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Finite amplitude elastic waves propagating in compressible solids

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Abstract

The paper studies the interaction of a longitudinal wave with transverse waves in general isotropic and unconstrained hyperelastic materials, including the possibility of dissipation. The dissipative term chosen is similar to the classical stress tensor describing a Stokesian fluid and is commonly used in nonlinear acoustics. The aim of this research is to derive the corresponding general equations of motion, valid for any possible form of the strain energy function and to investigate the possibility of obtaining some general and exact solutions to these equations by reducing them to a set of ordinary differential equations. Then the reductions can lead to some exact closed-form solutions for special classes of materials (here the examples of the Hadamard, Blatz-Ko, and power-law strain energy densities are considered, as well as fourth-order elasticity). The solutions derived are in a time/space separable form and may be interpreted as generalized oscillatory shearing motions and generalized sinusoidal standing waves. By means of standard methods of dynamical systems theory, some peculiar properties of waves propagating in compressible materials are uncovered, such as for example, the emergence of destabilizing effects. These latter features exist for highly nonlinear strain energy functions such as the relatively simple power-law strain energy, but they cannot exist in the framework of fourth-order elasticity.

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1   Introduction

Nonlinear elastic wave propagation is a subject of considerable interest for many natural and industrial applications such as seismology [55], soft tissue acoustics [50], or the dynamics of elastomers [41]. A detailed study of the theoretical issues associated with wave phenomena forms the basis of our understanding of important non-destructive and non-invasive techniques of investigation such as for instance, the technique of transient elastography for the analysis of soft solids [20]. The mathematics of wave phenomena is still an active subject of research where many outstanding problems are waiting for a definitive systematic treatment. Many papers are devoted to the study of nonlinear elastic wave propagation and a recent summary on the status of contemporary research on the subject can be found in the authoritative review by Norris [47], while an account of the mathematics of hyperbolic conservation laws can be found in a book by Dafermos [23].

It is important to note that in most studies of nonlinear acoustics, the investigation is usually restricted to *weakly nonlinear* waves. Indeed, for materials such as metals, rocks, or ceramics, the ratio of the dynamic displacement to the wavelength is a small parameter and so, the theories of third-order elasticity or of fourth-order elasticity seem to be sufficient to account for the nonlinear effects observed in experimental tests. The situation is however completely different when we consider elastomeric materials such as those used for vibration isolators or automobile tires, and when we consider biological materials such as arterial walls or glands, under physiological or pathological conditions. The deformation rates occurring in these materials are so large that their behavior will differ significantly not only from the behavior predicted by the linear theory of elasticity, but also from the behavior predicted by the weakly nonlinear theories. It should therefore be important to be able to derive and to investigate the general equations governing the propagation of finite amplitude longitudinal and transverse waves in the context of full nonlinear elasticity, which recovers the weakly nonlinear case as a special case. Moreover, recent researches in the constitutive behavior of rubber-like material indicate that the stiffening effect, a peculiar but real phenomenon occurring at very large strains, cannot be described accurately when the full polynomial strain energy function is approximated; also, high order polynomial theories may give rise to numerical difficulties in the fitting of the material parameters with the experimental data and to artificial instability phenomena (see Pucci and Saccomandi [51] and Ogden

et al. [49]).

These considerations have led to the present paper which is devoted to the study of the interaction of a longitudinal wave with transverse waves in *general isotropic* and *unconstrained hyperelastic* materials and this also when *dissipation* is taken into account. The aim of our research is first, to derive the corresponding general equations of motion, valid for any possible form of the strain energy function; second, to investigate the possibility of obtaining some general and exact solutions to these equations by reducing the specific field equations to a set of ordinary differential equations; and third, to provide, using these reductions, some exact closed-form solutions for special classes of materials.

To the best of our knowledge, previous studies taking systematically into account the full nonlinear equations of motion have generally been restricted to the formal theory of singular surfaces and of acceleration waves. These studies stem from the fundamental researches initiated by Hadamard [30] in 1903 and were advanced mainly by Ericksen, by Thomas, and by Truesdell in the 1950s and 1960s (see for example Truesdell [58] and the review by Chen [22]). They show that the conjunction of the nonlinearity in the material response and of the hyperbolic nature of the governing equations (in the purely elastic case) leads inevitably to shock formation. The results obtained are general and exact but they are formal results: in effect, no solution is found explicitly but rather, conditions are given that have to be fulfilled by solutions, if they exist.

In contrast, the present article focuses on the explicit determination of solutions. Our investigation is directly related to the celebrated finite amplitude elastic motions discovered by Carroll [9, 10, 11, 12, 13, 14, 15]. His exact solutions are versatile in their fields of application because they are valid not only for nonlinearly elastic solids, but also for general viscoelastic solids [14], Reiner-Rivlin fluids [14, 15], Stokesian fluids [14], Rivlin-Ericksen fluids [15], liquid crystals [16], dielectrics [17], magnetic materials [18], etc. They also come in a great variety of forms, as circularly-polarized harmonic progressive waves, as motions with sinusoidal time dependence, as motions with sinusoidal space dependence, etc. Recently the present authors [25] extended Carroll's solutions to the case of an incompressible hyperelastic body in rotation, by considering a new synthetic and very effective method based on complex variables. In doing so, they discovered a striking analogy between the equations of motion obtained for a motion general enough to include all of the above motions, and the equations obtained for the motion of a nonlin-

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ear string, as considered by Rosenau and Rubin [53]. The method used by Rosenau and Rubin shows in a simple and direct way that all (and more) of the different results obtained by Carroll are a direct consequence of material isotropy and of the Galilean invariance of the field equations. This is indeed an explanation for their ubiquity and versatility.

The paper is organized in the following manner. In Section II we lay out the general field equations of nonlinear elasticity. Because dissipation cannot be neglected for most problems in the dynamics of elastomeric materials and of soft tissues, we add a simple inelastic tensor of differential type to the hyperelastic Cauchy stress tensor. This additional term is similar to the classical stress tensor describing a Stokesian fluid; it is the dissipative term introduced by Landau and Lifshitz in their monograph on elasticity theory [43] and it is commonly used in nonlinear acoustics (see Norris [47]). We point out that this term is different from the one used in continuum mechanics as a first approximation of dissipative effects or as a regularization of the hyperbolic equations of nonlinear elastodynamics for example in numerical computations [19]. Then we specialize the equations of motion to the case of one longitudinal wave and two transverse waves. In Section III we seek solutions in time/space separable form and derive nonlinear systems of ordinary differential equations for generalized oscillatory shearing motions and for generalized sinusoidal standing waves. In Section IV, different classes of materials are considered (Hadamard, Blatz-Ko, power-law, fourth-order elasticity) and some exact solutions are provided. Concluding remarks are made in Section V, which recaps the main results. In particular, it is emphasized there that a weak nonlinear elasticity theory (up to the fourth order) cannot account for some wrinkling phenomenon found in the fully nonlinear theory (power-law strain energy).

2 Preliminaries

2.1 Equations of motion

Consider a hyperelastic body with strain energy density Σ. Let the initial and current coordinates of a point of the body, referred to the same fixed rectangular Cartesian system of axes, be denoted by $\mathbf{X}$ and $\mathbf{x}$, respectively. Hence a motion of the body is defined by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \tag{1}$$

The response of a homogeneous compressible isotropic elastic solid to deformations from an undistorted reference configuration is described by the constitutive relation [4],

$$\mathbf{T}^E = 2\left(\frac{I_2}{\sqrt{I_3}}\frac{\partial \Sigma}{\partial I_2} + \sqrt{I_3}\frac{\partial \Sigma}{\partial I_3}\right)\mathbf{1} + \frac{2}{\sqrt{I_3}}\frac{\partial \Sigma}{\partial I_1}\mathbf{B} - 2\sqrt{I_3}\frac{\partial \Sigma}{\partial I_2}\mathbf{B}^{-1}, \qquad (2)$$

where $\mathbf{T}^E$ is the (elastic) Cauchy stress tensor, $\mathbf{1}$ is the unit tensor, $\mathbf{B}$ is the left Cauchy-Green strain tensor defined by

$$\mathbf{B} := \mathbf{F}\mathbf{F}^T, \qquad (3)$$

$\mathbf{F} := \partial \mathbf{x}/\partial \mathbf{X}$ being the deformation gradient tensor, and I_1, I_2, I_3 are the first three invariants of $\mathbf{B}$,

$$I_1 := \operatorname{tr}\mathbf{B}, \quad I_2 := \tfrac{1}{2}[I_1^2 - \operatorname{tr}(\mathbf{B}^2)], \quad I_3 := \det\mathbf{B}. \qquad (4)$$

To describe the simultaneous effects of thermal and viscoelastic dissipation, we introduce the following viscous-like stress tensor [43],

$$\mathbf{T}^D = 2\eta[\dot{\mathbf{E}} - \tfrac{1}{3}(\operatorname{tr}\dot{\mathbf{E}})\mathbf{1}] + (\zeta + \chi)(\operatorname{tr}\dot{\mathbf{E}})\mathbf{1}, \qquad (5)$$

where $\mathbf{E}$ is the Green-Lagrange strain tensor,

$$\mathbf{E} := \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}), \qquad (6)$$

and the dot indicates the time derivative. In (5), the constant η (> 0) is the shear viscosity coefficient, the constant ζ (> 0) is the bulk viscosity coefficient, and the constant χ (> 0) is the coefficient related to the thermal properties of the solid (that is, ambient temperature, thermal expansion coefficient, and specific heat).

Let ρ and ρ_0 denote the mass densities of the body measured in the current configuration and in the reference configuration, respectively. Then the equation of motions, in the absence of body forces, are: div $(\mathbf{T}^E + \mathbf{T}^D) = \rho\ddot{\mathbf{x}}$ in current form, or equivalently,

$$\operatorname{Div}\,[\sqrt{I_3}(\mathbf{T}^E + \mathbf{T}^D)(\mathbf{F}^{-1})^T] = \rho_0\ddot{\mathbf{x}}, \qquad (7)$$

$$\frac{\partial}{\partial X_j}[\sqrt{I_3}(T_{ik}^E + T_{ik}^D)F_{jk}^{-1}] = \rho_0\frac{\partial^2 x_i}{\partial t^2},$$

in referential form.

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Remark on the dissipative stress tensor

For *incompressible* solids, det $\mathbf{F} = 1$ at all times (so that $I_3 = 1$ at all times) and an arbitrary spherical pressure term, to be determined from initial and boundary conditions, is introduced so that

$$\mathbf{T}^E + \mathbf{T}^D = -p\mathbf{1} + 2\frac{\partial \Sigma}{\partial I_1}\mathbf{B} - 2\frac{\partial \Sigma}{\partial I_2}\mathbf{B}^{-1} + 2\eta\dot{\mathbf{E}}, \tag{8}$$

where $p = p(\mathbf{x}, t)$ is a Lagrange multiplier. We note that in order to study the propagation of finite amplitude motions in viscoelastic *incompressible* materials, some authors [34, 35, 24, 36] chose to add to the elastic Cauchy stress tensor a viscous term linear in the stretching tensor $\mathbf{D}$, which is the symmetric part of the velocity gradient tensor $\dot{\mathbf{F}}\mathbf{F}^{-1}$. In other words, they chose to write

$$\mathbf{T}^E + \mathbf{T}^D = -p_*\mathbf{1} + 2\frac{\partial \Sigma}{\partial I_1}\mathbf{B} - 2\frac{\partial \Sigma}{\partial I_2}\mathbf{B}^{-1} + 2\nu\mathbf{D}, \tag{9}$$

for some Lagrange multiplier p_* and for some constant ν. However it must be recalled (e.g. Chadwick [21, p.146]) that $\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F} \neq \mathbf{D}$ in general. Thus we remark, for the time being, that the assumptions (8) and (9) are not equivalent a priori. We discuss these issues further at the end of the next Subsection.

2.2 Finite amplitude longitudinal and transverse plane waves

Now we consider the following class of motions,

$$x = u(X, t), \quad y = Y + v(X, t), \quad z = Z + w(X, t), \tag{10}$$

which describes the superposition of a transverse wave, polarized in the (YZ) plane and propagating in the X-direction with a longitudinal wave propagating in the X-direction. Then,

$$[\mathbf{F}]_{ij} = \begin{bmatrix} u_X & 0 & 0 \\ v_X & 1 & 0 \\ w_X & 0 & 1 \end{bmatrix}, \quad [\mathbf{F}^{-1}]_{ij} = \frac{1}{u_X}\begin{bmatrix} 1 & 0 & 0 \\ -v_X & u_X & 0 \\ -w_X & 0 & u_X \end{bmatrix}, \tag{11}$$

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and

$$\left[\dot{\mathbf{E}}\right]_{ij} = \frac{1}{2} \begin{bmatrix} (u_X^2 + v_X^2 + w_X^2)_t & v_{Xt} & w_{Xt} \\ v_{Xt} & 0 & 0 \\ w_{Xt} & 0 & 0 \end{bmatrix}. \tag{12}$$

Here and hereafter, a subscript letter denotes partial differentiation (thus $u_X = \partial u/\partial X$, $v_{tt} = \partial^2 v/\partial t^2$, etc.) It follows from the definitions (4) that the first three invariants of $\mathbf{B}$ are given by

$$I_1 = 2 + u_X^2 + v_X^2 + w_X^2, \quad I_2 = 1 + 2u_X^2 + v_X^2 + w_X^2, \quad I_3 = u_X^2, \tag{13}$$

and consequently that the strain-energy density $\Sigma = \Sigma(I_1, I_2, I_3)$ is here a function of u_X^2 and $v_X^2 + w_X^2$ alone,

$$\Sigma = \Sigma(u_X^2, v_X^2 + w_X^2). \tag{14}$$

With the expressions (11) and (12) substituted into the Cauchy stress (2) and into the viscous-like tensor (5), the equation of motions (7) reduce to

$$\rho_0 u_{tt} = [(Q_1 + Q_2)u_X + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)(u_X^2 + v_X^2 + w_X^2)_t]_X,$$
$$\rho_0 v_{tt} = [Q_1 v_X + \eta v_{Xt}]_X,$$
$$\rho_0 w_{tt} = [Q_1 w_X + \eta w_{Xt}]_X, \tag{15}$$

where the functions $Q_1 = Q_1(u_X^2, v_X^2 + w_X^2)$ and $Q_2 = Q_2(u_X^2, v_X^2 + w_X^2)$ are defined by

$$Q_1 := 2\left(\frac{\partial \Sigma}{\partial I_1} + \frac{\partial \Sigma}{\partial I_2}\right), \quad Q_2 := 2\left(\frac{\partial \Sigma}{\partial I_2} + \frac{\partial \Sigma}{\partial I_3}\right). \tag{16}$$

Eqs. (15) form a system of two coupled nonlinear partial differential equations, generalizing the system derived by Carroll in [9] for an elastic compressible material.

The mathematical treatment of initial-boundary-value problems for equations such as (15) is not trivial at all. There is a considerable literature on the subject and the most updated reference is the recent paper by Antman and Seidman [2], to which we refer for further information. Here we use a semi-inverse method and we shall consider the possibility to use our explicit results to solve initial-boundary-value problems only a posteriori. Obviously, the possibility of general uniqueness theorems (such as those established in [2]) is quite valuable to give a clear and definitive status of the solutions found by a semi-inverse method.

Now we differentiate (15) with respect to X, and we recast the resulting equations in the form

$$
\begin{aligned}
\rho_0 U_{tt} &= [(Q_1 + Q_2)U]_{XX} \\
&\quad + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)[U^2 + V^2 + W^2]_{XXt}, \\
\rho_0 V_{tt} &= [Q_1 V]_{XX} + \eta V_{XXt}, \\
\rho_0 W_{tt} &= [Q_1 W]_{XX} + \eta W_{XXt},
\end{aligned}
\tag{17}
$$

where we have introduced the new functions: $U := u_X$, $V := v_X$, $W := w_X$.

Further, these equations can be rewritten in an even more compact form, by introducing the complex function Λ, with modulus Ω and argument θ, defined by

$$
\Lambda(X, t) = \Omega(X, t)e^{i\theta(X,t)} := V + iW,
\tag{18}
$$

so that

$$
\begin{aligned}
V &= \Re(\Lambda) = \Omega \cos\theta, \quad W = \Im(\Lambda) = \Omega \sin\theta, \\
V^2 + W^2 &= \Omega^2.
\end{aligned}
\tag{19}
$$

Then the three equations (17) are

$$
\begin{aligned}
\rho_0 U_{tt} &= [(Q_1 + Q_2)U]_{XX} + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)[U^2 + \Omega^2]_{XXt}, \\
\rho_0 \Lambda_{tt} &= [Q_1 \Lambda]_{XX} + \eta \Lambda_{XXt}.
\end{aligned}
\tag{20}
$$

Now also, the invariants I_1, I_2, I_3 found in (13) are written as

$$
I_1 = 2 + U^2 + \Omega^2, \quad I_2 = 1 + 2U^2 + \Omega^2, \quad I_3 = U^2,
\tag{21}
$$

so that $Q_1 = Q_1(U^2, \Omega^2)$ and $Q_2 = Q_2(U^2, \Omega^2)$ or, more explicitly,

$$
Q_1 = 2\frac{\partial \Sigma}{\partial(\Omega^2)}, \quad Q_2 = 2\left(\frac{\partial \Sigma}{\partial(U^2)} - \frac{\partial \Sigma}{\partial(\Omega^2)}\right).
\tag{22}
$$

These equations have been developed in the *purely elastic* case of a *single* shear wave by Carman and Cramer [8], who determined some numerical and asymptotic solutions.

Remark on finite amplitude motions in incompressible materials

In *incompressible* solids, $\det \mathbf{F} = 1$ at all times and so by $(11)_1$, $u_X = 1$. Then we find

$$\left[\dot{\mathbf{E}}\right]_{ij} = \frac{1}{2} \begin{bmatrix} (v_X^2 + w_X^2)_t & v_{Xt} & w_{Xt} \\ v_{Xt} & 0 & 0 \\ w_{Xt} & 0 & 0 \end{bmatrix}, \tag{23}$$

and

$$[\mathbf{D}]_{ij} = \frac{1}{2} \begin{bmatrix} 0 & v_{Xt} & w_{Xt} \\ v_{Xt} & 0 & 0 \\ w_{Xt} & 0 & 0 \end{bmatrix}. \tag{24}$$

It then follows that

$$\text{Div}\,[(-p\mathbf{1} + 2\eta\dot{\mathbf{E}})(\mathbf{F}^{-1})^T] = -\text{Grad}\,\hat{p} + \eta v_{XXt}\mathbf{j} + \eta w_{XXt}\mathbf{k}, \tag{25}$$

where $\hat{p} := p - (v_X^2 + w_X^2)_t$, and that

$$\text{Div}\,[(-p_*\mathbf{1} + 2\nu\mathbf{D})(\mathbf{F}^{-1})^T] = -\text{Grad}\,p_* + \nu v_{XXt}\mathbf{j} + \nu w_{XXt}\mathbf{k}. \tag{26}$$

Now we see that the equations of motion (7) are the same whether the choice (8) or the choice (9) is made. The two approaches in modeling the dissipative effects are reconciled, at least as far as finite amplitude motions in incompressible materials are concerned.

3   Solutions in separable form

The general solution of (20) may be found only via numerical methods. To obtain some simple analytical information here, we look for reductions of the governing equations (20) to a set of ordinary differential equations. The underlying idea is to search for $\Lambda(X,t)$, defined in (18), in a separable form such as

$$\Omega(X,t) = \Omega_1(X)\Omega_2(t), \quad \theta(X,t) = \theta_1(X) + \theta_2(t), \tag{27}$$

where Ω_1, θ_1 are functions of space only and Ω_2, θ_2 are functions of time only. Some lengthy computations, not reproduced here, show that in only two general cases can equations (20) be separated in such way for *any* material response.

The results of this Section generalize the results of Carroll [10, 13] by considering more general types of displacement fields and by including viscoelastic effects. We also acknowledge that the solutions presented here are a

generalization of the non-linear three-dimensional motions of an elastic string found by Rosenau and Rubin [53] (see [25] for a different generalization, to finite amplitude waves in rotating incompressible elastic bodies.)

3.1 Generalized oscillatory shearing motions

The first general type of separable solution is of the form,

$$\Lambda(X,t) = [\psi(t) + i\phi(t)]k\mathrm{e}^{i(kX - \theta(t))}, \tag{28}$$

where k is a constant and ψ, ϕ, θ are arbitrary real functions of time. Here Λ is indeed of the separable form given by (18), with

$$\begin{aligned}
\Omega_1(X) &= k = \text{const.}, & \Omega_2(t) &= [\phi(t)^2 + \psi(t)^2]^{\frac{1}{2}}, \\
\theta_1(X) &= kX, & \theta_2(t) &= \theta(t) + \tan^{-1}[\phi(t)/\psi(t)].
\end{aligned} \tag{29}$$

Separating real and imaginary parts, we find that the transverse displacement field (v, w) derived from (28) is

$$\begin{aligned}
v(X,t) &= \phi(t)\cos(kX - \theta(t)) + \psi(t)\sin(kX - \theta(t)), \\
w(X,t) &= \phi(t)\sin(kX - \theta(t)) - \psi(t)\cos(kX + \theta(t)).
\end{aligned} \tag{30}$$

In this case,

$$\Omega^2 = k^2[\phi^2(t) + \psi^2(t)], \tag{31}$$

is a function of time only.

Now the first equation of motion $(20)_1$ reduces to

$$\rho_0 U_{tt} = [(Q_1 + Q_2)U]_{XX} + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)[U^2]_{XXt}, \tag{32}$$

where

$$Q_1 = Q_1(U^2(X,t), \Omega^2(t)), \quad Q_2 = Q_2(U^2(X,t), \Omega^2(t)). \tag{33}$$

On the other hand, the second equation of motion $(20)_2$ reduces to

$$\rho_0 \Lambda_{tt} = Q_1 \Lambda_{XX} + \eta \Lambda_{XXt}. \tag{34}$$

Using the separable form (28) for Λ, factoring out the exponential term, and separating the real and imaginary parts yields

$$\begin{aligned}
\rho_0(\phi\theta'' - \psi\theta'^2 + 2\phi'\theta' + \psi'') + k^2 Q_1 \psi + k^2 \eta(\phi\theta' + \psi') &= 0, \\
\rho_0(\phi'' - 2\psi'\theta' - \phi\theta'^2 - \psi\theta'') + k^2 Q_1 \phi + k^2 \eta(\phi' - \psi\theta') &= 0.
\end{aligned} \tag{35}$$

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This is a nonlinear system of two *ordinary* differential equations in the three unknowns $\phi(t)$, $\theta(t)$, and $\psi(t)$ either: when Q_1 is a constant independent of its arguments $U^2(X,t)$ and $\Omega^2(t)$, or: when $Q_1 = Q_1(t)$. By $(16)_1$, this latter condition is satisfied if and only if $U(X,t) = U(t)$, a function of time only. It then follows by $(33)_2$ that $Q_2 = Q_2(t)$ also. By substitution into the first equation of motion $(20)_1$, we obtain $\rho_0 U'' = 0$. Then the longitudinal displacement field is given by

$$U(t) = C_1 t + C_2 = u_X, \tag{36}$$

so that

$$u(X,t) = (C_1 t + C_2)X + C_3, \tag{37}$$

where C_1, C_2, C_3 are constants. This is a *homogeneous accelerationless* motion.

Hence we showed that for any compressible elastic material with viscoelastic dissipative part defined as in (5), maintained in a state of longitudinal simple extension, the special transverse waves (30) may always propagate.

Because (35) is a system of only two equations in three unknowns, there is a certain freedom in considering special classes of solutions. One such class of special solutions is obtained by considering $\psi := 0$ in (30), leading to

$$
\begin{aligned}
v(X,t) &= \phi(t)\cos(kX - \theta(t)), \\
w(X,t) &= \phi(t)\sin(kX - \theta(t)),
\end{aligned}
\tag{38}
$$

a direct generalization of the classical damped harmonic circularly-polarized wave solution,

$$
\begin{aligned}
v(X,t) &= A\mathrm{e}^{-ht}\cos(kX - \omega t), \\
w(X,t) &= A\mathrm{e}^{-ht}\sin(kX - \omega t),
\end{aligned}
\tag{39}
$$

where A, h, and ω are suitable constants. For these special motions at $\psi := 0$, the system (35) reduces to

$$
\begin{aligned}
\rho_0(\phi\theta'' + 2\phi'\theta') + \eta k^2 \phi\theta' &= 0, \\
\rho_0(\phi'' - \phi\theta'^2) + k^2 Q_1 \phi + \eta k^2 \phi' &= 0.
\end{aligned}
\tag{40}
$$

The equation $(40)_1$ admits the first integral

$$\phi^2 \theta' = E\mathrm{e}^{-\frac{\eta k^2}{\rho_0}t}, \tag{41}$$

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where E is a constant of integration.

If $E = 0$, then $\theta(t) = \theta_0$, a constant. We end up with standing waves,

$$v(X,t) = \phi(t)\cos(kX - \theta_0), \quad w(X,t) = \phi(t)\sin(kX - \theta_0), \qquad (42)$$

where, according to $(40)_2$, ϕ is a solution to

$$\rho_0\phi'' + k^2\eta\phi' + k^2 Q_1(k^2\phi^2, t)\phi = 0, \qquad (43)$$

which is the equation of a damped nonlinear oscillator.

If $E \neq 0$, then solving (41) for θ', we end up with the single equation $(40)_2$, which reduces to

$$\rho_0\phi'' + k^2\eta\phi' + k^2 Q_1\phi - \frac{\rho_0 E^2}{\phi^3}e^{-\frac{2\eta k^2}{\rho_0}t} = 0. \qquad (44)$$

In the elastic case ($\eta = 0$), this equation describes the plane motion of a particle in a central force field. In general ($\eta \neq 0$), it is a nonlinear and non-autonomous differential equation.

For *harmonic wave propagation*, $\theta(t)$ is taken as $\theta(t) = \omega t$. By (41), ϕ is proportional to $e^{-\frac{\eta k^2}{2\rho_0}t}$ and describes the usual linear damping function. However by substitution in $(40)_2$, we see that this situation is possible only if the material response is such that Q_1 is a constant,

$$Q_1 = 2\left(\frac{\partial\Sigma}{\partial I_1} + \frac{\partial\Sigma}{\partial I_2}\right) = \text{const.} = \rho_0\frac{\omega^2}{k^2} + \frac{\eta^2 k^2}{4\rho_0}. \qquad (45)$$

In this Subsection we analyzed the solutions (38) which are the analogue of the Class I solutions presented by Rosenau and Rubin [54] for purely elastic strings. Qualitatively the solutions describe particles moving in a helical path which lies on a cylindrical surface of time-varying radius.

3.2 Generalized sinusoidal standing waves

We now consider the second class of solutions in separable form, namely

$$\Lambda(X,t) = [(i\phi(X) + \psi(X))\theta'(X) + (\phi'(X) - i\psi'(X))]e^{i(\omega t + \theta(X))}, \qquad (46)$$

where k is a constant and $\psi(X)$, $\phi(X)$, $\theta(X)$ are arbitrary functions of space. Here $\Lambda(X,t)$ is indeed of the separable form (18), with

$$\Omega_1(X) = [(\phi' + \psi\theta')^2 + (\phi\theta' - \psi')^2]^{\frac{1}{2}}, \quad \Omega_2(t) = 1 = \text{const.},$$
$$\theta_1(X) = \theta + \tan^{-1}[(\phi\theta' - \psi')/(\psi\theta' + \phi')], \quad \theta_2(t) = \omega t. \qquad (47)$$

The transverse displacement field (v, w) corresponding to (46) is

$$v(X, t) = \phi(X)\cos(\omega t + \theta(X)) + \psi(X)\sin(\omega t + \theta(X)),$$
$$w(X, t) = \phi(X)\sin(\omega t + \theta(X)) - \psi(X)\cos(\omega t + \theta(X)). \qquad (48)$$

In this case, $\Omega = \Omega_1(X)$, a function of X only, and the first equation of motion $(20)_1$ reduces to

$$\rho_0 U_{tt} = [(Q_1 + Q_2)U]_{XX} + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)[U^2]_{XXt}, \qquad (49)$$

where

$$Q_1 = Q_1(U^2, \Omega_1^2(X)), \quad Q_2 = Q_2(U^2, \Omega_1^2(X)). \qquad (50)$$

Now, introducing (46) into the second equation of motion $(20)_2$ and separating the real part from the imaginary part, we obtain

$$-\rho_0\omega^2(\psi\theta' + \phi') = (\psi\theta' + \phi')(Q_1)_{XX} + 2(2\psi'\theta' + \psi\theta'' + \phi'' - \phi\theta'^2)(Q_1)_X$$
$$+ (3\psi''\theta' + 3\psi'\theta'' + \psi\theta''' - \psi\theta'^3 - 3\phi'\theta'^2 - 3\phi\theta'\theta'' + \phi''')Q_1$$
$$- \eta\omega(3\phi''\theta' + 3\phi'\theta'' + \phi\theta''' - \phi\theta'^3 + 3\psi'\theta'^2 + 3\psi\theta'\theta'' - \psi'''), \quad (51)$$

and

$$-\rho_0\omega^2(\phi\theta' - \psi') = (\phi\theta' - \psi')(Q_1)_{XX} + 2(2\phi'\theta' + \phi\theta'' - \psi'' + \psi\theta'^2)(Q_1)_X$$
$$+ (3\phi''\theta' + 3\phi'\theta'' + \phi\theta''' - \phi\theta'^3 + 3\psi'\theta'^2 + 3\psi\theta'\theta'' - \psi''')Q_1$$
$$+ \eta\omega(3\psi''\theta' + 3\psi'\theta'' + \psi\theta''' - \psi\theta'^3 - 3\phi'\theta'^2 - 3\phi\theta'\theta'' + \phi'''). \quad (52)$$

A sufficient condition for the equations (51) and (52) to compose a nonlinear system of two *ordinary* differential equations in the three unknowns ϕ, ψ, and θ is that the longitudinal field depends only on X: $U(X, t) \equiv U(X)$. In this case, $Q_1 = Q_1(X)$, $Q_2 = Q_2(X)$ and (49) reduces to

$$[(Q_1(X) + Q_2(X))U(X)]'' = 0. \qquad (53)$$

Further progress is made by fixing one of the three unknown functions, $\theta := 0$, say. Then the governing equations (51) and (52) reduce to

$$(Q_1\phi')' + \rho_0\omega^2\phi = -\eta\omega\psi'', \quad (Q_1\psi')' + \rho_0\omega^2\psi = \eta\omega\phi''. \qquad (54)$$

At $\eta = 0$, these equations are formally equivalent to the equations derived by Carroll [11], with the difference that the materials response functions Q_1

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and Q_2 depend not only on Ω^2 (the amount of shear) but also on U. In Carroll [15] (see system 5.12 in that reference), the attention is restricted to *incompressible* fluids and solids, for which $U(X)$ must be constant.

The solutions considered in this sub-Section contain the Class II solutions of Rosenau and Rubin [54].

4   Specific materials

Now that we have established the possibility of reducing the general nonlinear partial differential equations to ordinary differential equations, we must specify constitutive relations in order to make progress.

4.1   Hadamard materials

Finite amplitude motions in *elastic* Hadamard materials have been thoroughly studied, see John [42] and Boulanger et al. [6]. The analysis conducted in this paper allows for a treatment of a *dissipative* Hadamard solid.

The Hadamard strain energy function is defined by

$$2\Sigma = C(I_1 - 3) + D(I_2 - 3) + G(I_3), \tag{55}$$

where $G(I_3)$ is a material function and C, D are material constants such that [6] $C > 0$, $D \geq 0$, or $C \geq 0$, $D > 0$. The connection with the Lamé constants λ and μ of the linear theory of isotropic elasticity is made through the relations:

$$C = 2\mu + G'(1), \quad D = -\mu - G'(1), \quad 2G''(1) = \lambda + 2\mu. \tag{56}$$

For this material, the functions Q_1 and Q_2 defined in (16) are

$$Q_1 = C + D, \quad Q_2 = D + G'(U^2), \tag{57}$$

and the equations of motion (20) reduce to

$$\begin{aligned}
\rho_0 U_{tt} &= \{[C + 2D + G'(U^2)]U\}_{XX} \\
&\quad + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)(U^2 + \Omega^2)_{tXX}, \\
\rho_0 \Lambda_{tt} &= (C + D)\Lambda_{XX} + \eta\Lambda_{tXX}.
\end{aligned} \tag{58}$$

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We point out that the equation governing the propagation of the transverse waves $(58)_2$ is always linear, for all Hadamard materials, elastic *or* dissipative. Hence we see at once that classical harmonic (damped and attenuated) transverse waves are always possible for these materials.

The material function $G(I_3)$ accounts for the effects of compressibility. For example, Levinson and Burgess [44] proposed the following explicit form,

$$G(I_3) = (\lambda + \mu)(I_3 - 1) - 2(\lambda + 2\mu)(\sqrt{I_3} - 1). \qquad (59)$$

This function leads here to a remarkably simple system of equations:

$$\rho_0 U_{tt} = (\lambda + 2\mu)U_{XX} + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)(U^2 + \Omega^2)_{tXX},$$
$$\rho_0 \Lambda_{tt} = \mu \Lambda_{XX} + \eta \Lambda_{tXX}. \qquad (60)$$

Now we take a look at some of the solutions investigated in the previous Section. Classical *damped harmonic circularly polarized waves* such as (39) are possible in a dissipative Hadamard material because then $\Omega = kAe^{-ht}$, a function of time only, and $Q_1 = C + D = \mu$, a constant. The corresponding dispersion equation (45) is here

$$\mu = \rho_0 \frac{\omega^2}{k^2} + \frac{\eta^2 k^2}{4\rho_0}. \qquad (61)$$

The associated longitudinal motion may be either the homogeneous accelerationless motion (36) or any solution to the nonlinear equation $(58)_1$, here:

$$\rho_0 U_{tt} = \{[G'(U^2) - G'(1)]U\}_{XX} + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)(U^2)_{tXX}, \qquad (62)$$

which for the Levinson and Burgess choice (59) of G is:

$$\rho_0 U_{tt} = (\lambda + 2\mu)U_{XX} + \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)(U^2)_{tXX}. \qquad (63)$$

Now consider the *sinusoidal standing waves* of Eq.(48) at $\theta(X) = 0$. When the longitudinal field U depends on X only, their behavior is governed by Eqs.(54), here:

$$\mu \phi'' = \rho_0 \omega^2 \phi + \eta \omega \psi'', \quad \mu \psi'' = \rho_0 \omega^2 \psi - \eta \omega \phi'', \qquad (64)$$

or equivalently, by the single complex equation

$$(\mu - i\eta\omega)(\phi + i\psi)'' = \rho\omega^2(\phi + i\psi). \qquad (65)$$

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This equation possesses the following class of attenuated solutions,

$$\phi(X) = e^{-\alpha X}[k_1 \sin(\beta X) + k_2 \cos(\beta X)],$$
$$\psi(X) = e^{-\alpha X}[k_3 \cos(\beta X) + k_4 \sin(\beta X)], \tag{66}$$

where k_1, k_2 are disposable constants and

$$\alpha, \beta = \sqrt{\frac{\rho_0 \omega^2}{2(\mu^2 + \eta^2 \omega^2)} \left[\sqrt{\mu^2 + \eta^2 \omega^2} \pm \mu\right]}. \tag{67}$$

These solutions may be used to solve some simple boundary value problems. For example, consider the case of semi-infinite body bounded by a vibrating rigid plate at $X = 0$, and take the velocity components of the plate as

$$\dot{x} = 0, \quad \dot{y} = V\omega \cos(\omega t), \quad \dot{z} = V\omega \sin(\omega t). \tag{68}$$

Then $\phi(0) = 0$, $\psi(0) = V$, and the associated transverse displacements are

$$v(X, t) = e^{-\alpha X}[\sin(\beta X) \cos(\omega t) + V \cos(\beta X) \sin(\omega t)],$$
$$w(X, t) = e^{-\alpha X}[\sin(\beta X) \sin(\omega t) - V \cos(\beta X) \cos(\omega t)]. \tag{69}$$

The associated longitudinal displacement independently satisfies (53), here:

$$\{[G'(U^2) - G'(1)]U\}'' = 0, \tag{70}$$

which for the Levinson and Burgess choice (59) of G is simply: $U'' = 0$. Then the basic displacement (69) may be superimposed upon an axial static stretch or a homogeneous motion in the axial direction. In a similar way, it is possible to solve the case of a slab fixed at $X = 0$ and oscillating at $X = L$, or it is possible to use the various integration constants to fix the stress traction or shear stress at the boundary of a vibrating half-space or of a slab.

4.2 Blatz-Ko materials

Experiments on compressible polyurethane rubber lead Blatz and Ko [3] to propose some strain energy functions which have since received much attention, see for instance, Beatty [4]. In particular, two reduced forms of the Blatz-Ko general constitutive equation were deemed appropriate to model

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certain polyurethane rubber samples. One is the Blatz-Ko strain energy function for a solid, polyurethane rubber,

$$\Sigma = \frac{\mu}{2}[I_1 - 3 + \beta(I_3^{1/\beta} - 1)], \tag{71}$$

where μ and β are constants. Direct comparison with (55) shows that this material is a special Hadamard material, with the identifications: $C = \mu$, $D = 0$, $G(I_3) = -\mu\beta I_3^{1/\beta}$. The other is the *Blatz-Ko strain energy function for a foamed, polyurethane elastomer,*

$$\Sigma = \frac{\mu}{2}\left[\frac{I_2}{I_3} - 3 + \beta(I_3^{1/\beta} - 1)\right], \tag{72}$$

where μ and β are constants. The continuity with linear elasticity requires that μ is the infinitesimal shear modulus and that $\beta = (1 - 2\nu)/\nu$, where ν is the infinitesimal Poisson ratio. Blatz and Ko's experiments showed that typically, $\nu = 1/4$ (and so, $\beta = 2$).

For this latter material, the functions Q_1 and Q_2 defined in (16) are

$$Q_1 = \mu U^{-2}, \quad Q_2 = -\mu U^{-2} - \mu(1 + \Omega^2)U^{-4} + \mu U^{\frac{2(\beta-1)}{\beta}}, \tag{73}$$

and the equations of motion (20) reduce to

$$\rho_0 U_{tt} = -\mu\left[(1 + \Omega^2)U^{-3} - U^{\frac{3\beta-2}{\beta}}\right]_{XX}$$
$$+ \tfrac{1}{2}(\zeta + \chi + \tfrac{4}{3}\eta)(U^2 + \Omega^2)_{tXX},$$
$$\rho_0 \Lambda_{tt} = \mu(\Lambda U^{-2})_{XX} + \eta\Lambda_{tXX}. \tag{74}$$

Here, and in contrast with the case of Hadamard materials, the transverse wave is coupled to the longitudinal wave.

First, consider the *generalized oscillatory shearing motions* of Section III.A. As seen there, they are governed by nonlinear ordinary differential equations when the longitudinal displacement u is the homogeneous accelerationless motion of $(36)_2$. Then the standing waves of (42) are governed by

$$\rho_0 \phi'' + \eta k^2 \phi' + \frac{\mu k^2}{(C_1 t + C_2)^2}\phi = 0. \tag{75}$$

At $C_1 = 0$, this is the classical equation of a damped oscillator, whose type of motion (exponential and/or sinusoidal) is decided by the sign of the

quantity: $(\eta k C_2)^2 - 4\rho_0\mu$; hence the nature of the transverse motion depends on the longitudinal extension via the parameter C_2.

At $C_1 \neq 0$ (linear stretch rate), we perform the change of variables $\zeta = C_1 t + C_2$, so that (75) reduces to

$$\zeta^2 \frac{d^2\phi}{d\zeta^2} + 2\Upsilon\zeta^2 \frac{d\phi}{d\zeta} + \frac{\Gamma}{4}\phi = 0. \tag{76}$$

where the quantities Υ, Γ are defined by

$$\Upsilon := \frac{\eta k^2}{2\rho_0 C_1}, \quad \Gamma := \frac{4\mu k^2}{\rho_0 C_1^2}. \tag{77}$$

In the purely elastic case ($\eta = 0$), Υ vanishes and the solution to this equation is quite simple: we find that for

$$\Gamma < 1, \quad \phi(\zeta) = k_1\zeta^{\lambda_+} + k_2\zeta^{\lambda_-}, \quad \text{where} \quad \lambda_\pm = (1 \pm \sqrt{1-\Gamma})/2;$$
$$\Gamma = 1, \quad \phi(\zeta) = k_1\sqrt{\zeta} + k_2\sqrt{\zeta}\ln\zeta;$$
$$\Gamma > 1, \quad \phi(\zeta) = k_1\sqrt{\zeta}\cos(\sqrt{\Gamma-1}\ln\zeta) + k_2\sqrt{\zeta}\sin(\sqrt{\Gamma-1}\ln\zeta); \tag{78}$$

where k_1 and k_2 are integration constants. We note that all these solutions *blow up* as $t \to \infty$ (equivalent to $\zeta \to \infty$); that for $\Gamma \leq 1$, the solutions are monotonic increasing; and that for $\Gamma > 1$, the solutions have a slight oscillatory character. In the viscoelastic case ($\eta \neq 0$), equation (76) is solvable in terms of special functions and a richer variety of solutions emerges, because now the solutions are *not necessarily unbounded*. Moreover, although they are damped solutions, they do *not necessarily vanish with time*. In order to solve equation (76) explicitly, we introduce the function f defined by

$$\phi(\zeta) = e^{-\Upsilon\zeta}f(2\Upsilon\zeta), \tag{79}$$

and we find that it satisfies a Whittaker equation:

$$f''(2\Upsilon\zeta) + \left(-\frac{1}{4} + \frac{\Gamma}{4(2\Upsilon\zeta)^2}\right)f(2\Upsilon\zeta) = 0. \tag{80}$$

It follows that the solution to (76) is expressed in terms of Whittaker's functions as:

$$\phi(\zeta) = e^{-\Upsilon\zeta}\left[k_1 W_{0,\frac{1}{2}\sqrt{1-\Gamma}}(2\Upsilon\zeta) + k_2 M_{0,\frac{1}{2}\sqrt{1-\Gamma}}(2\Upsilon\zeta)\right], \tag{81}$$
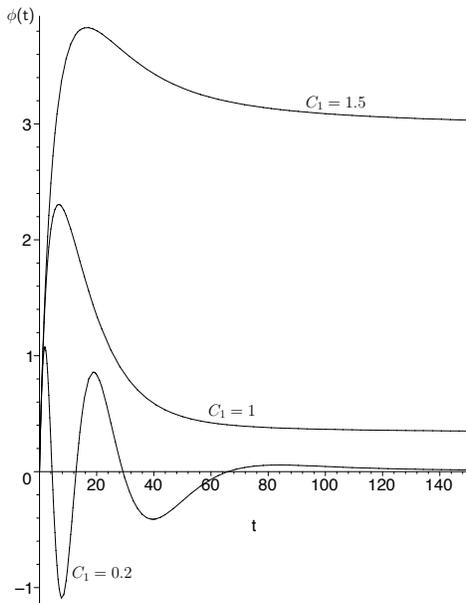
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Figure 1: Influence of the stretch rate C_1 in the X-direction on the shearing motion solutions for a Blatz-Ko material (foamed, polyurethane elastomer).

where k_1 and k_2 are integration constants. To illustrate the influence of the stretch rate in the X-direction (through the constant C_1) upon the behavior of the solutions, we consider that the material is unstretched at $t = 0$ (so that $C_2 = 1$ and $\phi(t = 0) = 0$); we take $\phi'(t = 0) = 1$, $\mu k^2/\rho_0 = 1$, $\eta = \mu/10$ (so that $\Upsilon = 1/(20C_1)$ and $\Gamma = 4/C_1^2$); and we let $C_1 = 0.2$, 1, 1.5. Figure 1 shows that for a "small" rate, the solution undergoes a few oscillations before it vanishes completely; as C_1 increases, the oscillations disappear but the solution tends to an increasing finite asymptotic value. We argue that this behavior can be related to the problem of a tensile impact: when the material is plucked at "low" speed, then a few oscillations take place before the material returns to its original state; when the material is plucked at "high" speed, then the material can accomodate an asymptotic transverse wave.

Now, consider the *generalized sinusoidal standing waves* of Section III.B. As seen there, they are governed by nonlinear ordinary differential equations when the longitudinal displacement u is such that $u_X = U = U(X)$, a

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function of X only. This condition leads to (53), here integrated twice as

$$(\phi'^2 + \psi'^2 - 1)U^{-3} + U^{\frac{(\beta-2)}{\beta}} = C_3 X + C_4, \tag{82}$$

where C_3, C_4 are disposable constants. Taking the typical value $\beta = 2$ corresponding to Poisson ratio $\nu = 1/4$, we find

$$U = \left[\frac{\phi'^2 + \psi'^2 - 1}{C_3 X + C_4 - 1}\right]^{\frac{1}{3}}. \tag{83}$$

Hence for this material, the governing equations (54) for the transverse solutions of type (48) at $\theta(X) = 0$ are, using (83) and $(73)_1$,

$$\left[\mu\left(\frac{\phi'^2 + \psi'^2 - 1}{C_3 X + C_4 - 1}\right)^{-\frac{2}{3}}\phi'\right]' + \rho_0\omega^2\phi = -\eta\omega\psi'',$$

$$\left[\mu\left(\frac{\phi'^2 + \psi'^2 - 1}{C_3 X + C_4 - 1}\right)^{-\frac{2}{3}}\psi'\right]' + \rho_0\omega^2\psi = \eta\omega\phi''. \tag{84}$$

The final governing equations are highly nonlinear, even with the choice $\beta = 2$. The possibility of solving them in closed form seems quite remote. Of course, numerical and qualitative analyses may be performed with the usual methods of dynamical systems theory.

4.3   A separable strain energy density

Recall that the strain energy function of an *incompressible* material, Σ_{inc} say, corresponds to the restriction to the subspace $(I_1, I_2, 1)$ of a strain-energy $\widetilde{\Sigma}$ say, defined in the full space (I_1, I_2, I_3):

$$\Sigma_{\text{inc}}(I_1, I_2) := \widetilde{\Sigma}(I_1, I_2, 1). \tag{85}$$

Hence the classical incompressible neo-Hookean form,

$$\Sigma_{\text{inc}} = \mu(I_1 - 3)/2, \tag{86}$$

may be associated for example with the full-space strain energy

$$\widetilde{\Sigma} = \frac{\mu}{2}(I_1 - 3 - 2\ln J), \quad J := \sqrt{I_3}, \tag{87}$$

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often used in the classical molecular theory of rubber [26].

Now, constitutive equations for *compressible* hyperelastic materials come in many formulations. One of them consists in adding a purely volumetric term $\Sigma_{\text{vol}}(J)$ say, to a basic strain energy density function $\widetilde{\Sigma}$, whose restriction (85) is the strain energy function of an incompressible material Σ_{inc}. Then the final strain energy density for a compressible hyperelastic material can be written as

$$\Sigma = \widetilde{\Sigma}(I_1, I_2, I_3) + \Sigma_{\text{vol}}(J). \tag{88}$$

Several models have have been proposed in the literature for this *pure volumetric part* (or *bulk term*) of the strain energy function. Ogden [48] proposed the form:

$$\Sigma_{\text{vol}}^{I}(J) = \lambda \beta^{-2}(\beta \ln J + J^{-\beta} - 1), \tag{89}$$

where λ is the first Lamé modulus and β (> 0) is an empirical parameter. Flory [27] proposed the form

$$\Sigma_{\text{vol}}^{II}(J) = (c/2)(\ln J)^2, \tag{90}$$

and Simo and Pister [56], the form

$$\Sigma_{\text{vol}}^{III}(J) = (c/2)[(J^2 - 1) - \ln J], \tag{91}$$

which is (89) at $\beta = -2$. Recently Bischoff et al. [5] proposed the form

$$\Sigma_{\text{vol}}^{IV}(J) = \frac{c}{\beta^2}[\cosh \beta(J - 1) - 1]. \tag{92}$$

In this manner, several strain energy functions may be constructed. As an example, consider the power-law function of Knowles [40],

$$\Sigma_{\text{inc}} = \frac{C}{b}\left[\left(1 + \frac{b}{n}(I_1 - 3)\right)^n - 1\right], \tag{93}$$

where C, b, and n are constitutive parameters, all three assumed positive. This simple strain energy captures several important features of rubber-like materials. At $n = 1$, it recovers the neo-Hookean form (86); at $n > 1$ it describes strain-stiffening materials; at $n < 1$, strain-softening materials. These hyperelastic properties prove crucial to the understanding of complex phenomena such as dynamic fracture, as shown recently by Buehler et al [7].

Then, using the formulation exposed above, we may construct the following compressible strain energy function

$$\Sigma = \frac{C}{b}\left[\left(1 + \frac{b}{n}(I_1 - 3)\right)^n - 1\right] - C\ln J + \Sigma_{\text{vol}}(J). \tag{94}$$

Turning back to the finite amplitude motions of Section II, we find that $J = \sqrt{I_3} = u_X = U$, so that $\Sigma_{\text{vol}}(J) = \Sigma_{\text{vol}}(U)$, and the functions Q_1, Q_2 defined in (16) are here,

$$Q_1 = 2C[1 + \frac{b}{n}(U^2 + \Omega^2 - 1)]^{n-1},$$
$$Q_2 = U^{-1}\Sigma'_{\text{vol}}(U) - CU^{-2}. \tag{95}$$

In general, the corresponding equations of motion are coupled and rather involved. Consider the special case of the standing waves (42), with the choice $U(t) = C_2$ ($C_1 = 0$ in (36)). Then $\phi(t)$ satisfies (43), which is here

$$\phi'' + \frac{k^2\eta}{\rho_0}\phi' + 2\frac{k^2C}{\rho_0}[1 + \frac{b}{n}(C_2^2 - 1 + k^2\phi^2)]^{n-1}\phi = 0. \tag{96}$$

We notice that this equation may admit not only the static solution: $\phi := 0$, but also the nontrivial static solutions:

$$\phi = \pm(1/k)\sqrt{1 - C_2^2 - n/b}. \tag{97}$$

These solutions are real if and only if $b(1 - C_2^2) - n > 0$: when the body is compressed in the X-direction ($u = C_2X$, $C_2 < 1$), then the solution is real if $b > n/(1 - C_2^2) > 0$; when the body is stretched ($u = C_2X$, $C_2 > 1$), the solution is not real (recall that Knowles [40] assumed that $b > 0$ in (93)).

Another important general property of equation (96) is that it can be derived [57] from Lagrangian

$$\mathcal{L} = \tfrac{1}{2}e^{\frac{k^2\eta}{\rho_0}t}[\phi'^2 - \mathcal{V}(\phi)], \tag{98}$$

where

$$\mathcal{V}(\phi) = 2\frac{k^2C}{\rho_0}\int_0^\phi [1 + \frac{b}{n}(C_2^2 - 1 + k^2\zeta^2)]^{n-1}\zeta\,\mathrm{d}\zeta$$
$$= \frac{C}{\rho_0 b}[1 + \frac{b}{n}(C_2^2 - 1 + k^2\phi^2)]^n. \tag{99}$$

As an example we consider a *strain-stiffening material*, by taking $n = 2$ in (94). Then (96) is a Duffing equation with damping,

$$\phi'' + \delta\phi' - \beta\phi + \alpha\phi^3 = 0, \tag{100}$$

where

$$\delta := \frac{k^2\eta}{\rho_0}, \quad \beta := \frac{k^2 C}{\rho_0}[b(1 - C_2^2) - 2], \quad \alpha := b\frac{k^4 C}{\rho_0}. \tag{101}$$

Clearly, $\alpha > 0$, $\delta > 0$. To make the connection with results by Holmes [37], we impose $\beta > 0$ also, which happens only when the body is *compressed* in the X-direction ($u = C_2 X$, $C_2 < 1$) and when $b > 2/(1 - C_2^2)$. As seen for (100), this equation possesses three fixed points in the phase plane, namely: $(0, 0)$, which is a saddle, and $(\pm\sqrt{\beta/\alpha}, 0)$, which are two sinks. Holmes [37] showed that equation (101) is locally and globally stable. It follows that as $t \to \infty$, our standing waves must approach one of the four wrinkled configurations

$$\lim_{t\to\infty} v(X, t) = \pm\sqrt{\beta/\alpha}\cos(kX - \theta_0),$$
$$\lim_{t\to\infty} w(X, t) = \pm\sqrt{\beta/\alpha}\sin(kX - \theta_0), \tag{102}$$

for almost any initial conditions. As any one of the four final configurations is equivalent to another, we conclude that the material is destabilized by the waves (only some special choices of the initial conditions will lead to the unstressed deformation corresponding to the saddle point (0,0); these special initial conditions are found by computing the corresponding stable manifold in the phase plane, and we refer to Holmes [37] for the details.)

The situation described here can be extended to all positive integers n, because the leading of nonlinearity of the potential $\mathcal{V}(\phi)$ in (99) is ϕ^{2n} so that the leading order of nonlinearity in the resulting differential equation is always to an odd integer power. We also point out that for a given b, an increasing n requires a decreasing C_2 for the appearance of the fixed points (97) that is, a greater compressive stretch in the X-direction. This is in agreement with the "physical" expectation that the stiffer a material is, the harder it is to destabilize it.

For non-integer powers n, the situation is more complex because destabilizing configurations may or may not appear in the case of strain-hardening materials ($n > 1$) *and* in the case of strain-softening materials ($n < 1$). Obviously the dynamical systems uncovered here are susceptible to a more complete and more careful analysis than the one provided, which may reveal

even richer behaviors than those already described, such as for example the appearance of parametric resonance; these aspects are however beyond the scope of the present paper.

Finally, we note that the instability evoked in this Subsection is somewhat reminiscent of the buckling of an elasticita under a compressive load, but we point out several differences. The buckling of an elastica is a phenomenon related to geometrical non-linearities, whereas our instability is due to constitutive non-linearities; in an elastica, more than one inflexion may occur whereas here, we could multiply instabilities only by considering a multiple well strain energy function (and such a strain energy is not usual for the modeling of soft tissues and elastomers); for an elastica, the stability of the various fixed points must be examined thoroughly, whereas here, the stability of such points comes out directly from our dynamical approach.

4.4 Fourth-order elasticity

In the linearized theory of "small-but-finite" amplitude waves, the strain energy function must be expanded up to the fourth order in the strain, in order to reveal nonlinear shear waves [59]. In that framework, Murnaghan's expansion [46] is often used (see for instance Porubov [52]):

$$\Sigma = \frac{\lambda + 2\mu}{2}i_1^2 - 2\mu i_2 + \frac{l + 2m}{3}i_1^3 - 2m i_1 i_2 + n i_3 + \nu_1 i_1^4 + \nu_2 i_1^2 i_2 + \nu_3 i_1 i_3 + \nu_4 i_2^2,$$
(103)

where λ, μ are the Lamé moduli, l, m, n are the third-order moduli, and ν_1, ν_2, ν_3, ν_4 are the fourth-order moduli (other notations exist: see Norris [47] for the connections). In this expansion, we used the first three principal invariants i_1, i_2, i_3 of $\mathbf{E}$, the Green-Lagrange strain tensor; they are related to the first three principal invariants I_1, I_2, I_3 of $\mathbf{B}$ defined in (4) by the relations [4],

$$
\begin{aligned}
I_1 &= 2i_1 + 3, \\
I_2 &= 4i_1 + 4i_2 + 3, \\
I_3 &= 2i_1 + 4i_2 + 8i_3 + 1.
\end{aligned}
$$
(104)

By chain rule differentiation, we find that the functions Q_1 and Q_2 defined

in (16) are here,

$$Q_1 = \mu + (\lambda + 2\mu + m)i_1 + (l + 2m - \frac{\nu_2}{2})i_1^2$$
$$- (2m + \nu_4)i_2 + 4\nu_1 i_1^3 + 2\nu_2 i_1 i_2 + \nu_3 i_3,$$
$$Q_2 = -\mu - mi_1 + \frac{\nu_2}{2}i_1^2 + \nu_4 i_2, \tag{105}$$

where now i_1, i_2, i_3 are found from (104) and (21) as

$$i_1 = (U^2 + \Omega^2 - 1)/2, \quad i_2 = -\Omega^2/4, \quad i_3 = 0. \tag{106}$$

Several comments are in order at this point. First of all, we recall that the equations corresponding to (15) for third-order elasticity (without dissipation) were first derived by Gold'berg [29] in 1960.

Then we note that recently there has been a renewed interested in fourth-order elasticity following experiments on the nonlinear acoustic properties of soft tissue-like solids [20]. Hence Hamilton et al. [32] proposed the following reduced version of the expansion (103), suitable when the relative portion of energy stored in compression is negligible, as is expected for shear deformations and motions of soft tissues,

$$\Sigma = -2\mu i_2 + ni_3 + \nu_4 i_2^2. \tag{107}$$

In this case, the functions Q_1 and Q_2 defined in (16) simplify to

$$Q_1 = -Q_2 = \mu - \nu_4 i_2 = \mu + \nu_4 \Omega^2/4, \tag{108}$$

and we check at once that the longitudinal equation of motion $(15)_1$ becomes a trivial identity in the purely elastic case ($\zeta = \chi = \eta = 0$), because here $Q_1 + Q_2 = 0$.

Finally, we also notice that the theories of third- and fourth-order elasticity have been used to study bulk solitons propagation in elastic materials. For instance, Hao and Marris [33] discuss the issue of acoustic solitons from both experimental and theoretical points of view; they consider the possibility to produce KdV solitons by adding a fourth-order spatial derivative to the longitudinal wave equation and by performing some *ad hoc* approximations.

The general expressions (103), (105), and (106) yield quite complicated equations of motion. If for example we restrict our attention to the standing waves (42), we find that the polynomial nonlinearity of the resulting ordinary

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differential equation is of the fifth order for third-order elasticity, and of seventh order for fourth-order elasticity. If we further restrict our attention to the reduced expansion (107) we find that equation (43) reduces to a damped Duffing equation,

$$\phi'' + \frac{\eta k^2}{\rho_0}\phi' + \frac{k^2}{4\rho_0}(4\mu + k^2\nu_4\phi^2)\phi = 0. \tag{109}$$

This equation is characterized by a positive linear stiffness and therefore the peculiar behavior found in the previous sub-section for some power-law materials is *ruled out*. For completeness, we point out that Holmes and Rand [38] computed an approximate solution of equation (109), using the method of averaging.

5   Concluding Remarks

We studied the propagation of finite amplitude waves in a general nonlinear elastic material, with dissipation taken into account by means of a simple mechanism of differential type. We showed that the general equations of motion admit some beautiful and most interesting reductions to ordinary differential equations, using a direct method based on complex functions. Clearly, these reductions are a consequence of the inherent symmetrical structure of the balance and constitutive equations and may therefore be recovered also by the standard methods of group analysis; we argue however that our direct method is more simple and more revealing from a mechanical point of view. Moreover, our results complement and generalize the well-known results of Carroll in various directions, as already pointed out in the Introduction; they also emphasize the analogy between Carroll's solutions and some exact solutions proposed for nonlinear strings by Rosenau and Rubin [53].

The solutions provided here can be used not only as benchmarks for the more complicated numerical analysis required in "real life" applications, but also for a better understanding of the mechanical properties and of the mathematical structure of various usual models. These advantages were highlighted in the discussion on standing waves solutions conducted in Section III. There, we focused much of the discussion on the special case of standing waves for the sake of simplicity and brevity, but it is clear that our general methods of investigation can be applied to all the ordinary differential equations that were derived.

One of the important findings uncovered by the analysis of our solutions is the possibility to predict the appearance of highly symmetric wrinkles in an elastic medium compressed longitudinally. This phenomenon has been observed experimentally by several researchers when, for example, a hard film is deposited on a soft material and put under compressive strain [39]. Our analysis showed that these wrinkles may indeed appear when the compression parameter satisfies some simple inequalities; it also showed that the detailed viscoelastic behavior of the material is unimportant because of the highly attracting nature of the wrinkled states of deformations. We note of course that our solutions have a higher symmetry than the patterns usually observed in experiments, probably because our solutions are characterized by strong invariance and because they are bulk solutions, in contrast to the half-space geometry generally used in experiments. Despite these limitations, we found that these solutions are compatible with the relatively simple non-linear power-law strain energy density whereas weak non-linear theories (up to the fourth-order) are not adequate to predict these features.

Our approach allowed a direct and complete comparison of the models used in nonlinear acoustics and in continuum mechanics. It was seen that although the popular weak nonlinear theories are often used to study some special features of wave propagation, a consequence of their polynomial nature is that they cannot contain all the necessary information of the general constitutive theory.

Last but not least, we gave some explicit solutions to some models of compressible hyperelasticity. New exact solutions are always welcomed and valuable additions to the short list of exact solutions found in the literature.

We conclude by pointing out that several generalizations of our results are possible in principle. For instance, the consideration in polar coordinates of a wave propagating in the radial direction coupled to two shear waves in the axial and azimuthal direction is one of such possible generalizations. In this case the following class of motions is considered

$$r = r(R, t), \quad \theta = \Theta + g(R, t), \quad z = Z + u(R, t), \tag{110}$$

where (R, Θ, Z) are cylindrical polar coordinates associated with the unde-formed configuration and (r, θ, z) are cylindrical polar coordinates associated with the deformed configuration. It is possible to show that from (110) and the corresponding kinematical quantities of interest, the equations of motion may be reduced to a set of three partial differential equations, similar to

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what we did for the motions (10) in (15). Unfortunately, it seems that in this case the equations of motion do not allow general solutions in separable form, as was the case in Section III for the equations (15)). Of course, we do not exclude that for special classes of materials, it is possible to find some classes of wave solutions (see for example [31] for compressible materials and [28] for incompressible materials).

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