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# Completions of Partial Matrices

by

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A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy  
(Mathematics)

School of Mathematics, Statistics and Applied Mathematics  
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## ABSTRACT

Completions of partial matrices

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A *partial matrix* over a field  $\mathbb{F}$  is a matrix whose entries are either elements of the field or independent indeterminates. A *completion* of a partial matrix is any matrix that results from assigning a field element to each indeterminate. The set of completions of an  $m \times n$  partial matrix forms an affine subspace of  $M_{m \times n}(\mathbb{F})$ .

This thesis investigates partial matrices whose sets of completions satisfy particular rank properties - specifically partial matrices whose completions all have ranks that are bounded below and partial matrices whose completions all have the same rank. The maximum possible number of indeterminates in such partial matrices are determined and the partial matrices that attain these bounds are fully characterized for all fields. These characterizations utilize a duality between properties of affine spaces of matrices that are related by the trace bilinear form.

Precise conditions (based on field order, rank and size) are provided to determine if a partial matrix whose completions all have the same rank  $r$  must possess an  $r \times r$  partial sub-matrix whose completions are all nonsingular.

Finally a characterization of maximal nonsingular partial matrices is provided – a maximal nonsingular partial matrix is a square partial matrix each of whose completions has full rank, with the property that replacement of any constant entry with an indeterminate results in a partial matrix having a singular completion.

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## CHAPTER I

### Introduction

Consider this question:

For what value(s) of  $x$  does the matrix  $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & x \\ 2 & 3 & 3 \end{pmatrix}$  have rank 2?

Many people who have studied Linear Algebra at undergraduate level have encountered questions like this. Without perhaps being explicitly aware of it they have encountered a *partial matrix* and are being asked to solve a partial matrix *completion problem*, albeit a relatively simple one. A *partial matrix* over a field  $\mathbb{F}$  is a matrix whose entries are either field elements or independent indeterminates (i.e. any indeterminate can only occur in one position within a partial matrix). A *completion* of a partial matrix is any matrix that results by assigning a field element to each indeterminate in that partial matrix.

This thesis, as the title suggests, is primarily concerned with completions of partial matrices, specifically characterizing partial matrices whose completions all possess some property concerning *rank*. The *rank* of a matrix is the dimension of the space spanned by its rows (or its columns). Rank is a theme of fundamental importance within linear algebra and indeed within any field in which linear transformations or matrix encoding of data are of interest.

In summary this thesis resolves the following problems:

**Problem 1.1.** *Determine the maximum possible number of indeterminates that a partial matrix can possess if all of its completions have ranks that are greater than or equal to some specified lower bound, and characterize all such partial matrices that attain this number of indeterminates.*

**Problem 1.2.** *Determine the maximum possible number of indeterminates that a partial matrix can possess if all of its completions have the same rank, and characterize all such partial matrices that attain this number of indeterminates.*

**Problem 1.3.** *Must a partial matrix whose completions all have rank  $r$  possess an  $r \times r$  partial submatrix whose completions all have rank  $r$ ?*

**Problem 1.4.** *Characterize square partial matrices whose completions are all non-singular, in which the replacement of any constant entry with an indeterminate results in a partial matrix having a singular completion.*

Issues concerning the ranks of completions of partial matrices have attracted much attention in linear algebra literature. In [15] (1984), Hartfiel and Loewy provide conditions for all the completions of a square partial matrix to be singular. In [9], Cohen, Johnson, Rodman and Woerdeman investigate the minimum and maximum ranks of the completions of a partial matrix. For an arbitrary partial matrix they characterize the maximum rank of the completions in terms of the ranks of maximal rectangular constant submatrices (i.e. submatrices consisting entirely of constant entries). The minimum rank is not in general determined by the ranks of constant submatrices, however they discuss patterns of constant entries for which it is.

More recently a number of articles have been published concerning partial matrices whose completions possess some common rank property. They have provided motivation for some of the work in this thesis. In Section 1.1 we discuss these articles in some detail and relate how the work in this thesis contributes to this particular area of research.

Rank is a property which behaves somewhat unpredictably under matrix addition. For example, if we know the ranks of two  $m \times n$  matrices  $A, B$ , we can only place bounds on the rank of their sum, i.e.

$$|\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A + B) \leq \min\{m, n, \text{rank}(A) + \text{rank}(B)\}.$$

Hence spaces of matrices in which rank behaves predictably are noteworthy. The set of completions of an  $m \times n$  partial matrix constitutes an *affine subspace* of the space of  $m \times n$  (constant) matrices. An *affine subspace* is a translate of a linear subspace – i.e. if  $U$  is a linear subspace of a vector space  $V$ , then  $v + U = \{v + u : u \in U\}$  is an affine subspace for any  $v \in V$ . So the case of a partial matrix whose completions possess some rank property is a particular case of an affine space of matrices satisfying

that same property. Hence research on partial matrices whose completions satisfy a particular rank property is part of a broader vein of research into spaces of matrices with special rank properties. The following paragraphs give a flavour of activity in this area.

In [14] (1962) Flanders investigates linear subspaces of matrices where the ranks of elements are bounded above. He shows that the dimension of a linear subspace of  $m \times n$  matrices where the maximum rank present is  $r$  cannot exceed  $rp$  where  $p = \max(m, n)$ . He also characterizes linear spaces of matrices that realize this dimension bound. These results of Flanders are subject to a condition that the field order exceeds the rank bound.

In [20] (1985) Meshulam revisits this topic and proves the same results as Flanders by independent means, succeeding in removing the field order restrictions required by the proofs of Flanders. In [21], Meshulam determines an upper bound for a linear subspace of *symmetric* matrices whose ranks are bounded above. This result is subject to the restriction that the field order exceeds the upper rank bound. A *symmetric matrix* is a square matrix that is equal to its transpose. For matrices defined over the real numbers or an *algebraically closed* field Meshulam also determines the maximum dimension of an *affine* subspace of square matrices whose ranks are bounded below. An *algebraically closed* field is a field that has a root for every non-constant polynomial over that field (e.g. the complex numbers).

In [1] (1962) and [2] (1965), for matrices defined over the real numbers, the complex numbers or the *skew field* of quaternions, Adams, Lax and Phillips determine an upper bound for the number of square matrices whose real nonzero linear combinations are all nonsingular. A *skew field* obeys the same axioms as a field with the exception that the multiplication is not required to be commutative. For matrices defined over the real numbers this is equivalent to determining an upper bound on the dimension of a linear subspace of square matrices whose nonzero elements are all nonsingular.

In [10] (2011) de Seguins Pazzis determines conditions for a linear subspace of matrices to be spanned by elements of a specified rank. In [12] he characterizes affine spaces of nonsingular matrices that are of maximum possible dimension for all fields other than the field of two elements. In [11] de Seguins Pazzis determines an upper bound for the dimension of an affine subspace in which the ranks of elements are bounded below and he characterizes those affine spaces that have maximum possible dimension.

## 1.1 Partial matrices with special rank properties

In recent years there have been a number of articles published on similar themes (relating to partial matrices) as discussed in this thesis. Of particular significance (and in fact motivational) to the work documented here is an article by Brualdi, Huang and Zhan [5] that was published in 2010. In this paper they introduce a more general family of matrices known as *affine column independent (ACI)* matrices and use them to prove results about partial matrices. An *affine column independent* matrix over a field  $\mathbb{F}$  is a matrix whose entries are  $\mathbb{F}$ -linear combinations of indeterminates and field elements with the restriction that any single indeterminate can only appear in one column. A partial matrix is an ACI matrix and ACI matrices typically arise when elementary row operations are performed on partial matrices. However not all ACI matrices can be converted to partial matrices via elementary row operations as evident from this  $2 \times 3$  ACI matrix over the real numbers:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & 2x_2 & 3x_3 \end{pmatrix}.$$

ACI matrices have subsequently become a focus of research interest in their own right.

In [5] Brualdi et al. characterize ACI matrices whose completions have ranks that are bounded (above) by a specified rank. They also prove the following result about square ACI matrices whose completions are all nonsingular. This result is subject to a field order condition that is a result of a scheme of elementary row operations that is employed in its proof.

**Theorem 1.5. (Brualdi, Huang, Zhan) [5]** *Let  $\mathbb{F}$  be a field with at least  $n + 1$  elements. Let  $A$  be an  $n \times n$  ACI matrix over  $\mathbb{F}$ . Then every completion of  $A$  is nonsingular if and only if there exists a nonsingular constant matrix  $T \in M_n(\mathbb{F})$  and a permutation matrix  $Q \in M_n(\mathbb{F})$  such that  $TAQ$  is an upper triangular matrix with nonzero constant diagonal entries.*

Brualdi et al. use Theorem 1.5 to determine the maximum number of indeterminates that a partial matrix can contain if all of its completions are nonsingular and to characterize such partial matrices that attain this number of indeterminates. Again their result requires that the field order be greater than the order of the partial matrix.

**Theorem 1.6. (Brualdi, Huang, Zhan) [5]** *Let  $\mathbb{F}$  be a field with at least  $n + 1$  elements. Let  $A$  be an  $n \times n$  partial matrix over  $\mathbb{F}$  whose completions are all nonsingular. Then the number of indeterminates of  $A$  is less than or equal to  $\frac{n(n-1)}{2}$ . This maximum number is attained if and only if there exist permutation matrices  $P, Q$  such that  $PAQ$  is upper triangular with nonzero constant diagonal entries, with all entries above the diagonal being indeterminates.*

Theorem 1.6 is a result about square partial matrices of order  $n$  whose completions all have rank  $n$ . It may also be interpreted as a result about square partial matrices of order  $n$  whose completions have ranks bounded below by  $n$  (of course  $n$  is the maximum possible rank attainable). Hence it has connections with Problems 1.1 and 1.2. In Chapter IV (and [17]) we prove the following theorem about partial matrices whose completions all have ranks that are greater than or equal to some specified lower bound:

**Theorem 4.2.** *Let  $\mathbb{F}$  be a field and let  $A$  be an  $m \times n$  partial matrix with the property that every completion of  $A$  has rank at least  $r$  for some fixed  $r \leq \min(m, n)$ . Then the number of indeterminates in  $A$  is at most  $mn - \frac{r(r+1)}{2}$ . This bound is attained if and only if  $A$  can be transformed by row and column permutations to a partial matrix  $A'$  of the following form:*

- *The upper left  $r \times r$  submatrix of  $A'$  is an upper triangular matrix with nonzero constant entries on the main diagonal and independent indeterminates above the main diagonal.*
- *All entries of  $A'$  outside the upper left  $r \times r$  region are independent indeterminates.*

Theorem 4.2 resolves Problem 1.1 and can be viewed as a generalization of Theorem 1.6. Whereas Theorem 1.6 characterizes  $n \times n$  partial matrices whose completions all have rank  $n$ , or equivalently all have ranks that are bounded below by  $n$ , Theorem 4.2 characterizes (rectangular) partial matrices whose completions all have ranks that are bounded below by some specified (arbitrary) number. In addition Theorem 4.2 does not require any restriction on the field over which the partial matrices are defined.

In [3] Borobia and Canogar extend the results of Brualdi et al. in another direction. They investigate square partial matrices over integral domains whose completions are all nonsingular. In [5] Brualdi et al. show that all the completions of

a square partial matrix over a field have full rank if and only if the determinant of the partial matrix is a nonzero constant. This is not the case with square partial matrices over integral domains as Borobia and Canogar illustrate with this simple example that is defined over the integers. The completions of

$$\begin{pmatrix} 2x + 1 & 3y + 2 \\ 1 & 3 \end{pmatrix}$$

all have full rank yet its determinant is  $6x - 3y + 1$ . They characterize square ACI matrices over an integral domain whose completions have the same nonzero constant determinant.

**Theorem 1.7. (Borobia, Canogar) [3]** *Let  $\mathbb{D}$  be an integral domain. Let  $A$  be an  $n \times n$  ACI matrix over  $\mathbb{D}$ . The following three statements are equivalent:*

- (a) *The determinant of  $A$  is a nonzero constant polynomial.*
- (b) *All the completions of  $A$  have the same nonzero constant determinant;*
- (c) *There exists a nonsingular constant matrix  $T \in M_n(\mathbb{D})$  and a permutation matrix  $Q \in M_n(\mathbb{D})$  such that  $TAQ$  is an upper triangular matrix with nonzero constant diagonal entries.*

In [3] Borobia and Canogar also characterize ACI matrices over fields whose completions are all nonsingular.

**Theorem 1.8. (Borobia, Canogar) [3]** *Let  $\mathbb{F}$  be a field. Let  $A$  be an  $n \times n$  ACI matrix over  $\mathbb{F}$ . Then all of the completions of  $A$  are nonsingular if and only if there exists a nonsingular constant matrix  $T \in M_n(\mathbb{F})$  and a permutation matrix  $Q \in M_n(\mathbb{F})$  such that  $TAQ$  is an upper triangular matrix with nonzero constant diagonal entries.*

In [4] Borobia and Canogar characterize (rectangular) ACI matrices whose completions all have full rank. Their result applies to all fields. In [16], Huang and Zhan once again investigate partial matrices using ACI matrices as their means of analysis. They characterize ACI matrices whose completions all have the same rank.

**Theorem 1.9. (Huang, Zhan) [16]** *Let  $m, n$  be positive integers, let  $\mathbb{F}$  be a field with  $|\mathbb{F}| \geq \max(m, n + 1)$  and let  $A$  be an  $m \times n$  ACI matrix over  $\mathbb{F}$ . Then all*

completions of  $A$  have the same rank  $r$  if and only if there exists a nonsingular matrix  $T \in M_m(\mathbb{F})$  and a permutation matrix  $Q \in M_n(\mathbb{F})$  such that

$$TAQ = \begin{pmatrix} U_1 & \boxplus & \boxplus \\ 0 & 0 & \boxplus \\ 0 & 0 & U_2 \end{pmatrix},$$

where  $U_1$  and  $U_2$  are square upper triangular ACI matrices with nonzero constant diagonal entries whose orders sum to  $r$ , and  $\boxplus$  denotes an ACI matrix whose only restriction is that it be of appropriate dimension.

Huang and Zhan use their result to determine the maximum possible number of indeterminates that a (rectangular) partial matrix can possess if all of its completions have the same rank, and they characterize partial matrices of this type that attain this maximum possible number of indeterminates.

**Theorem 1.10. (Huang, Zhan)** [16] *Let  $m \leq n$  be positive integers, let  $\mathbb{F}$  be a field with  $|\mathbb{F}| \geq \max(m, n + 1)$  and let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  all of whose completions have the same rank  $r$ . Then the number of indeterminates of  $A$  is less than or equal to  $rn - \frac{r(r+1)}{2}$ . This maximum number is attained if and only if there exist permutation matrices  $P \in M_m(\mathbb{F})$  and  $Q \in M_n(\mathbb{F})$  such that*

$$PAQ = \begin{pmatrix} U_1 & \Delta_1 & \Delta_2 \\ 0 & 0 & \Delta_3 \\ 0 & 0 & U_2 \end{pmatrix},$$

where  $\Delta_1$ ,  $\Delta_2$  and  $\Delta_3$  are partial matrices all of whose entries are indeterminates,  $U_1$  and  $U_2$  are  $(r - s) \times (r - s)$  and  $s \times s$  upper triangular partial matrices with nonzero constant diagonal entries and with all entries above the diagonal being indeterminates, with  $s = 0$  when  $m < n$ .

Huang and Zhan's result provides a partial resolution of Problem 1.2 – however it does not resolve it for fields that do not satisfy the field order condition required by Theorem 1.10. This field order restriction is a result of a mechanism of elementary row operations that they employ in their proof. In Chapter V (and [18]), we use an inductive argument and apply our results from Chapter IV to extend Theorem 1.10 (of Huang and Zhan) to all fields.

Partial matrices satisfying the hypothesis of our extended version of Theorem 1.10 possess an  $r \times r$  submatrix each of whose completions is nonsingular. This motivated

our interest in Problem 1.3 – must a partial matrix whose completions all have the same rank  $r$  possess an  $r \times r$  submatrix each of whose completions is nonsingular? Theorem 1.9 (of Huang and Zhan) provides a partial answer to this question. A consequence of this result is that this question may be answered in the affirmative for such  $m \times n$  partial matrices over fields that satisfy the field order conditions (i.e.  $|\mathbb{F}| \geq \max(m, n + 1)$ ). In Chapter VI (and [18]) we fully resolve this question. We show that a partial matrix whose completions all have the same rank  $r$  must contain an  $r \times r$  submatrix whose completions are all nonsingular if the field order is at least as great as the rank in question. If the field order is less than the rank, we show that the answer to this question depends on the size of the matrix. We show that the answer is affirmative if  $\max(m, n) < r + |\mathbb{F}| - 1$ . We also provide examples of partial matrices of minimum size whose completions all have the same rank but do not possess an  $r \times r$  submatrix whose completions are all nonsingular.

In Chapter VII we investigate Problem 1.4 which was originally posed by Brualdi et al. in [5] – namely the characterization of *maximal nonsingular* partial matrices. A *maximal nonsingular* partial matrix is a square partial matrix whose completions are all nonsingular, with the property that replacement of any constant entry with an indeterminate results in a partial matrix having a singular completion. We show that the rows and columns of a maximal nonsingular partial matrix can be permuted so that all the indeterminates occupy a block strictly upper triangular region. In addition when in this form the inverse of every completion is block upper triangular with a band of “fixed” or “specified” entries above the block lower triangular region of zeros.

## 1.2 Summary of contents

Chapter II of this thesis introduces some notation and terminology, and discusses some general background theory. Chapter III discusses specific background theory and Chapters III through VII contain the original results that resolve Problems 1.1 through 1.4. Chapter VIII briefly discusses some suggestions for further research.

## CHAPTER II

### Notation, terminology and background theory

The purpose of this chapter is twofold: the introduction of notation and terminology, and the presentation and discussion of a miscellany of established theory that is relied upon in this thesis. This latter material is provided for any reader that has a cursory knowledge of Linear Algebra in an attempt to alleviate the need to consult other texts [24, 22, 13].

#### 2.1 Notation and terminology

As mentioned in Chapter I, an affine subspace is a translate of a linear subspace. Let  $V$  be a vector space, let  $U$  be a linear subspace of  $V$  ( $U \subseteq V$ ) and let  $w \in V$ . Then  $w + U := \{w + u : u \in U\}$  is an affine subspace of  $V$ . We say that  $w + U$  is a *non-linear affine space* if  $w \notin U$  – in this case  $0 \notin w + U$ . The dimension of an affine space is defined to be the dimension of the linear subspace of which it is a translate, i.e.  $\dim(w + U) := \dim(U)$ . We say that  $U$  is a *proper* linear subspace of  $V$  if  $U \neq \{0\}$  and  $U \neq V$ .

An arbitrary field is denoted by  $\mathbb{F}$ . For a prime power  $q$ ,  $\mathbb{F}_q$  denotes the finite field of  $q$  elements. The space of row vectors of length  $n$  with entries in  $\mathbb{F}$  is denoted by  $\mathbb{F}^n$ , and the space of column vectors by  $(\mathbb{F}^n)^T$ . The superscript  $T$  denotes transpose. We use  $e_i$  to denote the  $i^{\text{th}}$  standard basis vector in  $\mathbb{F}^n$  – i.e.  $e_i(i) = 1$  and all other entries are zero. A row (column) is called *singly nonzero* if it has exactly one nonzero entry, and is called *nowhere zero* if it has no zero entry. In a vector space of finite dimension, a *hyperplane* is a linear subspace of codimension 1 – i.e. in a vector space of dimension  $n$ , a hyperplane has dimension  $n - 1$ . An *endomorphism* of a vector space  $V$  is a linear map from  $V$  to itself. For a vector space  $V$ , we denote the space of all endomorphisms of  $V$  by  $\text{End}(V)$ . If  $X, Y \in \text{End}(V)$ , then  $X \circ Y$  denotes the composition of  $X$  and  $Y$ , i.e.  $X$  after  $Y$ .

The space of all  $m \times n$  matrices with entries in  $\mathbb{F}$  is denoted by  $M_{m \times n}(\mathbb{F})$ ; this is contracted to  $M_n(\mathbb{F})$  if  $m = n$ . A (0–1) matrix is a matrix whose entries are either 0 or 1. The  $n \times n$  identity matrix is denoted by  $I_n$ . We denote the entry in Row  $i$  and Column  $j$  of a matrix  $A$  by  $A(i, j)$ . In square matrix,  $A$ , of order  $n$  the *minor* of  $A(i, j)$  is the determinant of the  $(n - 1) \times (n - 1)$  matrix that remains when Row  $i$  and Column  $j$  are deleted from  $A$ . The *cofactor* of  $A(i, j)$  is the minor of  $A(i, j)$  multiplied by  $(-1)^{i+j}$ . We use  $A^{(i)}$  to refer to the  $i^{\text{th}}$  column of a matrix  $A$ . A *line* in a matrix is a row or a column of that matrix. A set of positions in a matrix is called *independent* if it contains no two positions that occur on the same line. The *diagonal* in a square matrix comprises all the entries indexed by  $\{(i, i)\}_{i=1}^n$ . A *diagonal matrix* is a square matrix in which all entries away from the diagonal are zero. An *upper triangular matrix* is a square matrix whose entries below the diagonal are all zero and a *strictly upper triangular matrix* is an upper triangular matrix whose diagonal entries are all zero. A *lower triangular matrix* is a square matrix whose entries above the diagonal are all zero. A *skew-symmetric* matrix is a square matrix that is equal to the negative of its transpose. A *unimodular* matrix is a square matrix whose determinant is either 1 or  $-1$ .

An  $m \times n$  matrix is said to be in *row echelon form* if it satisfies the following conditions:

- Any row that only contains zeros occurs below all nonzero rows.
- The first nonzero entry in a nonzero row is 1 – this position is referred to as a *pivot*. The pivot in a row must be in a column of greater index than the pivot in the row above it.

A matrix that is in row echelon form has zeros in all positions directly below its pivots. A matrix is said to be in *reduced row echelon form* if it is in row echelon form and if it also has zeros in all positions directly above its pivots. We say that a matrix is in *column echelon form* if its transpose is in row echelon form and we say that a matrix is in *reduced column echelon form* if its transpose is in reduced row echelon form .

All partial matrices from now on are defined over fields. A *proper* partial matrix possesses at least one indeterminate. If we refer to a submatrix of a partial matrix it is understood that the submatrix is itself a partial matrix. A column (row) of a partial matrix that has no indeterminate entry is referred to as a *constant column (row)*. A column (row) that includes at least one indeterminate entry is referred to as an *indeterminate column (row)*. The *zero completion* of a partial matrix is the matrix that results when the value zero is assigned to every indeterminate. A *partial*

*completion* of a partial matrix is any partial matrix that results by the assignment of a field element to some of the indeterminates. A partial matrix is said to have or be of *constant rank*  $r$  if all of its completions have the same rank  $r$ . A partial matrix is said to have or be of *constant full rank* if all of its completions have full rank. A square partial matrix is said to be *nonsingular* if all of its completions are nonsingular. We use  $\triangle$  generically to denote a partial matrix or partial submatrix whose entries are all indeterminate. The symbol  $\blacksquare$  is used generically to denote a matrix or submatrix whose entries are constant and has no special properties of immediate interest. Similarly  $\star$  is used generically to denote a partial matrix or partial submatrix that has no special properties of immediate interest.

## 2.2 Linear subspaces and dimension

Lemma 2.2 below details the dimension considerations when two subspaces are added.

**Definition 2.1.** Let  $V$  be a vector space over a field  $\mathbb{F}$  and let  $U, W$  be linear subspaces of  $V$ . Then

$$U + W := \{u + w \mid u \in U, w \in W\}.$$

**Lemma 2.2.** *Let  $V$  be a vector space of finite dimension over a field  $\mathbb{F}$ . Let  $U, W$  be linear subspaces of  $V$ . Then*

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

*Proof.* The intersection of linear subspaces is itself a linear subspace and so  $U \cap W$  is a linear subspace of  $V$ . If  $U \cap W = \{0\}$  then it is clear that a basis for  $U + W$  is formed by the union of any two bases for  $U$  and  $W$ , hence  $\dim(U + W) = \dim(U) + \dim(W)$ . In this case we denote  $U + W$  by  $U \oplus W$ .

If  $U \cap W \neq \{0\}$  then it has some finite dimension  $k$ . Let  $B_{U \cap W} = \{u_1, \dots, u_k\}$  be a basis for  $U \cap W$ . It is possible to extend  $B_{U \cap W}$  to a basis for  $U$ , denoted  $B_U$ , by appending  $\dim(U) - k$  linearly independent vectors (from  $U \setminus W$ ). Similarly it is possible to extend  $B_{U \cap W}$  to a basis for  $W$ , denoted  $B_W$ , by appending  $\dim(W) - k$  linearly independent vectors (from  $W \setminus U$ ). Let  $\dim(W) = k + l$  and let  $B_W = \{u_1, \dots, u_k, w_1, \dots, w_l\}$ . Let  $W' = \text{span}\{w_1, \dots, w_l\}$ . Then  $U \cap W' = \{0\}$  and  $U + W = U + W' = U \oplus W'$ . It follows that

$$\dim(U + W) = \dim(U) + \dim(W') = \dim(U) + \dim(W) - \dim(U \cap W).$$

□

Lemma 2.3 below shows that in a vector space of finite dimension  $n \geq 2$ , the intersection of  $n - 1$  hyperplanes is a subspace of dimension greater than zero.

**Lemma 2.3.** *Let  $V$  be a vector space of finite dimension  $n > 1$  over a field  $\mathbb{F}$ . Let  $\{H_i\}_{i=1}^{n-1}$  be  $n - 1$  linear subspaces of  $V$  each having dimension at least  $n - 1$ . Then  $\dim(\bigcap_{i=1}^{n-1} H_i) \geq 1$ .*

*Proof.* We employ induction on the dimension of the vector space. As the base case consider a vector space of dimension 2. The lemma only requires us to consider one subspace in this case and it must have dimension at least 1 so the lemma clearly holds. We now assume that the lemma is true for all vector spaces of dimension less than  $n$  and we consider the case of a vector space  $V$  of dimension  $n$ . Let  $H_1, \dots, H_{n-1}$  be  $n - 1$  subspaces of  $V$  – each of dimension at least  $n - 1$ . If  $H_1 = \dots = H_{n-1} = V$  then clearly  $\bigcap_{i=1}^{n-1} H_i = V$  and we are done. So suppose  $H_1$  has dimension  $n - 1$  and for  $i = 2, \dots, n - 1$ , let  $H'_i = H_1 \cap H_i$ . Note that by Lemma 2.2 each  $H'_i$  has dimension at least  $n - 2$ . Now by the inductive hypothesis  $\bigcap_{i=2}^{n-1} H'_i$  is a subspace of  $H_1$  of dimension at least 1, hence  $\dim(\bigcap_{i=1}^{n-1} H_i) \geq 1$ .  $\square$

Lemma 2.4 shows how to construct a basis for a vector space of finite dimension  $n$  given  $n$  hyperplanes (of that space) whose intersection is the zero vector.

**Lemma 2.4.** *Let  $V$  be a vector space of finite dimension  $n$ . Let  $H_1, \dots, H_n$  be hyperplanes of  $V$  such that  $\bigcap_{i=1}^n H_i = \{0\}$ . For each  $i = 1, \dots, n$ , let  $v_i$  be a nonzero vector that spans  $\bigcap_{j=1, j \neq i}^n H_j$ . Then  $\{v_1, \dots, v_n\}$  is a basis for  $V$ .*

*Proof.* We employ induction on  $n$ . As the base case we take a vector space of dimension 1 for which the lemma is easily seen to be true. We now assume that the lemma holds for vector spaces of dimension less than  $n$ . Let  $H'_k = H_1 \cap H_k$  for  $2 \leq k \leq n$ . Each  $H'_k$  is a hyperplane of  $H_1$  and hence by the induction hypothesis  $v_2, \dots, v_n$  form a basis for  $H_1$ . As  $v_1 \notin H_1$ , then  $\{v_1, \dots, v_n\}$  is a linearly independent set and hence constitutes a basis for  $V$ .  $\square$

### 2.3 The rank–nullity theorem

The rank–nullity theorem (Theorem 2.5) is a hugely important result in Linear Algebra and is particularly relevant in Chapter III.

**Theorem 2.5. (The rank–nullity theorem)** *Let  $\mathbb{F}$  be a field and let  $U$  and  $V$  be vector spaces over  $\mathbb{F}$ . Let  $U$  be of finite dimension over  $\mathbb{F}$  and let  $f : U \rightarrow V$  be a linear mapping. Then*

$$\dim(U) = \dim(\ker(f)) + \dim(\operatorname{im}(f)).$$

*Proof.* As  $U$  has finite dimension, then so has any subspace of  $U$  such as  $\ker(f)$ . Let  $r = \dim(\ker(f))$  and let  $\{u_1, \dots, u_r\}$  be a basis for  $\ker(f)$ . As the image of any spanning set for  $U$  must span the image of  $f$ , it follows that  $\operatorname{im}(f)$  is also of finite dimension. Let  $\dim(\operatorname{im}(f)) = s$  and let  $\{v_1, \dots, v_s\}$  be a basis for  $\operatorname{im}(f)$ . As each element of  $\{v_1, \dots, v_s\}$  is in  $\operatorname{im}(f)$  there exists a set  $\{w_1, \dots, w_s\} \subset U$  such that  $f(w_i) = v_i$  for  $1 \leq i \leq s$ . We claim that  $\{u_1, \dots, u_r, w_1, \dots, w_s\}$  is a basis for  $U$ .

Firstly we will show that  $\{u_1, \dots, u_r, w_1, \dots, w_s\}$  spans  $U$ . Let  $u$  be an arbitrary vector in  $U$ . Now  $f(u) \in \operatorname{im}(f)$  and so there exist  $b_1, \dots, b_s \in \mathbb{F}$  such that

$$f(u) = b_1v_1 + \dots + b_sv_s.$$

Let  $u' = u - b_1w_1 - \dots - b_sw_s$ . Then

$$\begin{aligned} f(u') &= f(u - b_1w_1 - \dots - b_sw_s), \\ &= f(u) - b_1v_1 - \dots - b_sv_s, \\ &= b_1v_1 + \dots + b_sv_s - b_1v_1 - \dots - b_sv_s = 0. \end{aligned}$$

This means that  $u' \in \ker(f)$  and so there exist  $a_1, \dots, a_r \in \mathbb{F}$  such that

$$\begin{aligned} u' &= u - b_1w_1 - \dots - b_sw_s = a_1u_1 + \dots + a_ru_r \\ \Rightarrow u &= a_1u_1 + \dots + a_ru_r + b_1w_1 + \dots + b_sw_s. \end{aligned}$$

As we can write any  $u \in U$  as a linear combination of  $\{u_1, \dots, u_r, w_1, \dots, w_s\}$  we have that  $\{u_1, \dots, u_r, w_1, \dots, w_s\}$  spans  $U$ .

Next we need to show that  $\{u_1, \dots, u_r, w_1, \dots, w_s\}$  is linearly independent. Suppose that

$$a_1u_1 + \dots + a_ru_r + b_1w_1 + \dots + b_sw_s = 0. \tag{2.1}$$

As  $f$  is linear,  $f(0) = 0$  and so

$$\begin{aligned} 0 &= f(a_1u_1 + \dots + a_ru_r + b_1w_1 + \dots + b_sw_s) \\ &= a_1f(u_1) + \dots + a_rf(u_r) + b_1f(w_1) + \dots + b_sf(w_s) \\ &= b_1v_1 + \dots + b_sv_s. \end{aligned}$$

As  $\{v_1, \dots, v_s\}$  is a basis for  $\text{im}(f)$  it follows immediately that  $b_1 = \dots = b_s = 0$ . Substituting this into Equation 2.1 gives

$$a_1u_1 + \dots + a_ru_r = 0.$$

As  $\{u_1, \dots, u_s\}$  is a basis for  $\ker(f)$  it follows that  $a_1 = \dots = a_s = 0$ . Hence  $\{u_1, \dots, u_r, w_1, \dots, w_s\}$  is linearly independent and is a basis for  $U$  and it is immediate that

$$\dim(U) = \dim(\ker(f)) + \dim(\text{im}(f)).$$

□

## 2.4 Bilinear Forms

*Bilinear forms* permit a definition of *orthogonality*. In Chapter III we make extensive use of a particular *bilinear form* that we discuss in Section 2.4.1 below.

**Definition 2.6.** Let  $V$  be a vector space over a field  $\mathbb{F}$ . A *bilinear form* on  $V$  is a map  $\theta : V \times V \rightarrow \mathbb{F}$  such that

- (i)  $\theta(u + v, w) = \theta(u, w) + \theta(v, w) \forall u, v, w \in V$ ,
- (ii)  $\theta(u, v + w) = \theta(u, v) + \theta(u, w) \forall u, v, w \in V$  and
- (iii)  $\theta(au, v) = a\theta(u, v) = \theta(u, av) \forall a \in \mathbb{F}, \forall u, v \in V$ .

A bilinear form,  $\theta : V \times V \rightarrow \mathbb{F}$  is said to be *reflexive* if

$$\theta(u, v) = 0 \text{ for all } u, v \in V \text{ for which } \theta(v, u) = 0.$$

A bilinear form,  $\theta : V \times V \rightarrow \mathbb{F}$  is said to be *symmetric* if

$$\theta(u, v) = \theta(v, u) \forall u, v \in V.$$

A bilinear form  $\theta : V \times V \rightarrow \mathbb{F}$  is said to be *nondegenerate* if

$$\theta(u_1, v_1) = 0 \forall v_1 \in V \Leftrightarrow u_1 = 0 \text{ and } \theta(u_2, v_2) = 0 \forall u_2 \in V \Leftrightarrow v_2 = 0.$$

We now define the notion of *orthogonality* with respect to a bilinear form.

**Definition 2.7.** Let  $V$  be a vector space over a field  $\mathbb{F}$  and let  $\theta : V \times V \rightarrow \mathbb{F}$  be a reflexive bilinear form. Then  $u, v \in V$  are said to be *orthogonal* with respect to  $\theta$ , denoted  $u \perp_\theta v$ , if  $\theta(u, v) = 0$ . Let  $U \subseteq V$ , then the *orthogonal complement* of  $U$  with respect to  $\theta$ ,  $U^{\perp_\theta} := \{v \in V \mid \theta(u, v) = 0 \forall u \in U\}$ .

*Remark 2.8.* The usual scalar product on  $\mathbb{F}^n$  is an example of a reflexive nondegenerate bilinear form – it can be easily verified that it satisfies all the requirements. We denote orthogonality with respect to the usual scalar product by  $\perp$ .

Theorem 2.10 shows that in a vector space of finite dimension equipped with a nondegenerate reflexive bilinear form, orthogonal complements have complementary dimension. Before that we will prove a related result that is employed in the proof of the more general Theorem 2.10.

**Proposition 2.9.** *Let  $V$  be a non-trivial vector space of finite dimension  $n$  over a field  $\mathbb{F}$  and let  $\theta : V \times V \rightarrow \mathbb{F}$  be a nondegenerate reflexive bilinear form on  $V$ . Then for any hyperplane  $H$  of  $V$ ,  $\dim(H^\perp) \geq 1$ .*

*Proof.* If  $\dim(V) = 1$  the only hyperplane to consider is  $\{0\}$  and the proposition clearly holds. So suppose that  $\dim(V) > 1$  and let  $\{v_1, \dots, v_{n-1}\}$  be a basis for some hyperplane  $H$ . Define  $f_i : V \rightarrow \mathbb{F}$  by  $f_i(v) := \theta(v_i, v)$  for  $1 \leq i \leq n-1$  and for  $v \in V$ . Note that  $\ker(f_i) = v_i^\perp$  for  $1 \leq i \leq n-1$ . None of the  $f_i$  has  $V$  as its kernel as  $\theta$  is a nondegenerate form. So each  $f_i$  is surjective and by the rank-nullity theorem  $\dim(\ker(f_i)) = n-1$  (for  $1 \leq i \leq n$ ) – i.e. the kernel of each  $f_i$  is a hyperplane of  $V$ . The orthogonal complement of  $H$  is the intersection of these hyperplanes, i.e.  $H^\perp = \bigcap_{i=1}^{n-1} \ker(f_i)$ . Application of Lemma 2.3 to this situation gives that  $\dim(H^\perp) \geq 1$ .  $\square$

**Theorem 2.10.** *Let  $V$  be a vector space of finite dimension over a field  $\mathbb{F}$  and let  $\theta : V \times V \rightarrow \mathbb{F}$  be a nondegenerate reflexive bilinear form on  $V$ . Then for any linear subspace  $U \subseteq V$ ,*

$$\dim(U) + \dim(U^\perp) = \dim(V).$$

*Proof.* Let  $U$  be an arbitrary linear subspace of  $V$ , let  $\dim(U) = k$  and let  $\{u_1, \dots, u_k\}$  be a basis for  $U$ . Define  $f_i : V \rightarrow \mathbb{F}$  by  $f_i(v) := \theta(u_i, v)$  for  $1 \leq i \leq k$  and for  $v \in V$ . Note that  $\ker(f_i) = u_i^\perp$  for  $1 \leq i \leq k$ . Consider the map  $f : V \rightarrow \mathbb{F}^k$  defined by  $f(v) := (f_1(v) \dots f_k(v))$  for  $v \in V$ . The kernel of  $f$  is the intersection of the kernels of the individual  $f_i$ , i.e.  $\ker(f) = \bigcap_{i=1}^k \ker(f_i) = \bigcap_{i=1}^k u_i^\perp = U^\perp$ . Application of the rank-nullity theorem gives that

$$\dim(U^\perp) = \dim(V) - \dim(\operatorname{im}(f)).$$

We claim that  $f$  is surjective – i.e.  $\dim(\operatorname{im}(f)) = k = \dim(U)$ . Suppose it were not – then as the image of  $f$  is a linear subspace of  $\mathbb{F}^k$  it would be contained in

some hyperplane  $H \subset \mathbb{F}^k$ . As the usual scalar product is a reflexive bilinear form, it follows from Proposition 2.9 that there exists a nonzero vector  $(a_1 \dots a_k) \in H^\perp$  ( $H^\perp$  is the orthogonal complement of  $H$  with respect to the usual scalar product). As  $\text{im}(f) \subseteq H$  this means

$$\begin{aligned} & (a_1 \dots a_k) \cdot f(v) = 0 \quad \forall v \in V, \\ \Rightarrow & (a_1 \dots a_k) \cdot (f_1(v) \dots f_k(v)) = 0 \quad \forall v \in V, \\ \Rightarrow & (a_1 \dots a_k) \cdot (\theta(u_1, v) \dots \theta(u_k, v)) = 0 \quad \forall v \in V, \\ \Rightarrow & \theta(a_1 u_1, v) + \dots + \theta(a_k u_k, v) = 0 \quad \forall v \in V, \\ \Rightarrow & \theta(a_1 u_1 + \dots + a_k u_k, v) = 0 \quad \forall v \in V. \end{aligned}$$

Hence  $(a_1 u_1 + \dots + a_k u_k)^\perp = V$  contradicting the nondegeneracy of  $\theta$ . Thus we have that

$$\dim(U) + \dim(U^\perp) = \dim(V).$$

□

Note that any bilinear form that is symmetric is of course reflexive (but not vice versa). Hence Theorem 2.10 applies in particular to nondegenerate symmetric bilinear forms.

#### 2.4.1 The trace bilinear form

Before discussing this form we need to define the *trace* of a square matrix and discuss some of its properties.

**Definition 2.11.** Let  $X$  be an  $n \times n$  matrix over  $\mathbb{F}$ . The *trace* of  $X$  is the sum of its diagonal entries, i.e.

$$\text{trace}(X) := \sum_{i=1}^n X(i, i).$$

*Remark 2.12.*

- (i) As the diagonal entries of any square matrix are the same as that of its transpose it immediately follows that the trace of any square matrix equals the trace of its transpose.

- (ii) Let  $X, Y \in M_n(\mathbb{F})$ . Then

$$\text{trace}(XY) = \sum_{i=1}^n \sum_{j=1}^n X(i, j)Y(j, i) = \sum_{i=1}^n \sum_{j=1}^n Y(i, j)X(j, i) = \text{trace}(YX).$$

(iii) Let  $X \in M_n(\mathbb{F})$  and let  $Y \in M_n(\mathbb{F})$  be a nonsingular matrix. Then

$$\text{trace}(Y^{-1}XY) = \text{trace}(Y^{-1}(XY)) = \text{trace}((XY)Y^{-1}) = \text{trace}(X).$$

(iv) Let  $X, Y \in M_{m \times n}(\mathbb{F})$ . Then

$$\text{trace}(XY^T) = \sum_{i=1}^m \sum_{j=1}^n X(i, j)Y(i, j) = \sum_{i=1}^n \sum_{j=1}^m Y(j, i)X(j, i) = \text{trace}(Y^T X).$$

(v) For  $u, v \in \mathbb{F}^n$ ,  $\text{trace}(u^T v) = u \cdot v$ , where  $u \cdot v$  denotes the usual scalar product of  $u$  and  $v$ .

**Definition 2.13.** For positive integers  $m$  and  $n$ , the *trace bilinear form*  $\tau$  is defined on  $M_{m \times n}(\mathbb{F})$  by

$$\tau(X, Y) := \text{trace}(X^T Y),$$

for  $X, Y \in M_{m \times n}(\mathbb{F})$ .

It follows from the properties of the trace of a matrix that for all  $X, Y \in M_{m \times n}(\mathbb{F})$

$$\tau(X, Y) = \text{trace}(X^T Y) = \text{trace}((X^T Y)^T) = \text{trace}(Y^T X) = \tau(Y, X),$$

and we have that  $\tau$  is a symmetric form. It is straight-forward to verify that  $\tau$  is also a nondegenerate form. Let  $X$  be any nonzero  $m \times n$  matrix. Suppose that  $X(i, j) \neq 0$ . Let  $Y \in M_{m \times n}(\mathbb{F})$  be such that all its entries are zero except that  $Y(i, j) = 1$ , then

$$\tau(X, Y) = \text{trace}(X^T Y) = \sum_{p=1}^n \sum_{q=1}^m X(q, p)Y(q, p) = X(i, j)Y(i, j) = X(i, j) \neq 0.$$

As the trace bilinear form is symmetric and nondegenerate it follows from Theorem 2.10 that subspaces of  $M_{m \times n}(\mathbb{F})$  and their orthogonal complements (with respect to the trace form) have complementary dimension.

*Remark 2.14.* Let  $X, Y \in M_{m \times n}$ . Let  $v_X, v_Y \in \mathbb{F}^{mn}$  such that

$$\begin{aligned} v_X &= (X(1, 1) \dots X(m, 1) X(2, 1) \dots X(2, n) \dots X(m, 1) \dots X(m, n)), \\ v_Y &= (Y(1, 1) \dots Y(m, 1) Y(2, 1) \dots Y(2, n) \dots Y(m, 1) \dots Y(m, n)). \end{aligned}$$

Then

$$\tau(X, Y) = \text{trace}(X^T Y) = v_X \cdot v_Y.$$

So the trace bilinear form is really the usual scalar product applied to matrices.

**Definition 2.15.** For a linear subspace  $A$  of  $M_{m \times n}(\mathbb{F})$ , we use the term *trace complement* of  $A$  to refer to the linear subspace of  $M_{n \times m}(\mathbb{F})$  consisting of all those matrices  $Y \in M_{n \times m}(\mathbb{F})$  for which  $\text{trace}(YX) = 0$  for all  $X \in A$ . We denote the trace complement of  $A$  by  $A^*$ .

Thus the trace complement of  $A$  is the transpose of the orthogonal complement of  $A$  with respect to the trace bilinear form  $\tau$ . It follows that for any subspace  $A$  of  $M_{m \times n}(\mathbb{F})$ ,  $\dim A + \dim A^* = mn$ . For an element  $X$  of  $M_{m \times n}(\mathbb{F})$ , the notation  $X^*$  is used for the trace complement of the one-dimensional space  $\langle X \rangle$ . In the case  $n = 1$ , the trace complement of a subspace of  $\mathbb{F}^m$  is just the transpose of the orthogonal complement with respect to the usual scalar product.

## 2.5 König's Theorem

König's Theorem [6] is a beautiful result that dates from 1936. It is a keystone in our resolution of Problem 1.2 in Chapter V.

**Theorem 2.16. (König's Theorem)** *Let  $X$  be an  $m \times n$  (0–1) matrix. The minimum number of lines required to cover all the 1's of  $X$  is equal to the maximum number of 1's in independent positions in  $X$ .*

*Proof.* For an  $m \times n$  (0–1) matrix,  $X$ , let  $a$  denote the minimal number of lines to cover all of the 1's of  $X$  and let  $b$  denote the maximum number of 1's in independent positions in  $X$ . It must be that  $a \geq b$ , if not a contradiction would arise as  $a$  lines would not be sufficient to cover  $b$  ( $> a$ ) 1's in independent positions (in  $X$ ).

To show that  $a \leq b$ , we employ induction on the minimum of the number of rows and the number columns in the matrix. As the base case consider a matrix having only one row or one column. Such a matrix could be the zero matrix in which case  $a = b = 0$ . If it is not the zero matrix then it has only a single 1 in an independent position (i.e.  $a = 1$ ) and this 1 is covered by the single line that covers all the entries of the matrix (i.e.  $b = 1$ ). In either case  $a = b$ . We now assume that the theorem holds for (0–1) matrices with fewer rows than  $\min(m, n)$  or fewer columns than  $\min(m, n)$ , and we return to  $X$ , an  $m \times n$  (0–1) matrix. If  $X$  is the zero matrix, then  $a = b = 0$ , so we assume that  $X$  is nonzero.

We define a minimal covering of the 1's of  $X$  to be *proper* if it does not consist of all  $m$  rows or of all  $n$  columns. We now consider two cases:

Case 1:  $X$  does not have a proper minimal covering

In this case  $a = \min(m, n)$ . Suppose that  $X(i, j) = 1$  and consider the  $(m-1) \times (n-1)$  matrix  $X'$  that remains when Row  $i$  and Column  $j$  are deleted from  $X$ . The minimum number of lines required to cover all of the 1's in  $X'$  must be  $\min(m-1, n-1)$ . If it were not then by taking the corresponding lines in  $X$  along with Row  $i$  and Column  $j$  we would have a proper covering of  $X$  of no more than  $\min(m, n)$  lines, in contradiction of our hypothesis. Applying the induction hypothesis to  $X'$  gives that it has a maximum of  $\min(m-1, n-1)$  1's in independent positions. As the 1 in Position  $(i, j)$  cannot belong to the  $\min(m-1, n-1)$  lines covering the 1's in  $X'$  (in  $X$ ), it follows that  $b \geq \min(m, n)$ , and so  $a \leq b$ .

Case 2:  $X$  has a proper minimal covering

As  $X$  has a proper minimal covering, let us assume that the 1's in  $X$  are covered by some  $e$  rows and some  $f$  columns with  $e < m$  and  $f < n$ , with  $a = e + f$ . As permutation of rows and columns does not affect the quantities of interest here, let us assume that these are the first  $e$  rows and  $f$  columns. Then  $X$  has the following form:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & 0 \end{pmatrix}.$$

Note that the lower right  $(m-e) \times (n-f)$  region of  $X$  is zero. The submatrix  $X_2$  has  $e$  rows and  $n-f$  columns. It follows that the minimum number of lines to cover the 1's of  $X_2$  is  $e$  – if it were less than  $e$  then it would be possible to cover the 1's of  $X$  with fewer than  $e+f$  lines. Hence  $e \leq n-f$ . As  $e = \min(e, n-f) < \min(m, n)$  the inductive hypothesis applies, giving that  $X_2$  has  $e$  1's in independent positions. Applying the same rationale to  $X_3$  gives that it has  $f$  1's in independent positions. As  $X_2$  and  $X_3$  are covered by disjoint set of lines of  $X$  it follows immediately that  $X$  has at least  $e+f$  1's in independent positions and so  $a \leq b$ .  $\square$

## CHAPTER III

### Duality

The theme of this chapter is a characterization of affine spaces of matrices in which the ranks of elements are greater than or equal to some specified lower bound. The characterization is in terms of a dual property possessed by affine spaces that are related by orthogonality with respect to the trace bilinear form. The dual property is that the related affine subspaces, when considered as sets of linear transformations operating on column (or row) vectors, contain elements annihilating every subspace of a certain dimension. This duality is the foundation on which a large part of the research in this thesis stands. It is explicitly employed in the proof of the main result of Chapter IV and implicitly in Chapter V.

In Section 3.1, we use this duality to characterize affine spaces of square matrices in which every element is nonsingular and we determine an upper bound on the dimension of such an affine space. The space of endomorphisms of a vector space of finite dimension  $n$  is isomorphic to  $M_n(\mathbb{F})$  by the assignment of a basis to that vector space. So for convenience within this section, we use the notions of endomorphisms and square matrices interchangeably. To establish the dimension bound we use arguments first devised by Quinlan in [23]. Quinlan introduced a special case of this duality in [23] where she used it to show that the maximum dimension of a linear subspace of  $M_n(\mathbb{F})$  in which no element has a nonzero eigenvalue belonging to the field  $\mathbb{F}$  is  $\frac{n(n-1)}{2}$ . In Section 3.2 we extend the duality to characterize affine spaces of rectangular matrices whose ranks are greater than or equal to some specified lower bound. Again we determine an upper bound on the dimension of such an affine space.

The material in this chapter is drawn from an article of Quinlan [23] and from [17], an article written by Quinlan and the author – both articles have been published in *Linear Algebra and its Applications*.

### 3.1 Affine spaces of nonsingular matrices

To begin we define the *hyperplane annihilation property* which is central to the discussion in this section.

**Definition 3.1.** Let  $V$  be a vector space of finite dimension  $n$ . A subset of  $\text{End}(V)$  has the *hyperplane annihilation property* (on  $V$ ) if for every linear subspace of  $V$  of dimension  $n - 1$ , the subset contains an element whose kernel contains that linear subspace.

Correspondingly a subset of  $M_n(\mathbb{F})$  has the *hyperplane annihilation property* if for every linear subspace of  $(\mathbb{F}^n)^T$  of dimension  $n - 1$ , the subset contains an element whose right null-space contains that linear subspace. This occurs precisely if for every nonzero vector  $v$  in  $\mathbb{F}^n$ , the subset contains an element whose row-space is contained by  $\langle v \rangle$ .

Any subset of  $M_n(\mathbb{F})$  that contains the zero matrix has the hyperplane annihilation property. In this section our interest is in subsets of  $M_n(\mathbb{F})$  that do not contain the zero matrix and yet possess the hyperplane annihilation property. Such subsets have the hyperplane annihilation property if and only if for every nonzero vector  $v \in \mathbb{F}^n$  they contain an element whose row-space is precisely  $\langle v \rangle$ .

Theorem 3.2 is the main theorem of this section. It establishes that two affine subspaces, related to each other by orthogonality with respect to the trace bilinear form possess the properties that one has the hyperplane annihilation property (on  $(\mathbb{F}^n)^T$ ) whereas the other is such that every element is nonsingular.

**Theorem 3.2. (Quinlan)** [23, 17] *Let  $A$  be a proper subspace of  $M_n(\mathbb{F})$  and let  $X$  be an element of  $M_n(\mathbb{F}) \setminus A$ . Let  $Y \in A^*$  such that  $\text{trace}(YX) = 1$ . Then every element of the affine space  $X + A$  is nonsingular if and only if the affine space  $Y + \langle X, A \rangle^*$  has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$ .*

*Proof.* Suppose that every element of  $X + A$  is nonsingular. We need to show that  $Y + \langle X, A \rangle^*$  has the hyperplane annihilation property. Note that as  $\text{trace}(YX) = 1$ ,  $Y \notin \langle X, A \rangle^* \subset \langle X \rangle^*$ . Hence  $0 \notin Y + \langle X, A \rangle^*$  and so we need to show that for every nonzero vector  $v \in \mathbb{F}^n$ , there is a matrix in  $Y + \langle X, A \rangle^*$  whose row-space is spanned by  $v$ . As every matrix in the affine space  $X + A$  is nonsingular it follows that for every  $Z \in A$  and for every nonzero vector  $v \in \mathbb{F}^n$ ,  $v(X + Z) \neq 0$  or  $vX \neq -vZ$ . Equivalently  $vX \notin vA$ . Hence  $(vA)^* \not\subset (vX)^*$ . This means that there exists  $u \in \mathbb{F}^n$

such that

$$\text{trace}(u^T(vZ)) = 0 \text{ for all } Z \in A \text{ but } \text{trace}(u^T(vX)) = 1.$$

Hence

$$\text{trace}((u^T v)Z) = 0 \text{ for all } Z \in A \text{ but } \text{trace}((u^T v)X) = 1.$$

So the matrix  $u^T v$  is an element of  $Y + \langle A, X \rangle^*$  – i.e. it belongs to  $A^*$  (the trace complement of  $A$ ) and  $\text{trace}((u^T v)X) = 1$ . It has rank 1 and its row-space is spanned by  $v$ . So we have shown that  $Y + \langle A, X \rangle^*$  has the hyperplane annihilation property.

On the other hand assume that  $Y + \langle A, X \rangle^*$  has the hyperplane annihilation property. We now want to show that every element of  $X + A$  is nonsingular. For every  $v \in \mathbb{F}^n$  there is an element  $Y_v \in Y + \langle A, X \rangle^*$  whose row-space is spanned by  $v$ . Let  $Z$  be an arbitrary element of  $A$  so that  $X + Z$  is an arbitrary element of  $X + A$ . Hence  $\text{trace}(Y_v(X + Z)) = 1$ . As  $Y_v$  has a one-dimensional row-space, this means that there is at least one column of  $X + Z$  that does not belong to the trace complement of  $v$ . However as  $Y + \langle A, X \rangle^*$  has the hyperplane annihilation property this is true for every nonzero  $v \in \mathbb{F}^n$ . So the columns of  $X + Z$  must span all of  $(\mathbb{F}^n)^T$ , so  $X + Z$  is nonsingular. We chose  $X + Z$  arbitrarily ( $Z$  being an arbitrary element of  $A$ ), so we have shown that every element of  $X + A$  is nonsingular.  $\square$

Theorem 3.3 shows that the hyperplane annihilation property is preserved by transposition for the sets of matrices of interest. This fact will be used to establish a lower bound on the dimension of a non-linear affine space possessing the hyperplane annihilation property.

**Theorem 3.3.** [23] *Let  $B$  be a proper linear subspace of  $M_n(\mathbb{F})$  and let  $Y$  be an element of  $M_n(\mathbb{F}) \setminus B$  such that the affine subspace  $Y + B$  of  $M_n(\mathbb{F})$  has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$ . Then the affine subspace  $Y^T + B^T$  of  $M_n(\mathbb{F})$  also has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$ .*

*Proof.* Let  $X \in B^*$  such that  $\text{trace}(XY) = 1$ . Then by Theorem 3.2 every element of  $X + \langle Y, B \rangle^*$  is nonsingular since  $Y + B$  has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$ . As rank is preserved by transposition, it follows that every element of

$X^T + (\langle Y, B \rangle^*)^T$  is also nonsingular. Now

$$\begin{aligned}
(\langle Y, B \rangle^*)^T &= \{Z \in M_n(\mathbb{F}) : \text{trace}(Z^T Y') = 0, \forall Y' \in \langle Y, B \rangle\}, \\
&= \{Z \in M_n(\mathbb{F}) : \text{trace}((Y')^T Z) = 0, \forall Y' \in \langle Y, B \rangle\}, \\
&= \{Z \in M_n(\mathbb{F}) : \text{trace}(Z(Y')^T) = 0, \forall Y' \in \langle Y, B \rangle\}, \\
&= \{Z \in M_n(\mathbb{F}) : \text{trace}(ZY') = 0, \forall Y' \in \langle Y, B \rangle^T\}, \\
&= (\langle Y, B \rangle^T)^* = \langle Y^T, B^T \rangle^*.
\end{aligned}$$

Similarly  $X^T \in (B^T)^*$  and  $\text{trace}(X^T Y^T) = \text{trace}(XY) = 1$ . As every element of the affine space  $X^T + \langle Y^T, B^T \rangle^*$  is nonsingular, it follows from Theorem 3.2 that the affine space  $Y^T + B^T$  has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$ .  $\square$

Theorem 3.3 can be interpreted as saying that our characterization of the hyperplane annihilation property in terms of row-spaces of elements of  $Y + \langle X, A \rangle^*$  could just as well be expressed in terms of column spaces. This observation is already implicit in Theorem 3.2 and its proof, since the arguments there could be framed in terms of columns instead of rows. So if  $Y + \langle X, A \rangle^*$  has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$  (by multiplication on the left), it also possesses the hyperplane annihilation property on  $\mathbb{F}^n$  (by multiplication on the right).

Let  $Y + B$  be a non-linear affine subspace of  $M_n(\mathbb{F})$  having the hyperplane annihilation property. Theorem 3.4 shows that there exists a nonsingular matrix  $X$  so that the elements of  $\langle XY, XB \rangle$  of trace 1 have the hyperplane annihilation property. We will use such a set (the set of elements of trace 1 of a linear subspace of square matrices, having the hyperplane annihilation property) as a convenient object with which to establish the minimum possible dimension of a non-linear affine subspace having the hyperplane annihilation property.

**Theorem 3.4.** [23] *Let  $B$  be a proper linear subspace of  $M_n(\mathbb{F})$  and let  $Y \in M_n(\mathbb{F}) \setminus B$  such that  $Y + B$  has the hyperplane annihilation property. Then there exists a nonsingular matrix  $X$  so that*

- $\text{trace}(XY) = 1$ ,
- every element of  $XB$  has trace zero, and
- $XY + XB$  has the hyperplane annihilation property.

*Proof.* Let  $X \in B^*$  such that  $\text{trace}(XY) = 1$ . As  $Y + B$  has the hyperplane annihilation property, Theorem 3.2 gives that every element of the affine subspace  $X + \langle Y, B \rangle^*$

is nonsingular. In particular  $X$  is nonsingular and it follows that  $XY + XB$  has the hyperplane annihilation property.  $\square$

We can also express Theorem 3.4 in terms of endomorphisms.

**Theorem 3.5.** [23] *Let  $V$  be a vector space of finite dimension. Let  $B$  be a proper linear subspace of  $\text{End}(V)$  and let  $Y \in \text{End}(V) \setminus B$  such that  $Y + B$  has the hyperplane annihilation property on  $V$ . Then there exists an isomorphism  $X \in \text{End}(V)$  so that*

- $\text{trace}(X \circ Y) = 1$ ,
- every element of  $B' = \{X \circ Z : Z \in B\}$  has trace zero, and
- $(X \circ Y) + B'$  has the hyperplane annihilation property.

### 3.1.1 Dimension Bound

We now use the duality to determine the maximum possible dimension of an affine space of  $M_n(\mathbb{F})$  in which every element is nonsingular. First we determine the minimum possible dimension of an affine subspace of  $M_n(\mathbb{F})$ , consisting of matrices of trace 1, that possesses the hyperplane annihilation property. By Theorem 3.4, this is equivalent to determining the minimum possible dimension of a non-linear affine space of  $M_n(\mathbb{F})$ , possessing the hyperplane annihilation property. Using the fact that orthogonal complements have complementary dimension, this puts an upper bound on the dimension of an affine space in which every element is nonsingular.

Theorem 3.7 below establishes the minimum possible dimension of a non-linear affine subspace of endomorphisms (of a vector space of finite dimension) that has the hyperplane annihilation property – it relies on the following theorem.

**Theorem 3.6.** [23] *Let  $V$  be a vector space over  $\mathbb{F}$  of finite dimension  $n$ . Let  $B$  be a linear subspace of  $\text{End}(V)$  such that the affine subspace of  $B$  consisting of elements of trace 1 has the hyperplane annihilation property on  $V$ . Then there exists a vector  $v \in V$  such that  $\{f(v) : f \in B\} = V$ .*

*Proof.* By choosing a basis for  $V$  we can write the elements of  $\text{End}(V)$  as elements of  $M_n(\mathbb{F})$ . Let  $C_B$  be the subspace of  $M_n(\mathbb{F})$  corresponding to  $B \subseteq \text{End}(V)$  by this choice of basis. The affine subspace of  $C_B$  consisting of elements of trace 1 has the hyperplane annihilation property on  $(\mathbb{F}^n)^T$ . As established in Theorem 3.3, this affine subspace of  $C_B$  also has the hyperplane annihilation property on  $\mathbb{F}^n$ .

Let  $L_1$  be an element of  $C_B$  of trace 1 with rank 1. As  $L_1$  has rank 1, its row-space is a one-dimensional subspace of  $\mathbb{F}^n$ . Say that  $v_1$  spans the row-space of  $L_1$ .

It follows from the fact that  $L_1$  has trace 1 that  $v_1 L_1 = v_1$ . We extend  $\{v_1\}$  to a basis of  $\mathbb{F}^n$  in the following way. Assume that  $v_1, \dots, v_{i-1}$  have been identified and that they form a linearly independent set that spans a subspace of  $\mathbb{F}^n$  of dimension  $i - 1 \leq n - 1$ . As the affine subspace of  $C_B$ , consisting of elements of trace 1, has the hyperplane annihilation property on  $\mathbb{F}^n$  there is an element, say  $L_i$  of  $C_B$ , of rank 1 and trace 1, whose left nullspace contains  $\text{span}(v_1, \dots, v_{i-1})$ . Say that the row-space of  $L_i$  is spanned by  $u_i$ . Again it follows from the fact that  $L_i$  has trace 1 that  $u_i L_i = u_i$ . Hence  $u_i$  does not belong to the left nullspace of  $L_i$  and so is linearly independent of  $\{v_1, \dots, v_{i-1}\}$ . Let  $v_i = u_i - v_1$ . Iterating this process yields a basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{F}^n$ . Note that  $v_i L_i = (u_i - v_1) L_i = u_i L_i - v_1 L_i = u_i L_i = u_i = v_1 + v_i$ .

Let  $X$  denote the change of basis matrix from  $\{v_1, \dots, v_n\}$  to the standard basis of  $\mathbb{F}^n$ . Let  $L'_i = X L_i X^{-1}$  and let  $C'_B = X C_B X^{-1}$ . Note that  $C'_B$  still corresponds with  $B \subseteq \text{End}(V)$  but by an alternate basis. Then the first row of  $L'_1$  is  $e_1 \in \mathbb{F}^n$ . The first  $i - 1$  rows of  $L'_i$  are zero and Row  $i$  of  $L'_i$  is  $e_1 + e_i$  for  $2 \leq i \leq n$ . This means that the first columns of  $\{L'_i\}_{i=1}^n$  form a spanning set of  $(\mathbb{F}^n)^T$  and every element of  $(\mathbb{F}^n)^T$  is the first column of some element of  $C'_B$ . Hence the image of  $e_1^T$  under all the elements of  $C'_B$  is all of  $(\mathbb{F}^n)^T$ . This corresponds to some element  $v \in V$  whose image under all the elements of  $B$  is  $V$ .  $\square$

Refer to Examples 3.8 and 3.9 for examples of linear subspaces and vectors satisfying Theorem 3.6.

**Theorem 3.7.** [23] *Let  $V$  be a vector space of finite dimension  $n$  over  $\mathbb{F}$ . Let  $B$  be a subspace of  $\text{End}(V)$  such that the affine subspace of  $B$ , consisting of matrices of trace 1, has the hyperplane annihilation property on  $V$ . Then  $B$  has dimension at least  $\frac{n(n+1)}{2}$ .*

*Proof.* We employ induction on the dimension of the vector space. For the base case, let  $W$  be a vector space of dimension 1. The only hyperplane in a vector space of dimension 1 is the trivial subspace (i.e. comprising the zero vector). The only choice of subspace of  $\text{End}(W)$  that contains elements of trace 1 is  $\text{End}(W)$ . This has dimension  $1 = \frac{1(2)}{2}$  and so satisfies the theorem. We assume that given a vector space of dimension  $k < n$ , the minimum dimension of a linear subspace of endomorphisms (of that vector space) whose elements of trace 1 possess the hyperplane annihilation property is  $\frac{k(k+1)}{2}$ . We now consider the minimum dimension of a linear subspace  $B$  of  $\text{End}(V)$  whose elements of trace 1 possess the hyperplane annihilation property on  $V$  where  $\dim(V) = n$ .

From Theorem 3.6 we know there is a vector  $v \in V$  such that  $\{fv : f \in B\} = V$ . Choose a hyperplane  $U$  of  $V$ , such that  $\langle v \rangle \oplus U = V$ . We can write any vector  $w \in V$  as a sum  $w = av + u$  for some  $a \in \mathbb{F}$  and some  $u \in U$ . Consider the subspace of  $B$  consisting of all elements of  $B$  that annihilate  $v$  – call it  $B_v$ . For every hyperplane that contains  $v$  there must be an element of trace 1 of  $B_v$  that annihilates that hyperplane. Let  $\text{pr}_U : V \rightarrow U$  be the projection of  $V$  onto  $U$  with respect to the decomposition  $V = \langle v \rangle \oplus U$ . Then the subspace of  $\text{End}(U)$ ,  $\{\text{pr}_U \circ f|_U : f \in B_v\}$ , is such that its elements of trace 1 have the hyperplane annihilation property on  $U$ . By the induction hypothesis the dimension of this subspace is at least  $\frac{(n-1)n}{2}$ . Hence the dimension of  $B_v$  is at least  $\frac{(n-1)n}{2}$ .

Consider the map  $\theta : B \rightarrow (\mathbb{F}^n)^T$  defined by  $\theta(Z) = Zv$  for all  $Z \in B$ . The kernel of the map  $\theta$  is the subspace  $B_v$ . Since the space of images of  $v$  under elements of  $B$  is all of  $V$ , the dimension of the image of the map  $\theta$  is  $n$ . Applying the rank–nullity theorem gives that  $B_v$  has codimension  $n$  in  $B$ . Hence the dimension of  $B$  is at least  $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ .  $\square$

Quinlan provides the following two examples of linear subspaces of matrices of minimum possible dimension that contain affine subspaces, consisting of elements of trace 1, that possess the hyperplane annihilation property:

**Example 3.8 (Quinlan).** [23] The upper triangular matrices of trace 1 form an affine subspace having the hyperplane annihilation property over any field. The space of  $n \times n$  upper triangular matrices has dimension  $\frac{n(n+1)}{2}$ . Any hyperplane of  $(\mathbb{F}^n)^T$ , say  $H$ , is the orthogonal complement of some one–dimensional subspace of  $(\mathbb{F}^n)^T$ , with respect to the ordinary scalar product. Let  $v^T$  be a vector that spans this one–dimensional space such that its first nonzero entry is 1. Say that the first nonzero entry of  $v^T$  occurs in Position  $i$ . The upper triangular matrix containing zeros in all rows other than Row  $i$  and having  $v$  as Row  $i$  annihilates  $H$  and has trace 1.

With reference to Theorem 3.6, note that the image of  $e_n^T$  (the  $n^{\text{th}}$  standard basis vector) under the upper triangular matrices is  $(\mathbb{F}^n)^T$ .

**Example 3.9 (Quinlan).** [23] The symmetric matrices of trace 1 form an affine subspace having the hyperplane annihilation property so long as the field is *formally real*. In a *formally real* field  $-1$  cannot be written as a sum of squares in the

field. Again the space of  $n \times n$  symmetric matrices has dimension  $\frac{n(n+1)}{2}$ . Any hyperplane of  $(\mathbb{F}^n)^T$ , say  $H$ , is the orthogonal complement of some one-dimensional subspace of  $(\mathbb{F}^n)^T$ , with respect to the ordinary scalar product. Let  $v^T$  be a vector that spans this one-dimensional space. As the field is formally real the matrix  $v^T v$  has nonzero trace. Note that  $v^T v$  is symmetric and has rank 1, so annihilating  $H$ . Multiplying by an appropriate scalar results in a matrix of trace 1.

With reference to Theorem 3.6, note that for  $i = 1, \dots, n$  the image of  $e_i^T$  (the  $i^{\text{th}}$  standard basis vector) under the symmetric matrices is  $(\mathbb{F}^n)^T$ .

Corollary 3.10 follows from Theorems 3.2 and 3.7.

**Corollary 3.10. (Quinlan)** [23] *For any field  $\mathbb{F}$ , an affine subspace of  $M_n(\mathbb{F})$  in which every element is nonsingular can have dimension at most  $\frac{n(n-1)}{2}$ .*

*Proof.* Let  $A$  be a proper linear subspace of  $M_n(\mathbb{F})$  and let  $X \in M_n(\mathbb{F}) \setminus A$  such that every element of  $X + A$  is nonsingular. Let  $Y \in A^*$  such that  $\text{trace}(YX) = 1$ . Then by Theorem 3.2,  $Y + \langle X, A \rangle^*$  has the hyperplane annihilation property. By Theorem 3.7, the linear subspace  $\langle X, A \rangle^*$  must have dimension at least  $\frac{n(n+1)}{2} - 1$ . As noted earlier a subspace and its trace complement have complementary dimension and so  $\dim(\langle X, A \rangle)$  cannot exceed  $\frac{n(n-1)}{2} + 1$ . Hence the dimension of the affine subspace  $X + A$  cannot exceed  $\frac{n(n-1)}{2}$ .  $\square$

Quinlan provides the following examples of affine spaces satisfying the condition and dimension bound of Corollary 3.10:

**Example 3.11 (Quinlan).** [23] Let  $SUT_n \subseteq M_n(\mathbb{F})$  denote the subspace of strictly upper triangular matrices. Then  $I_n + SUT_n$  has dimension  $\frac{n(n-1)}{2}$  and all of its elements are nonsingular. Note that the subspace of strictly upper triangular  $n \times n$  matrices is the trace complement of the subspace of upper triangular  $n \times n$  matrices.

**Example 3.12 (Quinlan).** [23] Let  $Skew_n$  denote the subspace of  $n \times n$  skew-symmetric matrices over a formally real field. Then  $I_n + Skew_n$  has dimension  $\frac{n(n-1)}{2}$  and all of its elements are nonsingular. Note that the subspace of skew-symmetric  $n \times n$  matrices is the trace complement of the subspace of symmetric  $n \times n$  matrices.

### 3.2 Affine spaces with ranks bounded below

The duality considered in Theorem 3.2 involves a connection between affine subspaces of  $M_n(\mathbb{F})$  comprising nonsingular elements, and affine subspaces of  $M_n(\mathbb{F})$  possessing the hyperplane annihilation property. In this section we extend Theorem 3.2 to the case of affine subspaces of rectangular matrices in which the ranks that appear are greater than or equal to some specified lower bound. In order to state the extended duality theorem we need to generalize the notion of the hyperplane annihilation property.

**Definition 3.13.** A subset  $B$  of  $M_{n \times m}(\mathbb{F})$  has the *dimension  $k$  annihilation property* on  $(\mathbb{F}^m)^T$  if every  $k$ -dimensional subspace of  $(\mathbb{F}^m)^T$  is contained in the right null-space of some element of  $B$ .

Any subset of  $M_{n \times m}(\mathbb{F})$  that contains the zero matrix has the dimension  $k$  annihilation property. In this section our interest is in subsets of  $M_{n \times m}(\mathbb{F})$  that do not contain the zero matrix and yet possess the dimension  $k$  annihilation property.

**Example 3.14.** Consider the subspace of  $(k+1) \times (k+1)$  upper triangular matrices. In Section 3.1 we showed that the elements of trace 1 of this subspace have the hyperplane annihilation property on  $(\mathbb{F}^{k+1})^T$ . Let  $B$  be the linear subspace of  $M_{n \times m}(\mathbb{F})$  consisting of all matrices that have a  $(k+1) \times (k+1)$  upper triangular matrix in the upper left  $(k+1) \times (k+1)$  region and are otherwise filled with zeros. Then the elements of  $B$  whose first  $k+1$  diagonal entries sum to 1 constitute an affine space having the dimension  $k$  annihilation property on  $(\mathbb{F}^m)^T$ .

Theorem 3.15 is our main duality theorem. Its proof is an adaptation of the proof of the special case Theorem 3.2, and it uses the following notation. For a matrix  $L \in M_{m \times n}(\mathbb{F})$  and a positive integer  $r$ , we let  $L^{\oplus r}$  denote the *Kronecker product* of  $I_r$  and  $L$ , i.e.  $L^{\oplus r} := I_r \otimes L$ . Note that  $L^{\oplus r}$  is an  $rm \times rn$  matrix that is defined by

$$\begin{aligned} L^{\oplus r}(tm+i, tn+j) &= L(i, j) \text{ for } 0 \leq t \leq r-1, 1 \leq i \leq m, 1 \leq j \leq n, \\ L^{\oplus r}(p, q) &= 0 \text{ otherwise.} \end{aligned}$$

Thus  $L^{\oplus r}$  has  $r$  appearances of  $L$  as blocks arranged diagonally from upper left to lower right, and zeros elsewhere.

**Theorem 3.15.** *Let  $A$  be a proper linear subspace of  $M_{m \times n}(\mathbb{F})$  and let  $X$  be an element of  $M_{m \times n}(\mathbb{F}) \setminus A$ . Let  $Y \in A^* \subset M_{n \times m}(\mathbb{F})$  such that  $\text{trace}(YX) = 1$ . Then*

every element of the affine subspace  $X+A$  has rank  $k$  if and only if the affine subspace  $Y + \langle X, A \rangle^*$  has the dimension  $k - 1$  annihilation property on  $(\mathbb{F}^m)^T$ .

*Proof.* Assume that every element of  $X + A$  has rank  $k$ . We need to show that  $Y + \langle X, A \rangle^*$  has the dimension  $k - 1$  annihilation property. Let  $V$  be an arbitrary subspace of  $\mathbb{F}^m$  of dimension  $m - k + 1$ . We need to show that there is a matrix in  $Y + \langle X, A \rangle^*$  whose row-space is contained in  $V$ . Let  $\{v_1, \dots, v_{m-k+1}\}$  be a basis for  $V$ . As every matrix in the affine space  $X + A$  has rank  $k$  it follows that for all  $Z \in A$  there is some  $i \in \{1, \dots, m - k + 1\}$  such that  $v_i(X + Z) \neq 0$ . Equivalently for all  $Z \in A$  there is some  $i \in \{1, \dots, m - k + 1\}$  such that  $v_i X \neq -v_i Z$ . Let  $v \in \mathbb{F}^{m(m-k+1)}$  be the concatenation of  $v_1, \dots, v_m$ . So for all  $Z \in A$

$$vX^{\oplus m-k+1} \neq -vZ^{\oplus m-k+1} \Rightarrow vX^{\oplus m-k+1} \notin \{vZ^{\oplus m-k+1} : Z \in A\}.$$

Hence

$$\{vZ^{\oplus m-k+1} : Z \in A\}^* \not\subseteq (vX^{\oplus m-k+1})^*.$$

So there exists  $u \in \mathbb{F}^{n(m-k+1)}$  such that

$$\text{trace}(u^T(vZ^{\oplus m-k+1})) = 0 \text{ for all } Z \in A \text{ and } \text{trace}(u^T(vX^{\oplus m-k+1})) = 1.$$

This means

$$\text{trace}((u^T v)Z^{\oplus m-k+1}) = 0 \text{ for all } Z \in A \text{ and } \text{trace}((u^T v)X^{\oplus m-k+1}) = 1.$$

Let  $u_i \in \mathbb{F}^n$  be the vector whose entries are in Positions  $n(i - 1) + 1$  through  $ni$  of  $u$ . Given the structure of  $Z^{\oplus m-k+1}$  we have that

$$\sum_{i=1}^{m-k+1} \text{trace}((u_i^T v_i)Z) = 0 \text{ for all } Z \in A \text{ and } \sum_{i=1}^{m-k+1} \text{trace}((u_i^T v_i)X) = 1.$$

Equivalently

$$\text{trace} \left( \left( \sum_{i=1}^{m-k+1} u_i^T v_i \right) Z \right) = 0 \text{ for all } Z \in A \text{ and } \text{trace} \left( \left( \sum_{i=1}^{m-k+1} u_i^T v_i \right) X \right) = 1.$$

So the matrix  $\sum_{i=1}^{m-k+1} u_i^T v_i$  is an element of  $Y + \langle X, A \rangle^*$  - i.e. it belongs to the trace complement of  $A$  and  $\text{trace} \left( \left( \sum_{i=1}^{m-k+1} u_i^T v_i \right) X \right) = 1$ . Its row-space is contained in  $V = \text{span}(v_1, \dots, v_{m-k+1})$ . So we have shown that  $Y + \langle X, A \rangle^*$  has the dimension  $k - 1$  annihilation property.

We now want to show that every element of  $X + A$  has rank  $k$  if  $Y + \langle X, A \rangle^*$  has the dimension  $k - 1$  annihilation property. Suppose (anticipating contradiction) that for some  $Z \in A$ ,  $X + Z$  has rank less than  $k$ . Let  $U^T \subset (\mathbb{F}^n)^T$  be the column-space of  $X + Z$  and let  $V^T \subset (\mathbb{F}^n)^T$  be a subspace of dimension  $k - 1$  such that  $U^T \subseteq V^T$ . The product of any matrix in  $Y + \langle X, A \rangle^*$  with any matrix in  $X + A$  has trace 1, hence the product of any matrix in  $Y + \langle X, A \rangle^*$  with the element  $X + Z$  has trace 1. However a contradiction arises as this would mean that  $Y + \langle X, A \rangle^*$  could not contain an element that annihilates  $V^T \subset (\mathbb{F}^n)^T$  (i.e. have row-space contained by  $(V^T)^*$ ). Hence every element of  $X + A$  must have rank at least  $k$ .  $\square$

As with the hyperplane annihilation property, the (general dimension) annihilation property is preserved by transposition for the sets of matrices of interest, as stated in Lemma 3.16. The proof is omitted as it is similar to that of Theorem 3.3.

**Lemma 3.16.** *Let  $B$  be a proper subspace of  $M_{n \times m}(\mathbb{F})$  and let  $Y$  be an element of  $M_{n \times m}(\mathbb{F}) \setminus B$ . Suppose that the affine subspace,  $Y + B$  of  $M_{n \times m}(\mathbb{F})$ , has the dimension  $k$  annihilation property on  $(\mathbb{F}^m)^T$ . Then the affine subspace,  $Y^T + B^T$  of  $M_{m \times n}(\mathbb{F})$ , also has the dimension  $k$  annihilation property on  $(\mathbb{F}^n)^T$ .*

Lemma 3.16 is used in Chapter IV, where we apply this result to the problem of classifying partial matrices whose completions all have ranks that are greater than or equal to some specified lower bound.

### 3.2.1 Dimension Bound

To bound the dimension of affine spaces of matrices (over any field) whose elements have ranks that are greater than or equal to some specified lower bound, we refer to a theorem of Meshulam [21]. He shows that if  $\mathbb{K}$  is an algebraically closed field or if  $\mathbb{K} = \mathbb{R}$ , then the maximum possible dimension of an affine subspace of  $M_{m \times n}(\mathbb{K})$  in which every element has rank at least  $k$ , for some fixed  $k \leq \min(m, n)$ , is  $mn - \frac{k(k+1)}{2}$ . The only property of algebraically closed fields and of  $\mathbb{R}$  that is used in Meshulam's proof is the fact that Corollary 3.10 holds for these fields. Since it has now been established that Corollary 3.10 holds for all fields, Meshulam's proof of the dimension bound for affine spaces in which ranks are bounded below extends unproblematically to all fields. This proof is included below for completeness.

**Theorem 3.17.** *Let  $A$  be a proper subspace of  $M_{m \times n}(\mathbb{F})$  and let  $X$  be an element of  $M_{m \times n}(\mathbb{F}) \setminus A$ . If every element of  $X + A$  has rank at least  $k \leq \min(m, n)$ , the maximum possible dimension of  $X + A$  is  $mn - \frac{k(k+1)}{2}$ .*

*Proof.* Suppose that the minimum rank of any matrix in  $X + A$  is  $l \geq k$  and suppose that  $Y \in X + A$  has rank  $l$ . Then there exist nonsingular matrices  $P \in \text{GL}_m(\mathbb{F})$  and  $Q \in \text{GL}_n(\mathbb{F})$  such that  $PYQ$  has the  $l \times l$  identity matrix in its upper left  $l \times l$  region and has zeros elsewhere. Let  $X' = PXQ$  and let  $A' = PAQ$ . The affine space  $X' + A'$  has the same dimension as  $X + A$  and every element of  $X' + A'$  has rank at least  $l$ . Let  $B$  be the linear subspace of  $A'$  comprising elements that only contain zeros outside the upper left  $l \times l$  region. Every element of  $X' + A'$  has rank at least  $l$ , consequently every element of  $X' + B$  has rank at least  $l$ . As elements of  $X' + B$  have no more than  $l$  nonzero rows or columns they have rank exactly  $l$ . By Corollary 3.10 the maximum possible dimension of  $B$  is  $\frac{l(l-1)}{2}$ .

Let  $C$  be the linear subspace of  $A'$  comprising elements whose upper left  $l \times l$  region is zero. The maximum possible dimension of  $C$  is  $mn - l^2$ . The space  $A' = B \oplus C$  and so the maximum possible dimension of  $A'$  is  $mn - l^2 + \frac{l(l-1)}{2}$ . Hence the maximum possible dimension of  $A'$  (and hence  $A$ ) is  $mn - \frac{l(l+1)}{2}$ . This expression attains its maximum when  $l$  is at a minimum, i.e.  $l = k$ . Hence the dimension of  $X + A$  cannot exceed  $mn - \frac{k(k+1)}{2}$ .

It remains to show that this dimension bound is attainable. Consider the linear subspace  $A_1 \subset M_{m \times n}(\mathbb{F})$  comprising matrices whose upper-left  $k \times k$  region is strictly upper triangular but is otherwise unconstrained. This is a linear subspace of dimension  $mn - \frac{k(k+1)}{2}$ . Let  $X_1 \in M_{m \times n}(\mathbb{F})$  be the matrix whose entries are all zeros except that  $X_1(1, 1) = \dots = X_1(k, k) = 1$ . Then the affine space  $X_1 + A_1$  has dimension  $mn - \frac{k(k+1)}{2}$  and all of its elements have rank at least  $k$ .  $\square$

The following extension of Theorem 3.7 is immediate from Theorems 3.15 and 3.17.

**Theorem 3.18.** *Let  $k \in \{1, \dots, n\}$ . Let  $B$  be a linear subspace of  $M_{n \times m}(\mathbb{F})$  that contains a non-linear affine subspace that has the dimension  $(k-1)$  annihilation property on  $(\mathbb{F}^m)^T$ . Then*

$$\dim(B) \geq \frac{k(k+1)}{2}.$$

Examples of subspaces of  $M_{n \times m}(\mathbb{F})$  realizing the conditions and dimension bound of Theorem 3.18 may be constructed as follows. Let  $B'$  be a linear subspace of  $M_k(\mathbb{F})$  of dimension  $\frac{k(k+1)}{2}$ , whose elements of trace 1 constitute an affine subspace having the hyperplane annihilation property on  $(\mathbb{F}^k)^T$ . For example  $B'$  could be the space of upper triangular matrices in  $M_k(\mathbb{F})$ . Let  $B$  be the linear subspace of  $M_{n \times m}(\mathbb{F})$

consisting of all matrices that have an element of  $B'$  in the upper left  $k \times k$  region and are otherwise filled with zeros. Then the elements of  $B$  whose first  $k$  diagonal entries sum to 1 constitute an affine space having the dimension  $k - 1$  annihilation property on  $(\mathbb{F}^m)^T$ . To see this, let  $U$  be a subspace of  $(\mathbb{F}^m)^T$  of dimension  $k - 1$ . The trace complement in  $\mathbb{F}^m$  of  $U$  has dimension  $m - k + 1$  and therefore intersects every  $k$ -dimensional subspace of  $\mathbb{F}^m$  non-trivially. Thus every  $U$  is annihilated by some nonzero element  $v$  of  $\mathbb{F}^m$  whose nonzero entries are all in the first  $k$  positions, and  $v$  spans the row-space of some element of  $A$  of trace 1. In Chapter IV we will discuss at length affine spaces of matrices whose elements all have ranks that are greater than or equal to some specified lower bound.

## CHAPTER IV

### Ranks bounded below

Our work in Chapter III reveals a dual property relationship that exists between affine spaces of matrices related by the trace bilinear form. This chapter documents the application of this duality to resolve Problem 1.1.

**Problem 1.1.** *Determine the maximum possible number of indeterminates that a partial matrix can possess if all of its completions have ranks that are greater than or equal to some specified lower bound, and characterize all such partial matrices that attain this number of indeterminates.*

In [5], Brualdi et al. address this problem for the particular case of  $n \times n$  partial matrices whose completions are all nonsingular – i.e. their ranks are bounded below by  $n$ . However their result, stated below, is subject to a restriction on the field order.

**Theorem 4.1.** *Let  $\mathbb{F}$  be a field with at least  $n + 1$  elements, and let  $A$  be an  $n \times n$  partial matrix all of whose completions are nonsingular. Then the number of indeterminates in  $A$  is at most equal to  $\frac{n(n-1)}{2}$ . This bound is attained if and only if there exist row and column permutations that transform  $A$  to an upper triangular partial matrix having nonzero constant entries on the main diagonal, and independent indeterminates above the main diagonal.*

Our approach to resolving Problem 1.1 exploits the duality discussed in Chapter III. This allows us to resolve the problem in its full generality without any constraint on the numbers of rows or columns and without any field dependencies. Any  $m \times n$  partial matrix,  $A$ , can be considered as the sum of a *constant component* ( $C$ ) and an *indeterminate component* ( $B$ ) – i.e.  $A = C + B$ . The *constant component*,  $C$ , is the matrix that results when all the indeterminates in the partial matrix are assigned the value zero. The *indeterminate component*,  $B$ , is what remains when we subtract

the constant component from the partial matrix – i.e.  $B = A - C$ . The completions of this indeterminate component,  $B$ , form a linear subspace of  $M_{m \times n}(\mathbb{F})$  – we call it  $X \subseteq M_{m \times n}(\mathbb{F})$ . Hence the completions of  $A = C + B$  form an affine subspace of  $M_{m \times n}(\mathbb{F})$ ,  $C + X$ . Thus a partial matrix whose completions all have ranks that are greater than or equal to a specified lower bound corresponds precisely to an affine space of matrices whose ranks are at least as great as that specified lower bound. This being the case we know from Theorem 3.15 that there is a related affine space possessing a specified annihilation property. In this way the duality provides an avenue to investigate partial matrices whose completions all have ranks that are greater than or equal to some specified lower bound. The material in this chapter is drawn from a paper [17] by Quinlan and the author that has been published in *Linear Algebra and its Applications*.

We prove the following theorem to resolve Problem 1.1.

**Theorem 4.2.** *Let  $\mathbb{F}$  be a field and let  $A$  be an  $m \times n$  partial matrix with the property that every completion of  $A$  has rank at least  $r$  for some fixed  $r \leq \min(m, n)$ . Then the number of indeterminates in  $A$  is at most  $mn - \frac{r(r+1)}{2}$ . This bound is attained if and only if  $A$  can be transformed by row and column permutations to a partial matrix  $A'$  of the following form:*

- *The upper left  $r \times r$  submatrix of  $A'$  is an upper triangular matrix with nonzero constant entries on the main diagonal and independent indeterminates above the main diagonal.*
- *All entries of  $A'$  outside the upper left  $r \times r$  region are independent indeterminates.*

It is clear that the completions of a partial matrix having the structure described in Theorem 4.2 all have ranks that are bounded below by  $r$ . The upper bound on the number of indeterminates of Theorem 4.2 is a consequence of Theorem 3.17. So it is the characterization of partial matrices that attain the maximum possible number of indeterminates that requires additional effort.

#### 4.1 Partial matrices of constant full rank

A key ingredient in the characterization is the case of an  $m \times n$  partial matrix whose completions all have constant full rank, i.e.  $r = \min(m, n)$ . We begin by considering this case. For the purpose of this characterization there is no loss in generality in

assuming that the number of rows is less than or equal to the number of columns, and so we work with this assumption throughout this section. If the number of columns exceeds the number of rows the same rationale can be applied to the transpose of the partial matrix.

**Theorem 4.3.** *Let  $A$  be an  $m \times n$  (with  $m \leq n$ ) partial matrix of constant full rank having  $mn - \frac{m(m+1)}{2}$  indeterminates. Then there exist permutation matrices  $P \in M_m(\mathbb{F})$  and  $Q \in M_n(\mathbb{F})$  for which the partial matrix  $A' = PAQ$  has the following form:*

- *The  $m \times m$  matrix consisting of the first  $m$  columns of  $A'$  is upper triangular, with nonzero constants on the main diagonal and independent indeterminates above the main diagonal.*
- *Columns  $m + 1, \dots, n$  of  $A'$  are fully occupied by independent indeterminates.*

The following elementary lemma is required for the proof of Theorem 4.3.

**Lemma 4.4.** *Let  $U$  be a subspace of  $\mathbb{F}^m$  of dimension  $r$ , where  $1 \leq r \leq m$ . Then  $U$  contains an element with at least  $r$  nonzero entries.*

*Proof.* The subspace  $U$  of  $\mathbb{F}^m$  may be represented as the left null-space of a matrix in  $M_{m \times (m-r)}(\mathbb{F})$  that has rank  $m - r$  and is in reduced row echelon form. Thus in order to specify an element of  $U$ , we have a free choice for the entries in  $r$  positions.  $\square$

For the sake of clarity and comprehensibility, we present the proof of Theorem 4.3 as a series of three propositions. We begin by writing  $C$  and  $B$  respectively for the constant and indeterminate parts of  $A$ , and by writing  $X$  for the linear subspace comprising the completions of  $B$ . Note that  $X$  has dimension  $mn - \frac{m(m+1)}{2}$ . Let  $Y \subseteq M_{n \times m}(\mathbb{F})$  denote the trace complement of  $X$ . It has dimension  $\frac{m(m+1)}{2}$ . It is the linear subspace comprising the completions of an  $n \times m$  partial matrix that we shall call  $E$ . Note that  $E$  has independent indeterminates precisely in the positions occupied by zeros in  $B^T$ , and has zeros in the positions occupied by indeterminates in  $B^T$ . Our first step is concerned with the distribution of the  $\frac{m(m+1)}{2}$  indeterminates among the rows of  $E$ .

**Proposition 4.5.** *There exists a permutation matrix  $P \in M_n(\mathbb{F})$  for which the  $n \times m$  partial matrix  $PE$  has  $m - i + 1$  indeterminates in Row  $i$  for  $i = 1, \dots, m$  and has only zeros outside the first  $m$  rows.*

*Proof.* The affine subspace  $C + X$  has the property that all of its elements have rank  $m$ . Let  $E_1 \in Y$  (i.e. a completion of the partial matrix  $E$ ) such that  $\text{trace}(E_1 C) = 1$ . It follows from Theorem 3.15 that the affine subspace  $E_1 + \langle C, X \rangle^*$  of  $Y \subseteq M_{n \times m}(\mathbb{F})$  has the hyperplane annihilation property (i.e. the dimension  $m - 1$  annihilation property) on  $(\mathbb{F}^m)^T$ . This affine subspace comprises the completions of  $E$  that result in a product of trace 1 when multiplied by  $C$ . This means that every subspace of  $(\mathbb{F}^m)^T$  of dimension  $m - 1$  is annihilated by some element of  $E_1 + \langle C, X \rangle^*$ , or equivalently that every nonzero element of  $\mathbb{F}^m$  occurs as a basis of the row-space of an element of  $E_1 + \langle C, X \rangle^*$  of rank 1. In fact, in view of the special structure of the subspace  $Y$  of  $M_{n \times m}(\mathbb{F})$ , we may interpret the statement that  $E_1 + \langle C, X \rangle^*$  has the hyperplane annihilation property on  $\mathbb{F}^m$  to mean that every nonzero element of  $\mathbb{F}^m$  occurs as the unique nonzero row of some element of  $E_1 + \langle C, X \rangle^*$ .

In particular every element of  $\mathbb{F}^m$  that has no nonzero entries must occur as a row in some completion of  $E$ , and so there is some  $j_1 \in \{1, \dots, n\}$  for which Row  $j_1$  of  $E$  contains  $m$  indeterminates. There exists a hyperplane  $H_1$  of  $\mathbb{F}^m$  consisting of elements that are in the trace complement of Column  $j_1$  of  $C$ . If  $v$  is a nonzero element of  $H_1$ , then  $v$  must span the row-space of some completion of  $E$  that has nonzero entries outside Row  $j_1$ . Since  $H_1$  has elements with at least  $m - 1$  nonzero entries by Lemma 4.4, there is some  $j_2 \neq j_1$  for which Row  $j_2$  of  $E$  has at least  $m - 1$  indeterminates. Now the trace complements of Columns  $j_1$  and  $j_2$  of  $C$  intersect in a subspace  $H_2$  of  $\mathbb{F}^m$  of dimension at least  $m - 2$ . By Lemma 4.4,  $H_2$  contains an element with at least  $m - 2$  nonzero entries, hence there exists some  $j_3 \notin \{j_1, j_2\}$  for which Row  $j_3$  of  $E$  contains at least  $m - 2$  indeterminates.

This iterative argument concludes with a list  $j_1, j_2, \dots, j_m$  of indices of distinct rows of  $E$ , with the property that for  $i = 1, \dots, m$ , Row  $j_i$  of  $E$  contains at least  $m - i + 1$  indeterminates. Since the total number of indeterminates in  $E$  is  $\frac{m(m+1)}{2}$ , it must be that Row  $j_i$  of  $E$  contains exactly  $m - i + 1$  indeterminates for  $i = 1, \dots, m$ , and that all entries of  $E$  outside these  $m$  rows are zeros.

We reach the desired conclusion by permuting the rows of  $E$  so that the first  $m$  rows are the nonzero rows, and they are arranged in decreasing order of number of indeterminates.  $\square$

Note that in the situation of Proposition 4.5,  $PY$  is the trace complement of  $XP^{-1}$ , and that  $XP^{-1}$  is the linear space of completions of  $BP^{-1}$ . This partial matrix has independent indeterminates in the positions where  $(PE)^T$  has zeros, and has zeros elsewhere. Thus the number of indeterminates in Column  $j$  of  $BP^{-1}$ , or

equivalently of  $AP^{-1}$ , is  $j - 1$  if  $j \leq m$ , and is  $m$  if  $n > m$  and  $j > m$ . We have shown that the partial matrix  $A$  may be transformed by a column permutation to one with this pattern of indeterminate positions. Now  $CP^{-1} + XP^{-1}$  is an affine subspace of  $M_{m \times n}(\mathbb{F})$  in which every element has rank  $m$ , so  $PE_1 + \langle CP^{-1}, XP^{-1} \rangle^*$  is an affine subspace of  $M_{n \times m}(\mathbb{F})$  that has the hyperplane annihilation property on  $(\mathbb{F}^m)^T$ . Note that all nonzero entries of  $CP^{-1}$  occur in the first  $m$  columns.

The next step in our proof is the application of Lemma 3.16 to the concluding situation of Proposition 4.5, to describe the distribution of indeterminates amongst the columns of  $PE$ .

**Proposition 4.6.** *There exists a permutation matrix  $Q \in M_m(\mathbb{F})$  for which the entry in the  $(i, j)$  position of  $PEQ$  is an indeterminate if  $j \geq i$  and is zero otherwise.*

*Proof.* Since  $PE_1 + \langle CP^{-1}, XP^{-1} \rangle^*$  has the dimension  $(m - 1)$  annihilation property on  $(\mathbb{F}^m)^T$ , it follows from Lemma 3.16 that the affine subspace  $PE_1 + \langle CP^{-1}, XP^{-1} \rangle^*$  of  $M_{n \times m}(\mathbb{F})$  has the dimension  $(m - 1)$  annihilation property on  $\mathbb{F}^n$ . This means that every subspace of  $\mathbb{F}^n$  of dimension  $m - 1$  is annihilated by some completion of  $PE$  that results in a product of trace 1 when multiplied by  $CP^{-1}$ .

All entries of  $PE$  outside the first  $m$  rows are zeros. Let  $\mathbb{F}_m^n$  denote the  $m$ -dimensional subspace of  $\mathbb{F}^n$  consisting of all elements having zeros outside the first  $m$  positions. The dimension  $(m - 1)$  annihilation property on  $\mathbb{F}^n$  implies the same property on  $\mathbb{F}_m^n$ , and it follows that every element of  $\mathbb{F}_m^n$  must span the column-space of a completion of  $PE$  of rank 1 that results in a product of trace 1 when multiplied by  $CP^{-1}$ . Thus we are essentially back in the situation of Proposition 4.5, and the argument employed there can be used to show that the indeterminates of  $PE$  are confined to  $m$  columns, of which one contains  $m$  indeterminates, another contains  $m - 1$  indeterminates, and so on.

Thus the numbers of indeterminates in the  $m$  columns of  $PE$  are  $1, 2, \dots, m$ , in some order. Then there exists a permutation matrix  $Q \in M_m(\mathbb{F})$  for which the columns of  $PEQ$  are arranged in increasing order of number of indeterminates. Since Row  $i$  of  $PE$ , hence also of  $PEQ$ , has exactly  $m - i + 1$  indeterminates for  $i = 1, \dots, m$ , it follows that  $PEQ$  has the required form.  $\square$

Now  $PYQ$  is the trace complement of  $Q^{-1}XP^{-1}$  – this is the set of completions of the indeterminate part of the partial matrix  $A'$  obtained by applying the permutations represented by  $Q^{-1}$  and  $P^{-1}$  respectively to the rows and columns of  $A$ . Let  $C'$  denote the constant part of  $A'$ . Since the  $mn - \frac{m(m + 1)}{2}$  indeterminates of  $A'$

occupy all the positions  $(i, j)$  with  $i < j$ , the nonzero entries of  $C'$  must all be located in those positions  $(i, j)$  with  $i \geq j$ . The final step in our proof of Theorem 4.3 is to identify the positions of the nonzero entries of  $C'$ .

**Proposition 4.7.** *The constant part  $C'$  of  $A'$  has nonzero entries precisely in the positions  $(i, i)$ , for  $i = 1, \dots, m$ .*

*Proof.* As observed above,  $C'$  can have nonzero entries only in positions  $(i, j)$  with  $i \geq j$  – this is apparent from the locations of the indeterminate entries of  $A'$ . We note also that  $C'$  has at least one nonzero entry in each row and in each of the first  $m$  columns, since it has rank  $m$ . Now write  $E'$  for the matrix  $PEQ$ , which has  $\frac{m(m+1)}{2}$  independent indeterminates in the positions  $(i, j)$  with  $i \leq j$ , and zeros elsewhere, and write  $Y'$  for the space of completions of  $E'$ . Every nonzero element of  $\mathbb{F}^m$  arises as a basis for the row-space of some completion of  $E'$ . If  $v_1 \in \mathbb{F}^m$  has a nonzero entry in its first position, then a completion of  $E'$  can have  $\langle v_1 \rangle$  as its row-space only if its nonzero entries are all in Row 1. It follows that an element of  $\mathbb{F}^m$  whose first entry is not zero cannot be in the trace complement of the first column of  $C'$ . Then the first column of  $C'$  must have a unique nonzero entry, in its first position.

Thus the trace complement of Column 1 of  $C'$  is the hyperplane of  $\mathbb{F}^m$  consisting of all elements whose first entry is zero. Let  $v_2$  be an element of  $\mathbb{F}^m$  whose first nonzero entry is in the second position. Then  $v_2$  can occur only as the first or second row of a completion of  $E'$ , and it must span the row-space of some completion  $E'_2$  of  $E'$  for which  $\text{trace}(E'_2 C') = 1$ . As the first entry of  $v_2$  is nonzero,  $v_2$  lies in the trace complement of Column 1 of  $C'$ . Thus  $v_2$  cannot be in the trace complement of Column 2 of  $C'$ . It follows that the entry in Position  $(2, 2)$  of  $C'$  is not zero, and that all subsequent entries of Column 2 of  $C'$  are zeros.

Now the trace complements of Columns 1 and 2 of  $C'$  in  $\mathbb{F}^m$  intersect in the subspace of  $\mathbb{F}^m$  consisting of all elements with zeros in the first two positions; the argument above can be applied successively to Columns 3 through  $m$  of  $C'$ , establishing Positions  $(1, 1), \dots, (m, m)$  as the only locations of nonzero entries of  $C'$ .  $\square$

This completes our proof of Theorem 4.3: the partial matrix  $A' = Q^{-1}AP^{-1}$  was obtained from  $A$  by row and column permutations and has the required form.

## 4.2 Partial matrices whose completions have ranks that are bounded below

We now turn our attention to the more general Theorem 4.2, our characterization of rectangular partial matrices with the maximum possible number of indeterminates subject to a specific lower bound for the ranks of their completions. The strategy of our proof of Theorem 4.2 is to reduce the problem to the situation of Theorem 4.3, again using the duality discussed in Chapter III. The principal tool needed to effect this reduction is the following lemma.

**Lemma 4.8.** *Let  $k \leq \min(m, n)$ . Let  $C + Y_1$  be a non-linear affine subspace of  $M_{n \times m}(\mathbb{F})$  of dimension  $\frac{k(k+1)}{2} - 1$  that has the dimension  $k-1$  annihilation property on  $(\mathbb{F}^m)^T$ . Let  $Y$  denote the linear subspace of  $M_{n \times m}(\mathbb{F})$  spanned by  $Y_1$  and  $C$ . Let  $v \in (\mathbb{F}^m)^T$  be a nonzero vector. Then the subspace  $Yv = \{Bv : B \in Y\}$  of  $(\mathbb{F}^n)^T$  can have dimension at most  $k$ .*

*Proof.* If  $k = 1$  then  $Y$  is a one-dimensional subspace of  $M_{n \times m}(\mathbb{F})$  and the theorem is clearly true. So assume that  $k \geq 2$ .

We know that there exists a linear subspace  $Y_1 \subset Y$  and  $C \in Y \setminus Y_1$  such that  $C + Y_1$  has the dimension  $k-1$  annihilation property. Let  $Y_v$  denote the subspace of  $Y$  consisting of all those elements whose right null-space contains  $v$ . By the rank-nullity theorem a complement of  $Y_v$  in  $Y$  must have dimension equal to that of  $Yv$ , we have

$$\dim(Y) = \frac{k(k+1)}{2} = \dim(Y_v) + \dim(Yv).$$

Let  $H$  be a complement of  $\langle v \rangle$  in  $(\mathbb{F}^m)^T$ , so  $(\mathbb{F}^m)^T = \langle v \rangle \oplus H$ . Every  $(k-1)$ -dimensional subspace of  $(\mathbb{F}^m)^T$  that contains  $v$  intersects  $H$  in a subspace of dimension  $k-2$ , and every subspace of  $H$  of dimension  $k-2$  is contained in a unique  $(k-1)$ -dimensional subspace of  $(\mathbb{F}^m)^T$  containing  $v$ . Since  $C + Y_1$  has the dimension  $(k-1)$  annihilation property on  $(\mathbb{F}^m)^T$ , every  $(k-1)$ -dimensional subspace of  $(\mathbb{F}^m)^T$  that contains  $v$ , hence every  $(k-2)$ -dimensional subspace of  $H$ , is annihilated by an element of  $Y_v$  that also belongs to  $C + Y_1$ . Restricting to  $H$ , we can identify  $Y_v$  with a subspace  $Y^H$  of linear transformations from  $H$  to  $(\mathbb{F}^n)^T$ , and  $(C + Y_1) \cap Y_v$  with an affine subspace  $C' + Y_1^H$  of  $Y^H$ , that possesses the dimension  $(k-2)$  annihilation property on  $H$ . Then  $Y_v$  must have dimension at least  $\frac{(k-1)k}{2}$  by Theorem 3.18, and so

$$\dim(Yv) = \frac{k(k+1)}{2} - \dim(Y_v) \leq k,$$

as required. □

**Lemma 4.9.** *Let  $k \leq \min(m, n)$ . Let  $E$  be an  $n \times m$  partial matrix with all constant entries being zeros containing  $\frac{k(k+1)}{2}$  indeterminates such that the linear subspace comprising its completions contains a non-linear affine subspace possessing the dimension  $k - 1$  annihilation property on  $(\mathbb{F}^m)^T$ . Then the indeterminates of  $E$  are confined to at most  $k$  rows and to at most  $k$  columns.*

*Proof.* Let  $Y \subseteq M_{n \times m}(\mathbb{F})$  be the linear subspace comprising the completions of  $E$ . Suppose (anticipating contradiction) that there are  $k + 1$  rows of  $E$  that contain indeterminates. For convenience we can assume that they are the first  $k + 1$  rows. For any element  $(a_1, a_2, \dots, a_{k+1})$  of  $\mathbb{F}^{k+1}$ , let  $E_{(a_1, a_2, \dots, a_{k+1})}$  be the completion of  $E$  in which the first indeterminate in Row  $i$  is assigned the value  $a_i$  for  $i = 1, \dots, k + 1$ , and all other indeterminates are assigned the value 0. Let  $v$  be the element of  $\mathbb{F}^m$  whose entries are all equal to 1. Then

$$E_{(a_1, a_2, \dots, a_{k+1})} v^T = (a_1, \dots, a_{k+1}, 0, \dots, 0)^T \in (\mathbb{F}^n)^T,$$

and so  $Yv$  has dimension at least  $k + 1$ , contrary to Lemma 4.8.

That the indeterminates of  $E$  are also confined to  $k$  columns is now an immediate consequence of Lemma 3.16, since  $Y^T$  contains a non-linear affine subspace that possesses the dimension  $(k - 1)$  annihilation property on  $(\mathbb{F}^n)^T$ , and has minimum possible dimension  $\frac{k(k+1)}{2}$  subject to this condition.  $\square$

We are now in a position to prove Theorem 4.2. As in the proof of Theorem 4.3, we write  $C$  and  $B$  for the constant and indeterminate parts of our  $m \times n$  partial matrix  $A$ ,  $X$  for the space of completions of  $B$ ,  $Y$  for the trace complement of  $X$  in  $M_{n \times m}(\mathbb{F})$ , and  $E$  for the  $n \times m$  partial matrix of which  $Y$  is the space of completions.

**Proposition 4.10.** *Let  $r$  be a positive integer,  $r \leq \min(m, n)$ . Let  $A$  be an  $m \times n$  partial matrix with  $mn - \frac{r(r+1)}{2}$  indeterminates, all of whose completions have rank at least  $r$ . Then  $A$  can be transformed by row and column permutations to a partial matrix in which all constant entries occur in the first  $r$  rows and in the first  $r$  columns.*

*Proof.* Let  $E_1 \in Y$  such that  $\text{trace}(E_1 C) = 1$ . It follows from Theorem 3.15 that the affine subspace  $E_1 + \langle C, X \rangle^*$  of  $M_{n \times m}(\mathbb{F})$  has the dimension  $r - 1$  annihilation property on  $(\mathbb{F}^m)^T$ . By Lemma 4.9 the  $\frac{r(r+1)}{2}$  indeterminates of  $E$  collectively occupy at most  $r$  rows and at most  $r$  columns. Thus there exist permutation matrices  $P \in M_n(\mathbb{F})$  and  $Q \in M_m(\mathbb{F})$  for which  $PEQ$  is fully occupied by zeros outside its

upper left  $r \times r$  region, and for which  $Q^{-1}BP^{-1}$  and hence  $Q^{-1}AP^{-1}$  are fully occupied by indeterminates outside their upper left  $r \times r$  regions.  $\square$

The concluding step in the proof of Theorem 4.2 is a direct application of Theorem 4.3. The matrix  $A' = Q^{-1}AP^{-1}$  is fully occupied by indeterminates outside its upper left  $r \times r$  region, which contains  $\frac{r(r-1)}{2}$  indeterminates and  $\frac{r(r+1)}{2}$  constants. Let  $A'_r$  denote the  $r \times r$  partial matrix whose entries are those of the upper left  $r \times r$  region of  $A'$ . Every completion of  $A'$  in which all entries outside this region are zeros has rank at least  $r$ , hence exactly  $r$ . Thus  $A'_r$  is an  $r \times r$  partial matrix with  $\frac{r(r-1)}{2}$  indeterminates, whose completions all have rank  $r$ . By Theorem 4.3,  $A'_r$  can be transformed by row and column permutations to an upper triangular matrix in which the entries on the main diagonal are nonzero constants and the entries above the main diagonal are indeterminates. Since these permutations can be extended to permutations of the rows and columns of  $A'$  that affect only the first  $r$  rows and first  $r$  columns, our proof of Theorem 4.2 is complete.

## CHAPTER V

### Constant rank

The remaining chapters of this thesis investigate partial matrices of constant rank – i.e. partial matrices whose completions all have the same rank. We begin in this chapter by resolving the following problem:

**Problem 1.2.** *Determine the maximum possible number of indeterminates that a partial matrix of constant rank  $r$  can possess, and characterize all partial matrices of constant rank  $r$  that attain this number of indeterminates.*

This problem was addressed previously by Huang and Zhan in [16] for fields of sufficiently large order. In [16], Huang and Zhan investigate *affine column independent (ACI)* matrices that are defined over fields and have constant rank. This characterization is used to determine the maximum possible number of indeterminates in an  $m \times n$  partial matrix whose completions all have the same rank  $r$ , and to describe those examples in which this maximum is attained, over a field of at least  $\max(m, n + 1)$  elements. In this chapter we show that for these latter results of [16], the restriction on the field order is not necessary. The material in this chapter is drawn from [18] – a paper written by Quinlan and the author that has been published in *Linear Algebra and its Applications*.

#### 5.1 The maximum possible number of indeterminates

The following theorem provides an upper bound on the number of indeterminates that a partial matrix of constant rank can possess:

**Theorem 5.1.** *Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$ , with  $m \leq n$ , whose completions all have the same rank  $r$ . Then the number of indeterminates in  $A$  is at most*

$$rn - \frac{r(r+1)}{2}.$$

Corollary 5.2 is an immediate consequence of Theorem 5.1. For a partial matrix of constant rank and specified dimension, it determines the rank that admits the maximum possible number of indeterminates in that partial matrix.

**Corollary 5.2.** *The number of indeterminates in an  $m \times n$  partial matrix (with  $m \leq n$ ) of constant rank is at most  $mn - \frac{m(m+1)}{2}$ , and this number is attained only if the constant rank is  $m$  or  $n - 1$  (i.e.  $m = n$  and  $r = m - 1$ ).*

Our proof of Theorem 5.1 is presented as a series of propositions which identify the maximum possible number of indeterminates that exist in distinct regions of  $A$ . Throughout these proofs, the matrices are often row and column permuted to facilitate the ready identification of *distributed* submatrices (i.e. the intersection of non-consecutive rows and columns). We note that these permutations do not affect the properties of interest, namely the rank or the number of indeterminates of the partial matrices under discussion.

A fundamental and crucial fact upon which the proofs of these propositions rest is that the indeterminates of a partial matrix, whose completions all have the same rank  $r$ , must be covered by at most  $r$  lines of that partial matrix. Lemma 5.3 establishes this fact.

**Lemma 5.3.** *Let  $A$  be an  $m \times n$  partial matrix, all of whose completions have rank at most  $r$ . There is a set of at most  $r$  lines covering all the indeterminates of  $A$ .*

*Proof.* If  $r = \min(m, n)$ , then the lemma is trivially true. So it is the case of  $r < \min(m, n)$  that merits further attention. Assume, anticipating contradiction, that the minimum number of lines to contain all of the indeterminates of  $A$  is greater than  $r$ . König's Theorem [6] states that in a (0–1) matrix, the minimum number of lines to cover all of the 1's is equal to the maximum number of 1's in independent positions. Applying this result to the indeterminates of  $A$ , means that there are at least  $r + 1$  indeterminates in independent positions in  $A$ .

Let  $S$  be a set of  $r + 1$  independent positions of  $A$  that contain indeterminates. A position in  $A$  is identified by the intersection of a row and column. The positions in  $S$  are all independent so it requires  $r + 1$  rows and  $r + 1$  columns to identify the  $r + 1$  positions of  $S$ . Let  $A_S$  be the partial matrix defined by the intersection of the  $r + 1$  rows and the  $r + 1$  columns identifying the  $r + 1$  independent positions of  $S$ . We will now show that there is a completion of  $A_S$  of rank  $r + 1$ . As  $A_S$  has  $r + 1$  independent positions containing indeterminates, it can be row permuted so that all the diagonal entries are indeterminates. We apply such a row permutation,

and then let  $B$  be the partial matrix that results from assigning the value zero to all indeterminates in off-diagonal positions. Let  $B(i, i) = x_i$ . Assign  $x_1 = 1$  and subtract the appropriate multiples of Row 1 from Rows  $2, \dots, m$  to give zeros in Positions  $2, \dots, m$  of Column 1. After this the second diagonal entry is  $x_2 + c$  for some  $c \in \mathbb{F}$ . Assign the value  $1 - c$  to  $x_2$  and subtract the appropriate multiples of Row 2 from Rows  $3, \dots, m$  to give zeros in Positions  $3, \dots, m$  of Column 2. Iterating this process for successive columns eventually yields a unimodular upper triangular matrix. This matrix is a completion of  $A_S$  that has undergone a series of elementary row operations. Elementary row operations preserve the rank of a matrix, so there is clearly a completion of  $A_S$  of rank  $r + 1$ . Hence there is a completion of  $A$  that has rank at least  $r + 1$ . This contradicts the hypothesis that all the completions of  $A$  have rank not exceeding  $r$ . Hence the minimum number of lines to contain all of the indeterminates of  $A$  is at most  $r$ .  $\square$

The following corollary which is immediate from the proof Lemma 5.3 is also employed in the proof of Theorem 5.1.

**Corollary 5.4.** *Let  $A$  be an  $m \times n$  partial matrix having  $r$  indeterminates in independent positions. Then  $A$  has a completion of rank at least  $r$ .*

An  $m \times n$  partial matrix whose completions all have full rank may equally well be described as a partial matrix whose completions all have rank bounded below by  $\min(m, n)$ . Hence the case  $r = m$  of Theorem 5.1 corresponds with the case  $r = \min(m, n)$  of Theorem 4.2, which has already been proven in Chapter IV.

The full rank case of Theorem 4.2 plays a crucial role in our proof of the more general case  $r < m$  for which we employ induction on the constant rank. We consider the case  $r = 0$  as a base case for Theorem 5.1. The partial matrix in this case is the  $m \times n$  zero matrix which certainly satisfies Theorem 5.1. We now assume that Theorem 5.1 holds when the constant rank  $r < q$ . Note that as a consequence we can assume that Corollary 5.2 holds when there are fewer than  $q$  rows. Let  $A$  be an  $m \times n$  partial matrix (with  $m \leq n$ ) of constant rank  $q < \min(m, n)$ . We will prove that the maximum possible number of indeterminates in  $A$  is  $qn - \frac{q(q+1)}{2}$ .

By Lemma 5.3 we know that the minimum number of lines to contain all the indeterminates of  $A$  is at most  $q$ . Hence the indeterminates of  $A$  are contained by some  $e$  rows and  $f$  columns, with  $e + f \leq q$ , and no collection of fewer than  $e + f$  lines contains all the indeterminates of  $A$ . We can arrange that these are the first  $e$  rows and the last  $f$  columns by appropriately permuting the rows and columns of  $A$ .

**Proposition 5.5.** *Let  $m \leq n$  and let  $A$  be an  $m \times n$  partial matrix such that every completion of  $A$  has rank  $q$ . Let all the indeterminates of  $A$  reside in the first  $e$  rows and the last  $f$  columns, with  $e + f \leq q$ , and with no fewer than  $e + f$  lines of  $A$  possessing all of the indeterminates. Then the number of indeterminates in the upper left  $e \times (n - f)$  region of  $A$  cannot exceed*

$$e(n - f) - \frac{e(e + 1)}{2}.$$

*Proof.* Let  $Y_1$  denote the  $(m - e) \times f$  submatrix in the lower right region of  $A$ . Then the lower  $m - e$  rows of  $A$  are of the form:

$$\left( \blacksquare \ Y_1 \right), \text{ where } Y_1 \text{ is an } (m - e) \times f \text{ partial matrix.}$$

It must be that the  $f$  columns of  $Y_1$  contain indeterminates in  $f$  independent positions. If this were not the case, then by König's Theorem it would be possible to cover all the indeterminates of  $Y_1$  with fewer than  $f$  lines, and so it would be possible to cover all the indeterminates of  $A$  with fewer than  $e + f$  lines.

After row permutation (if required) we may assume that the first  $f$  rows of  $Y_1$  contain  $f$  indeterminates in independent positions. Hence by Corollary 5.4 there is a completion of  $Y_1$ , say  $C_1 \in M_{(m-e) \times f}(\mathbb{F})$ , such that the first  $f$  rows are linearly independent. Consider the partial completion of  $A$  that results from the completion of  $Y_1$  to  $C_1$ . The submatrix formed by the lower  $m - e$  rows of  $A$  has rank  $f + g$ , where  $0 \leq g \leq q - f$ . After suitably permuting Rows  $e + f + 1, \dots, m$  and Columns  $1, \dots, n - f$  of  $A$ , we can assume that the first  $f + g$  rows of this submatrix are linearly independent and the last  $f + g$  columns are linearly independent. Consider the partial completion of this partial matrix that results from this completion of the lower  $m - e$  rows. Using elementary row operations that do not involve the upper  $e$  rows, the lower  $m - e$  rows of our partial completion can be converted into a form described by the following partial matrix – note that any zero rows in the lower  $m - e$  rows have been omitted:

$$\begin{pmatrix} Y_2 & \star & \star \\ \blacksquare & 0 & I_f \\ \blacksquare & I_g & 0 \end{pmatrix}.$$

The first  $e$  rows of this partial matrix are the same as those of  $A$  except for some permutation of Columns  $1, \dots, n - f$ . Note that  $Y_2$  is an  $e \times (n - f - g)$  partial matrix. Assigning a value of zero to all the indeterminates in the last  $f + g$  columns

results in the following partial matrix (again, any zero rows at the bottom have been omitted):

$$\begin{pmatrix} Y_2 & \blacksquare & \blacksquare \\ \blacksquare & 0 & I_f \\ \blacksquare & I_g & 0 \end{pmatrix}.$$

Any nonzero constants that appear in the intersection of Rows  $e + 1, \dots, m$  and Columns  $1, \dots, n - f - g$  can be cleared by elementary column operations that only affect Columns  $1, \dots, n - f - g$ . This results in the following partial matrix

$$B_1 = \begin{pmatrix} Y'_2 & \blacksquare & \blacksquare \\ 0 & 0 & I_f \\ 0 & I_g & 0 \end{pmatrix}.$$

Each entry of  $Y'_2$  is either constant or has the form  $x + a$  for an indeterminate  $x$  and a constant  $a \in \mathbb{F}$ . Entries of the latter form can still be regarded as indeterminates, and since  $Y'_2$  differs from  $Y_2$  by a constant matrix, indeterminates in different positions of  $Y'_2$  are independent. Hence  $Y'_2$  is a partial matrix with the same number (and pattern) of indeterminate positions as  $Y_2$ .

As elementary row and column operations preserve rank, the completions of  $B_1$  have the same rank as the completions of  $A$ , namely  $q$ . As the last  $f + g$  columns of  $B_1$  are linearly independent, every completion of  $Y'_2$  must have rank  $q - f - g$ . We now deal separately with two cases that are dependent on the value of  $f + g$ .

Case 1:  $f + g = 0$

In this case (i.e. the lower  $m - e$  rows of  $A$  are all zeros),  $A = B_1$  and is of the form

$$A = \begin{pmatrix} Y_2 \\ 0 \end{pmatrix}.$$

Every completion of  $A$  has rank  $q$  and  $e \leq q$ , so it must be that  $e = q$  and that every completion of  $Y_2$  has rank  $q$ . It is clear that Theorem 4.2 can be applied directly to  $Y_2$ , and that the maximum possible number of indeterminates that  $Y_2$  can possess is

$$en - \frac{e(e+1)}{2} = qn - \frac{q(q+1)}{2}.$$

Note that this proves the special case of Theorem 5.1 where all the nonzero entries of  $A$  belong to  $r$  rows.

Case 2:  $f + g > 0$

In this case  $q - f - g < q$ . Also  $e + f + g \leq m \leq n \Rightarrow e \leq n - f - g$ , so the  $e \times (n - f - g)$  partial matrix,  $Y'_2$ , has at least as many columns as rows. So by the inductive hypothesis Corollary 5.2 applies to  $Y'_2$ , giving that the maximum possible number of indeterminates that  $Y'_2$  (and hence  $Y_2$ ) can possess is

$$e(n - f - g) - \frac{e(e + 1)}{2}.$$

Note that this number of indeterminates can only be attained if

- $e = q - f - g$ , or
- $e - 1 = q - f - g$  and  $e = n - f - g$  (which demands that  $m = n$  and  $q = m - 1$ ).

Note that this latter possibility gives rise to a special case when we characterize partial matrices that attain the maximum number of indeterminates such that all their completions have fixed rank – specifically  $n \times n$  partial matrices whose completions all have rank  $n - 1$ . This special case is addressed by Proposition 5.10 in Section 5.2.

In all cases the number of indeterminates in the upper left  $e \times (n - f - g)$  region of  $B_1$  (and hence of  $A$ ) cannot exceed  $e(n - f - g) - \frac{e(e + 1)}{2}$ . Allowing that the next  $g$  columns to the right of this in the upper  $e$  rows could be fully populated with indeterminates gives that the number of indeterminates in the upper left  $e \times (n - f)$  region of  $A$  cannot exceed

$$e(n - f) - \frac{e(e + 1)}{2}.$$

□

**Proposition 5.6.** *Let  $m \leq n$  and let  $A$  be an  $m \times n$  partial matrix such that every completion of  $A$  has rank  $q$  and such that  $A$  has the maximum possible number of indeterminates subject to this property. Let all the indeterminates of  $A$  reside in the first  $e$  rows and the last  $f$  columns, with  $e + f \leq q$ , and with no fewer than  $e + f$  lines of  $A$  possessing all of the indeterminates. Then the number of indeterminates in the lower right  $(m - e) \times f$  region of  $A$  cannot exceed*

$$f(m - e) - \frac{f(f + 1)}{2}.$$

*Proof.* A similar argument (to the one used to bound the number of indeterminates in the upper left  $e \times (n - f)$  region of  $A$ ) can be applied to bound the maximum possible number of indeterminates in the lower right  $(m - e) \times f$  region of  $A$ . This results in a

partial matrix  $B_2$ , that is a partial completion of  $A$  that has had elementary row and column operations applied to it. The form of  $B_2$  is as follows, where  $0 \leq h \leq q - e$  and any zero columns in the leftmost  $n - f$  columns have been omitted:

$$B_2 = \begin{pmatrix} I_e & 0 & \blacksquare \\ 0 & I_h & \blacksquare \\ 0 & 0 & Y_3 \end{pmatrix}, \text{ where } Y_3 \text{ is } (m - e - h) \times f \text{ partial matrix.}$$

Note that  $Y_3$  contains the same number of indeterminates as the lower right  $(m - e - h) \times f$  region of  $A$ . It is not immediately obvious how the number of rows of  $Y_3$  (i.e.  $m - e - h$ ) relates to the number of its columns ( $f$ ). Some additional effort is required to determine this relationship as it affects the maximum possible number of indeterminates that the region can contain.

As elementary row and column operations preserve rank, the completions of  $B_2$  have the same rank as the completions of  $A$ , namely  $q$ . As the first  $e + h$  rows of  $B_2$  are linearly independent, every completion of  $Y_3$  must have rank  $q - e - h$ . Once again two cases arise – on this occasion being dependent on the value of  $e + h$ .

Case 1:  $e + h = 0$

In this case the leftmost  $n - f$  columns of  $A$  contain only zeros and so  $A = B_2$  and is of the form

$$A = \begin{pmatrix} 0 & Y_3 \end{pmatrix}.$$

Every completion of  $A$  has rank  $q$  and  $f \leq q$ , so it must be that  $f = q$  and that every completion of  $Y_3$  has rank  $f$ . As  $f < m$  it is clear that Theorem 4.2 can be applied directly to the transpose of  $Y_3$ , and that the number of indeterminates that  $Y_3$  can possess cannot exceed

$$fm - \frac{f(f+1)}{2} = qm - \frac{q(q+1)}{2}.$$

Case 2:  $e + h > 0$

In this case  $q - e - h < q$  and so by the inductive hypothesis Corollary 5.2 applies to the  $(m - e - h) \times f$  partial matrix  $Y_3$  or its transpose (if  $f \leq m - e - h$ ). We claim that  $f \leq m - e - h$ . We know that  $e + f \leq q < m$ , so  $e + f + 1 \leq m \Rightarrow f \leq m - e - h$  for  $h \leq 1$ . If  $h \geq 2$ , considering only dimension and rank constraints it would be possible that  $f > m - e - h$ . Suppose that this is the case (anticipating contradiction), and define an  $m \times n$  partial matrix  $H$  as follows :  $H_{ij} = 0$  if  $i > q$ , and for  $1 \leq i \leq q$

$$H_{ij} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \end{cases},$$

with the remaining positions of  $H$  ( $1 \leq i \leq q$  and  $i < j$ ) occupied by independent indeterminates. Note that every completion of  $H$  has rank  $q$ . Let  $k_1$ ,  $k_2$  and  $k_3$  respectively denote the total numbers of indeterminates in Rows  $1, \dots, e$  of  $A$ , Rows  $e + 1, \dots, e + h$  of  $A$  and Rows  $e + h + 1, \dots, m$  of  $A$  – note that we allow that the upper right  $e \times f$  region of  $A$  could be full of indeterminates. Let  $k'_1$ ,  $k'_2$  and  $k'_3$  denote the corresponding numbers for  $H$ .

(i)  $k_1 \leq en - \frac{e(e+1)}{2} = k'_1$ , so  $k_1 \leq k'_1$ .

(ii) All the completions of the lower right  $(m - e - h) \times f$  part of  $A$  have rank  $q - e - h$ , so by the induction hypothesis and the hypothesis that  $m - e - h < f$ ,

$$k_3 \leq (q - e - h)f - \frac{(q - e - h)(q - e - h - 1)}{2}.$$

We also have

$$k'_3 = (q - e - h)(n - e - h) - \frac{(q - e - h)(q - e - h - 1)}{2}.$$

Since  $f + e + h \leq n$ , it is immediate that  $k_3 \leq k'_3$ .

(iii) Now look at rows  $e + 1, \dots, e + h$ .

$$k_2 \leq hf, \text{ and } k'_2 = h(n - e) - \frac{h(h+1)}{2} = h(n - e - h) + \frac{h(h-1)}{2}.$$

Now  $f \leq n - e - h$  and  $\frac{h(h-1)}{2}$  is positive since  $h > 1$ , so  $k_2 < k'_2$ .

Thus the total number  $k_1 + k_2 + k_3$  of indeterminates in  $A$  is strictly less than the number in  $H$ . This contradicts the hypothesis that  $A$  has the maximum possible number of indeterminates for a partial  $m \times n$  matrix whose completions all have rank  $q$ . We conclude that  $f \leq m - e - h$  and so by the induction hypothesis Corollary 5.2 applies to the transpose of  $Y_3$  giving that the number of indeterminates in  $Y_3$  cannot exceed

$$f(m - e - h) - \frac{f(f+1)}{2}.$$

Note that this number of indeterminates can only be attained if

- $f = q - e - h$ , or
- $f - 1 = q - e - h$  and  $f = m - e - h$  (which demands that  $q = m - 1$ ).

In all cases the number of indeterminates in the lower right  $(m - e - h) \times f$  region of  $B_2$  (and hence of  $A$ ) cannot exceed  $f(m - e - h) - \frac{f(f + 1)}{2}$ . Allowing that the next  $h$  rows above this in the last  $f$  columns could be fully populated with indeterminates (this is the maximum possible number of indeterminates in these positions) gives that the number of indeterminates in the lower right  $(m - e) \times f$  region of  $A$  cannot exceed

$$f(m - e) - \frac{f(f + 1)}{2}.$$

□

We are now in a position to complete our proof of Theorem 5.1. We have that  $A$  is an  $m \times n$  partial matrix over  $\mathbb{F}$ , with  $m \leq n$ , whose completions all have the same rank  $q$ . Application of Lemma 5.3 gives that all the indeterminates of  $A$  can be contained by a set of at most  $q$  lines. Hence there exists some  $e$  rows and some  $f$  columns that contain all of the indeterminates of  $A$ , such that  $e + f \leq q$  and such that no collection of fewer than  $e + f$  lines contains all the indeterminates of  $A$ . We can rewrite  $A$  by permuting its rows and columns so that these are the first  $e$  rows and the last  $f$  columns. Proposition 5.5 and Proposition 5.6 identify the maximum possible number of indeterminates in the upper left  $e \times (n - f)$  and lower right  $(m - e) \times f$  regions of  $A$  respectively. The upper right  $e \times f$  region is the only other region of  $A$  that can contain indeterminates. If we allow that this region could be fully populated with indeterminates we have that the maximum possible number of indeterminates in  $A$  is

$$ef + e(n - f) - \frac{e(e + 1)}{2} + f(m - e) - \frac{f(f + 1)}{2}. \quad (5.1)$$

For  $z \in \mathbb{N}$  and  $k \in \mathbb{N}_0$ , we define

$$\alpha_z(k) = kz - \frac{k(k + 1)}{2} = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{i=1}^k (z - i) & \text{if } k > 0 \end{cases}.$$

Note that  $\alpha_z(k_1) < \alpha_z(k_2)$  for  $0 \leq k_1 < k_2 \leq z - 1$  and  $\alpha_z(z - 1) = \alpha_z(z)$ .

Rewriting (5.1) yields

$$en + fm - ef - \left[ \frac{e(e + 1)}{2} + \frac{f(f + 1)}{2} \right]$$

$$\begin{aligned}
&= (e+f)n - f(n-m) - ef - \left[ \frac{(e+f)(e+f+1)}{2} - ef \right] \\
&= \alpha_n(e+f) - f(n-m).
\end{aligned}$$

We know that  $e+f \leq q < n$ , and as noted earlier  $\alpha_n(k)$  is a strictly increasing function of  $k$  when  $k < n$ . So  $\alpha_n(e+f) - f(n-m)$  is maximized when  $e+f$  is as large as possible (i.e.  $e+f = q$ ) and when  $f(n-m) = 0$  (i.e.  $f = 0$  if  $m < n$ ). This gives that the maximum possible number of indeterminates in  $A$  is

$$\alpha_n(e+f) = \alpha_n(q) = qn - \frac{q(q+1)}{2}. \quad (5.2)$$

We conclude our proof of Theorem 5.1 by noting that this upper bound on the number of indeterminates of  $A$  is attainable, as demonstrated by this example:

$$\begin{pmatrix} U_1 & \Delta & \Delta \\ 0 & 0 & \Delta \\ 0 & 0 & U_2 \end{pmatrix}. \quad (5.3)$$

Here  $U_1$  and  $U_2$  are  $(r-s) \times (r-s)$  and  $s \times s$  upper triangular partial matrices having nonzero constant diagonal entries and all indeterminate entries above the diagonal, with  $s = 0$  if  $m < n$  and  $0 \leq s \leq n$  if  $m = n$ .

## 5.2 Characterization

Having determined the maximum possible number of indeterminates that a partial matrix of constant rank can possess, the task remains to fully resolve Problem 1.2 and so characterize the partial matrices of constant rank that attain this bound. It transpires that this results in showing that any such partial matrix can be transformed by row and column permutations into a partial matrix of a form described by (5.3) above.

**Theorem 5.7.** *Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$ , with  $m \leq n$ , having constant rank  $r$ . Then  $A$  has  $rn - \frac{r(r+1)}{2}$  indeterminates if and only if it can be transformed by row and column permutations into a matrix of the form described by (5.3) above.*

As previously with Theorem 5.1 the case  $r = m$  of Theorem 5.7 corresponds with the case  $r = \min(m, n)$  of Theorem 4.2, which has already been proven in Chapter IV. It remains to prove Theorem 5.7 for the case  $r < m$ .

We suppose now that  $r < m$ , and note that if  $A$  can be transformed by row and column permutations into a form described by (5.3) above, it is clear that every

completion of  $A$  has rank  $r$  and a count verifies that it contains  $rn - \frac{r(r+1)}{2}$  indeterminates. The weightier task is to prove that if  $A$  is an  $m \times n$  partial matrix, with  $m \leq n$ , having constant rank  $r < m$  and possessing  $rn - \frac{r(r+1)}{2}$  indeterminates, then  $A$  can be transformed by row and column permutation into such a form. We present this proof as a series of propositions that focus on particular cases. Proposition 5.8 deals with the case  $m < n$  and Proposition 5.9 deals with the case  $m = n$  and  $r < n - 1$ . The case that  $m = n$  and  $r = n - 1$  is a more exceptional case and this is addressed by Proposition 5.10. Before delving into these propositions we note the following facts that are directly relevant to these propositions – these facts were elucidated in the proof of Theorem 5.1.

- All the indeterminates of  $A$  are contained by some  $e$  rows and some  $f$  columns with  $e + f = r$  and no collection of fewer than  $r$  lines cover all of the indeterminates of  $A$ . Hence we can rewrite  $A$  by permuting its row and columns so that these are the first  $e$  rows and the last  $f$  columns.
- The upper right  $e \times f$  region of  $A$  is fully populated with indeterminates.

The following items are only relevant to Propositions 5.8 and 5.9 (i.e. they may not be true for the case that  $m = n$  and  $r = n - 1$  that is addressed by Proposition 5.10).

- Every completion of the upper left  $e \times (n - f)$  region of  $A$  has rank  $e$ . Hence by Theorem 4.2, we can rewrite  $A$  by a permutation of Rows  $1, \dots, e$  and Columns  $1, \dots, n - f$  so that:
  - The upper right  $e \times (n - e)$  region is fully populated with indeterminates.
  - The upper left  $e \times e$  region is an upper triangular partial matrix with nonzero constant diagonal entries and all entries above the diagonal being indeterminate.
- Every completion of the lower right  $(m - e) \times f$  region of  $A$  has rank  $f$ . Hence by Theorem 4.2, we can rewrite  $A$  by a permutation of Rows  $e + 1, \dots, m$  and Columns  $n - f + 1, \dots, n$  so that the form of the upper  $e$  rows (as just described) is not disturbed and such that:
  - The upper right  $(m - f) \times f$  region of  $A$  only contains indeterminates.
  - The lower right  $f \times f$  region of  $A$  is an upper triangular partial matrix with nonzero constant diagonal entries and all entries above the diagonal being indeterminate.

**Proposition 5.8.** *Let  $m < n$  and let  $A$  be an  $m \times n$  partial matrix containing  $rn - \frac{r(r+1)}{2}$  indeterminates and having the property that all its completions have rank  $r < m$ . Then  $A$  conforms to the description set out in the statement of Theorem 5.7.*

*Proof.* We begin by rewriting  $A$  by permuting its rows and columns so that the indeterminates are arranged as detailed above. As  $m < n$ , we know from (5.2) that  $f = 0$ . Hence all of the indeterminates reside in the first  $e = r$  rows of  $A$ . The lower  $m - e$  rows of  $A$  are fully populated with zero entries – otherwise a completion of rank greater than  $r$  would be possible. Hence

$$A = \begin{pmatrix} U_1 & \Delta \\ 0 & 0 \end{pmatrix},$$

where  $U_1$  is an  $r \times r$  upper triangular partial matrix having nonzero constant diagonal entries and all indeterminate entries above the diagonal.  $\square$

**Proposition 5.9.** *Let  $A$  be an  $n \times n$  partial matrix with  $rn - \frac{r(r+1)}{2}$  indeterminates and having the property that all its completions have rank  $r < n - 1$ . Then  $A$  conforms to the description set out in the statement of Theorem 5.7.*

*Proof.* We begin by rewriting  $A$  by permuting its rows and columns so that it is of the form

$$A = \begin{pmatrix} U_1 & \Delta_1 & \Delta_2 \\ C_1 & C_2 & \Delta_3 \\ C_3 & C_4 & U_2 \end{pmatrix}, \quad (5.4)$$

where  $U_1$  and  $U_2$  are  $e \times e$  and  $f \times f$  upper triangular partial matrices having nonzero constant diagonal entries and all indeterminate entries above the diagonal,  $\Delta_1$ ,  $\Delta_2$  and  $\Delta_3$  are partial matrices all of whose entries are indeterminate, and  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  are constant matrices.

From the fact that the zero completion of  $A$  has rank  $r$ , it is immediate that  $C_2 = 0$ . Assigning the value 0 to all the indeterminates in  $U_1$ ,  $U_2$ ,  $\Delta_2$  and  $\Delta_3$  in the matrix above results in the partial matrix

$$A' = \begin{pmatrix} D_1 & \Delta_1 & 0 \\ C_1 & 0 & 0 \\ C_3 & C_4 & D_2 \end{pmatrix},$$

where  $D_1$  and  $D_2$  are diagonal matrices whose diagonal entries are nonzero constants. In every completion of  $A'$ , each of rows  $e + 1, \dots, n - f$  must be a linear combination of the first  $e$  rows. If Row  $i$  of  $C_1$  is not zero, values can be assigned to the indeterminates in  $\Delta_1$  that yield a completion of  $A'$  in which Row  $e + i$  is not a linear combination of the first  $e$  rows. The rank of this completion of  $A$  would exceed  $r$ , hence  $C_1 = 0$ . By assigning the value 0 to all the indeterminates in  $U_1$ ,  $U_2$ ,  $\Delta_1$  and  $\Delta_2$  and applying a similar argument it becomes apparent that  $C_4 = 0$ .

It remains to show that  $C_3 = 0$ . Every completion of  $A$  and hence of the  $r \times r$  partial matrix

$$\begin{pmatrix} U_1 & \Delta_2 \\ C_3 & U_2 \end{pmatrix}$$

has rank  $r$ . Assigning the value 0 to all the indeterminates in  $U_1$  and  $U_2$  gives the partial matrix

$$\begin{pmatrix} D_1 & \Delta_2 \\ C_3 & D_2 \end{pmatrix}.$$

If Row  $i$  of  $C_3$  is nonzero, it is a linear combination of the rows of  $D_1$ , and a suitable assignment of values to the indeterminates of  $\Delta_2$  would yield a completion (of this partial matrix and of  $A$ ) of rank less than  $r$ . Hence

$$A = \begin{pmatrix} U_1 & \Delta & \Delta \\ 0 & 0 & \Delta \\ 0 & 0 & U_2 \end{pmatrix}.$$

□

All that remains now is to address the case  $m = n$  and  $r < n - 1$ .

**Proposition 5.10.** *Let  $A$  be an  $n \times n$  partial matrix with  $\frac{n(n-1)}{2}$  indeterminates and having the property that all its completions have rank  $n - 1$ . Then  $A$  conforms to the description set out in the statement of Theorem 5.7.*

*Proof.* We begin by noting that the form referred to in the statement of this proposition is such that  $A$  can be converted by row and column permutations into an upper triangular partial matrix with constant diagonal entries of which exactly one is zero, with all entries above the diagonal being indeterminate.

The proposition holds for  $1 \times 1$  matrices; we assume that it holds for square matrices with fewer than  $n$  rows. From Lemma 5.3 and the proof of Theorem 5.1, we know that  $A$  has the following properties:

- All the indeterminates in  $A$  occur in some  $e$  rows and some  $f$  columns, where  $e + f = n - 1$  and no collection of fewer lines contain all the indeterminates of  $A$ . We rewrite  $A$  by permuting its rows and columns so that these are the first  $e$  rows and the last  $f$  columns.
- All entries in the upper right  $e \times f$  region of  $A$  are indeterminate and there are a further  $\frac{e(e+1)}{2}$  indeterminates in the upper left  $e \times (e+1)$  region of  $A$ .
- Every completion of the lower  $n-e$  rows of  $A$  has rank at least  $f$  (otherwise some completion of  $A$  would have rank less than  $n-1 = e+f$ ). If every completion of the lower  $n-e$  rows of  $A$  has rank exactly  $f$ , then every completion of the upper left  $e \times (e+1)$  region must have rank  $e$ . Hence Theorem 4.2 applies to this region giving that  $A$  can be rewritten (by a permutation of Rows  $1, \dots, e$  and Columns  $1, \dots, n-f$ ) such that the upper right  $e \times (f+1)$  region only contains indeterminates and such that the upper left  $e \times e$  region is an upper triangular matrix with nonzero constant diagonal entries with all entries above the diagonal being indeterminate.
- However if the lower  $n-e$  rows of  $A$  admit a completion of rank  $f+1 = n-e$ , then we can rewrite  $A$  (by a permutation of Columns  $1, \dots, n-f$ ) such that every completion of the upper left  $e \times e$  region has rank  $e-1$ . As  $e < n$ , the inductive hypothesis applies to this submatrix, giving that  $A$  can be rewritten (by a permutation of Rows  $1, \dots, e$  and Columns  $1, \dots, n-f-1$ ) so that the upper right  $e \times (f+1)$  region only contains indeterminates and so that the upper left  $e \times e$  region is an upper triangular matrix with constant diagonal entries (of which exactly one is zero) with all entries above the diagonal being indeterminate.
- In either case  $A$  can be rewritten (by the row and column permutations described) so that Row  $i$  (for  $1 \leq i \leq e$ ) contains indeterminates in Columns  $i+1, \dots, n$  and constants otherwise.
- A similar (but “transposed”) argument shows that further permutations of Rows  $e+1, \dots, n$ , and Columns  $n-f, \dots, n$ , give that Row  $i$  (for  $1 \leq i \leq e$ ) contains indeterminates in Columns  $i+1, \dots, n$  and constants otherwise, and that Column  $j$  (for  $n-f+1 \leq j \leq n$ ) contains indeterminates in Rows  $1, \dots, j-1$  and constants otherwise.

This means that the rows and columns of  $A$  can be permuted so that all entries above the diagonal are indeterminate and so we can assume that  $A$  is of this form. Immediately we conclude that there is at least one zero entry on the diagonal, otherwise the zero completion of  $A$  would have full rank. We now show that there is exactly one zero entry on the main diagonal of  $A$ . Suppose that there are at least two, and let  $p$  and  $q$  be respectively the least and greatest indices for which  $A(p, p) = 0$  and  $A(q, q) = 0$ . In the zero completion of  $A$ , Row  $p$  is either the zero vector of length  $n$  (if  $p = 1$ ) or else is a linear combination of Rows  $1, \dots, p - 1$ . The zero completion of  $A$  has rank  $n - 1$ , so Rows  $1, \dots, p - 1, p + 1, \dots, n$  of the zero completion of  $A$  must be linearly independent and hence span the  $(n - 1)$ -dimensional row-space of this matrix. Since every completion of  $A$  has rank  $n - 1$ , it then follows that *every* completion of Row  $p$  of  $A$  is a linear combination of the other  $n - 1$  rows of the zero completion. Consider the completion  $A_1$  of  $A$  that coincides with the zero completion except that the indeterminate in the  $(p, q)$  position is assigned the value 1. We claim this matrix has rank  $n$ . This is clear if  $q = n$ , since in this case Row  $p$  is the only one to have a nonzero entry in Column  $n$ . If  $q < n$ , an expression for Row  $p$  as a linear combination of the other  $n - 1$  rows must involve at least one of Rows  $q + 1, \dots, n$ , since these are the only available rows that can contain nonzero entries in Column  $q$ . However, since the lower right  $(n - q) \times (n - q)$  region of  $A_1$  is a nonsingular lower triangular matrix and the first  $n - q$  columns are otherwise full of zeros, any linear combination of the rows of  $A_1$  that involves some of Rows  $q + 1, \dots, n$  must include at least one nonzero entry among its last  $n - q$  entries. No such linear combination can be equal to Row  $p$ . This contradiction establishes that there is precisely one zero entry on the diagonal of  $A$ .

All that remains now is to show that all of the constant entries below the diagonal of  $A$  are zero. Suppose that the zero entry on the main diagonal of  $A$  occurs in Position  $(t, t)$ . Write  $e' = t - 1$  and  $f' = n - t$ . Then  $e' + f' = n - 1$  and all the indeterminates of  $A$  are contained in the first  $e'$  rows and the last  $f'$  columns. We assign the value zero to all indeterminates of  $A$  except for the  $e'$  indeterminates in Column  $n - f'$ . In every completion of the resulting matrix, Row  $e' + 1$  must be a linear combination of the first  $e'$  rows, and it follows that all the constant entries in Row  $e' + 1$  are zeros. By an analogous argument the constant entries of Column  $n - f'$  are all zeros. Since Row  $e' + 1$  and Column  $f' + 1$  can be completed to the zero row and zero column, the  $(n - 1) \times (n - 1)$  partial matrix obtained by deleting this row and column from  $A$  has all its completions of full rank  $n - 1$ . Since all entries above the main diagonal are indeterminates and all entries on the main diagonal

are nonzero constants, it follows from Theorem 4.2 that this partial matrix is upper triangular. Hence we have shown that  $A$  has the stated form.  $\square$

This completes our proof of Theorem 5.7.

## CHAPTER VI

### Constant rank submatrices

In Chapter V we saw that a partial matrix of constant rank  $r$  that has the maximum possible number of indeterminates must possess an  $r \times r$  partial submatrix of full constant rank. This observation gave rise to the following question:

**Problem 1.3.** *Must a partial matrix of constant rank  $r$  possess an  $r \times r$  nonsingular partial submatrix?*

Clearly this question also has an affirmative answer at the other end of the spectrum, in the case of a partial matrix that does not contain any indeterminates (i.e. a constant matrix). An  $m \times n$  constant matrix of rank  $r$  has at least one set of  $r$  rows that span its row-space. Consider the  $r \times n$  submatrix formed by any linearly independent set of  $r$  rows – it has at least one set of  $r$  columns that span its column-space. Any linearly independent set of  $r$  columns from this submatrix form a nonsingular  $r \times r$  submatrix of the original matrix.

That this question also has an affirmative answer in many other cases is a consequence of Corollary 6.1 that follows from this result of Huang and Zhan [16].

**Theorem 1.9. (Huang, Zhan) [16]** *Let  $m, n$  be positive integers, let  $\mathbb{F}$  be a field with  $|\mathbb{F}| \geq \max(m, n + 1)$  and let  $A$  be an  $m \times n$  ACI matrix over  $\mathbb{F}$ . Then all completions of  $A$  have the same rank  $r$  if and only if there exists a nonsingular matrix  $T \in M_m(\mathbb{F})$  and a permutation matrix  $Q \in M_n(\mathbb{F})$  such that*

$$TAQ = \begin{pmatrix} U_1 & \boxplus & \boxplus \\ 0 & 0 & \boxplus \\ 0 & 0 & U_2 \end{pmatrix},$$

where  $U_1$  and  $U_2$  are  $s \times s$  and  $(r - s) \times (r - s)$  upper triangular ACI matrices with nonzero constant diagonal entries with  $0 \leq s \leq r$ . The symbol  $\boxplus$  denotes an ACI matrix whose only restriction is that it be of appropriate dimension.

**Corollary 6.1.** *Let  $\mathbb{F}$  be a field with at least  $\max(m, n + 1)$  elements. Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  of constant rank  $r$ . Then  $A$  contains an  $r \times r$  partial submatrix of constant full rank.*

*Proof.* As  $A$  is of constant rank  $r$ , we know from Theorem 1.9 that there exists a nonsingular  $m \times m$  matrix  $T$  and an  $n \times n$  permutation matrix  $Q$  so that  $TAQ$  matches the description of Theorem 1.9. From the structure of  $TAQ$  it is clear that Columns  $1, \dots, s, n - r - s + 1, \dots, n$  form a linearly independent set of columns in every completion of  $TAQ$ , for some  $0 \leq s \leq r$ . It follows that in every completion of  $TA$  for some indices  $j_1, \dots, j_r$ , Columns  $j_1, \dots, j_r$  of  $TA$  form a linearly independent set of columns. As  $T$  is a nonsingular matrix it follows that in every completion of  $A$ , Columns  $j_1, \dots, j_r$  of  $A$  also form a linearly independent set of columns.

Now consider the  $m \times r$  matrix formed by Columns  $j_1, \dots, j_r$  of  $A$ . It is a partial matrix (hence an ACI matrix) of constant rank  $r$  as is its transpose. Applying the same procedure (as just outlined) to this transpose results in an  $r \times r$  partial submatrix of constant full rank.  $\square$

The restriction on the field order in Theorem 1.9 is required for the mechanism of its proof in [16]. There is no discussion in [16] of cases where this condition is not satisfied. However, it is not entirely dispensable as the following example shows:

**Example 6.2.** This partial matrix over the field  $\mathbb{F}_2$  of two elements has constant rank 3 but each of its four  $3 \times 3$  partial submatrices admits a completion of rank 2 – i.e. none of these  $3 \times 3$  partial submatrices have constant full rank.

$$\begin{pmatrix} 1 & 1 & x_2 & 0 \\ 1 & 0 & 0 & x_3 \\ 1 & x_1 & 1 & 1 \end{pmatrix}$$

The material in this chapter is drawn from [19] – a paper written by Quinlan and the author that has been published in *Linear Algebra and its Applications*. Section 6.1 details some features of finite vector spaces that are relevant to our analysis. In Section 6.2 we prove the following theorem.

**Theorem 6.11.** *Let  $\mathbb{F}$  be any field with at least  $r$  elements. Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  of constant rank  $r$ . Then  $A$  possesses an  $r \times r$  partial submatrix of constant full rank. Moreover, this submatrix may be chosen so that it involves all the elements of any maximal linearly independent set of constant columns of  $A$ .*

Section 6.3 discusses the case of fields whose order is less than the constant rank. The main result from this section is the following theorem.

**Theorem 6.13.** *Let  $A$  be an  $m \times n$  partial matrix of constant rank  $r$  over the field  $\mathbb{F}_q$  of  $q$  elements. Suppose that  $A$  possesses no  $r \times r$  partial submatrix of constant rank  $r$ . Then  $q < r$  and*

$$\max(m, n) \geq r + q - 1.$$

It is shown also that for any prime power  $q$  and positive integers  $m, n$  and  $r$  satisfying the conclusion of Theorem 6.13, there exists a partial  $m \times n$  submatrix over  $\mathbb{F}_q$  that has constant rank  $r$  and possesses no  $r \times r$  partial submatrix of constant rank  $r$ .

## 6.1 Properties of finite vector spaces

This section consists of some elementary observations about finite vector spaces that play a key role in our resolution of Problem 1.3, and may have some independent interest. We first recall a basic fact about writing a finite vector space as a union of (linear) hyperplanes.

**Lemma 6.3.**  *$\mathbb{F}_q^l$  is not the union of  $q$  or fewer hyperplanes (for any positive integer  $l$ ).*

*Proof.* A hyperplane of  $\mathbb{F}_q^l$  has  $q^{l-1} - 1$  nonzero elements. Thus the union of  $q$  hyperplanes of  $\mathbb{F}_q^l$  cannot contain more than  $q(q^{l-1} - 1) = q^l - q$  nonzero elements. Since  $\mathbb{F}_q^l$  has  $q^l - 1$  nonzero elements, it cannot be the union of  $q$  hyperplanes.  $\square$

We note that provided that  $l \geq 2$ ,  $\mathbb{F}_q^l$  is the union of the  $q + 1$  hyperplanes whose orthogonal complements are the  $q + 1$  lines of a two-dimensional subspace of  $\mathbb{F}_q^l$ .

**Corollary 6.4.** *Let  $v_1, \dots, v_q$  be a set of  $q$  nonzero vectors in  $\mathbb{F}_q^l$  (for any  $l$ ). Then there exists a vector  $u \in \mathbb{F}_q^l$  for which  $v_i u^T$  is nonzero for all  $i$  in the range 1 to  $q$ .*

*Proof.* By Lemma 6.3, there is a vector  $u \in \mathbb{F}_q^l$  that does not belong to any of the hyperplanes  $v_1^\perp, \dots, v_q^\perp$ .  $\square$

If  $v_1, \dots, v_q$  of Corollary 6.4 are written as the rows of a  $q \times l$  matrix, the corollary asserts that there is a linear combination of the columns of this matrix that has no entry equal to zero. This is the context in which we will use Corollary 6.4.

The remainder of this section is concerned with a property possessed by some finite vector spaces, which will feature in Section 6.3.

**Definition 6.5.** For a field  $\mathbb{F}$  and positive integer  $l$ , a subspace  $U$  of  $\mathbb{F}^l$  has the *distributed zero property* if every element of  $U$  has at least one entry equal to zero, and there is no position in which every element of  $U$  has a zero entry (so no element of  $U^\perp$  has exactly one nonzero entry).

If  $\mathbb{F}$  is infinite, no subspace of  $\mathbb{F}^l$  can have the distributed zero property. However, it is easily verified that the two dimensional subspace of  $\mathbb{F}_3^4$  spanned by the vectors  $(0\ 1\ 1\ 1)$  and  $(1\ 0\ 1\ 2)$  has the distributed zero property. Our immediate goal is to prove Lemma 6.7, which states that a subspace of  $\mathbb{F}_q^l$  having the distributed zero property must have codimension at least  $q - 1$  in  $\mathbb{F}_q^l$ . This is a consequence of the following observation.

**Lemma 6.6.** *Let  $v_1, \dots, v_{q-2}$  be a set of  $q-2$  nonzero vectors in  $\mathbb{F}_q^l$  (for any  $l$ ). Then there exists a nowhere zero vector  $\mathbb{F}_q^l$  that is not orthogonal to any  $v \in \{v_1, \dots, v_{q-2}\}$ .*

*Proof.* There is nothing to do if  $q = 2$ , so assume that  $q > 2$ . Then there exists a nowhere zero vector  $u_1$  that is not orthogonal to  $v_1$ . For  $i = 2, \dots, q - 2$ , define  $u_i$  as follows.

- $u_i = u_{i-1}$  if  $u_{i-1}$  is not orthogonal to  $v_i$ .
- If  $u_{i-1} \perp v_i$ , suppose that the first nonzero entry of  $v_i$  is in Position  $k$ . Adjust the entry in position  $k$  of  $u_{i-1}$  to produce a vector  $u_i$  that is not orthogonal to any of  $v_1, \dots, v_{i-1}, v_i$ . There are  $q - 1$  nonzero elements in  $\mathbb{F}_q$ , of which we have at least  $q - 1 - i$  to choose from in defining each component of  $u_i$ .

After  $q - 3$  steps, this process produces a nowhere zero vector  $u_{q-2}$  that is not orthogonal to any  $v \in \{v_1, \dots, v_{q-2}\}$ .  $\square$

**Lemma 6.7.** *Let  $U$  be a subspace of dimension  $k$  of  $\mathbb{F}_q^l$  that has the distributed zero property. Then  $l - k \geq q - 1$ .*

*Proof.* Write the elements of a basis of  $U$  as the columns of an  $l \times k$  matrix  $B$ , and apply elementary column operations to convert  $B$  to the reduced column echelon form  $B'$ . Since the column space of  $B'$  has the distributed zero property, every vector in  $\mathbb{F}_q^k$  is orthogonal (with respect to the usual scalar product) to some row of  $B'$ . Some  $k$  rows of  $B'$  form the  $k \times k$  identity matrix, and so every element of  $\mathbb{F}_q^k$  that has no zero entry is orthogonal to at least one of the remaining  $l - k$  rows of  $B'$ . It follows from Lemma 6.6 that  $l - k \geq q - 1$ .  $\square$

## 6.2 Large Fields ( $|\mathbb{F}| \geq r$ )

In this section we show that a partial matrix of constant rank  $r$  over a field  $\mathbb{F}$  satisfying  $|\mathbb{F}| \geq r$  must possess an  $r \times r$  submatrix of constant rank  $r$ . The corresponding statement in the case  $|\mathbb{F}| < r$  is false as evidenced by Example 6.2; this is the subject of Section 6.3. The dichotomy is a consequence of Lemma 6.7. Our first observation concerns the orthogonal complement (with respect to the ordinary scalar product) of the span of the constant columns in a partial matrix of constant rank.

**Lemma 6.8.** *Let  $\mathbb{F}$  be any field. Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  of constant rank  $r$ . Let  $C$  be the subspace of  $(\mathbb{F}^m)^T$  spanned by the constant columns of  $A$  and let  $\dim(C) = k$ . If  $A$  has more than  $r - k$  indeterminate columns then every element of  $C^\perp$  has a zero entry in a position in which some column of  $A$  has an indeterminate.*

*Proof.* Assume (anticipating contradiction) that  $C^\perp$  does contain an element,  $v^T$ , that has nonzero entries in every position in which a column of  $A$  has an indeterminate entry. Then every indeterminate column has a completion that is orthogonal to  $v^T$  and one that is not orthogonal to  $v^T$ . Let  $A_v$  be a completion of  $A$  in which every column is orthogonal to  $v^T$ . So  $A_v$  is a completion of  $A$  whose left null-space contains  $v$ . If  $r = m$ , this contradicts the hypothesis that every completion of  $A$  has rank  $r$ . If  $r < m$ , select a set of  $r$  linearly independent columns of  $A_v$  that includes  $k$  of the original constant columns - call this  $S$ . As there are more than  $r - k$  indeterminate columns in  $A$ , there is some indeterminate column whose completion in  $A_v$  is not in  $S$  - say Column  $j$ . Form a completion of  $A$  that is identical to  $A_v$  except that the completion of Column  $j$  is not orthogonal to  $v^T$ , and is thus outside the span of  $S$ . The rank of this completion of  $A$  exceeds  $r$ . □

We note the following immediate consequence of Lemma 6.8.

**Corollary 6.9.** *Let  $\mathbb{F}$  be any field. Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  that has constant rank  $r$ . If every nonzero column of  $A$  has an indeterminate entry, then  $A$  has precisely  $r$  nonzero columns.*

We have shown that a partial matrix  $A$  satisfying the hypotheses of Lemma 6.8 must possess a nonzero constant column. Let  $C$  denote the span of these constant columns. Then every element of  $C^\perp$  has at least one zero entry. This means either that  $C$  contains an element that has exactly one nonzero entry, so that there is a

position in which every element of  $C^\perp$  has a zero entry, or that  $C^\perp$  has the distributed zero property. In view of Lemma 6.7, the latter case can arise only if  $\mathbb{F}$  is finite and  $\dim(C) \geq |\mathbb{F}| - 1$ . Since  $\dim(C) \leq r$ , this means that  $C^\perp$  can have the distributed zero property only if  $|\mathbb{F}| \leq r + 1$ .

Our next lemma, which is needed for the proof of the main theorem of Section 6.3, establishes that (over any field) a partial matrix of constant rank  $r$  whose constant columns span a subspace of dimension greater than  $r - 2$  must possess an  $r \times r$  submatrix of constant full rank. Then Theorem 6.11 shows that over a field whose order is at least equal to  $r$ , a partial matrix of constant rank  $r$  must possess an  $r \times r$  submatrix of constant full rank.

**Lemma 6.10.** *Let  $\mathbb{F}$  be any field. Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  of constant rank  $r$ . Let  $C$  denote the subspace of  $(\mathbb{F}^m)^T$  spanned by the constant columns of  $A$ . Then if  $\dim(C) \geq r - 1$ ,  $A$  possesses an  $r \times r$  partial submatrix of constant full rank. Moreover, this submatrix may be chosen so that it involves all the elements of any maximal linearly independent set of constant columns of  $A$ .*

*Proof.* If  $\dim(C) = r$ , then we can select any set of  $r$  linearly independent constant columns and identify an  $r \times r$  constant submatrix of rank  $r$  within the selected columns.

If  $\dim(C) = r - 1$ , there is an indeterminate column in  $A$ , each of whose completions lies outside  $C$ . Let  $B$  be the  $m \times r$  submatrix of  $A$  composed of this indeterminate column and any  $r - 1$  linearly independent constant columns. If  $r = m$ , then  $B$  is the required submatrix. If  $r < m$ , consider the  $r \times m$  partial matrix  $B^T$ , which has constant rank  $r$  and whose indeterminates are confined to one row, say the first. If  $B^T$  has only  $r$  nonzero columns, these form an  $r \times r$  submatrix of constant rank  $r$ . If  $B^T$  has more than  $r$  nonzero columns, then by Corollary 6.9 it includes both constant and indeterminate columns. Let  $v^T$  be a nonzero vector of length  $r$  that is orthogonal to every constant column of  $B^T$ . Then  $v$  cannot be in the left null-space of any completion of  $B^T$ . It follows that the first entry of  $v$  must be zero, since every indeterminate column could otherwise be completed to be orthogonal to  $v^T$ . Hence the vector  $e_1^T = (1 \ 0 \ \dots \ 0)^T$  belongs to the span of the constant columns of  $B^T$ . Now choose an  $r \times r$  submatrix  $B'$  of  $B^T$  that includes a maximal linearly independent set of constant columns and admits a completion of rank  $r$ . Since all indeterminates of  $B'$  are in Row 1 and the vector  $e_1^T$  belongs to the span of the constant columns, all completions of  $B'$  have the same column-space. Hence all completions of  $B'$  have rank  $r$ , and  $(B')^T$  is an  $r \times r$  submatrix of  $A$  of constant full rank.  $\square$

**Theorem 6.11.** *Let  $\mathbb{F}$  be any field with at least  $r$  elements. Let  $A$  be an  $m \times n$  partial matrix over  $\mathbb{F}$  of constant rank  $r$ . Then  $A$  possesses an  $r \times r$  partial submatrix of constant full rank. Moreover, this submatrix may be chosen so that it involves all the elements of any maximal linearly independent set of constant columns of  $A$ .*

*Proof.* We begin by showing that  $A$  must possess an  $m \times r$  submatrix of constant full rank. To this end we employ induction on  $r$ , the base case  $r = 1$  being unproblematic. Let  $C$  denote the span of the constant columns of  $A$ , and let  $k = \dim(C)$ .

If  $k = 0$ , then by Corollary 6.9 there are only  $r$  nonzero columns and they form the required  $m \times r$  submatrix. If  $k \geq r - 1$  then by Lemma 6.10,  $A$  possesses the required  $m \times r$  partial submatrix.

If  $1 \leq k \leq r - 2$  and there are only  $r - k$  indeterminate columns in  $A$ , any selection of  $k$  linearly independent constant columns and the  $r - k$  indeterminate columns gives the required  $m \times r$  partial submatrix.

Suppose then that  $1 \leq k \leq r - 2$  and the number of indeterminate columns in  $A$  exceeds  $r - k$ . By Lemma 6.8 every element of  $C^\perp$  possesses at least one zero entry. Since  $\mathbb{F}$  has at least  $r$  elements and  $k \leq r - 2$ , Lemma 6.7 ensures that  $C^\perp$  does not have the distributed zero property. Thus  $C$  contains a singly nonzero vector.

We may assume (after permuting the rows and permuting the columns of  $A$  if necessary) that the first  $k$  columns of  $A$  are constant and linearly independent, and that  $(1 \ 0 \ \dots \ 0)^T$  is a linear combination of (all of) the first  $k'$  columns, where  $k' \leq k$ . Let  $A'$  denote the partial matrix obtained by replacing the first column with  $(1 \ 0 \ \dots \ 0)^T$ . Deleting the first row and column of  $A'$  yields an  $(m - 1) \times (n - 1)$  partial matrix of constant rank  $r - 1$  in which the first  $k - 1$  columns are constant and linearly independent. By the induction hypothesis this partial matrix possesses an  $(m - 1) \times (r - 1)$  partial matrix that has constant rank  $r - 1$  and includes the first  $k - 1$  columns. The existence of the required  $m \times r$  submatrix of  $A$  follows.

Having established that every matrix satisfying the hypotheses of Theorem 6.11 possesses an  $m \times r$  submatrix of constant rank  $r$ , we complete the proof by applying this fact to the transpose of this submatrix.  $\square$

Before concluding this section we return to Theorem 1.9 (of Huang and Zhan [16]). We remark that it is possible to marginally relax the field order constraint of this theorem to  $|\mathbb{F}| \geq \max(m, n)$  rather than  $|\mathbb{F}| \geq \max(m, n + 1)$ . This constraint is due in part to Lemma 4 of [16]. Lemma 6.12 has the same hypothesis as Lemma 4 (of [16]) except that the field order is only required to be at least as great as the number of columns (instead of being required to exceed the number of columns).

**Lemma 6.12.** *Let  $\mathbb{F}$  be a field with order  $n$  or greater. Let  $A$  be an  $m \times n$  ACI matrix over  $\mathbb{F}$ . If all completions of  $A$  have the same rank  $r < n$ , then  $A$  has at least one column with only constant entries.*

*Proof.* Suppose (anticipating contradiction) that each column of  $A$  possesses at least one indeterminate. There are two cases to consider.

- (1) There is a row in  $A$  that has an indeterminate in each entry.
- (2) Every row of  $A$  has at least one constant entry.

Case 1: Say that Row  $i$  of  $A$  possesses an indeterminate in each of its entries. Complete  $A$  so that Row  $i$  is all zero – call this completion  $A_1$ . Convert  $A_1$  into reduced row echelon form. Say that the resulting nonzero rows have pivots in Columns  $j_1, \dots, j_r$ . Complete  $A$  so that Row  $i$  is all zero except that it has 1 in some column outside of Columns  $j_1, \dots, j_r$ . This results in a completion of  $A$  with rank  $r + 1$ .

Case 2: Our strategy in this case is to use elementary row operations to create a row equivalent matrix that has a row that possesses an indeterminate in each of its entries. As elementary row operations do not affect rank, the same rationale as employed in Case 1 will then apply. Our assumption is that every column of  $A$  possesses an indeterminate, so say that  $A$  possesses the indeterminate  $x_j$  in Column  $j$ . Our aim is to add rows of  $A$  to get a row vector that possesses  $x_j$  in Position  $j$ .

Each row possesses at least one constant entry, so in particular there exists  $1 \leq j_1, j_2 \leq n$  such that  $A(1, j_1)$  does not involve  $x_{j_1}$  and  $A(2, j_2)$  does not involve  $x_{j_2}$ . As  $|\mathbb{F}| \geq n$ , it is possible to add some multiple of Row 2 to some multiple of Row 1 so that Position  $j$  of the resulting row vector possesses  $x_j$  if it existed in Position  $j$  of Row 1 or Position  $j$  of Row 2. If this results in a row vector that possesses  $x_j$  in Position  $j$ , we are done. If not we iterate this procedure with succeeding rows until we do. □

### 6.3 Small Fields ( $|\mathbb{F}| < r$ )

The goal of this section is to prove the following theorem.

**Theorem 6.13.** *Let  $A$  be an  $m \times n$  partial matrix of constant rank  $r$  over the field  $\mathbb{F}_q$  of  $q$  elements. Suppose that  $A$  possesses no  $r \times r$  partial submatrix of constant rank  $r$ . Then  $q < r$  and*

$$\max(m, n) \geq r + q - 1.$$

It is immediate from Theorem 6.11 that the matrix  $A$  of Theorem 6.13 can exist only if  $q < r$ . We will prove the second part of the statement under the additional hypothesis that  $m \leq n$ ; there is no loss of generality here since we may work equally well with  $A$  or its transpose. Let  $C$  denote the subspace of  $(\mathbb{F}^m)^T$  spanned by the constant columns of  $A$ . The case where  $C^\perp$  possesses the distributed zero property will be considered initially. An induction argument on  $r$  will then be applied to the alternative situation, where  $C$  contains a singly nonzero vector. The base for this induction argument is the case  $r = q + 1$  which is relevant to both cases and is presented in Theorem 6.15 for this reason. First we need to establish the following technicality.

**Lemma 6.14.** *Let  $A$  be a partial matrix of constant rank  $r$  over a field  $\mathbb{F}$ . Let  $k$  and  $k'$  respectively denote the dimensions of the span of the constant columns of  $A$  in  $(\mathbb{F}^m)^T$  and of the constant rows of  $A$  in  $\mathbb{F}^n$ . If no  $r \times r$  partial submatrix of  $A$  has constant rank  $r$ , then either the number of indeterminate columns in  $A$  exceeds  $r - k$  or the number of indeterminate rows in  $A$  exceeds  $r - k'$ .*

*Proof.* Suppose that the number of indeterminate columns in  $A$  is equal to  $r - k$ . Retaining these columns and a set of  $k$  linearly independent constant columns, form a  $m \times r$  partial submatrix  $A'$  of  $A$  of constant rank  $r$ . A set of linearly independent constant rows of  $A'$  can have at most  $k'$  elements, so  $A'$  must possess more than  $r - k'$  indeterminate rows in order to avoid a  $r \times r$  partial submatrix of constant full rank.  $\square$

We now address the base case  $r = q + 1$  for the induction component of our proof of Theorem 6.13.

**Theorem 6.15.** *Let  $B$  be an  $m \times n$  partial matrix, with  $m \leq n$ , of constant rank  $q + 1$  over  $\mathbb{F}_q$ . If  $B$  possesses no  $(q + 1) \times (q + 1)$  partial submatrix whose completions are all nonsingular, then  $n \geq 2q$ .*

*Proof.* Let  $C$  denote the subspace of  $(\mathbb{F}^m)^T$  spanned by the constant columns of  $B$ .

Case 1: The number of indeterminate columns of  $B$  exceeds  $q + 1 - \dim(C)$ .

Suppose (anticipating contradiction) that  $C$  contains a vector with exactly one nonzero entry. After permuting rows and permuting constant columns if necessary, we may assume that  $(1 \ 0 \ \dots \ 0)^T \in C$  and that Column 1 of  $B$  belongs to a minimal set  $S$  of constant columns of  $B$  whose span includes  $(1 \ 0 \ \dots \ 0)^T$ . Deleting Row 1 and Column 1 from  $B$ , we obtain an  $(m - 1) \times (n - 1)$  partial matrix whose completions

all have rank  $q$ . By Theorem 6.11, this matrix possesses a  $q \times q$  partial submatrix that has constant rank  $q$  and involves all those columns of  $B$  that belong to  $S$  (except Column 1). Restoring (the relevant entries of) the first row and column then results in a  $(q + 1) \times (q + 1)$  submatrix of  $B$  of constant rank  $q + 1$ , contrary to the hypotheses.

This contradiction leads to the conclusion that  $C$  contains no element with exactly one nonzero entry. Then from Lemma 6.8 it follows that  $C^\perp$  must have the distributed zero property, and by Lemma 6.7 the number of constant columns in  $B$  is at least  $q - 1$ .

It remains to show that the number of indeterminate columns of  $B$  must be at least  $q + 1$ . Form a partial completion  $B'$  of  $B$  by assigning a value to all but one of the indeterminates in each indeterminate column. Take any basis for  $C^\perp$  and write that basis as the columns of a matrix. As there is no position in which every element of  $C^\perp$  has a zero entry, all the rows of the resulting matrix are nonzero. It follows from Corollary 6.4 that given any  $q$  of the  $m$  coordinate positions in  $(\mathbb{F}_q^m)^T$ , there exists an element of  $C^\perp$  that has nonzero entries in all  $q$  of these positions. It is then immediate from Lemma 6.8 that the indeterminates of  $B'$  cannot be confined to  $q$  rows. Since  $B'$  has only one indeterminate in each indeterminate column, it follows that  $B'$ , hence  $B$ , has at least  $q + 1$  indeterminate columns.

We conclude that  $B$  has at least  $2q$  columns in all, since it has at least  $q - 1$  constant columns and at least  $q + 1$  indeterminate columns.

Case 2: The number of indeterminate columns in  $B$  is equal to  $q + 1 - \dim(C)$ .

Let  $B_1$  denote the  $m \times (q + 1)$  matrix formed from a set of  $\dim(C)$  constant columns of  $B$  and all the indeterminate columns. Then  $B_1$  has constant rank  $q + 1$  and by Lemma 6.14 the above argument applies to its transpose, hence  $m \geq 2q$ . Since  $n \geq m$ , the desired conclusion follows.  $\square$

Now let  $A$  be an  $m \times n$  (with  $m \leq n$ ) partial matrix of constant rank  $r$  over  $\mathbb{F}_q$ , and suppose that  $A$  contains no  $r \times r$  partial submatrix of constant full rank. Let  $C$  denote the span of the constant columns of  $A$ . Assume that the number of indeterminate columns of  $A$  exceeds  $r - \dim(C)$ ; the alternative situation may be converted to this one as in Case 2 in the proof of Theorem 6.15 above. By Lemma 6.8, every element of  $C^\perp$  possesses at least one zero entry. We now turn our attention to the case where  $C^\perp$  has the distributed zero property, with the aim of proving the following theorem.

**Theorem 6.16.** *Suppose that  $C^\perp$  has the distributed zero property. Then  $n \geq r + q - 1$ .*

Let  $k$  be the dimension of  $C$ , so that the number of constant columns of  $A$  is at least  $k$ . Then  $m - k$  is the dimension of  $C^\perp$ , and every completion of  $A$  has a left null-space that is a subspace of  $C^\perp$  of dimension  $m - r$ . We wish to show that the number of indeterminate columns in  $A$  is at least  $r - k + q - 1$ .

Let  $A'$  be a partial matrix obtained from  $A$  by assigning a value to all but one of the indeterminates in each indeterminate column. Let  $s$  denote the number of indeterminate columns in  $A$  (or  $A'$ ), and let  $t_i$  denote the position of the indeterminate in the  $i^{\text{th}}$  indeterminate column of  $A'$ . For each  $i$ , let  $H_i$  denote the subspace of  $C^\perp$  consisting of all those elements that have 0 in position  $t_i$ . Then, since there is no position in which every element of  $C^\perp$  has a zero entry, each  $H_i$  is a hyperplane of  $C^\perp$ . Also, it follows from Lemma 6.8 that

$$C^\perp = \bigcup_{i=1}^s H_i.$$

For  $i = 1, \dots, s$ , let  $w_i$  denote the completion of the  $i^{\text{th}}$  indeterminate column of  $A'$  obtained by assigning the value 0 to the indeterminate in that column. Define

$$W_i = \langle w_i \rangle^\perp \cap C^\perp.$$

Then each  $W_i$  is either equal to  $C^\perp$  or is a hyperplane of  $C^\perp$ . Write  $W = \bigcap_{i=1}^s W_i$  and note that

$$\dim(W) = m - r,$$

since this intersection is the left null-space of a completion of  $A'$ . It follows from Lemma 6.10 that  $\dim(C) \leq r - 2$ , hence  $\dim(C^\perp) \geq m - r + 2$  and so it follows that  $\dim(W) \leq \dim(C^\perp) - 2$ . This detail will be needed in the final step of our proof of Theorem 6.16.

For each  $i$ ,  $H_i \setminus W_i$  is the set of elements of  $C^\perp$  that are not orthogonal to *any* completion of the  $i^{\text{th}}$  indeterminate column of  $A'$ , and  $H_i \cap W_i$  is the set of elements of  $C^\perp$  that are orthogonal to *every* completion of the  $i^{\text{th}}$  indeterminate column of  $A'$ .

We write

$$S = \bigcup_{i=1}^s H_i \setminus W_i.$$

**Lemma 6.17.** *Every subspace of  $C^\perp$  of dimension  $m - r + 1$  has non-empty intersection with  $S$ .*

*Proof.* Let  $U$  be a subspace of  $C^\perp$  of dimension  $m - r + 1$ , and suppose (anticipating contradiction) that  $U \cap S = \emptyset$ . This would mean that  $A$  has a completion whose left null-space contains  $U$  and so this completion would have rank less than  $r$ . To show this it is sufficient to show that each indeterminate column has a completion that is orthogonal to all elements of  $U$ . Fix  $i$ , and note that  $U \cap H_i$  is contained in  $W_i$ , and that this subspace is either equal to  $U$  or has codimension 1 in  $U$ . In the first case, every element of  $U$  is orthogonal to every completion of the  $i^{\text{th}}$  indeterminate column of  $A'$ . In the second case, let  $x_i$  be an element of  $U$  that does not belong to  $H_i$ . Then the  $i^{\text{th}}$  indeterminate column of  $A'$  has a completion that is orthogonal to  $x_i$ . Since  $U \subseteq \langle x_i \rangle + (H_i \cap W_i)$ , it follows that every element of  $U$  is orthogonal to this completion of the  $i^{\text{th}}$  indeterminate column of  $A'$ . Thus  $A'$ , hence  $A$ , has a completion whose left null-space has dimension exceeding  $m - r$ , contrary to the hypothesis of constant rank  $r$ .  $\square$

To complete the proof of Theorem 6.16, we are required to show that  $s \geq r - k + q - 1$ . This last step is an immediate consequence of Lemma 6.17 and the following theorem.

**Theorem 6.18.** *Let  $V$  be a vector space of finite dimension  $t$  over the field  $\mathbb{F}_q$  of  $q$  elements, and suppose that for hyperplanes  $H_1, \dots, H_s, W_1, \dots, W_s$  of  $V$ , the following conditions are satisfied.*

- (i)  $\bigcup_{i=1}^s H_i = V$ .
- (ii)  $W = \bigcap_{i=1}^s W_i$  has dimension  $l \leq t - 2$ .
- (iii) The set  $S = \bigcup_{i=1}^s H_i \setminus W_i$  has non-empty intersection with every subspace of  $V$  of dimension  $l + 1$ .

Then  $s \geq t - l + q - 1$ .

*Proof.* We consider the following two cases separately.

- Case 1: There is a  $W_i$  that does not coincide with any of the  $H_j$ .
- Case 2: For each  $i$  there is some  $j$  for which  $W_i = H_j$ .

Case 1: Suppose that  $W_1$  is not equal to any  $H_j$ . Then each  $H_j$  intersects  $W_1$  in a hyperplane of  $W_1$ . In this case we use induction on  $t - l$ .

The base case for the induction is  $t - l = 2$ . In this case the desired conclusion is that  $s \geq q + 1$ , which follows immediately from Lemma 6.3 and the fact that  $V$  is the union of the  $H_i$ .

Returning to the general situation of Case 1, we now write  $H'_i = H_i \cap W_1$  and  $W'_i = W_i \cap W_1$ , for  $i = 2, \dots, s$ . Each  $H'_i$  is a hyperplane of  $W_1$ , and each  $W'_i$  is either a hyperplane of  $W_1$  or is equal to  $W_1$  (in which case  $H'_i \setminus W'_i$  is empty). Now

$$\bigcap_{i=2}^s W'_i = \bigcap_{i=1}^s W_i$$

is a subspace of  $W_1$  of dimension  $l$ . If  $U$  is a subspace of  $W_1$  of dimension  $l + 1$ , the intersection of  $U$  with  $S$  must be contained in the union  $\bigcup_{i=2}^s H_i \setminus W_i$ . Thus  $U$  has non empty intersection with the set  $\bigcup_{i=2}^s H'_i \setminus W'_i$ . Since  $\dim(W_1) = t - 1$  and  $\dim(\bigcap_{i=2}^s W'_i) = l$ , the inductive hypothesis applies to the space  $W_1$  and the set  $\bigcup_{i=2}^s H'_i \setminus W'_i$ . Thus

$$s - 1 \geq t - 1 - l + q - 1 \implies s \geq t - l + q - 1,$$

as required.

Case 2: Since some selection of  $t - l$  of the  $W_i$  intersect in a space of dimension  $l$ , suppose that each of  $H_1, \dots, H_{t-l}$  is one of the  $W_i$  and that these hyperplanes intersect in  $W$ . If  $l > 0$ , extend  $\{H_1, \dots, H_{t-l}\}$  to a set,  $\{H_1, \dots, H_t\}$ , of  $t$  hyperplanes of  $V$  whose intersection is 0. Let  $v_1, \dots, v_t$  be nonzero vectors of  $V$  that span the  $t$  one-dimensional subspaces obtained by taking the intersection of  $t - 1$  elements of  $\{H_1, \dots, H_t\}$ . Arrange the ordering of the  $v_i$  so that for  $i = 1, \dots, t - l$ , the element  $v_i$  does not belong to  $H_i$ . By Lemma 2.4  $\{v_1, \dots, v_t\}$  is a basis of  $V$ . Let  $\zeta$  denote the nondegenerate symmetric bilinear form on  $V$  with respect to which  $\{v_1, \dots, v_t\}$  is an orthonormal basis. For  $i = 1, \dots, t - l$ , the hyperplane  $H_i$  is exactly the orthogonal complement of  $\langle v_i \rangle$  with respect to  $\zeta$ . Thus the union  $\bigcup_{i=1}^{t-l} H_i$  contains precisely those elements of  $V$  whose expression in terms of the basis  $\{v_1, \dots, v_t\}$  includes at least one zero amongst the first  $t - l$  coordinate positions. Thus elements of  $V$  whose coordinates with respect to  $\{v_1, \dots, v_t\}$  are all nonzero are not included in the union of the first  $t - l$  of the  $H_i$ . From Lemma 6.6 it follows that at least a further  $q - 1$  hyperplanes are needed to accommodate all of these elements. Thus  $s \geq t - l + q - 1$ .  $\square$

Finally, we complete the proof of Theorem 6.16 by applying Theorem 6.18 to the situation of the matrix  $A'$ . From Theorem 6.18 it follows that  $s \geq m - k - (m - r) + q - 1$ , hence since  $n = k + s$  we have

$$n \geq r + q - 1,$$

as required.

The following general theorem about partial matrices of constant rank  $r$  possessing  $r \times r$  partial submatrices of constant full rank relies upon Theorem 6.16 and is needed for the completion of the proof of Theorem 6.13.

**Theorem 6.19.** *Let  $A$  be an  $m \times n$  partial matrix of constant rank  $r$  over any field  $\mathbb{F}$ . Let  $S$  be a maximal set of linearly independent constant columns of  $A$ . If  $A$  possesses an  $r \times r$  submatrix of constant rank  $r$ , then it possesses one that involves all the columns of  $S$ .*

*Proof.* Suppose that  $B$  is an  $r \times r$  partial submatrix of  $A$  that has constant rank  $r$  and does not involve all the columns of  $S$ . After permuting rows and permuting columns of  $A$  if necessary, we may assume that  $B$  occupies the upper left  $r \times r$  region of  $A$ , and that Column  $r + 1$  of  $A$  belongs to  $S$ . Let  $A'$  be the partial matrix comprising the upper  $r$  rows of  $A$ . We will show that an  $r \times r$  partial submatrix of  $A'$  of constant full rank may be obtained by exchanging Column  $r + 1$  with a column of  $B$  that does not belong to  $S$ . Let  $u^T$  denote the element of  $(\mathbb{F}^r)^T$  that is Column  $r + 1$  of  $A'$ . Note that  $u^T \neq 0$ , otherwise the matrix consisting of the first  $r + 1$  columns of  $A$  would have constant rank  $r + 1$ .

If  $u^T$  is a linear combination of constant columns of  $B$ , then one of these constant columns does not belong to  $S$  and may be exchanged with  $u^T$  to produce the required  $r \times r$  submatrix. If not, then let  $C_B$  denote the span of the constant columns of  $B$ . From the fact that the left null-space of a completion of  $B$  must be trivial (and in particular cannot contain a nowhere zero vector) it follows that  $\dim(C_B) \geq 1$  and that every element of  $C_B^\perp$  possesses at least one zero entry. Since  $B$  has constant rank  $r$  and has exactly  $r$  columns, it follows from Theorem 6.16 that  $C_B^\perp$  does not have the distributed zero property. Then  $C_B$  must contain some standard basis vectors of  $(\mathbb{F}^r)^T$ . After permuting the first  $r$  rows of  $A$  we may suppose that  $e_1^T, \dots, e_k^T$  are the standard basis vectors that belong to  $C_B$ . Now apply elementary column operations to the constant columns of  $B$ , to form a matrix  $B'$  in which the vectors  $e_1^T, \dots, e_k^T$  appear in the first  $k$  columns and any remaining constant columns are unchanged from  $B$  (though possibly permuted). Let  $B''$  be the  $(r - k) \times (r - k)$  submatrix of  $B'$  obtained by deleting the first  $k$  rows and first  $k$  columns. It has constant rank  $r - k$ , and by the above argument the span of its constant columns must contain some standard basis vector in  $(\mathbb{F}^{r-k})^T$ . It follows that the span of the constant columns of  $B''$  strictly contains the projection of  $C_B$  on the last  $r - k$  coordinate positions, hence that some indeterminate column of  $B$  possesses indeterminate entries amongst

the first  $k$  positions only.

Iterating this argument, we construct an ascending chain of subspaces of  $(\mathbb{F}^r)^T$  as follows.

- $C_B (= C_B^0)$  is the span of the constant columns of  $B$ .
- $C_B^1$  is the column space of any completion of the submatrix of  $B$  consisting of all those columns having no indeterminate entries outside the first  $k$  positions. That these completions all have the same column space follows from the fact that  $e_i^T \in C_B$  for  $i \leq k$ .
- For  $i \geq 1$ ,  $C_B^{i+1}$  is the column space of any completion of the submatrix of  $B$  consisting of all those columns that have a constant entry in position  $j$  whenever  $e_j \notin C_B^i$ .

This chain of subspaces is strictly ascending until it reaches  $(\mathbb{F}^r)^T$  after at most  $r - 1$  extensions from the original  $C_B$ . There is some  $k$  for which

$$u^T \in C_B^{k+1} \text{ and } u^T \notin C_B^k.$$

Then there is a column, say Column  $l$ , of  $B$  that contributes to  $C_B^{k+1}$  but not to  $C_B^k$  and is involved in the expression for  $u^T$  as a linear combination of the columns of the completion  $B_0$  in which all indeterminates are set to 0. If  $B_1$  is another completion of  $B$ , then the same coefficients determine a linear combination of the columns of  $B_1$  that differs from  $u^T$  only in those positions  $j$  where  $e_j^T \in C_B^k$ . So it differs from  $u^T$  by a linear combination of those columns that contribute to  $C_B^k$ . Thus Column  $l$  is involved in the expression for  $u^T$  as a linear combination of the columns of every completion of  $B$ , and hence its entries may be replaced with those of  $u^T$  to obtain the required  $r \times r$  submatrix of  $A$  of constant full rank.  $\square$

We are nearing the completion of Theorem 6.13, which is:

**Theorem 6.13.** *Let  $A$  be an  $m \times n$  partial matrix of constant rank  $r$  over the field  $\mathbb{F}_q$  of  $q$  elements. Suppose that  $A$  possesses no  $r \times r$  partial submatrix of constant rank  $r$ . Then  $q < r$  and*

$$\max(m, n) \geq r + q - 1.$$

To recap, Theorem 6.11 establishes that  $q < r$ . Theorem 6.16 (relying upon Theorem 6.15, Lemma 6.17 and Theorem 6.18) establishes that  $\max(m, n) \geq r + q - 1$  in the case where the span of the constant columns of  $A$  does not contain a singly nonzero vector. To complete the proof of Theorem 6.13, it remains to consider the case where  $C$  (the span of the constant columns of  $A$ ) contains a singly nonzero

vector. We apply induction on  $r$ ; the base case is Theorem 6.15 and the inductive hypothesis is the statement that in an  $m \times n$  partial matrix over  $\mathbb{F}_q$  of constant rank  $l$ , where  $m \leq n$ ,  $q + 1 \leq l < r$ , and there is no  $l \times l$  submatrix of constant rank  $l$ , the number  $n$  of columns is at least  $l + q - 1$ .

We may perform an elementary column operation that only involves a linearly independent set of constant columns and replaces a constant column, Column  $j$ , of  $A$  with a column having exactly one nonzero entry, in Position  $i$ . Let  $A_1$  be the  $(m - 1) \times (n - 1)$  partial matrix obtained by deleting Column  $j$  and Row  $i$ . Then  $A_1$  has constant rank  $r - 1$ . Suppose (anticipating contradiction) that  $A_1$  possesses an  $(r - 1) \times (r - 1)$  submatrix of constant full rank. Then by Theorem 6.19 it possesses one that involves all the columns of some maximal linearly independent (that includes Column  $j$ ) except Column  $j$ . Restoring the relevant entries of Column  $j$  and Row  $i$ , this submatrix extends to an  $r \times r$  submatrix of  $A$  of constant rank  $r$ .

By this contradiction we conclude that  $A_1$  does not possess an  $(r - 1) \times (r - 1)$  submatrix of constant full rank. Then the induction hypothesis applies to  $A_1$ , and

$$n - 1 \geq (r - 1) + (q - 1) \implies n \geq r + q - 1,$$

as required.

We finish by observing that if  $q < r$  and  $\max(m, n) \geq r + q - 1$ , then there exists a  $m \times n$  matrix of constant rank  $r$  over  $\mathbb{F}_q$  that possesses no  $r \times r$  submatrix of constant full rank. To see this it is sufficient to show that there exists a  $(q + 1) \times 2q$  partial matrix of constant rank  $q + 1$  that has no  $(q + 1) \times (q + 1)$  submatrix of constant rank  $q + 1$ . Such a matrix (or its transpose) can be extended to one of the desired size by taking its direct sum with the identity matrix  $I_{r-q-1}$  and adding zero rows and columns as required.

In  $(\mathbb{F}_q^{q+1})^T$ , let  $v_1^T$  be the vector that has 0 in position 1 and 1 in all other positions, and let  $v_2^T$  be the vector that has 1 and 0 as its first two entries, and whose remaining entries are the nonzero elements of  $\mathbb{F}_q$  in some order. Then  $\langle v_1^T, v_2^T \rangle$  has the distributed zero property and each of its nonzero elements has exactly one zero entry. Construct a  $(q + 1) \times 2q$  partial matrix  $A$  as follows.

- Column 1 through  $q - 1$  of  $A$  are constant columns forming a basis of  $\langle v_1^T, v_2^T \rangle^\perp$ .
- For  $i = 1, \dots, q + 1$ , Column  $q + i - 1$  has an indeterminate in position  $i$ , a 1 in position  $i + 1$  (or in position 1 in the case of Column  $2q$ ) and zeros in all other positions.

Then every completion of  $A$  has rank  $q + 1$ , since the left null-space of any completion of  $A$  is easily seen to be trivial. Moreover, no proper submatrix of  $A$  has constant

rank  $q + 1$ , for suppose that  $A'$  is obtained from  $A$  by deleting one column. If the deleted column is a constant one, then by Lemma 6.7 there is a nowhere zero vector in  $(\mathbb{F}^{q+1})^T$  that is orthogonal to every constant column of  $A'$ , and this vector is in the left null-space of some completion of  $A'$ . If the deleted column has an indeterminate in Position  $i$ , then some element  $v$  of  $\langle v_1^T, v_2^T \rangle$  has a zero in Position  $i$  only, and some completion of  $A'$  has  $v$  in its left null-space.

## CHAPTER VII

### Maximal nonsingular partial matrices

In this chapter our focus remains on partial matrices of constant rank, the theme here being a characterization of *maximal nonsingular* partial matrices (and resolution of Problem 1.4). The term *maximal nonsingular partial matrix* was first coined by Brualdi et al. in [5]. They give us the following definition and pose the problem of characterizing them.

**Definition 7.1.** A *maximal nonsingular* partial matrix is a partial matrix with the following properties:

- (i) Every completion is nonsingular.
- (ii) Replacing any constant entry with an indeterminate results in a partial matrix having a singular completion.

Brualdi et al. give the following example of a maximal nonsingular partial matrix:

**Example 7.2.** This partial matrix is maximal nonsingular over any field.

$$A = \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 1 & 1 & 0 & x_3 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The nonsingularity of the partial matrix  $A$  is apparent from the fact that it is row equivalent to the following ACI matrix, each of whose completions has determinant 1:

$$\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & -x_1 & -x_2 + x_3 \\ 0 & 0 & 1 & -x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To verify that  $A$  is maximal nonsingular consider the partial matrix that results by replacing any constant entry of  $A$  with an indeterminate. For example, consider

$$A' = \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 1 & 1 & 0 & x_3 \\ 1 & 1 & 1 & 0 \\ x_4 & 1 & 1 & 1 \end{pmatrix}.$$

Consider the partial completion of  $A'$ , call it  $B$ , that results by assigning  $x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 0$ .

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then  $B^{(1)} = B^{(3)} - B^{(4)}$ . It can be verified that replacing any other constant entry with an indeterminate results in a partial matrix having a singular completion.

Theorem 7.7 below is the main result of this chapter. Before we state this proposition, it is necessary to introduce the following terminology which allows for a more succinct discussion.

**Definition 7.3.** Let  $A$  be a partial matrix. We say that  $A$  is in *indeterminate echelon form* if:

- (i) No row of  $A$  has more indeterminates than any row of lesser index.
- (ii) If  $A(i, j)$  is an indeterminate then  $A(i, k)$  is an indeterminate for  $k \geq j$ .

If a partial matrix is in indeterminate echelon form the *indeterminate pivots* are the entries of least index that are indeterminate in any row that has a different number of indeterminates to the row after it. For convenience we shall prepend the “position”  $(0, 1)$  and append the “position”  $(n, n+1)$  to our lists of indeterminate pivot positions (even though they are not positions of an  $n \times n$  matrix). So for the remainder of this chapter if we refer to a set of indeterminate pivot positions  $\{(i_t, j_t)\}_{t=1}^k$ , it is understood that  $(i_1, j_1) = (0, 1)$  and  $(i_k, j_k) = (n, n+1)$ .

It should be noted that it is not possible to convert every partial matrix (by row and column permutations) to indeterminate echelon form. If a partial matrix is in indeterminate echelon form, its list of indeterminate pivot positions minimally

describes its indeterminate positions (ignoring the two extraneous additions). Note that if the indeterminate pivot positions are listed in terms of (strictly) increasing row index, then the column indices are also strictly increasing.

**Example 7.4.** The following partial matrix is in indeterminate echelon form, the entries in the indeterminate pivot positions are in bold.

$$\begin{pmatrix} 0 & 0 & \mathbf{x_1} & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 & x_5 & x_6 \\ 0 & 0 & 0 & 0 & \mathbf{x_7} & x_8 \\ 0 & 0 & 0 & 0 & 0 & x_9 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{x_{10}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Definition 7.5.** Let  $\{(i_t, j_t)\}_{t=1}^k$  be the set of indeterminate pivot positions for an  $n \times n$  partial matrix that is in indeterminate echelon form. Let  $A$  be an  $n \times n$  matrix. We say that  $A$  is in *staircase form with nowhere zero blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$* , if the following conditions are satisfied:

- $A(i, j) = 0$  if  $i \geq j_t$  and  $j \leq i_t$  for some  $t \in \{2, \dots, k-1\}$ .
- $A(i, j) \neq 0$  if  $j_{t-1} \leq i < j_t$  and  $i_{t-1} < j \leq i_t$  for some  $t \in \{2, \dots, k\}$ .

If a square matrix is in staircase form then it has a “staircase” pattern of zeros (rising from right to left) in the lower left region. We refer to the nowhere zero rectangular blocks (outlined above) that occur above this staircase of zeros as *nowhere zero diagonal blocks*.

**Example 7.6.** The following matrix is in staircase form with nowhere zero blocks delimited by the transposed indeterminate pivot positions of the partial matrix in Example 7.4 – the entries (zeros) in these positions are in bold. The entries in the nowhere zero blocks are all ones.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 1 \end{pmatrix}.$$

We can now state Theorem 7.7. Section 7.1 details its proof.

**Theorem 7.7.** *Let  $A$  be an  $n \times n$  partial matrix over a field  $\mathbb{F}$ . Then  $A$  is maximal nonsingular if and only if it satisfies the following conditions:*

- (i) *It is possible to convert  $A$  to indeterminate echelon form,  $A'$ , by a permutation of its rows and a permutation of its columns so that the indeterminate pivot positions of  $A'$ ,  $\{(i_t, j_t)\}_{t=1}^k$ , satisfy  $i_t < j_t$ . If  $\mathbb{F} = \mathbb{F}_2$  then in addition  $i_t \leq j_{t-1}$  for  $2 \leq t \leq k$ .*
- (ii) *The inverse of the zero completion of  $A'$  are in staircase form with nowhere zero blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$ .*

*Remark 7.8.* In Theorem 7.7 we could replace the condition about the inverse of the zero completion of  $A$  with a similar condition stated about the inverse of any particular completion of  $A$ . All the inverses of the completions of  $A'$  share the same staircase form and have identical nowhere zero diagonal blocks. They differ only in their entries above the nowhere zero diagonal blocks.

## 7.1 Characterization

Lemma 7.9 through Lemma 7.12 pertain to nonsingular partial matrices. Brualdi et al. prove the Lemma 7.9 in [5]. It provides valuable insight to nonsingular partial matrices.

**Lemma 7.9. (Brualdi, Huang, Zhan) [5]** *Let  $A$  be an  $n \times n$  partial matrix over a field  $\mathbb{F}$ . Then  $A$  has full constant rank if and only if all the completions of  $A$  have the same nonzero determinant.*

*Proof.* All the completions of  $A$  have the same nonzero determinant if and only if the determinant of  $A$  is a nonzero constant. As  $A$  is a partial matrix,  $\det(A)$  is a polynomial in its constituent indeterminates whose degree in each indeterminate is at most 1. If  $\det(A)$  involves any indeterminate then it would be possible to assign values to the indeterminates to give a determinant of zero. This would mean that the corresponding completion of  $A$  is rank deficient.  $\square$

Note that the following lemma is an adaptation of Lemma 6.8 in Chapter VI.

**Lemma 7.10.** *Let  $\mathbb{F}$  be any field and let  $A$  be a nonsingular partial matrix over  $\mathbb{F}$ . Then  $A$  possesses at least one constant column.*

*Proof.* Assume (anticipating contradiction) that  $A$  has no constant column. Then every column of  $A$  contains an indeterminate and it is possible to complete  $A$  so that every column is orthogonal to  $(1 \dots 1)^T \in (\mathbb{F}^n)^T$ . This results in a singular completion of  $A$ .  $\square$

The following corollary is a consequence of Theorems 6.11 and 6.13 in Chapter VI.

**Corollary 7.11.** *Let  $\mathbb{F}$  be any field and let  $A$  be a nonsingular partial matrix over  $\mathbb{F}$ . Then the span of the constant columns of  $A$  contains a singly nonzero vector.*

**Lemma 7.12.** *Let  $A$  be a nonsingular partial matrix. In any completion of  $A$  the set of columns required to write  $e_i^T \in (\mathbb{F}^n)^T$  does not contain any column that possesses an indeterminate in Row  $i$  of  $A$ .*

*Proof.* Assume (anticipating contradiction) that there is a completion of  $A$ , call it  $A_1$ , in which  $e_i^T \in (\mathbb{F}^n)^T$  can be written as a linear combination of columns that involves Column  $j$  and that  $A(i, j)$  is an indeterminate. Say that this indeterminate is assigned the value  $a$  in  $A_1$ . Let  $\{A_1^{(1)}, \dots, A_1^{(n)}\}$  denote the columns of  $A_1$ . Then

$$e_i^T = \sum_{l \neq j} \alpha_l A_1^{(l)} + \alpha_j A_1^{(j)}, \quad \alpha_j \neq 0.$$

Consider the completion  $A_2$  which differs from  $A_1$  only by virtue of the fact that the indeterminate in  $A(i, j)$  is assigned the value  $a - \frac{1}{\alpha_j}$  (rather than  $a$ ). Taking the same (nonzero) linear combination of columns of the completion  $A_2$  now yields the zero vector:

$$\sum_{l \neq j} \alpha_l A_2^{(l)} + \alpha_j A_2^{(j)} = \sum_{l \neq j} \alpha_l A_1^{(l)} + \alpha_j A_1^{(j)} - \alpha_j \left( \frac{1}{\alpha_j} e_i^T \right) = e_i^T - e_i^T = 0.$$

This singular completion of  $A$  is the anticipated contradiction.  $\square$

Application of Lemma 7.9 leads to the following lemma about maximal nonsingular partial matrices. It is an important component of our characterization.

**Lemma 7.13.** *Let  $A$  be an  $n \times n$  maximal nonsingular partial matrix. Then  $A(i, j)$  is an indeterminate if and only if the minor of  $A(i, j)$  is zero.*

*Proof.* Suppose that the minor of  $A(i, j)$  is nonzero. Consider the calculation of the determinant of  $A$  using the method of minors along Row  $i$ . Suppose (anticipating contradiction) that  $A(i, j)$  is an indeterminate,  $x$ . As the minor of  $A(i, j)$  is nonzero the resulting expression for  $\det(A)$  involves  $x$  and it is possible to assign values to the indeterminates in the expression so that it evaluates to zero. This contradicts the hypothesis that all the completions of  $A$  are nonsingular. So if the minor of  $A(i, j)$  is nonzero,  $A(i, j)$  must be a constant.

Suppose now that the minor of  $A(i, j)$  is zero. Further suppose (anticipating contradiction) that  $A(i, j)$  is a constant entry. Let  $A'$  be a partial matrix that is identical to  $A$  except that it contains an indeterminate in Position  $(i, j)$ . Again consider the calculation of  $\det(A')$  using the method of minors along Row  $i$ . The minor of  $A'(i, j)$  is equal to the minor of  $A(i, j)$  and so is zero. It follows that  $\det(A') = \det(A)$  is a nonzero constant and so  $A'$  is a nonsingular partial matrix. This contradicts the hypothesis that  $A$  is maximal nonsingular.  $\square$

Lemma 7.13 gives rise to the following corollary about maximal nonsingular constant matrices.

**Corollary 7.14.** *The inverse of a maximal nonsingular constant matrix has no zero entry.*

*Proof.* Let  $A$  be a maximal nonsingular matrix. Then  $A^{-1} = \frac{1}{\det(A)} (\text{cof}(A))$  where  $\text{cof}(A)$  is the matrix whose entry in Position  $(i, j)$  is the cofactor of  $A(i, j)$  (i.e. the minor of  $A(i, j)$  multiplied by  $(-1)^{i+j}$ ). As  $A$  only has constant entries, Lemma 7.13 gives that the minor of each position of  $A$  is nonzero. Hence  $A^{-1}$  has no zero entries.  $\square$

Note that Corollary 7.14 is a special case of Theorem 7.7. As a maximal nonsingular constant matrix has no indeterminates, it is already in indeterminate echelon form (having no indeterminate pivots). The inverse of such a matrix (having no zero entries) is an extreme example of a matrix in staircase form. The following corollary is a consequence of Corollary 7.14 – it deals with a peculiarity of maximal nonsingular matrices that are posed over the field of two elements ( $\mathbb{F}_2$ ).

**Corollary 7.15.**

- (i) *There is only one maximal nonsingular constant matrix over the field of two elements, the  $1 \times 1$  matrix  $\begin{pmatrix} 1 \end{pmatrix}$ .*

(ii) *Maximal nonsingular constant matrices of any order exist for all fields with at least three elements.*

*Proof.*

- (i) Corollary 7.14 establishes that the inverse of a maximal nonsingular constant matrix has no zero entry. Every square constant matrix over  $\mathbb{F}_2$  that has no zero entries has rank 1. So the only such nonsingular matrix is  $\begin{pmatrix} 1 \end{pmatrix}$ . It follows that this matrix, which is its own inverse, is the only maximal nonsingular constant matrix over  $\mathbb{F}_2$ .
- (ii) It is possible to construct a nonsingular constant matrix with no zero entries of any order over any field with at least three elements. The generic matrix below illustrates one such possibility for some  $a \in \mathbb{F}$  such that  $a \notin \{0, 1\}$ . The inverse of such a matrix is a maximal nonsingular constant matrix.

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & a & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & a & 1 \\ 1 & 1 & \dots & 1 & a \end{pmatrix}$$

□

We now introduce *quasi-maximal nonsingular* partial matrices. This is a class of partial matrices that includes maximal nonsingular partial matrices. We use *quasi-maximal nonsingular* partial matrices as a vehicle to prove Theorem 7.7.

**Definition 7.16.** A *quasi-maximal nonsingular* partial matrix is a partial matrix with the following properties:

- (i) Every completion is nonsingular.
- (ii) Replacing any constant entry in an indeterminate column with an indeterminate results in a partial matrix having a singular completion.

A maximal nonsingular partial matrix is a quasi-maximal nonsingular partial matrix with the additional constraint that the second property above applies to all columns (rather than just indeterminate columns). Quasi-maximal nonsingular partial matrices that are not maximal nonsingular may arise as submatrices within maximal nonsingular partial matrices. We note the following corollary which is a consequence of Lemma 7.13.

**Corollary 7.17.** *Let  $A$  be an  $n \times n$  quasi-maximal nonsingular partial matrix. Then an entry in an indeterminate column of  $A$  is indeterminate if and only if its minor is zero.*

Lemma 7.19 below shows that it is possible to convert any quasi-maximal nonsingular partial matrix to indeterminate echelon form by a permutation of its rows and a permutation of its columns. Lemma 7.19 also shows that when a quasi-maximal nonsingular partial matrix is in indeterminate echelon form no indeterminate is located on or below the main diagonal. The proof of Lemma 7.19 relies upon the following lemma.

**Lemma 7.18.** *Let  $A$  be a quasi-maximal nonsingular partial matrix over a field  $\mathbb{F}$ . Let  $K$  be the set of indices of the indeterminate columns of  $A$ . Let  $L$  be a set of indices of columns whose span in every completion of  $A$  contains (the singly nonzero unit vector)  $e_i^T$ . Then the entries of Row  $i$  indexed by the elements of  $K \setminus L$  are all indeterminates.*

*Proof.* Every completion of  $A$  is nonsingular, so Lemma 7.9 gives that the determinant of  $A$  is a nonzero constant. Consider the calculation of the determinant of  $A$  using the method of minors along Row  $i$ . Let  $A'_{(i,j)}$  be the  $(n-1) \times (n-1)$  partial submatrix that remains when Row  $i$  and Column  $j$  are deleted from  $A$ . The minor of  $A(i, j)$  is the determinant of  $A'_{(i,j)}$ . As  $e_i^T$  lies in the span of columns indexed by the elements of  $L$  in every completion of  $A$ , it follows that if  $j \notin L$  the zero vector lies in the span of the corresponding columns in every completion of  $A'_{(i,j)}$ . Hence the determinant of  $A'_{(i,j)}$  is zero for any  $j \in \{1, \dots, n\} \setminus L$ . Hence the minor of any position in Row  $i$  that is not indexed by an element of  $L$  is zero. It follows from Corollary 7.17 that the entries of Row  $i$  that are indexed by the elements of  $K \setminus L$  must be indeterminates.  $\square$

**Lemma 7.19.** *Let  $A$  be a quasi-maximal nonsingular  $n \times n$  partial matrix over a field  $\mathbb{F}$ . Then it is possible to convert  $A$  to indeterminate echelon form,  $A'$ , by a permutation of its rows and a permutation of its columns. Furthermore the indeterminate pivot positions of  $A'$ ,  $\{i_t, j_t\}_{t=1}^k$ , are such that  $i_t < j_t$  for  $1 \leq t \leq k$ .*

*Proof.* We employ induction – as the base case consider a  $1 \times 1$  quasi-maximal nonsingular partial matrix. Such a matrix must be a constant matrix so it trivially satisfies Lemma 7.19. We now assume that Lemma 7.19 is true for quasi-maximal

nonsingular partial matrices of order less than  $n$  and we consider  $A$ , an  $n \times n$  quasi-maximal nonsingular partial matrix.

By Lemma 7.10,  $A$  must contain at least one constant column - say that it possesses  $g$  constant columns. By Corollary 7.11, the subspace of  $(\mathbb{F}^n)^T$  spanned by the constant columns of  $A$  contains at least one singly nonzero unit vector. We can permute the rows of  $A$  so that the span of the constant columns of the resulting partial matrix contains  $e_1^T, \dots, e_l^T$  and no other singly nonzero unit vectors. Note that  $g \geq l$ . Permute the columns of this partial matrix so that the indeterminate columns are on the right and the constant columns are on the left with Column  $s$  being required to describe  $e_s^T$  for  $1 \leq s \leq l$ . We call the resulting partial matrix  $A'$  - note that it is a quasi-maximal nonsingular partial matrix as row and columns permutation do not affect any of the properties of interest. As  $e_1^T, \dots, e_l^T$  are linear combinations of the constant columns of  $A'$ , it follows from Lemma 7.18 that the indeterminate columns of  $A'$  possess indeterminates in their first  $l$  positions. It is possible to perform elementary column operations that only involve the constant columns and that only affect Columns  $1, \dots, l$  to give:

$$A'' = \begin{pmatrix} I_l & X \\ 0 & A_1 \end{pmatrix}.$$

We denote the lower right  $(n-l) \times (n-l)$  partial submatrix of  $A''$  by  $A_1$ . It is the same as the lower right  $(n-l) \times (n-l)$  partial submatrix of  $A'$ . All of its completions must be nonsingular - if not there would be a singular completion of  $A''$  (and hence  $A'$ ). Any column that possesses an indeterminate in  $A_1$  certainly is part of a column possessing an indeterminate in  $A'$ . Changing a constant entry to an indeterminate in any indeterminate column of  $A'$  admits a singular completion of the resulting partial matrix. So it follows that it would result in a singular completion of  $A_1$ . Hence  $A_1$  is a quasi-maximal nonsingular partial matrix of order less than  $n$  and so the inductive hypothesis applies to it. This means that it is possible to convert  $A_1$  to indeterminate echelon form ( $A'_1$ ) by permuting its rows and permuting its columns. Furthermore the indeterminate pivots of  $A'_1$  occur above the main diagonal. The required result follows immediately.  $\square$

We note that Lemma 7.18 and Lemma 7.19 apply to maximal nonsingular partial matrices in particular.

**Lemma 7.20.** *Let  $A$  be a maximal nonsingular matrix in indeterminate echelon form and let  $\{(i_t, j_t)\}_{t=1}^k$  be the set of indeterminate pivot positions of  $A$ .*

- (i) The expression for each of the standard basis vectors  $\{e_s^T\}_{s=1}^{i_2}$  as a linear combination of the columns of  $A$  involves all of Columns  $1, \dots, j_2 - 1$  (i.e. the constant columns) and no columns of higher index.
- (ii) The expression for each of the standard basis vectors  $\{e_s\}_{s=j_{k-1}}^n$  as a linear combination of the rows of  $A$  involves all of Rows  $i_{k-1} + 1, \dots, n$  (i.e. the constant rows) and no rows of lower index.
- (iii) In every completion of  $A$  for  $i_{t-1} < s \leq i_t$ , the expression for  $e_s^T$  as a linear combination of the columns of that completion involves all of Columns  $j_{t-1}, \dots, j_t - 1$  and no columns of higher index, for  $3 \leq t \leq k-1$ . Furthermore in every completion of  $A$ , the coefficients of Columns  $j_{t-1}, \dots, j_t - 1$  in the linear combination of columns to write  $e_s^T$  are fixed.

*Proof.*

- (i) By Lemma 7.12, any  $e_s^T$  (for  $1 \leq s \leq i_2$ ) must be a linear combination of some selection of Columns  $1, \dots, j_2 - 1$  only. As  $A$  is maximal nonsingular, Lemma 7.18 gives that all of Columns  $1, \dots, j_2 - 1$  are required to write each  $e_s^T$ . Furthermore as these columns are constant columns the linear combination for each  $e_s^T$  is fixed in every completion of  $A$ .
- (ii) The desired result follows by applying the same rationale to the rows of  $A$ .
- (iii) To begin take the case  $t = 3$ . By Lemma 7.12, in any completion  $e_s^T$  (for  $i_2 < s \leq i_3$ ) must be a linear combination of some selection of Columns  $1, \dots, j_3 - 1$ . Suppose (anticipating contradiction) that there is a completion of  $A$  in which some vector  $e_s^T$  could be written as a linear combination of columns that did not involve Column  $l$  where  $j_2 \leq l < j_3$ . Then by Lemma 7.18,  $A(s, l)$  is an indeterminate. This is a contradiction as Column  $l$  only has indeterminates in Rows  $1, \dots, i_2$  and  $s > i_2$ .

Columns  $j_2, \dots, j_3 - 1$  possess indeterminates in Rows  $1, \dots, i_2$  only. Also  $e_1^T, \dots, e_{i_2}^T$  are linear combinations of Columns  $1, \dots, j_2 - 1$  only. Hence it follows that in every completion of  $A$ , the coefficients of Columns  $j_2, \dots, j_3 - 1$  in the linear combination of columns to write  $e_s^T$  are fixed.

Extending this rationale for  $4 \leq i \leq k - 1$  completes the proof.

□

*Remark 7.21.* Let  $A$  be a nonsingular  $n \times n$  matrix. Column  $i$  of  $A^{-1}$ , i.e. the inverse of  $A$ , indicates the linear combination of the columns of  $A$  that is required to write

the column vector  $e_i^T$ . Similarly, Row  $i$  of  $A^{-1}$  indicates the linear combination of the rows of  $A$  that is required to write the row vector  $e_i$ .

Corollary 7.22 follows by applying Remark 7.21 to Lemma 7.20.

**Corollary 7.22.** *Let  $A$  be an  $n \times n$  maximal nonsingular partial matrix in indeterminate echelon form with indeterminate pivot positions  $\{(i_t, j_t)\}_{t=1}^k$ . Then the inverse of every completion of  $A'$  is in staircase form with nowhere zero blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$ . Furthermore the inverses of all completions of  $A$  have identical nowhere zero diagonal blocks.*

The indeterminate pivot positions of maximal nonsingular proper partial matrices over  $\mathbb{F}_2$  are subject to restrictions that do not apply to such matrices posed over any other field.

**Lemma 7.23.** *Let  $A$  be a maximal nonsingular partial matrix over  $\mathbb{F}_2$  in indeterminate echelon form and let  $\{(i_t, j_t)\}_{t=1}^k$  be the set of indeterminate pivot positions of  $A$  (with  $(i_1, j_1) = (0, 1)$  and  $(i_k, j_k) = (n, n + 1)$ ). Then  $i_t \leq j_{t-1}$  for  $2 \leq t \leq k$ .*

*Proof.* There is only one nowhere zero  $\mathbb{F}_2$ -linear combination of Columns  $1, \dots, c_1 - 1$ . Hence it follows from Lemma 7.20 that  $i_2 = j_1 = 1$ . Applying similar rationale to the constant rows of  $A$  gives that  $j_{k-1} = i_k = n$ .

For  $2 \leq t \leq k$ , let  $u_i^T$  be the sum of Columns  $j_{t-1}, \dots, j_t - 1$  of the zero completion of  $A$ . By Lemma 7.20, the subspace spanned by  $u_i^T$  and Columns  $1, \dots, j_{t-1} - 1$  of the zero completion of  $A$  contains  $e_1^T, \dots, e_{i_t}^T$ . Hence  $i_t \leq j_{t-1}$ .  $\square$

We have now concluded the “only if” direction of Theorem 7.7. Lemma 7.24 below proves the “if” direction of Theorem 7.7.

**Lemma 7.24.** *Let  $A$  be an  $n \times n$  partial matrix in indeterminate echelon form with indeterminate pivot positions  $\{(i_t, j_t)\}_{t=1}^k$  (with  $(i_1, j_1) = (0, 1)$  and  $(i_k, j_k) = (n, n + 1)$ ) with the following properties:*

- $i_t < j_t$  and additionally  $i_t \leq j_{t-1}$  if  $\mathbb{F} = \mathbb{F}_2$ .
- The zero completion of  $A$ , denoted  $A_0$ , is nonsingular.
- The inverse of  $A_0$  is in staircase form with nowhere zero diagonal blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$ .

*Then  $A$  is a maximal nonsingular partial matrix.*

*Proof.* Let  $X$  be the indeterminate part of  $A$ , i.e.  $A = A_0 + X$ . Consider the matrix product  $A_0^{-1}A = A_0^{-1}(A_0 + X) = I + A_0^{-1}X$ . As  $A_0^{-1}$  is in staircase form with nowhere zero blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$ , it has zeros where  $A^T$  (and hence  $X^T$ ) has indeterminates. As  $i_t < j_t$  (for  $1 \leq t \leq k$ ) all of the indeterminates of  $A$  are located above the main diagonal and so  $X$  is strictly upper triangular. Hence  $A_0^{-1}X$  is strictly upper triangular which means that  $A_0^{-1}A$  is an upper triangular matrix each of whose completions has determinant 1. As  $\det(A_0^{-1}A) = \det(A_0^{-1})\det(A)$ , this gives that  $A$  has nonzero constant determinant and by Lemma 7.9 every completion of  $A$  is nonsingular.

It remains to show that  $A$  is maximal nonsingular – i.e. if we replace a constant with a new indeterminate the resulting partial matrix has a singular completion. As  $A$  is in indeterminate echelon form and  $A_0^{-1}$  is in staircase form with nowhere zero diagonal blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$  (the transposed indeterminate pivot positions of  $A$ ), it follows from Lemma 7.12 that placing an indeterminate in any position of  $A$  having coordinates  $(p, q)$  such that

$$i_{t-1} < p \leq i_t \text{ and } j_{t-1} \leq q < j_t \text{ for } 2 \leq t \leq k,$$

would result in a partial matrix having a singular completion.

So suppose that  $(p, q)$  does not satisfy the condition above. Suppose that  $i_{s-1} < p \leq i_s$  and  $j_{r-1} \leq q < j_r$  for  $r, s \in \{2, \dots, k\}$  with  $r < s$ . Then as  $A_0^{-1}$  is in staircase form with nowhere zero blocks delimited by  $\{(j_t, i_t)\}_{t=1}^k$ , we know that Column  $j_{s-1}$  is required to write  $e_p^T$  as a linear combination of the columns of  $A_0$ . We also know that Column  $q$  is required to write  $e_{i_r}^T$  as a linear combination of the columns of  $A_0$ . Note that this linear combination (for  $e_{i_r}^T$ ) cannot involve Column  $j_{s-1}$  as it contains an indeterminate in Position  $i_r$ . Hence

$$e_{i_r}^T = \alpha_q A_0^{(q)} + \sum_{t \notin \{q, j_{s-1}\}} \alpha_t A_0^{(t)} \quad \text{with } \alpha_q \neq 0.$$

Let  $A(p, q) = a \in \mathbb{F}$ . Let  $B$  be a partial matrix that is identical to  $A$  except that it contains an indeterminate in Position  $(p, q)$ . So  $A$  is a partial completion of  $B$  and  $A_0$  is a completion of  $B$ . Our aim is to show that  $B$  has a singular completion.

If Column  $q$  is required to write  $e_p^T$  as a linear combination of the columns of  $A_0$ , then by Lemma 7.12 a singular completion of  $B$  exists and we are done.

If Column  $q$  is not required to write  $e_p^T$  as a linear combination of the columns of  $A_0$ , then

$$e_p^T = \beta_{j_{s-1}} A_0^{(j_{s-1})} + \sum_{t \notin \{j_{s-1}, q\}} \beta_t A_0^{(t)} \quad \text{with } \beta_{j_{s-1}} \neq 0.$$

Now consider the completion of  $B$ , call it  $B_1$ , that differs from  $A_0$  only in respect of the fact that we assign  $a - \frac{1}{\alpha_q}$  to the indeterminate in Position  $(p, q)$  and we assign the value  $-\frac{1}{\beta_{j_{s-1}}}$  to the indeterminate in Position  $(i_r, j_{s-1})$ . Taking the following nonzero linear combination of the columns of  $B_1$  gives

$$\begin{aligned} \alpha_q B_1^{(q)} + \sum_{t \notin \{j_{s-1}, q\}} \alpha_t B_1^{(t)} + \beta_{j_{s-1}} B_1^{(j_{s-1})} + \sum_{t \notin \{j_{s-1}, q\}} \beta_t B_1^{(t)} \\ = \alpha_q A_0^{(q)} + \sum_{t \notin \{j_{s-1}, q\}} \alpha_t A_0^{(t)} - e_p^T + \beta_{j_{s-1}} A_0^{(j_{s-1})} + \sum_{t \notin \{j_{s-1}, q\}} \beta_t A_0^{(t)} - e_{i_r}^T \\ = e_{i_r}^T - e_p^T + e_p^T - e_{i_r}^T = 0. \end{aligned}$$

Hence  $B_1$  is a singular completion of  $B$ . □

This concludes our proof of Theorem 7.7.

Returning to Example 7.2

$$A = \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 1 & 1 & 0 & x_3 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

we can see that it satisfies Theorem 7.7 – it is in indeterminate echelon form and the inverse of its zero completion ( $A_0$ ) is in staircase form with nowhere zero blocks delimited by the transposed indeterminate pivot positions of  $A$ .

$$A_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

## CHAPTER VIII

### Conclusion

We conclude with some general comments and some suggestions for further work. All the results in this thesis pertain to partial matrices. As noted partial matrices belong to the more general class of ACI matrices. It would be interesting to see to what extent these results about partial matrices can be extended to ACI matrices. In particular we have provided precise conditions for the necessity of a partial matrix of constant rank  $r$  to possess an  $r \times r$  nonsingular partial submatrix. There are presently no published results for ACI matrices in this particular regard, despite related contributions from Huang and Zhan [16] and Borobia and Canogar [4]. It is not possible to apply Theorem 1.9 to this end for ACI matrices as we do for partial matrices in Corollary 6.1. In Corollary 6.1 we appeal to the fact that the transpose of a partial matrix is also a partial matrix (hence an ACI matrix). However the transpose of an ACI matrix is not necessarily an ACI matrix.

Chapter VII deals with a characterization of maximal nonsingular partial matrices. It would be interesting to extend this characterization to *maximal constant rank* partial matrices – i.e. partial matrices of constant rank  $r$  with the property that the replacement of any constant entry with an indeterminate results in a partial matrix having a completion with rank other than  $r$ . It is likely that it will not be possible to get a universal characterization for such partial matrices. We know from Chapter VI that if  $|\mathbb{F}| < r$  that a partial matrix of constant rank  $r$  need not necessarily possess an  $r \times r$  nonsingular partial submatrix. It would also be of interest to characterize partial matrices of constant rank  $r$  that do not possess an  $r \times r$  nonsingular partial submatrix.

The problem of determining the minimum rank among the completions of an arbitrary partial matrix is still not fully resolved despite contributions from authors such as Cohen, Johnson, Rodman and Woerdeman [9]. In recent times there has been much interest in computational approaches to this problem for real and complex

matrices [7, 8]. An  $m \times n$  partial matrix for which the minimum rank among its completions is  $k$  corresponds precisely with an affine subspace of  $M_{m \times n}(\mathbb{F})$  with the property that all its elements have rank at least  $k$ . This being the case we know from Theorem 3.15 that there is a related affine space possessing a specified annihilation property. In this way the duality provides an alternate avenue to explore this problem and it may allow the identification of some families of partial matrices for which this problem can be resolved (either by direct or computational means).

In conclusion we note that we have achieved the following:

- Determination of the maximum number of indeterminates that a partial matrix can possess if all of its completions have ranks bounded below, and characterized such partial matrices that attain this bound.
- Determination of the maximum number of indeterminates that a partial matrix can possess if all of its completions have the same rank, and characterized such partial matrices that attain this bound.
- Determination of the precise conditions for a partial matrix of constant rank  $r$  to possess an  $r \times r$  nonsingular partial submatrix.
- Characterization of maximal nonsingular partial matrices.

We hope that these results and the techniques employed to derive them will lead to further progress in this area.

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