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Characteristic classes of complexified bundles

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Abstract

We examine the topological characteristic cohomology classes of complexified vector bundles[1]. In particular, all the classes coming from real vector bundles are computed. We use characteristic classes with the axioms of Milnor and Stasheff [6].

Introduction

Definition (Real generator bundles). We consider a real vector bundle $F \to B$ and a complex vector bundle $E \to B$ over the same base space B. If the fibre-wise constructed complexification $F \otimes_{\mathbb{R}} \mathbb{C} =: F^{\mathbb{C}}$ is isomorphic to E, we'll call F a *real generator bundle* of E.

We want to attribute topological characteristic classes $c(F)$ of the real generator bundles to the complexified bundles $F^{\mathbb{C}}$. Not every complex vector bundle admits a real generator bundle, as we shall see in a moment. So, supplementary cohomological information might be gathered when restricting attention to the subcategory of complex vector bundles that admit one.

Obstruction to real generator bundles. Consider a real vector bundle $F \to B$. By reflection on the real axes given by F, $F^{\mathbb{C}}$ is isomorphic to its complex conjugate bundle $\overline{F^{\mathbb{C}}}$. So, any complex bundle $E \to B$ that admits F as a real generator bundle must be isomorphic to its own conjugate bundle:

$$E \cong F^{\mathbb{C}} \cong \overline{F^{\mathbb{C}}} \cong \overline{E}.$$

The odd Chern classes c_{2k+1} have the property $c_{2k+1}(E) = -c_{2k+1}(\bar{E})$ ([6, lemma 14.9]), so

$$c_{2k+1}(E) = -c_{2k+1}(\bar{E}) \ = -c_{2k+1}(E) \ \epsilon \ H^{4k+2}(B, \mathbb{Z})$$

$\Rightarrow 2c_{2k+1}(E) = 0$. Consequently, no complex vector bundle with some nonzero and non-torsion odd Chern class can admit a real generator bundle.

We are interested in all attributions of topological characteristic classes $c(F)$ of the real generator bundles to the complexified bundles $F^{\mathbb{C}}$. For such an attribution to be well-defined, we need that real generator bundles F, G of the same complex bundle provide the same class $c(F) = c(G)$. For short, we get the
Basic requirement $F^{\mathbb{C}} \cong G^{\mathbb{C}} \Rightarrow c(F) = c(G)$.

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CLASSES FULFILLING THE BASIC REQUIREMENT

Theorem 1. *Let c be a polynomial in the Stiefel-Whitney classes w_i. Then the following two conditions are equivalent:*
(i) *c is an element of the polynomial sub-ring $\mathbb{Z}_2[w_i^2]_{i \,\epsilon\, \mathbb{N}\cup\{0\}}$*
(ii) *c satisfies the basic requirement.*

As the cohomology ring $H^*(B\mathcal{O}_n, \mathbb{Z}_2)$ of the classifying space $B\mathcal{O}_n$ is generated by the Stiefel-Whitney classes, this gives the entire information about modulo 2- classes. We find corresponding results for integral cohomology classes.

Theorem 2. *The basic requirement holds for polynomials in V_I^2, with I arbitrary, $V_{\{\frac{1}{2}\}}$ and the Pontrjagin classes p_i.*

For the convenience of the reader, we give a description of the integral cohomology classes V_I in the appendix on page 12.

Theorem 3. *Let $C \,\epsilon\, H^*(B\mathcal{O}, \mathbb{Z})$ be a characteristic class fulfilling the basic requirement. Then for any bundle ξ, $C(\xi)$ is completely determined by some Chern classes $c_k(\xi^{\mathbb{C}})$, $k \,\epsilon\, \mathbb{N}$.*

USING $\mathbb{Z}_2$-COEFFICIENTS

We'll restrain ourselves to $\mathbb{Z}_2$-coefficients in the following, in order to prove theorem 1. Let $F \to B$ be a real vector bundle.

Lemma 1. *A polynomial $c = \sum \bigcup w_i$ in the Stiefel-Whitney classes fulfilling the basic requirement satisfies the* ***Transferred stable invariance*** *property*:

$$F^{\mathbb{C}} \cong B \times \mathbb{C}^n \Rightarrow c(F \oplus G) = c(G).$$

Proof. Let c fulfill the basic requirement, and let $F^{\mathbb{C}} \cong B \times \mathbb{C}^n$. Let $G \to B$ be a real bundle. Then $c(F \oplus G) = c((B \times \mathbb{R}^n) \oplus G)$ because $(B \times \mathbb{R}^n)^{\mathbb{C}} = B \times \mathbb{C}^n \cong F^{\mathbb{C}}$, so the basic requirement can be applied. Thus, $c(F \oplus G) = \sum \bigcup w_i((B \times \mathbb{R}^n) \oplus G)$ and with the stability [5, p. 81] due to the Whitney-sum axiom of the Stiefel-Whitney classes, this term equals $\sum \bigcup w_i(G) = c(G)$. □

Remark. *If the base space B is compact Hausdorff, transferred stable invariance of c provides the basic requirement.*

Proof. Let $F \to B$, $G \to B$ be real bundles with $F^{\mathbb{C}} \cong G^{\mathbb{C}}$. Forgetting the complex structure, that's $F \oplus F \cong G \oplus G$. As B is compact Hausdorff, there is an inverse bundle $F^{-1} \to B$, such that $F \oplus F^{-1} \cong B \times \mathbb{R}^N$ for some N.
As seen in the last proof, $c(F) = c(F \oplus (B \times \mathbb{R}^N))$. And that's, in turn, $c(F \oplus F \oplus F^{-1}) = c(G \oplus G \oplus F^{-1})$. Now,

$$(G \oplus F^{-1})^{\mathbb{C}} = G^{\mathbb{C}} \oplus (F^{-1})^{\mathbb{C}} \cong F^{\mathbb{C}} \oplus (F^{-1})^{\mathbb{C}} = (F \oplus (F^{-1}))^{\mathbb{C}} \cong B \times \mathbb{C}^n.$$

That's why we can apply the transferred stable invariance and obtain

$$c(F) = c(G \oplus (G \oplus F^{-1})) = c(G).$$

□

Proof of theorem 1, (i)⇒(ii). Let $F \to B$, $G \to B$ be real bundles with $F^{\mathbb{C}} \cong G^{\mathbb{C}}$. Forgetting the complex structure, that's $F \oplus F \cong G \oplus G$. A consequence of working in $\mathbb{Z}_2$-coefficients is that all terms that appear twice in a sum vanish, just like

$$\sum_{k=1}^{2i} w_k w_{2i-k} = w_i^2.$$

Knowing these two facts, and the naturality of Stiefel-Whitney classes under bundle isomorphisms, we just need to apply the Whitney sum axiom to check that w_i^2 fulfills the basic requirement:

$$w_i^2(F) = \sum_{k=1}^{2i} w_k(F) w_{2i-k}(F) = w_{2i}(F \oplus F)$$

$$= w_{2i}(G \oplus G) = \sum_{k=1}^{2i} w_k(G) w_{2i-k}(G) = w_i^2(G).$$

This equation being valid for all $i \,\epsilon\, \mathbb{N} \cup \{0\}$, it just remains to check polynomials $\sum \bigcup w_i^2$. And this has become now only a question of commuting brackets (they commute because $2 = 0$ in $\mathbb{Z}_2$-coefficients):

$$(\sum \bigcup w_i^2)(F) = \sum \bigcup (w_i^2(F)) = \sum \bigcup (w_i^2(G)) = (\sum \bigcup w_i^2)(G).$$

□

Proof of theorem 1, (ii)⇒(i). Let c be a polynomial in the Stiefel-Whitney classes w_i fulfilling the basic requirement. From lemma 1 we see that it is transferred stable invariant.

Let $\mathcal{O}$ be the direct limit of the orthogonal groups, U the direct limit of the unitary groups and EU the universal total space to the classifying space BU for stable complex vector bundles. Let $B\mathcal{O} := EU/\mathcal{O}$, via the inclusion $\mathcal{O} \subset U$ induced by the canonical inclusion $\mathbb{R} \subset \mathbb{C}$.
According to Cartan [3, p. 17-22], there then is the Hopf spaces fibration

$$U/\mathcal{O} \xrightarrow{\quad f \quad} B\mathcal{O} \xrightarrow{\quad p \quad} BU,$$

where the projection p is the rest class map to dividing the whole group U out of EU; and $f : U/\mathcal{O} \to B\mathcal{O}$ embeds a fibre. $H^*(B\mathcal{O}, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1, \omega_2, ...]$ is the polynomial algebra with generators the Stiefel-Whitney classes $\omega_i := w_i(\gamma(\mathbb{R}^\infty))$ ([2, theorem B.2]). Cartan [3, p. 17-22] has shown that f^* maps these generators ω_i to the generators $v_i := w_i(f^*\gamma(\mathbb{R}^\infty))$ of the exterior algebra

$$H^*(U/\mathcal{O}, \mathbb{Z}_2) = \bigwedge(\mathbb{Z}_2[v_1, v_2, ...]),$$

which is obtained by dividing the ideal $\langle v_i^2 \rangle_{i \,\epsilon\, \mathbb{N}\setminus\{0\}}$ out of the polynomial algebra $\mathbb{Z}_2[v_1, v_2, ...]$. Hence, exactly the ideal $\langle \omega_i^2 \rangle_{i \,\epsilon\, \mathbb{N}\setminus\{0\}}$ is mapped to zero. So to write

$$\langle \omega_i^2 \rangle_{i \,\epsilon\, \mathbb{N}\setminus\{0\}} = \ker f^*. \tag{1}$$

Composing f with the projection $p : B\mathcal{O} \to BU$, we obtain a constant map (the whole fibre is mapped to its basepoint) and therefore a trivial bundle $(p \circ f)^*\gamma(\mathbb{C}^\infty)$. This pullback of the complex universal bundle is the complexification of $f^*\gamma(\mathbb{R}^\infty)$:

$$(p \circ f)^*\gamma(\mathbb{C}^\infty) = f^*p^*EU \times_U \mathbb{C}^\infty = f^*E\mathcal{O} \times_{\mathcal{O}} \mathbb{C}^\infty = f^*(E\mathcal{O} \times_{\mathcal{O}} \mathbb{R}^\infty)^{\mathbb{C}}$$
$$= f^*\gamma(\mathbb{R}^\infty)^{\mathbb{C}} = (f^*\gamma(\mathbb{R}^\infty))^{\mathbb{C}}.$$

So, $f^*\gamma(\mathbb{R}^\infty)$ admits a trivial complexification, and all of the transferred stable invariant classes c must treat it like the trivial bundle ε: $c(f^*\gamma(\mathbb{R}^\infty)) = c(\varepsilon)$. A pullback of the trivial bundle is trivial too, so

$0 = c(f^*\gamma(\mathbb{R}^\infty)) - c(f^*\varepsilon) = f^*(c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon))$ by naturality.

$$\Rightarrow c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \;\epsilon\; \ker f^* =^{(1)} \langle \omega_i^2 \rangle_{i \,\epsilon\, \mathbb{N}\setminus\{0\}}.$$

Goal. We want to get a decomposition $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$

$$= \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup \sum_{j_2=1}^{m_{j_1}} \omega_{i_{(j_1,j_2)}}^2 \cup \sum ... \cup \sum_{j_k=1}^{m_{(j_1,...,j_{k-1})}} \omega_{i_{(j_1,...,j_k)}}^2 \cup r_{(j_1,...,j_k)}(\gamma(\mathbb{R}^\infty))$$

$$+ \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + ... + \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup \sum ... \cup \sum_{j_{k-1}=1}^{m_{(j_1,...,j_{k-2})}} \omega_{i_{(j_1,...,j_{k-1})}}^2 \cup r_{(j_1,...,j_{k-1})}(\varepsilon)$$

for some $m, m_{j_1}, ..., m_{(j_1,...,j_{k-1})} \;\epsilon\; \mathbb{N} \cup \{0\}$, some $i_{j_1}, ..., i_{(j_1,...,j_k)} \;\epsilon\; \mathbb{N} \setminus \{0\}$,

some $r_{(j_1,...,j_k)}(\gamma(\mathbb{R}^\infty)) \;\epsilon\; H^*(B\mathcal{O}, \mathbb{Z}_2)$,

and some coefficients $r_{j_1}(\varepsilon), ..., r_{(j_1,...,j_{k-1})}(\varepsilon) \;\epsilon\; \{0, 1\}$,

in a way that $\forall \vec{j} := (j_1, ..., j_k)$: $\quad 2 \sum_{p \,\epsilon\, I(\vec{j})} p > \deg c$,

where $I(\vec{j}) := \{i_{j_1}, ..., i_{(j_1,...,j_k)}\}$.

Being arrived at this goal and knowing that the degree must be the same on both sides of the equation, the sum over all terms containing a factor $\bigcup_{p \,\epsilon\, I(\vec{j})} \omega_p^2$ of too high degree $2 \sum_{p \,\epsilon\, I(\vec{j})} p$, for any $\vec{j}$, must vanish.

So, a polynomial $c(\gamma(\mathbb{R}^\infty))$ in some squares ω_p^2 , $p \,\epsilon\, \mathbb{N} \cup \{0\}$ will remain[2]:

$c(\gamma(\mathbb{R}^\infty)) = c(\varepsilon) + \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + ... + ...$

$+ \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup \sum ... \cup \sum_{j_{k-1}=1}^{m_{(j_1,\dots,j_{k-2})}} \omega^2_{i_{(j_1,\dots,j_{k-1})}} \cup r_{(j_1,\dots,j_{k-1})}(\varepsilon).$

Before beginning, we should introduce two notions just to make the proof more readable:

Definition. An index vector $\vec{j}$ ***"appears"*** in a given decomposition of

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$$

if there is a summand $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \,\epsilon\, I(\vec{j})} \omega_p^2$ visible in this decomposition, and if $2 \sum_{p \,\epsilon\, I(\vec{j})} p \leq \deg c$.

Remark. The terms $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \cup \bigcup_{p \,\epsilon\, I(\vec{j})} \omega_p^2$ with $2 \sum_{p \,\epsilon\, I(\vec{j})} p > \deg c$ must vanish in any decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. That's why we don't let them contribute in the last definition.

Definition. Set $l := \min_{\vec{j} \text{ appears}} \max I(\vec{j})$. Consider an index vector $\vec{j}$ appearing in a given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. If $\max I(\vec{j}) = l$, then call $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)$ a ***"low situated rest term".***

As seen so far, $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) \,\epsilon\, \ker f^* = \langle \omega_i^2 \rangle_{i \epsilon \mathbb{N} \setminus \{0\}}$, so there is a decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\gamma(\mathbb{R}^\infty)),$$

for some $m \,\epsilon\, \mathbb{N} \cup \{0\}$, some $i_{j_1} \,\epsilon\, \mathbb{N} \setminus \{0\}$, and some $r_{j_1}(\gamma(\mathbb{R}^\infty)) \,\epsilon\, H^*(B\mathcal{O}, \mathbb{Z}_2)$. We will show that there's a low situated rest term $r_{j_1}(\gamma(\mathbb{R}^\infty)) - r_{j_1}(\varepsilon)$ in this decomposition that lies in $\ker f^*$. Then, that low situated rest term admits a decomposition as a linear combination of squares $\omega^2_{i_{(j_1,j_2)}}$ with coefficients $r_{(j_1,j_2)}(\gamma(\mathbb{R}^\infty)) \,\epsilon\, H^*(B\mathcal{O}, \mathbb{Z}_2)$, leading to a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$. So, inductively, we will replace a low situated rest term in any given decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$ by a linear combination whose coefficients are rest terms with longer index vectors. That's why after a finite number of these steps, the index vectors $\vec{j}$ won't "appear" no more, because the sums $2 \sum_{p \,\epsilon\, I(\vec{j})} p$ will exceed the degree of c. That's the moment when all low situated rest terms are eliminated and the decomposition described in our goal is achieved.

To do all this, we first need to introduce a procedure that shall be called:

[2] The classes $c(\varepsilon)$, $r_{\vec{j}}(\varepsilon)$ of the trivial bundle ε are just coefficients in $H^0(B\mathcal{O}, \mathbb{Z}_2) = \{0, 1\}$.

"Cutting the equation $c(F \oplus G) = c(G)$ at the dimension l". Define the bundles

$$F := pr_1^* f^* \gamma(\mathbb{R}^\infty) \longrightarrow U/\mathcal{O} \times B\mathcal{O} \text{ and}$$

$$G := pr_2^* \gamma(\mathbb{R}^\infty) \longrightarrow U/\mathcal{O} \times B\mathcal{O},$$

where pr_i shall be the projection on the i-th factor of the base space $U/\mathcal{O} \times B\mathcal{O}$. Let $l \;\epsilon\; \mathbb{N}$. Consider the map

$$(id, emb_l) : (U/\mathcal{O} \times B\mathcal{O}_l) \hookrightarrow (U/\mathcal{O} \times B\mathcal{O})$$

where $emb_l : B\mathcal{O}_l \hookrightarrow B\mathcal{O}$ shall be the natural embedding, recalling that $B\mathcal{O}$ is the direct limit over all $B\mathcal{O}_l$, $l \;\epsilon\; \mathbb{N}$. Then the bundle $G_l := (id, emb_l)^* G$ admits Stiefel-Whitney classes that are in bijective correspondence with those of the l-dimensional universal bundle $\gamma_l(\mathbb{R}^\infty) \to B\mathcal{O}_l$.

(To be precise, $G_l \cong pr_{B\mathcal{O}_l}{}^* \gamma_l(\mathbb{R}^\infty)$, the situation being

$$\begin{array}{ccccccc}
\gamma_l(\mathbb{R}^\infty) & & G_l \cong pr_{B\mathcal{O}_l}{}^*\gamma_l(\mathbb{R}^\infty) & & G := pr_2^*\gamma(\mathbb{R}^\infty) & & \gamma(\mathbb{R}^\infty) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B\mathcal{O}_l & \xleftarrow{pr_{B\mathcal{O}_l}} & (U/\mathcal{O} \times B\mathcal{O}_l) & \xhookrightarrow{(id, emb_l)} & (U/\mathcal{O} \times B\mathcal{O}) & \xrightarrow{pr_2} & B\mathcal{O}
\end{array}$$

).

Especially, $w_p(G_l)$ vanishes for $p > l$.

The bundle F inherits from $f^*\gamma(\mathbb{R}^\infty)$ the property to admit a trivial complexification. Therefore, the transferred stable invariance of c applies:

$$c(F \oplus G) = c(G).$$

Thus, applying the induced cohomology map $(id, emb_l)^*$ gives

$$(id, emb_l)^* c(F \oplus G) = (id, emb_l)^* c(G)$$

$$\Leftrightarrow c(id^* F \oplus emb_l^* G) = c(emb_l^* G)$$

$$\Leftrightarrow c(F \oplus G_l) = c(G_l).$$

By the universality of $\gamma(\mathbb{R}^\infty)$, and the naturality of all characteristic classes towards the classifying maps of G_l and $F \oplus G_l$, any given decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \bigcup_{p \;\epsilon\; I(\vec{j})} \omega_p^2$$

gives analogous decompositions

$$c(G_l) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(G_l) \bigcup_{p \;\epsilon\; I(\vec{j})} w_p^2(G_l)$$

and

$$c(F \oplus G_l) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \;\epsilon\; I(\vec{j})} w_p^2(F \oplus G_l).$$

Theorem 1, (i)⇒(ii) gives the transferred stable invariance of w_p^2, making it invariant under adding the bundle F, whose complexification is trivial :

$$w_p^2(F \oplus G_l) = w_p^2(G_l).$$

Thus, the equation $c(F \oplus G_l) = c(G_l)$ can be rewritten as:

$$\sum_{\vec{j}} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l) = \sum_{\vec{j}} r_{\vec{j}}(G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l)$$

where all summands containing a factor $w_p(G_l)$ with $p > l$ vanish:

$$\Leftrightarrow \sum_{\vec{j}}^{\max I(\vec{j}) \,\leq\, l} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l) = \sum_{\vec{j}}^{\max I(\vec{j}) \,\leq\, l} r_{\vec{j}}(G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l)$$

For not to exceed the degree of c, also all terms with $2 \sum_{p \,\epsilon\, I(\vec{j})} p > \deg c$ must vanish:

$$\Rightarrow \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\leq\, l} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l) = \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\leq\, l} r_{\vec{j}}(G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l)$$

So, it's this last expression that we'll call "the equation $c(F \oplus G) = c(G)$ cut at the dimension l".

<u>Induction over the index vector pointing at a low situated rest term</u>

Induction's beginning. Recall $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\gamma(\mathbb{R}^\infty))$.
Rename $i_1, ..., i_m$ such that $i_1 < i_2 < ... < i_m$.
Cut the equation $c(F \oplus G) = c(G)$ at i_1, and get

$$\sum_{j_1 \text{ appears}}^{i_{j_1} \,\leq\, i_1} r_{j_1}(F \oplus G_{i_1}) \cup w_{i_{j_1}}^2(G_{i_1}) = \sum_{j_1 \text{ appears}}^{i_{j_1} \,\leq\, i_1} r_{j_1}(G_{i_1}) \cup w_{i_{j_1}}^2(G_{i_1}).$$

As $i_1 < i_2 < ... < i_m$, this is just $r_1(F \oplus G_{i_1}) \cup w_{i_1}^2(G_{i_1}) = r_1(G_{i_1}) \cup w_{i_1}^2(G_{i_1})$.

Injectivity of the multiplication map $\cup w_{i_1}^2(G_{i_1})$ in $H^*(U/\mathcal{O} \times B\mathcal{O}_{i_1}, \mathbb{Z}_2)$ then holds $r_1(F \oplus G_{i_1}) = r_1(G_{i_1})$. Then pull this back with

$$(id \times const) : U/\mathcal{O} \to (U/\mathcal{O} \times B\mathcal{O}_{i_1}),$$

(where the map *const* takes just one, arbitrary, value), to get

$$r_1(f^*\gamma(\mathbb{R}^\infty) \oplus \varepsilon) = r_1(\varepsilon).$$

Due to the stability of the Stiefel-Whitney classes [5, p. 81], that's

$$r_1(f^*\gamma(\mathbb{R}^\infty)) = r_1(\varepsilon).$$

Using naturality of characteristic classes towards pullbacks, this gives

$$f^*(r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon)) = 0.$$

Or, $r_1(\gamma(\mathbb{R}^\infty)) - r_1(\varepsilon)$ lies in $\ker f^*$. So, we can replace it with a linear combination of quadratic terms, providing a new decomposition,

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \omega_{i_1}^2 \cup \sum_{j_2=1}^{m_1} \omega_{i_{(1,j_2)}}^2 \cup r_{(1,j_1)}(\gamma(\mathbb{R}^\infty)) + \omega_{i_1}^2 \cup r_1(\varepsilon)$$
$$+ \sum_{j_1=2}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\gamma(\mathbb{R}^\infty)).$$

Induction's prerequisite.

Consider a given decomposition

$$c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon) = \sum_{\vec{j}} r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) \bigcup_{p \,\epsilon\, I(\vec{j})} \omega_p^2$$

$$+ \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + \ldots + \sum_{j_1=1}^{m} \omega_{i_{j_1}}^2 \cup \sum \ldots \cup \sum_{j_{k-1}=1}^{m_{(j_1,\ldots,j_{k-2})}} \omega_{i_{(j_1,\ldots,j_{k-1})}}^2 \cup r_{(j_1,\ldots,j_{k-1})}(\varepsilon).$$

Induction's claim. There's a low situated rest term in this given decomposition that lies in $\ker f^*$.

Induction's step. Cut the equation $c(F \oplus G) = c(G)$ at the dimension

$$l := \min_{\vec{j} \text{ appears}} \max I(\vec{j}).$$

Then the remaining terms of $c(G_l) - c(\varepsilon)$ do all have the common factor $w_l^2(G_l)$. This is no zero divisor in $H^*(U/\mathcal{O} \times B\mathcal{O}_l, \mathbb{Z}_2)$ and further its multiplication map $\cup w_l^2(G_l)$ is injective. Now, in $c(F \oplus G_l) = c(G_l)$

$$\Rightarrow \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\le\, l} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l) = \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\le\, l} r_{\vec{j}}(G_l) \bigcup_{p \,\epsilon\, I(\vec{j})} w_p^2(G_l),$$

this injectivity delivers

$$\Rightarrow \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\le\, l} r_{\vec{j}}(F \oplus G_l) \bigcup_{p \,\epsilon\, I(\vec{j}) \setminus \{l\}} w_p^2(G_l) = \sum_{\vec{j} \text{ appears}}^{\max I(\vec{j}) \,\le\, l} r_{\vec{j}}(G_l) \bigcup_{p \,\epsilon\, I(\vec{j}) \setminus \{l\}} w_p^2(G_l).$$

$\diamondsuit$ If there is just one low situated rest term $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)$, then use the injectivity of the multiplication map $\cup \bigcup_{p \,\epsilon\, I(\vec{j}) \setminus \{l\}} w_p^2(G_l)$

in $H^*(U/\mathcal{O} \times B\mathcal{O}_l, \mathbb{Z}_2)$ to obtain $r_{\vec{j}}(F \oplus G_l) = r_{\vec{j}}(G_l)$. Then pull this back with $(id \times const) : U/\mathcal{O} \to (U/\mathcal{O} \times B\mathcal{O}_l)$ to get $r_{\vec{j}}(f^*\gamma(\mathbb{R}^\infty) \oplus \varepsilon) = r_{\vec{j}}(\varepsilon)$

$$\Rightarrow r_{\vec{j}}(f^*\gamma(\mathbb{R}^\infty)) = r_{\vec{j}}(\varepsilon).$$

Using naturality, this means

$$f^*(r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)) = 0.$$

$\Rightarrow$ The low situated rest term $r_{\vec{j}}(\gamma(\mathbb{R}^\infty)) - r_{\vec{j}}(\varepsilon)$ lies in $\ker f^*$.

$\diamondsuit$ Else cut the remaining equation again at the dimension

$$l' := \min_{\vec{j} \text{ appears}}^{\max I(\vec{j})=l} \max(I(\vec{j}) \setminus \{l\}),$$

such as to obtain

$$\sum_{\vec{j} \text{ appears}}^{\max(I(\vec{j})\setminus\{l\}) \leq l'} r_{\vec{j}}(F \oplus G_{l'}) \bigcup_{p \,\epsilon\, (I(\vec{j})\setminus\{l\})} w_p^2(G_{l'}) = \sum_{\vec{j} \text{ appears}}^{\max(I(\vec{j})\setminus\{l\}) \leq l'} r_{\vec{j}}(G_{l'}) \bigcup_{p \,\epsilon\, (I(\vec{j})\setminus\{l\})} w_p^2(G_{l'}).$$

Now proceed analogously with the choice marked with the "$\diamondsuit$" signs on this page, and after finitely many steps, find a low situated rest term in $\ker f^*$. This low situated rest term can be replaced by a linear combination of squares, holding a new decomposition of $c(\gamma(\mathbb{R}^\infty)) - c(\varepsilon)$.

This completes the induction.

By the universality of $\gamma(\mathbb{R}^\infty)$,

$$c = c(\varepsilon) + \sum_{j_1=1}^{m} w_{i_{j_1}}^2 \cup r_{j_1}(\varepsilon) + ... + ...$$

$$+ \sum_{j_1=1}^{m} w_{i_{j_1}}^2 \cup \sum ... \cup \sum_{j_{k-1}=1}^{m_{(j_1,\dots,j_{k-2})}} w_{i_{(j_1,\dots,j_{k-1})}}^2 \cup r_{(j_1,\dots,j_{k-1})}(\varepsilon).$$

As $c(\varepsilon), r_{j_1}(\varepsilon), ..., r_{(j_1,\dots,j_{k-1})}(\varepsilon) \,\epsilon\, \{0, 1 = w_0 = w_0^2\}$, c is in the sub-ring $\mathbb{Z}_2[w_i^2]_{i \,\epsilon\, \mathbb{N}\cup\{0\}}$ of the polynomial ring of Stiefel-Whitney classes.

So, theorem 1 is proved. □

USING INTEGRAL COEFFICIENTS

We will lean on the obtained results for $\mathbb{Z}_2$-coefficients and use the mod 2 - reduction homomorphism

$$\rho : H^*(B\mathcal{O}, \mathbb{Z}) \to H^*(B\mathcal{O}, \mathbb{Z}_2)$$

to prove the theorems with $\mathbb{Z}$-coefficients. Define V_I as in appendix A.

Lemma 2. $\rho(V_I^2(\xi)) = \sum_{i \,\epsilon\, I\cap\{\frac{1}{2}\}} w_1^2(\xi\oplus\xi) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(\xi\oplus\xi)$
$+ \sum_{i \,\epsilon\, I\backslash\{\frac{1}{2}\}} (w_{4i+2}(\xi\oplus\xi) + w_2(\xi\oplus\xi)\cup w_{4i}(\xi\oplus\xi)) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(\xi\oplus\xi),$
for any real bundle ξ.

Proof. Apply the reduction homomorphism:

$$\rho[V_I^2(\xi)] = (\rho[V_I(\xi)])^2 = (Sq^1[\bigcup_{i \,\epsilon\, I} w_{2i}(\xi)])^2 = (\sum_{i \,\epsilon\, I} Sq^1[w_{2i}(\xi)] \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{2j}(\xi))^2$$

$$= [\sum_{i \,\epsilon\, I\cap\{\frac{1}{2}\}} w_1^2(\xi) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{2j}(\xi) + \sum_{i \,\epsilon\, I\backslash\{\frac{1}{2}\}} (w_{2i+1}(\xi) + w_1(\xi)\cup w_{2i}(\xi)) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{2j}(\xi)]^2.$$

As $2 = 0$ in $H^*(B\mathcal{O}, \mathbb{Z}_2)$, this equals

$$= \sum_{i \,\epsilon\, I\cap\{\frac{1}{2}\}} w_1^4(\xi) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{2j}^2(\xi) + \sum_{i \,\epsilon\, I\backslash\{\frac{1}{2}\}} (w_{2i+1}^2(\xi) + w_1^2(\xi)\cup w_{2i}^2(\xi)) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{2j}^2(\xi).$$

Using the Whitney sum axiom and symmetry,

$$w_{4i}(\xi\oplus\xi) = \sum_{k=1}^{4i} w_k(\xi) w_{4i-k}(\xi) = w_{2i}^2(\xi).$$

Hence, the above term equals

$$\sum_{i \,\epsilon\, I\cap\{\frac{1}{2}\}} w_1^2(\xi\oplus\xi) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(\xi\oplus\xi)$$

$$+ \sum_{i \,\epsilon\, I\backslash\{\frac{1}{2}\}} (w_{4i+2}(\xi\oplus\xi) + w_2(\xi\oplus\xi)\cup w_{4i}(\xi\oplus\xi)) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(\xi\oplus\xi)$$

□

Proof of theorem 2. For $V_{\{\frac{1}{2}\}}$ and the Pontrjagin classes p_i, the result is trivial. Now let $F \to B$, $G \to B$ be real bundles with $F^{\mathbb{C}} \cong G^{\mathbb{C}}$. Forgetting the complex structure, that's $F\oplus F \cong G\oplus G$. By naturality of the Stiefel-Whitney classes,

$$\sum_{i \,\epsilon\, I\cap\{\frac{1}{2}\}} w_1^2(F\oplus F) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(F\oplus F)$$

$$+ \sum_{i \,\epsilon\, I\backslash\{\frac{1}{2}\}} (w_{4i+2}(F\oplus F) + w_2(F\oplus F)\cup w_{4i}(F\oplus F)) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(F\oplus F)$$

$$= \sum_{i \,\epsilon\, I\cap\{\frac{1}{2}\}} w_1^2(G\oplus G) \cup \bigcup_{j \,\epsilon\, I\backslash\{i\}} w_{4j}(G\oplus G)$$

$$+\sum_{i\,\epsilon\, I\setminus\{\frac{1}{2}\}}(w_{4i+2}(G\oplus G)+w_2(G\oplus G)\cup w_{4i}(G\oplus G))\cup\bigcup_{j\,\epsilon\, I\setminus\{i\}}w_{4j}(G\oplus G)$$

for any finite nonempty index set $I\subset(\{\frac{1}{2}\}\cup\mathbb{N}\setminus\{0\})$. Applying lemma 2, this means $\rho(V_I^2(F))=\rho(V_I^2(G))$.
As V_I^2 is in the torsion of $H^*(B\mathcal{O},\mathbb{Z})$, restricted on which ρ is injective [4, p. 513], this proves the theorem: $V_I^2(F)=V_I^2(G)$. □

Corollary from theorem 1
Let $C\,\epsilon\, H^*(B\mathcal{O},\mathbb{Z})$ fulfill the basic requirement. Then $\rho(C)\,\epsilon\,\mathbb{Z}_2[w_i^2]_{i\,\epsilon\,\mathbb{N}\cup\{0\}}$.

Proof. Let $F\to B$, $G\to B$ be real bundles with $F^{\mathbb{C}}\cong G^{\mathbb{C}}$. The reduction

$$\rho(C)\,\epsilon\, H^*(B\mathcal{O},\mathbb{Z}_2)$$

also satisfies the basic requirement:

$$\rho(C)(F)=\rho(C(F))=\rho(C(G))=\rho(C)(G).$$

Theorem 1 now gives the result. □

Proof of theorem 3.

Feshbach [4, p. 513] tells that $H^*(B\mathcal{O},\mathbb{Z})=\mathbb{Z}[p_i]_{i\,\epsilon\,\mathbb{N}}\oplus$ 2-Torsion.
$\Rightarrow C=\sum\bigcup p_i+T$ with some torsion element $T\,\epsilon\, H^*(B\mathcal{O},\mathbb{Z})$ ($\sharp$).
So for every real bundle ξ, $\rho(C)(\xi)=\sum\rho(\bigcup p_i(\xi))+\rho(T)(\xi)$.
$\Rightarrow^3$ $\rho(C)(\xi)=\sum\bigcup\rho((-1)^i c_{2i}(\xi^{\mathbb{C}}))+\rho(T)(\xi)$.
$\Rightarrow^4$ $\rho(C)(\xi)=\sum\bigcup w_{4i}(\xi\oplus\xi)+\rho(T)(\xi)$.
$\Rightarrow^5$ $\rho(C)(\xi)=\sum\bigcup w_{2i}^2(\xi)+\rho(T)(\xi)$.
Inserting the polynomial of the corollary from theorem 1, another polynomial in squares is produced: $\Rightarrow\sum\bigcup w_j^2(\xi)=\rho(T)(\xi)$.

As according to [4, p. 513], ρ is injective on the torsion elements, there is a local inverse ${\rho|_{2-\text{Torsion}}}^{-1}$ lifting $\rho(T)$ back to T.

$\Rightarrow {\rho|_{2-\text{Torsion}}}^{-1}(\sum\bigcup w_j^2(\xi))=T(\xi)$.

$\Rightarrow^5$ ${\rho|_{2-\text{Torsion}}}^{-1}(\sum\bigcup w_{2j}(\xi\oplus\xi))=T(\xi)$.

$\Rightarrow^4$ ${\rho|_{2-\text{Torsion}}}^{-1}(\sum\bigcup\rho(c_j(\xi^{\mathbb{C}})))=T(\xi)$.

$\Rightarrow^{\sharp,\,3}$ $C(\xi)=\sum\bigcup(-1)^i c_{2i}(\xi^{\mathbb{C}})+{\rho|_{2-\text{Torsion}}}^{-1}(\sum\bigcup\rho(c_j(\xi^{\mathbb{C}})))$. □

[3]By definition of the Pontrjagin classes.
[4]See [5, proposition 3.8] and use $(\xi^{\mathbb{C}})_{\mathbb{R}}=\xi\oplus\xi$.
[5]By the Whitney sum axiom and symmetry.

Appendix A

The cohomology ring of $B\mathcal{O}$ with $\mathbb{Z}$ -coefficients is known with all relations between its generators since Brown [1] and can be obtained as follows:
Define the set of generators of $H^*(B\mathcal{O}_n, \mathbb{Z})$ as in [4, definition 1]:
It consists of the Pontrjagin classes p_i of the universal bundle over $B\mathcal{O}_n$, and classes V_I with I ranging over all finite nonempty subsets of

$$\{\frac{1}{2}\} \cup \{k \in \mathbb{Z} \mid 0 < k < \frac{n+1}{2}\}$$

with the proviso that I does not contain both $\frac{1}{2}$ and $\frac{n}{2}$, for $n > 1$.
According to [4, theorem 2], $H^*(B\mathcal{O}_n, \mathbb{Z})$ is for all $n \leq \infty$ isomorphic to the polynomial ring over $\mathbb{Z}$ generated by the above specified elements modulo the ideal generated by the following six types of relations.
In all relations except the first, the cardinality of I is less than or equal to that of J and greater than one. (Most of the restrictions on I and J are to avoid repeating relations). By convention, $p_{\frac{1}{2}}$ where it occurs means $V_{\{\frac{1}{2}\}}$. Also, if $\{\frac{n}{2}, \frac{1}{2}\} \subset I \cup J$, then $V_{I \cup J}$ shall mean $V_{\{\frac{n}{2}\}} V_{(I \cup J) \setminus \{\frac{n}{2}, \frac{1}{2}\}}$.

1) $2V_I = 0$.
2) $V_I V_J + V_{I \cup J} V_{I \cap J} + V_{I \setminus J} V_{J \setminus I} \prod_{i \in I \cap J} p_i = 0$ (for $I \cap J \neq \emptyset$, $I \nsubseteq J$).
3) $V_I V_J + \sum_{i \in I} V_{\{i\}} V_{(J \setminus I) \cup \{i\}} \prod_{j \in I \setminus \{i\}} p_j = 0$ (for $I \subset J$).
4) $V_I V_J + \sum_{i \in I} V_{\{i\}} V_{(I \cup J) \setminus \{i\}} = 0$ (for $I \cap J = \emptyset$; if I and J have the same cardinality, then the smallest element of I is less than that of J).
5) $\sum_{i \in I} V_{\{i\}} V_{I \setminus \{i\}} = 0$.
6) $V_{\{\frac{1}{2}\}} p_{\frac{n}{2}} + V^2_{\{\frac{n}{2}\}} = 0$, if n is even.

Then $\rho(V_I) = Sq^1(\bigcup_{i \in I} w_{2i})$, with the Steenrod squaring operation Sq^1.

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