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# Explicit secular equation for Scholte waves over a monoclinic crystal.

Michel Destrade

2004

## 1 Introduction

Scholte waves are acoustic waves propagating at a fluid/solid interface. They are localized in the neighborhood of the phase boundary in the sense that they decay exponentially in both directions along the normal to the interface. Johnson [1] established the explicit secular equation for Scholte waves over an orthorhombic crystal. In his case, the crystal is cut along a plane  $x_2 = 0$  containing *two* crystallographic axes  $Ox_1$  and  $Ox_3$ ; the wave propagates with speed  $v$  in the  $x_1$  direction; the solid is characterized by a mass density  $\rho_s$  and relevant elastic stiffnesses  $C_{11}$ ,  $C_{12}$ ,  $C_{22}$ , and  $C_{66}$ ; the fluid by a mass density  $\rho_f$  and speed of sound  $c$ . The secular equation is

$$Z\sqrt{C_{11}C_{22} - C_{12}^2 - 2C_{12}C_{66} - (C_{22} + C_{66})X + 2\sqrt{C_{22}C_{66}(C_{11} - X)(C_{66} - X)}} - \sqrt{\frac{C_{66} - X}{C_{11} - X}}(C_{11}C_{22} - C_{12}^2 - C_{22}X) + X\sqrt{C_{22}C_{66}} = 0, \quad (1.1)$$

where

$$X = \rho_s v^2, \quad Z = \frac{\rho_f v^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (1.2)$$

For instance, consider a frozen lake with a layer of ice assumed thick enough to be considered as a semi-infinite body. At  $0.01^\circ\text{C}$  under 1 bar, the density of water is [2]:  $\rho_f = 999.84 \text{ kg/m}^3$  and sound propagates at  $c = 1402.4 \text{ m/s}$ ; the second line of Table 1 lists the elastic stiffnesses and density of ice [3]; according to (1.1), Scholte waves propagate for this model at speed  $v_S = 1237.6 \text{ m/s}$ . Ice however has the special property of being transversally isotropic, which means that any plane containing the  $x_3$  axis is a symmetry

plane and so the speed  $v_S$  is the same for any orientation of the water/ice interface plane containing the  $x_3$  axis.

The aim of this Letter to the Editor is to derive explicitly the secular equation for Scholte waves at the interface between a fluid and an anisotropic crystal cut along a plane containing the normal to a single symmetry plane, that is containing only *one* crystallographic axis. In effect, the crystal may be a monoclinic crystal with symmetry plane at  $x_3 = 0$ , or a rhombic, tetragonal, or cubic crystal cut along a plane containing  $x_3$  and making an angle  $\theta \neq 0$  with the other crystallographic planes; the higher symmetry cases ( $\theta = 0$  or transversally isotropic and isotropic crystals) are covered by (1.1). For cases with less symmetries, one can turn to approximate solutions [4] as long as the anisotropy is weak .

## 2 Equations of motion and boundary conditions

Consider two half-spaces delimited by the plane  $x_2 = 0$ ; the upper one  $x_2 < 0$  is filled with an inviscid fluid, the lower one  $x_2 > 0$  is made of a monoclinic crystal with symmetry plane at  $x_3 = 0$  whose relevant non-zero reduced compliances are  $s'_{11}$ ,  $s'_{22}$ ,  $s'_{12}$ ,  $s'_{16}$ ,  $s'_{26}$ , and  $s'_{66}$ . At the interface, an inhomogeneous plane wave travels with speed  $v$  and wave number  $k$  in the  $x_1$  direction, and decays rapidly in the  $x_2 \rightarrow \pm\infty$  directions.

In the *solid*, the corresponding equations of motion are written as a first-order differential system for the 4-component displacement-traction vector,

$$\boldsymbol{\xi}' = i\mathbf{N}\boldsymbol{\xi}, \quad \boldsymbol{\xi}(kx_2) = [U_1(kx_2), U_2(kx_2), t_{12}(kx_2), t_{22}(kx_2)]^T, \quad (2.1)$$

where the functions  $U_i$  and  $t_{i2}$  are related to the in-plane mechanical displacements  $u_1$ ,  $u_2$  and in-plane tractions  $\sigma_{12}$ ,  $\sigma_{22}$  through

$$u_i(x_1, x_2, x_3, t) = U_i(kx_2)e^{ik(x_1-vt)}, \quad \sigma_{i2}(x_1, x_2, x_3, t) = ikt_{i2}(kx_2)e^{ik(x_1-vt)}. \quad (2.2)$$

In (2.1), the  $4 \times 4$  matrix  $\mathbf{N}$  is given by [5, 6],

$$\mathbf{N} = \begin{bmatrix} -r_6 & -1 & n_{66} & n_{26} \\ -r_2 & 0 & n_{26} & n_{66} \\ X - \eta & 0 & -r_6 & -r_2 \\ 0 & X & -1 & 0 \end{bmatrix}, \quad (2.3)$$

where  $X = \rho_s v^2$  and

$$\eta = \frac{1}{s'_{11}}, \quad r_i = -\frac{s'_{1i}}{s'_{11}}, \quad n_{ij} = \frac{1}{s'_{11}} \begin{vmatrix} s'_{11} & s'_{1j} \\ s'_{1i} & s'_{ij} \end{vmatrix}. \quad (2.4)$$

These equations also cover the case of a wave (2.2) travelling in a crystal of rhombic, tetragonal, or cubic symmetry, with acoustic axes  $XYx_3$  and reduced compliances  $S'_{ij}$ , cut along the plane  $x_2 = 0$  containing the  $x_3$  axis and making an angle  $\theta$  with the crystallographic  $XY$  plane (see Figure 1). In that case, the reduced compliances  $s'_{ij}$  along the  $x_i$  axes are given in terms of those along the crystallographic axes  $XYx_3$  by (see Ting [7]),

$$\begin{aligned}
s'_{11} &= S'_{11} \cos^4 \theta + (2S'_{12} + S'_{66}) \cos^2 \theta \sin^2 \theta + S'_{22} \sin^4 \theta, \\
s'_{22} &= S'_{22} \cos^4 \theta + (2S'_{12} + S'_{66}) \cos^2 \theta \sin^2 \theta + S'_{11} \sin^4 \theta, \\
s'_{12} &= S'_{12} + (S'_{11} + S'_{22} - 2S'_{12} - S'_{66}) \cos^2 \theta \sin^2 \theta, \\
s'_{66} &= S'_{66} + 4(S'_{11} + S'_{22} - 2S'_{12} - S'_{66}) \cos^2 \theta \sin^2 \theta. \\
s'_{16} &= [2S'_{22} \sin^2 \theta - 2S'_{11} \cos^2 \theta + (2S'_{12} + S'_{66})(\cos^2 \theta - \sin^2 \theta)] \cos \theta \sin \theta, \\
s'_{26} &= [2S'_{22} \cos^2 \theta - 2S'_{11} \sin^2 \theta - (2S'_{12} + S'_{66})(\cos^2 \theta - \sin^2 \theta)] \cos \theta \sin \theta.
\end{aligned} \tag{2.5}$$

Note that for transversally isotropic crystals, the following relationships hold,  $S'_{11} = S'_{22}$ ,  $S'_{66} = 2(S'_{11} - S'_{12})$ , and the rotation does not affect the values of the compliances ( $s'_{ij} = S'_{ij}$ ). This author [8] recently showed that for waves vanishing with increasing distance from the plane  $x_2 = 0$ , the following fundamental relationships hold for any positive or negative integer power  $n$  of the matrix  $\mathbf{N}$ ,

$$\bar{\boldsymbol{\xi}}(0) \cdot \hat{\mathbf{I}} \mathbf{N}^n \boldsymbol{\xi}(0) = 0, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.6}$$

Because of the Cayley-Hamilton theorem, only three consecutive powers of  $\mathbf{N}$  are linearly independent so that (2.6) reduces to only three linearly independent equations.

In the *fluid*, the normal displacement and the normal stress component are connected, as recalled by Barnett et al. [9], by the (real) normal impedance  $Z$  defined in (1.2)<sub>2</sub>,

$$\sigma_{22} = kZu_2. \tag{2.7}$$

At the *solid/fluid interface*, the normal displacement and the normal stress component are continuous, and the shear stress component is zero. It follows from these boundary conditions and from (2.1)<sub>2</sub>, (2.2), (2.7), that the displacement-traction vector at the interface  $x_2 = 0^+$  is of the form,

$$\boldsymbol{\xi}(0^+) = U_2(0)[\alpha, 1, 0, -iZ]^T, \tag{2.8}$$

where  $\alpha = U_1(0^+)/U_2(0)$ .

Now the fundamental equations (2.6) read

$$(N^n)_{32}(\alpha + \bar{\alpha}) + iZ(N^n)_{21}(\alpha - \bar{\alpha}) + (N^n)_{31}\alpha\bar{\alpha} = -(N^n)_{42} - Z^2(N^n)_{24}. \quad (2.9)$$

Writing  $\alpha$  as  $\alpha = \alpha_1 + i\alpha_2$  and taking in turn  $n = -1, 1, 2$ , a non-homogeneous linear system of equations follows,

$$\mathbf{A}\mathbf{b} = \mathbf{d}, \quad \mathbf{A} = \begin{bmatrix} N_{32}^* & ZN_{22}^* & N_{31}^* \\ 0 & ZN_{22} & N_{31} \\ (N^2)_{32} & Z(N^2)_{22} & (N^2)_{31} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 2\alpha_1 \\ -2\alpha_2 \\ \alpha_1^2 + \alpha_2^2 \end{bmatrix}, \quad \mathbf{d} = - \begin{bmatrix} N_{42}^* + Z^2N_{24}^* \\ N_{42} + Z^2N_{24} \\ Z^2(N^2)_{24} \end{bmatrix}, \quad (2.10)$$

where  $\mathbf{N}^*$  denotes the adjoint of  $\mathbf{N}$ . The unique solutions to the system are  $b_k = \Delta_k/\Delta$ , where  $\Delta = \det \mathbf{A}$  and  $\Delta_k$  is the determinant of the matrix derived from  $\mathbf{A}$  by replacing the  $k$ -th column with  $\mathbf{d}$ . However, the  $b_k$  are linked by  $b_1^2 + b_2^2 = 4b_3$ , which is the explicit secular equation for Scholte wave over a monoclinic crystal with symmetry plane at  $x_3 = 0$ ,

$$\Delta_1^2 + \Delta_2^2 = 4\Delta\Delta_3. \quad (2.11)$$

As a check, the limit case of a solid/vacuum interface is examined. When the density of the fluid  $\rho_f$  is taken as zero, then by (1.2)<sub>2</sub>  $Z = 0$ , and so  $\Delta = \Delta_1 = \Delta_3 = 0$ . The secular equation reduces to  $\Delta_2 = 0$  (written at  $Z = 0$ ), that is the following quartic in  $X = \rho_s v^2$  [10, 5, 6],

$$\begin{vmatrix} X[r_2 r_6 - n_{26}(X - \eta)] & (X - \eta)(1 + n_{66}X) + r_6^2 X & X[r_2^2 - n_{66}(X - \eta)] \\ 0 & X & X - \eta \\ (1 + r_2)X - \eta & 0 & 2r_6(X - \eta) \end{vmatrix} = 0. \quad (2.12)$$

### 3 Examples

Calculations for usual combinations of a solid and a fluid show that in general the speed of a Scholte wave is very close to the speed of sound in the fluid. Hence, consider water ( $\rho_f = 1025 \text{ kg/m}^3$ ,  $c = 1531 \text{ m/s}$  at  $25^\circ\text{C}$  [11]) over gypsum (monoclinic,  $\rho_s$  and  $C_{ij}$  in Table 1 [12]): the secular equation (2.11) yields a Scholte wave speed within the interval [1519 m/s, 1526 m/s] (depending on the orientation of the cut plane), which is within less than

0.8% of the speed of sound in the water and beyond reasonable accuracy for measurements.

Yet for certain choices, the Scholte wave speed moves away from the speed of sound in the fluid. One example is the combination ice/water presented in the Introduction. A second example is the combination of pure water ( $\rho_f = 998 \text{ kg/m}^3$ ,  $c = 1498 \text{ m/s}$  at  $25^\circ\text{C}$  [11]) and Terpine Monohydrate (orthorhombic,  $\rho_s$  and  $C_{ij}$  in Table 1 [3]): at  $\theta = 0^\circ$  and  $\theta = 90^\circ$  (crystal cut along a plane containing two crystallographic axes) the wave propagates at  $1228.3 \text{ m/s}$  and  $1249.5 \text{ m/s}$ , respectively; Figure 2(a) shows how the Scholte wave speed varies between these two extremes as a function of  $\theta$ . Another way of separating distinctly the Scholte wave speed from the sound speed is to increase the pressure, and hence the speed of sound, in the fluid. Crowhurst [13] et al. recently measured the Scholte wave speed for Methanol over Germanium in a diamond anvil cell: as the pressure increases from  $0.56 \text{ GPa}$  to  $2.2 \text{ GPa}$ , so does the speed of sound in Methanol, from about  $2500 \text{ m/s}$  to  $3500 \text{ m/s}$ . In Table 1, the stiffnesses and density of Germanium (cubic) at  $20^\circ$  are recalled [3]; the density of Methanol is  $791.4 \text{ kg/m}^3$  at  $20^\circ$  [11]. Figure 2(b) shows, in agreement with their results, the combined influence of orientation and speed of sound on Scholte wave propagation; each curve corresponds to a different speed of sound in Methanol, from  $c = 2000 \text{ m/s}$  (bottom curve) to  $c = 4000 \text{ m/s}$  (top curve) by  $500 \text{ m/s}$  increments.

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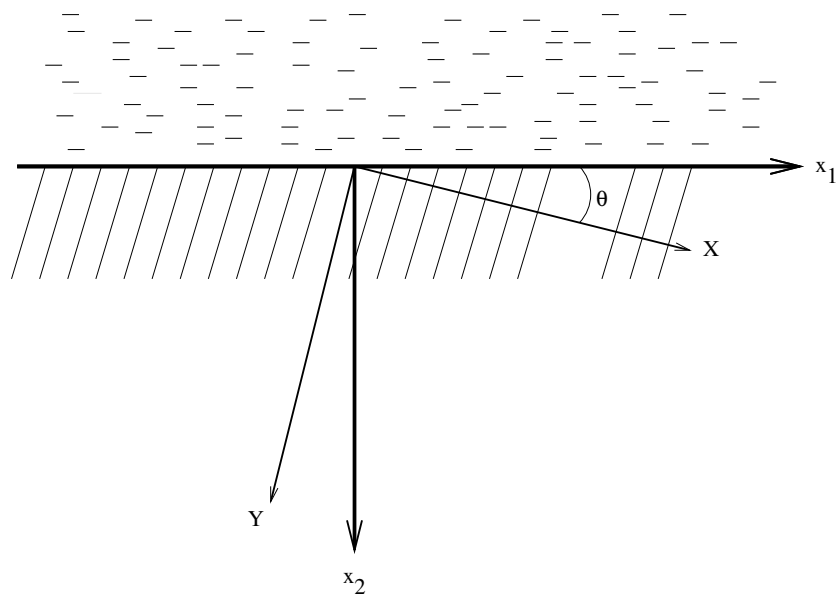


Figure 1: Fluid/solid interface



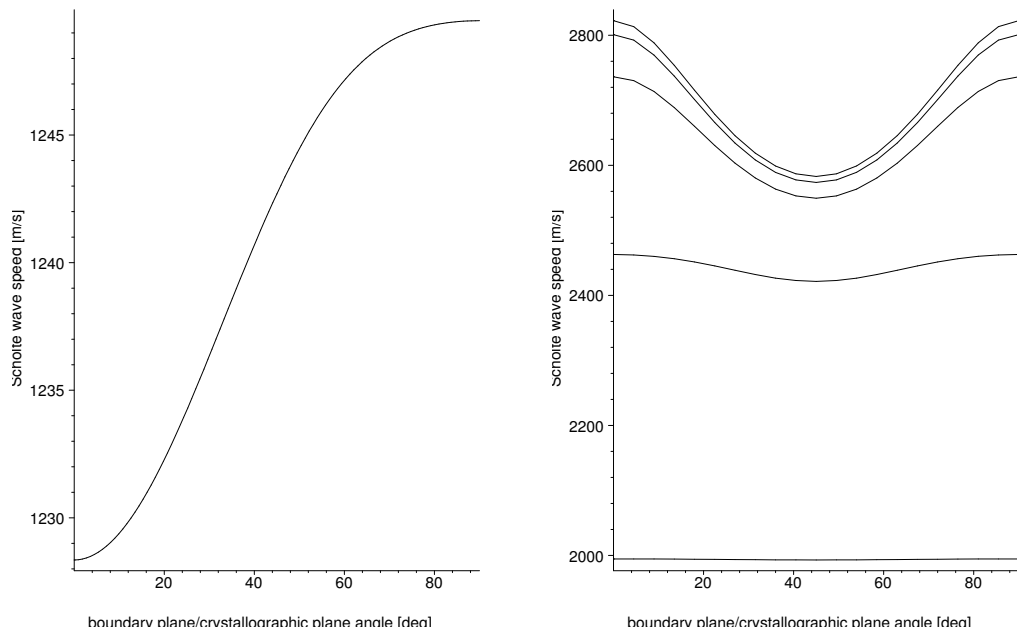


Figure 2: Scholte wave speeds for (a) Water/Terpine interface and (b) Methanol/Germanium interface, where the speed of sound in the fluid is [m/s]: 2000 (bottom curve), 2500, 3000, 3500, 4000 (top curve).

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**Figure 1: Fluid/solid interface.**

**Figure 2: Scholte wave speeds (a) Water/Terpine interface and (b) Methanol/Germanium interface, where the speed of sound in the fluid is [m/s]: 2000 (bottom curve), 2500, 3000, 3500, 4000 (top curve).**

**Figure 2(a):**

Legend on graduated horizontal axes: “boundary plane/crystallographic plane angle [deg].”

Legend on graduated vertical axis: “Scholte wave speed [m/s].”

**Figure 2(b):**

Legend on graduated horizontal axes: “boundary plane/crystallographic plane angle [deg].”

Legend on graduated vertical axis: “Scholte wave speed [m/s].”

Table 1. *Values of the elastic stiffnesses ( $10^{10}$  N/m<sup>2</sup>), density (kg/m<sup>3</sup>), and surface (Rayleigh) wave speed (m/s) for 3 crystals.*

crystal	$C_{11}$	$C_{22}$	$C_{12}$	$C_{16}$	$C_{26}$	$C_{66}$	$\rho_s$	$v_R$
ice ( $-5^\circ\text{C}$ )	1.38	1.38	0.707	0	0	0.3365	940	1766
gypsum	50.2	94.5	28.2	-7.5	-11.0	32.4	2310	3011
terpine	1.25	0.99	0.38	0	0	0.346	1110	1644
germanium	12.92	12.92	4.79	0	0	6.70	5320	2936