Title	Higher torsion in the Abelianization of the full Bianchi groups
Author(s)	Rahm, Alexander D.
Publication Date	2013-09
Publication Information	Rahm, AD (2013) 'Higher torsion in the Abelianization of the full Bianchi groups'. Lms Journal Of Computation And Mathematics, 16 :344-365.
Publisher	Cambridge University Press (Cambridge Journals Online)
Link to publisher's version	http://dx.doi.org/10.1112/S1461157013000168
Item record	http://hdl.handle.net/10379/4123
DOI	http://dx.doi.org/DOI 10.1112/S1461157013000168

HIGHER TORSION IN THE ABELIANIZATION
OF THE FULL BIANCHI GROUPS

ALEXANDER D. RAHM

ABSTRACT. Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}_{-m}$ its ring of integers. The *Bianchi groups* are the groups $\mathrm{SL}_2(\mathcal{O}_{-m})$. In the literature, there has been so far no example of p-torsion in the integral homology of the full Bianchi groups, for p a prime greater than the order of elements of finite order in the Bianchi group, which is at most 6.

However, extending the scope of the computations, we can observe examples of torsion in the integral homology of the quotient space, at prime numbers as high as for instance $p = 80737$ at the discriminant -1747.

1. INTRODUCTION

The Bianchi groups $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$ may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which date back to work of Henri Poincaré [Poi83]. In fact, each non-co-compact arithmetic Kleinian group is commensurable with some Bianchi group [MR03]. A wealth of information on the Bianchi groups can be found in the monographs [Fin89], [EGM98], [MR03]. In the literature, there has been so far no example of p-torsion in the integral homology of the full Bianchi groups, for p a prime greater than the order of elements of finite order in the Bianchi group (a recent survey of relevant calculations has been given in [Şen12]). In fact, the numerical studies that have been made so far, were carried out in the range where the quotient space of hyperbolic 3-space $\mathcal{H}$ by the Bianchi group is often homotopy equivalent to a wedge sum of 2-spheres, 2-tori and Möbius bands [Vog85].

We make use of Serre's decomposition [Ser70] of the homology group $\mathrm{H}_1(\Gamma\backslash\mathcal{H}; \mathbb{Z})$ into the direct sum of the free Abelian group with one generator for each element of the class group of $\mathcal{O}_{-m}$ and the group $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H}; \mathbb{Z})$ computed in figures 1 and 2. The first compuations of $\mathrm{H}_1(\Gamma; \mathbb{Z}) \supset \mathrm{H}_1(\Gamma\backslash\mathcal{H}; \mathbb{Z})$ by Swan [Swa71] were on a range of Bianchi groups with vanishing cuspidal homology $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H}; \mathbb{Z})$. The first example where $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H}; \mathbb{Z})$ is non-zero, occurred in an unpublished calculation of Mennicke. Swan's manual computations of group presentations have been extended on the computer by Riley [Ril83]; and later Vogtmann [Vog85] and Scheutzow [Sch92] systematically computed $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H}; \mathbb{Q})$ for a large range of Bianchi groups. But they were still in in the range where $\Gamma\backslash\mathcal{H}$ admits no homological torsion. Aranes [Ara10] has computed cell complexes for the Bianchi groups for all $m \leqslant 100$, and Yasaki [Yas10] has obtained $\mathrm{GL}_2(\mathcal{O}_{-m})$-cell complexes (with the Voronoï model) for the same range as well as all cases where $\mathcal{O}$ is of class number 1 or 2. This includes two cases, $m = 74$ and $m = 86$, where some 2-torsion appears in $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H}; \mathbb{Z})$, but the latter two authors have not yet provided homology computations. When the absolute

value of the discriminant gets greater, torsion in the integral homology of the quotient space appears (see figure 2) at prime numbers as high as for instance 80737 at the discriminant -1747, whereas the order of elements of finite order in $\mathrm{SL}_2(\mathcal{O}_{-m})$ is at most 6. A growth of the torsion in the Abelianization of the Bianchi groups with respect to the covolume can be observed, which is in concordance with the predictions of [BV12]. We can also observe that the occurring torsion subgroups are quite likely to occur as squares, but this is no general principle, because the discriminant -431 produces a counterexample to this phenomenon.

In order to obtain the results of figures 1 and 2, in section 7 we fill out Swan's concept [Swa71] and elaborate algorithms to compute a fundamental polyhedron for the action of the Bianchi groups on hyperbolic 3-space. Other algorithms based on the same concept have independently been implemented by Cremona [Cre84] for the five cases where $\mathcal{O}_{-m}$ is Euclidean, and by his students Whitley [Whi90] for the non-Euclidean principal ideal domain cases, Bygott [Byg98] for a case of class number 2 and Lingham ([Lin05], used in [CL07]) for some cases of class number 3; and finally Aranés [Ara10] for arbitrary class numbers. The algorithms presented in subsection 7 come with an implementation [Rah10] for all Bianchi groups; and we make explicit use of the cell complexes it produces. The provided implementation [Rah10] has been validated by the project PLUME of the CNRS, and is subject to the certificate C3I of the GENCI and the CPU. Other results obtained with the employed implementation are described in [Rah11] and [RS12]. On the computing clusters of the Weizmann Institute of Science, this implementation has been applied to establish a database of cell complexes for over 180 Bianchi groups, using over fifty processor-months. This database includes all the cases of ideal class numbers 3 and 5, most of the cases of ideal class number 4 and all of the cases of discriminant absolute value bounded by 500.

A computational advantage is the shortcut that we obtain in section 4 by linking the Borel–Serre compactification of the quotient space with Flöge's compactification in a long exact sequence, based on the recent paper [Rah12b]. Flöge's compactification admits a computationally easier cell structure, and we can explicitly calculate the equivariant Leray–Serre spectral sequence associated to it. In section 5, we describe how to assemble the homology of the Borel–Serre compactified quotient space and the Farrell cohomology of a Bianchi group to its full group homology with trivial $\mathbb{Z}$–coefficients. Here, we divide by the center of $\mathrm{SL}_2(\mathcal{O}_{-m})$, consisting of plus and minus the identity matrix, yielding $\mathrm{PSL}_2(\mathcal{O}_{-m})$. As the center of $\mathrm{SL}_2(\mathcal{O}_{-m})$ is the kernel of its action on hyperbolic 3-space, this does not change the quotient space. And for $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$, general formulae for its Farrell cohomology have been given [Rah12] (based on [Rah11b]).

1.1. **Organization of the paper.** We print the isomorphism types of $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma \backslash \mathcal{H}; \mathbb{Z})$ that were obtained in figures 1 and 2. The homology group $\mathrm{H}_1(\Gamma \backslash \mathcal{H}; \mathbb{Z})$ is a direct sum of the former homology group and the free Abelian group with rank the cardinality of the class group of $\mathcal{O}_{-m}$, which we also print. The group homology $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is an extension of $\mathrm{H}_1(\Gamma \backslash \mathcal{H}; \mathbb{Z})$ by a quotient of the Farrell supplement that has been computed and printed in a separate column. In section 2, we define the Bianchi fundamental polyhedron, which induces our cell structure on $\Gamma \backslash \mathcal{H}$. We use it in section 3 to obtain the Flöge cellular complex, which we connect in section 4 to the Borel–Serre compactification of $\Gamma \backslash \mathcal{H}$. Then we proceed to $\mathrm{H}_1(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z})$ in section 5, describe Swan's concept in section 6 and its realization in section 7.

Δ	m	class group	H_1^{cusp}	Farrell supplement	Δ	m	class group	H_1^{cusp}	Farrell supplement
-7	7	$\{1\}$	0	$\mathbb{Z}/2$	-219	219	$\mathbb{Z}/4$	$\mathbb{Z}^9$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-8	2	$\{1\}$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-223	223	$\mathbb{Z}/7$	$\mathbb{Z}^8$	$(\mathbb{Z}/2)^3$
-11	11	$\{1\}$	0	$\mathbb{Z}/3$	-227	227	$\mathbb{Z}/5$	$\mathbb{Z}^7$	$\mathbb{Z}/3$
-15	15	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-228	57	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{12}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-19	19	$\{1\}$	0	0	-231	231	$\mathbb{Z}/6 \times \mathbb{Z}/2$	$\mathbb{Z}^9$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-20	5	$\mathbb{Z}/2$	0	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-232	58	$\mathbb{Z}/2$	$\mathbb{Z}^{10}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-23	23	$\mathbb{Z}/3$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-235	235	$\mathbb{Z}/2$	$\mathbb{Z}^{11}$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$
-24	6	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-239	239	$\mathbb{Z}/15$	$\mathbb{Z}^3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-31	31	$\mathbb{Z}/3$	0	$\mathbb{Z}/2$	-244	61	$\mathbb{Z}/6$	$\mathbb{Z}^9$	$(\mathbb{Z}/2)^2$
-35	35	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-247	247	$\mathbb{Z}/6$	$\mathbb{Z}^8$	$\mathbb{Z}/2$
-39	39	$\mathbb{Z}/4$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-248	62	$\mathbb{Z}/8$	$\mathbb{Z}^8$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-40	10	$\mathbb{Z}/2$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-251	251	$\mathbb{Z}/7$	$\mathbb{Z}^7$	$\mathbb{Z}/3$
-43	43	$\{1\}$	$\mathbb{Z}$	0	-255	255	$\mathbb{Z}/6 \times \mathbb{Z}/2$	$\mathbb{Z}^{11}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$
-47	47	$\mathbb{Z}/5$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-259	259	$\mathbb{Z}/4$	$\mathbb{Z}^{10}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-51	51	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/3$	-260	65	$\mathbb{Z}/4 \times \mathbb{Z}/2$	$\mathbb{Z}^{12}$	$(\mathbb{Z}/2)^5 \oplus (\mathbb{Z}/3)^2$
-52	13	$\mathbb{Z}/2$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$	-263	263	$\mathbb{Z}/13$	$\mathbb{Z}^5$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-55	55	$\mathbb{Z}/4$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-264	66	$\mathbb{Z}/4 \times \mathbb{Z}/2$	$\mathbb{Z}^{12}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$
-56	14	$\mathbb{Z}/4$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-267	267	$\mathbb{Z}/2$	$\mathbb{Z}^{13}$	$\mathbb{Z}/3$
-59	59	$\mathbb{Z}/3$	$\mathbb{Z}$	$\mathbb{Z}/3$	-271	271	$\mathbb{Z}/11$	$\mathbb{Z}^6$	$\mathbb{Z}/2$
-67	67	$\{1\}$	$\mathbb{Z}^2$	0	-276	69	$\mathbb{Z}/4 \times \mathbb{Z}/2$	$\mathbb{Z}^{15}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-68	17	$\mathbb{Z}/4$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-280	70	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{15}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-71	71	$\mathbb{Z}/7$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-283	283	$\mathbb{Z}/3$	$\mathbb{Z}^{10}$	0
-79	79	$\mathbb{Z}/5$	$\mathbb{Z}$	$(\mathbb{Z}/2)^3$	-287	287	$\mathbb{Z}/14$	$\mathbb{Z}^7$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-83	83	$\mathbb{Z}/3$	$\mathbb{Z}^2$	$\mathbb{Z}/3$	-291	291	$\mathbb{Z}/4$	$\mathbb{Z}^{13}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-84	21	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^3$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$	-292	73	$\mathbb{Z}/4$	$\mathbb{Z}^{12}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-87	87	$\mathbb{Z}/6$	$\mathbb{Z}^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-295	295	$\mathbb{Z}/8$	$\mathbb{Z}^{11}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-88	22	$\mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-296	74	$\mathbb{Z}/10$	$\mathbb{Z}^9 \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-91	91	$\mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}/2$	-299	299	$\mathbb{Z}/8$	$\mathbb{Z}^{10}$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^4$
-95	95	$\mathbb{Z}/8$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-303	303	$\mathbb{Z}/10$	$\mathbb{Z}^{12}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-103	103	$\mathbb{Z}/5$	$\mathbb{Z}^2$	$\mathbb{Z}/2$	-307	307	$\mathbb{Z}/3$	$\mathbb{Z}^{11}$	0
-104	26	$\mathbb{Z}/6$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$	-308	77	$\mathbb{Z}/4 \times \mathbb{Z}/2$	$\mathbb{Z}^{15}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-107	107	$\mathbb{Z}/3$	$\mathbb{Z}^3$	$(\mathbb{Z}/3)^3$	-311	311	$\mathbb{Z}/19$	$\mathbb{Z}^4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-111	111	$\mathbb{Z}/8$	$\mathbb{Z}^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-312	78	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{18}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-115	115	$\mathbb{Z}/2$	$\mathbb{Z}^5$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-319	319	$\mathbb{Z}/10$	$\mathbb{Z}^{10}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-116	29	$\mathbb{Z}/6$	$\mathbb{Z}^3$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-323	323	$\mathbb{Z}/4$	$\mathbb{Z}^{12}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-119	119	$\mathbb{Z}/10$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-327	327	$\mathbb{Z}/12$	$\mathbb{Z}^{12}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-120	30	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^6$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^3$	-328	82	$\mathbb{Z}/4$	$\mathbb{Z}^{13}$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$
-123	123	$\mathbb{Z}/2$	$\mathbb{Z}^5$	$\mathbb{Z}/3$	-331	331	$\mathbb{Z}/3$	$\mathbb{Z}^{12}$	$\mathbb{Z}/3$
-127	127	$\mathbb{Z}/5$	$\mathbb{Z}^3$	$\mathbb{Z}/2$	-335	335	$\mathbb{Z}/18$	$\mathbb{Z}^8$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-131	131	$\mathbb{Z}/5$	$\mathbb{Z}^3$	$\mathbb{Z}/3$	-339	339	$\mathbb{Z}/6$	$\mathbb{Z}^{15}$	$\mathbb{Z}/3$
-132	33	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^6$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^4$	-340	85	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{19}$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2$
-136	34	$\mathbb{Z}/4$	$\mathbb{Z}^4$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$	-344	86	$\mathbb{Z}/10$	$\mathbb{Z}^{11} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-139	139	$\mathbb{Z}/3$	$\mathbb{Z}^4$	0	-347	347	$\mathbb{Z}/5$	$\mathbb{Z}^{12}$	$\mathbb{Z}/3$
-143	143	$\mathbb{Z}/10$	$\mathbb{Z}^2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$	-355	355	$\mathbb{Z}/4$	$\mathbb{Z}^{16}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-148	37	$\mathbb{Z}/2$	$\mathbb{Z}^6$	$(\mathbb{Z}/2)^4$	-356	89	$\mathbb{Z}/12$	$\mathbb{Z}^{12}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-151	151	$\mathbb{Z}/7$	$\mathbb{Z}^3$	$\mathbb{Z}/2$	-359	359	$\mathbb{Z}/19$	$\mathbb{Z}^6 \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$
-152	38	$\mathbb{Z}/6$	$\mathbb{Z}^4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-367	367	$\mathbb{Z}/9$	$\mathbb{Z}^{11} \oplus (\mathbb{Z}/3)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-155	155	$\mathbb{Z}/4$	$\mathbb{Z}^6$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-371	371	$\mathbb{Z}/8$	$\mathbb{Z}^{14}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-159	159	$\mathbb{Z}/10$	$\mathbb{Z}^4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-372	93	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{23}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-163	163	$\{1\}$	$\mathbb{Z}^6$	0	-376	94	$\mathbb{Z}/8$	$\mathbb{Z}^{14}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-164	41	$\mathbb{Z}/8$	$\mathbb{Z}^4$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-379	379	$\mathbb{Z}/3$	$\mathbb{Z}^{14}$	0
-167	167	$\mathbb{Z}/11$	$\mathbb{Z}^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-383	383	$\mathbb{Z}/17$	$\mathbb{Z}^8$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-168	42	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^9$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$	-388	97	$\mathbb{Z}/4$	$\mathbb{Z}^{17}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-179	179	$\mathbb{Z}/5$	$\mathbb{Z}^5$	$\mathbb{Z}/3$	-391	391	$\mathbb{Z}/14$	$\mathbb{Z}^{11}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-183	183	$\mathbb{Z}/8$	$\mathbb{Z}^6$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-395	395	$\mathbb{Z}/8$	$\mathbb{Z}^{16} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-184	46	$\mathbb{Z}/4$	$\mathbb{Z}^7$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-399	399	$\mathbb{Z}/8 \times \mathbb{Z}/2$	$\mathbb{Z}^{17}$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2$
-187	187	$\mathbb{Z}/2$	$\mathbb{Z}^7$	$\mathbb{Z}/3$	-403	403	$\mathbb{Z}/2$	$\mathbb{Z}^{17}$	$\mathbb{Z}/2$
-191	191	$\mathbb{Z}/13$	$\mathbb{Z}^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-404	101	$\mathbb{Z}/14$	$\mathbb{Z}^{14}$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$
-195	195	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{11}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$	-407	407	$\mathbb{Z}/16$	$\mathbb{Z}^{13}$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$
-199	199	$\mathbb{Z}/9$	$\mathbb{Z}^4$	$\mathbb{Z}/2$	-408	102	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{23}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^6$
-203	203	$\mathbb{Z}/4$	$\mathbb{Z}^8$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	-411	411	$\mathbb{Z}/6$	$\mathbb{Z}^{19}$	$\mathbb{Z}/3$
-211	211	$\mathbb{Z}/3$	$\mathbb{Z}^7$	0	-415	415	$\mathbb{Z}/10$	$\mathbb{Z}^{18}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-212	53	$\mathbb{Z}/6$	$\mathbb{Z}^8$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	-419	419	$\mathbb{Z}/9$	$\mathbb{Z}^{13}$	$(\mathbb{Z}/3)^3$
-215	215	$\mathbb{Z}/14$	$\mathbb{Z}^4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$					

FIGURE 1. The cuspidal homology $H_1^{cusp}(\Gamma \backslash \mathcal{H}; \mathbb{Z})$ for the absolute values of the discriminant Δ fulfilling $|\Delta| \leqslant 415$.

Discriminant	m	class group	$\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H};\mathbb{Z})$	Farrell supplement
-420	105	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{33}$	$(\mathbb{Z}/2)^8 \oplus (\mathbb{Z}/3)^4$
-424	106	$\mathbb{Z}/6$	$\mathbb{Z}^{17} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-427	427	$\mathbb{Z}/2$	$\mathbb{Z}^{19}$	$(\mathbb{Z}/2)^3$
-431	431	$\mathbb{Z}/21$	$\mathbb{Z}^8 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-435	435	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{27}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^6$
-436	109	$\mathbb{Z}/6$	$\mathbb{Z}^{19} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
-439	439	$\mathbb{Z}/15$	$\mathbb{Z}^{11}$	$(\mathbb{Z}/2)^5$
-440	110	$\mathbb{Z}/6 \times \mathbb{Z}/2$	$\mathbb{Z}^{20}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-443	443	$\mathbb{Z}/5$	$\mathbb{Z}^{16}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-447	447	$\mathbb{Z}/14$	$\mathbb{Z}^{18}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-451	451	$\mathbb{Z}/6$	$\mathbb{Z}^{17}$	$\mathbb{Z}/3$
-452	113	$\mathbb{Z}/8$	$\mathbb{Z}^{19} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
-455	455	$\mathbb{Z}/10 \times \mathbb{Z}/2$	$\mathbb{Z}^{19} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-456	114	$\mathbb{Z}/4 \times \mathbb{Z}/2$	$\mathbb{Z}^{24} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^4$
-463	463	$\mathbb{Z}/7$	$\mathbb{Z}^{16}$	$\mathbb{Z}/2$
-467	467	$\mathbb{Z}/7$	$\mathbb{Z}^{16}$	$\mathbb{Z}/3$
-471	471	$\mathbb{Z}/16$	$\mathbb{Z}^{18}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-472	118	$\mathbb{Z}/6$	$\mathbb{Z}^{19} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-479	479	$\mathbb{Z}/25$	$\mathbb{Z}^8 \oplus (\mathbb{Z}/3)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-483	483	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{29}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-487	487	$\mathbb{Z}/7$	$\mathbb{Z}^{17} \oplus (\mathbb{Z}/13)^2$	$\mathbb{Z}/2$
-488	122	$\mathbb{Z}/10$	$\mathbb{Z}^{18} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-491	491	$\mathbb{Z}/9$	$\mathbb{Z}^{16}$	$\mathbb{Z}/3$
-499	499	$\mathbb{Z}/3$	$\mathbb{Z}^{19} \oplus (\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^2$
-520	130	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{28} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2$
-523	523	$\mathbb{Z}/5$	$\mathbb{Z}^{19}$	0
-532	133	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{29}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-547	547	$\mathbb{Z}/3$	$\mathbb{Z}^{21} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/3)^2$
-555	555	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{35}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-568	142	$\mathbb{Z}/4$	$\mathbb{Z}^{25} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6 \oplus \mathbb{Z}/3$
-571	571	$\mathbb{Z}/5$	$\mathbb{Z}^{23}$	0
-595	595	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{33}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^4$
-619	619	$\mathbb{Z}/5$	$\mathbb{Z}^{23} \oplus (\mathbb{Z}/3)^2$	0
-627	627	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{35}$	$(\mathbb{Z}/3)^2$
-643	643	$\mathbb{Z}/3$	$\mathbb{Z}^{27}$	$\mathbb{Z}/3$
-667	667	$\mathbb{Z}/4$	$\mathbb{Z}^{28}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-683	683	$\mathbb{Z}/5$	$\mathbb{Z}^{26}$	$\mathbb{Z}/3$
-691	691	$\mathbb{Z}/5$	$\mathbb{Z}^{26} \oplus (\mathbb{Z}/7)^2$	0
-696	174	$\mathbb{Z}/6 \times \mathbb{Z}/2$	$\mathbb{Z}^{38}$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^3$
-715	715	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{39}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$
-723	723	$\mathbb{Z}/4$	$\mathbb{Z}^{37} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-739	739	$\mathbb{Z}/5$	$\mathbb{Z}^{28}$	0
-760	190	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{42} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2$
-763	763	$\mathbb{Z}/4$	$\mathbb{Z}^{34}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-787	787	$\mathbb{Z}/5$	$\mathbb{Z}^{30}$	0
-795	795	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}^{51}$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$
-883	883	$\mathbb{Z}/3$	$\mathbb{Z}^{35}$	0
-907	907	$\mathbb{Z}/3$	$\mathbb{Z}^{36} \oplus (\mathbb{Z}/13)^2$	0
-947	947	$\mathbb{Z}/5$	$\mathbb{Z}^{37} \oplus (\mathbb{Z}/89)^2$	$\mathbb{Z}/3$
-955	955	$\mathbb{Z}/4$	$\mathbb{Z}^{46} \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1003	1003	$\mathbb{Z}/4$	$\mathbb{Z}^{44} \oplus (\mathbb{Z}/3)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1027	1027	$\mathbb{Z}/4$	$\mathbb{Z}^{44} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3$
-1051	1051	$\mathbb{Z}/5$	$\mathbb{Z}^{43} \oplus (\mathbb{Z}/13)^2$	0
-1123	1123	$\mathbb{Z}/5$	$\mathbb{Z}^{44} \oplus (\mathbb{Z}/7)^2$	0
-1227	1227	$\mathbb{Z}/4$	$\mathbb{Z}^{65} \oplus (\mathbb{Z}/2^2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1243	1243	$\mathbb{Z}/4$	$\mathbb{Z}^{54} \oplus (\mathbb{Z}/3)^4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1387	1387	$\mathbb{Z}/4$	$\mathbb{Z}^{58} \oplus (\mathbb{Z}/167)^2$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3$
-1411	1411	$\mathbb{Z}/4$	$\mathbb{Z}^{60} \oplus (\mathbb{Z}/2^4)^2 \oplus (\mathbb{Z}/43)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1507	1507	$\mathbb{Z}/4$	$\mathbb{Z}^{66} \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/5)^4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1555	1555	$\mathbb{Z}/4$	$\mathbb{Z}^{76} \oplus (\mathbb{Z}/2^2)^8 \oplus (\mathbb{Z}/11)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
-1723	1723	$\mathbb{Z}/5$	$\mathbb{Z}^{69} \oplus (\mathbb{Z}/7)^2 \oplus (\mathbb{Z}/23)^2 \oplus (\mathbb{Z}/883)^2$	0
-1747	1747	$\mathbb{Z}/5$	$\mathbb{Z}^{70} \oplus (\mathbb{Z}/80737)^2$	$(\mathbb{Z}/3)^2$
-1867	1867	$\mathbb{Z}/5$	$\mathbb{Z}^{75} \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/7^2)^2 \oplus (\mathbb{Z}/137)^2$	0

FIGURE 2. $\mathrm{H}_1^{\mathrm{cusp}}(\Gamma\backslash\mathcal{H};\mathbb{Z})$, with its torsion decomposed into prime power factors, for some greater absolute values of the discriminant.

2. The Bianchi fundamental polyhedron

Let m be a squarefree positive integer and consider the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$ with ring of integers $\mathcal{O}_{-m}$, which we also just denote by $\mathcal{O}$. Consider the familiar action by fractional linear transformations (we give an explicit formula for it in lemma 22) of the group $\Gamma := \mathrm{SL}_2(\mathcal{O}) \subset \mathrm{GL}_2(\mathbb{C})$ on hyperbolic three-space, for which we will use the upper-half space model $\mathcal{H}$. As a set,

$$\mathcal{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}.$$

The Bianchi–Humbert theory [Bia92], [Hum15] gives a fundamental domain for this action. We will start by giving a geometric description of it, and the arguments why it is a fundamental domain.

Definition 1. A pair of elements $(\mu, \lambda) \in \mathcal{O}^2$ is called *unimodular* if the ideal sum $\mu\mathcal{O} + \lambda\mathcal{O}$ equals $\mathcal{O}$.

The boundary of $\mathcal{H}$ is the Riemann sphere $\partial\mathcal{H} = \mathbb{C} \cup \{\infty\}$ (as a set), which contains the complex plane $\mathbb{C}$. The totally geodesic surfaces in $\mathcal{H}$ are the Euclidean vertical planes (we define *vertical* as orthogonal to the complex plane) and the Euclidean hemispheres centred on the complex plane.

Notation 2. Given a unimodular pair $(\mu, \lambda) \in \mathcal{O}^2$ with $\mu \neq 0$, let $S_{\mu,\lambda} \subset \mathcal{H}$ denote the hemisphere given by the equation $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$.

This hemisphere has centre λ/μ on the complex plane $\mathbb{C}$, and radius $1/|\mu|$. Let
$B := \big\{(z, \zeta) \in \mathcal{H}:$ The inequality $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$

is fulfilled for all unimodular pairs $(\mu, \lambda) \in \mathcal{O}^2$ with $\mu \neq 0$ $\big\}$.

Then B is the set of points in $\mathcal{H}$ which lie above or on all hemispheres $S_{\mu,\lambda}$.

Lemma 3 ([Swa71]). *The set B contains representatives for all the orbits of points under the action of $\mathrm{SL}_2(\mathcal{O})$ on $\mathcal{H}$.*

The action extends continuously to the boundary $\partial\mathcal{H}$, which is a Riemann sphere.
In $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$, consider the stabiliser subgroup Γ_∞ of the point $\infty \in \partial\mathcal{H}$. In the cases $m = 1$ and $m = 3$, the latter group contains some rotation matrices like $\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$, which we want to exclude. These two cases have been treated in [Men79], [SV83] and others, and we assume $m \neq 1$, $m \neq 3$ throughout the remainder of this article. Then,

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathcal{O} \right\},$$

which performs translations by the elements of $\mathcal{O}$ with respect to the Euclidean geometry of the upper-half space $\mathcal{H}$.

Notation 4. A fundamental domain for Γ_∞ in the complex plane (as a subset of $\partial\mathcal{H}$) is given by the rectangle

$$D_0 := \begin{cases} \{x + y\sqrt{-m} \in \mathbb{C} \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}, & m \equiv 1 \text{ or } 2 \mod 4, \\ \{x + y\sqrt{-m} \in \mathbb{C} \mid \frac{-1}{2} \leqslant x \leqslant \frac{1}{2},\ 0 \leqslant y \leqslant \frac{1}{2}\}, & m \equiv 3 \mod 4. \end{cases}$$

And a fundamental domain for Γ_∞ in $\mathcal{H}$ is given by

$$D_\infty := \{(z, \zeta) \in \mathcal{H} \mid z \in D_0\}.$$

Definition 5. We define the *Bianchi fundamental polyhedron* as

$$D := D_\infty \cap B.$$

It is a polyhedron in hyperbolic space up to the missing vertex ∞, and up to missing vertices at the singular points if $\mathcal{O}$ is not a principal ideal domain (see subsection 6.2). As Lemma 3 states $\Gamma \cdot B = \mathcal{H}$, and as $\Gamma_\infty \cdot D_\infty = \mathcal{H}$ yields $\Gamma_\infty \cdot D = B$, we have $\Gamma \cdot D = \mathcal{H}$. We observe the following notion of strictness of the fundamental domain: the interior of the Bianchi fundamental polyhedron contains no two points which are identified by Γ. Swan proves the following theorem, which implies that the boundary of the Bianchi fundamental polyhedron consists of finitely many cells.

Theorem 6 ([Swa71]). *There is only a finite number of unimodular pairs (λ, μ) such that the intersection of $S_{\mu,\lambda}$ with the Bianchi fundamental polyhedron is non-empty.*

Swan further proves a corollary, from which it can be deduced that the action of Γ on $\mathcal{H}$ is properly discontinuous.

3. THE FLÖGE CELLULAR COMPLEX

In order to obtain a cell complex with compact quotient space, we proceed in the following way due to Flöge [Flö83]. The boundary of $\mathcal{H}$ is the Riemann sphere $\partial\mathcal{H}$, which, as a topological space, is made up of the complex plane $\mathbb{C}$ compactified with the cusp ∞. The totally geodesic surfaces in $\mathcal{H}$ are the Euclidean vertical planes (we define *vertical* as orthogonal to the complex plane) and the Euclidean hemispheres centred on the complex plane. The action of the Bianchi groups extends continuously to the boundary $\partial\mathcal{H}$. Consider the cellular structure on $\mathcal{H}$ induced by the Γ-images of the Bianchi fundamental polyhedron. The cellular closure of this cell complex in $\mathcal{H} \cup \partial\mathcal{H}$ consists of $\mathcal{H}$ and $\left(\mathbb{Q}(\sqrt{-m}) \cup \{\infty\}\right) \subset (\mathbb{C} \cup \{\infty\}) \cong \partial\mathcal{H}$. The $\mathrm{SL}_2(\mathcal{O}_{-m})$–orbit of a cusp $\frac{\lambda}{\mu}$ in $\left(\mathbb{Q}(\sqrt{-m}) \cup \{\infty\}\right)$ corresponds to the ideal class $[(\lambda, \mu)]$ of $\mathcal{O}_{-m}$. It is well-known that this does not depend on the choice of the representative $\frac{\lambda}{\mu}$. We extend our cell complex to a cell complex $\widetilde{X}$ by joining to it, in the case that $\mathcal{O}_{-m}$ is not a principal ideal domain, the $\mathrm{SL}_2(\mathcal{O}_{-m})$–orbits of the cusps $\frac{\lambda}{\mu}$ for which the ideal (λ, μ) is not principal. At these cusps, we equip $\widetilde{X}$ with the "horoball topology" described in [Flö83]. This simply means that the set of cusps, which is discrete in $\partial\mathcal{H}$, is located at the hyperbolic extremities of $\widetilde{X}$: No neighbourhood of a cusp, except the whole $\widetilde{X}$, contains any other cusp.

We retract $\widetilde{X}$ in the following, $\mathrm{SL}_2(\mathcal{O}_{-m})$–equivariant, way. On the Bianchi fundamental polyhedron, the retraction is given by the vertical projection (away from the cusp ∞) onto its facets which are closed in $\mathcal{H} \cup \partial\mathcal{H}$. The latter are the facets which do not touch the cusp ∞, and are the bottom facets with respect to our vertical direction. The retraction is continued on $\mathcal{H}$ by the group action. It is proven in [Flö80] that this retraction is continuous. We denote by X the retract of $\widetilde{X}$.

Notation 7. We will call this 2–dimensional retract X the *retracted Flöge cellular complex*.

So in the principal ideal domain cases, X is a retract of the original cellular structure on $\mathcal{H}$, obtained by contracting the Bianchi fundamental polyhedron onto its cells which do

not touch the boundary of $\mathcal{H}$. In [RF11], it is checked that the retracted Flöge cellular complex is contractible.

4. Connecting Flöge cell complex and Borel–Serre compactification

Let Γ be a Bianchi group with $\mathcal{O}$ admitting as only units $\{\pm 1\}$, i.e. we suppose $\mathcal{O}$ not to be the Gaussian or Eisenstein integers. In the latter two cases, the problem of the singular cusps treated here does not occur in any case. We make use of the Borel–Serre compactification [Ser70] for $_\Gamma\backslash\mathcal{H}$. Recall that in this case, the Borel–Serre compactification joins a 2-torus $\mathbb{T}$ to $_\Gamma\backslash\mathcal{H}$ at every cusp. Details are given in [Rah12b]. Let $\mathbb{T}_i$ be the torus attached at the cusp i of Γ. We decompose $\mathbb{T}_i$ in the classical way into a 2–cell, two 1–cells and a vertex. Let x_i and y_i denote the cycles generating $H_1(\mathbb{T}_i)$; they are given by the two 1–cells. Let P be the Bianchi fundamental polyhedron of Γ. Write "hyp. cells" for cells in the interior of hyperbolic space. Denote by $\widetilde{\partial}$ the boundary operator for the cell complex $\widetilde{X}$. Consider following the short exact sequence of chain complexes that we obtain from collapsing the singular tori.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \langle P \rangle & \longrightarrow & \langle P \rangle & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\partial_3} & & \downarrow{\widetilde{\partial}_3} & & \\
0 & \longrightarrow & \overset{\text{singular}}{\underset{s}{\bigoplus}} \langle \mathbb{T}_s \rangle & \overset{\beta}{\longrightarrow} & \overset{\text{any cusp}}{\underset{c}{\bigoplus}} \langle \mathbb{T}_c \rangle \oplus \langle \text{hyp. 2-cells} \rangle & \longrightarrow & \langle \mathbb{T}_\infty \rangle \oplus \langle \text{hyp. 2-cells} \rangle & \longrightarrow & 0 \\
& & \downarrow{0} & & \downarrow{0 \oplus \partial_2} & & \downarrow{0 \oplus \widetilde{\partial}_2} & & \\
0 & \longrightarrow & \overset{\text{singular}}{\underset{s}{\bigoplus}} \langle x_s, y_s \rangle & \overset{\beta}{\longrightarrow} & \overset{\text{any cusp}}{\underset{c}{\bigoplus}} \langle x_c, y_c \rangle \oplus \langle \text{hyp. edges} \rangle & \longrightarrow & \langle x_\infty, y_\infty \rangle \oplus \langle \text{hyp. edges} \rangle & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{0 \oplus \partial_1} & & \downarrow{0 \oplus \partial_1} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \overset{\text{any cusp}}{\underset{c}{\bigoplus}} \langle c \rangle \oplus \langle \text{hyp. vertices} \rangle & \longrightarrow & \overset{\text{any cusp}}{\underset{c}{\bigoplus}} \langle c \rangle \oplus \langle \text{hyp. vertices} \rangle & \longrightarrow & 0
\end{array}
$$

Poincaré's theorem on fundamental polyhedra tells us that $\partial_3(P) = \left\langle \overset{\text{any cusp}}{\underset{c}{\bigcup}} \mathbb{T}_c \right\rangle$, and hence $\widetilde{\partial}_3(P) = \langle \mathbb{T}_\infty \rangle$. From [Rah12b], we see that for every cusp c, there is a chain of hyperbolic 2-cells that we denote by $ch(x_c)$ and which is mapped to the cycle x_c by ∂_2. And furthermore, y_c is in the cokernel of ∂_2 (of course, this holds up to the appropriate permutation of the labels x_c and y_c). This implies that $\widetilde{\partial}_2(ch(x_\infty)) = x_\infty$ and y_∞ is in the cokernel of $\widetilde{\partial}_2$. As the quotient space is path-wise connected, the cokernel of ∂_1 is isomorphic to $\mathbb{Z}$. The above information tells us that the long exact sequence induced on integral homology by the map β concentrates in

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \overset{\text{singular}}{\underset{s}{\bigoplus}} \langle \mathbb{T}_s \rangle & \overset{\beta_2}{\longrightarrow} & \big(\overset{\text{any cusp}}{\underset{c}{\bigoplus}} \langle \mathbb{T}_c \rangle \big) /_{\langle \cup_c \mathbb{T}_c \rangle} \oplus H_2^{\text{cusp}} & \longrightarrow & H_2^{\text{cusp}} \underset{s}{\bigoplus} \langle ch(x_s) \rangle \longrightarrow \\[2mm]
& & \underset{s}{\bigoplus} \langle x_s, y_s \rangle & \overset{\beta_1}{\longrightarrow} & \underset{c}{\bigoplus} \langle y_c \rangle \oplus H_1^{\text{cusp}} & \longrightarrow & H_1^{\text{cusp}} \oplus \langle y_\infty \rangle \longrightarrow \\[2mm]
& & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \longrightarrow 0,
\end{array}
$$

where the maps without labels are the obvious restriction maps making the sequence exact; and where H_1^{cusp} and H_2^{cusp} are generated by cycles from the interior of $_\Gamma\backslash\mathcal{H}$.

Note that $H_2^{\mathrm{cusp}}\bigoplus_s\langle ch(x_s)\rangle$ is non-naturally isomorphic to $\left(\overset{\mathrm{any\ cusp}}{\underset{c}{\bigoplus}}\langle\mathbb{T}_c\rangle\right)\Big/_{\langle\cup_c\mathbb{T}_c\rangle}\oplus H_2^{\mathrm{cusp}}$, namely collapsing a torus $\mathbb{T}_s$ moves its 2-cycle into a bubble $ch(x_s)$ emerging adjacent to the singular cusp s in the Flöge complex.

5. The equivariant spectral sequence to group homology

Let $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$, and let X be the retracted Flöge cellular complex of section 3, the cell structure of which we subdivide until the cells are fixed pointwise by their stabilisers. We describe now how to assemble the homology of the Borel–Serre compactified quotient space (issue of the previous section) and the Farrell cohomology of Γ, for which general formulae have been given in [Rah12] (based on [Rah11b]), to the full group homology of Γ with trivial $\mathbb{Z}$–coefficients. We proceed following [Bro82, VII] and [SV83]. Let us consider the homology $H_*(\Gamma; C_\bullet(X))$ of Γ with coefficients in the cellular chain complex $C_\bullet(X)$ associated to X; and call it the Γ-equivariant homology of X. As X is contractible, the map $X \to pt.$ to the point $pt.$ induces an isomorphism

$$H_*(\Gamma;\ C_\bullet(X)) \to H_*(\Gamma;\ C_\bullet(pt.)) \cong H_*(\Gamma;\ \mathbb{Z}).$$

Denote by X^p the set of p-cells of X, and make use of that the stabiliser Γ_σ in Γ of any p-cell σ of X fixes σ pointwise. Then from

$$C_p(X) = \bigoplus_{\sigma\in X^p}\mathbb{Z} \cong \bigoplus_{\sigma\,\in\,_\Gamma\backslash X^p}\mathrm{Ind}_{\Gamma_\sigma}^{\Gamma}\mathbb{Z},$$

Shapiro's lemma yields

$$H_q(\Gamma;\ C_p(X)) \cong \bigoplus_{\sigma\,\in\,_\Gamma\backslash X^p}H_q(\Gamma_\sigma;\ \mathbb{Z});$$

and the equivariant Leray/Serre spectral sequence takes the form

$$E_{p,q}^1 = \bigoplus_{\sigma\,\in\,_\Gamma\backslash X^p}H_q(\Gamma_\sigma;\ \mathbb{Z}) \implies H_{p+q}(\Gamma;\ C_\bullet(X)),$$

converging to the Γ-equivariant homology of X, which is, as we have already seen, isomorphic to $H_{p+q}(\Gamma;\ \mathbb{Z})$ with the trivial action on the coefficients $\mathbb{Z}$.

As in degrees above the virtual cohomological dimension, which is 2 for the Bianchi groups, the group homology is isomorphic to the Farrell cohomology, we obtain the isomorphism type from the above mentioned general formulae.

In the lower degrees $q \in \{0, 1, 2\}$, the following terms remain on the E^2-page, which is concentrated in the columns $p = 0, 1, 2$:

$q = 2$	$\bigoplus\limits_{s\ \mathrm{singular}}\mathbb{Z}\oplus 2\text{-torsion}\oplus 3\text{-torsion}$	$2\text{-torsion}\oplus 3\text{-torsion}$	0
$q = 1$	$\bigoplus\limits_{s\ \mathrm{singular}}\mathbb{Z}^2\oplus\mathrm{Farrell\ supplement}$	$2\text{-torsion}\oplus 3\text{-torsion}$	0
$q = 0$	$\mathbb{Z}$	$H_1(_\Gamma\backslash X;\ \mathbb{Z})$	$H_2(_\Gamma\backslash X;\ \mathbb{Z})$

with the arrow $d_{2,0}^2$ from $H_2(_\Gamma\backslash X;\ \mathbb{Z})$ to the $q=1$, $p=0$ position.

where the "Farrell supplement" is the cokernel of the map

$$\bigoplus_{\sigma \in {}_\Gamma \backslash X^0} \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}) \xleftarrow{\ d^1_{1,1}\ } \bigoplus_{\sigma \in {}_\Gamma \backslash X^1} \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}).$$

induced by inclusion of finite cell stabilisers. As the cells are fixed pointwise by their stabilisers, we see that for $q > 0$, the $E^1_{p,q}$-terms are concentrated in the two columns $p = 0$ and $p = 1$. We compute the bottom row ($q = 0$) of the above spectral sequence as the homology of the quotient space ${}_\Gamma \backslash X$. Then we infer from section 4 that the rational rank of the differential $d^2_{2,0}$ is the number of non-trivial ideal classes of $\mathcal{O}_{-m}$.

Let us use Serre's decomposition of the homology group $\mathrm{H}_1({}_\Gamma \backslash \mathcal{H}; \mathbb{Z})$ into the direct sum of $\mathrm{H}^{\mathrm{cusp}}_1({}_\Gamma \backslash \mathcal{H}; \mathbb{Z})$ and the free Abelian group with one generator for each element of the class group of $\mathcal{O}_{-m}$. Then using the long exact sequence of section 4, we see that $\mathrm{H}_1({}_\Gamma \backslash X; \mathbb{Z}) \cong \mathrm{H}^{\mathrm{cusp}}_1({}_\Gamma \backslash \mathcal{H}; \mathbb{Z}) \oplus \mathbb{Z}$. This has made it possible to compute $\mathrm{H}^{\mathrm{cusp}}_1({}_\Gamma \backslash \mathcal{H}; \mathbb{Z})$ from the quotient space of the retracted Flöge cellular complex in figures 1 and 2. Finally, the group homology $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is an extension of $\mathrm{H}_1({}_\Gamma \backslash \mathcal{H}; \mathbb{Z})$ by a quotient of the Farrell supplement.

6. Swan's concept to determine the Bianchi fundamental polyhedron

This section recalls Richard G. Swan's work [Swa71], which gives a concept — from the theoretical viewpoint — for an algorithm to compute the Bianchi fundamental polyhedron. The set B which determines the Bianchi fundamental polyhedron has been defined using infinitely many hemispheres. But we will see that only a finite number of them are significant for this purpose and need to be computed. We will state a criterion for what is an appropriate choice that gives us precisely the set B. This criterion is easy to verify in practice. Suppose we have made a finite selection of n hemispheres. The index i running from 1 through n, we denote the i-th hemisphere by $S(\alpha_i)$, where α_i is its centre and given by a fraction $\alpha_i = \frac{\lambda_i}{\mu_i}$ in the number field $\mathbb{Q}(\sqrt{-m})$. Here, we require the ideal (λ_i, μ_i) to be the whole ring of integers $\mathcal{O}$. This requirement is just the one already made for all the hemispheres in the definition of B. Now, we can do an approximation of notation 2, using, modulo the translation group Γ_∞, a finite number of hemispheres.

Notation 8. Let $B(\alpha_1, \ldots, \alpha_n) := \big\{(z, \zeta) \in \mathcal{H}$: The inequality $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geqslant 1$ is fulfilled for all unimodular pairs $(\mu, \lambda) \in \mathcal{O}^2$ with $\frac{\lambda}{\mu} = \alpha_i + \gamma$, for some $i \in \{1, \ldots, n\}$ and some $\gamma \in \mathcal{O}\big\}$. Then $B(\alpha_1, \ldots, \alpha_n)$ is the set of all points in $\mathcal{H}$ lying above or on all hemispheres $S(\alpha_i + \gamma)$, $i = 1, \ldots, n$; for any $\gamma \in \mathcal{O}$.

The intersection $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ with the fundamental domain D_∞ for the translation group Γ_∞, is our candidate to equal the Bianchi fundamental polyhedron.

6.1. **Convergence of the approximation.** We will give a method to decide when $B(\alpha_1, \ldots, \alpha_n) = B$. This gives us an effective way to find B by adding more and more elements to the set $\{\alpha_1, \ldots, \alpha_n\}$ until we find $B(\alpha_1, \ldots, \alpha_n) = B$. We consider the boundary $\partial B(\alpha_1, \ldots, \alpha_n)$ of $B(\alpha_1, \ldots, \alpha_n)$ in $\mathcal{H} \cup \mathbb{C}$. It consists of the points $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ satisfying all the non-strict inequalities $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geqslant 1$ that we have used to define $B(\alpha_1, \ldots, \alpha_n)$, and satisfy the additional condition that at least one of these non-strict inequalities is an equality. We will see below that $\partial B(\alpha_1, \ldots, \alpha_n)$ carries a natural cell structure. This, together with the following definitions, makes it possible to state the criterion which tells us when we have found all the hemispheres relevant for the Bianchi fundamental polyhedron.

Definition 9. We shall say that the hemisphere $S_{\mu,\lambda}$ is *strictly below* the hemisphere $S_{\beta,\alpha}$ at a point $z \in \mathbb{C}$ if the following inequality is satisfied:

$$\left| z - \frac{\alpha}{\beta} \right|^2 - \frac{1}{|\beta|^2} < \left| z - \frac{\lambda}{\mu} \right|^2 - \frac{1}{|\mu|^2}.$$

This is, of course, an abuse of language because there may not be any points on $S_{\beta,\alpha}$ or $S_{\mu,\lambda}$ with coordinate z. However, if there is a point (z, ζ) on $S_{\mu,\lambda}$, the right hand side of the inequality is just $-\zeta^2$. Thus the left hand side is negative and so of the form $-(\zeta')^2$. Clearly, $(z, \zeta') \in S_{\beta,\alpha}$ and $\zeta' > \zeta$. We will further say that a point $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ is *strictly below* a hemisphere $S_{\mu,\lambda}$, if there is a point $(z, \zeta') \in S_{\mu,\lambda}$ with $\zeta' > \zeta$.

6.2. **Singular points.** We call *cusps* the elements of the number field $K = \mathbb{Q}(\sqrt{-m})$ considered as points in the boundary of hyperbolic space, via an embedding $K \subset \mathbb{C} \cup \{\infty\} \cong \partial\mathcal{H}$. We write $\infty = \frac{1}{0}$, which we also consider as a cusp. It is well-known that the set of cusps is closed under the action of $\mathrm{SL}_2(\mathcal{O})$ on $\partial\mathcal{H}$; and that we have the following bijective correspondence between the $\mathrm{SL}_2(\mathcal{O})$-orbits of cusps and the ideal classes in $\mathcal{O}$. A cusp $\frac{\lambda}{\mu}$ is in the $\mathrm{SL}_2(\mathcal{O})$-orbit of the cusp $\frac{\lambda'}{\mu'}$, if and only if the ideals (λ', μ') and (λ, μ) are in the same ideal class. It immediately follows that the orbit of the cusp $\infty = \frac{1}{0}$ corresponds to the principal ideals. Let us call *singular* the cusps $\frac{\lambda}{\mu}$ such that (λ, μ) is not principal. And let us call *singular points* the singular cusps which lie in ∂B. It follows from the characterisation of the singular points by Bianchi that they are precisely the points in $\mathbb{C} \subset \partial\mathcal{H}$ which cannot be strictly below any hemisphere. In the cases where $\mathcal{O}$ is a principal ideal domain, $K \cup \{\infty\}$ consists of only one $\mathrm{SL}_2(\mathcal{O})$-orbit, so there are no singular points. We use the following formulae derived by Swan, to compute representatives modulo the translations by Γ_∞, of the singular points.

Lemma 10 ([Swa71]). *The singular points of K, $\mod \mathcal{O}$, are given by* $\frac{p(r+\sqrt{-m})}{s}$, *where $p, r, s \in \mathbb{Z}$, $s > 0$, $\frac{-s}{2} < r \leqslant \frac{s}{2}$, $s^2 \leqslant r^2 + m$, and*
- *if $m \equiv 1$ or $2 \mod 4$,*
  $s \neq 1$, $s \mid r^2 + m$, the numbers p and s are coprime, and p is taken $\mod s$;
- *if $m \equiv 3 \mod 4$,*
  s is even, $s \neq 2$, $2s \mid r^2 + m$, the numbers p and $\frac{s}{2}$ are coprime; p is taken $\mod \frac{s}{2}$.

The singular points need not be considered in Swan's termination criterion, because they cannot be strictly below any hemisphere $S_{\mu,\lambda}$.

6.3. **Swan's termination criterion.** We observe that the set of $z \in \mathbb{C}$ over which some hemisphere is strictly below another is $\mathbb{C}$ or an open half-plane. In the latter case, the boundary of this is a line.

Notation 11. Denote by $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ the set of $z \in \mathbb{C}$ over which neither $S_{\beta,\alpha}$ is strictly below $S_{\mu,\lambda}$ nor vice versa.

This line is computed by turning the inequality in definition 9 into an equation. Swan calls it the line over which the two hemispheres *agree*, and we will see later that the most important edges of the Bianchi fundamental polyhedron lie on the preimages of such lines. We now restrict our attention to a set of hemispheres which is finite modulo the translations in Γ_∞. Consider a set of hemispheres $S(\alpha_i + \gamma)$, where the index i runs from 1 through n, and γ runs through $\mathcal{O}$. We call this set of hemispheres a *collection*, if every non-singular

point $z \in \mathbb{C} \subset \partial\mathcal{H}$ is strictly below some hemisphere in our set. Now consider a set $B(\alpha_1, \ldots, \alpha_n)$ which is determined by such a collection of hemispheres.

Theorem 12 (Swan's termination criterion [Swa71]). *We have $B(\alpha_1, \ldots, \alpha_n) = B$ if and only if no vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$ can be strictly below any hemisphere $S_{\mu,\lambda}$.*

In other words, no vertex v of $\partial B(\alpha_1, \ldots, \alpha_n)$ can lie strictly below any hemisphere $S_{\mu,\lambda}$. Let us call *height* the coordinate ζ of the upper-half space model introduced at the beginning of section 2. With this criterion, it suffices to compute the cell structure of $\partial B(\alpha_1, \ldots, \alpha_n)$ to see if our choice of hemispheres gives us the Bianchi fundamental polyhedron. This has only to be done modulo the translations of Γ_∞, which preserve the height and hence the situations of being strictly below. Thus our computations only need to be carried out on a finite set of hemispheres.

6.4. **Computing the cell structure in the complex plane.** We will in a first step compute the image of the cell structure under the homeomorphism from $\partial B(\alpha_1, \ldots, \alpha_n)$ to $\mathbb{C}$ given by the vertical projection. For each 2-cell of this structure, there is an associated hemisphere $S_{\mu,\lambda}$. The interior of this 2-cell consists of the points $z \in \mathbb{C}$, where all other hemispheres in our collection are strictly below $S_{\mu,\lambda}$. Swan shows that this is the interior of a convex polygon. The edges of these polygons lie on real lines in $\mathbb{C}$ specified in notation 11.

A vertex is an intersection point z of any two of these lines involving the same hemisphere $S_{\mu,\lambda}$, if all other hemispheres in our collection are strictly below, or agree with, $S_{\mu,\lambda}$ at z.

6.5. **Lifting the cell structure back to hyperbolic space.** Now we can lift the cell structure back to $\partial B(\alpha_1, \ldots, \alpha_n)$, using the projection homeomorphism onto $\mathbb{C}$. The preimages of the convex polygons of the cell structure on $\mathbb{C}$, are totally geodesic hyperbolic polygons each lying on one of the hemispheres in our collection. These are the 2-cells of $\partial B(\alpha_1, \ldots, \alpha_n)$.

The edges of these hyperbolic polygons lie on the intersection arcs of pairs of hemispheres in our collection. As two Euclidean 2-spheres intersect, if they do so non-trivially, in a circle centred on the straight line which connects the two 2-sphere centres, such an intersection arc lies on a semicircle centred in the complex plane. The plane which contains this semicircle must be orthogonal to the connecting line, hence a vertical plane in $\mathcal{H}$. We can alternatively conclude the latter facts observing that an edge which two totally geodesic polygons have in common must be a geodesic segment. Lifting the vertices becomes now obvious from their definition. This enables us to check Swan's termination criterion.

We will now sketch Swan's proof of this criterion. Let P be one of the convex polygons of the cell structure on $\mathbb{C}$. The preimage of P lies on one hemisphere $S(\alpha_i)$ of our collection. Now the condition stated in theorem 12 says that at the vertices of P, the hemisphere $S(\alpha_i)$ cannot be strictly below any other hemisphere. The points where $S(\alpha_i)$ can be strictly below some hemisphere constitute an open half-plane in $\mathbb{C}$, and hence cannot lie in the convex hull of the vertices of P, which is P. Theorem 12 now follows because $\mathbb{C}$ is tessellated by these convex polygons.

7. Algorithms realizing Swan's concept

From now on, we will work on putting Swan's concept into practice. We can reduce the set of hemispheres on which we carry out our computations, with the help of the following notion.

Definition 13. A hemisphere $S_{\mu,\lambda}$ is said to be *everywhere below* a hemisphere $S_{\beta,\alpha}$ when:
$$\left|\frac{\lambda}{\mu} - \frac{\alpha}{\beta}\right| \leqslant \frac{1}{|\beta|} - \frac{1}{|\mu|}.$$

Note that this is also the case when $S_{\mu,\lambda} = S_{\beta,\alpha}$. Any hemisphere which is everywhere below another one, does not contribute to the Bianchi fundamental polyhedron, in the following sense.

Proposition 14. *Let $S(\alpha_n)$ be a hemisphere everywhere below some other hemisphere $S(\alpha_i)$, where $i \in \{1, \ldots, n-1\}$.*
Then $B(\alpha_1, \ldots, \alpha_n) = B(\alpha_1, \ldots, \alpha_{n-1})$.

Proof. Write $\alpha_n = \frac{\lambda}{\mu}$ and $\alpha_i = \frac{\theta}{\tau}$ with $\lambda, \mu, \theta, \tau \in \mathcal{O}$. We take any point (z, ζ) strictly below $S_{\mu,\lambda}$ and show that it is also strictly below $S_{\tau,\theta}$. In terms of notation 8, this problem looks as follows: we assume that the inequality $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 < 1$ is satisfied, and show that this implies the inequality $|\tau z - \theta|^2 + |\tau|^2 \zeta^2 < 1$. The first inequality can be transformed into
$\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2 < \frac{1}{|\mu|^2}$. Hence, $\sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2} < \frac{1}{|\mu|}$. We will insert this into the triangle inequality for the Euclidean distance in $\mathbb{C} \times \mathbb{R}$ applied to the three points (z, ζ), $(\frac{\lambda}{\mu}, 0)$ and $(\frac{\theta}{\tau}, 0)$, which is
$$\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2}.$$
So we obtain $\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \frac{1}{|\mu|}$. By definition 13, the expression on the right hand side is smaller than or equal to $\frac{1}{|\tau|}$. Therefore, we take the square and obtain $\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2 < \frac{1}{|\tau|^2}$, which is equivalent to the claimed inequality. $\square$

Another notion that will be useful for our algorithm, is the following.

Definition 15. Let $z \in \mathbb{C}$ be a point lying within the vertical projection of $S_{\mu,\lambda}$. Define the *lift on the hemisphere $S_{\mu,\lambda}$* of z as the point on $S_{\mu,\lambda}$ the vertical projection of which is z.

Notation 16. Denote by the *hemisphere list* a list into which we will record a finite number of hemisphere s $S(\alpha_1)$, ..., $S(\alpha_n)$. Its purpose is to determine a set $B(\alpha_1, \ldots, \alpha_n)$ in order to approximate, and finally obtain, the Bianchi fundamental polyhedron.

7.1. The algorithm computing the Bianchi fundamental polyhedron.

We now describe the algorithm that we have realized using Swan's description; it is decomposed into algorithms 1 through 3 below.

Initial step. We begin with the smallest value which the norm of a non-zero element $\mu \in \mathcal{O}$ can take, namely 1. Then μ is a unit in $\mathcal{O}$, and for any $\lambda \in \mathcal{O}$, the pair (μ, λ) is unimodular. And we can rewrite the fraction $\frac{\lambda}{\mu}$ such that $\mu = 1$. We obtain the unit hemispheres (of radius 1), centred at the imaginary quadratic integers $\lambda \in \mathcal{O}$. We record into the hemisphere list the ones which touch the Bianchi fundamental polyhedron, i.e. the ones the centre of which lies in the fundamental rectangle D_0 (of notation 4) for the action of Γ_∞ on the complex plane.

Step A. Increase $|\mu|$ to the next higher value which the norm takes on elements of $\mathcal{O}$. Run through all the finitely many μ which have this norm. For each of these μ, run through all the finitely many λ with $\frac{\lambda}{\mu}$ in the fundamental rectangle D_0. Check that $(\mu, \lambda) = \mathcal{O}$ and that the hemisphere $S_{\mu,\lambda}$ is not everywhere below a hemisphere $S_{\beta,\alpha}$ in the hemisphere list. If these two checks are passed, record (μ, λ) into the hemisphere list.

We repeat step **A** until $|\mu|$ has reached an expected value. Then we check if we have found all the hemispheres which touch the Bianchi fundamental polyhedron, as follows.

Step B. We compute the lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ of definition 11, over which two hemispheres agree, for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another.
Then, for each hemisphere $S_{\beta,\alpha}$, we compute the intersection points of each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$.
We drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list. We erase the hemispheres from our list, for which less than three intersection points remain. We can do this because a hemisphere which touches the Bianchi fundamental polyhedron only in two vertices shares only an edge with it and no 2-cell.
Now, the vertices of $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ are the lifts of the remaining intersection points. Thus we can check Swan's termination criterion (theorem 12), which we do as follows. We pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift inside Hyperbolic Space of a remaining intersection point z.
If $\zeta \geqslant \frac{1}{|\mu|}$, then all (infinitely many) remaining hemispheres have radius equal or smaller than ζ, so (z, ζ) cannot be strictly below them. So Swan's termination criterion is fulfilled, we have found the Bianchi fundamental polyhedron, and can proceed by determining its cell structure.
Else, ζ becomes the new expected value for $\frac{1}{|\mu|}$. We repeat step **A** until $|\mu|$ reaches $\frac{1}{\zeta}$ and then proceed again with step **B**.

Algorithm 1 Computation of the Bianchi fundamental polyhedron

---

Input: A square-free positive integer m.
Output: The hemisphere list, containing entries $S(\alpha_1)$,...,$S(\alpha_n)$ such that $B(\alpha_1, \ldots, \alpha_n) = B$.

Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{-m})$.
Let $h_{\mathcal{O}}$ be the class number of $\mathcal{O}$. Compute $h_{\mathcal{O}}$.
Estimate the highest value for $|\mu|$ which will occur in notation 8 by
the formula $E := \begin{cases} \frac{5m}{2}h_{\mathcal{O}} - 2m + \frac{1}{2}, & m \equiv 3 \mod 4, \\ 21mh_{\mathcal{O}} - 19m, & \text{else.} \end{cases}$
$\mathcal{N} := 1$.
Swan's_cancel_criterion_fulfilled := false.

while Swan's_cancel_criterion_fulfilled $=$ false, **do**
    while $\mathcal{N} \leqslant E$ **do**
        Execute algorithm 2 with argument $\mathcal{N}$.
        Increase $\mathcal{N}$ to the next greater value in
        the set $\{\sqrt{n^2 m + j^2} \mid n, j \in \mathbb{N}\}$ of values of the norm on $\mathcal{O}$.
    end while
    Compute ζ with algorithm 3.
    if $\zeta \geqslant \frac{1}{\mathcal{N}}$, **then**
        `All (infinitely many) remaining hemispheres have radius`
        `smaller than` ζ`,`
        `so` (z, ζ) `cannot be strictly below any of them.`
        Swan's_cancel_criterion_fulfilled := true.
    else
        ζ `becomes the new expected lowest value for` $\frac{1}{\mathcal{N}}$`:`
        $E := \frac{1}{\zeta}$.
    end if
end while

Proposition 17. *The hemisphere list, as computed by algorithm 1, determines the Bianchi fundamental polyhedron. This algorithm terminates within finite time.*

Proof.

- The value ζ is the minimal height of the non-singular vertices of the cell complex $\partial B(\alpha_1, \ldots, \alpha_n)$ determined by the hemisphere list $\{S(\alpha_1), \ldots, S(\alpha_n)\}$.
    All the hemispheres which are not in the list, have radius smaller than $\frac{1}{\mathcal{N}}$. By remark 19, the inequality $\zeta \geqslant \frac{1}{\mathcal{N}}$ will become satisfied; and then no non-singular vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$ can be strictly below any of them. Hence by theorem 12, $B(\alpha_1, \ldots, \alpha_n) = B$; and we obtain the Bianchi fundamental polyhedron as $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$.
- We now consider the run-time. By theorem 6, the set of hemispheres

$$\{S_{\mu,\lambda} \mid S_{\mu,\lambda} \text{ touches the Bianchi Fundamental Polyhedron}\}$$

is finite. So, there exists an $S_{\mu,\lambda}$ for which the norm of μ takes its maximum on this finite set. The variable $\mathcal{N}$ reaches this maximum for $|\mu|$ after a finite number of steps; and then Swan's termination criterion is fulfilled. The latter steps require a finite run-time because of propositions 20 and 21.

$\square$

Swan explains furthermore how to obtain an a priori bound for the norm of the $\mu \in \mathcal{O}$ occurring for such hemispheres $S_{\mu,\lambda}$. But he states that this upper bound for $|\mu|$ is much too large. So instead of the theory behind theorem 6, we use Swan's termination criterion (theorem 12 above) to limit the number of steps in our computations. We then get the following.

Observation 18. We can give bounds for $|\mu|$ in the cases where K is of class number 1 or 2 (there are nine cases of class number 1 and eighteen cases of class number 2, and we have done the computation for all of them). They are the following:

$$\begin{cases} K \text{ of class number 1:} \qquad |\mu| \leqslant \frac{|\Delta|+1}{2}, \\[2em] K \text{ of class number 2:} \begin{cases} |\mu| \leqslant 3|\Delta|, & m \equiv 3 \mod 4, \\ |\mu| \leqslant (5 + \frac{61}{116})|\Delta|, & \text{else,} \end{cases} \end{cases}$$

where Δ is the discriminant of $K = \mathbb{Q}(\sqrt{-m})$, i.e., $|\Delta| = \begin{cases} m, & m \equiv 3 \mod 4, \\ 4m, & \text{else.} \end{cases}$

Remark 19. In algorithm 1, we have chosen the value E by an extrapolation formula for observation 18. If this is greater than the exact bound for $|\mu|$, the algorithm computes additional hemispheres which do not contribute to the Bianchi fundamental polyhedron. On the other hand, if E is smaller than the exact bound for $|\mu|$, it will be increased in the outer while loop of the algorithm, until it is sufficiently large. But then, the algorithm performs some preliminary computations of the intersection lines and vertices, which cost additional run-time. Thus our extrapolation formula is aimed at choosing E slightly greater than the exact bound for $|\mu|$ we expect.

Proposition 20. *Algorithm 2 finds all the hemispheres of radius $\frac{1}{N}$, on which a 2-cell of the Bianchi fundamental polyhedron can lie. This algorithm terminates within finite time.*

Proof.

- Directly from the definition of the hemispheres $S_{\mu,\lambda}$, it follows that the radius is given by $\frac{1}{|\mu|}$. So our algorithm runs through all μ in question. By construction of the Bianchi fundamental polyhedron D, the hemispheres on which a 2-cell of D lies must have their centre in the fundamental rectangle D_0. By proposition 14, such hemispheres cannot be everywhere below some other hemisphere in the list.
- Now we consider the run-time of the algorithm. There are finitely many $\mu \in \mathcal{O}$ the norm of which takes a given value. And for a given μ, there are finitely many $\lambda \in \mathcal{O}$ such that $\frac{\lambda}{\mu}$ is in the fundamental rectangle D_0. Therefore, this algorithm consists of finite loops and terminates within finite time.

$\square$

Algorithm 2 Recording the hemispheres of radius $\frac{1}{\mathcal{N}}$

Input: The value $\mathcal{N}$, and the hemisphere list (empty by default).
Output: The hemisphere list with some hemispheres of radius $\frac{1}{\mathcal{N}}$ added.

for a running from 0 through $\mathcal{N}$ within $\mathbb{Z}$, **do**
    for b in $\mathbb{Z}$ such that $|a + b\omega| = \mathcal{N}$, **do**
        Let $\mu := a + b\omega$.
        for all the $\lambda \in \mathcal{O}$ with $\frac{\lambda}{\mu}$ in the fundamental rectangle D_0, **do**
            if the pair (μ, λ) is unimodular, **then**
                Let $\mathcal{L}$ be the length of the hemisphere list.
                everywhere_below := false, $j := 1$.
                while everywhere_below = false and $j \leqslant \mathcal{L}$, **do**
                    Let $S_{\beta, \alpha}$ be the j'th entry in the hemisphere list;
                    if $S_{\mu, \lambda}$ is everywhere below $S_{\beta, \alpha}$, **then**
                        everywhere_below := true.
                    end if
                    Increase j by 1.
                end while
                if everywhere_below = false, **then**
                    Record $S_{\mu, \lambda}$ into the hemisphere list.
                end if
            end if
        end for
    end for
end for

We recall that the notion "everywhere below" has been made precise in
definition 13; and that the fundamental rectangle D_0 has been specified
in notation 4.

Proposition 21. *Algorithm 3 finds the minimal height occurring amongst the non-singular vertices of $\partial B(\alpha_1, \dots, \alpha_n)$. This algorithm erases only such hemispheres from the list, which do not change $\partial B(\alpha_1, \dots, \alpha_n)$. It terminates within finite time.*

Proof.

- The heights of the points in $\mathcal{H}$ are preserved by the action of the translation group Γ_∞, so we only need to consider representatives in the fundamental domain D_∞ for this action. Our algorithm computes the entire cell structure of $\partial B(\alpha_1, \dots, \alpha_n) \cap D_\infty$, as described in subsection 6.4. The number of lines to intersect is smaller than the square of the length of the hemisphere list, and thus finite. As a consequence, the minimum of the height has to be taken only on a finite set of intersection points, whence the first claim.
- If a cell of $\partial B(\alpha_1, \dots, \alpha_n)$ lies on a hemisphere, then its vertices are lifts of intersection points. So we can erase the hemispheres which are strictly below some other hemispheres at all the intersection points, without changing $\partial B(\alpha_1, \dots, \alpha_n)$.

- Now we consider the run-time. This algorithm consists of loops running through the hemisphere list, which has finite length. Within one of these loops, there is a loop running through the set of pairs of lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$. A (far too large) bound for the cardinality of this set is given by the fourth power of the length of the hemisphere list.

  The steps performed within these loops are very delimited and easily seen to be of finite run-time.

$\square$

Algorithm 3 Computing the minimal proper vertex height

Input: The hemisphere list $\{S(\alpha_1), \ldots, S(\alpha_n)\}$.
Output: The lowest height ζ of a non-singular vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$. And the hemisphere list with some hemispheres removed which do not touch the Bianchi fundamental polyhedron.

for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another, **do**
    compute the line $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ of notation 11.
end for

for each hemisphere $S_{\beta,\alpha}$ in the hemisphere list, **do**
    for each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$, **do**
        Compute the intersection point of $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$, if it exists.
    end for
end for
Drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list.
Erase the hemispheres from our list, for which no intersection points remain.
Now the vertices of $B(\alpha_1,\ldots,\alpha_n)$ $\cap$ D_∞ are the lifts (specified in definition 15) on the appropriate hemispheres of the remaining intersection points.
Pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift on some hemisphere of a remaining intersection point z.
Return ζ.

7.2. **The cell complex and its orbit space.** With the method described in subsection 6.4, we obtain a cell structure on the boundary of the Bianchi fundamental polyhedron. The cells in this structure which touch the cusp ∞ are easily determined: they are four 2-cells each lying on one of the Euclidean vertical planes bounding the fundamental domain D_∞ for Γ_∞ specified in notation 4; and four 1-cells each lying on one of the intersection lines of these planes. The other 2-cells in this structure lie each on one of the hemispheres determined with our realization of Swan's algorithm.

As the Bianchi fundamental polyhedron is a hyperbolic polyhedron up to some missing cusps, its boundary cells can be oriented as its facets. Once the cell structure is subdivided until the cells are fixed pointwise by their stabilisers, this cell structure with orientation is transported onto the whole hyperbolic space by the action of Γ.

7.3. **Computing the vertex stabilisers and identifications.** Let us state explicitly the Γ-action on the upper-half space model $\mathcal{H}$, in the form in which we will use it rather than in its historical form.

Lemma 22 (Poincaré). *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, *the action of* γ *on* $\mathcal{H}$ *is given by* $\gamma \cdot (z, \zeta) = (z', \zeta')$, *where*

$$\zeta' = \frac{|\det \gamma| \zeta}{|cz - d|^2 + \zeta^2 |c|^2}, \qquad z' = \frac{\left(\overline{d - cz} \right)(az - b) - \zeta^2 \bar{c} a}{|cz - d|^2 + \zeta^2 |c|^2}.$$

From this operation formula, we establish equations and inequalities on the entries of a matrix sending a given point (z, ζ) to another given point (z', ζ') in $\mathcal{H}$. We will use them in algorithm 4 to compute such matrices. For the computation of the vertex stabilisers, we have $(z, \zeta) = (z', \zeta')$, which simplifies the below equations and inequalities as well as the pertinent algorithm. First, we fix a basis for $\mathcal{O}$ as the elements 1 and

$$\omega := \begin{cases} \sqrt{-m}, & m \equiv 1 \text{ or } 2 \mod 4, \\ -\frac{1}{2} + \frac{1}{2}\sqrt{-m}, & m \equiv 3 \mod 4. \end{cases}$$

As we have put $m \neq 1$ and $m \neq 3$, the only units in the ring $\mathcal{O}$ are ± 1. We will use the notations $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geqslant x\}$ and $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leqslant x\}$ for $x \in \mathbb{R}$.

Lemma 23. *Let* $m \equiv 3 \mod 4$. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ *be a matrix sending* (z, r) *to* $(\zeta, \rho) \in \mathcal{H}$. *Write* c *in the basis as* $j + k\omega$, *where* $j, k \in \mathbb{Z}$. *Then* $|c|^2 \leqslant \frac{1}{r\rho}$, $|j| \leqslant \sqrt{\frac{1 + \frac{1}{m}}{r\rho}}$ *and*

$$\frac{2j}{m + 1} - 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m + 1} \leqslant k \leqslant \frac{2j}{m + 1} + 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m + 1}.$$

Proof. From the operation equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = (\zeta, \rho)$, we deduce $|cz - d|^2 + r^2 |c|^2 = \frac{r}{\rho}$ and conclude $r^2 |c|^2 \leqslant \frac{r}{\rho}$, whence the first inequality. We insert $|c|^2 = \left(j - \frac{k}{2} \right)^2 + m \left(\frac{k}{2} \right)^2 = j^2 + \frac{m+1}{4} k^2 - jk$ into it, and obtain

$$0 \geqslant k^2 - \frac{4j}{m + 1} k + \frac{4}{m + 1} \left(j^2 - \frac{1}{r\rho} \right) =: f(k).$$

We observe that $f(k)$ is a quadratic function in $k \in \mathbb{Z} \subset \mathbb{R}$, taking its values exclusively in $\mathbb{R}$. Hence its graph has the shape of a parabola, and the negative values of $f(k)$ appear exactly on the interval where k is between its two zeroes,

$$k_\pm = \frac{2j}{m+1} \pm 2\frac{\sqrt{\Delta}}{m+1}, \qquad \text{where } \Delta = \frac{m+1}{r\rho} - j^2 m.$$

This implies the third and fourth claimed inequalities. As k is a real number, Δ must be non-negative in order that $f(k)$ be non-positive. Hence $j^2 \leqslant \frac{1 + \frac{1}{m}}{r\rho}$, which gives the second claimed inequality. $\qquad\square$

Algorithm 4 Computation of the matrices identifying two points in $\mathcal{H}$.

Input: The points (z, r), (ζ, ρ) in the interior of $\mathcal{H}$, where $z, \zeta \in K$ and $r^2, \rho^2 \in \mathbb{Q}$.
Output: The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_{-m})$, $m \equiv 3 \mod 4$, with nonzero entry c, sending the first of the input points to the second one.

`c will run through` $\mathcal{O}$ `with` $0 < |c|^2 \leqslant \frac{1}{r\rho}$.
Write c in the basis as $j + k\omega$, where $j, k \in \mathbb{Z}$.

for j running from $-\left\lceil \sqrt{\frac{1 + \frac{1}{m}}{r\rho}} \right\rceil$ through $\left\lceil \sqrt{\frac{1 + \frac{1}{m}}{r\rho}} \right\rceil$ **do**

$\quad k_{\text{limit}}^{\pm} := 2\frac{j}{m+1} \pm 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1}$.

$\quad$ **for** k running from $\lfloor k_{\text{limit}}^{-} \rfloor$ through $\lceil k_{\text{limit}}^{+} \rceil$ **do**
$\quad\quad c := j + k\omega$;
$\quad\quad$ **if** $|c|^2 \leqslant \frac{1}{r\rho}$ and c nonzero, **then**
$\quad\quad\quad$ Write cz in the basis as $R(cz) + W(cz)\omega$ with $R(cz), W(cz) \in \mathbb{Q}$.
$\quad\quad\quad$ d `will run through` $\mathcal{O}$ `with` $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$.
$\quad\quad\quad$ Write d in the basis as $q + s\omega$, where $q, s \in \mathbb{Z}$.
$\quad\quad\quad$ $s_{\text{limit}}^{\pm} := W(cz) \pm 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}}$.
$\quad\quad\quad$ **for** s running from $\lfloor s_{\text{limit}}^{-} \rfloor$ through $\lceil s_{\text{limit}}^{+} \rceil$ **do**

$\quad\quad\quad\quad \Delta := \frac{r}{\rho} - r^2|c|^2 - m\left(\frac{W(cz)}{2} - \frac{s}{2}\right)^2$;
$\quad\quad\quad\quad$ **if** Δ is a rational square, **then**
$\quad\quad\quad\quad\quad q_{\pm} := R(cz) - \frac{W(cz)}{2} + \frac{s}{2} \pm \sqrt{\Delta}$.
$\quad\quad\quad\quad\quad$ Do the following for both $q_{\pm} = q_{+}$ and $q_{\pm} = q_{-}$ if $\Delta \neq 0$.
$\quad\quad\quad\quad\quad$ **if** $q_{\pm} \in \mathbb{Z}$, **then**
$\quad\quad\quad\quad\quad\quad d := q_{\pm} + s\omega$;
$\quad\quad\quad\quad\quad\quad$ **if** $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$ and (c, d) unimodular, **then**
$\quad\quad\quad\quad\quad\quad\quad a := \frac{\rho}{r}\overline{d} - \frac{\rho}{r}\overline{cz} - c\zeta$.
$\quad\quad\quad\quad\quad\quad\quad$ **if** a is in the ring of integers, **then**
$\quad\quad\quad\quad\quad\quad\quad\quad$ b `is determined by the determinant 1`:
$\quad\quad\quad\quad\quad\quad\quad\quad$ $b := \frac{ad-1}{c}$.
$\quad\quad\quad\quad\quad\quad\quad\quad$ **if** b is in the ring of integers, **then**
$\quad\quad\quad\quad\quad\quad\quad\quad\quad$ Check that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = (\zeta, \rho)$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad$ Return $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
$\quad\quad\quad\quad\quad\quad\quad\quad$ **end if**
$\quad\quad\quad\quad\quad\quad\quad$ **end if**
$\quad\quad\quad\quad\quad\quad$ **end if**
$\quad\quad\quad\quad\quad$ **end if**
$\quad\quad\quad\quad$ **end if**
$\quad\quad\quad$ **end for**
$\quad\quad$ **end if**
$\quad$ **end for**
end for

Lemma 24. *Under the assumptions of lemma 23, write d in the basis as $q + s\omega$, where $q, s \in \mathbb{Z}$. Write cz in the basis as $R(cz) + W(cz)\omega$, where $R(cz), W(cz) \in \mathbb{Q}$. Then $W(cz) - 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}} \leqslant s \leqslant W(cz) + 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}}$, and*

$$q = R(cz) - \frac{W(cz)}{2} + \frac{s}{2} \pm \sqrt{\frac{r}{\rho} - r^2|c|^2 - m\left(\frac{W(cz)}{2} - \frac{s}{2}\right)^2}.$$

Proof. Recall that $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-m}$, so $\overline{q + s\omega} = q - \frac{s}{2} - \frac{s}{2}\sqrt{-m}$. The operation equation yields $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$. From this, we derive

$$
\begin{aligned}
\frac{r}{\rho} - r^2|c|^2 &= (cz - (q + s\omega))\left(\overline{cz} - (q - \tfrac{s}{2} - \tfrac{s}{2}\sqrt{-m})\right) \\
&= \left(\operatorname{Re}(cz) - q + \tfrac{s}{2}\right)^2 + \left(\operatorname{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2 \\
&= \operatorname{Re}(cz)^2 + q^2 - qs + \tfrac{s^2}{4} - 2\operatorname{Re}(cz)q + \operatorname{Re}(cz)s + \left(\operatorname{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2.
\end{aligned}
$$

We solve for q,

$$q^2 + (-2\operatorname{Re}(cz) - s)\,q + \left(\operatorname{Re}(cz) + \frac{s}{2}\right)^2 + \left(\operatorname{Im}(cz) - \frac{s}{2}\sqrt{m}\right)^2 - \frac{r}{\rho} + r^2|c|^2 = 0$$

and find

$$q_{\pm} = \operatorname{Re}(cz) + \tfrac{s}{2} \pm \sqrt{\Delta}, \qquad \text{where } \Delta = \tfrac{r}{\rho} - r^2|c|^2 - \left(\operatorname{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2.$$

We express this as

$$q_{\pm} = R(cz) - \tfrac{W(cz)}{2} + \tfrac{s}{2} \pm \sqrt{\Delta}, \qquad \text{where } \Delta = \tfrac{r}{\rho} - r^2|c|^2 - m\left(\tfrac{W(cz)}{2} - \tfrac{s}{2}\right)^2,$$

which is the claimed equation. The condition that q must be a rational integer implies $\Delta \geqslant 0$, which can be rewritten in the claimed inequalities. $\qquad\square$

We further state a simple inequality in order to prove that algorithm 4 terminates in finite time.

Lemma 25. *Let $K = \mathbb{Q}(\sqrt{-m})$ with $m \neq 3$. Let $c, z \in K$. Write their product cz in the $\mathbb{Q}$-basis $\{1, \omega\}$ for K as $R(cz) + W(cz)\omega$. Then the inequality $|W(cz)| \leqslant |c| \cdot |z|$ holds.*

Proof. Let $x + y\omega \in K$ with $x, y \in \mathbb{Q}$. Our first step is to show that $|y| \leqslant |x + y\omega|$. Consider the case $m \equiv 1$ or $2 \mod 4$. Then

$$|x + y\omega| = \sqrt{x^2 + my^2} \geqslant \sqrt{m}|y| \geqslant |y|,$$

and we have shown our claim. Else consider the case $m \equiv 3 \mod 4$. Then,

$$|x + y\omega| = \sqrt{(x + \omega y)(x + \overline{\omega}y)} = \sqrt{\left(x^2 - 2x\frac{y}{2} + \frac{y^2}{4}\right) + \frac{m}{4}y^2} \geqslant \frac{\sqrt{m}}{2}|y|,$$

and our claim follows for $m > 3$. Now we have shown that $|W(cz)| \leqslant |cz|$; and we use some embedding of K into $\mathbb{C}$ to verify the equation $|cz| = |c| \cdot |z|$. $\qquad\square$

Proposition 26. *Let $m \equiv 3 \mod 4$. Then algorithm 4 gives all the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ with $c \neq 0$, sending (z, r) to $(\zeta, \rho) \in \mathcal{H}$. It terminates in finite time.*

Proof.

- The first claim is easily established using the bounds and formulae stated in lemmata 23 and 24.

- Now we consider the run-time. This algorithm consists of three loops the limits of which are at most linear expressions in $\frac{1}{\sqrt{r\rho}}$. For $s^{\pm}_{\text{limit}}$, we use lemma 25 and $r^2|c|^2 \leqslant \frac{r}{\rho}$ to see this (we get a factor $|z|$ here, which we can neglect).

$\square$

Finally, it should be said that the scope of computations one can do with geometric models for the Bianchi groups does not stop once the integral homology of the full group is known. There is further interest in homology with twisted coefficients, congruence subgroups and modular forms (see for instance [Şen11], [ŞT09]). Currently, Page [Pag12] is working on optimizing algorithms in order to obtain more cell complexes for Bianchi groups and other Kleinian groups.

References

[Ara10] Maria T. Aranés, *Modular symbols over number fields*, Ph.D. thesis, University of Warwick, 2010.

[Bia92] Luigi Bianchi, *Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginarî*, Math. Ann. **40** (1892), no. 3, 332–412.

[BV12] Nicolas Bergeron and Akshay Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, published in the Journal of the Institute of Mathematics of Jussieu (2012), http://dx.doi.org/10.1017/S1474748012000667

[Bro82] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. **87**, Springer-Verlag, 1982.

[Byg98] Jeremy Bygott, *Modular forms and modular symbols over imaginary quadratic fields*, Ph.D. thesis, University of Exeter, 1998.

[CL07] John E. Cremona and Mark P. Lingham, *Finding all elliptic curves with good reduction outside a given set of primes*, Experiment. Math. **16** (2007), no. 3, 303–312. MR 2367320 (2008k:11057)

[Cre84] John E. Cremona, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Compositio Math. **51** (1984), no. 3, 275–324.

[EGM98] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[Fin89] Benjamin Fine, *Algebraic theory of the Bianchi groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 129, Marcel Dekker Inc., New York, 1989.

[Flö80] Dieter Flöge, *Zur Struktur der PSL_2 über einigen imaginär-quadratischen Zahlringen*, Dissertation, Johann-Wolfgang-Goethe-Universität, Fachbereich Mathematik, 1980.

[Flö83] ———, *Zur Struktur der PSL_2 über einigen imaginär-quadratischen Zahlringen*, Math. Z. **183** (1983), no. 2, 255–279.

[Hum15] Georges Humbert, *Sur la réduction des formes d'Hermite dans un corps quadratique imaginaire*, C. R. Acad. Sci. Paris **16** (1915), 189–196.

[Lin05] Mark Lingham, *Modular forms and elliptic curves over imaginary quadratic fields*, Ph.D. thesis, University of Nottingham, 2005.

[MR03] Colin Maclachlan and Alan W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol. 219, Springer-Verlag, New York, 2003.e

[Men79] Eduardo R. Mendoza, *Cohomology of PGL_2 over imaginary quadratic integers*, Bonner Mathematische Schriften, **128**, Dissertation, Rheinische Friedrich-Wilhelms-Universität, Mathematisches Institut, 1979.

[Pag12] Aurel Page, *Computing arithmetic Kleinian groups*, preprint, http://hal.archives-ouvertes.fr/hal-00703043, 2012.

[Poi83]   Henri Poincaré, *Mémoire*, Acta Math. **3** (1883), no. 1, 49–92, Les groupes Kleinéens.

[Rah10]   Alexander D. Rahm, *Bianchi.gp*, Open source program (GNU general public license), validated by the CNRS: `http://www.projet-plume.org/fiche/bianchigp`, subject to the Certificat de Compétences en Calcul Intensif (C3I) and part of the GP scripts library of Pari/GP Development Center, 2010.

[Rah11]   _____, *Homology and K-theory of the Bianchi groups*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 11-12, 615–619. MR 2817377 (2012e:20116)

[Rah11b] _____, *The homological torsion of* PSL₂ *of the imaginary quadratic integers*, Trans. Amer. Math. Soc. 365 (2013), no. 3, 1603–1635. MR 3003276

[Rah12]   _____, *Accessing the Farrell-Tate cohomology of discrete groups*, preprint, `http://hal.archives-ouvertes.fr/hal-00618167`, 2012.

[Rah12b] _____, *On a question of Serre*, C. R. Math. Acad. Sci. Paris 350 (2012), no. 15-16, 741–744. MR 2981344

[RF11]   Alexander D. Rahm and Mathias Fuchs, *The integral homology of* PSL₂ *of imaginary quadratic integers with non-trivial class group*, J. Pure Appl. Algebra 215 (2011), no. 6, 1443–1472. MR 2769243

[RS12]   Alexander D. Rahm and Mehmet Haluk Şengün, *On Level One Cuspidal Bianchi Modular Forms*, accepted for publication in LMS J. Comp. & Math., `http://hal.archives-ouvertes.fr/hal-00589184/en/`

[Ril83]   Robert Riley, *Applications of a computer implementation of Poincaré's theorem on fundamental polyhedra*, Math. Comp. **40** (1983), no. 162, 607–632. MR 689477 (85b:20064)

[Sch92]   Alexander Scheutzow, *Computing rational cohomology and Hecke eigenvalues for Bianchi groups*, J. Number Theory **40** (1992), no. 3, 317–328.

[SV83]   Joachim Schwermer and Karen Vogtmann, *The integral homology of* SL₂ *and* PSL₂ *of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), no. 4, 573–598.

[Şen11]   Mehmet Haluk Şengün, *On the integral cohomology of Bianchi groups*, Exp. Math. **20** (2011), no. 4, 487–505. MR 2859903

[Şen12]   Mehmet Haluk Şengün, *Arithmetic Aspects of Bianchi Groups*, preprint (2012), arXiv:1204.6697

[ŞT09]   Mehmet Haluk Şengün and Seyfi Turkelli, *Weight reduction for* mod ℓ *Bianchi modular forms*, J. Number Theory **129** (2009), no. 8, 2010–2019. MR 2522720 (2010c:11064)

[Ser70]   Jean-Pierre Serre, *Le problème des groupes de congruence pour SL(2)*, Ann. of Math. (2) **92** (1970), 489–527.

[Swa71]   Richard G. Swan, *Generators and relations for certain special linear groups*, Advances in Math. **6** (1971), 1–77.

[Vog85]   Karen Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419.

[Whi90]   Elise Whitley, *Modular symbols and elliptic curves over imaginary quadratic fields*, Ph.D. thesis, University of Exeter, 1990.

[Yas10]   Dan Yasaki, *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic Number Theory, 9th International Symposium, Nancy, France, ANTS-IX, July 19-23, 2010, Proceedings, 385–396. MR 2721434 (2012g:11069)

E-mail address: `Alexander.Rahm@nuigalway.ie`

URL: `http://www.maths.nuigalway.ie/~rahm/`

NATIONAL UNIVERSITY OF IRELAND AT GALWAY, DEPARTMENT OF MATHEMATICS