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The Bosonic Vertex Operator Algebra on a Genus $g$ Riemann Surface

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Abstract
We discuss the partition function for the Heisenberg vertex operator algebra on a genus $g$ Riemann surface formed by sewing $g$ handles to a Riemann sphere. In particular, it is shown how the partition can be computed by means of the MacMahon Master Theorem from classical combinatorics.

1 Introduction
In this paper we briefly sketch recent progress in defining and computing the partition function for the Heisenberg Vertex Operator Algebra (VOA) on a genus $g$ Riemann surface. The partition function and $n$-point correlation functions are familiar concepts at genus one and have recently been computed on genus two Riemann surfaces formed from sewing tori together [MT1],[MT2]. Here we discuss an alternative approach for computing these objects on a general genus $g$ Riemann surface formed by sewing $g$ handles onto a Riemann sphere. This approach includes the classical Schottky parameterisation and a related simpler canonical parameterisation for which we obtain the partition function for rank 2 Heisenberg VOA in terms of an explicit infinite determinant. This determinant is computed by means of the MacMahon Master Theorem in classical combinatorics [MM].

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2 A Generalized MacMahon Master Theorem

We begin with a review of the MacMahon Master Theorem and a recent generalization. We will provide a proof of this which gives some flavour of the combinatorial graph theory methods developed to compute higher genus partition functions [MT2], [TZ].

Let $A = (A_{ij})$ be an $n \times n$ matrix indexed by $i, j \in \{1, \ldots, n\}$. Consider the cycle decomposition of $\pi \in \Sigma_n$, the symmetric group on $\{1, \ldots, n\}$,

$$\pi = \sigma_1 \ldots \sigma_{C(\pi)}. \quad (1)$$

The $\beta$-extended Permanent of the matrix $A$ is defined by [FZ]

$$\text{perm}_\beta A = \sum_{\pi \in \Sigma_n} \beta^{C(\pi)} \prod_i A_{i\pi(i)}. \quad (2)$$

The standard permanent and determinant are the particular cases:

$$\text{perm} A = \text{perm}_{+1} A, \quad \det A = (-1)^n \text{perm}_{-1} A. \quad (3)$$

Consider a multiset $\{k_1, \ldots, k_m\}$ with $1 \leq k_1 \leq \ldots \leq k_m \leq n$ i.e. index repetition is allowed. We notate the multiset as the unrestricted partition

$$k = \{1^{r_1} 2^{r_2} \ldots n^{r_n}\}, \quad (4)$$

i.e. the index $i$ occurs $r_i \geq 0$ times and where $m = \sum_{i=1}^{n} r_i$. Let $A(k)$ denote the $m \times m$ matrix indexed by $k$ for a given matrix $A$ indexed by $\{1, \ldots, n\}$. We now describe a generalisation of the classic MacMahon Master Theorem (MMT) of combinatorics [MM]. Let $A$ be an $n \times n$ matrix indexed by $\{1, \ldots, n\}$. Let $A(k)$ denote the $m \times m$ matrix indexed by a multiset $k$ (4).

**Theorem 2.1 (Generalized MMT - Foata and Zeilberger [FZ])**

$$\sum_k \frac{\text{perm}_\beta A(k)}{r_1! r_2! \ldots r_n!} = \frac{1}{\det(I - A)^\beta}, \quad (5)$$

where the (infinite) sum ranges over all multisets $k = \{1^{r_1} 2^{r_2} \ldots n^{r_n}\}$. 

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For $\beta = 1$, Theorem 2.1 reduces to the classical MMT [MM]. For $\beta = -1$ we use (3) to find that the sum is restricted to proper subsets of $\{1, 2, \ldots, n\}$ resulting in the determinant identity

$$
\det(I + B) = \sum_{1 \leq k_1 < \ldots < k_m \leq n} \det B(k),
$$

for $B = -A$.

**Proof of Theorem 2.1.** We use a graph theory method applied in [MT2]. Define a set of oriented graphs $\Gamma$ with elements $\gamma \pi$ whose vertices are labelled by multisets $k = \{1^{r_1} \ldots n^{r_n}\}$ and directed edges $\{e_{ij}\}$ determined by permutations $\pi \in \Sigma(k)$ as follows

$$
e_{ij} = \overset{k_i}{\bullet} \overset{k_j}{\bullet} \text{ for } k_j = \pi(k_i)
$$

Define a $\beta$ dependent weight for each $\gamma \pi$

$$
w_\beta(e_{ij}) = A_{k_i k_j}, \quad w_\beta(\gamma \pi) = \beta^{C(\pi)} \prod_{e_{ij} \in \gamma \pi} w_\beta(e_{ij}), \quad (6)
$$

where $C(\pi)$ is the number of disjoint cycles in $\pi$. Then we may write

$$\perm_\beta A(k) = \sum_{\pi \in \Sigma(k)} w_\beta(\gamma \pi).$$

$\gamma \pi$ is invariant under permutations of the identical labels of $k$. Hence the left hand side of (5) can be rewritten as

$$\sum_{k} \perm_\beta A(k) r_1! r_2! \ldots r_n! = \sum_{\gamma \in \Gamma} \frac{w_\beta(\gamma)}{|\text{Aut}(\gamma)|},$$

where we sum over all inequivalent graphs in $\Gamma$. Each $\gamma \in \Gamma$ can be decomposed into disjoint connected cycle graphs $\gamma_\sigma \in \Gamma$

$$
\gamma = \gamma_1^{m_1} \ldots \gamma_K^{m_K}.
$$

Each cycle $\sigma$ corresponds to a disjoint connected cycle graph $\gamma_\sigma \in \Gamma$ with weight

$$
w_\beta(\gamma_\sigma) = \prod_i w_\beta(\gamma_\sigma_i)^{m_i}.
$$
Furthermore
\[ |\text{Aut}(\gamma_\sigma)| = \prod_i |\text{Aut}(\gamma_{\sigma_i})|^{m_i} m_i! \]

Let \( \Gamma_\sigma \) denote the set of inequivalent cycles. Then
\[
\sum_{g \in \Gamma} \frac{w_{\beta}(g)}{|\text{Aut}(g)|} = \prod_{\gamma_\sigma \in \Gamma_\sigma} \sum_{m \geq 0} \frac{w_{\beta}(\gamma_\sigma)^m}{|\text{Aut}(\gamma_\sigma)|^{m m!}} = \exp \left( \sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_{\beta}(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} \right). \tag{7}
\]

For a cycle \( \sigma \) of order \( |\sigma| = r \) then \( \text{Aut}(\gamma_\sigma) = \langle \sigma \rangle \) a cyclic group of order \( |\text{Aut}(\gamma_\sigma)| = \frac{r}{s} \). Using the trace identity
\[
\sum_{\gamma_\sigma \mid |\sigma| = r} s w_{\beta}(\gamma_\sigma) = \beta \text{Tr}(A^r),
\]
we find
\[
\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_{\beta}(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \beta \sum_{r \geq 1} \frac{1}{r} \text{Tr}(A^r) = -\beta \text{Tr}(\log(I - A)) = -\beta \log \det(I - A).
\]
Thus
\[
\sum_k \frac{\text{perm}_{\beta} A(k)}{r_1! r_2! \ldots r_n!} = \det(I - A)^{-\beta}. \quad \square
\]

Define a cycle to be primitive (or rotationless) if \( |\text{Aut}(\gamma_\sigma)| = 1 \). For a general cycle \( \sigma \) with \( |\text{Aut}(\gamma_\sigma)| = s \) we have for \( \beta = 1 \)
\[
w_1(\gamma_\sigma) = w_1(\gamma_\rho)^s,
\]
for some primitive cycle \( \rho \). Let \( \Gamma_\rho \) denote the set of all primitive cycles. Then
\[
\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_1(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} = \sum_{\gamma_\rho \in \Gamma_\rho} \sum_{s \geq 1} \frac{1}{s} w_1(\gamma_\rho)^s = -\sum_{\gamma_\rho \in \Gamma_\rho} \log \det(1 - w_1(\gamma_\rho)).
\]
Combining this with (7) implies [MT2]
Theorem 2.2
\[ \det(I - A) = \prod_{\gamma \rho \in \Gamma \rho} (1 - w_1(\gamma \rho)). \]

3 Riemann Surfaces from a Sewn Sphere

3.1 The Riemann torus

Consider the construction of a torus by sewing a handle to the Riemann sphere \( \hat{\mathbb{C}} \) by identifying annular regions centred at \( A_{\pm 1} \in \hat{\mathbb{C}} \) via a sewing condition with complex sewing parameter \( \rho \)

\[ (z - A_{-1})(z' - A_1) = \rho. \]  

We call \( \rho, A_{\pm} \) canonical parameters. The annuli do not intersect provided

\[ |\rho| < \frac{1}{4}|A_{-1} - A_1|^2. \]  

Inequivalent tori depend only on

\[ \chi = -\frac{\rho}{(A_{-1} - A_1)^2}, \]  

where (9) implies \( |\chi| < \frac{1}{4} \) [MT1].

Equivalently, we define \( q, a_{\pm 1} \), known as Schottky parameters, by

\[ a_i = \frac{A_i + qA_{-i}}{1 + q}, \]

\[ \frac{q}{(1 + q)^2} = \chi, \]  

(11)
for $i = \pm 1$. Inequivalent tori depend only on $q$ with $|q| < 1$. The canonical sewing condition (8) is equivalent to:

$$
\left( \frac{z - a_{-1}}{z - a_1} \right) \left( \frac{z' - a_1}{z' - a_{-1}} \right) = q.
$$

(12)

Inverting (11) we find that $q = C(\chi)$ for Catalan series

$$
C(\chi) = \frac{1 - (1 - 4\chi)^{1/2}}{2\chi} - 1 = \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n+1} \chi^n.
$$

(13)

### 3.2 Genus $g$ Riemann Surfaces

We may similarly construct a general genus $g$ Riemann surface by identifying $g$ pairs of annuli centred at $A_{\pm i} \in \hat{\mathbb{C}}$ for $i = 1, \ldots, g$ and sewing parameters $\rho_i$ satisfying

$$
(z - A_{-i})(z' - A_i) = \rho_i,
$$

(14)

provided no two annuli intersect. Equivalently, for $i = 1, \ldots, g$ we define Schottky parameters $a_{\pm i}, q_i$ by

$$
a_{\pm i} = \frac{A_{\pm i} + q_i A_{\mp i}}{1 + q_i},
$$


$$
q_i = \frac{\rho_i}{(A_i - A_{-i})^2},
$$

(15)

where $|q_i| < 1$ is again related to the Catalan series (13)

$$
q_i = C(\chi_i), \quad \chi_i = -\frac{\rho_i}{(A_i - A_{-i})^2}.
$$

The canonical sewing condition can then be rewritten as a standard Schottky sewing condition:

$$
\left( \frac{z - a_{-i}}{z - a_i} \right) \left( \frac{z' - a_i}{z' - a_{-i}} \right) = q_i.
$$

(16)

The Schottky sewing condition (16) determines a Möbius map $z' = \gamma_i(z)$ where

$$
\gamma_i = \sigma_i^{-1} \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix} \sigma_i,
$$

(17)
for Möbius map
\[ \sigma_i(z) = \frac{z - a_i}{z - a_{-i}}. \] (18)

We define the Schottky group \( \Gamma = \langle \gamma_i \rangle \) as the Kleinian group freely generated by \( \gamma_i \) for \( i = 1, \ldots, g \).

One can find explicit formulas for various objects defined on the Riemann surface such as the bilinear form of the second kind, a basis of \( g \) holomorphic 1-forms and the genus \( g \) period matrix in terms of either the Canonical or Schottky parametrizations [TZ]. In the Schottky case, these involve sums or products over the Schottky group or subsets thereof.

### 4 Vertex Operator Algebras

Consider a simple VOA with \( \mathbb{Z} \)-graded vector space \( V = \oplus_{n \geq 0} V^{(n)} \) and local vertex operators \( Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \) for \( a \in V \) e.g. [Ka],[FLM],[MN],[MT3]. We assume that \( V \) is of CFT type (i.e. \( V_0 = \mathbb{C}1 \)) with a unique symmetric invertible invariant bilinear form \( \langle , \rangle \) with normalization \( \langle 1, 1 \rangle = 1 \) where [FHL],[Li]
\[ \langle Y(a, z)b, c \rangle = \langle b, Y(e^{z L_1}(-\frac{1}{z^2}) L_0 a, \frac{1}{z}) c \rangle \] (19)

For a \( V \)-basis \( \{u^a\} \), we let \( \{\pi^a\} \) denote the dual basis. If \( a \in V^{(k)} \) is quasi-primary \( (L_1 a = 0) \) then (19) implies
\[ \langle a_n b, c \rangle = (-1)^k \langle b, a_{2k-n-2} c \rangle. \]

In particular:
\[ \langle a_n b, c \rangle = -\langle b, a_{-n} c \rangle \text{ for } a \in V^{(1)} \]
\[ \langle L_n b, c \rangle = \langle b, L_{-n} c \rangle \text{ for } \omega \in V^{(2)}, \] (20)
so that \( b, c \) with unequal weights are orthogonal.

#### 4.1 Genus Zero Correlation Functions

For \( u_1, u_2, \ldots, u_n \in V \) define the \( n \)-point (correlation) function by
\[ \langle 1, Y(u_1, z_1)Y(u_2, z_2) \ldots Y(u_n, z_n)1 \rangle. \] (21)
The locality property of vertex operators implies that this formal expression (21) coincides with the analytic expansion of a rational function of $z_1, z_2, \ldots, z_n$ in the domain $|z_1| > |z_2| > \ldots > |z_n|$. Thus the $n$-point function can taken to be a rational function of $z_1, z_2, \ldots, z_n \in \hat{\mathbb{C}}$, the Riemann sphere in the domain. For example [HT]

**Theorem 4.1** For a VOA of central charge $C$, the Virasoro $n$-point function is a $\beta$-extended permanent

$$
\langle 1, Y(\omega, z_1) \ldots Y(\omega, z_n) 1 \rangle = \text{perm}_{C} B,
$$

for $B_{ij} = \frac{1}{(z_i - z_j)^2}$, $i \neq j$ and $B_{ii} = 0$.

### 4.2 Rank Two Heisenberg VOA $M_2$

Consider the VOA generated by two Heisenberg vectors $a^\pm \in V^{(1)}$ whose modes satisfy non-trivial commutator

$$
[a_m^+, a_n^-] = m\delta_{m,-n}. \tag{22}
$$

$V$ has a Fock basis spanned by

$$
a_{k,l} = a_{-k_1}^+ \ldots a_{-k_m}^+ a_{-l_1}^- \ldots a_{-l_n}^- 1, \tag{23}
$$

labelled by multisets $k = \{k_1, \ldots, k_m\} = \{1^{r_1}, 2^{r_2}, \ldots\}$ and $l = \{l_1, \ldots, l_n\} = \{1^{s_1}, 2^{s_2}, \ldots\}$. The Fock vectors are orthogonal with respect to to the invariant bilinear form with dual basis

$$
\overline{a}_{k,l} = \prod_i \frac{1}{i^{r_i} r_i!} \prod_j \frac{1}{j^{s_j} s_j!} a_{l,k}. \tag{24}
$$

The basic Heisenberg 2-point function is

$$
\langle 1, Y(a^+, x) Y(a^-, y) 1 \rangle = \frac{1}{(x - y)^2}. \tag{25}
$$

This function provides all the necessary data for computing the Heisenberg partition and correlation functions on a genus $g$ surface! Thus the general rank 2 Heisenberg $2n$-point function is

$$
\langle 1, Y(a^+, x_1) \ldots Y(a^+, x_n) Y(a^-, y_1) \ldots Y(a^-, y_n) 1 \rangle = \text{perm} \left( \frac{1}{(x_i - y_j)^2} \right). \tag{26}
$$
This is a generating function for all rank two Heisenberg correlation functions by associativity of the VOA.

Let \( x_{-i} = x - A_{-i} \) and \( y_j = y - A_j \) be local coordinates in the neighborhood of canonical sewing parameters \( A_{-i}, A_j \) for \( i, j \in \{ \pm 1, \ldots, \pm g \} \) with \( i \neq -j \).

The 2-point function has expansion

\[
\frac{1}{(x - y)^2} = \sum_{k, l \geq 1} (-1)^{k+1} \frac{(k + l - 1)!}{(k-1)!(l-1)!} \frac{x_{-i}^{k-1} y_j^{l-1}}{(A_{-i} - A_j)^{k+l}}.
\]

Define the canonical moment matrix \( R^{\text{Can}} \), an infinite matrix indexed by \( k, l = 1, 2, \ldots \) and \( i, j \in \{ \pm 1, \ldots, \pm g \} \) with \( i \neq -j \).

\[
R^{\text{Can}}_{ij}(k, l) = \begin{cases} \frac{(-1)^k \sqrt{k/2} \rho_i^{l/2} (k+l-1)!}{(A_{-i} - A_j)^{k+l}}, & i \neq -j \\ 0, & i = -j \end{cases}
\]  

(30)

\((I - R^{\text{Can}})^{-1}\) plays a central role in computing the genus \( g \) period matrix and other structures.

We similarly have expansions in the Schottky parameters. Let

\[
x_{-i} = \sigma_{-i}(x) = \frac{x - a_{-i}}{x - a_i} \\
y_j = \sigma_j(x) = \frac{y - a_j}{y - a_{-j}}
\]

for \( i, j \in \{ 1, \ldots, g \} \) be local coordinates in the neighborhood of the Schottky points \( a_{-i} \) and \( a_j \) for \( i \neq -j \). The 2-point function expansion leads to the Schottky moment matrix with

\[
R^{\text{Sch}}_{ij}(k, l) = \begin{cases} q_i^{k/2} q_j^{l/2} D(k, l)(\sigma_i \sigma_j^{-1}), & i \neq -j \\ 0, & i = -j \end{cases}
\]

where for \( \gamma \in SL(2, \mathbb{C}) \)

\[
D(k, l)(\gamma) = \frac{1}{l!} \sqrt{\frac{l}{k}} \partial_z^l (\gamma(z)^k) |_{z=0}.
\]

D is an \( SL(2, \mathbb{C}) \) representation [Mo]. Then it follows

\[
\sum_{s \geq 1} R^{\text{Sch}}_{ij}(r, s) R^{\text{Sch}}_{jk}(s, t) = q_i^{r/2} q_k^{t/2} D(r, t)(\sigma_i \gamma_j \sigma_k^{-1}),
\]

for Schottky generator (17).
4.3 The Genus $g$ Partition Function - Canonical Parameters

We now define the genus $g$ partition function for a VOA $V$ in the canonical sewing scheme in terms of genus zero $2g$-point correlation functions as follows:

$$Z_V^{(g)}(\rho_i, A_{\pm i}) = \langle 1, \prod_{i=1}^g \sum_{n_i \geq 0} \rho_i^{n_i} \sum_{v_i \in V(n)} Y(v_i, A_{-i})Y(\tilde{v}_i, A_i)1 \rangle,$$

(33)

where $\tilde{v}_i$ is dual to $v_i$.

For genus one this reverts to the standard definition:

**Theorem 4.2 (Mason and T.)**

$$Z_V^{(1)}(\rho, A_{\pm 1}) = \text{Tr}_V(q^{L_0})$$

where $q = C(\chi)$, the Catalan series for $\chi = -\frac{\rho}{(A_{-1} - A_1)^2}$.

4.4 $Z_{M_2}^{(g)}(\rho_i, A_{\pm i})$ for Heisenberg VOA $M_2$

The genus $g$ partition function can be computed for the rank 2 Heisenberg VOA by means of the MacMahon Master Theorem where, schematically, we have:

- Sum over $g$ Fock bases $\rightarrow$ Sum over multisets
- $2g$-point function $\rightarrow$ Permanent of matrix
- Dual vector factorials $\rightarrow$ Multiset factorials
- $\rho_i$ and other dual vector factors $\rightarrow$ Absorbed into matrix definition

We then find that [TZ]

**Theorem 4.3**

$$Z_{M_2}^{(g)}(\rho_i, A_{\pm i}) = \frac{1}{\det(I - R_{\text{Can}})},$$

where $R_{\text{Can}}$ is the canonical moment matrix. Furthermore, $\det(I - R_{\text{Can}})$ is holomorphic and non-vanishing. In general, the genus $g$ Heisenberg generating function is expressed in terms of a permanent of genus $g$ bilinear forms of the second kind.
We may repeat this by using an alternative definition of the genus \( g \) partition function in terms of in Schottky parameters account must be taken of the Möbius maps \( \sigma_i \) of (18). We then find [TZ]

**Theorem 4.4** The genus \( g \) partition function is

\[
Z_{M_2}^{(g)}(q_i, a_{\pm i}) = \frac{1}{\det(I - R^{Sch})},
\]

where \( R^{Sch} \) is the Schottky moment matrix. Furthermore, \( \det(I - R^{Sch}) \) is holomorphic and non-vanishing and the genus \( g \) Heisenberg generating function is expressed in terms of a permanent of genus \( g \) bilinear forms of the second kind.

**Conjecture:** \( \det(I - R^{Can}) = \det(I - R^{Sch}) \). This is true for \( g = 1 \) [MT2].

### 4.5 The Montonen-Zograf Product Formula

\( \det(I - R^{Sch}) \) can be also re-expressed in terms of an infinite product formula originally calculated in physics by Montonen in 1974 [Mo]. A similar product formula was subsequently found by Zograf [Z]. This has been recently related by McIntyre and Takhtajan [McT] to Mumford’s theorem concerning the absence of a global section on moduli space for the canonical line bundle [Mu].

Recall that \( R^{Sch}_{ij}(k, l) \) is expressed in terms of an \( SL(2, \mathbb{C}) \) representation \( D \). This leads to

\[
\det(I - R^{Sch}) = \prod_{m \geq 1} \prod_{\gamma^\alpha \in \Gamma} (1 - q^m_{\alpha}),
\]

where the inner product ranges over the primitive elements \( \gamma^\alpha \) of the Schottky group \( \Gamma \) i.e. \( \gamma^\alpha \neq \gamma^k \) for any \( \gamma \in \Gamma \) for \( k > 1 \). Each such element has a multiplier \( q^m_{\alpha} \) where

\[
\gamma^\alpha \sim \begin{pmatrix} q^m_{\alpha} & 0 \\ 0 & 1 \end{pmatrix}.
\]
References


