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Author(s)	Park, Sejong
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ANALOGUES OF GOLDSCHMIDT'S THESIS FOR FUSION SYSTEMS

JUSTIN LYND AND SEJONG PARK

ABSTRACT. We extend the results of David Goldschmidt's thesis concerning fusion in finite groups to saturated fusion systems.

1. INTRODUCTION

Just recently, David Goldschmidt published his doctoral thesis [6] which had gone unpublished since 1968. In it he shows that if G is a finite simple group and $T \in \text{Syl}_2(G)$, then the exponent of $Z(T)$ (and hence of T) is bounded by a function of the nilpotence class of T . He also includes in the write-up a fusion factorization result for an arbitrary finite group involving $\mathcal{U}^1 Z$ and the Thompson subgroup. In this paper, we generalize these results to arbitrary saturated fusion systems. Throughout this paper, unless otherwise indicated, p denotes an arbitrary prime number, n a nonnegative integer, and P a nontrivial finite p -group.

Theorem 1. *Suppose P is of nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a saturated fusion system on P with $O_p(\mathcal{F}) = 1$. Then $Z(P)$ has exponent at most p^n .*

This bound is sharp for all n and p ; see Example 1 in Section 3. This also gives a bound on the exponent of P itself, which we certainly do not expect to be sharp.

Corollary 1. *Suppose P is of nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a saturated fusion system on P with $O_p(\mathcal{F}) = 1$. Then P has exponent at most $p^{n^2(p-1)+n}$.*

Proof. By Theorem 1, $Z(P)$ has exponent at most p^n . We claim that then every upper central quotient also has exponent at most p^n , and the proof is by induction. Let $k \geq 1$, and let $x \in Z^{k+1}(P)$. If x^{p^n} does not lie in $Z^k(P)$, then there exists $t \in P$ such that $[x^{p^n}, t]$ does not lie in $Z^{k-1}(P)$. But by a standard commutator identity, $[x^{p^n}, t] \equiv [x, t]^{p^n} \equiv 1$ modulo $Z^{k-1}(P)$, since by induction $Z^k(P)/Z^{k-1}(P)$ has exponent at most p^n . This contradiction establishes the claim. The nilpotence class of P is at most $n(p-1)+1$ by hypothesis, so the exponent of P is at most $p^{n(n(p-1)+1)}$. \square

Theorem 1 follows from the following, which we prove as Theorem 5 below.

Theorem 2. *Suppose P has nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a saturated fusion system on P . Then $\mathcal{U}^n(Z(P))$ is normal in \mathcal{F} .*

In the course of proving this last result in the group case for $p = 2$, Goldschmidt reduces to the situation in which a putative counterexample G has a weakly embedded 2-local

subgroup. Then his post-thesis classification [5] of such groups gives a contradiction. However, any weakly embedded 2-local M controls 2-fusion, and so the 2-subgroup $O_2(M)$ will show up as a normal subgroup in the fusion system, a shadow of the weakly embedded phenomenon. This allows the corresponding fusion result to hold for an arbitrary prime.

We note that Theorem 2 has the following corollary in the category of groups.

Theorem 3. *Let P be a nonabelian Sylow p -subgroup of a finite group G . Suppose that P has nilpotence class at most $n(p-1)+1$ and that G has no nontrivial strongly closed abelian p -subgroup. Then $Z(P)$ has exponent at most p^n .*

Proof. We can form the saturated fusion system $\mathcal{F}_P(G)$, and Theorem 2 then says that $\mathcal{U}^n(Z(P))$ is strongly \mathcal{F} -closed (see Proposition 1 below), that is, strongly closed in P with respect to G . Thus, $\mathcal{U}^n(Z(P))$ must be trivial. \square

Using a recent theorem of Flores and Foote [4], in which they use the Classification of Finite Simple Groups to describe all finite groups having a strongly closed p -subgroup, we get the following direct generalization of Goldschmidt's main theorem.

Corollary 2. *Let P be a nonabelian Sylow p -subgroup of a finite simple group G . If P has nilpotence class at most $n(p-1)+1$, then $Z(P)$ has exponent at most p^n .*

Proof. Suppose to the contrary that $A := \mathcal{U}^n(Z(P)) \neq 1$. Then by Theorem 2, A is a nontrivial strongly closed abelian subgroup of P . By inspection of the simple groups arising in the conclusion of the main theorem in [4], either P is abelian or $Z(P)$ has exponent p . Since P is nonabelian, we must have $n \geq 1$ and the corollary follows. \square

However, if the hypotheses of Corollary 2 are weakened slightly to assume only that $F^*(G)$ is simple, then the statement is false for all odd primes p , as the following example shows. Let $H = \text{PSL}(2, q)$ with $q = r^p$ for some prime power r and with the p -part of $q-1$ equal to p^e . Let σ be a field automorphism of \mathbf{F}_q of order p and $G = H\langle\sigma\rangle$. If P is a Sylow p -subgroup of G , then P has nilpotence class 2, while $Z(P)$ has exponent p^{e-1} , and we may take e as large as we like.

Recall the Thompson subgroup $J(P)$, defined as the group generated by the abelian subgroups of P of maximum order. We also prove the following

Theorem 4. *Let \mathcal{F} be a saturated fusion system on P . Then*

$$\mathcal{F} = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle.$$

2. DEFINITIONS AND NOTATION

We collect in this section the necessary information on fusion systems. Since there are by now many good sources of this knowledge [2], in particular in background sections of papers [3, 7] to which this one is similar, we will content ourselves to be brief.

Let P be a finite p -group. A *category on P* is a category \mathcal{F} with objects the subgroups of P and whose morphism sets $\text{Hom}_{\mathcal{F}}(Q, R)$ consist of injective group homomorphisms subject to the requirement that every morphism in \mathcal{F} is a composition of an isomorphism in \mathcal{F} and an inclusion.

Let \mathcal{F} be a category on the p -group P . Let Q and R be subgroups of P . We write $\text{Aut}_{\mathcal{F}}(Q)$ for $\text{Hom}_{\mathcal{F}}(Q, Q)$, $\text{Hom}_P(Q, R)$ for the set of group homomorphisms in \mathcal{F} from Q to R induced by conjugation by elements of P , and $\text{Out}_{\mathcal{F}}(Q)$ for $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)$.

We say Q is

- *fully \mathcal{F} -normalized* if $|N_P(Q)| \geq |N_P(Q')|$ for all Q' which are \mathcal{F} -isomorphic to Q ,
- *fully \mathcal{F} -centralized* if $|C_P(Q)| \geq |C_P(Q')|$ for all Q' which are \mathcal{F} -isomorphic to Q ,
- *\mathcal{F} -centric* if $C_P(Q') \leq Q'$ for all Q' which are \mathcal{F} -isomorphic to Q , and
- *\mathcal{F} -radical* if $O_p(\text{Out}_{\mathcal{F}}(Q)) = 1$.

For a morphism $\varphi : Q \rightarrow P$ in \mathcal{F} , let

$$N_{\varphi} = \{x \in N_P(Q) \mid \exists y \in N_P(\varphi(Q)), \forall z \in Q, \varphi(xzx^{-1}) = y\varphi(z)y^{-1}\}$$

Note that we have $QC_P(Q) \leq N_{\varphi}$ for all $\varphi : Q \rightarrow P$ in \mathcal{F} .

A *saturated fusion system* on P is a category \mathcal{F} on P whose morphism sets contain all group homomorphisms induced by conjugation by elements of P , and which satisfies the following two axioms.

- (Sylow axiom) $\text{Aut}_P(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$, and
- (Extension axiom) for every isomorphism $\varphi : Q \rightarrow Q'$ with Q' fully \mathcal{F} -normalized, there exists a morphism $\tilde{\varphi} : N_{\varphi} \rightarrow P$ such that $\tilde{\varphi}|_Q = \varphi$.

For the remainder of the paper, \mathcal{F} will denote a saturated fusion system on the finite p -group P , even though we will often drop the adjective “saturated”.

For $Q \leq P$, we define the following local subcategories of \mathcal{F} . The *normalizer* $N_{\mathcal{F}}(Q)$ of Q in \mathcal{F} is the category on $N_P(Q)$ such that for any $R_1, R_2 \leq N_P(Q)$, $\text{Hom}_{N_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \rightarrow R_2$ in \mathcal{F} for which there is an extension $\tilde{\varphi} : QR_1 \rightarrow QR_2$ of φ in \mathcal{F} such that $\tilde{\varphi}(Q) = Q$. The *centralizer* $C_{\mathcal{F}}(Q)$ of Q in \mathcal{F} is the category on $C_P(Q)$ such that for any $R_1, R_2 \leq C_P(Q)$, $\text{Hom}_{C_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \rightarrow R_2$ in \mathcal{F} for which there is an extension $\tilde{\varphi} : QR_1 \rightarrow QR_2$ of φ in \mathcal{F} such that $\tilde{\varphi}|_Q = \text{id}_Q$. Lastly, we define $N_P(Q)C_{\mathcal{F}}(Q)$ as we do the normalizer of Q , but only allow those $\varphi : R_1 \rightarrow R_2$ whose extensions $\tilde{\varphi}$ restrict to automorphisms in $\text{Aut}_P(Q)$.

If Q is fully \mathcal{F} -normalized, then $N_{\mathcal{F}}(Q)$ is a saturated fusion system. And if Q is fully \mathcal{F} -centralized, then both $C_{\mathcal{F}}(Q)$ and $N_P(Q)C_{\mathcal{F}}(Q)$ are saturated fusion systems.

A *characteristic functor* is a mapping from finite p -groups to finite p -groups which takes Q to a characteristic subgroup $W(Q)$ of Q such that for any group isomorphism $\varphi : Q \rightarrow Q'$, $\varphi(W(Q)) = W(Q')$. We say that a characteristic functor is *positive* provided $W(Q) \neq 1$ whenever $Q \neq 1$. The *center functor*, sending a finite p -group P to its center, is a positive characteristic p -functor.

A *conjugation family* for \mathcal{F} is a set \mathcal{C} of nonidentity subgroups of P such that \mathcal{F} is generated by compositions and restrictions of morphisms in $\text{Aut}_{\mathcal{F}}(Q)$ as Q ranges over \mathcal{C} . Alperin’s fusion theorem for saturated fusion systems says that the set of \mathcal{F} -centric, \mathcal{F} -radical subgroups is a conjugation family for \mathcal{F} , and we call this the *Alperin conjugation family*.

Recall that a subgroup W of P is said to be *weakly \mathcal{F} -closed* if for each $\varphi \in \text{Hom}_{\mathcal{F}}(W, P)$, $\varphi(W) = W$. The subgroup W is *strongly \mathcal{F} -closed* if for each subgroup W' of W and each

$\varphi \in \text{Hom}_{\mathcal{F}}(W', P)$, $\varphi(W') \leq W$. We say W is *normal in \mathcal{F}* if $\mathcal{F} = N_{\mathcal{F}}(W)$, and denote by $O_p(\mathcal{F})$ the largest such subgroup of P .

3. PROOFS

The following proposition is slightly misstated in [1, Proposition 1.6], where a normal W is claimed to be contained in every radical subgroup. For this reason, we state a correct version here, but the proof in [1] goes through with little modification.

Proposition 1. *Let \mathcal{F} be a fusion system on P and $W \leq P$. The following are equivalent.*

- (a) *W is normal in \mathcal{F} .*
- (b) *W is strongly \mathcal{F} -closed and is contained in every \mathcal{F} -centric, \mathcal{F} -radical subgroup of P .*
- (c) *W is weakly \mathcal{F} -closed and is contained in every subgroup of some conjugation family for \mathcal{F} .*

Lemma 1. *Suppose P has nilpotence class at most $n(p-1)+1$. If Q is a subgroup of P with $C_P(\mathcal{U}^n(Z(Q))) = Q$, then $Q = P$.*

Proof. This is Corollary 6 in [6]. □

Proposition 2. *Let W be a characteristic subfunctor of the center functor such that $W(P) \leq W(Q)$ for all $Q \leq P$ with $C_P(Q) \leq Q$. Then for any fusion system \mathcal{F} on P , either there exists a proper \mathcal{F} -centric subgroup Q of P such that $C_P(W(Q)) = Q$, or $W(P)$ is normal in \mathcal{F} .*

Proof. Suppose there is no proper \mathcal{F} -centric subgroup Q of P with $C_P(W(Q)) = Q$. We will show that $W(P)$ is weakly closed in \mathcal{F} . In this case, $W(P) \leq Z(P)$ is contained in every \mathcal{F} -centric subgroup of P , hence in every member of an Alperin conjugation family for \mathcal{F} . Thus, by Proposition 1, $W(P)$ is in fact normal in \mathcal{F} .

Let Q be a fully \mathcal{F} -normalized, \mathcal{F} -centric subgroup of P . Then by hypothesis, $W(P) \leq W(Q)$. Let $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$. By Alperin's fusion theorem, it suffices to show that $W(P)$ is invariant under α . We do this by induction on $|P : Q|$. If $Q = P$, then $\alpha(W(P)) = W(P)$ since $W(P)$ is a characteristic subgroup of P , so suppose that $Q < P$. Then $C_P(W(Q)) > Q$. Let $\beta : W(Q) \rightarrow R$ be an isomorphism in \mathcal{F} with R fully \mathcal{F} -normalized. Then by the extension axiom, β extends to a map $\tilde{\beta} : C_P(W(Q)) \rightarrow P$. By induction and Alperin's fusion theorem, we have that $\beta(W(P)) = \tilde{\beta}(W(P)) = W(P)$. But $\beta\alpha|_{W(Q)}$ also extends to $C_P(W(Q))$, and $\beta\alpha(W(P)) = W(P)$ by the same reasoning. Therefore $\alpha(W(P)) = \beta^{-1}\beta\alpha(W(P)) = W(P)$, and this completes the proof. □

We are now ready to prove Theorem 2.

Theorem 5. *Suppose P has nilpotence class at most $n(p-1)+1$ and \mathcal{F} is a fusion system on P . Then $\mathcal{U}^n(Z(P))$ is normal in \mathcal{F} .*

Proof. Let $W = \mathcal{U}^n Z$. If $C_P(Q) \leq Q \leq P$, then $Z(P) \leq Z(Q)$ and so $W(P) = \mathcal{U}^n(Z(P)) \leq \mathcal{U}^n(Z(Q)) = W(Q)$. Thus W satisfies the hypotheses of Proposition 2, and Lemma 1 says that there is no proper subgroup of P with $C_P(W(Q)) = Q$. Therefore by Proposition 2, $\mathcal{U}^n(Z(P))$ is normal in \mathcal{F} . □

Theorem 1 now follows immediately from Theorem 2. The following example generalizes a remark of Goldschmidt's in [6], and shows that the bound on the exponent of $Z(P)$ given in Theorem 1 is sharp.

Example 1. Let p be an odd prime, let $G = \text{SL}(p+1, q)$ with $|q-1|_p = p^n$, and let P be a Sylow p -subgroup of G . Then P is isomorphic to $C_{p^n} \wr C_p$. Let x be the wreathing element, a p -cycle permutation matrix, generating the C_p on top. Then $P' = [P, P]$ is isomorphic to $p-1$ copies of C_{p^n} . Let $P_0 = \langle P', x \rangle$. As $Z(P)$ has exponent p^n , the bound in Theorem 1 is sharp provided the class of P is $n(p-1) + 1$. For this it suffices to show that P_0 has class $n(p-1)$, that is, P_0 is of maximal class.

By an inductive argument, we quickly reduce to the case where $n = 2$. Suppose $n = 2$ and let a_1, \dots, a_{p-1} be generators for the $p-1$ cyclic groups of order p^2 . Then x sends a_i to a_{i+1} for $1 \leq i \leq p-2$ and a_{p-1} to $a_1^{-1} \cdots a_{p-1}^{-1}$. Factoring by $\Omega_1(P')$ we have that $[P'/\Omega_1(P'), x; p-1] = 1$ so that $[P', x; p-1] \leq \Omega_1(P')$. By direct computation,

$$[a_1, x; p-1] = \prod_{k=0}^{p-2} a_{k+1}^{(-1)^k \binom{p-1}{k} - 1}.$$

The sum of the exponents of the a_i in $[a_1, x; p-1]$ is

$$-p+1 + \sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} = -p+1 + (1-1)^{p-1} - \binom{p-1}{p-1} = -p.$$

This means that $[a_1, x; p-1]$ lies outside the sum-zero submodule (which is the unique maximal submodule) for the action of x on $\Omega_1(P')$, and so $[P', x; p-1] = \Omega_1(P')$. It follows that P_0 has class $2(p-1)$, as claimed.

Therefore P has class $n(p-1) + 1$ while $Z(P)$ has exponent p^n , and so the bound of Theorem 1 is sharp.

We now turn to the proof of Theorem 4. We will need a version of the Frattini argument due to Onofrei and Stancu [8, Proposition 3.7].

Proposition 3. *Let \mathcal{F} be a fusion system on P and suppose $Q \leq P$ is normal in \mathcal{F} . Then*

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle.$$

Lemma 2. *Suppose P is a p -group, $Q \trianglelefteq P$, and $C_P(\text{U}^1(Z(Q))) = Q$. Then $J(P) \leq Q$.*

Proof. This is Lemma 8 in [6]. □

The *Thompson ordering* on subgroups of P is defined by

$$Q \leq_P Q' \quad \text{iff} \quad |N_P(Q)| \leq |N_P(Q')| \quad \text{or} \quad |N_P(Q)| = |N_P(Q')| \quad \text{and} \quad |Q| \leq |Q'|.$$

We are now ready to prove

Theorem 6. *Let \mathcal{F} be a fusion system on P . Then*

$$\mathcal{F} = \langle C_{\mathcal{F}}(\text{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle.$$

Proof. Write $\mathcal{F}' = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle$. Since each \mathcal{F} -centric subgroup of P contains $Z(P)$, it suffices by Alperin's fusion theorem to prove that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ for all $Q \leq P$ with $Z(P) \leq Q$. We do this by induction on the Thompson ordering. If $Q = P$, then $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$, since $J(P)$ is a characteristic subgroup of P , so suppose that $Q <_P P$ with $Z(P) \leq Q$ and that $N_{\mathcal{F}}(Q') \subseteq \mathcal{F}'$ for all $Q' >_P Q$ with $Z(P) \leq Q'$.

First we reduce to the case where Q is fully \mathcal{F} -normalized. Suppose Q is not fully \mathcal{F} -normalized. By [7, Lemma 2.2], there exists $\alpha : N_P(Q) \rightarrow P$ such that $\alpha(Q)$ is fully \mathcal{F} -normalized. Note that $\alpha(Q) >_P Q$, and since $R >_P Q$ for every $R \leq P$ with $|N_P(Q)| \leq |R|$, we have by induction and Alperin's fusion theorem that α is in \mathcal{F}' . Also note that $\alpha(N_P(Q)) \leq N_P(\alpha(Q))$; we still denote by α the induced morphism $N_P(Q) \rightarrow N_P(\alpha(Q))$. Let $\varphi : R_1 \rightarrow R_2$ be a morphism in $N_{\mathcal{F}}(Q)$, and let $\tilde{\varphi}$ be an extension to $QR_1 \leq N_P(Q)$. Then $\alpha\tilde{\varphi}\alpha^{-1} : \alpha(Q)\alpha(R_1) \rightarrow \alpha(Q)\alpha(R_2)$ restricts to an automorphism of $\alpha(Q)$, whence is contained in \mathcal{F}' by induction. But α is in \mathcal{F}' , so φ is in \mathcal{F}' too. Thus $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$, so henceforth we assume Q is fully \mathcal{F} -normalized.

For brevity, set $W = \mathcal{U}^1(Z(Q))$, $N = N_P(Q)$, and $C = C_N(W)$. Then $C \trianglelefteq N$, so that $N_P(C) \geq N$. Suppose first that $C = Q$. Then by Lemma 2, we have $J(N) \leq Q$. As $J(N) \trianglelefteq N_P(N)$, either $J(N) >_P Q$ or $N = P$. In the first case, since $Z(P) \leq J(N)$ and $J(N) = J(Q)$ is a characteristic subgroup of Q , we apply induction to get $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(N)) \subseteq \mathcal{F}'$. In the second case we have $J(P) \leq Q$, so $J(P) = J(Q)$, and hence $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$ here as well.

Assume now that $C > Q$. Then $C >_P Q$ because $C \trianglelefteq N$. Looking to see that $W \trianglelefteq N_{\mathcal{F}}(Q)$, we apply Proposition 3 in this normalizer to get

$$N_{\mathcal{F}}(Q) = \langle NC_{N_{\mathcal{F}}(Q)}(W), N_{N_{\mathcal{F}}(Q)}(C) \rangle.$$

Since C contains $Z(P)$, we have by induction that $N_{N_{\mathcal{F}}(Q)}(C) \subseteq N_{\mathcal{F}}(C) \subseteq \mathcal{F}'$, so to complete the proof, it suffices to show that $NC_{N_{\mathcal{F}}(Q)}(W) \subseteq C_{\mathcal{F}}(\mathcal{U}^1(Z(P)))$. To see this, let $R_1, R_2 \leq N$, and let $\varphi : R_1 \rightarrow R_2$ be a morphism in $NC_{N_{\mathcal{F}}(Q)}(W)$. Then there exists $x \in N$ such that φ extends to an \mathcal{F} -map $\tilde{\varphi} : WR_1 \rightarrow WR_2$ with $\tilde{\varphi}|_W = c_x$, the conjugation map induced by x . But since Q contains $Z(P)$, it follows that $W = \mathcal{U}^1(Z(Q)) \geq \mathcal{U}^1(Z(P))$, and so $\tilde{\varphi}|_{\mathcal{U}^1(Z(P))} = c_x|_{\mathcal{U}^1(Z(P))} = \text{id}_{\mathcal{U}^1(Z(P))}$. Therefore, $\varphi \in C_{\mathcal{F}}(\mathcal{U}^1(Z(P)))$, as was to be shown. We conclude that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ and the result follows. \square

Remark 1. In [3, Theorem 4.1], the authors prove in part that for any fusion system \mathcal{F} on P , $\mathcal{U}^1(Z(P)) \cap Z(N_{\mathcal{F}}(J(P))) \leq Z(\mathcal{F})$ by reducing to the group case. Theorem 4 gives a reduction-free proof of this fact.

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