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Some results on finite amplitude elastic waves propagating in rotating media

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Abstract

Two questions related to elastic motions are raised and addressed. First: in which theoretical framework can the equations of motion be written for an elastic half-space put into uniform rotation? It is seen that nonlinear finite elasticity provides such a framework for incompressible solids. Second: how can finite amplitude exact solutions be generated? It is seen that for some finite amplitude transverse waves in rotating incompressible elastic solids with general shear response, the solutions are obtained by reduction of the equations of motion to a system of ordinary differential equations equivalent to the system governing the central motion problem of classical mechanics. In the special case of circularly-polarized harmonic progressive waves, the dispersion equation is solved in closed form for a variety of shear responses, including nonlinear models for rubberlike and soft biological tissues. A fruitful analogy with the motion of a nonlinear string is pointed out.

1 Introduction

The propagation of elastic waves in rotating media has been a subject of continuous interest in the last three decades or so. Ever since the publication of a seminal article by Schoenberg and Censor [1], numerous workers have studied how uniform rotation affects time-dependent solutions to the governing equations (pointers to such studies can be found in recent articles on waves in rotating media such as Refs. [2–5].) The starting point of these studies is the inclusion of the Coriolis and centrifugal accelerations into the equations of motion:

$$\text{div } \mathbf{T} = \rho \ddot{\mathbf{y}} + 2\rho \mathbf{\Omega} \times \dot{\mathbf{y}} + \rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{y}). \tag{1}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, ρ is the mass density, $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ denotes the current position of a particle in the material initially at $\mathbf{x}$ in the reference configuration, and $\mathbf{\Omega}$ is the constant rotation rate vector. Also, a dot denotes differentiation with respect to time t in a *fixed* (non-rotating) frame; in other words, if $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is one such frame, then $\mathbf{y} = y_i \mathbf{e_i}$ and $\dot{\mathbf{y}} := (\partial y_i / \partial t)\mathbf{e_i}$.

The second term on the right hand-side of Eq. (1) is the Coriolis force and the third term is the centrifugal force. This latter term is the source of an obvious concern in a linearly elastic material with infinite dimension(s) because it grows linearly with the distance between the particle and the axis of rotation. Most (and perhaps all) previous works on the subject have dealt with this potential problem simply by focusing on the so-called "time-dependent" part of the equations of motion. In this approach, the solution $\mathbf{y}$ is split into a "time-independent" part and a "time-dependent" part as $\mathbf{y}(\mathbf{x}, t) = \mathbf{y}^{\mathbf{s}}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t)$ (say). Then, the constitutive equation of the elastic material being linear, the Cauchy stress can also be split: $\mathbf{T}(\mathbf{x}, t) = \mathbf{T}^{\mathbf{s}}(\mathbf{x}) + \boldsymbol{\sigma}(\mathbf{x}, t)$ (say) and the linearization of the equations of motion allows for the *separate* resolution of a time-independent problem and of a time-dependent problem,

$$\text{div } \mathbf{T}^{\mathbf{s}}(\mathbf{x}) = \rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{y}^{\mathbf{s}}(\mathbf{x})), \tag{2}$$

$$\text{div } \boldsymbol{\sigma}(\mathbf{x}, t) = \rho \ddot{\mathbf{u}}(\mathbf{x}, t) + 2\rho \mathbf{\Omega} \times \dot{\mathbf{u}}(\mathbf{x}, t) + \rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}(\mathbf{x}, t)). \tag{3}$$

Although the resolution of Eq. (3) has generated a wealth of results in a variety of contexts, the resolution of Eq. (2) seems to have been left aside, at least as long as potentially infinite distances from the rotation axis are involved. This paper aims at providing a context in which the *global* equations

of motion in a rotating elastic media Eq. (1), possibly inclusive of finite strain effects and of a nonlinear constitutive equation, can be posed and solved.

Because large strains might appear in a rotating elastic solid, we place ourselves in the framework of finite nonlinear elasticity. We focus on materials subject to the internal constraint of incompressibility, first because many actual materials with a nonlinear elastic response such as rubber or biological soft tissue can be considered to be incompressible, and second because the inherent introduction of an arbitrary scalar quantity (the "pressure") leads to an immediate simplification of the equations of motion Eq. (1). Indeed, as we show in the next Section, the arbitrariness of the $p\mathbf{1}$ term in the constitutive equation of an incompressible body allows for the centrifugal force to be absorbed by this pressure term. Once this manipulation is done, the resolution of the equations of motion can be conducted quite naturally. As noted by Schoenberg and Censor [1], two features characterize waves in rotating bodies as opposed to waves in non-rotating bodies: a new direction of anisotropy (linked to the rotation axis) and more dispersion (linked to the rotation frequency). To illustrate these features, we revisit some classic results on finite amplitude elastic motion due to Carroll [6–9] and extend them to the case of a body in rotation.

The exact solutions of Carroll are versatile in their fields of application because they are valid not only for nonlinearly elastic solids, but also for viscoelastic solids [10], Reiner-Rivlin fluids [10,11], Stokesian fluids [10], Rivlin-Ericksen fluids [11], liquid crystals [12], dielectrics [13], magnetic materials [14], etc. They also come in a great variety of forms, as circularly-polarized harmonic progressive waves, as motions with sinusoidal time dependence, as motions with sinusoidal space dependence, etc. In our revisiting his findings, we note a striking analogy between the equations of motion obtained for a motion general enough to include all of the above motions, and the equations obtained in the problem of the motion of a nonlinear string, as considered by Rosenau and Rubin [15]. Then we show how the method of [15] can be used to derive all (and more) of the different results obtained by Carroll, which turn out to be a direct consequence of material isotropy and of the Galilean invariance of the field equations.

The paper is organized in the following manner. In the next Section the basic equations for motions in a rotating nonlinearly elastic incompressible solid and their specialization to finite amplitude transverse waves are given. In Section 3 we recast the determining equations in a general complex form and we show that they admit some special separable solutions. In Section

4 we investigate in detail the case of circularly-polarized harmonic progressive waves. We give the dispersion relation and solve it for Mooney-Rivlin materials and for some other strain energy density functions relevant to the modelling of rubberlike materials (some of these results are new even in the non-rotating case). Next we show that in rotating solids, motions with a sinusoidal time dependence (Section 5) and motions with a sinusoidal spatial dependence (Section 6) are determined by solving a reduced system of ordinary differential equations, equivalent to that of a central motion problem. The main difference with Carroll's results for the non-rotating case is that, for special values of the angular velocity, the central force may be repulsive; this possibility is ruled out in the non-rotating case by the empirical inequalities [16].

2 Preliminaries

2.1 Equations of motion in a rotating elastic solid

Let the initial and current coordinates of a point of the body, referred to the same fixed rectangular Cartesian system of axes, be denoted by x_i and y_i, respectively, where the indices take the values 1, 2, 3. A motion of the body is defined by

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, t). \tag{4}$$

The response of a homogeneous isotropic incompressible elastic solid to deformations from an undistorted reference configuration is described by the constitutive relation,

$$\mathbf{T} = -\widetilde{p}\mathbf{1} + \alpha\mathbf{B} - \beta\mathbf{B}^{-1}, \tag{5}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{1}$ is the unit tensor, and $\mathbf{B}$ is the left Cauchy-Green strain tensor, defined by

$$\mathbf{B} := \mathbf{F}\mathbf{F}^T, \tag{6}$$

$\mathbf{F} := \partial\mathbf{y}/\partial\mathbf{x}$ being the deformation gradient tensor. Also in Eq. (5), $\widetilde{p}$ is an arbitrary scalar function associated with the internal constraint of incompressibility

$$\det\mathbf{F} = 1, \tag{7}$$

to be determined from the equations of motion and eventual boundary/initial conditions. The response parameters α and β are functions of the first and

second invariants of $\mathbf{B}$: $\alpha = \alpha(I, II)$, $\beta = \beta(I, II)$, where

$$I = \text{tr } \mathbf{B}, \quad II = \text{tr } \mathbf{B}^{-1}. \tag{8}$$

For a hyperelastic material, a strain energy density per unit of volume $W = W(I, II)$ is defined and α, β are given by

$$\alpha = 2\frac{\partial W}{\partial I}, \quad \beta = 2\frac{\partial W}{\partial II}. \tag{9}$$

Now we consider that the elastic medium rotates with a uniform rotation vector $\mathbf{\Omega}$, about a given axis. In the absence of body forces, the equations of motions relative to a rotating frame (see for instance [17, pp.60–61]) are given by Eq.(1). Using the constitutive equation Eq. (5), we obtain

$$-\text{ grad } \widetilde{p} + \text{div } (\alpha\mathbf{B} - \beta\mathbf{B}^{-1}) = \rho\ddot{\mathbf{y}} + 2\rho\mathbf{\Omega} \times \dot{\mathbf{y}} + \rho\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{y}). \tag{10}$$

Now write $\widetilde{p}$ in the form

$$\widetilde{p} = p - \tfrac{1}{2}\rho[\mathbf{\Omega}\times(\mathbf{\Omega} \times \mathbf{y})]\cdot\mathbf{y}, \tag{11}$$

where $p = p(\mathbf{x}, t)$ is yet another arbitrary pressure scalar. Then Eq. (10) reduces to

$$-\text{ grad } p + \text{div } (\alpha\mathbf{B} - \beta\mathbf{B}^{-1}) = \rho\ddot{\mathbf{y}} + 2\rho\mathbf{\Omega} \times \dot{\mathbf{y}}. \tag{12}$$

Hence the equations of motion can be tackled independently of the centrifugal acceleration, which does not appear here. Once Eqs.(12) are solved, the solution $\mathbf{y}$ will lead to a pressure field $\widetilde{p}$ given by Eq. (11) which does depend on the centrifugal force.

2.2 Finite amplitude shearing motions

Following Carroll [6], we study for the remainder of the paper the propagation of plane transverse waves in a bi-axially deformed incompressible material. Thus we consider the following class of shearing motions,

$$y_1 = \mu x_1 + u(z, t), \quad y_2 = \mu x_2 + v(z, t), \quad y_3 = \lambda x_3 =: z, \tag{13}$$

that is, a transverse wave polarized in the $(x_1 x_2)$ plane and propagating in the x_3-direction of a material subject to a pure homogeneous pre-stretch with

constant principal stretch ratios μ, μ, λ ($\mu^2\lambda = 1$) in the x_1, x_2, x_3 directions, respectively. For these motions, we find

$$\mathbf{B} = \begin{bmatrix} \mu^2 + \lambda^2 u_z^2 & & \\ \lambda^2 u_z v_z & \mu^2 + \lambda^2 v_z & \\ \lambda^2 u_z & \lambda^2 v_z & \lambda^2 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} \lambda & & \\ 0 & \lambda & \\ -\lambda u_z & -\lambda v_z & \lambda(u_z^2 + v_z^2) + \mu^4 \end{bmatrix}. \tag{14}$$

Here and henceforward, a subscript letter denotes partial differentiation (i.e. $u_z := \partial u/\partial z$, $v_{tt} := \partial^2 v/\partial t^2$, etc.) It follows from Eq. (8) that

$$I = 2\mu^2 + \lambda^2(1 + u_z^2 + v_z^2), \quad II = \mu^4 + \lambda(2 + u_z^2 + v_z^2), \tag{15}$$

so that both invariants, and consequently the response parameters α, β, are functions of $u_z^2 + v_z^2$ alone,

$$\alpha = \alpha(u_z^2 + v_z^2), \quad \beta = \beta(u_z^2 + v_z^2). \tag{16}$$

Then the equations of motion Eq. (12) read

$$\begin{aligned} &- p_{y_1} + (Qu_z)_z = \rho(u_{tt} - 2\Omega_3 v_t), \\ &- p_{y_2} + (Qv_z)_z = \rho(v_{tt} + 2\Omega_3 u_t), \\ &- p_z + [\alpha\lambda^2 + \beta\mu^4 + \beta\lambda(u_z^2 + v_z^2)]_z = 2\rho(\Omega_1 v_t - \Omega_2 u_t), \end{aligned} \tag{17}$$

where the function $Q = Q(u_z^2 + v_z^2)$ is defined by

$$Q := \alpha\lambda^2 + \beta\lambda. \tag{18}$$

By inspection of Eqs. (17), we find that p can be taken in the form

$$p = p(z,t) = \alpha\lambda^2 + \beta\mu^4 - \beta\lambda(u_z^2 + v_z^2) - 2\rho\textstyle\int(\Omega_1 v_t - \Omega_2 u_t)\mathrm{d}z. \tag{19}$$

Then Eq. $(17)_3$ is satisfied and Eqs. $(17)_{1,2}$ reduce to

$$(Qu_z)_z = \rho(u_{tt} - 2\Omega_3 v_t), \quad (Qv_z)_z = \rho(v_{tt} + 2\Omega_3 u_t). \tag{20}$$

Eqs. (20) form a system of two coupled nonlinear hyperbolic partial differential equations, generalizing the system derived by Carroll in [6] for a non-rotating body.

3 Separable solutions

3.1 Link with another problem (string motion)

By inspection of the system Eqs. (20), an analogy can be drawn with the system of equations governing the motion of a nonlinear string, as treated by Rosenau and Rubin [15]. Indeed, if the position of a particle in a string is denoted by the rectangular Cartesian coordinates $x(\xi, t)$, $y(\xi, t)$, where ξ is a curvilinear abscissa, then the equations of motion of the string can be put in the form,

$$[(T/a)x_\xi]_\xi = \rho_0(x_{tt} - f_1), \quad [(T/a)y_\xi]_\xi = \rho_0(y_{tt} - f_2). \tag{21}$$

Here, T is the internal tension in the string (acting along the tangent to the string curve), f_1 and f_2 are the components of the body force per unit mass, $\rho_0 = \rho_0(\xi)$ is the mass density, and a is the metric associated with the stretch of the string: $a = \sqrt{x_\xi^2 + y_\xi^2}$. Finally, a constitutive equation $T = T(a)$ for the internal tension characterizes a the string material.

The similarity between the two systems Eqs. (20) and Eqs. (21) is striking. Accordingly we now adapt the analysis devised by Rosenau and Rubin [15] for a nonlinear string to our system of governing equations.

3.2 Separation of variables

Seeking some exact solutions, we follow Rosenau's and Rubin's [15] steps. First we differentiate Eqs. (20) with respect to z, and obtain

$$[QU]_{zz} = \rho(U_{tt} - 2\Omega_3 V_t), \quad [QV]_{zz} = \rho(V_{tt} + 2\Omega_3 U_t), \tag{22}$$

where $U := u_z$ and $V := v_z$. Next, we define the complex function Z as

$$Z(z, t) = \eta(z, t)\mathrm{e}^{i\xi(z,t)} := U + iV, \tag{23}$$

so that

$$U = \Re(Z) = \eta \cos \xi, \quad V = \Im(Z) = \eta \sin \xi. \tag{24}$$

Then, we rewrite the system Eqs. (22) as a single complex equation,

$$[Q(\eta^2)Z]_{zz} = \rho(Z_{tt} + 2i\Omega_3 Z_t). \tag{25}$$

To reduce further this equation to a set of ordinary differential equations, we look for a class of solutions admitting the separable forms:

$$\eta(z,t) = \eta_1(z)\eta_2(t), \quad \xi(z,t) = \xi_1(z) + \xi_2(t), \tag{26}$$

where η_1 and ξ_1 (η_2 and ξ_2) are functions of space (time) only. Then Eq. (25) can be cast in the form

$$\frac{[Q(\eta_1^2\eta_2^2)\eta_1\mathrm{e}^{i\xi_1}]_{zz}}{\eta_1\mathrm{e}^{i\xi_1}} = \rho\frac{(\eta_2\mathrm{e}^{i\xi_2})'' + 2i\Omega_3(\eta_2\mathrm{e}^{i\xi_2})'}{\eta_2\mathrm{e}^{i\xi_2}}, \tag{27}$$

where the prime denotes differentiation with respect to the argument of a single-variable function.

Rosenau and Rubin [15] noted that a sufficient condition to ensure complete separation of time functions from space functions in this equation is that the material response function Q be itself separable. Indeed if

$$Q(\eta_1^2\eta_2^2) = Q_1(\eta_1^2)Q_2(\eta_2^2), \tag{28}$$

(say) then we end up with the two ordinary differential equations,

$$\begin{aligned} &[Q_1(\eta_1^2)\eta_1\mathrm{e}^{i\xi_1}]'' = h\eta_1\mathrm{e}^{i\xi_1}, \\ &\rho[(\eta_2\mathrm{e}^{i\xi_2})'' + 2i\Omega_3(\eta_2\mathrm{e}^{i\xi_2})'] = hQ_2(\eta_2^2)\eta_2\mathrm{e}^{i\xi_2}, \end{aligned} \tag{29}$$

for some constant h.

The separation condition Eq. (28) is however rather strong and might be fulfilled only for very specific constitutive equations. Another possibility, not mentioned by Rosenau and Rubin, for the separation of space functions from time functions arises when either $\eta_1(z)$ or $\eta_2(t)$ are constant functions (independent of their argument). Hence, when $\eta_1 = k_1$ (say), Eq. (27) yields

$$(\mathrm{e}^{i\xi_1})'' = h\mathrm{e}^{i\xi_1}, \quad \rho[(\eta_2\mathrm{e}^{i\xi_2})'' + 2i\Omega_3(\eta_2\mathrm{e}^{i\xi_2})'] = hQ(k_1^2\eta_2^2)\eta_2\mathrm{e}^{i\xi_2}, \tag{30}$$

and when $\eta_2 = k_2$ (say), it yields

$$[Q(k_2^2\eta_1^2)\eta_1\mathrm{e}^{i\xi_1}]'' = h\eta_1\mathrm{e}^{i\xi_1}, \quad \rho[(\mathrm{e}^{i\xi_2})'' + 2i\Omega_3(\mathrm{e}^{i\xi_2})'] = h\mathrm{e}^{i\xi_2}. \tag{31}$$

The conditions η_1 =const. or η_2 =const. do not impose any restriction on the strain energy function. Thus, the solutions to Eqs. (30) or Eqs. (31) are valid for any type of material, in contrast to the solutions to Eqs. (29), which require Eq. (28) to be satisfied.

For instance, consider the solution

$$Z(z,t) = [\psi(t) + i\phi(t)]k\mathrm{e}^{i(kz+\theta(t))}, \tag{32}$$

where k is a constant and ψ, ϕ, θ are arbitrary real functions of time. A simple check shows that Z is indeed of the form given by Eqs. (23) and Eqs. (26), with the following identifications: $\eta_1(z) = k = \text{const.}$, $\eta_2(t) = [\phi^2 + \psi^2]^{\frac{1}{2}}$, $\xi_1(z) = kz$, and $\xi_2(t) = \theta + \tan^{-1}(\phi/\psi)$. Once the ordinary differential equations Eqs. (30) are solved, the displacement field is given by

$$\begin{aligned} u(z,t) &= \phi(t)\cos(kz + \theta(t)) + \psi(t)\sin(kz + \theta(t)), \\ v(z,t) &= \phi(t)\sin(kz + \theta(t)) - \psi(t)\cos(kz + \theta(t)). \end{aligned} \tag{33}$$

On the other hand, consider the solution

$$Z(z,t) = [(i\phi(z) + \psi(z))\theta'(z) + (\phi'(z) - i\psi'(z))]\mathrm{e}^{i(\omega t+\theta(z))}, \tag{34}$$

where k is a constant and ψ, ϕ, θ are arbitrary functions of space. Here Z is of the form given by Eqs. (23) and Eqs. (26), with the identifications $\eta_1(z) = [(\phi'+\psi\theta')^2+(\phi\theta'-\psi')^2]^{\frac{1}{2}}$, $\eta_2(t) = 1 = \text{const.}$, $\xi_1(z) = \theta+\tan^{-1}[(\phi\theta'-\psi')/(\psi\theta' + \phi')]$, and $\xi_2(t) = \omega t$. Once the ordinary differential equations Eqs. (31) are solved, the displacement field is given by

$$\begin{aligned} u(z,t) &= \phi(z)\cos(\omega t + \theta(z)) + \psi(z)\sin(\omega t + \theta(z)), \\ v(z,t) &= \phi(z)\sin(\omega t + \theta(z)) - \psi(z)\cos(\omega t + \theta(z)). \end{aligned} \tag{35}$$

The two sets of displacement fields Eqs. (33) and Eqs. (35) provide a great variety of possible finite amplitude motions, valid in every deformed incompressible nonlinearly elastic solid. They are inclusive of the solutions discovered and analyzed by Carroll over the years. Thus the motion Eqs. (33) written at $\psi(t) = 0$ corresponds to the "oscillatory shearing motions" treated in [7]; the motion Eqs. (35) written at $\psi(z) = 0$ corresponds to the "motions with time-independent invariants" treated in [7]; the motion Eqs. (35) written at $\theta(z) = 0$ corresponds to the "motions with sinusoidal time dependence" or "finite amplitude circularly-polarized standing waves" treated in [8,9]; the motion Eqs. (33) written at $\phi(z) =$const., $\psi(z) = 0$, $\theta(z) = -kz$, or equivalently the motion Eqs. (35) written at $\phi(t) =$const., $\psi(t) = 0$, $\theta(t) = -\omega t$, corresponds to the celebrated finite-amplitude circularly-polarized harmonic progressive waves of [6].

Before we consider in turn each of these finite-amplitude motions for a rotating body, we sum up the main results established in this Section. We used a formalism proposed by Rosenau and Rubin [15] for the plane motion of a nonlinear string to derive separable solutions to the equations of motion of a deformed rotating solid in which finite-amplitude shearing motions might propagate. In the process, we noticed that two classes of solutions, not considered by Rosenau and Rubin, were valid for any form of the strain energy function. Each class provided solutions which generalize those proposed by Carroll [6–14] and which put them into a wider context. On the other hand, the complex formalism makes it clear that the solutions considered here are related to natural symmetry properties of the coupled wave equations Eqs. 22. These properties are natural because they come out from material symmetries and frame indifference requirements [16]. We refer to the works of Olver [18] and of Vassiliou [19] for further information on the application of group analysis to coupled wave equations.

4 Circularly-polarized harmonic waves

First we consider a finite amplitude circularly-polarized harmonic progressive wave propagating in the z-direction,

$$u(z,t) = A\cos(kz-\omega t), \quad v(z,t) = \pm A\sin(kz-\omega t), \tag{36}$$

which is a subcase of Eqs. (33) or of Eqs. (35). Here the amplitude A, the wave number k, and the frequency ω are real positive constants, and the plus (minus) sign for $v(z,t)$ corresponds to a left (right) circularly-polarized wave. For the choice of motion Eqs. (36), we have

$$u_z^2 + v_z^2 = A^2k^2, \tag{37}$$

and Eqs. (20) reduce to the following dispersion equation,

$$k^2 Q(A^2k^2) = \rho(\omega^2 \mp 2\Omega_3\omega). \tag{38}$$

The actual explicit form of the dispersion depends on a given constitutive equation. However we recall that, according to considerations by Carroll [6] pertaining to the non-rotating case, $k^2Q(A^2k^2)$ must be a positive, monotonically increasing function tending to infinity with k^2. It follows from the dispersion equation Eq. (38), that for a given left circularly-polarized wave,

the rotation rate Ω_3 has a cut-off frequency of $\omega/2$ and the wave does not exist for rotation rates Ω_3 beyond that cut-off frequency.

We now treat in turn three types of constitutive equations, which have proved useful for the modelling of some incompressible rubberlike and soft biological materials.

4.1 Waves in deformed Mooney-Rivlin materials

As a first illustration we consider a Mooney-Rivlin hyperelastic material, with strain energy density,

$$W_{\text{MR}} = C(I-3)/2 + D(II-3)/2, \tag{39}$$

where C and D are constants, satisfying [20] $C > 0$, $D \geq 0$ or $C \geq 0$, $D > 0$. It follows at once from Eqs. (9) that $\alpha = C$, $\beta = D$, and by Eq. (18), that Q is also independent of z. Introducing the speed c of circularly-polarized waves in a bi-axially deformed, non-rotating Mooney-Rivlin material [6, 20],

$$\rho c^2 := Q = C\lambda^2 + D\lambda, \tag{40}$$

we find that the dispersion equation Eq. (38) reads here,

$$c^2 k^2 = \omega^2 \mp 2\Omega_3 \omega. \tag{41}$$

From this equation we easily deduce the phase speed $v_\varphi := \omega/k$ and the group speed $v_g := \partial\omega/\partial k$, as well as their Taylor expansion to third-order for small ratios of the rotation rate Ω_3 with respect to the wave frequency ω. Introducing δ, the ratio of these two frequencies, $\delta := \Omega_3/\omega$, we find

$$\begin{aligned}
\frac{v_\varphi}{c} &= \frac{1}{\sqrt{1 \mp 2\delta}} = 1 \pm \delta + \tfrac{3}{2}\delta^2 + O(\delta^3), \\
\frac{v_g}{c} &= \frac{\sqrt{1 \mp 2\delta}}{1 \mp \delta} = 1 + \tfrac{1}{2}\delta^2 + O(\delta^3).
\end{aligned} \tag{42}$$

Clearly, the right circularly-polarized wave is defined for any value of the rotation rate whereas the left circularly-polarized wave only exists for a limited range of Ω_3, with $\omega/2$ as a cut-off frequency. Note also that a left circularly-polarized wave is accelerated when the Mooney-Rivlin material is put into rotation and that a right circularly-polarized wave is slowed down.

To investigate further nonlinear stress-strain responses, we consider two types of incompressible materials belonging to the class of 'neo-Hookean generalized materials'. These are materials whose strain-energy function depends only on the first invariant: $W = W(I)$. For simplicity, we consider that the solids are not prestressed ($\lambda = \mu = 1$) prior to the rotation and wave propagation although this assumption is not essential.

4.2 Waves in undeformed Gent materials

Consider the following strain energy density:

$$W_{\mathrm{G}} = -\frac{CJ_m}{2} \ln\left(1 - \frac{I-3}{J_m}\right), \tag{43}$$

where $C(> 0)$ is the infinitesimal shear modulus and J_m is a material parameter. Gent [21] introduced the strain energy function W_{G} to take into account the effect of the finite chain length for the macromolecular chains composing elastomeric materials (see also [22]). Hence, the parameter J_m has a physical interpretation: it is the constant limiting value for $I - 3$, and it reflects the mesoscopic finite chain length limiting effect. As $J_m \to \infty$, the limiting effect vanishes and the strain energy density Eq. (43) tends to that of a neo-Hookean solid (Eq. (39) with $D = 0$.)

For the motion considered in this Section, $I = 3 + A^2k^2$ and so, the limiting chain condition imposes $A^2k^2 < J_m$. From the strain energy density Eq. (43) we find that the response parameters α and β defined in Eqs. (9) are:

$$\alpha = C\frac{J_m}{J_m - A^2k^2}, \quad \beta = 0. \tag{44}$$

It follows from the definition Eq. (18) of Q, written at $\lambda = 1$, that the dispersion equation Eq. (38) reads, for finite-amplitude circularly-polarized harmonic waves in a rotating undeformed Gent material, as

$$C\frac{J_m}{J_m - A^2k^2}k^2 = \rho(\omega^2 \mp 2\Omega_3\omega). \tag{45}$$

Introducing $\delta := \Omega_3/\omega$, we find that the phase velocity $v_\varphi := \omega/k$ is given by

$$\rho v_\varphi^2 = \frac{CJ_m + \rho\omega^2A^2(1 \mp 2\delta)}{J_m(1 \mp 2\delta)}, \tag{46}$$

and is defined everywhere for the right wave and only below the cut-off frequency for the right wave. The group velocity, $v_g := \partial\omega/\partial k$, is found as

$$v_g = \frac{\rho v_\varphi^3}{C}\frac{(1 \mp 2\delta)^2}{1 \mp \delta}. \tag{47}$$

In contrast to the case of a Mooney-Rivlin material, the waves are also dispersive when the body is not rotating; then $\Omega_3 = 0$ and

$$v_\varphi = \sqrt{\frac{CJ_m + \rho\omega^2 A^2}{\rho J_m}}, \quad v_g = \frac{\rho v_\varphi^3}{C}. \tag{48}$$

These latter results are worth mentioning because in [6], Carroll treated explicitly only the case of Mooney-Rivlin materials. Moreover they may be used as benchmarks for an acoustical determination of the limiting chain parameter J_m. Acoustical evaluation is non-invasive and non-destructive, and is therefore appropriate for an estimation *in vivo* of J_m, whose numerical value can be linked to the ageing and stiffening of a soft biological tissue such as an arterial wall [22].

4.3 Waves in undeformed power-law materials

Now consider the following strain energy density,

$$W_\mathrm{K} = \frac{C}{b}\left[\left(1 + \frac{b}{n}(I-3)\right)^n - 1\right], \tag{49}$$

where $C(>0)$, b, and n are constitutive parameters. Knowles [23] proposed that this strain energy could account for *strain softening* when $n < 1$ and for *strain hardening* when $n > 1$. These effects have been observed for many real materials.

Here we find that the dispersion equation Eq. (38) is given by

$$C\left(1 + \frac{b}{n}A^2k^2\right)^{n-1} k^2 = \rho(\omega^2 \mp 2\Omega_3\omega). \tag{50}$$

Taking $n = 2$ in Eq. (49) as an example of strain energy for a hardening material, we find that the corresponding phase and group velocities are given

by

$$\rho v_\varphi^2 = \frac{C}{2(1 \mp 2\delta)} \left[1 + \sqrt{1 + 2\frac{\rho\omega^2}{C} bA^2(1 \mp 2\delta)} \right],$$

$$v_g = \frac{C}{\rho v_\varphi (1 \mp \delta)} \sqrt{1 + 2\frac{\rho\omega^2}{C} bA^2(1 \mp 2\delta)}. \tag{51}$$

The choice $n = \frac{1}{2}$ in Eq. (49) provides an example of strain energy for a softening material. As pointed out by Knowles [23], this choice is a borderline value for n, as the material is elliptic but not uniformly elliptic. We compute the corresponding phase speed as

$$\rho v_\varphi^2 = \frac{C}{1 \mp 2\delta} \sqrt{1 + \left[\frac{\rho\omega^2}{C} bA^2(1 \mp 2\delta) \right]^2} - \rho\omega^2 bA^2, \tag{52}$$

and we omit to display the group speed because its expression is too cumbersome.

5 Motions with sinusoidal time dependence

In this section we consider finite-amplitude shearing motions with a sinusoidal time-independence,

$$u(z,t) = \phi(z)\cos(\omega t) + \psi(z)\sin(\omega t), \quad v(z,t) = \phi(z)\sin(\omega t) - \psi(z)\cos(\omega t), \tag{53}$$

which are a subcase of Eqs. (35). For these solutions we have

$$u_z^2 + v_z^2 = \phi'^2 + \psi'^2, \tag{54}$$

and so the strain invariants Eq. (15) are spatially nonuniform and constant in time [7]. The governing equations Eqs. (20) reduce to

$$(Q\phi')' = \rho(\omega^2 + 2\Omega_3\omega)\phi, \quad (Q\psi')' = \rho(\omega^2 + 2\Omega_3\omega)\psi. \tag{55}$$

These equations are consistent at $\Omega_3 = 0$ with those derived by Carroll [8] . Following his lead, we reduce them to a problem in central force motion.

We introduce the functions $\Phi(z)$ and $\Psi(z)$ defined by

$$\Phi := Q\phi', \quad \Psi := Q\psi'. \tag{56}$$

We assume that these latter equalities are invertible as

$$\phi' = \nu\Phi, \quad \psi' = \nu\Psi, \tag{57}$$

where [8, 9] the generalized shear compliance ν (> 0) is a function of the shear stress σ, itself given by $\sigma^2 = \Phi^2 + \Psi^2$. For example, in the case of a bi-axially deformed Mooney-Rivlin material with strain energy Eq. (39), ν is constant: $\nu_{\text{MR}} = 1/(C\lambda^2 + D\lambda)$; in the case of an undeformed Gent material with strain energy Eq. (43), we find that ν is given by $\nu_{\text{G}} = (CJ_m/2\sigma^2)(\sqrt{1 + (4\sigma^2)/(C^2 J_m)} - 1)$. Note that Carroll [9] proposed expressions for ν when the strain-energy density is expanded up to sixth-order in the invariants $(I - 3)$ and $(II - 3)$.

Substitution of Eq. (57) into the derivative with respect to z of Eqs. (55) leads to the system of coupled ordinary differential equations,

$$\Phi'' - \rho(\omega^2 + 2\Omega_3\omega)\nu\Phi = 0, \quad \Psi'' - \rho(\omega^2 + 2\Omega_3\omega)\nu\Psi = 0. \tag{58}$$

This system is formally equivalent to the one governing the motion of a particle in a plane under a field of central forces, after identification of Φ and Ψ with the rectangular Cartesian coordinates and of z with time. The usual change of variables from rectangular Cartesian to polar coordinates,

$$\Phi = r\cos\theta, \quad \Psi = r\sin\theta, \tag{59}$$

leads to

$$r'' - r\theta'^2 = \rho(\omega^2 + 2\Omega_3\omega)\nu(r^2)r, \quad r\theta'' + 2r'\theta' = 0. \tag{60}$$

These equations coincide at $\Omega_3 = 0$ with those of Carroll [8]. Eq. $(60)_2$ is integrated as $r^2\theta' = A$, a constant. Substituting this new equation into Eq. $(60)_1$, multiplying across by r', and integrating yields

$$r'^2 + Ar^{-2} - \rho(\omega^2 + 2\Omega_3\omega)\textstyle\int \nu(s)ds = B, \tag{61}$$

another constant. For a further treatment and discussions on the interpretation of the solution to this equation, we refer to the papers by Carroll [7–11], at least as long as $\Omega_3 > -\omega/2$. We note that the nature of this equation and of its solutions is dramatically altered as Ω_3 tends to $-\omega/2$ and beyond, where it is reasonable to expect that (for example) what was a periodic solution to Eq. (61) for $\Omega_3 > -\omega/2$ has turned into an unbounded solution for $\Omega_3 < -\omega/2$ because then, the central force of Eq. $(60)_1$ is repulsive instead of attractive.

6 Motions with sinusoidal spatial dependence

Finally, we consider a plane wave motion with sinusoidal spatial variations,

$$u(z,t) = \phi(t)\cos(kz) + \psi(t)\sin(kz), \quad v(z,t) = \phi(t)\sin(kz) - \psi(t)\sin(kz). \tag{62}$$

This standing wave [8, 9] generalizes the superposition of two circularly-polarized wave propagating in opposite directions. It is a subcase of Eqs. (33).

Here,

$$u_z^2 + v_z^2 = k^2(\phi^2 + \psi^2), \tag{63}$$

so that I, II, α, β, and Q are independent of z. The governing equations Eqs. (20) reduce to the system of ordinary differential equations,

$$\rho\phi'' + 2\rho\Omega_3\psi' + k^2Q\phi = 0, \quad \rho\psi'' - 2\rho\Omega_3\phi' + k^2Q\psi = 0. \tag{64}$$

This system coincides at $\Omega_3 = 0$ with the system established by Carroll [8]. It is worth noting that the change of variables,

$$k\phi(z) = r(z)\cos(\theta(z) + \Omega_3 z), \quad k\psi(z) = r(z)\sin(\theta(z) + \Omega_3 z), \tag{65}$$

leads to a *modified* central field problem

$$r'' - r\theta^{'2} + [(k^2/\rho)Q(r^2) + \Omega_3^2]r = 0, \quad r\theta'' + 2r'\theta' = 0. \tag{66}$$

Again, integration of the second equation Eq. $(66)_2$ leads to $r^2\theta' = A$, a constant. Then, substitution into Eq. $(66)_1$, multiplication, and integration leads to

$$r^{'2} + Ar^{-2} + (k^2/\rho)\textstyle\int Q(s)ds + \Omega_3 r^2 = B, \tag{67}$$

another constant. Here the presence of rotation $\Omega_3 \neq 0$ always alters the nature of the solution with respect to non-rotating case.

7 Concluding remarks

Incompressible nonlinear elasticity provided a coherent framework where the equations of motion could be written in full, and possibly solved, for a rotating elastic body, without having to be split into a "time-dependent" solution and a a hypothetical "time-independent" solution. The internal constraint of incompressibility played a crucial role in the writing of these equations,

because the arbitrary pressure term can englobe the possibly troublesome centrifugal force.

As an illustration, the equations of motion were solved using the finite amplitude motions introduced and developped by Carroll in non-rotating elastic bodies. Because his solutions constitute one of the few examples of finite amplitude exact solutions, much emphasis was placed on how to derive them. In particular, it was shown how the search for separable solutions could recover and extend Carroll's results. For circularly polarized harmonic waves, the dispersion equation was derived explicitly and solved for the Mooney-Rivlin, Gent, and power-law strain energy functions. For motions with sinusoidal time dependence and for motions with sinusoidal space dependence, the procedure of reduction to a set of ordinary differential equations was outlined. Their eventual resolution can be adapted from Carroll's works, but is beyond the scope of this contribution.

The resolution of the full equations of motion in a rotating hyperelastic *compressible* material is also left open.

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