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Characteristic classes of complexified bundles

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Abstract
We examine the topological characteristic cohomology classes of complexified vector bundles\(^1\). In particular, all the classes coming from real vector bundles are computed. We use characteristic classes with the axioms of Milnor and Stasheff [6].

Introduction

Definition (Real generator bundles). We consider a real vector bundle \(F \to B\) and a complex vector bundle \(E \to B\) over the same base space \(B\). If the fibre-wise constructed complexification \(F \otimes_{\mathbb{R}} \mathbb{C} =: F^C\) is isomorphic to \(E\), we'll call \(F\) a real generator bundle of \(E\).

We want to attribute topological characteristic classes \(c(F)\) of the real generator bundles to the complexified bundles \(F^C\). Not every complex vector bundle admits a real generator bundle, as we shall see in a moment. So, supplementary cohomological information might be gathered when restricting attention to the subcategory of complex vector bundles that admit one.

Obstruction to real generator bundles. Consider a real vector bundle \(F \to B\). By reflection on the real axes given by \(F\), \(F^C\) is isomorphic to its complex conjugate bundle \(F^C\). So, any complex bundle \(E \to B\) that admits \(F\) as a real generator bundle must be isomorphic to its own conjugate bundle:

\[ E \cong F^C \cong F^\mathbb{C} \cong E. \]

The odd Chern classes \(c_{2k+1}\) have the property \(c_{2k+1}(E) = -c_{2k+1}(\bar{E})\) ([6, lemma 14.9]), so

\[ c_{2k+1}(E) = -c_{2k+1}(\bar{E}) = -c_{2k+1}(E) \in H^{4k+2}(B, \mathbb{Z}) \]

\(\Rightarrow 2c_{2k+1}(E) = 0\). Consequently, no complex vector bundle with some nonzero and non-torsion odd Chern class can admit a real generator bundle.

We are interested in all attributions of topological characteristic classes \(c(F)\) of the real generator bundles to the complexified bundles \(F^C\). For such an attribution to be well-defined, we need that real generator bundles \(F\), \(G\) of the same complex bundle provide the same class \(c(F) = c(G)\). For short, we get the Basic requirement \(F^C \cong G^C \Rightarrow c(F) = c(G)\).

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**Theorem 1.** Let \( c \) be a polynomial in the Stiefel-Whitney classes \( w_i \). Then the following two conditions are equivalent:

(i) \( c \) is an element of the polynomial sub-ring \( \mathbb{Z}_2[w_i^2] \), \( i \in \mathbb{N}_0 \)

(ii) \( c \) satisfies the basic requirement.

As the cohomology ring \( H^*(BO_n, \mathbb{Z}_2) \) of the classifying space \( BO_n \) is generated by the Stiefel-Whitney classes, this gives the entire information about modulo 2-classes. We find corresponding results for integral cohomology classes.

**Theorem 2.** The basic requirement holds for polynomials in \( V^2_{2i} \), with \( i \) arbitrary, \( V \in \{1, 2\} \) and the Pontrjagin classes \( p_i \).

For the convenience of the reader, we give a description of the integral cohomology classes \( V^2_{2i} \) in the appendix on page 12.

**Theorem 3.** Let \( C \in H^*(BO, \mathbb{Z}) \) be a characteristic class fulfilling the basic requirement. Then for any bundle \( \xi \), \( C(\xi) \) is completely determined by some Chern classes \( c_k(\xi^C) \), \( k \in \mathbb{N} \).

Using \( \mathbb{Z}_2 \)-coefficients

We’ll restrain ourselves to \( \mathbb{Z}_2 \)-coefficients in the following, in order to prove theorem 1. Let \( F \to B \) be a real vector bundle.

**Lemma 1.** A polynomial \( c = \sum \bigcup w_i \) in the Stiefel-Whitney classes fulfilling the basic requirement satisfies the **Transferred stable invariance** property:

\[ F^C \cong B \times \mathbb{C}^n \Rightarrow c(F \oplus G) = c(G). \]

**Proof.** Let \( c \) fulfill the basic requirement, and let \( F^C \cong B \times \mathbb{C}^n \). Let \( G \to B \) be a real bundle. Then \( c(F \oplus G) = c((B \times \mathbb{R}^n) \oplus G) \) because \((B \times \mathbb{R}^n)^C = B \times \mathbb{C}^n \cong F^C \), so the basic requirement can be applied. Thus, \( c(F \oplus G) = \sum \bigcup w_i((B \times \mathbb{R}^n) \oplus G) \) and with the stability [5, p. 81] due to the Whitney-sum axiom of the Stiefel-Whitney classes, this term equals \( \sum \bigcup w_i(G) = c(G). \) \[ \square \]

**Remark.** If the base space \( B \) is compact Hausdorff, transferred stable invariance of \( c \) provides the basic requirement.

**Proof.** Let \( F \to B, \ G \to B \) be real bundles with \( F^C \cong G^C \). Forgetting the complex structure, that’s \( F \oplus F \cong G \oplus G \). As \( B \) is compact Hausdorff, there is an inverse bundle \( F^{-1} \to B \), such that \( F \oplus F^{-1} \cong B \times \mathbb{R}^N \) for some \( N \). As seen in the last proof, \( c(F) = c(F \oplus (B \times \mathbb{R}^N)) \). And that’s, in turn, \( c(F \oplus F \oplus F^{-1}) = c(G \oplus G \oplus F^{-1}) \). Now,

\[ (G \oplus F^{-1})^C = G^C \oplus (F^{-1})^C \cong F^C \oplus (F^{-1})^C = (F \oplus (F^{-1}))^C \cong B \times \mathbb{C}^n. \]
That’s why we can apply the transferred stable invariance and obtain
\[ c(F) = c(G \oplus (G \oplus F^{-1})) = c(G). \]

□

Proof of theorem 1, (i)⇒(ii). Let \( F \to B, G \to B \) be real bundles with \( F^C \cong G^C \). Forgetting the complex structure, that’s \( F \oplus F \cong G \oplus G \). A consequence of working in \( \mathbb{Z}_2 \)-coefficients is that all terms that appear twice in a sum vanish, just like
\[
\sum_{k=1}^{2i} w_k w_{2i-k} = w_i^2.
\]

Knowing these two facts, and the naturality of Stiefel-Whitney classes under bundle isomorphisms, we just need to apply the Whitney sum axiom to check that \( w_i^2 \) fulfills the basic requirement:
\[
w_i^2(F) = \sum_{k=1}^{2i} w_k(F) w_{2i-k}(F) = w_{2i}(F \oplus F) = w_{2i}(G \oplus G) = \sum_{k=1}^{2i} w_k(G) w_{2i-k}(G) = w_i^2(G).
\]

This equation being valid for all \( i \in \mathbb{N} \cup \{0\} \), it just remains to check polynomials \( \sum \bigcup w_i^2 \). And this has become now only a question of commuting brackets (they commute because \( 2 = 0 \) in \( \mathbb{Z}_2 \)-coefficients):
\[
(\sum \bigcup w_i^2)(F) = \sum \bigcup (w_i^2(F)) = \sum \bigcup (w_i^2(G)) = (\sum \bigcup w_i^2)(G).
\]

□

Proof of theorem 1, (ii)⇒(i). Let \( c \) be a polynomial in the Stiefel-Whitney classes \( w_i \) fulfilling the basic requirement. From lemma 1 we see that it is transferred stable invariant.

Let \( O \) be the direct limit of the orthogonal groups, \( U \) the direct limit of the unitary groups and \( EU \) the universal total space to the classifying space \( BU \) for stable complex vector bundles. Let \( BO := EU/O \), via the inclusion \( O \subset U \) induced by the canonical inclusion \( \mathbb{R} \subset \mathbb{C} \).

According to Cartan [3, p. 17-22], there then is the Hopf spaces fibration
\[
U/O \xrightarrow{f} BO \xrightarrow{p} BU,
\]
where the projection \( p \) is the rest class map to dividing the whole group \( U \) out of \( EU \); and \( f : U/O \to BO \) embeds a fibre. \( H^*(BO, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_2, ...] \) is the
polynomial algebra with generators the Stiefel-Whitney classes \( \omega_i := w_i(\gamma(\mathbb{R}^\infty)) \) ([2, theorem B.2]). Cartan [3, p. 17-22] has shown that \( f^* \) maps these generators \( \omega_i \) to the generators \( v_i := w_i(f^*\gamma(\mathbb{R}^\infty)) \) of the exterior algebra

\[
H^*(U/\mathcal{O}, \mathbb{Z}_2) = \bigwedge (\mathbb{Z}_2[v_1, v_2, \ldots]),
\]

which is obtained by dividing the ideal \( (v_i^2)_{i \in \mathbb{N}\setminus\{0\}} \) out of the polynomial algebra \( \mathbb{Z}_2[v_1, v_2, \ldots] \). Hence, exactly the ideal \( (\omega_i^2)_{i \in \mathbb{N}\setminus\{0\}} \) is mapped to zero. So to write

\[
(\omega_i^2)_{i \in \mathbb{N}\setminus\{0\}} = \ker f^*.
\]  

(1)

Composing \( f \) with the projection \( p : BO \to BU \), we obtain a constant map (the whole fibre is mapped to its basepoint) and therefore a trivial bundle \( (p \circ f)^*\gamma(\mathbb{C}^\infty) \). This pullback of the complex universal bundle is the complexification of \( f^*\gamma(\mathbb{R}^\infty) \):

\[
(p \circ f)^*\gamma(\mathbb{C}^\infty) = f^*p^*EU \times_U \mathbb{C}^\infty = f^*E\mathcal{O} \times_\mathcal{O} \mathbb{C}^\infty = f^*(E\mathcal{O} \times_\mathcal{O} \mathbb{R}^\infty)^\mathbb{C}.
\]

So, \( f^*\gamma(\mathbb{R}^\infty) \) admits a trivial complexification, and all of the transferred stable invariant classes \( c \) must treat it like the trivial bundle \( \varepsilon \):

\[
c(f^*\gamma(\mathbb{R}^\infty)) = c(\varepsilon). \quad \text{(A pullback of the trivial bundle is trivial too, so)}
\]

\[
0 = c(f^*\gamma(\mathbb{R}^\infty)) - c(f^*\varepsilon) = f^*(c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)) \quad \text{by naturality.}
\]

\[
\Rightarrow c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \in \ker f^* = \{\omega_i^2\}_{i \in \mathbb{N}\setminus\{0\}}.
\]

**Goal.** We want to get a decomposition \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \)

\[
= \sum_{j_1=1}^m \omega_{i_{j_1}}^2 \cup \sum_{j_2=1}^m \omega_{i_{(j_1,j_2)}}^2 \cup \sum_{j_3=1}^m \omega_{i_{(j_1,j_2,j_3)}}^2 \cup \bigcup_{j_k=1}^m \omega_{i_{(j_1,\ldots,j_k)}}^2 \cup r_{(j_1,\ldots,j_k)}(\gamma(\mathbb{R}^\infty))
\]

\[
+ \sum_{j_1=1}^m \omega_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + \sum_{j_1=1}^m \omega_{i_{j_1}}^2 \cup \sum_{j_2=1}^m \sum_{j_k=1}^m \omega_{i_{(j_1,\ldots,j_k)}}^2 \cup r_{(j_1,\ldots,j_k)}(\varepsilon)
\]

for some \( m, m_{j_1}, \ldots, m_{(j_1,\ldots,j_k-1)} \in \mathbb{N} \setminus \{0\} \), some \( i_{j_1}, \ldots, i_{(j_1,\ldots,j_k)} \in \mathbb{N} \setminus \{0\} \), some \( r_{(j_1,\ldots,j_k)}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2) \), and some coefficients \( r_{j_1}(\varepsilon), \ldots, r_{(j_1,\ldots,j_k-1)}(\varepsilon) \in \{0,1\} \),

in a way that \( \forall \vec{j} := (j_1, \ldots, j_k) : 2 \sum_{p \in I(\vec{j})} p > \deg, \)

\[
\text{where } I(\vec{j}) := \{i_{j_1}, \ldots, i_{(j_1,\ldots,j_k)}\}.
\]

Being arrived at this goal and knowing that the degree must be the same on both sides of the equation, the sum over all terms containing a factor \( \bigcup_{p \in I(\vec{j})} \omega_p^2 \)

of too high degree \( 2 \sum_{p \in I(\vec{j})} p \), for any \( \vec{j} \), must vanish.
An index vector \( \vec{j} \) are eliminated and the decomposition described in our goal is achieved. If \( r \) steps, the index vectors \( \vec{j} \) rest terms with longer index vectors. That’s why after a finite number of these decomposition that lies in \( \ker \gamma \), we will show that there’s a low situated rest term \( \omega^2 \) appearing in this decomposition, and if \( \sum_{p \in I(\vec{j})} p \leq \deg c \).

**Remark.** The terms \( r_j \gamma(\mathbb{R}^\infty) \cup \bigcup_{p \in I(\vec{j})} \omega^2 \) with \( \sum_{p \in I(\vec{j})} p \geq \deg c \) vanish in any decomposition of \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \). That’s why we don’t let them contribute in the last definition.

**Definition.** Set \( l := \min \max I(\vec{j}) \) appears. Consider an index vector \( \vec{j} \) appearing in a given decomposition of \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \). If \( \max I(\vec{j}) = l \), then call \( \gamma(\mathbb{R}^\infty) - r_j \varepsilon \) a "low situated rest term".

As seen so far, \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \in \ker f^* = \langle \omega^2 \rangle_{i \in \mathbb{N} \setminus \{0\}} \), so there is a decomposition
\[
c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j=1}^{m} \omega^2_{i_1} \cup r_{j_1}(\gamma(\mathbb{R}^\infty)),
\]
for some \( m \in \mathbb{N} \setminus \{0\} \), some \( i_{j_1} \in \mathbb{N} \setminus \{0\} \), and some \( r_{j_1}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2) \). We will show that there’s a low situated rest term \( r_{j_1}(\gamma(\mathbb{R}^\infty)) - r_{j_1}(\varepsilon) \) in this decomposition that lies in \( \ker f^* \). Then, that low situated rest term admits a decomposition as a linear combination of squares \( \omega^2_{i_{j_1}j_2} \), with coefficients \( r_{j_1j_2}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2) \), leading to a new decomposition of \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \). So, inductively, we will replace a low situated rest term in any given decomposition of \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \) by a linear combination whose coefficients are rest terms with longer index vectors. That’s why after a finite number of these steps, the index vectors \( \vec{j} \) won’t "appear" no more, because the sums \( \sum_{p \in I(\vec{j})} p \) will exceed the degree of \( c \). That’s the moment when all low situated rest terms are eliminated and the decomposition described in our goal is achieved.

To do all this, we first need to introduce a procedure that shall be called:

\[\text{Definition.} \quad \text{An index vector } \vec{j} \text{ "appears" in a given decomposition of } c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \text{ if there is a summand } r_j(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \in I(\vec{j})} \omega^2 \text{ visible in this decomposition, and if } \sum_{p \in I(\vec{j})} p \leq \deg c.\]

\[\text{Remark:} \quad \text{The terms } r_j(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \in I(\vec{j})} \omega^2 \text{ with } \sum_{p \in I(\vec{j})} p \geq \deg c \text{ must vanish in any decomposition of } c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon). \text{ That’s why we don’t let them contribute in the last definition.}\]

\[\text{Definition.} \quad \text{Set } l := \min \max I(\vec{j}) \text{ appears. Consider an index vector } \vec{j} \text{ appearing in a given decomposition of } c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon). \text{ If } \max I(\vec{j}) = l, \text{ then call } r_j(\gamma(\mathbb{R}^\infty)) - r_j(\varepsilon) \text{ a "low situated rest term".}\]

\[\text{As seen so far, } c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \in \ker f^* = \langle \omega^2 \rangle_{i \in \mathbb{N} \setminus \{0\}}, \text{ so there is a decomposition}\]
\[c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j=1}^{m} \omega^2_{i_1} \cup r_{j_1}(\gamma(\mathbb{R}^\infty)),\]
for some \( m \in \mathbb{N} \setminus \{0\} \), some \( i_{j_1} \in \mathbb{N} \setminus \{0\} \), and some \( r_{j_1}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2) \). We will show that there’s a low situated rest term \( r_{j_1}(\gamma(\mathbb{R}^\infty)) - r_{j_1}(\varepsilon) \) in this decomposition that lies in \( \ker f^* \). Then, that low situated rest term admits a decomposition as a linear combination of squares \( \omega^2_{i_{j_1}j_2} \), with coefficients \( r_{j_1j_2}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2) \), leading to a new decomposition of \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \). So, inductively, we will replace a low situated rest term in any given decomposition of \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \) by a linear combination whose coefficients are rest terms with longer index vectors. That’s why after a finite number of these steps, the index vectors \( \vec{j} \) won’t "appear" no more, because the sums \( \sum_{p \in I(\vec{j})} p \) will exceed the degree of \( c \). That’s the moment when all low situated rest terms are eliminated and the decomposition described in our goal is achieved.

\[\text{To do all this, we first need to introduce a procedure that shall be called:}\]
"Cutting the equation \( c(F \oplus G) = c(G) \) at the dimension \( l \)." Define the bundles

\[
F := pr_1^* \gamma_2(\mathbb{R}^\infty) \to U/O \times BO \quad \text{and} \quad G := pr_2^* \gamma_2(\mathbb{R}^\infty) \to U/O \times BO,
\]

where \( pr_i \) shall be the projection on the \( i \)-th factor of the base space \( U/O \times BO \). Let \( \ell \in \mathbb{N} \). Consider the map

\[
(id, emb_l) : (U/O \times BO) \hookrightarrow (U/O \times BO)
\]

where \( emb_l : BO_l \hookrightarrow BO \) shall be the natural embedding, recalling that \( BO \) is the direct limit over all \( BO_l, \ell \in \mathbb{N} \). Then the bundle \( G_l := (id, emb_l)^*G \) admits Stiefel-Whitney classes that are in bijective correspondence with those of the \( l \)-dimensional universal bundle \( \gamma_l(\mathbb{R}^\infty) \to BO_l \).

(To be precise, \( G_l \cong pr_{BO_l}^* \gamma_l(\mathbb{R}^\infty) \), the situation being

\[
\gamma_l(\mathbb{R}^\infty) \quad G_l \cong pr_{BO_l}^* \gamma_l(\mathbb{R}^\infty) \quad G := pr_2^* \gamma_2(\mathbb{R}^\infty) \quad \gamma(\mathbb{R}^\infty)
\]

\[
BO_l \quad (U/O \times BO) \quad (U/O \times BO) \quad BO
\]

\[
\xymatrix{BO_l & (U/O \times BO) \ar[l]_{pr_{BO_l}} & (U/O \times BO) \ar[r]^{pr_2} & BO \ar[l]^{(id, emb_l)}
}
\]

Especially, \( w_p(G_l) \) vanishes for \( p > l \).

The bundle \( F \) inherits from \( f^* \gamma(\mathbb{R}^\infty) \) the property to admit a trivial complexification. Therefore, the transferred stable invariance of \( c \) applies:

\[
c(F \oplus G) = c(G).
\]

Thus, applying the induced cohomology map \((id, emb_l)^*\) gives

\[
(id, emb_l)^*c(F \oplus G) = (id, emb_l)^*c(G)
\]

\[
\Leftrightarrow c(id^*F \oplus emb_l^*G) = c(emb_l^*G)
\]

\[
\Leftrightarrow c(F \oplus G_l) = c(G_l).
\]

By the universality of \( \gamma(\mathbb{R}^\infty) \), and the naturality of all characteristic classes towards the classifying maps of \( G_l \) and \( F \oplus G_l \), any given decomposition

\[
c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_j r_j(\gamma(\mathbb{R}^\infty)) \bigcup_{p \in I(j)} \omega_p^2
\]

gives analogous decompositions

\[
c(G_l) - c(\varepsilon) = \sum_j r_j(G_l) \bigcup_{p \in I(j)} w_p^2(G_l)
\]

and

\[
c(F \oplus G_l) - c(\varepsilon) = \sum_j r_j(F \oplus G_l) \bigcup_{p \in I(j)} w_p^2(F \oplus G_l).
\]
Theorem 1, (i)→(ii) gives the transferred stable invariance of \( w_p^2 \), making it invariant under adding the bundle \( F \), whose complexification is trivial:

\[
    w_p^2(F \oplus G_i) = w_p^2(G_i).
\]

Thus, the equation \( c(F \oplus G_i) = c(G_i) \) can be rewritten as:

\[
    \sum_j r_j(F \oplus G_i) \bigcup_{p \leq i(j)} w_p^2(G_i) = \sum_j r_j(G_i) \bigcup_{p \leq i(j)} w_p^2(G_i)
\]

where all summands containing a factor \( w_p(G_i) \) with \( p > l \) vanish:

\[
    \iff \sum_j r_j(F \oplus G_i) \bigcup_{p \leq i(j)} w_p^2(G_i) = \sum_j r_j(G_i) \bigcup_{p \leq i(j)} w_p^2(G_i)
\]

For not to exceed the degree of \( c \), also all terms with \( 2 \sum_p p > \deg c \) must vanish:

\[
    \Rightarrow \sum_j r_j(F \oplus G_i) \bigcup_{p \leq i(j)} w_p^2(G_i) = \sum_j r_j(G_i) \bigcup_{p \leq i(j)} w_p^2(G_i)
\]

So, it’s this last expression that we’ll call “the equation \( c(F \oplus G) = c(G) \) cut at the dimension \( l \)”.

Induction over the index vector pointing at a low situated rest term

**Induction’s beginning.** Recall

\[
    c(\gamma(\mathbb{R}^\infty)) - c(\epsilon) = \sum_{j=1}^{m} \omega_{i_j}^2 \cup r_j(\gamma(\mathbb{R}^\infty)).
\]

Rename \( i_1, \ldots, i_m \) such that \( i_1 < i_2 < \ldots < i_m \).

Cut the equation \( c(F \oplus G) = c(G) \) at \( i_1 \), and get

\[
    \sum_{j_1 \text{ appears}} r_{j_1}(F \oplus G_{i_1}) \cup w_{i_1}^2(G_{i_1}) = \sum_{j_1 \text{ appears}} r_{j_1}(G_{i_1}) \cup w_{i_1}^2(G_{i_1}).
\]

As \( i_1 < i_2 < \ldots < i_m \), this is just

\[
    r_1(F \oplus G_{i_1}) \cup w_{i_1}^2(G_{i_1}) = r_1(G_{i_1}) \cup w_{i_1}^2(G_{i_1}).
\]

Injectivity of the multiplication map \( \cup w_{i_1}^2(G_{i_1}) \) in \( H^*(U/O \times BO_{i_1}, \mathbb{Z}_2) \) then holds \( r_1(F \oplus G_{i_1}) = r_1(G_{i_1}) \). Then pull this back with

\[
    (id \times const) : U/O \to (U/O \times BO_{i_1}),
\]

(where the map \( const \) takes just one, arbitrary, value), to get

\[
    r_1(f^*\gamma(\mathbb{R}^\infty) \oplus \epsilon) = r_1(\epsilon).
\]

Due to the stability of the Stiefel-Whitney classes \( [5, \text{p. 81}] \), that’s

\[
    r_1(f^*\gamma(\mathbb{R}^\infty)) = r_1(\epsilon).
\]
Using naturality of characteristic classes towards pullbacks, this gives

\[ f^*(r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon)) = 0. \]

Or, \( r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon) \) lies in \( \ker f^* \). So, we can replace it with a linear combination of quadratic terms, providing a new decomposition,

\[ c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \omega^2_{i_1} \cup \sum_{j_2=1}^{m_1} \omega^2_{i_2,j_2} \cup r_{1,j_1}(\gamma(\mathbb{R}^\infty)) + \omega^2_{i_1} \cup r_1(\varepsilon) \]

\[ + \sum_{j_1=2}^m \omega^2_{i_1} \cup r_{j_1}(\gamma(\mathbb{R}^\infty)). \]

**Induction’s prerequisite.**

Consider a given decomposition

\[ c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \bigcup_{p \in I(\vec{j})} \omega^2_{p} \]

\[ + \sum_{j_1=1}^m \omega^2_{i_1} \cup r_{j_1}(\varepsilon) + \ldots + \sum_{j_1=1}^m \omega^2_{i_1} \cup \sum_{j_k=1}^{m_{i_1\ldots i_k-1}} \omega^2_{i_1\ldots i_k-1} \cup r_{(j_1\ldots j_{k-1})}(\varepsilon). \]

**Induction’s claim.** There’s a low situated rest term in this given decomposition that lies in \( \ker f^* \).

**Induction’s step.** Cut the equation \( c(F \oplus G) = c(G) \) at the dimension

\[ l := \min_{\vec{j} \text{ appears}} \max I(\vec{j}). \]

Then the remaining terms of \( c(G) - c(\varepsilon) \) do all have the common factor \( w^2_{l}(G) \). This is no zero divisor in \( H^*(U/O \times BO, \mathbb{Z}_2) \) and further its multiplication map \( \cup w^2_{l}(G) \) is injective. Now, in \( c(F \oplus G) = c(G) \)

\[ \Rightarrow \sum_{\vec{j} \text{ appears}} r_{\vec{j}}(F \oplus G) \bigcup_{p \in I(\vec{j})} w^2_{p}(G) = \sum_{\vec{j} \text{ appears}} r_{\vec{j}}(G) \bigcup_{p \in I(\vec{j})} w^2_{p}(G), \]

this injectivity delivers

\[ \Rightarrow \sum_{\vec{j} \text{ appears}} r_{\vec{j}}(F \oplus G) \bigcup_{p \in I(\vec{j}) \setminus \{l\}} w^2_{p}(G) = \sum_{\vec{j} \text{ appears}} r_{\vec{j}}(G) \bigcup_{p \in I(\vec{j}) \setminus \{l\}} w^2_{p}(G). \]

\[ \Diamond \text{ If there is just one low situated rest term } r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon), \text{ then use the injectivity of the multiplication map } \bigcup_{p \in I(\vec{j}) \setminus \{l\}} w^2_{p}(G) \]
in $H^*(U/O \times BO_l, \mathbb{Z}_2)$ to obtain $r_j(F \oplus G_l) = r_j(G_l)$. Then pull this back with $(id \times const) : U/O \to (U/O \times BO_l)$ to get 
$r_j(f^*\gamma(\mathbb{R}^\infty) \oplus \varepsilon) = r_j(\varepsilon)$

$\Rightarrow r_j(f^*\gamma(\mathbb{R}^\infty)) = r_j(\varepsilon)$.

Using naturality, this means

$f^*(r_j(\gamma(\mathbb{R}^\infty)) - r_j(\varepsilon)) = 0$.

$\Rightarrow$ The low situated rest term $r_j(\gamma(\mathbb{R}^\infty)) - r_j(\varepsilon)$ lies in $\ker f^*$.

$\Diamond$ Else cut the remaining equation again at the dimension

$l' := \max \{ \sum_{j \text{ appears}} I_j \} = \min_{j \text{ appears}} \max (I_j \setminus \{ l \})$,

such as to obtain

$\sum_{j \text{ appears}} r_j(F \oplus G_{l'}) \bigcup_{p \in (I(j) \setminus \{ l \})} w^2_p(G_{l'}) = \sum_{j \text{ appears}} r_j(G_{l'}) \bigcup_{p \in (I(j) \setminus \{ l \})} w^2_p(G_{l'})$.

Now proceed analogously with the choice marked with the "$\Diamond$" signs on this page, and after finitely many steps, find a low situated rest term in $\ker f^*$.

This low situated rest term can be replaced by a linear combination of squares, holding a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$.

This completes the induction.

By the universality of $\gamma(\mathbb{R}^\infty)$,

$c = c(\varepsilon) + \sum_{j_1=1}^{m} w^2_{j_1} \cup r_{j_1}(\varepsilon) + ... + ...$

$+ \sum_{j_1=1}^{m} w^2_{j_1} \cup \sum_{j_{k-1}=1}^{m(j_1,...,j_{k-2})} \sum_{j_{k-1}=1}^{m(j_1,...,j_{k-1})} w^2_{j_1,j_2,...,j_{k-1}} \cup r_{j_1,...,j_{k-1}}(\varepsilon)$.

As $c(\varepsilon), r_{j_1}(\varepsilon), ..., r_{j_1,...,j_{k-1}}(\varepsilon) \in \{ 0, 1 = w_0 = w_0^2 \}$, $c$ is in the sub-ring $\mathbb{Z}_2[w^2_{j_1} \epsilon \cup \{ 0 \}]$ of the polynomial ring of Stiefel-Whitney classes.

So, theorem 1 is proved. $\square$

**Using integral coefficients**

We will lean on the obtained results for $\mathbb{Z}_2$-coefficients and use the mod 2 - reduction homomorphism

$\rho : H^*(BO, \mathbb{Z}) \to H^*(BO, \mathbb{Z}_2)$

to prove the theorems with $\mathbb{Z}$-coefficients. Define $V_l$ as in appendix A.
Lemma 2. \( \rho(V_i^2(\xi)) = \sum_{i \in \mathbb{N}(\frac{1}{2})} w_i^2(\xi \oplus \xi) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(\xi \oplus \xi) + \sum_{i \in \mathbb{N}(\frac{1}{2})} (w_{4i+2}(\xi \oplus \xi) + w_2(\xi \oplus \xi) \cup w_{4i}(\xi \oplus \xi)) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(\xi \oplus \xi) \),

for any real bundle \( \xi \).

Proof. Apply the reduction homomorphism:

\[
\rho[V_i^2(\xi)] = (\rho[V_i(\xi)])^2 = (\sum_{i \in I} S^q[w_{2i}(\xi)])^2 = (\sum_{i \in I} S^q[w_{2i}(\xi)])^2
\]

and

As \( 2 = 0 \) in \( H^*(BO, \mathbb{Z}_2) \), this equals

\[
\sum_{i \in \mathbb{N}(\frac{1}{2})} w_i^2(\xi) \cup w_{2j}(\xi) + \sum_{i \in \mathbb{N}(\frac{1}{2})} (w_{2i+1}(\xi) + w_1(\xi) \cup w_{2i}(\xi)) \cup \bigcup_{j \in I \setminus \{i\}} w_{2j}(\xi).
\]

Using the Whitney sum axiom and symmetry,

\[
w_{4i}(\xi \oplus \xi) = \sum_{k=1}^{4i} w_k(\xi)w_{4i-k}(\xi) = w_{2i}(\xi).
\]

Hence, the above term equals

\[
\sum_{i \in \mathbb{N}(\frac{1}{2})} w_i^2(\xi \oplus \xi) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(\xi \oplus \xi) + \sum_{i \in \mathbb{N}(\frac{1}{2})} (w_{4i+2}(\xi \oplus \xi) + w_2(\xi \oplus \xi) \cup w_{4i}(\xi \oplus \xi)) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(\xi \oplus \xi)
\]

\( \square \)

Proof of theorem 2. For \( V_i^2(\frac{1}{2}) \) and the Pontrjagin classes \( p_i \), the result is trivial. Now let \( F \to B, G \to B \) be real bundles with \( F^C \cong G^C \). Forgetting the complex structure, that’s \( F \oplus F \cong G \oplus G \). By naturality of the Stiefel-Whitney classes,

\[
\sum_{i \in \mathbb{N}(\frac{1}{2})} w_i^2(F \oplus F) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(F \oplus F) + \sum_{i \in \mathbb{N}(\frac{1}{2})} (w_{4i+2}(F \oplus F) + w_2(F \oplus F) \cup w_{4i}(F \oplus F)) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(F \oplus F) = \sum_{i \in \mathbb{N}(\frac{1}{2})} w_i^2(G \oplus G) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(G \oplus G)
\]
for any finite nonempty index set \( I \subseteq ((\frac{1}{2}) \cup \mathbb{N} \setminus \{0\}) \). Applying lemma 2, this means \( \rho(V^2_F) = \rho(V^2_G) \).
As \( V^2_\rho \) is in the torsion of \( H^*(BO, \mathbb{Z}) \), restricted on which \( \rho \) is injective \([4, \text{p. 513}]\), this proves the theorem: \( V^2_F = V^2_G \).

\[
\sum_{i \in I \setminus \{\frac{1}{2}\}} \left( w_{2i+2}(G \oplus G) + w_2(G \oplus G) \cup w_{4i}(G \oplus G) \right) \cup \bigcup_{j \in I \setminus \{i\}} w_{4j}(G \oplus G)
\]

\text{Corollary from theorem 1}

\( C \in H^*(BO, \mathbb{Z}) \) fulfill the basic requirement. Then \( \rho(C) \in \mathbb{Z}[w^2_i]_{i \in N \setminus \{0\}} \).

\text{Proof.} Let \( F \rightarrow B, G \rightarrow B \) be real bundles with \( F^C \cong G^C \). The reduction

\[ \rho(C) \in H^*(BO, \mathbb{Z}) \]

also satisfies the basic requirement:

\[ \rho(C)(F) = \rho(C(F)) = \rho(C(G)) = \rho(C)(G). \]

Theorem 1 now gives the result.

\text{Proof of theorem 3.}

Feshbach \([4, \text{p. 513}]\) tells that \( H^*(BO, \mathbb{Z}) = \mathbb{Z}[p_i] \in N \oplus 2\)-Torsion.

\[ \Rightarrow C = \sum \bigcup p_i + T \text{ with some torsion element } T \in H^*(BO, \mathbb{Z}) \quad (\sharp). \]
So for every real bundle \( \xi \), \( \rho(C)(\xi) = \sum \rho(\bigcup p_i(\xi)) + \rho(T)(\xi). \)

\[ \Rightarrow^3 \rho(C)(\xi) = \sum \bigcup \rho((-1)^{i} e_{2i}^{(C)}) + \rho(T)(\xi). \]

\[ \Rightarrow^4 \rho(C)(\xi) = \sum \bigcup w_{4i}(\xi \oplus \xi) + \rho(T)(\xi). \]

\[ \Rightarrow^5 \rho(C)(\xi) = \sum \bigcup w_{2j}^{(C)}(\xi) + \rho(T)(\xi). \]

Inserting the polynomial of the corollary from theorem 1, another polynomial in squares is produced: \( \sum \bigcup w_j^2(\xi) = \rho(T)(\xi). \)

As according to \([4, \text{p. 513}]\), \( \rho \) is injective on the torsion elements, there is a local inverse \( \rho_{2\text{-Torsion}}^{-1} \) lifting \( \rho(T) \) back to \( T \).

\[ \Rightarrow \rho_{2\text{-Torsion}}^{-1}(\sum \bigcup w_j^2(\xi)) = T(\xi). \]

\[ \Rightarrow^5 \rho_{2\text{-Torsion}}^{-1}(\sum \bigcup w_j(\xi \oplus \xi)) = T(\xi). \]

\[ \Rightarrow^4 \rho_{2\text{-Torsion}}^{-1}(\sum \bigcup \rho(c_j(\xi)) = T(\xi). \]

\[ \Rightarrow^4 C(\xi) = \sum \bigcup (-1)^{i} e_{2i}(\xi) + \rho_{2\text{-Torsion}}^{-1}(\sum \bigcup \rho(c_j(\xi))). \]

\text{3}By definition of the Pontrjagin classes.

\text{4}See \([5, \text{proposition 3.8}]\) and use \((\xi^C)_b = \xi \oplus \xi). \)

\text{5}By the Whitney sum axiom and symmetry.
Appendix A

The cohomology ring of $BO$ with $\mathbb{Z}$-coefficients is known with all relations between its generators since Brown [1] and can be obtained as follows: Define the set of generators of $H^*(BO_n, \mathbb{Z})$ as in [4, definition 1]:

It consists of the Pontrjagin classes $p_i$ of the universal bundle over $BO_n$, and classes $V_I$ with $I$ ranging over all finite nonempty subsets of

$$\left\{\frac{1}{2}\right\} \cup \{k \in \mathbb{Z} \mid 0 < k < \frac{n+1}{2}\}$$

with the proviso that $I$ does not contain both $\frac{1}{2}$ and $\frac{n}{2}$, for $n > 1$.

According to [4, theorem 2], $H^*(BO_n, \mathbb{Z})$ is for all $n \leq \infty$ isomorphic to the polynomial ring over $\mathbb{Z}$ generated by the above specified elements modulo the following six types of relations.

In all relations except the first, the cardinality of $I$ is less than or equal to that of $J$ and greater than one. (Most of the restrictions on $I$ and $J$ are to avoid repeating relations). By convention, $p_{\frac{1}{2}}$ where it occurs means $V_{\{\frac{1}{2}\}}$. Also, if $\{\frac{n}{2}, \frac{1}{2}\} \subseteq I \cup J$, then $V_{I \cup J}$ shall mean $V_{\{\frac{n}{2}\}} V_{(I \cup J) \{\frac{1}{2}\}}$.

1) $2V_I = 0$.

2) $V_J V_I + V_{I \cap J} V_{I \setminus J} + V_{\Gamma \setminus J} V_{J \setminus I} \prod_{i \in I \cap J} p_i = 0$ (for $I \cap J \neq \emptyset$, $I \not\subseteq J$).

3) $V_J V_I + \sum_{i \in I} V_{(i)} V_{(J \cup I) \setminus (i)} \prod_{j \in I \setminus (i)} p_j = 0$ (for $I \subseteq J$).

4) $V_J V_I + \sum_{i \in I} V_{(i)} V_{(I \cup J) \setminus (i)} = 0$ (for $I \cap J = \emptyset$; if $I$ and $J$ have the same cardinality, then the smallest element of $I$ is less than that of $J$).

5) $\sum_{i \in I} V_{(i)} V_{\Gamma \setminus (i)} = 0$.

6) $V_{\{\frac{1}{2}\}} p_{\frac{1}{2}} + V_{\{\frac{1}{2}\}}^2 = 0$, if $n$ is even.

Then $\rho(V_I) = Sq^1(\bigcup_{i \in I} w_{2i})$, with the Steenrod squaring operation $Sq^1$.

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