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# ANALOGUES OF GOLDSCHMIDT'S THESIS FOR FUSION SYSTEMS

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ABSTRACT. We extend the results of David Goldschmidt's thesis concerning fusion in finite groups to saturated fusion systems.

## 1. INTRODUCTION

Just recently, David Goldschmidt published his doctoral thesis [6] which had gone unpublished since 1968. In it he shows that if  $G$  is a finite simple group and  $T \in \text{Syl}_2(G)$ , then the exponent of  $Z(T)$  (and hence of  $T$ ) is bounded by a function of the nilpotence class of  $T$ . He also includes in the write-up a fusion factorization result for an arbitrary finite group involving  $\mathcal{U}^1 Z$  and the Thompson subgroup. In this paper, we generalize these results to arbitrary saturated fusion systems. Throughout this paper, unless otherwise indicated,  $p$  denotes an arbitrary prime number,  $n$  a nonnegative integer, and  $P$  a nontrivial finite  $p$ -group.

**Theorem 1.** *Suppose  $P$  is of nilpotence class at most  $n(p-1)+1$  and  $\mathcal{F}$  is a saturated fusion system on  $P$  with  $O_p(\mathcal{F}) = 1$ . Then  $Z(P)$  has exponent at most  $p^n$ .*

This bound is sharp for all  $n$  and  $p$ ; see Example 1 in Section 3. This also gives a bound on the exponent of  $P$  itself, which we certainly do not expect to be sharp.

**Corollary 1.** *Suppose  $P$  is of nilpotence class at most  $n(p-1)+1$  and  $\mathcal{F}$  is a saturated fusion system on  $P$  with  $O_p(\mathcal{F}) = 1$ . Then  $P$  has exponent at most  $p^{n^2(p-1)+n}$ .*

*Proof.* By Theorem 1,  $Z(P)$  has exponent at most  $p^n$ . We claim that then every upper central quotient also has exponent at most  $p^n$ , and the proof is by induction. Let  $k \geq 1$ , and let  $x \in Z^{k+1}(P)$ . If  $x^{p^n}$  does not lie in  $Z^k(P)$ , then there exists  $t \in P$  such that  $[x^{p^n}, t]$  does not lie in  $Z^{k-1}(P)$ . But by a standard commutator identity,  $[x^{p^n}, t] \equiv [x, t]^{p^n} \equiv 1$  modulo  $Z^{k-1}(P)$ , since by induction  $Z^k(P)/Z^{k-1}(P)$  has exponent at most  $p^n$ . This contradiction establishes the claim. The nilpotence class of  $P$  is at most  $n(p-1)+1$  by hypothesis, so the exponent of  $P$  is at most  $p^{n(n(p-1)+1)}$ .  $\square$

Theorem 1 follows from the following, which we prove as Theorem 5 below.

**Theorem 2.** *Suppose  $P$  has nilpotence class at most  $n(p-1)+1$  and  $\mathcal{F}$  is a saturated fusion system on  $P$ . Then  $\mathcal{U}^n(Z(P))$  is normal in  $\mathcal{F}$ .*

In the course of proving this last result in the group case for  $p = 2$ , Goldschmidt reduces to the situation in which a putative counterexample  $G$  has a weakly embedded 2-local

subgroup. Then his post-thesis classification [5] of such groups gives a contradiction. However, any weakly embedded 2-local  $M$  controls 2-fusion, and so the 2-subgroup  $O_2(M)$  will show up as a normal subgroup in the fusion system, a shadow of the weakly embedded phenomenon. This allows the corresponding fusion result to hold for an arbitrary prime.

We note that Theorem 2 has the following corollary in the category of groups.

**Theorem 3.** *Let  $P$  be a nonabelian Sylow  $p$ -subgroup of a finite group  $G$ . Suppose that  $P$  has nilpotence class at most  $n(p-1)+1$  and that  $G$  has no nontrivial strongly closed abelian  $p$ -subgroup. Then  $Z(P)$  has exponent at most  $p^n$ .*

*Proof.* We can form the saturated fusion system  $\mathcal{F}_P(G)$ , and Theorem 2 then says that  $\mathcal{U}^n(Z(P))$  is strongly  $\mathcal{F}$ -closed (see Proposition 1 below), that is, strongly closed in  $P$  with respect to  $G$ . Thus,  $\mathcal{U}^n(Z(P))$  must be trivial.  $\square$

Using a recent theorem of Flores and Foote [4], in which they use the Classification of Finite Simple Groups to describe all finite groups having a strongly closed  $p$ -subgroup, we get the following direct generalization of Goldschmidt's main theorem.

**Corollary 2.** *Let  $P$  be a nonabelian Sylow  $p$ -subgroup of a finite simple group  $G$ . If  $P$  has nilpotence class at most  $n(p-1)+1$ , then  $Z(P)$  has exponent at most  $p^n$ .*

*Proof.* Suppose to the contrary that  $A := \mathcal{U}^n(Z(P)) \neq 1$ . Then by Theorem 2,  $A$  is a nontrivial strongly closed abelian subgroup of  $P$ . By inspection of the simple groups arising in the conclusion of the main theorem in [4], either  $P$  is abelian or  $Z(P)$  has exponent  $p$ . Since  $P$  is nonabelian, we must have  $n \geq 1$  and the corollary follows.  $\square$

However, if the hypotheses of Corollary 2 are weakened slightly to assume only that  $F^*(G)$  is simple, then the statement is false for all odd primes  $p$ , as the following example shows. Let  $H = \text{PSL}(2, q)$  with  $q = r^p$  for some prime power  $r$  and with the  $p$ -part of  $q-1$  equal to  $p^e$ . Let  $\sigma$  be a field automorphism of  $\mathbf{F}_q$  of order  $p$  and  $G = H\langle\sigma\rangle$ . If  $P$  is a Sylow  $p$ -subgroup of  $G$ , then  $P$  has nilpotence class 2, while  $Z(P)$  has exponent  $p^{e-1}$ , and we may take  $e$  as large as we like.

Recall the Thompson subgroup  $J(P)$ , defined as the group generated by the abelian subgroups of  $P$  of maximum order. We also prove the following

**Theorem 4.** *Let  $\mathcal{F}$  be a saturated fusion system on  $P$ . Then*

$$\mathcal{F} = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle.$$

## 2. DEFINITIONS AND NOTATION

We collect in this section the necessary information on fusion systems. Since there are by now many good sources of this knowledge [2], in particular in background sections of papers [3, 7] to which this one is similar, we will content ourselves to be brief.

Let  $P$  be a finite  $p$ -group. A *category on  $P$*  is a category  $\mathcal{F}$  with objects the subgroups of  $P$  and whose morphism sets  $\text{Hom}_{\mathcal{F}}(Q, R)$  consist of injective group homomorphisms subject to the requirement that every morphism in  $\mathcal{F}$  is a composition of an isomorphism in  $\mathcal{F}$  and an inclusion.

Let  $\mathcal{F}$  be a category on the  $p$ -group  $P$ . Let  $Q$  and  $R$  be subgroups of  $P$ . We write  $\text{Aut}_{\mathcal{F}}(Q)$  for  $\text{Hom}_{\mathcal{F}}(Q, Q)$ ,  $\text{Hom}_P(Q, R)$  for the set of group homomorphisms in  $\mathcal{F}$  from  $Q$  to  $R$  induced by conjugation by elements of  $P$ , and  $\text{Out}_{\mathcal{F}}(Q)$  for  $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)$ .

We say  $Q$  is

- *fully  $\mathcal{F}$ -normalized* if  $|N_P(Q)| \geq |N_P(Q')|$  for all  $Q'$  which are  $\mathcal{F}$ -isomorphic to  $Q$ ,
- *fully  $\mathcal{F}$ -centralized* if  $|C_P(Q)| \geq |C_P(Q')|$  for all  $Q'$  which are  $\mathcal{F}$ -isomorphic to  $Q$ ,
- *$\mathcal{F}$ -centric* if  $C_P(Q') \leq Q'$  for all  $Q'$  which are  $\mathcal{F}$ -isomorphic to  $Q$ , and
- *$\mathcal{F}$ -radical* if  $O_p(\text{Out}_{\mathcal{F}}(Q)) = 1$ .

For a morphism  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ , let

$$N_{\varphi} = \{x \in N_P(Q) \mid \exists y \in N_P(\varphi(Q)), \forall z \in Q, \varphi(xzx^{-1}) = y\varphi(z)y^{-1}\}$$

Note that we have  $QC_P(Q) \leq N_{\varphi}$  for all  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ .

A *saturated fusion system* on  $P$  is a category  $\mathcal{F}$  on  $P$  whose morphism sets contain all group homomorphisms induced by conjugation by elements of  $P$ , and which satisfies the following two axioms.

- (Sylow axiom)  $\text{Aut}_P(P)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ , and
- (Extension axiom) for every isomorphism  $\varphi : Q \rightarrow Q'$  with  $Q'$  fully  $\mathcal{F}$ -normalized, there exists a morphism  $\tilde{\varphi} : N_{\varphi} \rightarrow P$  such that  $\tilde{\varphi}|_Q = \varphi$ .

For the remainder of the paper,  $\mathcal{F}$  will denote a saturated fusion system on the finite  $p$ -group  $P$ , even though we will often drop the adjective “saturated”.

For  $Q \leq P$ , we define the following local subcategories of  $\mathcal{F}$ . The *normalizer*  $N_{\mathcal{F}}(Q)$  of  $Q$  in  $\mathcal{F}$  is the category on  $N_P(Q)$  such that for any  $R_1, R_2 \leq N_P(Q)$ ,  $\text{Hom}_{N_{\mathcal{F}}(Q)}(R_1, R_2)$  consists of those  $\varphi : R_1 \rightarrow R_2$  in  $\mathcal{F}$  for which there is an extension  $\tilde{\varphi} : QR_1 \rightarrow QR_2$  of  $\varphi$  in  $\mathcal{F}$  such that  $\tilde{\varphi}(Q) = Q$ . The *centralizer*  $C_{\mathcal{F}}(Q)$  of  $Q$  in  $\mathcal{F}$  is the category on  $C_P(Q)$  such that for any  $R_1, R_2 \leq C_P(Q)$ ,  $\text{Hom}_{C_{\mathcal{F}}(Q)}(R_1, R_2)$  consists of those  $\varphi : R_1 \rightarrow R_2$  in  $\mathcal{F}$  for which there is an extension  $\tilde{\varphi} : QR_1 \rightarrow QR_2$  of  $\varphi$  in  $\mathcal{F}$  such that  $\tilde{\varphi}|_Q = \text{id}_Q$ . Lastly, we define  $N_P(Q)C_{\mathcal{F}}(Q)$  as we do the normalizer of  $Q$ , but only allow those  $\varphi : R_1 \rightarrow R_2$  whose extensions  $\tilde{\varphi}$  restrict to automorphisms in  $\text{Aut}_P(Q)$ .

If  $Q$  is fully  $\mathcal{F}$ -normalized, then  $N_{\mathcal{F}}(Q)$  is a saturated fusion system. And if  $Q$  is fully  $\mathcal{F}$ -centralized, then both  $C_{\mathcal{F}}(Q)$  and  $N_P(Q)C_{\mathcal{F}}(Q)$  are saturated fusion systems.

A *characteristic functor* is a mapping from finite  $p$ -groups to finite  $p$ -groups which takes  $Q$  to a characteristic subgroup  $W(Q)$  of  $Q$  such that for any group isomorphism  $\varphi : Q \rightarrow Q'$ ,  $\varphi(W(Q)) = W(Q')$ . We say that a characteristic functor is *positive* provided  $W(Q) \neq 1$  whenever  $Q \neq 1$ . The *center functor*, sending a finite  $p$ -group  $P$  to its center, is a positive characteristic  $p$ -functor.

A *conjugation family* for  $\mathcal{F}$  is a set  $\mathcal{C}$  of nonidentity subgroups of  $P$  such that  $\mathcal{F}$  is generated by compositions and restrictions of morphisms in  $\text{Aut}_{\mathcal{F}}(Q)$  as  $Q$  ranges over  $\mathcal{C}$ . Alperin’s fusion theorem for saturated fusion systems says that the set of  $\mathcal{F}$ -centric,  $\mathcal{F}$ -radical subgroups is a conjugation family for  $\mathcal{F}$ , and we call this the *Alperin conjugation family*.

Recall that a subgroup  $W$  of  $P$  is said to be *weakly  $\mathcal{F}$ -closed* if for each  $\varphi \in \text{Hom}_{\mathcal{F}}(W, P)$ ,  $\varphi(W) = W$ . The subgroup  $W$  is *strongly  $\mathcal{F}$ -closed* if for each subgroup  $W'$  of  $W$  and each

$\varphi \in \text{Hom}_{\mathcal{F}}(W', P)$ ,  $\varphi(W') \leq W$ . We say  $W$  is *normal in  $\mathcal{F}$*  if  $\mathcal{F} = N_{\mathcal{F}}(W)$ , and denote by  $O_p(\mathcal{F})$  the largest such subgroup of  $P$ .

### 3. PROOFS

The following proposition is slightly misstated in [1, Proposition 1.6], where a normal  $W$  is claimed to be contained in every radical subgroup. For this reason, we state a correct version here, but the proof in [1] goes through with little modification.

**Proposition 1.** *Let  $\mathcal{F}$  be a fusion system on  $P$  and  $W \leq P$ . The following are equivalent.*

- (a)  *$W$  is normal in  $\mathcal{F}$ .*
- (b)  *$W$  is strongly  $\mathcal{F}$ -closed and is contained in every  $\mathcal{F}$ -centric,  $\mathcal{F}$ -radical subgroup of  $P$ .*
- (c)  *$W$  is weakly  $\mathcal{F}$ -closed and is contained in every subgroup of some conjugation family for  $\mathcal{F}$ .*

**Lemma 1.** *Suppose  $P$  has nilpotence class at most  $n(p-1)+1$ . If  $Q$  is a subgroup of  $P$  with  $C_P(\mathcal{U}^n(Z(Q))) = Q$ , then  $Q = P$ .*

*Proof.* This is Corollary 6 in [6]. □

**Proposition 2.** *Let  $W$  be a characteristic subfunctor of the center functor such that  $W(P) \leq W(Q)$  for all  $Q \leq P$  with  $C_P(Q) \leq Q$ . Then for any fusion system  $\mathcal{F}$  on  $P$ , either there exists a proper  $\mathcal{F}$ -centric subgroup  $Q$  of  $P$  such that  $C_P(W(Q)) = Q$ , or  $W(P)$  is normal in  $\mathcal{F}$ .*

*Proof.* Suppose there is no proper  $\mathcal{F}$ -centric subgroup  $Q$  of  $P$  with  $C_P(W(Q)) = Q$ . We will show that  $W(P)$  is weakly closed in  $\mathcal{F}$ . In this case,  $W(P) \leq Z(P)$  is contained in every  $\mathcal{F}$ -centric subgroup of  $P$ , hence in every member of an Alperin conjugation family for  $\mathcal{F}$ . Thus, by Proposition 1,  $W(P)$  is in fact normal in  $\mathcal{F}$ .

Let  $Q$  be a fully  $\mathcal{F}$ -normalized,  $\mathcal{F}$ -centric subgroup of  $P$ . Then by hypothesis,  $W(P) \leq W(Q)$ . Let  $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ . By Alperin's fusion theorem, it suffices to show that  $W(P)$  is invariant under  $\alpha$ . We do this by induction on  $|P : Q|$ . If  $Q = P$ , then  $\alpha(W(P)) = W(P)$  since  $W(P)$  is a characteristic subgroup of  $P$ , so suppose that  $Q < P$ . Then  $C_P(W(Q)) > Q$ . Let  $\beta : W(Q) \rightarrow R$  be an isomorphism in  $\mathcal{F}$  with  $R$  fully  $\mathcal{F}$ -normalized. Then by the extension axiom,  $\beta$  extends to a map  $\tilde{\beta} : C_P(W(Q)) \rightarrow P$ . By induction and Alperin's fusion theorem, we have that  $\beta(W(P)) = \tilde{\beta}(W(P)) = W(P)$ . But  $\beta\alpha|_{W(Q)}$  also extends to  $C_P(W(Q))$ , and  $\beta\alpha(W(P)) = W(P)$  by the same reasoning. Therefore  $\alpha(W(P)) = \beta^{-1}\beta\alpha(W(P)) = W(P)$ , and this completes the proof. □

We are now ready to prove Theorem 2.

**Theorem 5.** *Suppose  $P$  has nilpotence class at most  $n(p-1)+1$  and  $\mathcal{F}$  is a fusion system on  $P$ . Then  $\mathcal{U}^n(Z(P))$  is normal in  $\mathcal{F}$ .*

*Proof.* Let  $W = \mathcal{U}^n Z$ . If  $C_P(Q) \leq Q \leq P$ , then  $Z(P) \leq Z(Q)$  and so  $W(P) = \mathcal{U}^n(Z(P)) \leq \mathcal{U}^n(Z(Q)) = W(Q)$ . Thus  $W$  satisfies the hypotheses of Proposition 2, and Lemma 1 says that there is no proper subgroup of  $P$  with  $C_P(W(Q)) = Q$ . Therefore by Proposition 2,  $\mathcal{U}^n(Z(P))$  is normal in  $\mathcal{F}$ . □

Theorem 1 now follows immediately from Theorem 2. The following example generalizes a remark of Goldschmidt's in [6], and shows that the bound on the exponent of  $Z(P)$  given in Theorem 1 is sharp.

**Example 1.** Let  $p$  be an odd prime, let  $G = \text{SL}(p+1, q)$  with  $|q-1|_p = p^n$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $P$  is isomorphic to  $C_{p^n} \wr C_p$ . Let  $x$  be the wreathing element, a  $p$ -cycle permutation matrix, generating the  $C_p$  on top. Then  $P' = [P, P]$  is isomorphic to  $p-1$  copies of  $C_{p^n}$ . Let  $P_0 = \langle P', x \rangle$ . As  $Z(P)$  has exponent  $p^n$ , the bound in Theorem 1 is sharp provided the class of  $P$  is  $n(p-1) + 1$ . For this it suffices to show that  $P_0$  has class  $n(p-1)$ , that is,  $P_0$  is of maximal class.

By an inductive argument, we quickly reduce to the case where  $n = 2$ . Suppose  $n = 2$  and let  $a_1, \dots, a_{p-1}$  be generators for the  $p-1$  cyclic groups of order  $p^2$ . Then  $x$  sends  $a_i$  to  $a_{i+1}$  for  $1 \leq i \leq p-2$  and  $a_{p-1}$  to  $a_1^{-1} \cdots a_{p-1}^{-1}$ . Factoring by  $\Omega_1(P')$  we have that  $[P'/\Omega_1(P'), x; p-1] = 1$  so that  $[P', x; p-1] \leq \Omega_1(P')$ . By direct computation,

$$[a_1, x; p-1] = \prod_{k=0}^{p-2} a_{k+1}^{(-1)^k \binom{p-1}{k} - 1}.$$

The sum of the exponents of the  $a_i$  in  $[a_1, x; p-1]$  is

$$-p+1 + \sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} = -p+1 + (1-1)^{p-1} - \binom{p-1}{p-1} = -p.$$

This means that  $[a_1, x; p-1]$  lies outside the sum-zero submodule (which is the unique maximal submodule) for the action of  $x$  on  $\Omega_1(P')$ , and so  $[P', x; p-1] = \Omega_1(P')$ . It follows that  $P_0$  has class  $2(p-1)$ , as claimed.

Therefore  $P$  has class  $n(p-1) + 1$  while  $Z(P)$  has exponent  $p^n$ , and so the bound of Theorem 1 is sharp.

We now turn to the proof of Theorem 4. We will need a version of the Frattini argument due to Onofrei and Stancu [8, Proposition 3.7].

**Proposition 3.** *Let  $\mathcal{F}$  be a fusion system on  $P$  and suppose  $Q \leq P$  is normal in  $\mathcal{F}$ . Then*

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle.$$

**Lemma 2.** *Suppose  $P$  is a  $p$ -group,  $Q \trianglelefteq P$ , and  $C_P(\text{U}^1(Z(Q))) = Q$ . Then  $J(P) \leq Q$ .*

*Proof.* This is Lemma 8 in [6]. □

The *Thompson ordering* on subgroups of  $P$  is defined by

$$Q \leq_P Q' \quad \text{iff} \quad |N_P(Q)| \leq |N_P(Q')| \quad \text{or} \quad |N_P(Q)| = |N_P(Q')| \quad \text{and} \quad |Q| \leq |Q'|.$$

We are now ready to prove

**Theorem 6.** *Let  $\mathcal{F}$  be a fusion system on  $P$ . Then*

$$\mathcal{F} = \langle C_{\mathcal{F}}(\text{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle.$$

*Proof.* Write  $\mathcal{F}' = \langle C_{\mathcal{F}}(\mathcal{U}^1(Z(P))), N_{\mathcal{F}}(J(P)) \rangle$ . Since each  $\mathcal{F}$ -centric subgroup of  $P$  contains  $Z(P)$ , it suffices by Alperin's fusion theorem to prove that  $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$  for all  $Q \leq P$  with  $Z(P) \leq Q$ . We do this by induction on the Thompson ordering. If  $Q = P$ , then  $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$ , since  $J(P)$  is a characteristic subgroup of  $P$ , so suppose that  $Q <_P P$  with  $Z(P) \leq Q$  and that  $N_{\mathcal{F}}(Q') \subseteq \mathcal{F}'$  for all  $Q' >_P Q$  with  $Z(P) \leq Q'$ .

First we reduce to the case where  $Q$  is fully  $\mathcal{F}$ -normalized. Suppose  $Q$  is not fully  $\mathcal{F}$ -normalized. By [7, Lemma 2.2], there exists  $\alpha : N_P(Q) \rightarrow P$  such that  $\alpha(Q)$  is fully  $\mathcal{F}$ -normalized. Note that  $\alpha(Q) >_P Q$ , and since  $R >_P Q$  for every  $R \leq P$  with  $|N_P(Q)| \leq |R|$ , we have by induction and Alperin's fusion theorem that  $\alpha$  is in  $\mathcal{F}'$ . Also note that  $\alpha(N_P(Q)) \leq N_P(\alpha(Q))$ ; we still denote by  $\alpha$  the induced morphism  $N_P(Q) \rightarrow N_P(\alpha(Q))$ . Let  $\varphi : R_1 \rightarrow R_2$  be a morphism in  $N_{\mathcal{F}}(Q)$ , and let  $\tilde{\varphi}$  be an extension to  $QR_1 \leq N_P(Q)$ . Then  $\alpha\tilde{\varphi}\alpha^{-1} : \alpha(Q)\alpha(R_1) \rightarrow \alpha(Q)\alpha(R_2)$  restricts to an automorphism of  $\alpha(Q)$ , whence is contained in  $\mathcal{F}'$  by induction. But  $\alpha$  is in  $\mathcal{F}'$ , so  $\varphi$  is in  $\mathcal{F}'$  too. Thus  $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ , so henceforth we assume  $Q$  is fully  $\mathcal{F}$ -normalized.

For brevity, set  $W = \mathcal{U}^1(Z(Q))$ ,  $N = N_P(Q)$ , and  $C = C_N(W)$ . Then  $C \trianglelefteq N$ , so that  $N_P(C) \geq N$ . Suppose first that  $C = Q$ . Then by Lemma 2, we have  $J(N) \leq Q$ . As  $J(N) \trianglelefteq N_P(N)$ , either  $J(N) >_P Q$  or  $N = P$ . In the first case, since  $Z(P) \leq J(N)$  and  $J(N) = J(Q)$  is a characteristic subgroup of  $Q$ , we apply induction to get  $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(N)) \subseteq \mathcal{F}'$ . In the second case we have  $J(P) \leq Q$ , so  $J(P) = J(Q)$ , and hence  $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$  here as well.

Assume now that  $C > Q$ . Then  $C >_P Q$  because  $C \trianglelefteq N$ . Looking to see that  $W \trianglelefteq N_{\mathcal{F}}(Q)$ , we apply Proposition 3 in this normalizer to get

$$N_{\mathcal{F}}(Q) = \langle NC_{N_{\mathcal{F}}(Q)}(W), N_{N_{\mathcal{F}}(Q)}(C) \rangle.$$

Since  $C$  contains  $Z(P)$ , we have by induction that  $N_{N_{\mathcal{F}}(Q)}(C) \subseteq N_{\mathcal{F}}(C) \subseteq \mathcal{F}'$ , so to complete the proof, it suffices to show that  $NC_{N_{\mathcal{F}}(Q)}(W) \subseteq C_{\mathcal{F}}(\mathcal{U}^1(Z(P)))$ . To see this, let  $R_1, R_2 \leq N$ , and let  $\varphi : R_1 \rightarrow R_2$  be a morphism in  $NC_{N_{\mathcal{F}}(Q)}(W)$ . Then there exists  $x \in N$  such that  $\varphi$  extends to an  $\mathcal{F}$ -map  $\tilde{\varphi} : WR_1 \rightarrow WR_2$  with  $\tilde{\varphi}|_W = c_x$ , the conjugation map induced by  $x$ . But since  $Q$  contains  $Z(P)$ , it follows that  $W = \mathcal{U}^1(Z(Q)) \geq \mathcal{U}^1(Z(P))$ , and so  $\tilde{\varphi}|_{\mathcal{U}^1(Z(P))} = c_x|_{\mathcal{U}^1(Z(P))} = \text{id}_{\mathcal{U}^1(Z(P))}$ . Therefore,  $\varphi \in C_{\mathcal{F}}(\mathcal{U}^1(Z(P)))$ , as was to be shown. We conclude that  $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$  and the result follows.  $\square$

**Remark 1.** In [3, Theorem 4.1], the authors prove in part that for any fusion system  $\mathcal{F}$  on  $P$ ,  $\mathcal{U}^1(Z(P)) \cap Z(N_{\mathcal{F}}(J(P))) \leq Z(\mathcal{F})$  by reducing to the group case. Theorem 4 gives a reduction-free proof of this fact.

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#### REFERENCES

1. Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, and Bob Oliver, *Subgroup families controlling  $p$ -local finite groups*, Proc. London Math. Soc. (3) **91** (2005), no. 2, 325–354.
2. Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).

3. Antonio Díaz, Adam Glesser, Nadia Mazza, and Sejong Park, *Glauberman and Thomson's theorems for fusion systems*, Proc. Amer. Math. Soc. **137** (2009), 495–503.
4. Ramón Flores and Richard Foote, *Strongly closed subgroups of finite groups*, preprint (2009).
5. David M. Goldschmidt, *Weakly embedded 2-local subgroups of finite groups*, J. Algebra **21** (1972), 341–351.
6. ———, *On the 2-exponent of a finite group*, J. Algebra **319** (2008), 616–620.
7. Radha Kessar and Markus Linckelmann, *ZJ-theorems for fusion systems*, Trans. Amer. Math. Soc. **360** (2008), 3093–3206.
8. Silvia Onofrei and Radu Stancu, *A characteristic subgroup for fusion systems*, preprint (2008).