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Characteristic classes of complexified bundles

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June 14, 2007

Abstract
We examine the topological characteristic cohomology classes of complexified vector bundles\(^1\). In particular, all the classes coming from real vector bundles are computed. We use characteristic classes with the axioms of Milnor and Stasheff [6].

Introduction

Definition (Real generator bundles). We consider a real vector bundle \(F \to B\) and a complex vector bundle \(E \to B\) over the same base space \(B\). If the fibre-wise constructed complexification \(F \otimes_{\mathbb{R}} \mathbb{C} =: F^\mathbb{C}\) is isomorphic to \(E\), we’ll call \(F\) a real generator bundle of \(E\).

We want to attribute topological characteristic classes \(c(F)\) of the real generator bundles to the complexified bundles \(F^\mathbb{C}\). Not every complex vector bundle admits a real generator bundle, as we shall see in a moment. So, supplementary cohomological information might be gathered when restricting attention to the subcategory of complex vector bundles that admit one.

Obstruction to real generator bundles. Consider a real vector bundle \(F \to B\). By reflection on the real axes given by \(F\), \(F^\mathbb{C}\) is isomorphic to its complex conjugate bundle \(\overline{F}\). So, any complex bundle \(E \to B\) that admits \(F\) as a real generator bundle must be isomorphic to its own conjugate bundle:

\[
E \cong F^\mathbb{C} \cong \overline{F} \cong E.
\]

The odd Chern classes \(c_{2k+1}\) have the property \(c_{2k+1}(E) = -c_{2k+1}(\overline{E})\) ([6, lemma 14.9]), so

\[
c_{2k+1}(E) = -c_{2k+1}(\overline{E}) = -c_{2k+1}(E) \in H^{4k+2}(B, \mathbb{Z})
\]

\(\Rightarrow 2c_{2k+1}(E) = 0\). Consequently, no complex vector bundle with some nonzero and non-torsion odd Chern class can admit a real generator bundle.

We are interested in all attributions of topological characteristic classes \(c(F)\) of the real generator bundles to the complexified bundles \(F^\mathbb{C}\). For such an attribution to be well-defined, we need that real generator bundles \(F, G\) of the same complex bundle provide the same class \(c(F) = c(G)\). For short, we get the Basic requirement \(F^\mathbb{C} \cong G^\mathbb{C} \Rightarrow c(F) = c(G)\).

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Classes fulfilling the basic requirement

**Theorem 1.** Let $c$ be a polynomial in the Stiefel-Whitney classes $w_i$. Then the following two conditions are equivalent:

(i) $c$ is an element of the polynomial sub-ring $\mathbb{Z}_2[w^2_i]$, $i \in \mathbb{N}_0$

(ii) $c$ satisfies the basic requirement.

As the cohomology ring $H^*(BO_n, \mathbb{Z}_2)$ of the classifying space $BO_n$ is generated by the Stiefel-Whitney classes, this gives the entire information about modulo 2-classes. We find corresponding results for integral cohomology classes.

**Theorem 2.** The basic requirement holds for polynomials in $V^2_I$, with $I$ arbitrary, $V_{\{1,2\}}$ and the Pontrjagin classes $p_i$.

For the convenience of the reader, we give a description of the integral cohomology classes $V_I$ in the appendix on page 12.

**Theorem 3.** Let $C \in H^*(BO, \mathbb{Z})$ be a characteristic class fulfilling the basic requirement. Then for any bundle $\xi$, $C(\xi)$ is completely determined by some Chern classes $c_k(\xi^C)$, $k \in \mathbb{N}$.

Using $\mathbb{Z}_2$-coefficients

We’ll restrain ourselves to $\mathbb{Z}_2$-coefficients in the following, in order to prove theorem 1. Let $F \to B$ be a real vector bundle.

**Lemma 1.** A polynomial $c = \sum \bigcup w_i$ in the Stiefel-Whitney classes fulfilling the basic requirement satisfies the **Transferred stable invariance** property:

$$F^C \cong B \times \mathbb{C}^n \Rightarrow c(F \oplus G) = c(G).$$

**Proof.** Let $c$ fulfill the basic requirement, and let $F^C \cong B \times \mathbb{C}^n$. Let $G \to B$ be a real bundle. Then $c(F \oplus G) = c((B \times \mathbb{R}^n) \oplus G)$ because $(B \times \mathbb{R}^n)^C = B \times \mathbb{C}^n \cong F^C$, so the basic requirement can be applied. Thus, $c(F \oplus G) = \sum w_i((B \times \mathbb{R}^n) \oplus G)$ and with the stability [5, p. 81] due to the Whitney-sum axiom of the Stiefel-Whitney classes, this term equals $\sum \bigcup w_i(G) = c(G)$. \qed

**Remark.** If the base space $B$ is compact Hausdorff, transferred stable invariance of $c$ provides the basic requirement.

**Proof.** Let $F \to B$, $G \to B$ be real bundles with $F^C \cong G^C$. Forgetting the complex structure, that’s $F \oplus F \cong G \oplus G$. As $B$ is compact Hausdorff, there is an inverse bundle $F^{-1} \to B$, such that $F \oplus F^{-1} \cong B \times \mathbb{R}^N$ for some $N$.

As seen in the last proof, $c(F) = c(F \oplus (B \times \mathbb{R}^N))$. And that’s, in turn, $c(F \oplus F \oplus F^{-1}) = c(G \oplus G \oplus F^{-1})$. Now,

$$(G \oplus F^{-1})^C = G^C \oplus (F^{-1})^C \cong F^C \oplus (F^{-1})^C = (F \oplus (F^{-1}))^C \cong B \times \mathbb{C}^n.$$
That’s why we can apply the transferred stable invariance and obtain
\[ c(F) = c(G \oplus (G \oplus F^{-1})) = c(G). \]

Proof of theorem 1, (i)⇒(ii). Let \( F \to B, G \to B \) be real bundles with \( F^\mathbb{C} \cong G^\mathbb{C} \). Forgetting the complex structure, that’s \( F \oplus F \cong G \oplus G \). A consequence of working in \( \mathbb{Z}_2 \)-coefficients is that all terms that appear twice in a sum vanish, just like
\[ \sum_{k=1}^{2i} w_k w_{2i-k} = w_i^2. \]
Knowing these two facts, and the naturality of Stiefel-Whitney classes under bundle isomorphisms, we just need to apply the Whitney sum axiom to check that \( w_i^2 \) fulfills the basic requirement:
\[ w_i^2(F) = \sum_{k=1}^{2i} w_k(F)w_{2i-k}(F) = w_{2i}(F \oplus F) = w_{2i}(G \oplus G) = \sum_{k=1}^{2i} w_k(G)w_{2i-k}(G) = w_i^2(G). \]
This equation being valid for all \( i \in \mathbb{N} \cup \{0\} \), it just remains to check polynomials \( \sum \bigcup w_i^2 \). And this has become now only a question of commuting brackets (they commute because \( 2 = 0 \) in \( \mathbb{Z}_2 \)-coefficients):
\[ (\sum \bigcup w_i^2)(F) = \sum \bigcup (w_i^2(F)) = \sum \bigcup (w_i^2(G)) = (\sum \bigcup w_i^2)(G). \]

Proof of theorem 1, (ii)⇒(i). Let \( c \) be a polynomial in the Stiefel-Whitney classes \( w_i \) fulfilling the basic requirement. From lemma 1 we see that it is transferred stable invariant.

Let \( O \) be the direct limit of the orthogonal groups, \( U \) the direct limit of the unitary groups and \( EU \) the universal total space to the classifying space \( BU \) for stable complex vector bundles. Let \( BO := EU/O \), via the inclusion \( O \subset U \) induced by the canonical inclusion \( \mathbb{R} \subset \mathbb{C} \).
According to Cartan [3, p. 17-22], there then is the Hopf spaces fibration
\[ U/O \xrightarrow{f} BO \xrightarrow{p} BU, \]
where the projection \( p \) is the rest class map to dividing the whole group \( U \) out of \( EU \); and \( f : U/O \to BO \) embeds a fibre. \( H^*(BO, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_2, ...] \) is the
polynomial algebra with generators the Stiefel-Whitney classes \( \omega_i := w_i(\gamma(\mathbb{R}^\infty)) \) ([2, theorem B.2]). Cartan [3, p. 17-22] has shown that \( f^* \) maps these generators \( \omega_i \) to the generators \( v_i := w_i(f^*(\gamma(\mathbb{R}^\infty))) \) of the exterior algebra

\[
H^*(U/O, \mathbb{Z}_2) = \bigwedge(Z_2[v_1, v_2, \ldots]),
\]

which is obtained by dividing the ideal \( (v_i^2)_{i \in \mathbb{N}\setminus\{0\}} \) out of the polynomial algebra \( \mathbb{Z}_2[v_1, v_2, \ldots] \). Hence, exactly the ideal \( (\omega_i^2)_{i \in \mathbb{N}\setminus\{0\}} \) is mapped to zero. So to write

\[
(\omega_i^2)_{i \in \mathbb{N}\setminus\{0\}} = \ker f^*.
\]

Composing \( f \) with the projection \( p : BO \to BU \), we obtain a constant map (the whole fibre is mapped to its basepoint) and therefore a trivial bundle \((p \circ f)^*(\gamma(\mathbb{C}^\infty))\). This pullback of the complex universal bundle is the complexification of \( f^*(\gamma(\mathbb{R}^\infty)) \):

\[
(p \circ f)^*(\gamma(\mathbb{C}^\infty)) = f^*p^*EU \times_U \mathbb{C}^\infty = f^*EO \times_O \mathbb{C}^\infty = f^*(EO \times_O \mathbb{R}^\infty)^\mathbb{C}
= f^*(\mathbb{R}^\infty)^\mathbb{C} = (f^*(\gamma(\mathbb{R}^\infty)))^\mathbb{C}.
\]

So, \( f^*(\gamma(\mathbb{R}^\infty)) \) admits a trivial complexification, and all of the transferred stable invariant classes \( c \) must treat it like the trivial bundle \( \varepsilon \):

\[
c(f^*(\gamma(\mathbb{R}^\infty))) = c(\varepsilon). \]

A pullback of the trivial bundle is trivial too, so

\[
0 = c(f^*(\gamma(\mathbb{R}^\infty))) - c(f^*\varepsilon) = f^*(c(\gamma(\mathbb{R}^\infty))) - c(\varepsilon)
\]

by naturality.

\[
\Rightarrow c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \in \ker f^* = (\omega_i^2)_{i \in \mathbb{N}\setminus\{0\}}.
\]

**Goal.** We want to get a decomposition \( c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \)

\[
= \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup \sum_{j_2=1}^{m} \omega_{i_{j_1},j_2}^2 \cup \sum_{j_3=1}^{m} \sum_{j_4=1}^{m(j_1,\ldots,j_3)} \omega_{i_{j_1},\ldots,j_4} \cup r_{j_1,\ldots,j_k}(\gamma(\mathbb{R}^\infty))
+ \sum_{j_1=1}^{m} \omega_{i_{j_1}} \cup r_{j_1}(\varepsilon) + \cdots + \sum_{j_1=1}^{m} \omega_{i_{j_1}} \cup \sum_{j_2=1}^{m(j_1,\ldots,j_k-2)} \omega_{i_{j_1},\ldots,j_{k-1}} \cup r_{j_1,\ldots,j_k}(\varepsilon)
\]

for some \( m, m_{j_1}, \ldots, m_{j_1,\ldots,j_{k-1}} \in \mathbb{N}\setminus\{0\} \), some \( i_{j_1}, \ldots, i_{j_1,\ldots,j_k} \in \mathbb{N}\setminus\{0\} \), some \( r_{j_1,\ldots,j_k}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2) \),

and some coefficients \( r_{j_1}(\varepsilon), \ldots, r_{j_1,\ldots,j_k}(\varepsilon) \in \{0, 1\} \),

in a way that \( \forall \vec{j} := (j_1, \ldots, j_k) : 2 \sum_{p \in I(\vec{j})} p > \deg e \),

where \( I(\vec{j}) := \{i_{j_1}, \ldots, i_{j_1,\ldots,j_k}\} \).

Being arrived at this goal and knowing that the degree must be the same on both sides of the equation, the sum over all terms containing a factor \( \bigcup_{p \in I(\vec{j})} \omega_p^\mathbb{C} \) of too high degree \( 2 \sum_{p \in I(\vec{j})} p \), for any \( \vec{j} \), must vanish.
An index vector $\vec{j}$ "appears" in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ if there is a summand $r_j(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \in I(\vec{j})} \omega_p^2$ visible in this decomposition, and if $2 \sum_{p \in I(\vec{j})} p \leq \deg e$. 

To do all this, we first need to introduce a procedure that shall be called:

**Definition.** Set $l := \min_j \{ \max I(\vec{j}) \}$ appearing in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. If $\max I(\vec{j}) = l$, then call $r_j(\gamma(\mathbb{R}^\infty)) - r_j(\varepsilon)$ a "low situated rest term".

Before beginning, we should introduce two notions just to make the proof more readable:

**Definition.** An index vector $\vec{j}$ is said to "appear" in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ if $2 \sum_{p \in I(\vec{j})} p \leq \deg e$. 

**Remark.** The terms $r_j(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \in I(\vec{j})} \omega_p^2$ with $2 \sum_{p \in I(\vec{j})} p > \deg e$ must vanish in any decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. That’s why we don’t let them contribute in the last definition.

As seen so far, $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \in \ker f^* = \langle \omega_j^2 \rangle_{j \in \mathbb{N} \setminus \{0\}}$, so there is a decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j=1}^m \omega_{i_j}^2 \cup r_{i_j}(\gamma(\mathbb{R}^\infty)),$$

for some $m \in \mathbb{N} \setminus \{0\}$, some $i_j \in \mathbb{N} \setminus \{0\}$, and some $r_{i_j}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2)$. We will show that there’s a low situated rest term $r_{i_j}(\gamma(\mathbb{R}^\infty)) - r_{i_j}(\varepsilon)$ in this decomposition that lies in $\ker f^*$. Then, that low situated rest term admits a decomposition as a linear combination of squares $\omega_{i_{(i_j,j_2)}}^2$ with coefficients $r_{(i_j,j_2)}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2)$, leading to a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. So, inductively, we will replace a low situated rest term in any given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ by a linear combination whose coefficients are rest terms with longer index vectors. That’s why after a finite number of these steps, the index vectors $\vec{j}$ won’t "appear" no more, because the sums $2 \sum_{p \in I(\vec{j})} p$ will exceed the degree of $c$. That’s the moment when all low situated rest terms are eliminated and the decomposition described in our goal is achieved.

So, a polynomial $c(\gamma(\mathbb{R}^\infty))$ in some squares $\omega_p^2$, $p \in \mathbb{N} \cup \{0\}$ will remain:\n
$$c(\gamma(\mathbb{R}^\infty)) = c(\varepsilon) + \sum_{j=1}^m \omega_{i_j}^2 \cup r_{i_j}(\varepsilon) + ... + ...$$

$$+ \sum_{j=1}^m \omega_{i_j}^2 \cup \sum_{m(j_1,...,j_{k-2})} \omega_{i_{(j_1,...,j_{k-1})}}^2 \cup r_{(j_1,...,j_{k-1})}(\varepsilon).$$

2The classes $c(\varepsilon)$, $r_j(\varepsilon)$ of the trivial bundle $\varepsilon$ are just coefficients in $H^0(BO, \mathbb{Z}_2) = \{0, 1\}$. 

3So, a polynomial $c(\gamma(\mathbb{R}^\infty))$ in some squares $\omega_p^2$, $p \in \mathbb{N} \cup \{0\}$ will remain: $c(\gamma(\mathbb{R}^\infty)) = c(\varepsilon) + \sum_{j=1}^m \omega_{i_j}^2 \cup r_{i_j}(\varepsilon) + ... + ...$ 

$$+ \sum_{j=1}^m \omega_{i_j}^2 \cup \sum_{m(j_1,...,j_{k-2})} \omega_{i_{(j_1,...,j_{k-1})}}^2 \cup r_{(j_1,...,j_{k-1})}(\varepsilon).$$

Before beginning, we should introduce two notions just to make the proof more readable:

**Definition.** An index vector $\vec{j}$ "appears" in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ if there is a summand $r_j(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \in I(\vec{j})} \omega_p^2$ visible in this decomposition, and if $2 \sum_{p \in I(\vec{j})} p \leq \deg e$. 

**Remark.** The terms $r_j(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \in I(\vec{j})} \omega_p^2$ with $2 \sum_{p \in I(\vec{j})} p > \deg e$ must vanish in any decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. That’s why we don’t let them contribute in the last definition.

**Definition.** Set $l := \min_j \{ \max I(\vec{j}) \}$ appearing in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. If $\max I(\vec{j}) = l$, then call $r_j(\gamma(\mathbb{R}^\infty)) - r_j(\varepsilon)$ a "low situated rest term".

As seen so far, $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \in \ker f^* = \langle \omega_j^2 \rangle_{j \in \mathbb{N} \setminus \{0\}}$, so there is a decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j=1}^m \omega_{i_j}^2 \cup r_{i_j}(\gamma(\mathbb{R}^\infty)),$$

for some $m \in \mathbb{N} \setminus \{0\}$, some $i_j \in \mathbb{N} \setminus \{0\}$, and some $r_{i_j}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2)$. We will show that there’s a low situated rest term $r_{i_j}(\gamma(\mathbb{R}^\infty)) - r_{i_j}(\varepsilon)$ in this decomposition that lies in $\ker f^*$. Then, that low situated rest term admits a decomposition as a linear combination of squares $\omega_{i_{(i_j,j_2)}}^2$ with coefficients $r_{(i_j,j_2)}(\gamma(\mathbb{R}^\infty)) \in H^*(BO, \mathbb{Z}_2)$, leading to a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. So, inductively, we will replace a low situated rest term in any given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ by a linear combination whose coefficients are rest terms with longer index vectors. That’s why after a finite number of these steps, the index vectors $\vec{j}$ won’t "appear" no more, because the sums $2 \sum_{p \in I(\vec{j})} p$ will exceed the degree of $c$. That’s the moment when all low situated rest terms are eliminated and the decomposition described in our goal is achieved.

To do all this, we first need to introduce a procedure that shall be called:
"Cutting the equation \( c(F \oplus G) = c(G) \) at the dimension \( l \)." Define the bundles
\[
F := pr_1^* \gamma(\mathbb{R}^\infty) \to U/O \times BO \\
G := pr_2^* \gamma(\mathbb{R}^\infty) \to U/O \times BO,
\]
where \( pr_i \) shall be the projection on the \( i \)-th factor of the base space \( U/O \times BO \). Let \( l \in \mathbb{N} \). Consider the map
\[
(id, emb_l) : (U/O \times BO_l) \hookrightarrow (U/O \times BO)
\]
where \( emb_l : BO_l \hookrightarrow BO \) shall be the natural embedding, recalling that \( BO \) is the direct limit over all \( BO_l, l \in \mathbb{N} \). Then the bundle \( G_l := (id, emb_l)^* G \) admits Stiefel-Whitney classes that are in bijective correspondence with those of the \( l \)-dimensional universal bundle \( \gamma_l(\mathbb{R}^\infty) \to BO_l \).

(To be precise, \( G_l \cong pr_{BO_l}^* \gamma_l(\mathbb{R}^\infty) \), the situation being
\[
\begin{array}{cccc}
\gamma_l(\mathbb{R}^\infty) & G_l & G := pr_2^* \gamma(\mathbb{R}^\infty) & \gamma(\mathbb{R}^\infty) \\
BO_l & \leftarrow & (U/O \times BO_l) & \leftarrow (U/O \times BO) \xrightarrow{pr_2} BO
\end{array}
\]
)

Especially, \( w_p(G_l) \) vanishes for \( p > l \).

The bundle \( F \) inherits from \( f^* \gamma(\mathbb{R}^\infty) \) the property to admit a trivial complexification. Therefore, the transferred stable invariance of \( c \) applies:
\[
c(F \oplus G) = c(G).
\]

Thus, applying the induced cohomology map \( (id, emb_l)^* \) gives
\[
(id, emb_l)^* c(F \oplus G) = (id, emb_l)^* c(G)
\]
\[
\Leftrightarrow c(id^* F \oplus emb_l^* G) = c(emb_l^* G)
\]
\[
\Leftrightarrow c(F \oplus G_l) = c(G_l).
\]

By the universality of \( \gamma(\mathbb{R}^\infty) \), and the naturality of all characteristic classes towards the classifying maps of \( G_l \) and \( F \oplus G_l \), any given decomposition
\[
c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_j r_j(\gamma(\mathbb{R}^\infty)) \bigcup_{p \in I(j)} \omega_p^2
\]
gives analogous decompositions
\[
c(G_l) - c(\varepsilon) = \sum_j r_j(G_l) \bigcup_{p \in I(j)} w_p^2(G_l)
\]
and
\[
c(F \oplus G_l) - c(\varepsilon) = \sum_j r_j(F \oplus G_l) \bigcup_{p \in I(j)} w_p^2(F \oplus G_l).
\]
Theorem 1, (i)⇒(ii) gives the transferred stable invariance of \(w_p^2\), making it invariant under adding the bundle \(F\), whose complexification is trivial:

\[w_p^2(F \oplus G_i) = w_p^2(G_i)\]

Thus, the equation \(c(F \oplus G_i) = c(G_i)\) can be rewritten as:

\[
\sum_j r_j(F \oplus G_i) \bigcup_{p \in I(j)} w_p^2(G_i) = \sum_j r_j(G_i) \bigcup_{p \in I(j)} w_p^2(G_i)
\]

where all summands containing a factor \(w_p(G_i)\) with \(p > l\) vanish:

\[
\Rightarrow \sum_j r_j(F \oplus G_i) \bigcup_{p \in I(j)} w_p^2(G_i) = \sum_j r_j(G_i) \bigcup_{p \in I(j)} w_p^2(G_i)
\]

For not to exceed the degree of \(c\), also all terms with \(2 \sum_{p \in I(j)} p > \deg c\) must vanish:

\[
\Rightarrow \sum_j r_j(F \oplus G_i) \bigcup_{p \in I(j)} w_p^2(G_i) = \sum_j r_j(G_i) \bigcup_{p \in I(j)} w_p^2(G_i)
\]

So, it’s this last expression that we’ll call "the equation \(c(F \oplus G) = c(G)\) cut at the dimension \(l\)."

**Induction over the index vector pointing at a low situated rest term**

**Induction’s beginning.** Recall \(c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j=1}^{m} \omega_{i_j}^2 \cup r_{i_j}(\gamma(\mathbb{R}^\infty))\).

Rename \(i_1, \ldots, i_m\) such that \(i_1 < i_2 < \ldots < i_m\).

Cut the equation \(c(F \oplus G) = c(G)\) at \(i_1\), and get

\[
\sum_{j_1 \text{ appears}} r_{i_1}(F \oplus G_{i_1}) \cup w_{i_1}^2(G_{i_1}) = \sum_{j_1 \text{ appears}} r_{i_1}(G_{i_1}) \cup w_{i_1}^2(G_{i_1}).
\]

As \(i_1 < i_2 < \ldots < i_m\), this is just \(r_1(F \oplus G_{i_1}) \cup w_{i_1}^2(G_{i_1}) = r_1(G_{i_1}) \cup w_{i_1}^2(G_{i_1})\).

Injectivity of the multiplication map \(\cup w_{i_1}^2(G_{i_1})\) in \(H^*(U/O \times B\mathcal{O}_{i_1}, \mathbb{Z}_2)\) then holds \(r_1(F \oplus G_{i_1}) = r_1(G_{i_1})\). Then pull this back with

\[(id \times \text{const}) : U/O \to (U/O \times B\mathcal{O}_{i_1}),\]

(where the map \(\text{const}\) takes just one, arbitrary, value), to get

\[r_1(f^* \gamma(\mathbb{R}^{\infty}) \oplus \varepsilon) = r_1(\varepsilon)\]

Due to the stability of the Stiefel-Whitney classes [5, p. 81], that’s

\[r_1(f^* \gamma(\mathbb{R}^{\infty})) = r_1(\varepsilon)\]
Using naturality of characteristic classes towards pullbacks, this gives
\[ f^*(r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon)) = 0. \]

Or, \( r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon) \) lies in ker \( f^* \). So, we can replace it with a linear combination of quadratic terms, providing a new decomposition,
\[ c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \omega^2_{i_1} \cup \sum_{j_2=1}^{m_1} \omega^2_{i_1,j_2} \cup r_{(1,j_1)}(\gamma(\mathbb{R}^\infty)) + \omega^2_{i_1} \cup r_1(\varepsilon) \]
\[ + \sum_{j_1=2}^{m} \omega^2_{i_1} \cup r_{j_1}(\gamma(\mathbb{R}^\infty)). \]

**Induction’s prerequisite.**

Consider a given decomposition
\[ c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{\vec{j}}^N \omega^2_{\vec{j}} \cup \sum_{p \in I(\vec{j})} \omega^2_{p} \]
\[ + \sum_{j_1=1}^{m} \omega^2_{i_1} \cup r_{j_1}(\varepsilon) + \sum_{j_1=1}^{m} \omega^2_{i_1} \cup \sum_{j_k-1=1}^{m} \omega^2_{i_1,...,j_k-1} \cup r_{(j_1,...,j_{k-1})}(\varepsilon). \]

**Induction’s claim.** There’s a low situated rest term in this given decomposition that lies in ker \( f^* \).

**Induction’s step.** Cut the equation \( c(F \oplus G) = c(G) \) at the dimension
\[ l := \min_{\vec{j}} \max I(\vec{j}). \]
Then the remaining terms of \( c(G_l) - c(\varepsilon) \) do all have the common factor \( \omega^2_{l}(G_l) \).
This is no zero divisor in \( H^*(U/O \times B\mathcal{O}_l, \mathbb{Z}_2) \) and further its multiplication map \( \cup \omega^2_{l}(G_l) \) is injective. Now, in \( c(F \oplus G_l) = c(G_l) \)
\[ \Rightarrow \sum_{\vec{j}}^N \omega^2_{\vec{j}} \cup r_{\vec{j}}(F \oplus G_l) \cup \sum_{p \in I(\vec{j})} \omega^2_{p} = \sum_{\vec{j}}^N \sum_{p \in I(\vec{j})} \omega^2_{p} \]
this injectivity delivers
\[ \Rightarrow \sum_{\vec{j}}^N \omega^2_{\vec{j}} \cup r_{\vec{j}}(F \oplus G_l) \cup \sum_{p \in I(\vec{j})\setminus\{l\}} \omega^2_{p} = \sum_{\vec{j}}^N \sum_{p \in I(\vec{j})\setminus\{l\}} \omega^2_{p}. \]

\( \Diamond \) If there is just one low situated rest term \( r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon) \), then use the injectivity of the multiplication map \( \cup \sum_{p \in I(\vec{j})\setminus\{l\}} \omega^2_{p} \).
in $H^*(U/O \times BO_l, \mathbb{Z}_2)$ to obtain $r_j(F \oplus G) = r_j(G)$. Then pull this back with $(id \times const): U/O \to (U/O \times BO_l)$ to get $r_j(f^*(\mathbb{R}^\infty) \oplus \varepsilon) = r_j(\varepsilon)$

\[ \Rightarrow r_j(f^*(\mathbb{R}^\infty)) = r_j(\varepsilon). \]

Using naturality, this means

\[ f^*(r_j(\mathbb{R}^\infty)) - r_j(\varepsilon) = 0. \]

\[ \Rightarrow \text{The low situated rest term } r_j(\mathbb{R}^\infty) - r_j(\varepsilon) \text{ lies in } \ker f^*. \]

\[ \diamond \text{ Else cut the remaining equation again at the dimension } l' := \max_{j \text{ appears}} (I_j) \leq l \min_{j \text{ appears}} (I_j) \}

\[ \leq l' \sum_{j \text{ appears}} r_j(F \oplus G) \bigcup_{p \in (I_j \setminus \{l\})} w_p^2(G) = \sum_{j \text{ appears}} r_j(G) \bigcup_{p \in (I_j \setminus \{l\})} w_p^2(G). \]

Now proceed analogously with the choice marked with the ”◊” signs on this page, and after finitely many steps, find a low situated rest term in $\ker f^*$.

This low situated rest term can be replaced by a linear combination of squares, holding a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$.

This completes the induction.

By the universality of $\gamma(\mathbb{R}^\infty)$,

\[ c = c(\varepsilon) + \sum_{j_1 = 1}^{m} w_{i_1}^2 \cup r_{j_1}(\varepsilon) + ... + ... \]

\[ + \sum_{j_1 = 1}^{m} w_{i_1}^2 \cup \sum_{j_{k-1} = 1}^{m_{j_1,...,j_{k-1}}} w_{i_{(j_1,...,j_{k-1})}}^2 \cup r_{(j_1,...,j_{k-1})}(\varepsilon). \]

As $c(\varepsilon), r_{j_1}(\varepsilon), ..., r_{(j_1,...,j_{k-1})}(\varepsilon) \in \{0, 1 = w_0 = w_0^2\}$, $c$ is in the sub-ring $\mathbb{Z}_2[w_i^2]_{i \in \mathbb{N} \cup \{0\}}$ of the polynomial ring of Stiefel-Whitney classes.

So, theorem 1 is proved. \[ \square \]

**USING INTEGRAL COEFFICIENTS**

We will lean on the obtained results for $\mathbb{Z}_2$-coefficients and use the mod 2 - reduction homomorphism

\[ \rho: H^*(BO, \mathbb{Z}) \to H^*(BO, \mathbb{Z}_2) \]

to prove the theorems with $\mathbb{Z}$-coefficients. Define $V_I$ as in appendix A.
Lemma 2. \( \rho(V_i^2(\xi)) = \sum_{i \in \mathbb{I}_i(\frac{1}{2})} w_i^2(\xi \oplus \xi) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi \oplus \xi) \)

\[ + \sum_{i \in \mathbb{I}_i(\frac{1}{2})} (w_{ai+2}(\xi \oplus \xi) + w_2(\xi \oplus \xi) \cup w_{ai}(\xi \oplus \xi)) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi \oplus \xi), \]

for any real bundle \( \xi \).

Proof. Apply the reduction homomorphism:

\[ \rho(V_i^2(\xi)) = (\rho(V_i(\xi)))^2 = (\sum_{i \in I} \sum_{i \in \mathbb{I}_i(\frac{1}{2})} w_i^2(\xi) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi))^2 \]

As \( 2 = 0 \) in \( H^*(BO, \mathbb{Z}_2) \), this equals

\[ = \sum_{i \in \mathbb{I}_i(\frac{1}{2})} w_i^2(\xi) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi) + \sum_{i \in \mathbb{I}_i(\frac{1}{2})} (w_{ai+1}(\xi) + w_2(\xi) \cup w_{ai}(\xi)) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi). \]

Using the Whitney sum axiom and symmetry,

\[ w_{4i}(\xi \oplus \xi) = \sum_{k=1}^{4i} w_k(\xi)w_{4i-k}(\xi) = w_{2i}^2(\xi). \]

Hence, the above term equals

\[ \sum_{i \in \mathbb{I}_i(\frac{1}{2})} w_i^2(\xi \oplus \xi) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi \oplus \xi) \]

\[ + \sum_{i \in \mathbb{I}_i(\frac{1}{2})} (w_{ai+2}(\xi \oplus \xi) + w_2(\xi \oplus \xi) \cup w_{ai}(\xi \oplus \xi)) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(\xi \oplus \xi) \]

\[ \blacksquare \]

Proof of theorem 2. For \( V(\frac{1}{2}) \) and the Pontrjagin classes \( p_i \), the result is trivial. Now let \( F \to B, G \to B \) be real bundles with \( F^C \cong G^C \). Forgetting the complex structure, that’s \( F \oplus F \cong G \oplus G \). By naturality of the Stiefel-Whitney classes,

\[ \sum_{i \in \mathbb{I}_i(\frac{1}{2})} w_i^2(F \oplus F) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(F \oplus F) \]

\[ + \sum_{i \in \mathbb{I}_i(\frac{1}{2})} (w_{ai+2}(F \oplus F) + w_2(F \oplus F) \cup w_{ai}(F \oplus F)) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(F \oplus F) \]

\[ = \sum_{i \in \mathbb{I}_i(\frac{1}{2})} w_i^2(G \oplus G) \cup \bigcup_{j \in \mathbb{I}_{i}(\frac{1}{2})} w_{aj}(G \oplus G) \]
As according to □ Theorem 1 now gives the result.

Corollary from theorem 1
Let \( C \in H^*(BO, \mathbb{Z}) \) fulfill the basic requirement. Then \( \rho(C) \in \mathbb{Z}_2[\! \sum \! w^2_i]_i \in \mathbb{N}\{0\} \).

Proof. Let \( F \rightarrow B, G \rightarrow B \) be real bundles with \( F^C \cong G^C \). The reduction

\[ \rho(C) \in H^*(BO, \mathbb{Z}_2) \]

also satisfies the basic requirement:

\[ \rho(C)(F) = \rho(C(F)) = \rho(C(G)) = \rho(C)(G). \]

Theorem 1 now gives the result. □

Proof of theorem 3.
Feshbach [4, p. 513] tells that \( H^*(BO, \mathbb{Z}) = \mathbb{Z}[p_i]_i \in \mathbb{N} \oplus 2\)-Torsion.

\[ \Rightarrow C = \sum \bigcup p_i + T \text{ with some torsion element } T \in H^*(BO, \mathbb{Z}) \]

(\( \xi \)).

So for every real bundle \( \xi, \rho(C)(\xi) = \sum \rho(\bigcup p_i(\xi)) + \rho(T)(\xi). \)

\[ \Rightarrow^3 \rho(C)(\xi) = \sum \bigcup \rho((-1)^i e_{2i}(\xi^C)) + \rho(T)(\xi). \]

\[ \Rightarrow^4 \rho(C)(\xi) = \sum \bigcup w_{4i}(\xi \oplus \xi) + \rho(T)(\xi). \]

\[ \Rightarrow^5 \rho(C)(\xi) = \sum \bigcup w_{2j}(\xi) + \rho(T)(\xi). \]

Inserting the polynomial of the corollary from theorem 1, another polynomial in squares is produced: \( \Rightarrow \sum \bigcup w^2_j(\xi) = \rho(T)(\xi). \)

As according to [4, p. 513], \( \rho \) is injective on the torsion elements, there is a local inverse \( \rho|_{2\text{-Torsion}}^{-1} \) lifting \( \rho(T) \) back to \( T \).

\[ \Rightarrow \rho|_{2\text{-Torsion}}^{-1}(\sum \bigcup w^2_j(\xi)) = T(\xi). \]

\[ \Rightarrow^5 \rho|_{2\text{-Torsion}}^{-1}(\sum \bigcup w_{2j}(\xi \oplus \xi)) = T(\xi). \]

\[ \Rightarrow^4 \rho|_{2\text{-Torsion}}^{-1}(\sum \bigcup \rho(c_j(\xi^C))) = T(\xi). \]

\[ \Rightarrow^4.3 C(\xi) = \sum \bigcup (-1)^i e_{2i}(\xi^C) + \rho|_{2\text{-Torsion}}^{-1}(\sum \bigcup \rho(c_j(\xi^C))). \]

\(^3\)By definition of the Pontrjagin classes.
\(^4\)See [5, proposition 3.8] and use \( (\xi^C)_H = \xi \oplus \xi \).
\(^5\)By the Whitney sum axiom and symmetry.
Appendix A

The cohomology ring of $BO$ with $\mathbb{Z}$-coefficients is known with all relations between its generators since Brown [1] and can be obtained as follows:

Define the set of generators of $H^*(BO_n, \mathbb{Z})$ as in [4, definition 1]:

It consists of the Pontrjagin classes $p_i$ of the universal bundle over $BO_n$, and classes $V_I$ with $I$ ranging over all finite nonempty subsets of

$$\left\{\frac{1}{2}\right\} \cup \{k \in \mathbb{Z} \mid 0 < k < \frac{n+1}{2}\}$$

with the proviso that $I$ does not contain both $\frac{1}{2}$ and $\frac{n}{2}$, for $n > 1$.

According to [4, theorem 2], $H^*(BO_n, \mathbb{Z})$ is for all $n \leq \infty$ isomorphic to the polynomial ring over $\mathbb{Z}$ generated by the above specified elements modulo the ideal generated by the following six types of relations.

In all relations except the first, the cardinality of $I$ is less than or equal to that of $J$ and greater than one. (Most of the restrictions on $I$ and $J$ are to avoid repeating relations). By convention, $p_{\frac{1}{2}}$ where it occurs means $V_{\left\{\frac{1}{2}\right\}}$. Also, if $\{\frac{n}{2}, \frac{1}{2}\} \subset I \cup J$, then $V_{I \cup J}$ shall mean $V_{\left\{\frac{n}{2}\right\}}V_{I \cup \left\{\frac{1}{2}\right\}}V_{J \cup \left\{\frac{n}{2}\right\}}V_{I \cup \left\{\frac{1}{2}\right\}}\left\{\frac{n}{2}, \frac{1}{2}\right\}$.

1) $2V_I = 0$.
2) $V_I V_J + V_{I \cup J} V_{I \cap J} V_{I \setminus J} \prod_{i \in I \cap J} p_i = 0$ (for $I \cap J \neq \emptyset$, $I \not\subseteq J$).
3) $V_I V_J + \sum_{i \in I \setminus J} V_{\{i\}} V_{(J \setminus I) \cup \{i\}} \prod_{j \in (J \setminus I) \cup \{i\}} p_j = 0$ (for $I \not\subseteq J$).
4) $V_I V_J + \sum_{i \in I \setminus J} V_{\{i\}} V_{(I \cup J) \setminus \{i\}} = 0$ (for $I \cap J = \emptyset$; if $I$ and $J$ have the same cardinality, then the smallest element of $I$ is less than that of $J$).
5) $\sum_{i \in I} V_{\{i\}} V_{\{i\}} = 0$.
6) $V_{\left\{\frac{1}{2}\right\}} p_{\frac{1}{2}} + V_{\left\{\frac{1}{2}\right\}}^2 = 0$, if $n$ is even.

Then $\rho(V_I) = Sq^1(\bigcup_{i \in I} w_{2i})$, with the Steenrod squaring operation $Sq^1$.

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