E_1}, & \text{if } \text{sgn}(\mu_{E_1}) = \text{sgn}(\mu_M) \text{ and } \text{sgn}(\mu_{E_2}) \neq \text{sgn}(\mu_M), \\
\mu_{E_2}, & \text{if } \text{sgn}(\mu_{E_2}) = \text{sgn}(\mu_M) \text{ and } \text{sgn}(\mu_{E_1}) \neq \text{sgn}(\mu_M), \\
\max(|\mu_{E_1}|, |\mu_{E_2}|), & \text{if } \text{sgn}(\mu_{E_1}) = \text{sgn}(\mu_{E_2}) = \text{sgn}(\mu_M) \text{ and } \mu_s \notin (\mu_{E_1}, \mu_{E_2}), \\
\min(|\mu_{E_1}|, |\mu_{E_2}|), & \text{if } \text{sgn}(\mu_{E_1}) = \text{sgn}(\mu_{E_2}) = \text{sgn}(\mu_M) \text{ and } \mu_s \in (\mu_{E_1}, \mu_{E_2}), \\
\emptyset, & \text{if } \text{sgn}(\mu_{E_1}) = \text{sgn}(\mu_{E_2}) \neq \text{sgn}(\mu_M), 
\end{cases}$$

(3.22)

and

$$\mu_C = \begin{cases} 
\mu_C, & \text{sgn}(\mu_C) = \text{sgn}(\mu_M), \\
\emptyset, & \text{sgn}(\mu_C) \neq \text{sgn}(\mu_M). 
\end{cases}$$

(3.23)

Here the notation $\emptyset$ means that no value needs to be considered and thus $\mu_E$ is only considered when $\text{sgn}(\mu_{E_1}) = \text{sgn}(\mu_{E_2}) \neq \text{sgn}(\mu_M)$ and $\mu_C$ is only considered when $\text{sgn}(\mu_C) = \text{sgn}(\mu_M)$. This means that the tangential contact-point velocity can only go to zero from one direction for a given impact, so if the sign of $\mu_M$ does not correspond to this direction then $\mu_C$ does not need to be considered. This case was previously overlooked and is a further alteration to Brach’s original scheme.

In Table 3.1 and in the following text we have listed the eight different cases of possible $\mu$ values that can be chosen. In the table we have also highlighted where the singularity $\mu_s$ is located in relation to $\mu_{E_1}$ and $\mu_{E_2}$. To do this we define $I_E$ to
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<table>
<thead>
<tr>
<th>Case</th>
<th>sgn($\mu_{E_1}$)</th>
<th>sgn($\mu_{E_2}$)</th>
<th>sgn($\mu_M$)</th>
<th>$\mu_s \in I_E$</th>
</tr>
</thead>
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<td>+</td>
<td>+</td>
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<td>+</td>
<td>+</td>
<td>×</td>
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<td>-</td>
<td>-</td>
<td>✓</td>
</tr>
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<td>-</td>
<td>-</td>
<td>×</td>
</tr>
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<td>-</td>
<td>×</td>
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<td>×</td>
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</tr>
<tr>
<td>8</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 3.1: A list of the different cases in which $\mu_E$ can be chosen. The choice depends on sgn($\mu_{E_1}$), sgn($\mu_{E_2}$) and sgn($\mu_M$) and how they are located in relation to $\mu_s$.

be an open interval defined as

$$I_E := (\min(\mu_{E_1}, \mu_{E_2}), \max(\mu_{E_1}, \mu_{E_2}))$$

and we let ✓ indicate that $\mu_s$ lies in the set and × that it does not. The eight different cases can be summarised as follows.

**Cases 1 and 3:** Here $\mu_{E_1}$, $\mu_{E_2}$ and $\mu_M$ have the same sign and $\mu_s \in I_E$. For these cases almost all realistic values for $\mu$ will introduce energy into the system at impact. It is therefore necessary that $\mu_E = \min(|\mu_{E_1}|, |\mu_{E_2}|)$, which for most impacts is approx 0 (see Figure 3.5(a) and (b)).

**Cases 2 and 4:** Here $\mu_{E_1}$, $\mu_{E_2}$ and $\mu_M$ have the same sign but $\mu_s \notin I_E$. In this situation any value for $\mu$ in the interval $I_E$ is allowed, provided $\mu \notin (0, \min(|\mu_{E_1}|, |\mu_{E_2}|))$, and so $\mu_E = \max(|\mu_{E_1}|, |\mu_{E_2}|)$ (this example is not depicted here).

**Cases 5 and 6:** Here $\mu_{E_1}$ and $\mu_{E_2}$ have different sign, and $\mu_E$ is assigned with value as the one of $\mu_{E_1}$ and $\mu_{E_2}$ that has the same sign as $\mu_M$, as this is the solution of physical significance (see Figure 3.3(a) and (b)). This class of impact was considered in [61].

**Cases 7 and 8:** Here $\mu_{E_1}$ and $\mu_{E_2}$ have the same sign but a different sign to $\mu_M$. This can only occur when the singularity $\mu_s \in I_E$. For these cases any value for $\mu$ with the same sign as $\mu_M$ is allowed (see Figure 3.4(a) and (b)).
3.1.3 A Model Example: Slender Rod

In this section we will use a slender rod (see Figure 3.2) as a model example to highlight various aspects of the impact law introduced in Section 3.1.2. We will use this example to describe how the energy is changing in the system at impact for some of the different cases introduced in Section 3.1.2.

![Figure 3.2: Model example of a slender rod in contact with a non-compliant surface.](image)

We assume that one of the tips, \(P_1\) say, of a uniform slender rod is impacting a noncompliant surface. Without loss of generality we assume the distance from the centre of mass to the tip is \(L = 1\) and the mass \(m = 1\) so that the moment of inertia \(I = \frac{1}{3}\) and the radius of gyration \(k_r^2 = \frac{1}{3}\). A very similar setup was also studied in [1] and is analogous to the original Painlevé problem studied in [62].

**Kinetic Energy Loss at Impact**

In Section 3.1.2 we extended the impact law derived in [9] to be consistent with more possible scenarios than what was originally considered. The extension lead to eight different cases for the mapping and these cases depend on how the kinetic energy loss \(T_L(\mu)\) varies as the impulse ratio \(\mu\) varies (see (3.15)). In Figures 3.3-3.5 we have plotted \(\bar{T}_L(\mu)\) (normalised \(T_L\)) for three different values of the restitution coefficient.
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e, namely, \( e = 0 \), \( e = 0.5 \) and \( e = 1 \). The normalisation of \( T_L \) is given by

\[
\bar{T}_L(\mu) = \frac{T_L(\mu)}{\bar{T}_L(\mu)},
\]

where \( \bar{T}_L(\mu) \) is the kinetic energy of the rod at the beginning of the impact phase.

First, in Figures 3.3(a) and (b) we plot \( \bar{T}_L(\mu) \) for a set of parameter values such that Case 6 in Section 3.1.2 is fulfilled. Here this means that \( \mu_E^1 < 0 \) and \( \mu_E^2 > 0 \) and since \( \mu_M > 0 \) the scheme sets \( \mu_E = \mu_E^2 \). The impact is only physical if \( \mu_E^1 \leq \mu \leq \mu_E^2 \), which does not change for different values of \( e \).

![Figure 3.3: Plots of \( \bar{T}_L(\mu) \) for \( e = 0 \), \( e = 0.5 \) and \( e = 1 \), where \( \theta = \tan^{-1}(2) \), \( V_T^- = -2 \), \( V_N^- = -1.5 \) and \( \dot{\theta}^- = 1.9 \). All three curves correspond to Case 6 described in Section 3.1.2. In (a) \( \bar{T}_L(\mu) \) is plotted for \( -9 \leq \mu \leq 9 \) and (b) is a zoom-in of (a) to highlight where \( \mu_E^1 \) and \( \mu_E^2 \) as well as \( \mu_M \) are located.](image)

Second, in Figure 3.4(a) and (b) we plot \( \bar{T}_L(\mu) \) for a set of parameter values such that Case 7 in Section 3.1.2 is fulfilled for \( e = 0.5 \) and \( e = 1 \), and Case 3 is fulfilled for \( e = 0 \). Here this means that for \( e = 0.5 \) and \( e = 1 \) \( \mu_E^1 < 0 \) and \( \mu_E^2 < 0 \) have the same signs and since \( \mu_M > 0 \) the value of \( \mu_E \) does not need to be considered. An impact is always physical in these cases. However for \( e = 0 \) \( \mu_E^1 < 0 \) and \( \mu_E^2 < 0 \) and since \( \mu_M < 0 \) we get here that \( \mu_E = \max(\mu_E^1, \mu_E^2) \). The impact is only physical for \( \mu \leq \min(\mu_E^1, \mu_E^2) \).

Third, in Figure 3.5(a) and (b) we plot \( \bar{T}_L(\mu) \) for a set of parameters such that Case 3 in Section 3.1.2 is fulfilled. Here this means that \( \mu_E^1 < 0 \) and \( \mu_E^2 < 0 \) and since \( \mu_M < 0 \) the value of \( \mu_E \) does not need to be considered. An impact is always physical in this case.
In this Section, we have presented the extended Brach impact Law, necessary for the simulation of long term dynamics of impacting systems with friction. Further, we have used the model example of a slender rod impacting with a noncompliant surface to illustrate how the kinetic energy loss $T_L$ varies as a function of the impulse ratio $\mu$. The choice of initial conditions, which in reality reflects the type of impact in question, gives rise to eight different cases, some of which are very restrictive.
This point is illustrated further in Figure 3.5, where any physical choice of \( \mu \) will lead to energy gains in the system. In Figure 3.3 (a), we see that \( T_L \) approaches the singularity \( \mu_s \). This point will be revisited in Chapter 6, and discussed in terms of the Painlevé paradox. This motivates the need for a more developed impact law which describes the complexity of the impact phase in more detail and is thus guaranteed to not artificially introduce energy into the system. To this purpose, in the next Section we will present the Energetic impact law.

### 3.2 The Energetic Impact Law

In this Section the framework for deriving the equations of motion for a two rigid body collision in the general case is presented. Moreover, this framework is further used to derive one of the impact laws [1] which was used for the work in Chapters 5 and 6.

Consider two planar rigid bodies, \( H \) and \( H' \), whose configuration relative to an inertial reference frame can be described in terms of vectors of generalized coordinates. Further, impose that the bodies at any moment have a finite number of isolated contact points and contact cannot occur at two separate points simultaneously. For this purpose we derive the equations of motion for two separate cases, when the bodies are in free flight, see Figure 3.6 (a), and when they are in contact at an isolated point \( C \), see Figures 3.6 (b) and (c). The corresponding dynamics can be described using a Lagrangian formulation or using a Newtonian formulation (see Sections 2.1.2 and 2.1.1). In Sections 3.2.1 and 3.2.2 we consider a Newtonian formulation of the two bodies.

#### 3.2.1 Free Flight

Consider the two bodies \( H \) and \( H' \) in free flight as shown in Figure 3.6(a). The position and rotation of the centre of mass \( G \) of body \( H \) can be described in the \( X - Y \) plane by the coordinates \( q_X \) and \( q_Y \) and the angle \( \alpha \), and similarly \( q'_{X'} \) and \( q'_{Y'} \) are the coordinates and \( \alpha' \) the rotation of the centre of mass \( G' \) of body \( H' \) (see Figure 3.6(a)). Next we let

\[
\mathbf{r} = (q_X, q_Y, \alpha)^T, \quad \mathbf{r}' = (q'_{X'}, q'_{Y'}, \alpha')^T,
\]

52
be the position, velocity and acceleration vectors of the centre of mass of $H$ and $H'$ respectively. Then using Newton’s second law the equations of motion for the two bodies can now be written as

$$M\ddot{r} = F^T \quad \text{and} \quad M'\ddot{r}' = F'^T,$$

(3.25)

where $M$ and $M'$ are, respectively, the mass matrices for $H$ and $H'$ given by

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} m' & 0 & 0 \\ 0 & m' & 0 \\ 0 & 0 & I' \end{pmatrix}$$

and $F$ and $F'$ are, respectively, the external forces and torques acting on $H$ and $H'$ given by

$$F = (F_X, F_Y, R) \quad \text{and} \quad F' = (F'_X, F'_Y, R').$$
Here $m$ and $m'$ are the masses and $I$ and $I'$ are the moments of inertia of $H$ and $H'$, respectively. Further $F_X, F'_X$ and $F_Y, F'_Y$ represent the force components in the $X-Y$ plane and $R$ and $R'$ are the external torques acting on $H$ and $H'$, respectively, as shown in Figure 3.6(a).

### 3.2.2 Contact

Next we derive the equations of motion for the system when the two objects are in contact, as shown in Figure 3.6 (b) and (c). To do this we define a new coordinate system $n_1-n_2$, rotated an angle $\beta$ about the origin relative to the coordinate system $X-Y$, where $n_1$ is the tangent to the contact plane and $n_2$ is normal to the contact plane (see Figure 3.6(d)). We also define $\theta = \alpha - \beta$ and $\theta' = \alpha' - \beta$ and let

$$q = (q_1, q_2, \theta)^T = (q_X \cos(\beta) - q_Y \sin(\beta), q_X \sin(\beta) + q_Y \cos(\beta), \alpha - \beta)^T,$$

and

$$q' = (q'_1, q'_2, \theta')^T = (q'_X \cos(\beta) - q'_Y \sin(\beta), q'_X \sin(\beta) + q'_Y \cos(\beta), \alpha' - \beta)^T.$$

Further, defining $L$ as the distance from $G$ to $C$ and $L'$ as the distance form $G'$ to $C$, the positions $q_C$ and $q'_C$ of the contact point $C$ relative to both bodies in the $n_1-n_2$ frame can be written as

$$q_C := (q_1 - L \cos(\theta), q_2 - L \sin(\theta))^T$$

and

$$q'_C := (q'_1 - L' \cos(\theta'), q'_2 - L' \sin(\theta'))^T.$$

Note that we do not deal with the problem of finding the contact points in the general case but assume that there are well-defined contact points on each object. Let $d = q_C - q'_C$ be the relative distance between the contact points of the two bodies. Then the unilateral constraint between the two bodies is $d = (0, 0)^T$ or equivalently when $q_C = q'_C$.

This is a useful framework to work with, particularly when deriving an impact mapping. It is also straightforward to translate back positions and angles to the original $X-Y$ coordinate system. The derivation of the impact mapping introduced in [1] splits contact forces into tangential and normal components in relation to the
contact plane, i.e. using the $n_1 - n_2$ coordinate system. In order to relate how the contact force $F_c$ will affect the centre of mass in terms of translations and rotations we need to consider

$$\frac{\partial q_C}{\partial q} = \begin{pmatrix} 1 & 0 & L \sin(\theta) \\ 0 & 1 & -L \cos(\theta) \end{pmatrix} \quad \text{and} \quad \frac{\partial q_C}{\partial q'} = \begin{pmatrix} 1 & 0 & L' \sin(\theta') \\ 0 & 1 & -L' \cos(\theta') \end{pmatrix}$$

so that we can define

$$F_c = \left( \lambda \frac{\partial q_C}{\partial q} \right)^T \lambda^T$$

and

$$F_c' = \left( \lambda' \frac{\partial q_C}{\partial q'} \right)^T \lambda'^T,$$

where we define $\lambda$ and $\lambda'$ as the forces generated at impact of each body given by

$$\lambda = (\lambda_1, \lambda_2) \quad \text{and} \quad \lambda' = (-\lambda_1, -\lambda_2).$$

Now the equations of motion for $H$ and $H'$ during contact can respectively be written as

$$M \ddot{q} = \hat{F}_c^T + \left( \frac{\partial q_C}{\partial q} \right)^T \lambda^T \tag{3.28}$$

and

$$M' \ddot{q}' = \hat{F}_c'^T + \left( \frac{\partial q_C}{\partial q'} \right)^T \lambda'^T, \tag{3.29}$$

where $\hat{F}_c$ and $\hat{F}_c'$ are, respectively, the external forces and torques acting on $H$ and $H'$ given by

$$\hat{F}_c = (F_1, F_2, R) = (F_X \cos(\beta) - F_Y \sin(\beta), F_X \sin(\beta) + F_Y \cos(\beta), R),$$

and

$$\hat{F}_c' = (F'_1, F'_2, R') = (F'_X \cos(\beta) - F'_Y \sin(\beta), F'_X \sin(\beta) + F'_Y \cos(\beta), R'),$$

where the subscripts 1 and 2 respectively represent the components of the external forces acting in the tangential and normal direction, and $R$ and $R'$ are as above the external torques acting on the bodies as shown in Figure 3.6 (c).

This setup is general and does not specify the mechanism that generates the
tangential force $\lambda_1$. For this work however we assume that any tangential force arises due to friction at the contact point of the colliding bodies and here we use the Amontons-Coulomb friction law

$$\lambda_1 = \pm \mu \lambda_2$$

(3.30)

for some non-negative constant $\mu$ representing a coefficient of friction.

The energetic restitution coefficient allows for the various stick-slip processes that can occur throughout the impact phase and thus is a restitution coefficient that will not violate energy conservation, as discussed in Section 2.4.1. It is notable that the kinematic and Newtonian coefficients do not allow for situations where the direction of slip can vary throughout the impact phase, and the consequence of this is that the final impulse is not calculated correctly. Further, the energetic restitution coefficient forms the basis for the impact mapping derived in [1], a brief description of which will be given below. When we refer to the impact phase we are considering velocity changes which occur as a function of normal impulse. This impulse formulation is a natural framework to use given that we assume the impact is of infinitesimal contact duration. As mentioned above, this energetic coefficient depends on the work done by the normal component of the contact force during the impact phase. In order to correctly calculate the work done for each separate segment of slip Stronge [4] makes use of the following definition

**Definition 1** The partial work $W_{\tilde{n}}$ done on colliding bodies by the component of reaction impulse in direction $\tilde{n}_i$ during any period of unidirectional slip $\Delta t = t_2 - t_1$ equals the scalar product of this component of reaction impulse $\Delta P_{\tilde{n}}$ with half the sum of the components in direction $\tilde{n}_i$ of the initial and final relative velocities across the contact point

$$W_{\tilde{n}} = \Delta P_{\tilde{n}} \left( \dot{q}_{\tilde{n}}(t_2) + \dot{q}_{\tilde{n}}(t_1) \right),$$

$$\Delta P_{\tilde{n}} = \tilde{n}_i (P_i(t_2) - P_i(t_1)),$$

with $\dot{q}_{\tilde{n}}(t) = \tilde{n}_i \dot{q}_i(t)$ and $\tilde{n}_i$ a unit vector such that $\tilde{n}_i \cdot \tilde{n}_i = 1$.

Definition 1 is necessary for the derivation of the Energetic impact law as it allows for the calculation of the work, during the impact phase, to be split into the separate calculations for each of the various stick-slip segments.
In order to map pre-impact velocities to post-impact ones it is necessary to consider the terminal impulse $P_f$ for the given collision. Incorporating Amontons-Coulomb friction law (3.30) and the energetic restitution coefficient (2.37) allows for a variety of stick-slip processes, each of which need to be considered and the corresponding $P_f$ in each case determined. We will consider the equations of motion for a planar two-body collision. It is necessary to consider velocity changes as a function of normal impulse $P$ instead of the time variable $t$. Consider (3.28) translated to the contact point $q_C$ so that

$$\frac{dq_C}{dt} = \frac{\partial q_C}{\partial q} \ddot{q} = \frac{\partial q_C}{\partial q} M^{-1} \left( \frac{\partial q_C}{\partial q} \right)^T \lambda^T + \frac{\partial q_C}{\partial q} M^{-1} F^T$$

$$= w^{-1} \lambda^T + f(F_1, F_2, R, q, \dot{q})$$ (3.31)

and

$$\frac{dq_C}{dt} = \frac{\partial q_C}{\partial q'} \ddot{q'} = \frac{\partial q_C}{\partial q'} (M')^{-1} \left( \frac{\partial q_C}{\partial q'} \right)^T \lambda'^T + \frac{\partial q_C}{\partial q'} (M')^{-1} F'^T$$

$$= (w')^{-1} \lambda'^T + f'(F'_1, F'_2, R', q', \dot{q'})$$, (3.32)

which is the rate of change of the contact point velocities as a function of time and where $w^{-1}$ and $(w')^{-1}$ are the symmetric matrices given by

$$w^{-1} = \frac{\partial q_C}{\partial q} M^{-1} \left( \frac{\partial q_C}{\partial q} \right)^T = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$,

$$(w')^{-1} = \frac{\partial q_C}{\partial q'} (M')^{-1} \left( \frac{\partial q_C}{\partial q'} \right)^T = \begin{pmatrix} A' & B' \\ B' & C' \end{pmatrix}$$,

where

$$A = \frac{1}{m} + \frac{L^2 \sin^2(\theta)}{I}, \quad B = -\frac{\sin(\theta) \cos(\theta)}{I}, \quad C = \frac{1}{m} + \frac{L^2 \cos^2(\theta)}{I}$$

$$A' = \frac{1}{m'} + \frac{L'^2 \sin^2(\theta')}{I'}, \quad B' = -\frac{\sin(\theta') \cos(\theta')}{I'}, \quad C' = \frac{1}{m'} + \frac{L'^2 \cos^2(\theta')}{I'}$$.

In this context we do not need to consider terms which do not change throughout the impact phase as they are negligible in comparison to the contact forces and therefore we can neglect $f$ and $f'$. The analysis used in the construction of this
impact law involves considering changes over an infinitesimal time interval $\tilde{t} = t + \epsilon t$. It is therefore reasonable to assume $f$ and $f'$ do not change over the interval $\tilde{t}$ which gives

$$\frac{dq_C}{dt} = w^{-1} \chi^T,$$

(3.33)

and

$$\frac{dq_C}{dt} = (w')^{-1} \chi'^T.$$

(3.34)

Further, during the impact phase we also have that

$$\frac{dP_2}{dt} = \lambda_2$$

(3.35)

for $H$ and by Newton’s third law

$$\frac{d(-P_2)}{dt} = -\lambda_2$$

(3.36)

for $H'$, since the normal impulse is a uniformly increasing scalar function during contact. Now, (3.33), (3.34), (3.35) and (3.36) allow us to replace the independent variable $\tilde{t}$ with $P$ in (3.35) and (3.36) to give

$$\frac{dq_C}{dP_2} = \frac{dq_C}{dP_2} \frac{dP_2}{dt} = \frac{dq_C}{dP_2} \lambda_2 = w^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

(3.37)

$$\Rightarrow \frac{dq_C}{dP_2} = \frac{1}{\lambda_2} w^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

(3.38)

which is the rate of change of contact-point velocities with respect to $H$ as a function of normal impulse. In a similar way we also have

$$\frac{dq_C'}{dt} = \frac{dq_C'}{dP_2} \frac{dP_2}{dt} = \frac{dq_C'}{dP_2} \lambda_2 = (w')^{-1} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix} \Rightarrow \frac{dq_C'}{dP_2} = \frac{1}{\lambda_2} (w')^{-1} \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix}.$$  

(3.39)

Subtracting (3.39) from (3.38) yields

$$\frac{dq_C}{dP_2} := \frac{dq_C}{dP_2} - \frac{dq_C'}{dP_2} = \frac{1}{\lambda_2} (w^{-1} + (w')^{-1}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

(3.40)
or in the notation of [1]

\[
\frac{d\hat{q}_C}{dP_2} = \frac{1}{\lambda_2} \begin{pmatrix} \hat{A} & -\hat{B} \\ -\hat{B} & \hat{C} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},
\]

(3.41)

which is the relative change in contact point velocities as a function of normal impulse. Expanding (3.41) and isolating the tangential and normal components, respectively, gives

\[
\frac{d\hat{q}_{1C}}{dP_2} = \hat{A} \frac{\lambda_1}{\lambda_2} - \hat{B}, \quad \frac{d\hat{q}_{2C}}{dP_2} = -\hat{B} \frac{\lambda_1}{\lambda_2} + \hat{C},
\]

(3.42)

which will be used to define the rate constants used for the impact mappings presented below.

Stronge [4] describes four possible impact-phase processes and calculates the terminal impulse and the post-impact velocity components for each phase. The four phases are:

**Unidirectional slip during contact.** In this case slip does not cease throughout the impact phase, and the tangential forcing acts in a direction opposite to the motion of the body. For this case we have the condition that the normal component of the relative velocity goes to zero at the end of the compression phase.

**Slip reversal during compression.** In this situation initial sliding is brought to rest and then reverses direction. Again we have the condition that the normal component of the relative velocity goes to zero at the end of the compression phase.

**Slip-stick transition during compression.** The case whereby initial sliding is brought to rest which means that the contact point sticks if the friction coefficient \( \mu \) is sufficiently large or undergoes reverse slip if not. It is also required that the initial sliding velocity is sufficiently small, otherwise this motion can not occur. Here it is necessary that the normal impulse is such as to bring slip to a halt.

**Jam.** This is the process whereby there is an increase in normal acceleration at the contact point due to a large rotational acceleration. This motion occurs during an initial period of sliding and is generated mainly due to a large frictional force. The occurrence of jam is highly dependent on the system configuration.
at contact. It is necessary to have an orientation whereby the centre of mass is at a sufficiently small negative inclination angle relative to the common normal.

As with the previous phase, it is also a requirement that the normal impulse is such as to bring slip to a halt.

Nordmark et al. [1] extend this theory by describing 10 different impact regions from which an impact law consisting of three mappings is derived. The impact mappings in [1] map the relative tangential and normal contact point velocities before the impact phase \( \tilde{q}_C^- \) to the post-impact phase velocities \( \tilde{q}_C^+ \). Using (3.42) and following [1] we define the rate constants \( k_T^+ \), \( k_T^- \), \( k_N^+ \), \( k_N^- \), \( k_T^0 \) and \( k_N^0 \). These rate constants describe how stick and positive and negative slip can occur throughout the compression and restitution phase. For the various stick-slip processes described above Nordmark et al. [1] define the rate constants

\[
\begin{align*}
  k_T^+ &= -\hat{B} - \mu \hat{A}, & k_N^+ &= \hat{C} + \mu \hat{B}, \\
  k_T^- &= -\hat{B} + \mu \hat{A}, & k_N^- &= \hat{C} - \mu \hat{B}, \\
  k_T^0 &= 0, & k_N^0 &= \frac{\hat{A} \hat{C} - \hat{B}^2}{\hat{A}},
\end{align*}
\]

from which

\[
k_T = \begin{cases} 
  k_T^+, & \text{in positive slip}, \\
  k_T^-, & \text{in negative slip}, \\
  k_T^0, & \text{in stick},
\end{cases}
\]

(3.43)

\[
k_N = \begin{cases} 
  k_N^+, & \text{in positive slip}, \\
  k_N^-, & \text{in negative slip}, \\
  k_N^0, & \text{in stick},
\end{cases}
\]

(3.44)

can be determined, where \( \mu = \frac{\lambda_1}{\lambda_2} \). It is worth mentioning that \( \mu \) is taken as an absolute here and the rate constants described above consider all cases of positive and negative slip so it is not necessary to assign a sign to \( \mu \). Nordmark et al. [1] also defines the constants \( k'_T \) and \( k'_N \), which are assigned one of the values of \( k_T^+ \), \( k_T^- \), \( k_N^+ \), \( k_N^- \) and \( k_T^0 \), and \( k_N^0 \), and determined by the system parameters and pre-collision conditions, but for full details we refer to [1]. From this and using (2.37) the following three maps, for pre-impact to post-impact contact point velocities \( \tilde{q}_C^- \rightarrow \tilde{q}_C^+ \) can be derived:
Figure 3.7: Decision tree to determine what region a given impact corresponds to. Each of the regions 1 - 10 is determined by the pre impact parameters and rate constants. In the decision tree, if a statement is true the arrow to the right is followed. If a statement is false, the arrow to the left is followed [1].

Table 3.2: Values for each of the rate constants and the corresponding impact mapping for each of the ten regions.

Map I:

\[
\dot{q}_{1C}^{+} = \dot{q}_{1C} - (1 + e_{*}) \frac{k_{T}}{k_{N}} \dot{q}_{2C}^{+} \quad (3.45)
\]

\[
\dot{q}_{2C}^{+} = -e_{*} \dot{q}_{2C} \quad (3.46)
\]
Map II:

\[
\dot{q}_{1C}^+ = \frac{k'_T}{k'_N} \left( \frac{k_N}{k_T} \dot{q}_{1C} - \dot{q}_{2C} \right) \\
+ \frac{k'_T}{k'_N} \left( \sqrt{\left( 1 - \frac{k'_N}{k_N} \right) \left( \frac{k_N}{k_T} \dot{q}_{1C} - \dot{q}_{2C} \right)^2 + e^2 \frac{k'_N}{k_N} \left( \dot{q}_{2C} \right)^2} \right) \\
\dot{q}_{2C}^+ = \sqrt{\left( 1 - \frac{k'_N}{k_N} \right) \left( \frac{k_N}{k_T} \dot{q}_{1C} - \dot{q}_{2C} \right)^2 + e^2 \frac{k'_N}{k_N} \left( \dot{q}_{2C} \right)^2} 
\]

(3.47)

Map III:

\[
\dot{q}_{1C}^+ = \frac{k'_T}{k'_N} \left( \frac{k_N}{k_T} \dot{q}_{1C} - \dot{q}_{2C} \right) \\
+ \frac{k'_T}{k'_N} e_* \left( \sqrt{\left( 1 - \frac{k'_N}{k_N} \right) \left( \frac{k_N}{k_T} \dot{q}_{1C} - \dot{q}_{2C} \right)^2 + e^2 \frac{k'_N}{k_N} \left( \dot{q}_{2C} \right)^2} \right) \\
\dot{q}_{2C}^+ = e_* \sqrt{\left( 1 - \frac{k'_N}{k_N} \right) \left( \frac{k_N}{k_T} \dot{q}_{1C} - \dot{q}_{2C} \right)^2 + \frac{k'_N}{k_N} \left( \dot{q}_{2C} \right)^2} 
\]

(3.49)

(3.50)

for \( k_N \neq 0 \) and

\[
\dot{q}_{1C}^+ = 0 \\
\dot{q}_{2C}^+ = e_* \sqrt{\left( \dot{q}_{2C} \right)^2 + \frac{2k'_N \dot{q}_{1C}}{k_T}} 
\]

(3.51)

(3.52)

for \( k_N = 0 \).

As mentioned above, the different combinations of segments of stick and relative slip can be described by ten different regions, each of which corresponds to one of the three maps given above. The region and the choice of \( k_T, k_N, k'_T \) and \( k'_N \), are all determined by the system parameters and pre-collision conditions, as detailed in Figure 3.7 and shown in Table 3.2.

For full details we refer to [1]. It is worth noting that the authors [63] use an equivalent energetic coefficient of restitution together with the Amontons-Coulomb friction law to describe the contact and impact dynamics for the case of a bouncing
3.3 Comparing the Two Impact Laws

In this Section we will compare the Brach impact law described in Section 3.1 with the Energetic impact law described in Section 3.2. We will compare them in terms of a projected mapping of the tangential contact-point velocity. For this purpose we will again consider the model example of a slender rod impacting a non-compliant surface introduced in Section 3.1.3.

In Figures 3.8, 3.10 and 3.11 we have varied the tangential pre-impact contact velocity $V_{TC}^-$ in a number of different cases for both the extended Brach (Section 3.1.2) and Energetic impact laws and plotted this against tangential post-impact velocity $V_{TC}^+$. This can be seen as a projection of the full three-dimensional mapping. We also compared the two velocity mappings, which will highlight some unwanted behaviour in the more basic Brach impact law.

In Figure 3.8(a) and (b) we have plotted $V_{TC}^-$ versus $V_{TC}^+$ at impact using the Brach impact law and the Energetic impact law, respectively. The two figures show that the velocity mappings given by the two impact laws are not the same but sometimes they display the same output. For instance, when $\mu = \mu_f$ or $\mu = \mu_C$ in Figure 3.8(a) the same results as maps I and II or III, respectively, in Figure 3.8(b) are achieved. To show this, the two velocity mappings are superimposed in Figure 3.8(c), where solid curves show where the two velocity mappings give the same output. We also note that the point of transition to and from stick motion ($V_{TC}^+ = 0$) at $V_{TC}^- = -0.125$ is the same for both velocity mappings. In Figure 3.8(a) this corresponds to a transition between $\mu_f$ and $\mu_C$ and in Figure 3.8(b) this corresponds to the transition between map I and map II. There is a similar transition to and from stick at $V_{TC}^- = 0.012$, which corresponds to a transition between $\mu_f$ and $\mu_C$ in Figure 3.8(a) and a transition between map III and map I in Figure 3.8(b). Note that during stick in Figure 3.8(b) there is a transition between map II, map III and map III, all of which permit a stick regime. In Figure 3.8(d) we have also plotted the actual value of $\mu$ used in the impact law in Figure 3.8(a). It shows the discontinuities in $\mu$ that the Brach approach introduces, which can cause a lot of unexpected and unwanted behaviours. Furthermore, Figures 3.9(a) and (b) show how the tangential centre of mass velocities $V_T^-$ and $V_T^+$ vary for the two cases shown.
in Figures 3.8(a) and (b). It is clear that both mappings have discontinuities even if the ones in the Brach approach are more severe.

Figures 3.10(a) and (b) also display \(V^-_{TC}\) versus \(V^+_{TC}\) at an impact using the Brach mapping and the energetic mapping, respectively. The two mappings show the same behaviour when \(\mu = \mu_f\) in the Brach mapping and map I for the energetic mapping. The transition to and from stick also occurs at the same value, \(V^-_{TC} = -0.0713\), for the two mappings. However, in the energetic mapping in Figure 3.10(b) the stick regime remains for increasing \(V^-_{TC}\), whereas in the Brach mapping in Figure 3.10(a)
the sticking ends as $\mu = \mu_E$ and a non-zero tangential contact-point velocity ensues.

Figure 3.9: Plots of $V_T^-$ versus $V_T^+$ (centre of mass) corresponding to the two $V_{TC}^-$ versus $V_{TC}^+$ plots in Figure 3.8(a) and (b).

In Figure 3.11(a) and (b) we have also plotted $V_{TC}^-$ versus $V_{TC}^+$ at an impact using the Brach mapping and the energetic mapping, respectively. The two mappings appear to display very similar behaviour, however, key differences exist. For instance, in Figure 3.11(a) the transition between $\mu = \mu_f$ and $\mu = \mu_C$ occurs at $V_{TC}^- = 0$, whereas in Figure 3.11(b) there is a transition between map I and map III at the same point, however this does not correspond to a transition to and from stick. Although it would appear from Figure 3.11(b) that $V_{TC}^+ = 0$, it never actually reaches zero. Another discrepancy in Figure 3.11(a) is that the stick regime is briefly interrupted in a transition between $\mu = \mu_C$, $\mu = \mu_f$ and $\mu = \mu_C$. There is no correspondence to this behaviour in Figure 3.11(b).

It is worth mentioning that in Figure 3.8(a) it is observed that at $V_{TC}^- = 0.6421$ $\mu_f \rightarrow -\mu_E$. This transition is of particular interest as $|\mu_E| \approx 0$. This is also observed in Figure 3.9(a). In terms of the scheme (3.21) this represents the case $\text{sgn}(\mu_{E1}) = \text{sgn}(\mu_{E2}) = \text{sgn}(\mu_M), \mu_s \in (\mu_{E1}, \mu_{E2})$, meaning that almost all initial conditions will lead to energy gains in the system, unless this restrictive bound is imposed.
CHAPTER 3. COMPARISON OF TWO IMPACT LAWS

3.4 Discussion

In this Chapter we have introduced and compared a basic (Brach) and a more recent and advanced (Energetic) impact law for rigid body impacts with friction. The former is based on the impulse ratio $\mu$ and the latter on the fact that the contact point is allowed to slip and stick during the impact process.

The Brach scheme that was originally developed in [30] was shown to be incomplete for the purpose of simulation of multiple impacts as it did not cover all the different possible situations that can be encountered. This is of importance if the scheme is to be implemented in a computer code that uses impact mappings for resetting initial conditions of the numerical solver once an impact has occurred. This
Figure 3.11: Plots of $V_{TC}^-$ versus $V_{TC}^+$ for $\theta^- = -0.8$, $V_N^- = -1$, $\theta = 0.2$, $e = 0.6$ and $\mu_f = 0.5$ using (a) the Brach impact law, (b) the Energetic impact law and (c) the two impact laws in (a) and (b) superimposed. In (c) the solid curves correspond to intervals where the two velocity mappings are equal.

aspect will be discussed in detail in Chapters 4 and 5. Therefore, in order to be able to comprehensively compare the two impact laws, the scheme was extended in Section 3.1.2 to account for all situations and thus made more comprehensive. For this purpose, in Section 3.1.3 we presented a novel analysis of the energy gains and losses when varying the impulse ratio $\mu$. This analysis also formed the basis of the extended scheme for selecting a $\mu$-value to ensure physically sound impacts that do not increase the total kinetic energy in the system.

An Energetic impact law was presented in Section 3.2 to enable a comparison with the Brach impact law. It is well known that too basic impact laws have many shortcomings, mainly due to the inherent approximations in the impact-law formula-
CHAPTER 3. COMPARISON OF TWO IMPACT LAWS

tion itself. This is clearly seen when comparing the tangential post-impact velocities in two impact mappings in Section 3.3. However, for certain parameter regimes and rigid-body orientations similarities do exist, in particular in the transitions to and from stick motion. These bounds however can be incredibly restrictive and do not always make physical sense. For certain regions of parameter space, for example, it is necessary to impose a $\mu_E$ bound that is $\approx 0$. This situation is unphysical and does not reflect any realistic impact configuration. For these situations it is therefore necessary to use a more sophisticated impact law, like the Energetic impact law, to correctly model the impact process.

The main mechanism behind the energy gains and inconsistencies when using a more basic impact law, is that these formulations do not take into account the various stick-slip processes that occur on the impulse level throughout the duration of the impact phase. The Brach impact law, and other similar formulations, give the same velocity mappings as the more advanced Energetic impact law, only for region 1 and region 2 impacts, or when stick motion occurs. The reason for this is that a region 1 or region 2 impact, predicted by the Energetic impact law, does not have multiple segments of relative stick-slip motion and thus the velocity mappings are equivalent for both impact laws. For regions 3-10, different phases of stick-slip motion occur throughout the impact phase which can not be predicted by the Brach impact law. The consequence of this is that unrealistic velocity mappings manifest and, in many cases, energy gains ensue. The Brach impact law is able to predict the transition to stick motion, consistent with the Energetic impact law, as it is the same kinematic constraint ($V_{TC}^+ = 0$) that leads to the transition. However, the velocity mappings predicted by both impact are only equivalent for certain regions in parameter space. The main conclusion is that if one decides to use a basic impact law, like the Brach impact law, it is important to first explore the type of impacts that can occur (see the different cases introduced in Section 3.1.2) and second if the output makes physical sense.
Chapter 4

Numerical Methods

In this Chapter we will give a brief overview of the main numerical methods used for the analysis of rigid-body systems with impacts and friction. Further, we will illustrate some of the specific techniques employed throughout this thesis. This Chapter provides the background for some of the numerical methods employed in Chapters 3 and Chapter 6 but particularly for the work in Chapter 5. We will also present a simple example to give an understanding of some of the numerical techniques.

4.1 An Overview of ODEs for Numerical Simulation

The dynamics of rigid-body systems with impacts and friction is usually determined using numerical integration of systems of ordinary differential equations (ODEs) that describe the mechanical system under question. There are two main schemes for how this is usually done, namely, time-stepping and event-driven schemes [64]. How to choose one over the other depends on the class or type of mechanical system that is being analysed, how the impact law is resolved and the type of numerical analysis one would like to perform.

Time-stepping schemes (or event-capturing schemes) consist of a time discretisation of both the differential and algebraic dynamics in which each time step is advanced from step \( n \) to \( n + 1 \) by solving an appropriate complementarity problem, quadratic problem or a projection algorithm. The differential dynamics usually correspond to the equations of motion of the system and the algebraic dynamics typically
describe the impact law of choice. As discussed in Section 2.1, the equations of motion typically correspond to a system of second order ODE’s which can then be recast as a system of first order ODE’s and solved numerically. These complementarity problems are one-step nonsmooth problems which can be written in integral form which are then solved using a technique that depends on the choice of integral approximation used [8,35,64]. A common complementarity formulation is the **Linear Complementarity Problems** (LCP). The LCP is a well known formulation in mathematical programming theory. In an LCP($M,q$) there is no user defined function to be optimised, moreover the problem is to find vectors $w$ and $v$ such that

$$w = (w_1 \ldots w_n)^T, \quad v = (v_1 \ldots v_n)^T$$

subject to the constraints

$$w - Mv = q, \quad w \geq 0, \quad v \geq 0, \quad w_iv_i = 0, \quad M \in \mathbb{R}^{2\times2}, \quad q \in \mathbb{R} \quad \forall i.$$

For the case of rigid-body impact with Amontons-Coulomb friction, an LCP can be constructed for the various modes of stick and slip associated with Amontons-Coulomb friction [42]. In these schemes the moment of each collision or when changes in relative velocity between bodies occurs is not exactly located but instead some level of penetration can occur. This is the price to pay for using rigidly formulated time-stepping methods. These schemes are however very advantageous for the simulation of systems with a large number of degrees of freedom with multiple contacts, for example flows in a granular material or masonry structures [64].

Event-driven schemes (event-tracking) are also basically time-stepping schemes but the time for which a trajectory reaches a constraint or **discontinuity surface** is located as precisely as possible to avoid penetration and to preserve order. Between events (zero crossings of user defined functions or instances of non differentiability), the trajectories are smooth enough (differentiable everywhere) so that any standard high-order scheme is sufficient to use. Higher order schemes are numerical schemes, with sufficient accuracy, used to find an approximate solution to ordinary differential equations [65]. When we refer to standard higher order schemes we are typically referencing Runge-Kutta methods which will be discussed further in Section 4.2. This class of higher order scheme can in turn be divided into two separate categories, the **complementarity methods** and the **hybrid** methods. As the name suggests, in the
CHAPTER 4. NUMERICAL METHODS

complementarity method a complementarity problem is solved as an event is located, whereafter the standard time-stepping scheme continues as discussed above [8, 64].

In the hybrid methods, which are the focus of the numerics used in Chapters 3, 5 and 6, the integration is terminated when an event is located and a discrete map is applied to describe how the state changes at the event. When the map is applied the time-stepping scheme is restarted with the post-event states as the initial conditions with a new set of ODEs depending on the current state of the system. This technique can be best illustrated by means of a switching diagram, as seen in [37] and used in Chapter 5. A switching diagram consists of nodes that correspond to a specific state of the system and arrows showing the possible transitions between them. Hybrid methods have some obvious drawbacks but also some very important advantages that are used for the work in Chapter 5 for the computation of brute force bifurcation diagrams for the case of a slender rod impacting a noncompliant surface with friction. The main complication of hybrid methods is that since each event has to be identified and resolved individually the complexity of all different combinations of events and ODEs grows very quickly with the number of possible events. Another complication is that for each event a mapping has to be found that reflects, in the case of the work in this thesis, what the impact law dictates.

These issues make hybrid schemes only feasible for systems with relative few different discontinuity surfaces and with a small number of degrees of freedom but with multiple spatially and temporally separated contact points. [27, 66, 67]. However, on the positive side it is worth raising at least four different points. First, since the events are not included in the time stepping only ODEs with smooth dynamics have to be integrated and it is thus possible to use a suitable high-order integrator with well-know convergence properties so that trajectories can be found with high accuracy [64]. Second, since no events will be lost during simulation hybrid methods are useful for brute-force bifurcation analysis and in particular when discontinuity-induced bifurcations (DIBs) are involved [1, 25]. Third, hybrid methods make stability analysis of periodic orbits relatively straightforward since it is possible to calculate saltation matrices that "glue" fundamental solution matrices together for trajectories passing through regions between different events [37]. Fourth, there are methods born out of hybrid schemes for impacting systems that can deal with the accumulation of impacts, sometimes referred to as chatter or Zeno behaviour, and also calculate the corresponding saltation matrices [37].
4.2 Numerical Simulation

It is rare that nonlinear ordinary differential equations have analytic solutions which can be expressed in closed form. It is therefore necessary to find approximate solutions by using numerical techniques [64]. Due to advances in numerical analysis it is now possible to find approximate solutions to a high degree of accuracy at relatively low computational cost. The numerical schemes in question range from the simple methods like Euler’s method to the more advanced Runge–Kutta methods [65]. The technique chosen depends on the type of dynamics and modes of motion present in the system. It is therefore necessary to describe these motions in context with the work carried out throughout this thesis.

As the type of mechanical systems considered for the work in Chapters 3 and 6 are unconstrained, various possible modes of sustained motion are possible. To highlight this in Figure 4.1 we consider a schematic of the time history of the position of a point $p(t)$ on a rigid body that occasionally acts as a contact point during impacts with a non-compliant surface. We define three modes of motion, namely unconstrained free flight, chatter and constrained stick, all of which will be discussed throughout this Section. At an impact, after free flight, a system can evolve in a number of

![Figure 4.1: A schematic detailing the time history of a contact point $p(t)$ in an impacting system.](image-url)
different ways. It can continue in free flight motion, go into a stick regime or go through a chatter sequence. A feature which is also present in the schematic Figure 4.1 is an External impact, where another point of the rigid-body system impacts and thus causes a change in the dynamics of the point $p(t)$. For analysis purposes it is important to be able to distinguish between the different features and subtle changes in order to understand the mechanisms that cause them. There are typically also a number of system-specific long-term dynamical behaviours present. In Chapter 5 we discuss such examples for the case of a slender rod impacting an oscillating surface.

4.2.1 ode45

The numerical analysis of our systems under consideration is carried out primarily using MATLAB, specifically the ODE45 differential equation solver. This routine uses the variable step Runge-Kutta Method formula, the Dormand-Prince pair, to solve ODE’s numerically [68]. As input the solver requires a time span vector $t_{\text{span}} = [t_{\text{start}}, t_{\text{final}}]$ defining the interval of integration, and a vector of initial conditions. It is also necessary to rewrite the system of second order ODES as a system of first order ODEs [64].

For our purposes as input from the user the ODE solver requires at least two vector fields (one for the free-flight phase and one for stick, but see further Section 5.2 for a specific implementation), simulation times, initial conditions, error tolerances and integration step sizes. Finally, an impact law, like the ones described in Chapter 3 has to be defined together with a process that determines what impact law to use for the specific impact. To numerically implement an impact law in MATLAB varies vastly in difficulty depending on the impact law in question. In Chapter 5 we will describe the numerical implementation of the Energetic impact law [1] presented in detail in Chapter 3. In Appendix A we present a MATLAB script which describes the numerical implementation of the model system examined in Section 5.2.

4.2.2 Event Detection

When solving a system of ODE’s it is possible to detect specific events by locating crossings to, from, or through zeros of user-defined functions. Event detection is of particular importance for this work for the location of impacts. In general when dealing with rigid bodies it is more than likely that the overall system can have mul-
tiple contact points and a large number of events can occur (impacts or vector-field transitions). These two factors can generally give rise to a number of computational complications, making the analysis of long-term dynamics difficult [67].

In order to accurately locate the event surfaces, event functions need to be described that are derived from the geometries of the impacting rigid bodies. To increase the event detection accuracy it is also advantageous to detect zero crossings of other state variables of the system. For example, for most mechanical systems it is usual to detect crossings of relative position, for example when a point on a mechanical system makes contact with a constraint surface. However, to increase accuracy, zero crossings of the the contact point relative normal velocity can also be detected. When an accumulation of events over a short period of time occurs it is very possible that events can be missed. Detecting velocity crossings as well as position crossings increases the event detection accuracy and in most cases ensures events will not be missed. The methodology described here can equally well be implemented in any environment that has an ODE solver and event location capabilities.

4.2.3 Transitions

One of the many difficulties associated with using an event-driven strategy for finding the solution to a dynamical system with discontinuities is the accumulation of events, for instance when an incomplete or a complete chattering sequence is encountered (see Section 4.2 and [69]). To deal with this in a systematic manner it is advantageous to introduce the notion of system states in a similar way as was done in [37,67]. For a general system we can define $n$ discrete states $S_i$ in which one of the defined vector fields $\Phi_i$ is being used. Each vector field $\Phi_i$ either corresponds to free-flight or sticking motion, which has to be defined by the user. A transition diagram can be used to decide how the system transits from one state to another at an event. This also gives a way to handle the transition from a state corresponding to free flight to a state corresponding to stick after a complete chatter sequence.

The number of free-flight and stick states depends mainly on the geometry of the impacting rigid bodies. This point will be illustrated further in Chapter 5 for a model example showing a slender rod impacting an oscillating surface.

The mechanism for switching between the states $S_i$ at impact involves evaluating relative normal contact-point accelerations and velocities, whereafter a transition diagram together with a decision-tree can be used to evaluate what state the system
should be in after the impact. The contact-point velocities are mainly used to calculate post-impact velocities. This is in contrast to the contact-point accelerations that are used to determine when the system should release from stick to free flight. The specific implementations have to be assessed on a case-by-case basis.

4.2.4 Chatter

A problematic feature of impacting systems, with non-compliant impacts is chatter. Two types of chattering phenomena exist. Complete chatter: Phenomenon where a system exhibits an infinite number of impacts in a finite time. In this regime the velocity of the impacting rigid body goes uniformly to zero. Incomplete chatter: Series of impacts with behavior resembling complete chattering but with the chattering sequence ending after a large but finite number of impacts [37].

In order to simulate impacting systems for the analysis of long term dynamics it is necessary to numerically deal with chatter. This can be done by bypassing the tail of the complete chatter sequence by defining multiple system states and deriving an appropriate mapping to correspond [37].

Impacting systems typically enter a stick region via a complete chattering sequence in which an infinite sequence of impacts, with successively reducing velocity, converges towards a stick point. The evolution of the stick dynamics then depends on the system in question and the forcing involved. Chatter is a common feature of hybrid systems and requires special considerations for the correct calculation of the numerical solution.

4.2.5 Constraining the Sliding Vector Field

It is common for numerically constructed stick solutions to not follow the discontinuity surface exactly. Consequently, for a stick vector field $F$, there is the possibility of so called numerical drifting away from the surface. The underlying premise of a stick solution, is that one or multiple system state variables, which typically represent relative positions or relative velocities, are exactly zero. In reality, however, this is never the case and there will always be an error on the order of the error of the numerical solver, which for MATLAB is typically $\approx 10^{-15}$. This error often manifests as a linear push away from the surface with a measurable slope. A useful technique to avoid numerical drift is to redefine the specific stick vector field to make
the surface attracting [67]. Typically this is achieved by defining a new vector field which has an additional correction term to make the surface locally attractive.

For a given position constraint \( h(q) = k \), which is a function of the generalised system coordinates \( q \) and where \( k \) is some constant value, we can define our locally attractive term by defining \( \phi = h(q)\frac{\partial h(q)}{\partial q}^T \) where \( \frac{\partial h(q)}{\partial q}^T \) is the derivative of the position constraint with respect to the system state variables, and thus is the normal vector to the surface. It is clear that \( \phi \) is orthogonal to the discontinuity surface and further will always be directed away from the surface. We therefore define the new attracting vector field \( \hat{F} \) such that

\[
\hat{F} = F - \phi.
\]

The minus sign ensures the additional term will always be directed towards the surface. When the sliding solution is exact we see that \( \phi \) will be a zero vector which is necessary to ensure the additional term does not interfere with the sliding vector field.

### 4.2.6 Example

We will illustrate the various characteristics described above by means of a simple example. Consider a ball of mass \( m \) bouncing on a smooth non-compliant plane. We let \( (x(t), y(t)) \) be the position of the ball relative to an inertial origin, with initial position \((x_0, y_0)\) and initial velocity \((\dot{x}_0, \dot{y}_0)\). Further, for this example we consider the simple Newtonian restitution law \( \dot{y}^+ = -e\dot{y}^- \), where \( \dot{y}^- \) is the normal component of velocity at impact and \( e \) is the coefficient of restitution and where \( - \) denotes the value of the state variable before impact and \( + \) denotes the value immediately after an impact. The equations of motion for this system can then be written as

\[
\begin{align*}
    m\ddot{x} &= 0 \\
    m\ddot{y} &= -mg
\end{align*}
\]
This system can be written as four first order ODE’s by making the substitutions 
\( \dot{x} = v, \quad \ddot{x} = \dot{v}, \quad \dot{y} = w \) and \( \ddot{y} = \dot{w} \)

\[
\begin{align*}
\dot{x} &= v \quad (4.1) \\
m\dot{v} &= 0 \quad (4.2) \\
\dot{y} &= w \quad (4.3) \\
m\dot{w} &= -mg. \quad (4.4)
\end{align*}
\]

Let the impact plane be defined by \( y = 0 \) and using the notation introduced above we describe the event surface as \( \phi = y \). The MATLAB event location routine detects when \( \phi = 0 \) and stops the integration at this point. An impact law is then applied to calculate the initial conditions necessary for restarting the integration. Using the model notation we arrive at the following reset for the system

\[
\begin{align*}
x_0^+ &= x_0^- \\
v_0^+ &= v_0^- \\
y_0^+ &= y_0^- \\
w_0^+ &= -ew_0^- ,
\end{align*}
\]

This formulation assumes there is no change in generalised coordinates throughout the impact phase (i.e. the position of the contact point does not change throughout the duration of contact) and due to the nature of the impact (smooth and frictionless) there is no tangential force generation, moreover there is no change in the tangential component of the velocity after impact. See Section 2.5 for a full discussion on the assumptions of rigid-body impact theory. In Figure 4.2 (a) we see the position of the ball in the normal direction getting closer and closer to zero and in Figure 4.2 (b) we see the velocity of the ball in the normal direction approaching zero also. This system will eventually break down numerically if certain measures are not taken to deal with this as the number of impacts (events) grows infinite. It is therefore necessary to apply a numerical technique to deal with this chatter sequence. Here we will employ a hybrid technique as discussed in Section 4.1 and that will also be used for the model system analysed in Chapter 5. This technique involves constraining the ball to stick to the surface of the plane when a predefined velocity threshold is reached. This is achieved by calculating the normal forcing \( \lambda_N \) required to ensure
the acceleration of the ball in the normal direction is zero. Figure 4.3 shows the ball in static equilibrium and it is clear that the ball will remain constrained to the plane provided $\lambda_N = mg$. This allows us to write the new vector field when the system is in this constrained state as

$$\begin{align*}
\dot{x} &= v \\
m\dot{v} &= 0 \\
\dot{y} &= w \\
m\dot{w} &= \lambda_N - mg = 0
\end{align*}$$

We define the critical normal contact-point velocity as $V_{tol}$ and use this as criteria for deciding when the system is going through a chatter sequence. When the normal contact-point velocity is smaller than this threshold the system transitions to the stick state as illustrated in Figure 4.4. If we were to introduce periodic forcing for example it would be possible for the system to transition back to free flight motion, however, we do not consider this in the present example.

As discussed in Section 4.2.5 it is necessary to constrain the sliding vector field to ensure the solution does not drift away from the discontinuity surface. We can
define \( h(y) \) as the distance between the ball and the surface and is given by

\[
h(y) = y
\]  
\[(4.5)\]

and differentiating \( h(y) \) with respect to the system state variables \( q(x, y, v, w) \) gives

\[
\frac{\partial h(y)}{\partial q}^T = (0, 0, 1, 0)^T
\]

We can now define the following attracting term

\[
\phi = -h(y) \frac{\partial h(y)}{\partial q}^T = -(0, 0, y, 0)^T
\]

where \( \phi \) is very small, and further we can define the new constrained sliding vector field by introducing this attracting term to give

\[
\dot{x} = v
\]
\[(4.6)\]
\[
m\dot{v} = 0
\]
\[(4.7)\]
\[
\dot{y} = w - y
\]
\[(4.8)\]
\[
m\dot{w} = \lambda_N - mg = 0
\]
\[(4.9)\]

We now simulate this system with the condition that when the normal velocity \( w \) is less than the velocity threshold \( V_{\text{Tol}} \) the system will transition from free flight state \( f \), with vector field given by (4.1) - (4.4) to stick state \( s \), with vector field given by (4.6) - (4.9). We can summarise the possible transitions as follows.
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\begin{center}
\begin{tikzpicture}
\node [circle, draw] (f) at (0,0) {\textit{f}}; \node [circle, draw] (s) at (2,0) {\textit{s}}; \node [circle, draw, label=above:\textit{II}] (ii) at (1,1) {}; \node [circle, draw, label=above:\textit{I}] (i) at (1,-1) {}; \draw (f) to [bend right=45] (i); \draw (i) to (ii); \draw (ii) to (s); \draw (s) to [bend right=45] (f);
\end{tikzpicture}
\end{center}

Figure 4.4: State transition diagram illustrating the system states and possible transitions (\textit{I, II}) from free flight \textit{f} to stick \textit{s}.

**Transition I.** Impact with $|\dot{y}| > V_{\text{tol}}$. The system will remain in free flight State \textit{f}.

**Transition II.** Impact with $|\dot{y}| < V_{\text{tol}}$. The system will transition to stick State \textit{s}.

\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure4a.png}
\caption{(a)}
\end{subfigure}
\hfill
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure4b.png}
\caption{(b)}
\end{subfigure}
\caption{The (a) position of a bouncing ball versus time, with coefficient of restitution $e = 0.9$ and $V_{\text{tol}} = 10^{-6}$, and (b) the relative velocity of the bouncing ball in time. The transition from free flight state \textit{f} to stick state \textit{s} can be seen in both (a) and (b).}
\end{figure}

The additional attracting term given in (4.8) ensures that when in stick state \textit{s} the ball will remain constrained to the surface and will not drift away due to numerical error as seen in Figure 4.5 (a). Although this is a very simple example, it serves to highlight the techniques used to deal with chatter and other non smooth phenomena.
4.3 Bifurcation Diagram Computation

A useful way to analyse the dynamics of impacting systems, is to compute the periodic and chaotic orbits under parameter variation. This is a common way in which to make an initial assessment of the long-term dynamics of a mechanical system. Broadly speaking there are two classes of numerical methods used to examine the change in system dynamics under parameter variation, namely: Brute force numerical simulation and numerical continuation.

4.3.1 Brute Force Numerical Simulation

In brute force or direct numerical simulation the solution is calculated for a specific value of the parameter under investigation until all the transients have decayed, and thus deemed to have reached an attractor. The solution is recorded in a Poincaré section and the parameter is varied slightly and the process is repeated [67]. The time taken for all transients to decay is highly dependent on the system under consideration and can be determined by performing investigative simulations of the system. An approach used to minimise the transient time is to choose an initial condition which lies on the attractor for the previous parameter value. This technique ensures that the initial condition will always be close to the actual attractor. A disadvantage to this choice of initial condition is that other co-existing attractors may be missed, and consequently information about the system may be lost [35]. A way to avoid this is to choose random initial conditions for each new parameter, thus ensuring all attractors will be located. For the bifurcation diagrams in Chapter 5 random initial conditions were used to locate the system attractors. Initial conditions were then chosen on each attractor and the corresponding branch was followed in each case.

4.3.2 Continuation

Continuation or path-following is a useful tool for accurately pinpointing bifurcation points which often are missed by brute-force simulation. This involves numerical path following of the solution under parameter variation to locate bifurcation points, and possible continuation of these points in two or multiple parameters. These techniques can be used for regular bifurcations or for discontinuity induced bifurcations. Continuation involves semi analytically solving the periodic dynamics of the system in question as we will illustrate by example in Chapter 5.
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For the continuation techniques used in this thesis, Newton’s method was used to solve the equations describing the periodic dynamics of the systems. In the one dimensional case Newton’s Method is derived by considering an approximation of the function \( f \) to be solved. For higher dimensions it is necessary to derive a linear approximation for vector functions. It is therefore necessary to compute the Jacobian \( J \) of \( f \) which may be viewed as the multivariable analog of the derivative \( f' \). The Jacobian is the matrix of all first order partial derivatives of a vector-valued function such that for a function \( f(x) = (f_1, \cdots, f_n)^T, x = (x_1, \cdots, x_n) \) we have that

\[
J(x) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}.
\] (4.10)

If we consider the equation

\[
f(x) = 0,
\] (4.11)

then Newton’s method states that the solution \( x \) of (4.11) can be approximated by iteratively solving

\[
x_{i+1} = x_i - (J(x_i))^{-1} f(x_i),
\] (4.12)

where \( x_0 \) is the vector of the initial guess of the solution at the first step and then \( x_i \) is the vector of the initial guess of the solution at the \( i+1 \)th step. It is also necessary that \( x_0 \) must lie in a neighborhood of the solution [65].

Stability Analysis

In this Section we will discuss the various stability properties considered in this thesis. Stability analysis was only a small aspect of the work in this thesis and this Section, therefore, is only included for completeness. The stability of an equilibrium point or a trajectory is a measure of its sensitivity to perturbations. In other words it is a measure of the deviation of nearby trajectories or solutions from some reference equilibrium point or trajectory. When we refer to asymptotic stability of equilibrium points or periodic orbits, we mean that as the system moves forward in time, all nearby initial conditions will be attracted exponentially to the equilibrium value or trajectory. However, if the system is unstable then nearby initial conditions will deviate away from the equilibrium as the system moves forward in time [37].

First, we will discuss the stability of equilibrium points. Equilibrium points of a
dynamical system can be viewed as a static value of the dynamical system or constant solutions to the differential equations. Dynamical systems describe the evolution of a system in time and typically are described by a set of first order differential equations. As an example, consider the following dynamical system

$$\dot{x}(t) = f(x(t), t), \quad x(0) = x_0, \quad x \in \mathbb{R}^n,$$  (4.13)

where $x$ is the state vector. An equilibrium point $x_{eq}$ of the system is a solution such that

$$f(x_{eq}, t) = 0, \quad \forall t.$$  

To determine the stability of $x_{eq}$ we consider a local linearisation in the neighborhood of the point and we do this by evaluating the Jacobian of the system at the equilibrium point. In general then, the stability is determined by the sign of the real part of the eigenvalues of the Jacobian. If the real parts of all eigenvalues of the system are negative then the solution is asymptotically stable, otherwise the system is unstable.

Second, we will discuss the stability of periodic orbits, which are recurrent periodic motions or trajectories in a system. An important analysis of recurrent motions of impacting systems is to determine whether this motion is stable in state space. The methods by which the stability is determined involves examining the local flow in the neighborhood of the trajectory or solution which, for our purposes, involves a linear approximation of the flow about the reference trajectory [70]. This can be achieved by calculating the fundamental solution matrix which can be viewed as a local linearisation of the system along the periodic orbit. The stability is then determined by evaluating the eigenvalues of the fundamental solution matrix which is a solution of the first variational equations.

Consider the initial value problem given by (4.13) with solution

$$x(t) = v(x_0, t).$$  (4.14)

We consider the deviation of a nearby trajectory, for a smooth system, at the point $x_0$ from a reference point $x_r$ located close by. We assume that $x_0 \approx x_r$ and further we can define the deviation as

$$v(x_0, t) - v(x_r, t) = \frac{\partial v}{\partial x}(x_r, t)(x_0 - x_r),$$  (4.15)
where the Jacobian \( \frac{\partial \nu}{\partial x} (x_r, t) \) can be found by solving the first variational equation given by
\[
\frac{d}{dt} \frac{\partial \nu}{\partial x} (x_r, t) = \frac{\partial f}{\partial x} (\nu (x_r, t)) \frac{\partial \nu}{\partial x}, \quad \frac{\partial \nu}{\partial x} (0) = Id,
\]
where \( Id \) is the identity matrix. This analysis, however, may not work for nonsmooth systems since the vector field \( f \) is not guaranteed to be differentiable everywhere. We follow the approach of [27, 70] to extend the analysis to nonsmooth systems.
Consider again the dynamical system given by (4.13). We define an event surface \( h(x) \), such that when a zero crossing of a discontinuity surface \( S \) is detected, the incoming state of the system is mapped to another point in space by a function \( g(x) \). Now, the reference trajectory defined above starting at \( x_r \) will intersect \( S \) at a time \( t_r \) such that
\[
h (\nu_1 (x_r, t_r)) = 0,
\]
where \( \nu_1 \) denotes the flow function before intersection with the surface. At this point a map \( g(\nu) \) is applied and the system evolves from the new point.
After some time \( \tau > t_r \), the trajectory reaches the point \( x_1 \) given by
\[
x_1 = \nu_2 (g(\nu_1 (x_r, t_r)), \tau - t_r),
\]
where \( \nu_2 \) denotes the flow function after crossing with \( S \). Following the notation used in [27, 70], we let
\[
x_{\text{in}} = \nu_1 (x_r, t_r), \quad x_{\text{out}} = g (x_{\text{in}}),
\]
where the impact function \( g (x_{\text{in}}) \) is a function of the system state prior to intersecting with \( S \). Also, we define
\[
f_{\text{in}} = f_1 (x_{\text{in}}) = \nu_{1,t} (x_r, t_r) \quad f_{\text{out}} = f_2 (x_{\text{out}}) = \nu_{2,t} (x_1, t_r - \tau).
\]
A discontinuity mapping \( D(x) \) can be derived which takes a point close to \( x_{\text{in}} \), maps it along \( \nu_1 \) for a time \( t(x) \) until it reaches \( S \), at this point the map \( g \) is applied, and then flows along \( \nu_2 \) for a time \(-t(x)\) and ends at a point close to \( x_{\text{out}} \).
It can be shown using Taylor expansion of a discontinuity mapping for \( x \approx x_{\text{in}} \).
that the Jacobian of the discontinuity mapping is given by

\[
\frac{\partial D}{\partial x}(x_{in}) = \frac{\partial g}{\partial x}(x_{in}) + \left( f_{out} - \frac{\partial g}{\partial x}(x_{in}) f_{in} \right) \frac{\partial h}{\partial t}(x_{in}, t_r) + \frac{\partial h}{\partial x}(x_{in}, t_r) f_{in},
\]

(4.21)

which is also sometimes referred to as the saltation matrix.

The Jacobian of the discontinuity mapping \( \frac{\partial D}{\partial x}(x_{in}) \) is necessary to analyse the stability of nonsmooth systems. Between each crossing of the discontinuity surface the fundamental solution matrix is calculated and these can then be merged together using the saltation matrix in order to find the fundamental solution matrix for entire trajectories that include discontinuities.

Now we consider the general case of \( n \) crossings of the discontinuity surface \( S \).

For each \( n \) crossing of (4.17), a new initial value problem is solved and then the general solution to the first variational equation (4.16) is found by evaluating

\[
\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x_n} \prod_{i=1}^{n} \frac{\partial D}{\partial x}(x_{in}) \frac{\partial v}{\partial x_{n-1}}.
\]

(4.22)

The fundamental solution matrix will have different properties depending on whether the system is smooth or non-smooth [40]. For periodic solutions the stabilities are found by calculating the eigenvalues using (4.22). These methods have successfully been implemented in [27–29] and particularly in [37] which deals with stability analysis for impacting systems without friction but with complete chatter.
Chapter 5

A Hybrid Scheme for the Simulation of a Slender Rod with Impacts and Friction

5.1 Introduction

This Chapter follows on from the numerical techniques presented in Chapter 4 and introduces a hybrid scheme for an impacting mechanical system in the presence of Amontons-Coulomb friction. The system in question is an unconstrained slender rod that can impact with a periodically forced non-compliant surface. We mainly follow the methodologies described in [37] and [67] with some extensions and some simplifications, as we will describe below. The methodology we use here is referred to as the hybrid-system or event-driven approach, as described in detail in Chapter 4. The maps are used when a solution trajectory reaches an event surface defined by the system variables and parameters. In this context we consider the continuous dynamics as the motion between the impacts or other transitions and the maps correspond to the actual impacts or transitions. The transitions are typically changes from free-flight dynamics to stick or to chatter. The impact law we use here has the advantage that it can be used together with the unconstrained equations of motion for a mechanical system to form this hybrid system. Following the setup of system states presented in [37, 67] we can achieve a robust numerical code capable of simulating the system to examine long-term dynamics.

The emphasis of this chapter is thus two-fold. First we will show how mappings
for impacts with friction derived in [1] can be implemented for reliable simulations of systems with impacts and friction. Second we want to exploit the fact that we have reliable simulation routines to analyse the long-term qualitative behavior of an unconstrained mechanical system with impacts and friction. Since previous research have mainly considered long-term dynamics for systems with impacts but without friction [33,37,38] the analysis of the unconstrained object will show that it is feasible to also consider long-term simulations for mechanical system with impact and friction. The example of a slender rod impacting with an oscillating surface can be seen as a generalisation of a system where a machine element detaches and is free to vibrate in the presence of friction or an item that lies on a vibrating conveyor belt. We will show how rattle is affected by the presence of friction.

5.2 Dynamics of a Slender Rod Impacting a Periodically-oscillating Surface

In this Section we will use a basic planar model of rigid slender rod impacting a periodically-oscillating surface to describe how the modelling introduced in Chapters 2 and 3 can be done in practice. The model will also be used to describe how the numerical methods discussed in Chapter 4 can be implemented and what the dynamical features presented look like for this specific case. We will also use this setup to show how friction affects chaotic rattling behaviour of the rod.

\[ P_1 \]
\[ P_2 \]
\[ \lambda_1 \]
\[ \lambda_2 \]
\[ n_1 \]
\[ n_2 \]
\[ 2L \]
\[ G \]
\[ \theta \]
\[ D(t) \]

Figure 5.1: Model example of a slender rod in contact with a periodically forced surface.
We consider a slender rod of mass $m$, length $2L$ and moment of inertia $I$ and derive the equations of motion for the system when the slender rod is in contact, as shown in Figure 5.1. To do this we define the coordinate system $n_1 - n_2$, where $n_1$ is the tangential to the contact plane and $n_2$ is normal to the tangent plane (see Figure 5.1). We also define $\theta$ as the angle of inclination between the slender rod and the surface. This gives us the following generalised coordinates for our system

$$q = (q_1, q_2, \theta)^T,$$

where $q_1$, $q_2$ are respectively, the tangential and normal position of the centre of mass of the slender rod. Further, defining $L$ as the distance form $G$ to $C$ the position $q_C$ of the contact point $C$ relative to the $n_1 - n_2$ frame can be written as

$$q_C := (q_1 - L \cos(\theta), q_2 - L \sin(\theta))^T$$

(5.1)

Using the formulations developed in Chapters 2 and 3 the equations of motion for the system can be written as

$$M \ddot{q} = F^T + \left( \frac{\partial q_C}{\partial q} \right)^T \lambda^T,$$

(5.2)

where $M$ is, respectively, the mass matrix for the slender rod and is given by

$$M = \begin{pmatrix}
    m & 0 & 0 \\
    0 & m & 0 \\
    0 & 0 & I
\end{pmatrix}$$

and $F$ is, respectively, the external forces and torques acting on the slender rod and is given by

$$F = (F_1, F_2, R),$$

and where

$$\frac{\partial q_C}{\partial q} = \begin{pmatrix}
    1 & 0 & L \sin(\theta) \\
    0 & 1 & -L \cos(\theta)
\end{pmatrix}$$

and where the subscripts 1 and 2 respectively represent the components of the external forces acting in the tangential and normal direction, and $R$ is the external torque acting on the body. In a similar way we define $\lambda$ as the force generated at
impact given by

$$\lambda = (\lambda_1, \lambda_2).$$

When the slender rod is in free flight we have that $\lambda_1 = \lambda_2 = 0$ and the same
equations of motion apply.

5.2.1 The Model, System States and Vector Fields

As discussed above we consider a planar uniform slender rigid rod and let $q_1, q_2$
be the tangential and normal position of the centre of mass, relative to the contact
plane, and let $\theta$ be the angle of rotation of the rod (see Figure 5.2(a)). The rod is
subjected to gravity and where either of the two isolated end points, named $P_1$ and
$P_2$, can impact, get stuck to or slide along the periodically oscillating surface. The
slender rod can essentially be in four different states; free flight (Figure 5.2(a)), one
of the two end points is stuck to the surface (Figure 5.2(b) and (c)) or both end
points are stuck to the surface (Figure 5.2(d)). The fourth state here also allows for
the release of the two end points at the same time, which in effect leads to a lower-
dimensional dynamical system that can be treated as a simple impacting particle.
In Section 5.2.3 we will discuss the exact conditions for which the system will be in
a free flight or stick state.

![Figure 5.2](image)

Figure 5.2: The four possible main states of the slender rod. (a) Free-flight motion. (b) End point $P_1$ constrained to the surface. (c) End point $P_2$ constrained to the surface. (d) Symmetric free-flight motion (dashed) and symmetric stick motion (solid).

Without loss of generality and following the general setup in Chapter 3, we let the
mass of the rod be $m = 1$ and the distance from the centre of mass to either of the two
end points be $L = 1$. This gives that the moment of inertia is $I = \frac{1}{3}$ and the radius of
gyration is $k^2 = \frac{1}{3}$. We further assume that the oscillating surface is not affected by
the rod at impact and thus let $D(t)$ represent the oscillating surface with frequency
$\omega$, amplitude $A$ and where $t$ is time, so that $D(t) := A \sin(\omega t)$ (see Figure 5.2). This
means that we only need to consider one of the two impacting bodies introduced in Chapter 3 as the mass of the surface can be assumed to be much greater than that of the rod and thus only one of the two systems of differential equations, say (3.28), needs to be considered. For future reference we let $q_4 = \dot{q}_1, q_5 = \dot{q}_2, q_6 = \dot{\theta}$ and introduce $\tau$ as the phase of the oscillating floor. We also let $d_1$ be the distance between the end point $P_1$ and the surface in the normal direction and let $d_2$ be the distance between the end point $P_2$ and the surface in the normal direction, and thus using (5.1) we find

$$d_1 = q_2 - \sin(\theta) - A\sin(\omega t),$$

$$d_2 = q_2 + \sin(\theta) - A\sin(\omega t),$$

which gives that the relative velocity between the end point $P_1$ and the floor as

$$\dot{d}_1 = \dot{q}_2 - \cos(\theta)\dot{\theta} - A\omega \cos(\omega t),$$

the relative velocity between the end point $P_2$ and the floor as

$$\dot{d}_2 = \dot{q}_2 + \cos(\theta)\dot{\theta} - A\omega \cos(\omega t),$$

the relative acceleration between the end point $P_1$ and the floor as

$$\ddot{d}_1 = \ddot{q}_2 + \sin(\theta)\dot{\theta}^2 - \cos(\theta)\ddot{\theta} + A\omega^2 \sin(\omega t),$$

and the relative acceleration between the end point $P_2$ and the floor as

$$\ddot{d}_2 = \ddot{q}_2 - \sin(\theta)\dot{\theta}^2 + \cos(\theta)\ddot{\theta} + A\omega^2 \sin(\omega t).$$

Now we are ready to introduce the five systems states, which we will use for simulating this mathematical model of the planar rod, together with the corresponding vector fields.

**State 1 – Free flight.** Since we make the assumption that there is no external torque or no horizontally acting forcing present, we have that $R = 0$ and $F_1 = 0$ in (5.2). The only external force acting in the vertical direction is due to gravity, and so $F_2 = -g$. Since the slender rod is in free flight we also have that $\lambda_1 = \lambda_2 = 0.$
Following this the equations of motion of the rod at free flight are given by

\begin{align}
\ddot{q}_1 &= 0, \\
\ddot{q}_2 &= -g, \\
\ddot{\theta} &= 0,
\end{align}

and the corresponding dynamical system is

\[(\dot{q}_1, \dot{q}_2, \dot{\theta}, \dot{q}_4, \dot{q}_5, \dot{q}_6, \dot{\tau})^T = (q_4, q_5, q_6, 0, -g, 0, 1)^T := \Phi_1(t)\]

which will be used for the numerical simulation of the free-flight motion. Notice that we have included the phase \(\tau\) in the dynamical system in order to have better control of the periodic influence of time.

**State 2 – End point** \(P_1\) **is stuck to the floor.** We will derive a new vector field for the system when \(P_1\) is stuck to the floor. First, from (5.2) we get that the equations of motion for the constrained bar is

\begin{align}
\ddot{q}_1 &= \lambda_1, \\
\ddot{q}_2 &= -g + \lambda_2, \\
\ddot{\theta} &= 3 \sin(\theta) \lambda_1 - 3 \cos(\theta) \lambda_2.
\end{align}

Next we need to find the forces \(\lambda_1\) and \(\lambda_2\) needed to constrain \(P_1\) to the oscillating surface. Substituting (5.11) and (5.12) into (5.5) and using the Amontons-Coulomb friction law \(\lambda_1 = s \mu \lambda_2\) gives

\[\ddot{d}_1 = -g + \lambda_2 + \sin(\theta) \dot{\theta}^2 - 3 \cos \theta \sin(\theta) s \mu \lambda_2 + 3 \cos^2(\theta) \lambda_2 + A \omega^2 \sin(\omega t),\]

where \(s\) can be +1 or −1, depending on the relative tangential velocity between the initial sliding direction. Further, using the fact that \(d_1 = \ddot{d}_1 = \ddot{d}_1 = 0\) when the end point is in contact with the surface, and solving for \(\lambda_2\) gives

\[\lambda_2 = \frac{g - \sin(\theta) \dot{\theta}^2 - A \omega^2 \sin(\omega t)}{1 + 3 \cos^2(\theta) - 3 s \cos(\theta) \sin(\theta) \mu},\]

which is the normal forcing required to ensure the contact point will remain constrained to the plane.
Now we can write the vector field for the rod with end point \( P_1 \) stuck to the floor as
\[
\begin{pmatrix}
\dot{q}_1, \dot{q}_2, \dot{\theta}, \dot{q}_5, \dot{q}_6, \dot{\tau}
\end{pmatrix}^T = (q_1, q_5, q_6, \alpha_1, \alpha_2, \alpha_3, 1)^T := \Phi_2(t)
\]
where
\[
\alpha_1 = \frac{s\mu \left(g - \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)\right)}{1 + 3 \cos^2(\theta) - 3s\mu \cos(\theta) \sin(\theta)},
\]
\[
\alpha_2 = -g + \frac{g - \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)}{1 + 3 \cos^2(\theta) - 3s\mu \cos(\theta) \sin(\theta)},
\]
\[
\alpha_3 = \frac{(3s\mu \sin(\theta) - 3 \cos(\theta)) \left(g - \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)\right)}{1 + 3 \cos^2(\theta) - 3s\mu \cos(\theta) \sin(\theta)}.
\]

**State 3 – End point \( P_2 \) is stuck to the floor.** We will derive a new vector field for the system when \( P_1 \) is stuck to the floor. First, from (5.2) we get that the equations of motion for the constrained bar is
\[
\begin{align*}
\ddot{q}_1 &= \lambda_1, \\
\ddot{q}_2 &= -g + \lambda_2, \\
\ddot{\theta} &= 3 \sin(\theta) \lambda_1 - 3 \cos(\theta) \lambda_2.
\end{align*}
\]
Next we need to find the forces \( \lambda_1 \) and \( \lambda_2 \) needed to constrain \( P_2 \) to the oscillating surface. Substituting (5.16) and (5.17) into (5.6) and using the Amontons-Coulomb friction law \( \lambda_1 = s\mu \lambda_2 \) gives
\[
\ddot{d}_2 = -g + \lambda_2 - \sin(\theta)\dot{\theta}^2 + 3 \cos \theta \sin(\theta)s\mu \lambda_2 - 3 \cos^2(\theta) \lambda_2 + A\omega^2 \sin(\omega t),
\]
where \( s \) can be \( +1 \) or \( -1 \), depending on the relative tangential velocity between the initial sliding direction. Further, using the fact that \( d_2 = \dot{d}_2 = \ddot{d}_2 = 0 \) when the end point is in contact with the surface, and solving for \( \lambda_2 \) gives
\[
\lambda_2 = \frac{g + \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)}{1 - 3 \cos^2(\theta) + 3s \cos(\theta) \sin(\theta) \mu},
\]
which is the normal forcing required to ensure the contact point will remain constrained to the plane.

Now we can write the vector field for the rod with end point \( P_2 \) stuck to the floor
as

\[
(\dot{q}_1, \dot{q}_2, \dot{\theta}, \dot{q}_4, \dot{q}_5, \dot{q}_6, \dot{\tau})^T = (q_4, q_5, q_6, \alpha_4, \alpha_5, \alpha_6, 1)^T := \Phi_3(t)
\]

where

\[
\begin{align*}
\alpha_4 &= \frac{s\mu \left(-g + \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)\right)}{3s\mu \cos \theta \sin(\theta) - 1 - 3 \cos^2(\theta)}, \\
\alpha_5 &= -g + \frac{-g + \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)}{3s\mu \cos \theta \sin(\theta) - 1 - 3 \cos^2(\theta)}, \\
\alpha_6 &= \frac{(3s\mu \sin(\theta) - 3 \cos(\theta)) \left(-g + \sin(\theta)\dot{\theta}^2 - A\omega^2 \sin(\omega t)\right)}{3s\mu \cos \theta \sin(\theta) - 1 - 3 \cos^2(\theta)}.
\end{align*}
\]

**State 4 – Symmetric motion.** We define symmetric motion as one where \(\theta \mod \pi = q_6 = 0\) for all time, which means that the two end points will impact the floor at the same time. The dynamical system will be the same as in the free-flight case albeit the motion is heavily constrained, and thus the vector field \(\Phi_1\) can be used.

**State 5 – Both end points \(P_1\) and \(P_2\) stuck to the floor.** If both end points are stuck to the floor it means that the centre of mass will oscillate as \(S(\tau)\) and thus the vector field in this case is trivially

\[
(\dot{q}_1, \dot{q}_2, \dot{\theta}, \dot{q}_4, \dot{q}_5, \dot{q}_6, \dot{\tau})^T = (q_4, q_5, q_6, 0, -\omega^2 A \sin(\omega \tau), 0, 1)^T := \Phi_4.
\]

### 5.2.2 Making the Sliding Vector Field Locally Attractive

As discussed in Chapter 4 it is typical for numerically constructed stick solutions to not follow the discontinuity surface exactly. Consequently, for the numerical stick solutions constructed in Section 5.2.1, there is the possibility of so called numerical drifting away from the surface. We will define two locally attracting terms, one on the position of the contact point of the slender rod and one on the contact point velocity, and include them in the stick vector fields.

From Section 5.2.1 we have that \(d_1\) and \(d_2\) are the distances between the end points \(P_1\) and \(P_2\) and the surface in the direction normal to the surface respectively

\[
d_1 = q_2 - \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_2 = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_3 = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_4 = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_5 = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_6 = \sin(\theta),
\]

\[
d_7 = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_8 = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_9 = \sin(\theta),
\]

\[
d_{10} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{11} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{12} = \sin(\theta),
\]

\[
d_{13} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{14} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{15} = \sin(\theta),
\]

\[
d_{16} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{17} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{18} = \sin(\theta),
\]

\[
d_{19} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{20} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{21} = \sin(\theta),
\]

\[
d_{22} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{23} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{24} = \sin(\theta),
\]

\[
d_{25} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{26} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{27} = \sin(\theta),
\]

\[
d_{28} = \sin(\theta) + A \sin(\omega \tau),
\]

\[
d_{29} = \sin(\theta) - A \sin(\omega \tau),
\]

\[
d_{30} = \sin(\theta),
\]
when the end point $P_1$ is constrained to the plane and

$$d_2 = q_2 + \sin(\theta) - A \sin(\omega \tau), \quad (5.21)$$

when the end point $P_2$ is constrained to the plane. It is important to note the obvious symmetry present in the system when either contact point is in contact with the surface. However as we do not consider absolute angles it is essential from a numerical viewpoint that we distinguish between the two cases. For clarity observe that when we rotate the slender rod by $\pi$ radians we alternate between the state 2 and state 3 configuration and since sin is an odd function we have that $h(q_{0+\pi}) = g(q)$ as required. Differentiating (5.20) and (5.21) with respect to the system state variables gives

$$\frac{\partial d_1}{\partial q}^T = (0, 1, -\cos(\theta), 0, 0, 0, -\omega A \cos(\omega \tau))^T$$

for $P_1$ and

$$\frac{\partial d_2}{\partial q}^T = (0, 1, \cos(\theta), 0, 0, 0, -\omega A \cos(\omega \tau))^T$$

for $P_2$. We can now define the additional artificially attracting terms $\phi_1 = -d_1 \frac{\partial d_1}{\partial q}^T$ for $P_1$ and $\psi_1 = -d_2 \frac{\partial d_2}{\partial q}^T$ for $P_2$. When in the stick regime it can also happen that a non zero contact point normal velocity can occur due to build up of numerical errors. We can therefore also consider a constraint on the contact point normal velocity. Let $\dot{d}_1$ and $\dot{d}_2$ be the normal contact point velocities of the end points $P_1$ and $P_2$ in the direction normal to the surface respectively

$$\dot{d}_1 = q_5 - q_6 \cos(\theta) - A\omega \cos(\omega \tau) \quad (5.22)$$

for $P_1$ and

$$\dot{d}_2 = q_5 + q_6 \cos(\theta) - A\omega \cos(\omega \tau) \quad (5.23)$$

for $P_2$. Differentiating $\dot{d}_1$ and $\dot{d}_2$ with respect to the system state variables gives

$$\frac{\partial \dot{d}_1}{\partial q}^T = (0, 0, q_6 \sin(\theta), 0, 1, -\cos(\theta), \omega^2 A \sin(\omega \tau))^T$$

for $P_1$ and

$$\frac{\partial \dot{d}_2}{\partial q}^T = (0, 0, -q_6 \sin(\theta), 0, 1, -\cos(\theta), \omega^2 A \sin(\omega \tau))^T$$

95
for $P_2$. We can now define the additional artificially attracting terms as $\phi_2 = -\dot{d}_1 \frac{\partial \dot{d}_1}{\partial q}^T$ and $\psi_2 = -\dot{d}_2 \frac{\partial \dot{d}_2}{\partial q}^T$. Using $\phi_1$ and $\psi_1$ we can constrain the contact point $P_1$ to the surface by making the following adjustment to the vector field

$$(\dot{q}_1, \dot{q}_2, \dot{\theta}, \dot{q}_4, \dot{q}_5) = (q_4, q_5, q_6, \alpha_1, \alpha_2, \alpha_3, 1)^T + \phi_1 + \psi_1 := \hat{\Phi}_2(t)$$

for State 2, and using $\phi_2$ and $\psi_2$ we can constrain the contact point $P_2$ to the surface by making the following adjustment to the vector field

$$(\dot{q}_1, \dot{q}_2, \dot{\theta}, \dot{q}_4, \dot{q}_5) = (q_4, q_5, q_6, \alpha_1, \alpha_2, \alpha_3, 1)^T + \phi_2 + \psi_2 := \hat{\Phi}_3(t)$$

for state 3. It is clear that $\phi_1, \phi_2, \psi_1$ and $\psi_2$ are orthogonal to the discontinuity surface and further that the new vector field will always be directed towards the surface. When the sliding solution is exact we see that $\phi_1, \phi_2, \psi_1$ and $\psi_2$ are all zero vectors which is necessary to ensure the additional term does not interfere with the sliding vector field.

### 5.2.3 State Transitions and Impact Mappings

For the planar rod model described above in Section 5.2.1 we introduced five system states, two for free flight and three for stick. Our proposed scheme for dealing with chatter involves constraining the respective end point to the impact surface when that corresponding end point is going through a chatter sequence. The system then acts as a sliding hinge. For this purpose we define the critical normal contact point relative velocity as $V_{tol}$ and use this, along with the end point accelerations, as criteria for deciding when the system is going through a chatter sequence. $V_{tol}$ is chosen based on the system in question and what makes physical sense. In this paper we typically let $V_{tol}$ be $\leq 10^{-6}$. When an impact occurs the critical normal contact point relative velocities $\ddot{d}_1$ and $\ddot{d}_2$ together with the critical normal contact point relative accelerations $\dddot{d}_1$ and $\dddot{d}_2$ are evaluated. These values are then used with a decision tree to decide if the system should transition to another state. It is important to note that when calculating the relative contact point accelerations when in a stick state, we use the unconstrained values. This approach ensures that the system will naturally release from stick due to the change in relative acceleration. At impact the impact law needs to be applied to determine what type of transition
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should occur. For the model example described here the rate constants defined in Section 3.2.2 now take the form

Map I:

\[
\dot{\tilde{q}}_{1C} = \dot{\tilde{q}}_{1C} - (1 + e_*) \frac{k_T}{k_N} \dot{\tilde{q}}_{2C} \tag{5.24}
\]

\[
\dot{\tilde{q}}_{2C} = -e_\ast \dot{\tilde{q}}_{2C} \tag{5.25}
\]

Map II:

\[
\dot{\tilde{q}}_{1C} = \frac{k_T'}{k_N'} \left( \frac{k_N'}{k_T} \dot{\tilde{q}}_{1C} - \dot{\tilde{q}}_{2C} \right)
+ \frac{k_T'}{k_N'} \left( \sqrt{1 - \frac{k_N'}{k_N}} \left( \frac{k_N'}{k_T} \dot{\tilde{q}}_{1C} - \dot{\tilde{q}}_{2C} \right)^2 + e^2_\ast \frac{k_T'}{k_N} (\dot{\tilde{q}}_{2C})^2 \right) \tag{5.26}
\]

\[
\dot{\tilde{q}}_{2C} = \sqrt{1 - \frac{k_N'}{k_N}} \left( \frac{k_N'}{k_T} \dot{\tilde{q}}_{1C} - \dot{\tilde{q}}_{2C} \right)^2 + e^2_\ast \frac{k_T'}{k_N} (\dot{\tilde{q}}_{2C})^2 \tag{5.27}
\]

Map III:

\[
\dot{\tilde{q}}_{1C} = \frac{k_T'}{k_N'} \left( \frac{k_N'}{k_T} \dot{\tilde{q}}_{1C} - \dot{\tilde{q}}_{2C} \right)
+ \frac{k_T'}{k_N'} e_\ast \left( \sqrt{1 - \frac{k_N'}{k_N}} \left( \frac{k_N'}{k_T} \dot{\tilde{q}}_{1C} - \dot{\tilde{q}}_{2C} \right)^2 + e^2_\ast \frac{k_T'}{k_N} (\dot{\tilde{q}}_{2C})^2 \right) \tag{5.28}
\]

\[
\dot{\tilde{q}}_{2C} = e_\ast \sqrt{1 - \frac{k_N'}{k_N}} \left( \frac{k_N'}{k_T} \dot{\tilde{q}}_{1C} - \dot{\tilde{q}}_{2C} \right)^2 + \frac{k_T'}{k_N} (\dot{\tilde{q}}_{2C})^2 \tag{5.29}
\]

for \( k_N \neq 0 \) and

\[
\dot{\tilde{q}}_{1C} = 0 \tag{5.30}
\]

\[
\dot{\tilde{q}}_{2C} = e_\ast \sqrt{\dot{q}_{2C}^2 + \frac{2k_N' \dot{\tilde{q}}_{1C} \dot{\tilde{q}}_{2C}}{k_T}} \tag{5.31}
\]

for \( k_N = 0 \).

As described in Chapter 3, the different combinations of segments of stick and
relative slip can be described by ten different regions, each of which corresponds to one of the three maps given above. The region and the choice of $k_T$, $k_N$, $k'_T$ and $k'_N$, are all determined by the system parameters and pre-collision conditions. For the model example described here the rate constants defined in Chapter 3 now take the form

$$k_T, k'_T = -3 \sin(\theta) \cos(\theta) - \mu \left(1 + 3 \sin^2(\theta)\right), \quad k_N, k'_N = 1 + 3 \cos^2(\theta) + 3 \mu \sin(\theta) \cos(\theta)$$

for positive slip,

$$k_T, k'_T = -3 \sin(\theta) \cos(\theta) + \mu \left(1 + 3 \sin^2(\theta)\right), \quad k_N, k'_N = 1 + 3 \cos^2(\theta) - 3 \mu \sin(\theta) \cos(\theta)$$

for negative slip, and

$$k_T, k'_T = 0, \quad k_N, k'_N = \frac{4}{1 + 3 \sin^2(\theta)}$$

for the stick regime. These rate constants are used in the mappings for the numerical implementation together with a decision tree. The process involves deciding what region the impact corresponds to depending on initial conditions and assigning the corresponding rate constants and impact map accordingly, see further [1]. However, apart from deciding what will happen at a specific impact the system can also switch between the different states described in Section 5.2.1. The transition diagram describing what transitions are possible for the rod system are given in Fig. 5.3, where $S_1 - S_5$ are the states introduced in Section 5.2.1.

A summary of all states and transitions are given in Table 5.1 and brief descriptions of the 16 transitions (I - XVI) in Figure 5.3 is given here:

**Transition I.** Impact of end point $P_1$ with $|\dot{d}_1| > V_{tol}$. The system will remain in free flight State 1.

**Transition II.** Impact of end point $P_2$ with $|\dot{d}_2| > V_{tol}$. The system will remain in free flight State 1.

**Transition III.** If at impact $|\dot{d}_1| < V_{tol}$. The system will transition to stick State 2.

**Transition IV.** If at impact $|\dot{d}_2| < V_{tol}$. The system will transition to stick State 3.
Figure 5.3: A transitions diagram for the planar rod system showing the five states, $S_1 - S_5$ and the state transitions $I - XVI$. See also Table 5.1 for a list of event types and transition criteria.

<table>
<thead>
<tr>
<th>Transition</th>
<th>Type</th>
<th>Event</th>
<th>Transition Criteria</th>
<th>Contact angle $\theta$</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>$d_1 = 0$</td>
<td>$</td>
<td>d_1</td>
<td>&gt; V_{tol}$, $\theta \neq 0$</td>
</tr>
<tr>
<td>II</td>
<td>I</td>
<td>$d_2 = 0$</td>
<td>$</td>
<td>d_2</td>
<td>&gt; V_{tol}$</td>
</tr>
<tr>
<td>III</td>
<td>C</td>
<td>$d_1 = 0$</td>
<td>$</td>
<td>d_1</td>
<td>&lt; V_{tol}$, $\theta \neq 0$</td>
</tr>
<tr>
<td>IV</td>
<td>C</td>
<td>$d_2 = 0$</td>
<td>$</td>
<td>d_2</td>
<td>&lt; V_{tol}$</td>
</tr>
<tr>
<td>V</td>
<td>R</td>
<td>$\dot{d}_1 &gt; 0$</td>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$S_2 \rightarrow S_1$</td>
</tr>
<tr>
<td>VI</td>
<td>R</td>
<td>$\dot{d}_2 &gt; 0$</td>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>$S_3 \rightarrow S_1$</td>
</tr>
<tr>
<td>VII</td>
<td>L</td>
<td>$d_1, d_2 = 0$, $</td>
<td>\dot{d}_{1,2}</td>
<td>&gt; V_{tol}$, $\theta &lt; \theta_{Crit}$</td>
<td>$0$</td>
</tr>
<tr>
<td>VIII</td>
<td>TR</td>
<td>$\ddot{d}_2 &gt; 0$</td>
<td>$0$</td>
<td>$S_2 \rightarrow S_3$</td>
<td></td>
</tr>
<tr>
<td>IX</td>
<td>TR</td>
<td>$\ddot{d}_1 &lt; 0$</td>
<td>$0$</td>
<td>$S_3 \rightarrow S_2$</td>
<td></td>
</tr>
<tr>
<td>X</td>
<td>T</td>
<td>$d_2 = 0$</td>
<td>$</td>
<td>\dot{d}_2</td>
<td>&gt; V_{tol}$</td>
</tr>
<tr>
<td>XI</td>
<td>T</td>
<td>$d_1 = 0$</td>
<td>$</td>
<td>\dot{d}_1</td>
<td>&gt; V_{tol}$</td>
</tr>
<tr>
<td>XII</td>
<td>TC</td>
<td>$d_2 = 0$</td>
<td>$</td>
<td>d_2</td>
<td>&lt; V_{tol}$</td>
</tr>
<tr>
<td>XIII</td>
<td>TC</td>
<td>$d_1 = 0$</td>
<td>$</td>
<td>d_1</td>
<td>&lt; V_{tol}$</td>
</tr>
<tr>
<td>XIV</td>
<td>R</td>
<td>$\ddot{d}_{1,2} &gt; g$</td>
<td>$0$</td>
<td>$S_5 \rightarrow S_4$</td>
<td></td>
</tr>
<tr>
<td>XV</td>
<td>TC</td>
<td>$d_1, d_2 = 0$, $</td>
<td>\dot{d}_1</td>
<td>&lt; V_{tol}$, $</td>
<td>\dot{d}_2</td>
</tr>
<tr>
<td>XVI</td>
<td>T</td>
<td>$d_1, d_2 = 0$, $</td>
<td>\dot{d}_1</td>
<td>&gt; V_{tol}$, $</td>
<td>\dot{d}_2</td>
</tr>
</tbody>
</table>

Table 5.1: Table corresponding to the transition diagram in Fig. 5.3. The different types of transitions are I - Impact, C - Chatter, R - Release, L - Limit, T - Two-point impact, TR - Two-point impact with Release and TC - Two-point impact with Chatter.
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Transition V. If at any moment during the constrained motion $\ddot{d}_1 > 0$, then the system will transition to free flight State 1.

Transition VI. If at any moment during the constrained motion $\ddot{d}_2 > 0$ then the system will transition to free flight State 1.

Transition VII. In the limiting case where the impact angle $\theta$ approaches zero with each impact and eventually reaches a predefined threshold in which the angle can be assumed to be zero. The system transitions to the symmetric State 4.

Transition VIII. If at impact $\ddot{d}_1 > 0$ and $\ddot{d}_2 < 0$, then the system will transition to stick State 3.

Transition IX. If at impact $\ddot{d}_2 > 0$ and $\ddot{d}_1 < 0$, then the system will transition to stick State 2.

Transition X. If at impact $|\dot{d}_2| > V_{tol}$, then the system transitions to the symmetric State 4.

Transition XI. If at impact $|\dot{d}_1| > V_{tol}$, then the system transitions to the symmetric State 4.

Transition XII. If at impact $|\dot{d}_2| < V_{tol}$, then the system will transition to the symmetric stick State 5.

Transition XIII. If at impact $|\dot{d}_1| < V_{tol}$, then the system will transition to the symmetric stick State 5.

Transition XIV. The system will remain in State 5 unless the frequency of oscillation exceeds a critical value $\omega^*$ given by $\omega^* = \sqrt{\frac{g}{A \sin(\omega \tau)}}$. When this value is exceeded the system will release from stick and transition to the symmetric State 4. For a given $\omega$ value, $\omega^*$ will vary sinusoidally depending on the phase of oscillation $\tau$. When the floor acceleration is maximal, $\omega^*$ will be minimized, and when the floor acceleration is minimal, $\omega^*$ will be maximized.

Transition XV. If $\ddot{d}_{1,2} < 0$ or $|\dot{d}_{1,2}| < V_{tol}$, then the system will transition to the symmetric stick State 5.

Transition XVI. If at impact $\ddot{d}_{1,2} > 0$, $\dot{d}_1 > V_{tol}$ and $\dot{d}_2 > V_{tol}$ the system will remain in State 4.
5.3 Results

Here we will focus on numerical analysis of some aspects of long-term dynamics of the rod-surface system introduced in Section 5.2.1. The purpose of this is twofold. First, we want to show the robustness of the numerical techniques presented in Sections 5.2.1 and 5.2.3, and second we want to display some of the behaviour that one can expect from a rattling object where energy is dissipated both through the impact and friction. In this context we will focus both on steady-state dynamics and transients.

As discussed in Section 5.2.1 and shown in Figure 5.2 the system can essentially be in five different states between events (impacts or transitions). On top of this numerical experiments have shown that the more energy that is removed from the system at impact through friction the faster the system tends from asymmetric to symmetric motion (see Figure 5.4). In terms of the model example presented in this work, asymmetric chaos refers to chaotic motion when in State 1, and symmetric chaos refers to chaos when in State 4. This indicates that if the energy that is added into the system is acting in the normal direction relative to the impact then the friction, which only acts in the tangential direction, will reduce the rotational energy over time and only symmetric motion will remain. To highlight this the example in Figure 5.4(a) shows a time history of the angle $\theta$ when the system goes through a transition from asymmetric to symmetric motion. Recall that $\theta = 0$ or $\theta = \pi$ means that the rod is aligned parallel to the surface. In Figure 5.4(b) we see a close up of how the actual transition in this case happens over a very short time interval. In principle it is an accumulation of alternative impacts between the two ends, i.e. very similar to what we see in complete chatter, but here we do not necessarily have stick. In Figure 5.4(c) we see a similar example, where the time history of the position of each of the two end points are shown to highlight how the transition between transient asymmetric chaos and transient symmetric chaos can occur. We note that up to approximately $t = 302$ there are two separate trajectories, one for each end point, but suddenly the two trajectories converge and the two end points move in synchrony and the rotation of the rod ends. The figure also indicates that the system stays chaotic but where all the rotational energy has dissipated due to friction. This will not always be the case however, as seen in Figures 5.5 (a) and (b). After $t = 790$ the system still remains in asymmetric chaos. A comparison of Figure 5.5 (b) with Figure 5.4(c) shows the difference between asymmetric and symmetric chaos. In Figure 5.5 (b) the system has not yet reached symmetric chaos.
(transitioned to state 4) and it is possible that it never will. In Figure 5.4(c) the system has reached symmetric chaos (transitioned to state 4).

In Figure 5.6 a schematic transition diagram for transient motions found in the rod system is shown. Note that this schematic is based on observations from numerical experiments of the rod system and not on analytically derived conditions. In many cases, for general initial conditions, the rod system undergoes motion akin to asymmetric chaos until the rotational energy has dissipated and the symmetric (transient) chaos takes over. Depending on the frequency of the external forcing the symmetric chaos may persist or the motion turn periodic. Again, depending on the
Asymmetric transient

Symmetric transient

Chaos

Periodic orbits with stick

Periodic orbits without stick

Figure 5.5: Time histories showing (a) the rod angle $\theta \mod 2\pi$ and (b) the contact points $P_1(t)$ and $P_2(t)$ when for $\omega = 4.40572$, $e_* = 0.9$, $A = 1$ and $\mu = 0.5$.

Figure 5.6: A schematic detailing the possible transitions from transient asymmetric chaos to transient symmetric chaos and periodic orbits that can occur in the system.

frequency and the value of the restitution coefficient the periodic orbits may have periods of stick. The effect of this is that, at least for low frequencies $\omega$, all long term motions that we have come across, are symmetric and thus brute-force bifurcation diagrams only show stable solutions, where the dynamics is symmetric. It is worth noting that in the limit, where the impact times between the end points as well as the tangential impact velocities go to zero, Map I (see Section 3.2.2) is successively applied and once the transition to symmetric motion has occurred Map I reduces to the standard Newtonian restitution law, which is in-line with what is discussed in [1]. For the rod system in question this means that the long term behavior can simply be approximated by a one-dimensional system of a mass impacting an oscillating surface. While the general one-dimensional system has been analysed before, see particularly Holmes [71], we will present some specific results for the system anal-
used here in order to give us an idea on what we can predict regarding the long-term behaviour for specific parameter values.

Figure 5.7: Bifurcation diagrams showing how the steady-state of the position of the centre of mass \( q_2 \) varies as the frequency \( \omega \) is varied. In (a) \( e_* = 0.9, A = 1 \) and \( \mu = 0.05 \) and in (b) \( e_* = 0.8, A = 1 \) and \( \mu = 0.05 \). The period-1 and the period-3 orbits are labeled in (a) and the period-2 orbit is labeled in (b).

To describe how possible transitions between different types of long-term motion in the symmetric (one-dimensional) rod system occur we show in Figure 5.7 (a) and (b) two brute-force bifurcation diagrams, \( \omega \) vs. \( q_2 \), for two different values of the restitution coefficient, \( e_* = 0.9 \) and \( e_* = 0.8 \), respectively. We consider the Poincaré section corresponding to \( \dot{q}_2 = 0 \). Figure 5.7(a) shows three coexisting period-1 solutions that undergo period-doubling sequences, at three different values of \( \omega \), until the branches disappear in grazing bifurcations at \( \omega \approx 3.5 \). The figures also shows that regions of chaos start at \( \omega \approx 3.5 \), with two periodic windows (a period-1 and a period-3 orbit, see Figure 5.8(a)) also existing within the chaos. The bifurcation diagram in Figure 5.7(b), where the restitution coefficient is lower, we see that the onset of chaos occurs at \( \omega \approx 3.8 \) and thus the periodic orbits are sustained longer when more energy is taken out of the systems at impact. To show what some of the symmetric period-1 solutions look like in Fig. 5.8(a) a time series of the end points of the period-1 orbit highlighted as \( p_1 \) in Figure 5.7(a) \( (p_1) \) is shown. In Figure 5.8(b) a time series of the end points of the period-2 orbit highlighted as \( p_2 \) in Figure 5.7(b) is shown. Similarly in Figure 5.8(c) a time series of the end points of the period-3 orbit highlighted as \( p_3 \) in Figure 5.7(a) is shown. This shows that
freely rattling objects subject to periodic forcing have co-existing recurrent motions, periodic and/or chaotic, as has been shown before [71]. The three co-existing period-1 orbits in Figure 5.7(a) are reached from different initial conditions, i.e. the rod system is initially impacting the surface at different phases of the surface oscillation. The time history illustrated in Figure 5.8 (c) corresponds to initial conditions taken from the the period-1 orbit in Figure 5.7.

Figure 5.8: Time histories for $q_2$ showing (a) a period-3 solution (see label $p_3$ in Figure 5.7(a)), (b) a period-2 solution (see label $p_2$ in Figure 5.7(b)) and (c) a period-1 solution (see label $p_1$ in Figure 5.7(a)). $q_2$ varying in time for a specific frequency $\omega$. In (a) and (c) $\omega = 3.0$, $e_s = 0.9$, $A = 1$ and $\mu = 0.05$ and in (b) $\omega = 3.55085$, $e_s = 0.8$, $A = 1$ and $\mu = 0.05$.

5.3.1 Path Following of Period-1 Solutions

Of special interest for understanding the long-term behaviour in this system is to get a handle on how period-1 solutions behave under parameter variations. In Figure 5.10(a) we plot bifurcation diagrams using continuation for five different $e_s$ values under variation in $\omega$. Here we will give the details of the technique used to semi-analytically find solutions of these period-one orbits for the model example. This
technique was discussed in detail in Chapter 4. It is state 4 which is considered here and the dynamics of which can be described semi-analytically. This technique involves considering the periodic motion and describing it in a general form. We let $t_0$ denote the time at which the slender rod reaches the period-1 solution, let $t_i$ denote the time of the next impact and let $t_1$ denote the time at which the system returns to the period-1 solution again. We describe this in terms of two phases and apply the appropriate system conditions in each case as we describe below.

![Figure 5.9: Schematic illustrating the centre of mass position $q_2(t)$ of the slender rod and the oscillating surface in time. The points $t_0$, $t_i$ and $t_1$ are highlighted.](image)

**Phase 1, $t_0 \rightarrow t_i$:** The first phase of the motion, the rod moving from its maximum point, corresponding to the period-1 solution, to the impact point at time $t_i$, can be described as follows. Integration of (5.8) gives the position of the centre of mass $q_2(t)$ of the slender rod is given by

$$q_2(t) = -\frac{g(t - t_0)^2}{2} + \dot{q}_2(t_0)(t - t_0) + q_2(t_0) \tag{5.32}$$
and the corresponding velocity $\dot{q}_2(t)$ is given by

$$\dot{q}_2(t) = -g(t - t_0) + \dot{q}_2(t_0)$$

(5.33)

where $t_0$ is the time at which a relative first maximum of the system occurs (See Figure 5.9).

**Phase 2** $t_i \rightarrow t_1$: The second phase of the motion, describing the rod moving from the impact point back to the next maximum point at time $t_1$, can be defined as follows

$$q_2(t) = -\frac{g(t - t_i)^2}{2} + \dot{q}_2^+(t_i)(t - t_i) + q_2(t_i),$$

(5.34)

$$\dot{q}_2(t) = -g(t - t_i) + \dot{q}_2^+(t_i),$$

(5.35)

subject to the initial condition

$$q_2(t_i) = A \sin(\omega t_i),$$

(5.37)

which implies that the position of the slender rod at impact is given by the position of the surface at that time. Further, by periodicity we require that

$$q_2(t_0) = q_2 \left( t_0 + \frac{2\pi}{\omega} \right),$$

(5.38)

which means the rod will always return to the same maximum position after each impact. Also we know that at each maximum point the normal velocity of the rod is zero, thus given the following condition

$$\dot{q}_2(t_0) = \dot{q}_2 \left( t_0 + \frac{2\pi}{\omega} \right) = 0.$$  

(5.39)

As discussed in Section 5.2.1, in State 4 the system reduces to a simple impacting particle and the impact mapping given by (5.24) and (5.25) reduces to the simple form

$$\dot{q}_2(t_i)^+ = -e_a \left( \dot{q}_2(t_i)^- - A\omega \cos(\omega t_i) \right),$$

(5.40)

where the term in parenthesis on the right hand side of (5.40) is the relative velocity
of the rod at impact, and

\[ \dot{q}_i^+(t_i) = \dot{q}_i^-(t_i). \]

The second equation, which maps the tangential velocity, is irrelevant for the reduced one dimensional system. Further, it tells us no new information about the system.

Using (5.32) and (5.37) gives

\[ -\frac{g}{2} (t_i - t_0)^2 + q_{20} - A \sin(\omega t_i) = 0, \tag{5.41} \]

which is the first system equation. Further using (5.34) and (5.38) gives

\[ q_2(t_0) = -\frac{g}{2} \left( t_0 + \frac{2\pi}{\omega} - t_i \right)^2 + \dot{q}_2^+(t_i)(t_0 + \frac{2\pi}{\omega} - t_i) + q_2(t_i) = q_2(t_0 + \frac{2\pi}{\omega}) \]

and using (5.33)

\[ q_2(t_0) = -\frac{g}{2} \left( t_0 + \frac{2\pi}{\omega} - t_i \right)^2 + \dot{q}_2^+(t_i)(t_0 + \frac{2\pi}{\omega} - t_i) + A \sin(\omega t_i) \tag{5.42} \]

and using (5.42) together with the impact law given by (5.40) yields

\[ -q_2(t_0) - \frac{g}{2} \left( t_0 + \frac{2\pi}{\omega} - t_i \right)^2 + e_\ast \left( t_0 + \frac{2\pi}{\omega} - t_i \right) \left( g(t_i - t_0) + A \omega \cos(\omega t_i) \right) + A \sin(\omega t_i) = 0, \tag{5.43} \]

which is the second system equation. Using (5.39) and (5.35) gives

\[ \dot{q}_2(t_0 + \frac{2\pi}{\omega}) = -g \left( t_0 + \frac{2\pi}{\omega} - t_i \right) + \dot{q}_2(t_i)^+ = 0 \]

and again using (5.33) together with the impact law given by (5.40) gives

\[ -g \left( t_0 + \frac{2\pi}{\omega} - t_i \right) + e_\ast \left( g(t_i - t_0) + A \omega \cos(\omega t_i) \right) = 0, \tag{5.44} \]

which is the final system equation. Using (5.41), (5.43) and (5.44) we define the following nonlinear system \( f \) as follows

\[ f = f(t_0, t_i, t_1)^T = (f_1(t_0, t_i, t_1), f_2(t_0, t_i, t_1), f_3(t_0, t_i, t_1))^T, \tag{5.45} \]
where

\[ f_1 = -\frac{g (t_i - t_0)^2}{2} + q_2 - A \sin(\hat{\omega}), \quad (5.46) \]

\[ f_2 = -q_2(t_0) - \frac{gt^2}{2} + e_s \hat{t} (g (t_i - t_0) + A\omega \cos(\hat{\omega})) + A \sin(\hat{\omega}), \quad (5.47) \]

\[ f_3 = -g\hat{t} + e_s (g (t_i - t_0) + A\omega \cos(\hat{\omega})), \quad (5.48) \]

where \( \hat{\omega} = \omega(t_i), \ \hat{t} = t_0 + \frac{2\pi}{\omega} - t_i. \) The nonlinear system \( f = (f_1, f_2, f_3)^T = (0, 0, 0)^T \) has no closed form solution and thus has to be solved semi-analytically using Newton’s method as described in Chapter 4. A semi-analytical solution is a solution to a system of analytically derived equations that need to be solved numerically. We let \( t_i = x(1), t_0 = x(2) \) and \( t_1 = x(3), \) then \( x = (x(1), x(2), x(3))^T. \) Newton’s method, as described in detail in Chapter 4, then states that the solution \( x \) can be approximated by iteration of the following

\[ x_{i+1} = x_i - (J(x_i))^{-1} f(x_i), \quad (5.49) \]

where \( x_0 \) is the vector of the initial guess of the solution and \( J \) is the Jacobian given by

\[
J(t_0, t_i, t_1) = \begin{pmatrix}
\frac{\partial f_1}{\partial t_i} & \frac{\partial f_1}{\partial t_0} & \frac{\partial f_1}{\partial t_1} \\
\frac{\partial f_2}{\partial t_i} & \frac{\partial f_2}{\partial t_0} & \frac{\partial f_2}{\partial t_1} \\
\frac{\partial f_3}{\partial t_i} & \frac{\partial f_3}{\partial t_0} & \frac{\partial f_3}{\partial t_1}
\end{pmatrix},
\]
which for this example has the coefficients

\[
\begin{align*}
\frac{\partial f_1}{\partial t_i} &= -g(t_i - t_0) - A\cos(\omega t_i)\omega, \\
\frac{\partial f_1}{\partial t_0} &= g(t_i - t_0), \\
\frac{\partial f_1}{\partial t_1} &= 1, \\
\frac{\partial f_2}{\partial t_i} &= -e_\star g(t_i - t_0) - e_\star A\omega \cos(\omega t_i) + (t_0 + \frac{2\pi}{\omega} - t_i) \left(e_\star g - A\omega^2 \sin(\omega t_i) (e_\star - 1) + g\right), \\
\frac{\partial f_2}{\partial t_0} &= e_\star g(t_i - t_0) + e_\star A\omega \cos(\omega t_i) + A\cos(\omega t_i)\omega - (t_0 + \frac{2\pi}{\omega} - t_i) e_\star g - g(t_0 + \frac{2\pi}{\omega} - x(1)), \\
\frac{\partial f_2}{\partial t_1} &= -1, \\
\frac{\partial f_3}{\partial t_i} &= g + e_\star (g - A\sin(\omega t_i)\omega^2) - A\sin(\omega t_i)\omega^2, \\
\frac{\partial f_3}{\partial t_0} &= -g(e_\star + 1), \\
\frac{\partial f_3}{\partial t_1} &= 0.
\end{align*}
\]

Using (5.49) iteratively together with the above coefficients allows for the calculation of the period-one orbits as shown in Figure 5.10. The initial condition vector \(X_0\) was chosen by estimating values from the branch of the period-1 solution from the brute force bifurcation diagram labeled \(p_1\) in Figure 5.7 (a).

The figure shows that period-1 orbits are born at a smaller frequency \(\omega\) the bigger the \(e_\star\) value is. It seems obvious that if less energy is taken out at impact then less energy is needed from the oscillating surface to sustain a similar periodic orbit. In the inset I we highlight how the branches of periodic orbits retract as \(e_\star\) is increased. In Figure 5.10(b) we show a magnification of the region II in Figure 5.10(a) to highlight that the stable branches born at saddle-node bifurcations (SN) undergo period-doubling bifurcations (PD) for increasing \(\omega\), which we also see in Figure 5.7(a). Figure 5.10(b) also shows how the unstable periodic orbits born at SN bifurcations disappear at grazing bifurcations (G).

**Stability of the Period-1 Solutions**

For the period-1 solutions the stabilities are found by calculating the solution of the first variational equations of the piecewise-smooth system. For this purpose salutation
matrices are used to merge together fundamental solution matrices for trajectories that switch form one state to another as described in Section 4.3.2.

For this purpose, we use (4.22) such that

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x_2} \frac{\partial D}{\partial x_2} (x_{in}) \frac{\partial v}{\partial x_1}.$$  (5.50)

The matrix $\frac{\partial v}{\partial x_1}$ describes how the Jacobian changes along the trajectory moving from $t_0 \to t_i$. The matrix $\frac{\partial v}{\partial x_2}$ describes how the Jacobian changes along the trajectory moving from $t_i \to t_1$. The stability of the periodic orbit is then determined by
calculating the eigenvalues of (5.50), where the terms needed in (5.50) are given by

\[
\frac{\partial g}{\partial X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e^* & -(1 + e^*)\omega^2 \sin(\omega \tau) \\ 0 & 0 & 1 \end{pmatrix},
\]

(5.51)

\[
\frac{\partial h}{\partial X} = \begin{pmatrix} 1 & 0 & \omega \cos(\omega \tau) \end{pmatrix},
\]

(5.52)

\[
f_{in} = f_{out} = \begin{pmatrix} 0 & -g & 1 \end{pmatrix}^T,
\]

(5.53)

(5.54)

where \(X\) is the reduced one dimensional dynamical system.

### 5.4 Discussion

In this Chapter we have introduced a general framework for numerical simulation of rigid-body systems with impacts and friction, specifically using the energetic impact law first derived in [1]. Further we implemented the energetic impact law in MATLAB, allowing for simulation of long-term dynamics of a planar rod where the two end points can impact an oscillating surface. For this purpose we introduced a framework that includes an ODE solver, system states and a transition diagram. This allowed us to reliably simulate the system in free flight, through impacts, in sticking and through complete chatter. For instance, the approach involved switching vector fields when the system is transitioning between stick and free flight motion, where the vector field for stick was found by calculating the normal force required to constrain the contact point to the surface.

As mentioned above, we implemented the general framework for the model example of a slender rod impacting a periodically forced surface. This example highlighted some of the phenomena present in impacting systems with friction, and in particular symmetric systems. The model example can, in some sense, be likened to the problem of machine rattle, whereby a machine element becomes detached and is free to undergo unconstrained mechanical vibration. For the rod system we examined the dynamics under variation of the frequency of surface oscillation \(\omega\). We showed that for high values of the restitution coefficient and low frequencies multiple periodic solutions coexist, but they do not survive an increase in \(\omega\), but instead chaotic regimes.
take over. Another aspect that was highlighted for low frequencies $\omega$ is how friction removes rotational energy so that mainly symmetric motions (impacts of both end point of the rod at the same time) persists.
Chapter 6

The Painlevé Paradox

In this Chapter we will discuss the phenomenon of the Painlevé paradox and analyse under what conditions it can occur in two model problems. We will use two mechanical systems, namely, the slender rod system introduced in Chapter 3, and a double pendulum system which we will introduce here. We will show that it is possible to achieve the Painlevé paradox for relatively low coefficient of friction values.

6.1 Introduction

Dynamics involving impacts with friction can be extremely difficult to model due to inherent non-smoothness and discontinuities in the friction laws. Incorporating Amontons-Coulomb friction with a rigid body formulation gives rise to a number of configurations where a solution is non-unique or in some cases does not exist [72]. A major issue with rigid body impact theory is that, depending on the method used to model the interaction, certain configurations lead to unbounded solutions, and the space within which the solutions have to be defined to prevent crossings of the system into such inconsistent regions must be augmented to discontinuous velocities and distributional interaction forces even if no collision has occurred [9].

One special case which occurs in impacts with friction is that of the Painlevé paradox. The Painlevé paradox is the phenomenon whereby an increase in negative acceleration occurs from an initial period of sliding, i.e. an impact without collision. This occurs when the friction force is large enough to cause a moment about the centre of mass of the rigid body which is greater than the magnitude of the moment of the corresponding normal force. This results in a compressive normal relative ac-
acceleration at the contact point [72], sometimes referred to as Jam. The large friction force leads to a large rotational acceleration which in turn gives a negative normal relative acceleration at the contact point [72]. This can be seen when pushing a piece of chalk across a blackboard for example, or in the pin-on-disc braking mechanism in a car whereby the pin periodically jumps from the disc, resulting in a screeching noise [41]. The dependence of frictional dissipation on the direction of slip relative to the angle of inclination of the centre of mass is particularly noticeable. There is much more frictional dissipation when the bar strikes with the end that leads the centre of mass, much as a javelin does at the end of its flight [4].

Various numerical and experimental investigations of the Painlevé paradox have been carried out. Zhao et. al [73] consider the experimental setup of a two-link robotic arm in contact with a moving rail. A bouncing motion is seen which can be attributed to the Painlevé paradox. They also present the analytical model of the setup. Leine et. al [41] set up a numerical experiment to replicate the stick-slip motion experienced when a human finger is pushed over a rough surface. They make use of a frictional impact oscillator to do this. The occurrence of the Painlevé paradox in a rotating shaft mechanism [74] has also been considered. Lancioni et. al [75] investigate the physically observed phenomenon of the hopping motion of window wipers when in moving contact with the windscreen. They use a frictional impact oscillator model to capture this behaviour using an analysis that was very similar to the work done in [41].

6.2 Example: Slender Rod

The purpose of this Section is twofold. First, we will present the analysis of one of the classical Painlevé formulations and use this to illustrate the mathematical conditions under which the paradox will occur. Second, we will use this example to discuss the relevance of the Painlevé paradox to the Brach impact law derived in Chapter 3. To this purpose we will revisit the model problem, analysed in Chapter 3, of a slender rod in contact with a noncompliant plane.

The analysis in this Section again assumes that one of the tips of a uniform slender rod is in contact with a noncompliant surface at a point C and where we again let \( q_1 \), \( q_2 \) be the tangential and normal position of the centre of mass, relative to the contact plane, and let \( \theta \) be the angle of rotation of the rod. Without loss of
generality we assume the distance from the centre of mass to the tip is $L = 1$ and the mass $m = 1$ so that the moment of inertia $I = \frac{1}{3}$ and the radius of gyration $k_r^2 = \frac{1}{3}$. We will now analyse under what conditions the Painlevé paradox can occur for this model system.

Consider the general formulation derived in Section 3.2.2. Using (3.41) we can relate relative contact point velocity $\dot{q}_C$ changes in terms of the normal impulse of collision $P_2$, such that

$$\frac{d\dot{q}_C}{dP_2} = \frac{1}{\lambda_2} \left( \begin{array}{cc} \hat{A} & -\hat{B} \\ -\hat{B} & \hat{C} \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right).$$

For the model example considered here we have that

$$\hat{A} = 1 + 3 \sin^2(\theta), \quad \hat{B} = 3 \sin(\theta) \cos(\theta), \quad \hat{C} = 1 + 3 \cos^2(\theta), \quad (6.1)$$

to give

$$\frac{d\dot{q}_C}{dP_2} = \frac{1}{\lambda_2} \left( \begin{array}{cc} 1 + 3 \sin^2(\theta) & -3 \sin(\theta) \cos(\theta) \\ -3 \sin(\theta) \cos(\theta) & 1 + 3 \cos^2(\theta) \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right). \quad (6.2)$$

As discussed in Section 6.1, the Painlevé paradox will occur when there is anti-proportional forcing in the system or, in other words, when the rate of change of the normal contact point velocity with respect to the normal impulse becomes negative.
Therefore, to examine the conditions under which the Painlevé paradox can occur, we need to consider the normal components of (6.2) such that
\[
\frac{d\dot{\tilde{q}}_C}{dP_2} = -3\sin(\theta)\cos(\theta)\frac{\lambda_1}{\lambda_2} + 1 + 3\cos^2(\theta) = \hat{B}\frac{\lambda_1}{\lambda_2} + \hat{C},
\]
and imposing the Amontons-Coulomb friction law (2.38) gives
\[
\frac{d\dot{\tilde{q}}_C}{dP_2} = -3\sin(\theta)\cos(\theta)\mu + 1 + 3\cos^2(\theta) = \hat{B}\mu + \hat{C},
\]
where \(\mu = \frac{\lambda_1}{\lambda_2}\).

The Painlevé paradox occurs when tangential velocity is transferred to normal velocity at impact, even in the case where the normal velocity is zero at impact or in other words where an increase in negative acceleration occurs from an initial period of sliding, i.e. an impact without collision. This means that for the Painlevé paradox to occur there has to be a critical value of \(\mu = \mu_c\) for which the rate of change of normal velocity with respect to normal impulse is equal to 0. Using (6.4) we get that the critical value is
\[
\mu_c = -\frac{\hat{C}}{\hat{B}} = \frac{1 + 3\cos^2(\theta)}{3\sin(\theta)\cos(\theta)}.
\]

Any value of \(\mu\) exceeding \(\mu_c\) will result in a decrease in normal acceleration which leads to an increase in the normal velocity after impact, the Painlevé paradox. Further, the minimum value for \(\mu_c\) such that this can occur can be found by minimising (6.5), which gives that
\[
\frac{d\mu_c}{d\theta} = 0 \Rightarrow \frac{d\mu_c}{d\theta} = \left(4\frac{1}{\sin(4\theta)} + 3\right) - 1 = 0,
\]
and assuming \(\theta\) is positive we can write (6.7) as
\[
4\cot(\theta) = \tan(\theta),
\]
which has the solution \(\theta = \tan^{-1}(2)\), which corresponds to \(\mu_c = \frac{4}{3}\). This result is consistent with the work done in [1], [42] and [76]. Now we can compare \(\mu_c\) to what we got when we derived the Brach impact mapping in Section 3.1.2. There is a
singularity of the mapping when the denominator in (3.10)-(3.12) is 0 and as we saw in (3.13) it is given by

$$\mu_s = \frac{k_r^2 + x^2}{sxy},$$

where $x$ is the position of the centre of mass relative to the contact point in the tangential direction, $y$ is the position of the centre of mass relative to the contact point in the normal direction and where $s = \pm 1$ depends on the initial direction of sliding. For the case of the slender rod this ratio reduces to

$$\mu_s = \frac{1 + 3\cos^2(\theta)}{3\sin(\theta)\cos(\theta)}, \quad (6.9)$$

which is the same value as what we have in (6.5) for $s = 1$. This means that the minimum value such that the Painlevé paradox can occur corresponds to the same value at which the Brach impact mapping becomes singular. To emphasize this point, we consider again Figure 3.3 (a) from Chapter 3, see Figure 6.2. We see that

![Figure 6.2: Plots of $\bar{T}_L(\mu)$ for $e = 0$, $e = 0.5$ and $e = 1$, where $\theta = \tan^{-1}(2)$, $V_T^- = -2$, $V_N^- = -1.5$ and $\dot{\theta}^- = 1.9$. All three curves corresponds to Case 6 described in Section 3.1.2.](image)

the normalised kinetic energy loss $\bar{T}_L$ approaches $-\infty$ as $\mu$ approaches $\frac{4}{3}$. This plot is for the specific case of $\theta = \tan^{-1}(2)$, the minimum value derived above.

The conditions under which the Painlevé paradox will occur in this simple setup have been derived and analysed. It is only possible to achieve this phenomenon for a relatively high $\mu$ value which, for most collisions, is unphysical. This naturally leads
us to ask the question of will different mechanical systems give configurations which permit the Painlevé paradox for lower and more realistic $\mu$ values. In Section 6.3 we will construct one such setup.

6.3 Example: Double Pendulum

The aim of this Section is twofold. First, we will show that it is possible to achieve the Painlevé paradox for relatively low coefficient of friction values, and second we will use the Energetic impact law detailed in Chapter 3 to show, numerically, the occurrence of the Painlevé paradox. To this purpose we now consider a second model system. We will use the example of a double pendulum subject to external forcing due to gravity and an external torque $T$ as shown in Figure 6.3. Let $q = (\theta, \phi)^T$ be the generalized coordinates of the system, where $\theta$ is the absolute angle of the upper arm with length $l_1$ and $\phi$ is the absolute angle of the lower arm with length $l_2$. We assume that the mass of the arms is located at their end points and the arms act as massless rigid connections. The upper arm has mass $m_1$ and the lower arm has mass $m_2$. Both joints are completely rigid and frictionless and a constant torque $T$ is applied at the upper joint.

The equations of motion for the system can be determined using a Lagrangian formulation as discussed in Chapter 2. We assume that the lower mass can make contact with the plane at a point $C$ and we then introduce a coordinate system $n_1 - n_2$, as shown in Figure 6.3, where the subscript 1 denotes the direction tangential to the contact plane and the coordinate 2 denotes the direction normal to the tangent plane. For ease of modelling, we define the position $c$ of the contact point $C$ in terms of the $n_1 - n_2$ coordinate system as follows

$$
\mathbf{c} := (c_1, c_2)^T = (-l_1 \cos(\theta) - l_2 \cos(\phi), l_1 \sin(\theta) + l_2 \sin(\phi) + d)^T, \quad (6.10)
$$

where $d$ is the distance in the horizontal ($n_2$) direction between the upper joint of the double pendulum and the plane. The velocity components of the contact point are found by evaluating the time derivative of (6.10) to give

$$
\mathbf{\dot{c}} := (\dot{c}_1, \dot{c}_2)^T = \left( l_1 \sin(\theta) \dot{\theta} + l_2 \sin(\phi) \dot{\phi}, l_1 \cos(\theta) \dot{\theta} + l_2 \cos(\phi) \dot{\phi} \right)^T. \quad (6.11)
$$

It is useful to calculate the normal to the contact plane $\mathbf{N}$ as it required for translation.
Figure 6.3: A double pendulum in which the lower mass $m_2$ can impact with a non-compliant vertical plane. The distance between the upper joint and the plane is denoted by $d$.

of velocities and forces from the generalized coordinates frame $\theta - \phi$ to the $n_1 - n_2$ reference frame such that

$$N = \frac{\partial c}{\partial q} = \begin{pmatrix} l_1 \sin(\theta) & l_2 \sin(\phi) \\ l_1 \cos(\theta) & l_2 \cos(\phi) \end{pmatrix}.$$  (6.12)

We now follow the methodology described in Section 2.1.2 to determine the equations of motion for this model system. First, the kinetic energy $T$ can be determined by using (2.10) to give

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 \left( l_1^2 \ddot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2l_1 l_2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right).$$  (6.13)
CHAPTER 6. THE PAINLEVÉ PARADOX

Next, using (2.11) we can calculate the potential energy by finding a function \( V \) such that

\[
Q_i = -\nabla V = -\frac{\partial V}{\partial q_i},
\]

(6.14)

where \( Q_i \) are the conservative forces acting on the system which, for this system, is the force due to gravity only and is given by

\[
Q_i = -\left( \begin{array}{c} (m_1 + m_2) g \\ m_2 g \end{array} \right)^T N^T \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]

(6.15)

Integration of (6.14) with respect to the generalized coordinates gives

\[
V = -(m_1 + m_2) g l_1 \cos(\theta) - m_2 l_2 \cos(\phi),
\]

(6.16)

the potential energy of the system. Next, using (2.12) we can construct the Lagrangian \( \mathcal{L} \) for the system such that

\[
\mathcal{L} = T - V = \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 \left( l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right) + (m_1 + m_2) g l_1 \cos(\theta) + m_2 l_2 \cos(\phi).
\]

(6.17)

Finally, we can find the equations of motion of the system from (2.13) together with (6.17). For purpose of clarity, and for ease of analysis, we will express the equations of motion in the following form

\[
M(q)\ddot{q} + L(q, \dot{q}) + U(q) = F_c^T,
\]

(6.18)

where the mass matrix \( M(q) \) is given by

\[
M(q) = \left( \begin{array}{cc} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \cos(\theta - \phi) \\ m_2 l_1 l_2 \cos(\theta - \phi) & m_2 l_2^2 \end{array} \right),
\]

and the terms dependent on the generalised coordinates and velocities \( L(q, \dot{q}) \) are given by

\[
L(q, \dot{q}) = l_1 l_2 \sin(\theta - \phi) m_2 \left( \begin{array}{c} \dot{\theta}^2 \\ \dot{\phi}^2 \end{array} \right).
\]
Further, the potential terms \( U(q) \) are given by

\[
U(q) = \begin{pmatrix}
(m_1 + m_2) gl_1 \sin(\theta) - T \\
m_2 gl_2 \sin(\phi)
\end{pmatrix}
\]

and where \( F^T_c \) is the contact force which is generated when the lower mass is in contact with the plane where

\[
F^T_c = (\lambda N)^T = N^T \lambda^T
\]

for

\[
\lambda = (\lambda_2, \lambda_1),
\]

and where \( \lambda_2 \) is the normal component of the forcing and \( \lambda_1 \) is the tangential component of the forcing. For this problem we assume that the tangential force has a component which is due to Amontons-Coulomb friction. When the system is in free flight motion we have that \( \lambda = (0, 0) \), and the same equations of motion (6.18) can be used. Now that we have derived the equations of motion for the system when in free flight and in contact, we next need to consider an impact law to apply at contact.

### 6.3.1 Impact Law

We will use the Energetic impact law [1] presented in Chapter 3 to model the system when the lower arm impacts with the plane. As discussed in Chapter 3, the impact law requires the relative contact point velocities in directions tangential and normal to the contact plane. For this purpose the tangential and normal components of (6.11) can be used. The rate constants for this system are found by using (3.43) and (3.44) where the symmetric matrix \( m^{-1} \) is found by evaluating

\[
m^{-1} = NM^{-1}N^T = \begin{pmatrix}
\hat{A} & \hat{B} \\
\hat{B} & \hat{C}
\end{pmatrix},
\]

where

\[
M^{-1}(q) = \begin{pmatrix}
-1 & \cos(\theta-\phi)^2 m_2 - m_1 - m_2 \\
\frac{l_1}{l_1 l_2} \cos(\theta-\phi)^2 m_2 - m_1 - m_2 & \frac{\cos(\theta-\phi)}{l_1 l_2} m_1 + m_2 \\
\frac{1}{l_1 l_2} \cos(\theta-\phi)^2 m_2 - m_1 - m_2 & \frac{m_2 l_2}{l_2} \cos(\theta-\phi)^2 m_2 - m_1 - m_2
\end{pmatrix}
\]

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and where

\[
\hat{A} = \sin(\theta) - \sin(\theta + \phi) \sin(\phi) + \sin(\phi) \left( \frac{\cos(\theta - \phi) \sin(\theta)}{\cos^2(\theta - \phi)m_2 - m_1 - m_2} + \frac{(m_1 + m_2) \sin(\phi)}{m_2 (\cos^2(\theta - \phi)m_2 - m_1 - m_2)} \right),
\]

\[
\hat{B} = \sin(\theta) - \cos(\theta + \phi) \cos(\phi) + \cos(\phi) \left( \frac{\cos(\theta - \phi) \cos(\theta)}{\cos^2(\theta - \phi)m_2 - m_1 - m_2} + \frac{(m_1 + m_2) \cos(\phi)}{m_2 (\cos^2(\theta - \phi)m_2 - m_1 - m_2)} \right),
\]

and

\[
\hat{C} = \cos(\theta) - \cos(\theta + \phi) \cos(\phi) + \cos(\phi) \left( \frac{\cos(\theta - \phi) \cos(\theta)}{\cos^2(\theta - \phi)m_2 - m_1 - m_2} + \frac{(m_1 + m_2) \cos(\phi)}{m_2 (\cos^2(\theta - \phi)m_2 - m_1 - m_2)} \right).
\]

We next consider (6.18) and solve for \( \ddot{q} \) to give

\[
\ddot{q} = M^{-1} N^T \lambda^T + ..., \tag{6.21}
\]

where the terms which do not change throughout the impact phase have been neglected. For this analysis it is only necessary to consider velocity changes which occur throughout the impact phase. Multiplying across by the normal \( N \) gives

\[
N \ddot{q} = NM^{-1} N^T \lambda^T + ... = m^{-1} \lambda^T
\]

where \( m^{-1} \) is given by (6.20), and where we define

\[
N \ddot{q} = \frac{d\dot{q}_c}{dt},
\]

which is the contact point acceleration, and where

\[
\frac{d\dot{q}_c}{dt} = m^{-1} \lambda^T \Rightarrow \quad m \frac{d\dot{q}_c}{dt} = \lambda^T. \tag{6.22}
\]

Using (6.21) together with (6.22) allows us to write the velocity transformation between contact point accelerations and centre of mass accelerations as

\[
\ddot{q} = M^{-1} N^T m \frac{d\dot{q}_c}{dt}.
\]
This general transformation is necessary as the impact law maps contact point velocities before impact to the contact point velocities after impact. For numerical simulations we typically use centre of mass velocities.

### 6.3.2 Impulse Space Considerations

In this Section we examine under what conditions the Painlevé paradox can occur for the model example presented in Section 6.3. Taking the time derivative of (6.11) gives

\[
\ddot{c} := (\ddot{c}_1, \ddot{c}_2)^T,
\]

which is the acceleration of the contact point, where the tangential acceleration \(\ddot{c}_1\) is given by

\[
\ddot{c}_1 = l_1 \sin(\theta) \dot{\theta} + l_2 \sin(\phi) \dot{\phi} + l_1 \cos(\theta) \dot{\phi}^2 + l_2 \cos(\phi) \dot{\phi}^2,
\]

and where the normal acceleration \(\ddot{c}_2\) is given by

\[
\ddot{c}_2 = l_1 \cos(\theta) \dot{\theta} + l_2 \cos(\phi) \dot{\phi} - l_1 \sin(\theta) \dot{\theta}^2 - l_2 \sin(\phi) \dot{\phi}^2.
\]

In order to examine the exact conditions required for the Painlevé paradox to occur it is first necessary to express velocity changes as a function of the normal impulse \(P_2\) instead of time. Again using (3.41) we can relate contact point velocity changes in terms of the normal impulse of collision \(P_2\) such that

\[
\frac{d\dot{q}_C}{dP_2} = \frac{1}{\lambda_2} \begin{pmatrix} \hat{A} & -\hat{B} \\ -\hat{B} & \hat{C} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.
\]

Again this means that for the Painlevé paradox to occur there has to be a critical value of \(\mu = \mu_c\) for which the rate of change of normal velocity with respect to normal impulse is equal to 0. Using (6.4) we get that

\[
\mu_c = \frac{m_2 (-\cos^2(\theta) + 2 \cos(\theta - \phi) \cos(\theta) \cos(\phi))}{\sin(\theta) m_2 (\cos(\theta - \phi) \cos(\phi) - \cos(\theta)) + \sin(\phi) m_2 \cos(\theta - \phi) \cos(\theta) - \sin(\phi) \cos(\phi) \dot{m}} - \frac{\dot{m} \cos(\phi) \cos(\theta)}{\sin(\theta) m_2 (\cos(\theta - \phi) \cos(\phi) - \cos(\theta)) + \sin(\phi) m_2 \cos(\theta - \phi) \cos(\theta) - \sin(\phi) \cos(\phi) \dot{m}}
\]

where \(\dot{m} = (m_1 + m_2)\). In order to select an appropriate value for \(\mu\), we hold \(\phi\)
constant and find values for $\theta$ such that $\hat{B}\mu > -\hat{C}$. This allows us to find appropriate initial conditions under which the Painlevé paradox will occur for relatively low $\mu$ values. In Figures 6.4 (a) and 6.5 (a) we consider two such cases of this, where we are plotting $\hat{B}$ and $\hat{C}$ as functions of $\theta$. In Figures 6.4 (b) and 6.5 (b), we show a magnification of the region $I$ in (a) in which $\hat{B} > \hat{C}$. In these regions of state space it is possible to achieve the Painlevé paradox for $\mu$ values less than 1. With
this information in mind, we can now devise a numerical experiment to ensure, at impact, the system parameters will always lie in the desired region of state space.

Figure 6.6: The forced double pendulum in constrained contact with a non-compliant plane moving vertically with velocity $V_T$.

Consider the setup in Figure 6.6. For a given set of initial conditions, we can calculate the required torque $T$ and normal forcing $\lambda_2$ to ensure the double pendulum will remain in stationary contact with the plane. Now we suppose that the plane is a moving conveyor belt with velocity $V$ orientated in the vertical direction as shown in Figure 6.6, and at a certain point on this conveyor belt the friction coefficient $\mu$ changes such that $\mu \mapsto \mu_p$. We model this change in $\mu$ as an impact and apply the Energetic impact law accordingly. Now that we have described the model setup, the next step is to find equilibrium points to ensure the system remains in static equilibrium.
6.3.3 Constructing the Vector Field

In this Section we derive the vector fields required for the implementation of this numerical experiment. As we are only interested in a one-impact case, we do not need to derive a numerical scheme for switching between states as was required in Chapter 5. Using (6.18) together with the Amontons-Coulomb friction law (2.38) and recasting as a system of four ODE’s we arrive at the following dynamical system

\[
(\dot{\theta}, \dot{\phi}, \dot{q}_3, \dot{q}_4, \dot{\tau})^T = (q_3, q_4, \alpha_3, \alpha_4, 1)^T := (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5)^T = \Phi(\theta, \phi, q_3, q_4, \tau),
\]

(6.24)

where \(\Phi\) is the vector field and where

\[
\alpha_3 = \frac{(T - l_1 l_2) s_{12} m_2 \dot{\phi}^2 (1 - c_{12}^2)}{l_1^2 \hat{m}} + \frac{c_{12} (c_{12} m_2 l_1 K + l_1 \hat{m} (m_2 l_1 \dot{\phi}^2 s_{12} + \bar{F}))}{l_1^2 \hat{m} (m_2 (c_{12}^2 - 1) - m_1 + l_1 \bar{K})},
\]

\[
\alpha_4 = \frac{c_{12} (T - l_1 l_2 m_2 \dot{\phi}^2 + s_1 \lambda_1 - \dot{\hat{m}} g s_1 + c_1 \lambda_2) - \hat{m} (l_1 \dot{\phi}^2 s_{12} + \bar{F})}{l_2 (m_2 (c_{12}^2 - 1) - m_1)},
\]

where \(s_{12} = \sin(\theta - \phi), \quad c_{12} = \cos(\theta - \phi), \quad s_1 = \sin(\theta), \quad s_2 = \sin(\phi), \quad c_1 = \cos(\theta), \quad c_2 = \cos(\phi), \quad K = -s_1 \lambda_1 - c_1 \lambda_2 + g s_1 \hat{m}, \quad \bar{F} = c_2 \lambda_2 + s_2 \lambda_1 - m_2 g s_2.

Using the methodologies described in Chapter 4, (6.24) can then be solved numerically using the MATLAB ode45 solver.

6.3.4 Equilibrium Points

In this Section we will find equilibrium values for the system by calculating the torque \(T\) and the normal forcing \(\lambda_2\) such that the double pendulum will remain in static equilibrium when in contact with the plane. Consider again (6.18)

\[
\dot{\bar{q}} = -M^{-1} L(q, \bar{q}) - M^{-1} U(q) + M^{-1} F^T.
\]

(6.25)

We now solve the system for \(\lambda_2\) and \(T\) subject to the following static conditions

\[
\dot{\bar{q}} = (\dot{\theta}, \dot{\phi})^T = (0, 0)^T, \quad \dot{\theta} = \dot{\phi} = 0.
\]

(6.26)
Using (6.26) together with (6.25) gives the system

\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -M^{-1} \left( U(q) - F_c^T \right),
\]

and since \( M \) is invertible we get

\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = U(q) - F_c^T,
\]

The expression in (6.28) illustrates that the system will remain in equilibrium provided the potential terms \( U(q) \) are in balance with the contact forcing terms \( F_c^T \).

Now, using (6.19) and assuming sliding friction such that \( \lambda_1 = \mu \lambda_2 \) we have that

\[
F_c^T = \begin{pmatrix} l_1 \sin(\theta) \mu \lambda_2 + l_1 \cos(\theta) \lambda_2 \\ l_2 \sin(\phi) \mu \lambda_2 + l_2 \cos(\phi) \lambda_2 \end{pmatrix}.
\]

Solving (6.28) for \( \lambda_2 \) and \( T \) gives

\[
\lambda_2 = \frac{m_2 g \sin(\phi)}{\sin(\phi) \mu + \cos(\phi)}
\]

and

\[
T = (m_1 + m_2) gl_1 \sin(\theta) - \frac{m_2 g \sin(\phi)}{\sin(\phi) \mu + \cos(\phi)} (l_1 \sin(\theta) \mu + l_1 \cos(\theta)).
\]

The main reason for having \( T \) as an unknown is that it allows us to impose any \( \mu \) value for the system when in static equilibrium, and thus giving us more freedom in our parameter choice. The equilibrium point \( x^* = (x^*_1, x^*_2, x^*_3, x^*_4) \) will always be of the form

\[
x^* = (x^*_1, x^*_2, x^*_3, x^*_4) = (\theta^*, \phi^*, 0, 0),
\]

where \( \theta^* \) and \( \phi^* \) are chosen such that the lower mass will be in contact with the plane. Using (6.29) and (6.30) in the vector field (6.24) ensures that any value we choose for \( \theta \) and \( \phi \) will be an equilibrium value. Note that in (6.31) we have omitted the variable \( \tau \) as it is not physical to consider a stationary value for time for this system.
Stability of the Equilibrium Points

The stability of the equilibrium points $x^*$ calculated in Section 6.3.4 can be determined using the methodology described in Section 4.3.2. To determine the stability of the vector field at the point $x^*$ we consider a local linearisation in the neighborhood of the point and we do this by evaluating the Jacobian of the vector field at the equilibrium point. Using (4.10) together with (6.24) we can define the Jacobian of this system as

$$J(x^*) = \begin{pmatrix}
\frac{\partial \Phi_1}{\partial \theta} & \frac{\partial \Phi_1}{\partial \phi} & \frac{\partial \Phi_1}{\partial \dot{\theta}} & \frac{\partial \Phi_1}{\partial \dot{\phi}} \\
\frac{\partial \Phi_2}{\partial \theta} & \frac{\partial \Phi_2}{\partial \phi} & \frac{\partial \Phi_2}{\partial \dot{\theta}} & \frac{\partial \Phi_2}{\partial \dot{\phi}} \\
\frac{\partial \Phi_3}{\partial \theta} & \frac{\partial \Phi_3}{\partial \phi} & \frac{\partial \Phi_3}{\partial \dot{\theta}} & \frac{\partial \Phi_3}{\partial \dot{\phi}} \\
\frac{\partial \Phi_4}{\partial \theta} & \frac{\partial \Phi_4}{\partial \phi} & \frac{\partial \Phi_4}{\partial \dot{\theta}} & \frac{\partial \Phi_4}{\partial \dot{\phi}}
\end{pmatrix}$$

(6.32)

The stability of the equilibrium point is then determined by calculating the eigenvalues of (6.32). By numerical investigation it was found that it was not possible to find a stable configuration and therefore the equilibrium point was always unstable.

To illustrate this point we will calculate the eigenvalues $E$ for the system using the initial conditions that will be used for Figure 6.7 (a) and Figure 6.8 (a). For $\theta = 3.8572$, $\phi = 4.1154$ and $V = -0.96$ we have that

$$E_1 = 9.38, \quad E_2 = -9.38, \quad E_3 = 8.59i, \quad E_4 = -8.59i$$

and for the case of $\theta = 4.21$, $\phi = 4.3$ and $V = -0.98$ we have that that

$$E_1 = 9.44, \quad E_2 = -9.44, \quad E_3 = 8.50i, \quad E_4 = -8.50i.$$ 

In both cases we see that the real part of one of the four eigenvalues is positive, meaning the system is unstable. By numerical investigation we were unable to find a configuration in which all of the eigenvalues were negative.

### 6.3.5 Results

In this Section we perform some numerical simulations of the system when in stationary contact with the conveyor belt as described in Section 6.3.2. We will illustrate by example that it is possible to achieve the Painlevé paradox for this setup. The system is constructed such that the conveyor belt is moving with constant velocity in the negative tangential direction, and at a pre-defined point on the belt, the friction
coefficient changes from $\mu \mapsto \mu_p$. In Figures 6.7 and 6.8 we show the normal contact point velocity of the lower arm of the double pendulum as a function of time. The normal contact point velocity is zero until the the coefficient of friction changes. If the impact law was applied to the system before this point is reached, then the result would be that the normal contact point velocity would be mapped to zero, meaning

Figure 6.7: Plot of the normal contact point velocity $\dot{c}_2$ as a function of time for $\theta = 3.8572$, $\phi = 4.1154$, $V = -0.96$, $e = 0.7$. In (a) for $\mu_0 = 0.7$, $\mu_c = 0.82$ and in (b) $\mu_0 = 0.7$, $\mu_p = 1.3$.

Figure 6.8: Plot of the normal contact point velocity $\dot{c}_2$ as a function of time for $\theta = 4.21$, $\phi = 4.3$, $V = -0.98$, $e = 0.9$. In (a) for $\mu_0 = 0.3$, $\mu_c = 0.462$ and in (b) $\mu_0 = 0.7$, $\mu_p = 1.3$. 

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that the system will still remained constrained to the belt. Figure 6.7 (a) illustrates
the minimum value of $\mu = \mu_c$ such that the Painlevé paradox can occur for the
set of parameters chosen using Figure 6.4. Figure 6.8 (a) illustrates the minimum
value of $\mu = \mu_c$ such that the Painlevé paradox can occur for the set of parameters
chosen using Figure 6.5. The dashed line in both Figures corresponds to $\mu < \mu_c$
and the solid line corresponds to $\mu = \mu_c$. The solid line shows the occurrence of the
Painlevé paradox whereby a so called impact without collision occurs. For this case,
the friction coefficient is large enough to induce jam and the zero pre impact normal
velocity is mapped to a non zero normal velocity after the impact law is applied.
Further, in Figure 6.8 (a) we have shown that it is possible to achieve the Painlevé
paradox for $\mu = 0.462$, a relatively low value. It is also clear from Figure 6.7 (a)
and Figure 6.8 (a) that the velocity mapping can vary vastly with a slight change
in initial conditions. A very small increase in $\mu$ gives rise to the Painlevé paradox.
The normal contact point velocity mapping is discontinuous for Figures 6.7 and 6.8,
albeit it difficult to see this jump in (a) of both Figures.

We have chosen a significantly larger $\mu$ value in Figures 6.7 (b) and 6.8 (b).
The purpose of this is to illustrate how the dynamics changes with increasing $\mu$.
The effect in both cases is that the lower arm leaves the wall with a higher normal
contact point velocity. The discontinuous velocity mapping is much more visible for
these cases. In Figure 6.7 (b) the result of this is that the normal contact point
velocity is sustained for longer before it starts decreasing. The result in Figure 6.8
(b), however, is the opposite. The initial normal contact point velocity is significantly
higher, meaning that the lower arm will fall away from the wall faster than in Figure
6.7 (b) and 6.8 (a). In Figures 6.7 (a) and 6.8 (a), we have a continuous change in
the generalised coordinates $\theta$ and $\phi$ as the impact law is applied. In Figures 6.7 (b)
and 6.8 (b), however, we have a discontinuous change in the generalised velocities
$\dot{\theta}$ and $\dot{\phi}$ as the impact law is applied. If we consider again Figures 6.7 and 6.8 we
find that, in all cases, there is a continuous change in generalised coordinates and
a discontinuous change in velocities as the impact law is applied. This is to be
expected as the Painlevé paradox gives rise to an increased negative acceleration at
the contact point and therefore when the contact point velocities are translated back
to the generalised velocities, we expect the change to also be discontinuous.

In Figures 6.9 (a) and 6.10 (a) we show the generalised coordinates $\theta$ and $\phi$ of
the double pendulum as a function of time. In Figures 6.9 (b) and 6.10 (b) we show
the generalised velocities $\dot{\theta}$ and $\dot{\phi}$ of the double pendulum as a function of time. The
Figure 6.9: (a) Plot of the angles $\theta$ and $\phi$ as a function of time. (b) Plot of the angular velocity $\dot{\theta}$ and $\dot{\phi}$ as a function of time. Both plots for $\theta = 3.8572$, $V = -0.96$, $\phi = 4.1154$, $e = 0.7$, $\mu_0 = 0.7$, $\mu_c = 0.82$.

Figure 6.10: (a) Plot of the angles $\theta$ and $\phi$ as a function of time. (b) Plot of the angular velocity $\dot{\theta}$ and $\dot{\phi}$ as a function of time. Both plots for $\theta = 4.21$, $\phi = 4.3$, $V = -0.98$, $e = 0.9$, $\mu_0 = 0.3$, $\mu_c = 0.462$.

Initial conditions chosen for Figures 6.9 (a) and (b) correspond to the same initial conditions used in 6.7 (a). The initial conditions chosen for Figures 6.10 (a) and (b) correspond to the same initial conditions used in 6.8 (a). The purpose of these figures is to help in the visualisation of the dynamics of this system and to show how the normal contact point velocity translates to the generalised velocities.
6.4 Discussion

In this Chapter we have described the phenomenon known as the Painlevé paradox and further derived under what conditions it can occur. We considered two model systems and performed analysis and numerical simulation to show the conditions under which the Painlevé paradox can occur. In both cases, the occurrence of the paradox is entirely dependent on the initial configuration of the system together with the inertia of the system.

The first model system, the classical example of a slender rod in contact with a non compliant plane, was analysed using one of the original Painlevé paradox formulations. The minimum value for which the paradox can occur was derived and found to be consistent with previous works. An interesting comparison of this minimum value was made with the Brach impact law derived in Chapter 3. We have shown that the singular point in the Brach impact law corresponds to the minimum value for which the Painlevé paradox can occur meaning that for this system the paradox can never be achieved. The reason being that the system will always lie in the negative portion of the $T_L$ curve which is unphysical and not permissible for the extended Brach impact law. This has also opened up some interesting questions with regards to how the energy loss function $T_L$ is related to the Painlevé paradox in general.

The second model system, a double pendulum in constrained contact with a conveyor belt like surface, was used as a numerical experiment to show the occurrence of the Painlevé paradox. We have shown that it is possible to achieve the Painlevé paradox for relatively low $\mu$ values which are realistic for many materials.

The minimum friction coefficient value for which the Painlevé paradox can occur in both mechanisms varies vastly. The main reason for us being able to achieve the paradox for much lower $\mu$ values for the second model system is that we have much greater control over the mass matrix of the system. The occurrence of the Painlevé paradox is entirely dependent on the mass configuration of the system in question. Further, in both systems, we see that the system dynamics can change dramatically with a very small change in $\mu$. This opens up many questions in terms of the occurrence of this phenomenon in every day mechanical systems.
Chapter 7

Discussion

In this thesis we have explored the area of rigid body impacts with friction, an area which has far reaching applications in Engineering, Sports Science and every day life. We have concerned ourselves with the two main streams of this field, theoretical and numerical, and made a novel contribution to both. We will now summarise the main conclusions of this thesis and highlight some potential areas for future research.

7.1 Theoretical contribution

In Chapter 3 we derive two impact laws, the Brach impact law and the Energetic impact law. We presented the framework for a general rigid-body collision and showed how the Energetic impact law derived in [1] can be extended to allow for an impact between two unconstrained rigid bodies. This extension, albeit relatively straightforward, is useful for researchers who want to implement the Energetic impact law for a two-body collision, as opposed to a one-body collision with a surface of infinite mass. Although the two-body collision was never considered for the work in this thesis, we felt it was useful to derive for completeness. The Brach impact law for the two-body case is already well documented in the literature. The novel extension to the Brach impact law, although very restrictive, allowed for the direct comparison with the more sophisticated Energetic impact law.

This analysis has highlighted the importance of understanding what can happen in an implementation of an impact law model for a rigid-body with impacts and friction. What initially may appear to be a basic and easy-to-use methodology can cause more problems than expected. We have shown that a basic method may also
require more consistency checks than what is needed for a more advanced one. The main consequence of this is that if an engineer or other researcher chooses the wrong type of impact law for a certain application then the predictions achieved may not hold for all regions in parameter space and may be nonsensible. The most common feature observed in the misuse of an impact law is energy gains at impact as a result of an incorrect velocity mapping. Many configurations, as highlighted in this work, can lead to energy gains in the system unless certain measures are taken. Very often it is necessary to impose an impulse ratio which is very restrictive and may not reflect a physical situation in any sense. For these cases we recommend that a more sophisticated impact law is used instead.

This work has also highlighted the importance of an impulse space consideration as opposed to a force consideration. One of the fundamental assumptions of rigid body impact theory is that the impact occurs over an infinitesimal contact duration, which naturally leads to an impulse formulation. Using work-energy principles it is then possible to decompose this infinitesimal impact phase into different segments of relative stick and slip. It is this fundamental principle that makes the more sophisticated impact laws, like the Energetic impact law, energetically conservative and representative of reality. Impact laws such as the Brach impact law can not incorporate this level of complexity and therefore often result in inconsistencies.

In Chapter 6 we considered the Painlevé Paradox and analysed under what conditions it can occur in two model problems. The first model problem, a slender rod in contact with a non compliant plane, allowed for an interesting comparison with the extended Brach impact law derived in Chapter 3. We found that the value of the impulse ratio $\mu$ for which the impact mapping (3.10)-(3.12) is singular is the same as the minimum value for which the Painlevé paradox can occur. For the slender rod presented here, this value was found to be $\mu = \frac{4}{3}$, which is in agreement with previous works in the literature. However, in the extended Brach scheme this singularity, and thus the Painlevé paradox, is (correctly or wrongfully) avoided since impacts will have energy gains close to the singularity. The singularity will always occur where the kinetic energy-loss curve is negative, conflicting with the energetic principles inherent in correctly assigning a $\mu$ value. Although it is physically possible for the Painlevé paradox to occur for $\mu \geq \frac{4}{3}$ for the Energetic approach, in the Brach approach this value for $\mu$ will never be selected as a result of implementing the selection criteria for $\mu$ given by (3.21).

In Chapter 6 we also devise a model system capable of achieving the Painlevé
paradox for relatively low coefficient of friction values, albeit an unstable configuration. There is scope here to continue this research in order to investigate if the Painlevé paradox could occur in every day mechanical systems and it could therefore have significant engineering applications.

7.2 Numerical contribution

The extension to the Brach impact law presented in Chapter 3 also enabled the full numerical simulation of a rigid body system that is subject to multiple impacts using this formulation. It is worth mentioning however that Brach did not intend on this being a general impact law. This mapping was originally used on a case by case basis for specific vehicle collisions. The extension in Chapter 3 was derived to allow for the consideration of long-term dynamics using such a formalism.

In Chapter 5 we introduced the numerical techniques necessary for the implementation of an energetic impact law for rigid body impacts with friction. In particular these methods allow for long-term simulation with various behaviours such as dynamic transitions and chatter. The methodology used is known as a hybrid-system approach and involves deriving maps to deal with the various non-smooth events such as impact and chatter. To illustrate our techniques we used the example of a slender rod impacting with a non-compliant plane and showed that it is possible to achieve long term dynamics using numerical simulation. The advantage of the techniques presented in Chapter 5 is that they can be easily extended and implemented for other mechanical systems with few contact points, where impacts and stick motions can occur. Further, the numerical scheme does not need to handle large number of events at one time and it is possible to deal with complete chatter sequences. The overall method presented in Chapter 5 is also a first step in developing methods for bifurcation and stability analysis of systems with impacts and friction. In particular for systems in which discontinuity-induced bifurcations are present.

In Table 5.1 together with Figure 5.3 we presented a summary of what is needed to detect an event and what state transition the system will encounter. One of the shortcomings of our approach, and an area of future research, is that we end a complete chattering sequence when the contact point velocity reaches a pre-defined velocity threshold, albeit small. This is in contrast to the method that was derived in [37] for impacts without friction, where at the end of a complete chatter sequence
a jump in the states and time is done to minimize local numerical errors. An improvement on this would be to extrapolate through chatter sequences, using a similar approach as in [37], until the relative velocity is zero. Another related issue that is not fully resolved in Chapter 5 is in the limiting case whereby two impact events accumulate, and in our case where the rod angle \( \theta \) approaches zero. In this case we, again, pre-define a tolerance for when the transition from asymmetric to symmetric motion should occur. This was necessary to ensure robustness in the numerical scheme.

One of the main limitations of the symmetric rod system is that once it is in symmetric motion it cannot get back to asymmetric motion. If an external torque was included or if the bar was not completely uniform then such a transition could occur and most likely make the system even more unpredictable. Although both generalisations would be possible to implement, they have not been considered for this work since the main goal was to introduce a framework for simulating long-term dynamics for a rigid-body system with impacts and friction and to highlight some interesting features that can be observed through long-term simulations.

It is clear from the complexity of the transition diagram that incorporating more contact points, for an unconstrained body, would make these methodologies almost impossible. However, a problem for future consideration would be to consider a rigid body with multiple contact points but additional constraints. This would lead to a transition diagram of equal complexity to the one presented in Figure 5.3 and would no doubt lead to some interesting dynamics.

### 7.3 Future Research

This work has also opened up new research questions regarding numerical methods, as mentioned above, but also the exact role friction has in dissipating rotational energy in general rattling objects. It may be possible to find a relationship between the energy removed due to friction and the energy introduced into the system through the periodic forcing. Such analysis may be a useful predictive tool for engineers working with unconstrained impacting systems with friction.

The stability analysis considered in Section 5.3.1 can also be extended to examine the stability of all three maps in the Energetic impact law. Our analysis only considered the stability of period-one orbits using map I. Using the same techniques
it could also be possible to consider the stability of period-one orbits, which uses map II or III. This may be of interest for the improved understanding of the action of impact with friction on the impulse level and the various stability properties of these impact mappings.

It would be worth investigating if it is possible to achieve the Painlevé paradox in a mechanical system for a stable configuration. This could then be verified experimentally and perhaps compared with certain unexplained phenomena that have already been documented in mechanical engineering applications. This would lead to better control of such nonsmooth phenomenon and would allow engineers to more carefully choose parameters to avoid entering into these regions.
Appendix A

Matlab File

```
function out = enrgtic_impact_solver_omegasweep_v6(in)
% Physical Parameters
%__________________________________________________________________________
g = 9.81; % Mass.
m = 1; % Length.
L = 2; % Length.
I = m*(L^2)/12; % Inertia.
r = .8; % Coefficient of restitution.
my_tol = 10^-6; % Velocity tolerances for transitions.
my_tol_2 = 5^-5;
my_pos_tol = 10^-7;
st_prev = [];
%my_tol = 10^-15;
%__________________________________________________________________________
%Initial Conditions
omega_vec = [];
%omega_vec_1 = 0.5;
%omega_vec_end = 3.5;
%omega_vec_end = 4;
delta_omega = 0.0002;
mu_start = 0.05;
delta_mu = 0.05;
mu_end = 0.05;
omega_val = [omega_vec_1:delta_omega:omega_vec_end];
mu_vector = [mu_start:delta_mu:mu_end];

traj_1_max = [];
traj_2_max = [];
traj_3_max = [];
traj_4_max = [];
traj_5_max = [];
traj_6_max = [];
traj_7_max = [];
check = [];
traj_2_max_temp = [];
qe_max2_check = 0;
qu0_vec = [];
qu_prev_vec = [];
st_prev = [];
aw = [];
for i = 1:length(mu_vector)
tau = 0.705288553846737;
```

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APPENDIX A. MATLAB FILE

q0_k = [0; 7.145273725460764;0;0;0;0;tau]';
q0 = q0_k;
q_prevp = q0;
for k = 1:length(omega_val)

% if isempty(st_prev)
% q0_k = [q_prevp];
% end
if isempty(st_prev)
    switch 1
        case {st_prev(1)}
            % if isempty(q_prevp)
            q0 = [q_prevp];
            % else
            q0 = q0_k;
            % end
        case {st_prev(2)}
            q_1 = 0;
            q0 = q0_k;
        case {st_prev(3)}
            q0 = q0_k;
        case {st_prev(4)}
            % if isempty(q_prevp)
            q0 = [q_prevp];
            % else
            q0 = q0_k;
            % end
        case {st_prev(5)}
            q0 = q0_k;
        case {st_prev(6)}
            q0 = q0_k;
    otherwise disp('prob with previous state')
end
end
Amp = 1;
omega = omega_val(k)
mu = mu_vector(i)
disp('________')

%pos_check_init = [q_2];
tstart = 0; % Integration start time.
tfinal = 30000; % Integration end time.
tfinal = 10000; % Integration end time.
times = [];%tstart;
traj = [ ];%q0;
APPENDIX A. MATLAB FILE

traj_1 = 0;
traj_2 = [];
traj_3 = 0;
traj_4 = 0;
traj_5 = 0;
traj_6 = 0;

v1_p_init = 1;
v2_p_init = 1;

ie_curr_11 = 0;
ie_curr = 0;
st = check_state(tstart,q0);
t = 0;
impact_count = 0;
while tstart < tfinal
    impact_count = impact_count + 1;
    %options=odeset('Events',@(t,q)eventsb(t,q,v1_p_init,v2_p_init),'RelTol',1e-9,'AbsTol',1e-9,'maxstep',0.001);
    if t < 5
        options=odeset('Events',@(t,q)eventsb(t,q,v1_p_init,v2_p_init),'RelTol',1e-9,'AbsTol',1e-9,'maxstep',0.005);
    else
        options=odeset('Events',@(t,q)eventsb(t,q,v1_p_init,v2_p_init),'RelTol',1e-9,'AbsTol',1e-9);
    end
    %options=odeset('Events',@(t,q)eventsb(t,q,v1_p_init,v2_p_init),'RelTol',1e-9,'AbsTol',1e-9,'maxstep',0.1);
    clear t
    tspan = [tstart,tfinal]; % Time span vector
    st_ind = find(st == 1);
    % ie_curr
    % st_ind
    % impact_count
    if st_ind == 5
        break
    elseif (impact_count > 10 && st_ind == 4 && isempty(ie_curr_11) )
    disp('break1')
    %
    % break
    end
    [t,q,te,qe,ie]=ode45(@(t,q) vec_field(t,q,st_ind),tspan,q0,options) ; %Calls the ode solver.

    if isempty(ie) && st_ind ==4
        break
    end

    n_steps = length(t);
    times = [times t(1:n_steps)]; % Updates the times
    traj = [traj t(1:end,:)]; % Updates the entire trajectory
    traj_2 = [traj_2 q(:,2)];
    traj_3 = [traj_3 q(:,3)];
    omega_vec = [omega_vec omega];
    traj_2_max = [traj_2_max max(q(:,2))];

    Floor_vel = omega*Amp*cos(omega*q(:,7));
    Floor_position = Amp*sin(omega*q(:,7)); %defines the oscillating floor in the events function.
    Floor_acc = -(omega^2)*Amp*sin(omega*q(:,7));

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if t(end) < tfinal
    [vt, vn, thetadot, theta_unmod, A, B, C] = impact_param_bar(q, traj);
end

% [vt, vn, thetadot, theta_unmod, A, B, C, x1, x2, g1, g2, vnc_1, vnc_2] = impact_param_bar(q, traj);

switch 1
    case {st(1)}
        [vt0, vn0] = initvelcalc_s1(theta_unmod, thetadot, L, vt, vn, ie, Floor_vel);
    case {st(2)}
        [vt0, vn0] = initvelcalc_s2(theta_unmod, thetadot, L, vt, vn, ie, Floor_vel);
    case {st(3)}
        [vt0, vn0] = initvelcalc_s3(theta_unmod, thetadot, L, vt, vn, ie, Floor_vel);
    case {st(4)}
        [vt0, vn0] = initvelcalc_s4(theta_unmod, thetadot, L, vt, vn, ie, Floor_vel);
    case {st(5)}
        [vt0, vn0] = initvelcalc_s5(theta_unmod, thetadot, L, vt, vn, ie, Floor_vel);
end

s = s_decide(q, vt0);

[av] = avdecide(vt0, r, vn0, A, B, C);
[ratevect] = ratecon(ktp, knp, ktn, kn, kt0, kn0, av);
[zan] = map(vt0, r, kt, kn, ktprime, knprime, vn0, av);
[vnf, vtf] = tree_1(m, I, zan);

[vt, vnp, thetadot] = vel_translation(vn0, vt0, vnf, vtf, theta_unmod, L, I, m, A, B, C, vt, vn, thetadot);

else
    % disp('t end not < t final')
end

% Resetting initial conditions after impact and checking/adjusting current system state.

q0 = q(n_steps, 1:7);

if ~isempty(ie) % If there is an event.
    switch 1
        % State 1
        case {st(1)}
            switch ie(end)
                case {1}
                    v1 = (q(end, 5) - Floor_vel(end)) - (L/2)*cos(q(end, 3))*(q(end, 6));
                    a1 = -g + (L/2)*sin(q(end, 3))*(q(end, 6))^2 + (omega^2)*Amp*sin(omega*q(end, 7));
                    v2 = (q(end, 5) - Floor_vel(end)) + (L/2)*cos(q(end, 3))*(q(end, 6));
                    a2 = -g - (L/2)*sin(q(end, 3))*(q(end, 6))^2 + (omega^2)*Amp*sin(omega*q(end, 7));
                    v1_p = vnp - (L/2)*cos(q(end, 3))*thetadot;
                    v2_p = vnp + (L/2)*cos(q(end, 3))*thetadot;
                    if abs(v1(end)) < my_tol & a1 < 0
                        x_vec = (L/2)*cos(theta_unmod);
                        st = [0, 1, 0, 0, 0]';
                    end
                end
            end
        end
end

% Resetting initial conditions after impact and checking/adjusting current system state.

q0 = q(n_steps, 1:7);

if ~isempty(ie) % If there is an event.
    switch 1
        % State 1
        case {st(1)}
            switch ie(end)
                case {1}
                    v1 = (q(end, 5) - Floor_vel(end)) - (L/2)*cos(q(end, 3))*(q(end, 6));
                    a1 = -g + (L/2)*sin(q(end, 3))*(q(end, 6))^2 + (omega^2)*Amp*sin(omega*q(end, 7));
                    v2 = (q(end, 5) - Floor_vel(end)) + (L/2)*cos(q(end, 3))*(q(end, 6));
                    a2 = -g - (L/2)*sin(q(end, 3))*(q(end, 6))^2 + (omega^2)*Amp*sin(omega*q(end, 7));
                    v1_p = vnp - (L/2)*cos(q(end, 3))*thetadot;
                    v2_p = vnp + (L/2)*cos(q(end, 3))*thetadot;
                    if abs(v1(end)) < my_tol & a1 < 0
                        x_vec = (L/2)*cos(theta_unmod);
                        st = [0, 1, 0, 0, 0]';
                    end
                end
            end
        end
end
APPENDIX A. MATLAB FILE

```matlab
q0(1:6) = q(n_steps,1:6);
ie_ind1 = find(ie == 1);
q0(2) = qe(ie_ind1,2);
q0(5) = x_vec*q(n_steps,6) + Floor_vel(end);
elseif abs(v1(end)) < my_tol_2 & pos_check_1 < my_pos_tol
x_vec = (L/2)*cos(theta_unmod);
st = [0,1,0,0,0,0]';
q0(1:6) = q(n_steps,1:6);
ie_ind1 = find(ie == 1);
q0(2) = qe(ie_ind1,2);
q0(5) = x_vec*q(n_steps,6) + Floor_vel(end);
elseif abs(v2(end)) < my_tol & a2 < 0
x_vec = (L/2)*cos(theta_unmod);
st = [0,0,1,0,0,0]';
q0(1:6) = q(n_steps,1:6);
q0(3) = q(n_steps,3);
q0(4) = q(n_steps,4);
q0(5) = x_vec*q(n_steps,6) + Floor_vel(end);
q0(6) = q(n_steps,6);
elseif abs(v2(end)) < my_tol_2 & pos_check_2 < my_pos_tol
x_vec = (L/2)*cos(theta_unmod);
st = [0,0,1,0,0,0]';
q0(1:6) = q(n_steps,1:6);
q0(3) = 0;
q0(4) = vtf;
q0(5) = vnf;
q0(6) = 0;
```

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APPENDIX A. MATLAB FILE

```matlab
st = [0,0,0,1,0]';

elseif omega < 3 & mod(abs(q(end,3)),pi) < 0.02
    q0(1:6) = q(n_steps,1:6);
    q0(3) = 0;
    q0(4) = vtf;
    q0(5) = vnf;
    q0(6) = 0;
    st = [0,0,0,1,0]';

else
    q0(1:6) = q(n_steps,1:6);
    q0(4) = vtp;
    q0(5) = vnp;
    q0(6) = thetadotp;
end

case (2)

% Pre impact contact point velocities and accelerations
v1 = q(end,5) - Floor_vel(end) - (L/2)*cos(q(end,3))*q(end,6);
ai = -g - (L/2)*sin(q(end,3))*q(end,6)^2 + (omega^2)*Amp*sin(omega*q(end,7));
v2 = (q(end,5) - Floor_vel(end)) + (L/2)*cos(q(end,3))*q(end,6);
ai. = -g - (L/2)*sin(q(end,3))*q(end,6)^2 + (omega^2)*Amp*sin(omega*q(end,7));

% Post impact contact point velocities and accelerations
v1_p = vnp - (L/2)*cos(q(end,3))*thetadotp;
v2_p = vnp + (L/2)*cos(q(end,3))*thetadotp;

if abs(v2(end)) < my_tol & a2 < 0
    x_vec = (L/2)*cos(theta_unmod);
st = [0,0,1,0,0]';
    q0(1:6) = q(n_steps,1:6);
    ie_ind2 = find(ie == 2);
    q0(2) = qe(ie_ind2,2);
    q0(5) = -x_vec*q(n_steps,6) + Floor_vel(end);
end

elseif abs(v2(end)) < my_tol_2 & pos_check < my_pos_tol
    x_vec = (L/2)*cos(theta_unmod);
st = [0,0,1,0,0]';
    q0(1:6) = q(n_steps,1:6);
    ie_ind2 = find(ie == 2);
    q0(5) = -x_vec*q(n_steps,6) + Floor_vel(end);
end
```

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elseif abs(v1) < my_tol & a1 < 0
    st = [0,1,0,0,0]';
    x_vec = (L/2)*cos(theta_unmod);
    q0(1) = q(n_steps,1);
    q0(2) = Floor_position(end) + (L/2)*sin(q(n_steps,3));
    q0(3) = q(n_steps,3);
    q0(4) = q(n_steps,4);
    q0(5) = x_vec*q(n_steps,6) + Floor_vel(end);
    q0(6) = q(n_steps,6);
elseif abs(v1(end)) < my_tol_2 & pos_check_1 < my_pos_tol
    st = [0,1,0,0,0]';
    x_vec = (L/2)*cos(theta_unmod);
    q0(1) = q(n_steps,1);
    q0(2) = Floor_position(end) + (L/2)*sin(q(n_steps,3));
    q0(3) = q(n_steps,3);
    q0(4) = q(n_steps,4);
    q0(5) = x_vec*q(n_steps,6) + Floor_vel(end);
    q0(6) = q(n_steps,6);
elseif mod(abs(q(end,3)),pi) < 0.003
    q0(1:6)= q(n_steps,1:6);
    q0(3) = 0;
    q0(4) = vtf;
    q0(5) = vnf;
    q0(6) = 0;
    st = [0,0,0,1,0]';
elseif omega < 3 & mod(abs(q(end,3)),pi) < 0.02
    q0(1:6)= q(n_steps,1:6);
    q0(3) = 0;
    q0(4) = vtf;
    q0(5) = vnf;
    q0(6) = 0;
    st = [0,0,0,1,0]';
else
    q0(1:3)= q(n_steps,1:3);
    q0(4) = vtp;
    q0(5) = vnp;
APPENDIX A. MATLAB FILE

q0(6) = thetadotp;

end
case {3}
    if mod(abs(q(end,3)),pi) < 0.003
        q0(1:6)= q(n_steps,1:6);
        q0(3) = 0;
        q0(4) = vtf;
        q0(5) = vnf;
        q0(6) = 0;
        st = [0,0,0,1,0]';
    elseif omega < 3 & mod(abs(q(end,3)),pi) < 0.02
        q0(1:6)= q(n_steps,1:6);
        q0(3) = 0;
        q0(4) = vtf;
        q0(5) = vnf;
        q0(6) = 0;
        st = [0,0,0,1,0]';
    end
end
case {4}
    if mod(abs(q(end,3)),pi) < 0.003
        q0(1:6)= q(n_steps,1:6);
        q0(3) = 0;
        q0(4) = vtf;
        q0(5) = vnf;
        q0(6) = 0;
        st = [0,0,0,1,0]';
    elseif omega < 3 & mod(abs(q(end,3)),pi) < 0.02
        q0(1:6)= q(n_steps,1:6);
        q0(3) = 0;
        q0(4) = vtf;
        q0(5) = vnf;
        q0(6) = 0;
        st = [0,0,0,1,0]';
    end
end

% State 2______________________________________________________________
%____________________________________________________________________

switch ie(end)
    case {2,6}
        if vnf - Floor_vel(end) >0
            q0(1:6)= q(n_steps,1:6);
            q0(2) = Floor_position(end);
        end
    end

end
APPENDIX A. MATLAB FILE

```matlab
q0(3) = 0;
q0(4) = vtf;
q0(5) = vnf;
q0(6) = 0;
st = [0,0,0,1,0]';
elseif vnf - Floor_vel(end) <0
q0(1:6)= q(n_steps,1:6);
q0(3) = 0;
q0(2) = Floor_position(end) ;
q0(5) = Floor_vel(end);
st = [0,0,0,0,1]';
else disp('prob state 2')
end

 case 5
% Switches from state 2 to state 1.
st = [1,0,0,0,0]';
q0(1:6)= q(n_steps,1:6);
otherwise
 disp('prob state 2')
end

% State 3________________________________________________________________
%________________________________________________________________________
case {st(3)}
switch ie(end)
  case 1
    if vnf >0
q0(1:6)= q(n_steps,1:6);
q0(2) = Floor_position(end) ;
q0(3) = 0;
q0(4) = vtf;
q0(5) = vnf;
q0(6) = 0;
st = [0,0,0,1,0]';
    else
q0(1:6)= q(n_steps,1:6);
q0(3) = 0;
q0(2) = Floor_position(end) ;
q0(5) = Floor_vel(end);
st = [0,0,0,0,1]';
end

 case 5
if vnf >0
q0(1:6)= q(n_steps,1:6);
q0(2) = Floor_position(end) ;
q0(3) = 0;
q0(4) = vtf;
q0(5) = vnf;
q0(6) = 0;
st = [0,0,0,1,0]';
else
```

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APPENDIX A. MATLAB FILE

```matlab
q0(1:6)= q(n_steps,1:6);
q0(3) = 0;
q0(2) = Floor_position(end) ;
q0(5) = Floor_vel(end);
st = [0,0,0,0,1]';
end

case {6}
% Switches from state 3 to state 1.
st = [1,0,0,0,0]';
q0(1:6)= q(n_steps,1:6);
otherwise
    disp('prob state 3')
end

% State 4________________________________________________________________
%________________________________________________________________________
case(st(4))

switch ie(end)

case {1}
if abs(vn0) < 0.00001
    q0(1:6)= q(n_steps,1:6);
    q0(3) = 0;
    q0(2) = Floor_position(end) ;
    q0(5) = Floor_vel(end);
    st = [0,0,0,0,1]';
else
    q0(1:3)= q(n_steps,1:3);
    q0(3) = 0;
    q0(4) = vtp;
    q0(5) = vnp + Floor_vel(end);
    q0(6) = 0;
end

case (2)
if abs(vn0) < 0.00001
    disp('1')
    q0(1:6)= q(n_steps,1:6);
    q0(3) = 0;
    q0(2) = Floor_position(end) ;
    q0(5) = Floor_vel(end);
    st = [0,0,0,0,1]';
else
    disp('2')
    q0(1:3)= q(n_steps,1:3);
    q0(3) = 0;
    q0(4) = vtp;
    q0(5) = vnp + Floor_vel(end);
```

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```matlab
q0(6) = 0;
end

case (5)
    if abs(vn0) < 0.00001
        q0(1:6) = q(n_steps,1:6);
        q0(3) = 0;
        q0(2) = Floor_position(end);
        q0(5) = Floor_vel(end);
        st = [0,0,0,0,1]';
    else
        q0(1:3) = q(n_steps,1:3);
        q0(3) = 0;
        q0(4) = vtp;
        q0(5) = vnp + Floor_vel(end);
        q0(6) = 0;
    end

    case (11)
        break
    end

% State 5________________________________________________________________
%________________________________________________________________________

    case{st(5)}
        if ~isempty(ie)
            switch ie(end)
                case (9)
                    q0(1:6) = q(n_steps,1:6);
                    q0(3) = 0;
                    q0(2) = Floor_position(end);
                    q0(5) = q(n_steps,5);
                    q0(6) = 0;
                    st = [0,0,0,1,0]';
                end
            else
                break
                % q0(1:6) = q(n_steps,1:6);
                % st = [0,0,0,0,1]';
            end
        else
            break
        end
    end

end
```

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ie_curr = ie(end);

ie_curr_11 = find(ie == 11);

tstart = t(n_steps); %Resets the integration start time.

v1_n = q(:,5) - L/2*q(:,6).*cos(q(:,3))-Floor_vel;
v2_n = q(:,5) + L/2*q(:,6).*cos(q(:,3))-Floor_vel;

st_new = st;

if isempty(ie)
  switch ie(end)
    case 1
      v1_p_init = v1_p(end);
v2_p_init = 1;
    case 2
      v1_p_init = 1;
v2_p_init = v2_p(end);
  end

  q_prev = q(n_steps,1:6).';
endif

ie_ind = find(ie == 11);

st_ind = find(st == 1);

if t > 1000*2*pi/omega & st_ind ==1 | t > 1000*2*pi/omega & st_ind ==4

  if isempty(ie_ind)
    qe_max1 = qe(ie_ind,1);
    qe_max2 = qe(ie_ind,2);
    qe_max3 = qe(ie_ind,3);
    qe_max4 = qe(ie_ind,4);
    qe_max5 = qe(ie_ind,5);
    qe_max6 = qe(ie_ind,6);
    qe_max7 = qe(ie_ind,7);

    omega_vec = [omega_vec;omega];
    traj_1_max = [traj_1_max;qe_max1];
    traj_2_max = [traj_2_max;qe_max2];
    traj_3_max = [traj_3_max;qe_max3];
    traj_4_max = [traj_4_max;qe_max4];
    traj_5_max = [traj_5_max;qe_max5];
    traj_6_max = [traj_6_max;qe_max6];
    traj_7_max = [traj_7_max;qe_max7];

    a1 = traj_2_max(end) - qe_max2;
    a2 = traj_3_max(end) - qe_max3;
    a3 = traj_6_max(end) - qe_max6;

    bound = abs(sqrt(a1^2 + a2^2 +a3^2));
  end
endif

q0_vec = [q0_vec;q0_k];
APPENDIX A. MATLAB FILE

q_prev_vec = [q_prev_vec; q_prevp];

check = abs( traj_2_max(end) - traj_2_max) < 10^-6;
% qe_max2
% sin(omega*qe_max7)
if (qe_max2) > (sin((omega+ delta_omega)*qe_max7))
  qe_max7_mod = mod(qe_max7,2*pi/omega);
  aw = [qe_max1;qe_max2;qe_max3;qe_max4;qe_max5;qe_max6;qe_max7_mod].'
end
if sum(check) >0
  %if qe_max2 > 4
  % else
  % aw = [];
  %end
  break
end

end

elseif t > 1000*2*pi/omega & st_ind == 2 | t > 1000*2*pi/omega & st_ind == 3 ;

omega_vec = [omega_vec;omega];
traj_1_max = [traj_1_max;0];
traj_2_max = [traj_2_max;0];
traj_3_max = [traj_3_max;0];
traj_4_max = [traj_4_max;0];
traj_5_max = [traj_5_max;0];
traj_6_max = [traj_6_max;0];
traj_7_max = [traj_7_max;qe_max7];
q0_vec = [q0_vec;q0_k];
%q_prevp = (qe_max1,qe_max2,qe_max3,qe_max4,qe_max5,qe_max6,qe_max7).'
%q_prev_vec = [q_prev_vec; q_prevp];
break
elseif t > 1000*2*pi/omega & ( isempty(ie_ind) & st_ind == 4) 

omega_vec = [omega_vec;omega];
traj_1_max = [traj_1_max;0];
traj_2_max = [traj_2_max;0];
traj_3_max = [traj_3_max;0];
traj_4_max = [traj_4_max;0];
traj_5_max = [traj_5_max;0];
traj_6_max = [traj_6_max;0];
traj_7_max = [traj_7_max;0];
st_prev = [0,0,0,0,0,1]';
break
elseif st_ind == 5

omega_vec = [omega_vec;omega];
traj_1_max = [traj_1_max;0];
traj_2_max = [traj_2_max;0];
traj_3_max = [traj_3_max;0];
traj_4_max = [traj_4_max;0];
traj_5_max = [traj_5_max;0];
traj_6_max = [traj_6_max;0];
traj_7_max = [traj_7_max;0];
end
APPENDIX A. MATLAB FILE

```
st_prev = st;
end
if ~isempty(aw)
q_prevp = aw;
end
customname = ['omega_sweep_mu100cc_';num2str(100*mu),'_';num2str(100*r),'.mat'];
%customname = ['0611_omega_sweep_mu100_';num2str(100*mu),'_';num2str(100*r),'.mat'];
save(customname,'omega_vec','traj_1_max','traj_2_max','traj_3_max','traj_4_max','traj_5_max','traj_6_max','traj_7_max')
%plot(times,traj_2)
end

end
% figure(1), hold on
% plot(times*2604.5785,traj_2,'r')
% hold on
% plot(times(end)*2604.5785, qe_max2,'bo')
%__________________________________________________________________________
%function [vt,vn,thetadot,theta_unmod,A,B,C,x1,x2,g1,g2,vnc_1,vnc_2] = impact_param_bar(q,traj)
function [vt,vn,thetadot,theta_unmod,A,B,C] = impact_param_bar(q,traj)
% Calculation of impact paramters for bar.
theta_unmod = q(end,3);
vt = q(end,4);
vn = q(end,5);
theta = mod(theta_unmod,pi);
% x1 = traj(:,1) - (L/2)*cos(traj(:,3)); %x Position of left end of bar.
% g1 = traj(:,2) - (L/2)*sin(traj(:,3)); %y Position of left end of bar.
% x2 = traj(:,1) + (L/2)*cos(traj(:,3)); %x Position of right end of bar.
% g2 = traj(:,2) + (L/2)*sin(traj(:,3)); %y Position of right end of bar.
% vnc_1 = traj(:,5) - (L/2)*cos(traj(:,3)).*traj(:,6); %
% vnc_2 = traj(:,5) + (L/2)*cos(traj(:,3)).*traj(:,6);
A = 1+ 3*(sin(theta_unmod)^2);
B = -3*sin(theta_unmod)*cos(theta_unmod);
C = 1+ 3*(cos(theta_unmod)^2);
end
%__________________________________________________________________________
function [vt0,vn0] = initvelcalc_s5( theta_unmod, thetadot, L, vt, vn,ie, Floor_vel)
% Contact point initial velocity calculation.
y_vec = (L/2)*sin(theta_unmod);
x_vec = (L/2)*cos(theta_unmod);
vt0 = vt;
vn0 = vn - Floor_vel(end);
if ~isempty(ie)
switch ie(end)
 case 1
disp('error1')
 case 2
disp('error2')
 case 3
disp('error3')
 case 4
disp('error4')
 case 5
disp('error5')
end
end
end
function [vt0,vn0] = initvelcalc_s1( theta_unmod, thetadot, L, vt, vn,ie, Floor_vel)
% Contact point initial velocity calculation.
```

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y_vec = (L/2)*sin(theta_unmod);
x_vec = (L/2)*cos(theta_unmod);

if ~isempty(ie)
    switch ie(end)
        case 1
            vt0 = vt + y_vec*thetadot;
            vn0 = vn - x_vec*thetadot - Floor_vel(end);
        case 2
            vt0 = vt - y_vec*thetadot;
            vn0 = vn + x_vec*thetadot - Floor_vel(end);
        case 3
            vt0 = vt + y_vec*thetadot;
            vn0 = vn - x_vec*thetadot - Floor_vel(end);
            disp('error3')
        case 4
            vt0 = vt - y_vec*thetadot;
            vn0 = vn + x_vec*thetadot - Floor_vel(end);
            disp('error4')
        case {5,10}
            vt0 = vt;
            vn0 = vn - Floor_vel(end);
    end
end
end

function [vt0,vn0] = initvelcalc_s4( theta_unmod, thetadot, L, vt, vn,ie, Floor_vel)
% Contact point initial velocity calculation.

y_vec = (L/2)*sin(theta_unmod);
x_vec = (L/2)*cos(theta_unmod);

if ~isempty(ie)
    switch ie(end)
        case 1
            vt0 = vt + y_vec*thetadot;
            vn0 = vn - x_vec*thetadot - Floor_vel(end);
        case 2
            vt0 = vt - y_vec*thetadot;
            vn0 = vn + x_vec*thetadot - Floor_vel(end);
        case 3
            vt0 = vt + y_vec*thetadot;
            vn0 = vn - x_vec*thetadot - Floor_vel(end);
        case 4
            vt0 = vt - y_vec*thetadot;
            vn0 = vn + x_vec*thetadot - Floor_vel(end);
        case 6
            vt0 = vt;
        end
    end
end

% Contact point initial velocity calculation.

function [vt0,vn0] = initvelcalc_s2( theta_unmod, thetadot, L, vt, vn,ie, Floor_vel)

x_vec = (L/2)*cos(theta_unmod);
y_vec = (L/2)*sin(theta_unmod);
vtc_1 = vt - (L/2)*sin(theta_unmod)*thetadot;

% vt0 = vtc_1 + 2*y_vec*thetadot; % approx vt
vt0 = vt; % approx vt
vn0 = - Floor_vel(end) ;
end

function [vt0,vn0] = initvelcalc_s3( theta_unmod, thetadot, L, vt, vn, ie, Floor_vel)
% Contact point initial velocity calculation.
x_vec = (L/2)*cos(theta_unmod);
y_vec = (L/2)*sin(theta_unmod);
vtc_2 = vt + (L/2)*sin(theta_unmod)*thetadot;
vt0 = vt - y_vec*thetadot;
v0 = -Floor_vel(end);
end

function s = s_decide(q,vt0)
y_vec = (L/2)*sin(theta_unmod);
if vt0 > 0
    s = -1;
elseif vt0 < 0
    s = 1;
else
    disp('error')
end
end

function s_c = check_state(t,q)
s_c = zeros(1,7)
q2 = q(2);
q5 = q(5);
q6 = q(6);
q7 = q(7);
s3 = sin(q(3));
c3 = cos(q(3));
somt = sin(omega*q7);
comt = cos(omega*q7);

floor = Amp*sin(omega*q7); %defines the oscillating floor in the events function.
Floor_vel = omega*Amp*cos(omega*q7);
v1 = q5 - (L/2)*c3*q6 - Floor_vel; % Zero crossings of relative velocity of end 1.
v2 = q5 + (L/2)*c3*q6 - Floor_vel; % Crossings of end 2 of the bar rel_vel
if abs(v1(end)) > 0 & abs(v2(end)) > 0 & abs(q(3)) > 0.000001
    s_c = [1;0;0;0;0];
elseif abs(v1(end)) < 0.00001 & abs(q(3)) > 0.000001
    s_c = [0;1;0;0;0];
elseif abs(v2(end)) < 0.00001 & abs(q(3)) > 0.000001
    s_c = [0;0;1;0;0];
elseif abs(q(3)) < 0.0000001 & abs(q2 - Floor_vel) < 0.000001
    s_c = [0;0;0;1];
end

end
elseif abs(q(3)) < 0.0000001 & abs(q2 - Floor_vel) > 0.000001
s_c = [0;0;0;1;0];
else
disp('cant find state')
end
end

function av = avdecide(vt0,r,vn0,A,B,C)
%Decision tree to map impact region.

ktp = B - mu*A;
knm = C - mu*B;
ktm = B + mu*A;
knt = C + mu*B;
k0 = 0;
knt0 = (A*C -(B^2))/A ;
v_p = (vt0 > 0);
ktp_p = (ktp > 0);
ktm_p = (ktm > 0);
k1_temp = (knp*vt0 - (1+r)*ktp*vn0);

k1t_p = (k1_temp > 0);
k2_temp = knp*vt0- ktp*vn0;
k2t_p = (k2_temp > 0);

if v_p
    if k1t_p
        av = 1;
    else
        if k2t_p
            if ktm_p
                av = 10;
            else
                av = 7;
            end
        else
            if ktm_p
                av = 9;
            else
                av = 2;
            end
        end
    end
else
    k1_temm = (knm*vt0 - (1+r)*ktm*vn0);
k1t_m = (k1_temm > 0);
    if k1t_m
        av = 2;
    else
        k2_temm = knm*vt0 -ktm*vn0;
k2t_m = (k2_temm > 0);
        if k2t_m
            if ktm_p
                av = 10;
            else
                av = 7;
            end
        else
            if ktm_p
                av = 9;
            else
                av = 2;
            end
        end
    end
end

%__________________________________________________________________________

%chris randolph
%chris randolph
else
    av = 6;
end
else
    if ktm_p
        if ktp_p
            av = 8;
        else
            av = 4;
        end
    end
end
end

end
%---------------------------------------------------------------------------

function [ratevect] = ratecon(ktp,knp,ktm,knm,kt0,kn0,av)

switch av
%* These parameters are not technically 0. This was done for ease of
%* programming.
    case 1
        %disp('kt = ktp , kn = knp');
        kt = ktp;
        kn = knp;
        ktprime = nan; %*
        knprime = nan; %*
    case 2
        %disp('kt = ktm , kn = knm');
        kt = ktm;
        kn = knm;
        ktprime = nan; %*
        knprime = nan; %*
    case {3,5}
        %disp('ktprime = kt0 ,knprime = kn0');
        % disp('kt = ktp , kn = knp');
        ktprime = kt0;
        knprime = kn0;
        kt = ktp;
        kn = knp;
    case {4,6}
        %disp('ktprime = kt0 ,knprime = kn0');
        %disp('kt = ktm , kn = knm');
        ktprime = kt0;
        knprime = kn0;
        kt = ktm;
        kn = knm;
    case {7, 9}
        %disp('ktprime = ktm ,knprime = knm');
        %disp('kt = ktp , kn = knp');
        ktprime = ktm;
        knprime = knm;
        kt = ktp;
        kn = knp;
    case {8 , 10}
        %disp('ktprime = ktp , knprime = knp');
        %disp('kt = ktp , kn = knp');
        ktprime = ktp;
        knprime = knp;
        kt = ktm;
        kn = knm;
end

ratevect = [kt;kn;ktprime;knprime];

end
% This function evaluates the tangential and normal components of each impact mapping.
impIt = vt0 - (1+r)*(kt/kn)*vn0;
impIn = -r*vn0;
impIIt = (ktprime/knprime)*((kn/kt)*vt0 - vn0 +sqrt(((1-(knprime/kn))*((kn/kt)*vt0 - vn0)^2)+ (r^2)*(knprime/kn)*(vn0^2)));
impIIn = r*sqrt((1-(knprime/kn))*(((kn/kt)*vt0 - vn0)^2)+ (knprime/kn)*(vn0^2));
if kn == 0
impIIIt = 0;
impIIIn = r*sqrt((vn0^2) + 2*(knprime*vt0*vn0/k0) );
else
impIIIt = (ktprime/knprime)*((kn/kt)*vt0 - vn0 +r*sqrt(((1-(knprime/kn))*((kn/kt)*vt0 - vn0)^2)+ (knprime/kn)*(vn0^2)));
impIIIn = r*sqrt((1-(knprime/kn))*(((kn/kt)*vt0 - vn0)^2)+ (knprime/kn)*(vn0^2));
end
zan = [impIt;impIn;impIIt;impIIn];
% Assigns values to vtf, vnf depending on impact region.
switch av
  case 1
    vtf = zan(1,:);
    vnf = zan(2,:);
  case 2
    vtf = zan(1,:);
    vnf = zan(2,:);
  case 3
    vtf = zan(3,:);
    vnf = zan(4,:);
  case 4
    vtf = zan(3,:);
    vnf = zan(4,:);
  case 5
    vtf = zan(5,:);
    vnf = zan(6,:);
  case 6
    vtf = zan(5,:);
    vnf = zan(6,:);
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```matlab
vtf = zan(5,:);
vnf = zan(6,:);

case 7
  disp(['The region is 7 and the corresponding impact map is II']);
  vtf = zan(3,:);
  vnf = zan(4,:);

case 8
  disp(['The region is 8 and the corresponding impact map is II']);
  vtf = zan(3,:);
  vnf = zan(4,:);

case 9
  disp(['The region is 9 and the corresponding impact map is III']);
  vtf = zan(5,:);
  vnf = zan(6,:);

case 10
  disp(['The region is 10 and the corresponding impact map is III']);
  vtf = zan(5,:);
  vnf = zan(6,:);
end
end

function [vtp,vnp,thetadotp] = vel_translation(vn0,vt0,vnf,vtf,theta_unmod,L,I,m,A,B,C,vt,vn,thetadot);
theta = mod(theta_unmod,pi);
M = [m,0,0
     0,m,0
     0,0,I];
M_inv = inv(M);
m_inv = [A,B
         B,C];
m_0 = inv(m_inv);
del_x1 = [1, 0
           0,1
           (L/2)*sin(theta), -(L/2)*cos(theta)];
delta_v = [(vtf - vt0)
          (vnf - (vn0 ))];
  delta_v = [(vtf - vt0)
      (vnf - (vn0 + Floor_vel(end)) ));
if theta == 0
  disp('zeroangle')
  vtp = vtf;
  vnp = vnf;
  thetadotp = 0;
else
  deltaq1 = (M_inv)*(del_x1)*(m_0)*delta_v;
end
```

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vtp = vt + deltaq1(1,:);
vnp = vn + deltaq1(2,:);
theta_d = theta_d + deltaq1(3,:);
end
end

function [vtp_s23,vnp_s23,theta_d_s23] = vel_translation_s23(vn0,vt0,vnf,vtf,theta_unmod,L,I,m,A,B,C,vt,vn,theta)
theta = mod(theta_unmod,pi);
M = [m,0,0;
     0,m,0;
     0,0,I];
M_inv = inv(M);

m_inv = [A,B;
         B,C];
m_0 = inv(m_inv);
del_x1 = [1, 0;
          0,1;
          (L/2)*sin(theta), (-L/2)*cos(theta)];
delta_v = [(vtf - vt0);
          (vnf - (vn0 ) )];

% delta_v = [(vtf - vt0);
%            (vnf - (vn0 + Floor_vel(end)) )];
deltaq1 = (M_inv)*(del_x1)*(m_0)*delta_v;

vtp_s23 = vt + deltaq1(1,:);
vnp_s23 = vn + deltaq1(2,:);
theta_d_s23 = theta_d + deltaq1(3,:);
end

%__________________________________________________________________________

function [value,isterminal,direction] = eventsb(t,q,v1_p_init,v2_p_init)
%This function defines our event surface.

% Floor_vel = omega*Amp*cos(omega*q(7));
% Floor_position = Amp*sin(omega*q(7)); %defines the oscillating floor in the events function.
% Floor_acc = -(omega^2)*Amp*sin(omega*q(7));
% % NB........V1,V2 NEED TO BE DEFINED FOR STATE 2 AND 3
% v1_p_init;
% v2_p_init;

s3 = sin(q(3));
c3 = cos(q(3));
q5 = q(5);
q6 = q(6);
q2 = q(2);
s_q1 = (q5 + L*s3*q6)/abs(q5 + L*s3*q6);
s_q2 = (q5 - L*s3*q6)/abs(q5 - L*s3*q6);

% y1 = (q(2) - (L/2)*s3 - Floor_position); %Zero crossings of the left end of the bar.
% y2 = (q(2) + (L/2)*s3 - Floor_position); %Zero crossings of the right end of the bar.
% y1_uns = (q(2) - (L/2)*s3 - Floor_position); %Zero crossings of the left end of the bar.
% y2_uns = (q(2) + (L/2)*s3 - Floor_position); %Zero crossings of the right end of the bar.
% y_com = q(2) - Floor_position;
% y1 = (q(2) - (L/2)*s3 - Floor_position)/v1_p_init; %Zero crossings of the left end of the bar.
% y2 = (q(2) + (L/2)*s3 - Floor_position)/v2_p_init; %Zero crossings of the right end of the bar.
% Free flight contact velocities

%__________________________________________________________________________________

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\[ v_1 = (q_5 - \text{Floor vel}) - (L/2) \cdot c_3 \cdot (q_6); \]
\[ v_2 = (q_5 - \text{Floor vel}) + (L/2) \cdot c_3 \cdot (q_6); \]

Contact point contact velocities
\[ \% \text{v1}_c = (-\text{Floor vel}) - (L) \cdot c_3 \cdot (q(6)); \]
\[ \% \text{v2}_c = (-\text{Floor vel}) + (L) \cdot c_3 \cdot (q(6)); \]

Free accelerations
\[ \% a_1 = -g + (L/2) \cdot s_3 \cdot q(6)^2 + (\omega^2) \cdot \text{Amp} \cdot \sin(\omega \cdot q(7)); \]
\[ \% a_2 = -g - (L/2) \cdot s_3 \cdot q(6)^2 + (\omega^2) \cdot \text{Amp} \cdot \sin(\omega \cdot q(7)); \]

Contact accelerations
\[ v_{\text{com}} = (q_5 - \text{Floor vel}); \]

switch 1
    case {st(1)} % Value 1,0. Free flight
        value = [y1; y2; v1; v2; 1; y_com; 1; 1; v_com; q5];
        isterminal = [1; 1; 1; 0; 1; 0; 0; 0; 0; 0];
        direction = [-1; -1; 0; 0; -1; 0; 0; 0; -1; -1];
    case {st(2)} % Value 1,0. End 1 constrained
        a1_c = -g + (L/2) \cdot s_3 \cdot q(6)^2 - 3g \cdot (c_3 \cdot s_3 \cdot s_q1 \cdot \mu - c_3^2) + (\omega^2) \cdot \text{Amp} \cdot \sin(\omega \cdot q(7)); %event 5
        value = [y1; y2; v1; a1_c; v_com; 1; 1; 1; v_com; 1];
        isterminal = [1; 1; 0; 1; 1; 0; 0; 0; 0; 0];
        direction = [-1; -1; 0; 0; 1; -1; 0; 0; 0; -1; 0];
    case {st(3)} % Value 1,0. End 2 constrained
        a2_c = -g - (L/2) \cdot s_3 \cdot q(6)^2 + 3g \cdot (c_3 \cdot s_3 \cdot s_q2 \cdot \mu - c_3^2) + (\omega^2) \cdot \text{Amp} \cdot \sin(\omega \cdot q(7)); %event 6
        value = [y1; v1; y_com; a2_c; 1; 1; 1; v_com; 1];
        isterminal = [1; 1; 0; 1; 1; 0; 0; 0; 0; 0];
        direction = [-1; -1; 0; 0; -1; 1; 0; 0; -1; 0; 0];
    case {st(4)} % Value 1,0. Free flight
        value = [1; 1; 1; y_com; 1; 1; 1; 1; q5];
        isterminal = [0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
        direction = [-1; -1; -1; -1; 0; 0; 0; -1; -1; 0];
    case {st(5)}
        F_acc_con = g - \text{Floor acc} ;
        value = [1; 1; 1; 1; 1; 1; 1; F_acc_con; 1; 1];
        isterminal = [0; 0; 0; 0; 0; 0; 0; 0; 1; 0];
        direction = [0; 0; 0; 0; 0; 0; 0; 0; -1; 0];
    otherwise
        error('Wrong state!');
    end
end

%__________________________________________________________________________

function qdot = vec_field(t,q,st_ind)
\% Vector field for a slender rod, free flight.
\% c3 = \cos(q(3));
\% s3 = \sin(q(3));

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if st_ind == 1
qdot = [q(4);q(5);q(6);0;-g ;0.0;1];
elseif st_ind == 2
co7 = cos(omega*q(7));
s07 = sin(omega*q(7));
ch_1 = g - ((L*m/2)*s3*q(6)^2 - (omega^2)*Amp*s07;
ch_2 = m*(1 + 3*c3^2 - 3*c3*s3*s*mu);
ch_3 = 6*s*mu*s3 - 6*c3;
q_const_1 = (q(2)-(L/2)*s3-Amp*s07)*[0;1;-(L/2)*c3;0;0;0,-omega*Amp*co7];
q_const_2 = (q(5) - q(6)*c3 - Amp*omega*q(7))*[0;0;q(6)*s3;0;1;-c3;Amp*(omega^2)*s07];
qdot = [q(4);q(5);q(6);s*mu*ch_1/(ch_2);-g/m + ch_1/(ch_2);ch_3*ch_1/(L*ch_2);1] - 100*q_const_1 - 100*q_const_2;
elseif st_ind == 3
co7 = cos(omega*q(7));
s07 = sin(omega*q(7));
ch_1 = g + ((L*m/2)*s3*q(6)^2 - (omega^2)*Amp*s07;
ch_2 = m*(1 - 3*(c3)^2 + 3*(c3)*(s3)*s*mu);
ch_3 = 6*s*mu*s3 - 6*(c3);
q_const_1 = (q(2)-(L/2)*s3-Amp*s07)*[0;1;-(L/2)*c3;0;0;0,-omega*Amp*co7];
q_const_2 = (q(5) - q(6)*c3 - Amp*omega*q(7))*[0;0;q(6)*s3;0;1;-c3;Amp*(omega^2)*s07];
qdot = [q(4);q(5);q(6);s*mu*ch_1/(ch_2);-g/m + ch_1/(ch_2);ch_3*ch_1/(L*ch_2);1] - 100*q_const_1 - 100*q_const_2;
elseif st_ind == 4
qdot = [q(4);q(5);q(6);0;-g ;0.0;1];
elseif st_ind == 5
co7 = cos(omega*q(7));
s07 = sin(omega*q(7));
g_const_1 = 50*(q(2) - Amp*s07)*[0;1;0;0;0;0,-omega*Amp*co7];
qdot = [q(4);q(5);q(6);0;-(omega^2)*Amp*sin(omega*q(7));0.0;1] - g_const_1;
else
disp('problem in vector field')
end
end

%__________________________________________________________________________

end

end


Bibliography


