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Author(s)	Destrade, Michel
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Inhomogeneous "longitudinal" plane waves in a deformed elastic arterial.

Michel Destrade & Michael Hayes

Abstract

By definition, a homogeneous isotropic compressible Hadamard material has the property that an infinitesimal longitudinal homogeneous plane wave may propagate in every direction when the material is maintained in a state of arbitrary finite static homogeneous deformation. Here, as regards the wave, 'homogeneous' means that the direction of propagation of the wave is parallel to the direction of eventual attenuation; and 'longitudinal' means that the wave is linearly polarized in a direction parallel to the direction of propagation. In other words, the displacement is of the form $\mathbf{u} = \mathbf{n} \cos k(\mathbf{n} \cdot \mathbf{x} - ct)$, where $\mathbf{n}$ is a real vector.

It is seen that the Hadamard material is the most general one for which a 'longitudinal' *inhomogeneous* plane wave may also propagate in any direction of a predeformed body. Here, 'inhomogeneous' means that the wave is attenuated, in a direction distinct from the direction of propagation; and "longitudinal" means that the wave is elliptically polarized in the plane containing these two directions, and that the ellipse of polarization is similar and similarly situated to the ellipse for which the real and imaginary parts of the complex wave vector are conjugate semi-diameters. In other words, the displacement is of the form $\mathbf{u} = \Re\{\mathbf{S} \exp i\omega(\mathbf{S} \cdot \mathbf{x} - ct)\}$, where $\mathbf{S}$ is a complex vector (or bivector).

Then a Generalized Hadamard material is introduced. It is the most general homogeneous isotropic compressible material which allows the propagation of infinitesimal "longitudinal" inhomogeneous plane circularly polarized waves for all choices of the isotropic directional bivector.

Finally, the most general forms of response functions are found for homogeneously deformed isotropic elastic materials in which "longitudinal" inhomogeneous plane waves may propagate with a circular polarization in each of the two planes of central circular section of

the $\mathbb{B}^n$-ellipsoid, where $\mathbb{B}$ is the left Cauchy-Green strain tensor corresponding to the primary pure homogeneous deformation.

1 Introduction.

Infinitesimal longitudinal homogeneous plane waves play a special role in classical linearized theory. For such waves the amplitude is parallel to the direction of propagation $\mathbf{n}$ so that all the particles oscillate along the direction $\mathbf{n}$. Such waves may propagate in every direction in an isotropic compressible elastic body. This is no longer the case for propagation in an elastic anisotropic crystal. Possible directions of propagation of longitudinal homogeneous plane waves, called 'specific directions' by Borgnis [1], may be as few as three in an elastic anisotropic crystal. Of course, by assuming certain restrictions on the elastic constants, it is possible, as Hadamard [3] did, to have a special model anisotropic material in which longitudinal waves are possible in every direction.

A similar situation holds for infinitesimal motions of an isotropic homogeneous elastic material held in an (arbitrary) state of static (finite) homogeneous deformation – specific directions are limited in number. However, it is possible to place conditions on the basic strain energy density, or on the response coefficients defining the material, such that every direction is specific for arbitrary (finite) homogeneous deformation. The resulting model was called a 'Hadamard material' by John [2].

Here we consider the propagation of infinitesimal *inhomogeneous* plane waves in an isotropic homogeneous compressible elastic body held in an arbitrary state of finite static homogeneous deformation. Such waves may be described in terms of bivectors – complex vectors – the amplitude bivector $\mathbf{A}$ and the slowness bivector $\mathbf{S}$, which is written $\mathbf{S} = N\mathbf{C}$, where the directional bivector $\mathbf{C}$ is written $\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}$ ($\hat{\mathbf{m}} \cdot \hat{\mathbf{n}} = 0$, $m \geq 1$, $|\hat{\mathbf{m}}| = |\hat{\mathbf{n}}| = 1$). Once the directional bivector $\mathbf{C}$ is prescribed, the slowness $\mathbf{S}$ and the amplitude $\mathbf{A}$ are determined from the equations of motion. Prescribing $\mathbf{C}$ is equivalent to prescribing an ellipse with principal semi-axes $m\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$; this directional ellipse for inhomogeneous plane waves corresponds to the direction of propagation $\mathbf{n}$ for homogeneous plane waves [4]. We borrow the adjective 'longitudinal' to describe an inhomogeneous plane wave for which $\mathbf{A}$ and $\mathbf{S}$ (and $\mathbf{C}$) are parallel: $\mathbf{A} \times \mathbf{S} = \mathbf{0}$. What this means is that [5] the ellipses of $\mathbf{A}$ and of $\mathbf{S}$ (and of $\mathbf{C}$) are all parallel, similar (same aspect ratio) and similarly situated (parallel major axes and parallel minor axes). We determine the most general form of strain-energy density so that such waves may propagate for all choices of the directional bivector $\mathbf{C}$. It turns out to be a

Hadamard material.

Then we consider the possibility of having ‘circularly polarized longitudinal’ waves. For such waves both $\mathbf{C}$ and $\mathbf{A}$ are isotropic and coplanar: the corresponding ellipses are coplanar circles. Such waves are a subclass of longitudinal waves. The corresponding class of materials for which circularly polarized longitudinal waves are possible is called a ‘Generalized Hadamard material’. Clearly they include Hadamard materials.

Next we obtain constitutive equations for materials which allow the propagation of two special infinitesimal longitudinal circularly polarized inhomogeneous plane waves for all choices of the basic static homogeneous deformation. The circles of polarization are to lie in the planes of central circular section of the $\mathbb{B}^n$-ellipsoid: $\mathbf{x}\cdot\mathbb{B}^n\mathbf{x} = 1$, where $\mathbb{B}$ is the left Cauchy-Green strain tensor corresponding to the basic deformation. In the particular cases where $n = \pm 1$, we determine the corresponding forms of the strain-energy function. In every case, the slownesses are presented explicitly. Finally we present the very simple results when $n = \frac{1}{2}$.

2 Inhomogeneous longitudinal plane waves.

Here we recall some basic properties of inhomogeneous plane waves and then introduce inhomogeneous *longitudinal* plane waves. Finally we introduce inhomogeneous longitudinal *circularly* polarized plane waves.

The displacement field corresponding to an infinite train of infinitesimal inhomogeneous plane waves is $\epsilon\mathbf{u}$ (where ϵ is an infinitesimal constant: $\epsilon^2 \ll |\epsilon|$) and $\mathbf{u}$ is given by

$$\begin{aligned}\mathbf{u} &= [\mathbf{A}\exp i\omega(\mathbf{S}\cdot\mathbf{x} - t)]^+ \\ &= \{\mathbf{A}^+\cos\omega(\mathbf{S}^+\cdot\mathbf{x} - t) - \mathbf{A}^-\sin\omega(\mathbf{S}^+\cdot\mathbf{x} - t)\}\exp -\omega\mathbf{S}^-\cdot\mathbf{x}. \qquad (2.1)\end{aligned}$$

Here $\mathbf{A} = \mathbf{A}^+ + i\mathbf{A}^-$ is the amplitude bivector; $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$ is the slowness bivector; and the period of the waves is $2\pi/\omega$. The particle initially at $\mathbf{x}$ is displaced to $\mathbf{x} + \epsilon\mathbf{u}$ at time t, so that it moves on an ellipse, centre $\mathbf{x}$, with conjugate radii $\epsilon\mathbf{A}^+\exp -\omega\mathbf{S}^-\cdot\mathbf{x}$ and $\epsilon\mathbf{A}^-\exp -\omega\mathbf{S}^-\cdot\mathbf{x}$. When the particle is at $\mathbf{x} + \epsilon\mathbf{A}^+\exp -\omega\mathbf{S}^-\cdot\mathbf{x}$ it is moving parallel to $\mathbf{A}^-$, and when it is at $\mathbf{x} + \epsilon\mathbf{A}^-\exp -\omega\mathbf{S}^-\cdot\mathbf{x}$ it is moving parallel to $\mathbf{A}^+$. The sense of description of the ellipse is from the tip of $\mathbf{x} + \epsilon\mathbf{A}^+\exp -\omega\mathbf{S}^-\cdot\mathbf{x}$ towards the tip of $\mathbf{x} + \epsilon\mathbf{A}^-\exp -\omega\mathbf{S}^-\cdot\mathbf{x}$ that is, from the tip of $\mathbf{A}^+$ towards the tip of $\mathbf{A}^-$.

It has been pointed out [4] that the slowness $\mathbf{S}$ may not be prescribed a priori. Rather, $\mathbf{S}$ is written as

$$\mathbf{S} = N\mathbf{C} = Te^{i\phi}(m\hat{\mathbf{m}} + i\hat{\mathbf{n}}), \qquad (2.2)$$

where

$$\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}, \quad m \geq 1, \quad \hat{\mathbf{m}} \cdot \hat{\mathbf{n}} = 0, \quad |\hat{\mathbf{m}}| = |\hat{\mathbf{n}}| = 1. \tag{2.3}$$

In the directional ellipse approach [4], what is prescribed is the directional bivector $\mathbf{C}$, or equivalently, the directional ellipse associated with $\mathbf{C}$ that is, $(\hat{\mathbf{m}} \cdot \mathbf{x})^2/m^2 + (\hat{\mathbf{n}} \cdot \mathbf{x})^2 = 1$, $\hat{\mathbf{m}} \times \hat{\mathbf{n}} \cdot \mathbf{x} = 0$. On choosing $\mathbf{C}$, then N, and thus $\mathbf{S} = N\mathbf{C}$, are determined from the equations of motion, and $\mathbf{A}$ follows as an eigenbivector of the acoustical tensor [4]. Let $\hat{\mathbf{p}} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$ be the unit normal to the plane of the directional bivector $\mathbf{C}$. There is an infinity of choices of directional ellipses, first by varying the magnitude of m, keeping $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ held fixed and orthogonal, and then by rotating $(\hat{\mathbf{m}}, \hat{\mathbf{n}})$ into another orthogonal pair $(\hat{\mathbf{m}}^*, \hat{\mathbf{n}}^*)$ (say) with $\hat{\mathbf{p}} = \hat{\mathbf{m}}^* \times \hat{\mathbf{n}}^*$ and repeating the procedure. Finally $\hat{\mathbf{p}}$ is rotated and the process continued. In this way every possible inhomogeneous plane wave solution is obtained.

For a 'longitudinal' inhomogeneous plane wave the displacement field $\epsilon\mathbf{u}$ may be written

$$\mathbf{u} = [\delta \mathbf{S} \exp i\omega(\mathbf{S} \cdot \mathbf{x} - t)]^+, \tag{2.4}$$

where δ is a constant. Writing

$$\delta = |\delta| e^{i\psi}, \quad \mathbf{S} = N\mathbf{C} = N(m\hat{\mathbf{m}} + i\hat{\mathbf{n}}) = Te^{i\phi}(m\hat{\mathbf{m}} + i\hat{\mathbf{n}}), \tag{2.5}$$

we find

$$\begin{aligned}\mathbf{u} = |\delta| T \exp -\omega T[(\sin\phi) m\hat{\mathbf{m}} \cdot \mathbf{x} + (\cos\phi)\hat{\mathbf{n}} \cdot \mathbf{x}] \\ \times \{ m\hat{\mathbf{m}} \cos[\omega T(\cos\phi) m\hat{\mathbf{m}} \cdot \mathbf{x} - \omega T(\sin\phi)\hat{\mathbf{n}} \cdot \mathbf{x} - \omega t + \psi + \phi] \\ - \hat{\mathbf{n}} \sin[\omega T(\cos\phi) m\hat{\mathbf{m}} \cdot \mathbf{x} - \omega T(\sin\phi)\hat{\mathbf{n}} \cdot \mathbf{x} - \omega t + \psi + \phi]\}. \end{aligned} \tag{2.6}$$

The planes of constant phase are

$$(\cos\phi) m\hat{\mathbf{m}} \cdot \mathbf{x} - (\sin\phi)\hat{\mathbf{n}} \cdot \mathbf{x} = \text{constant}, \tag{2.7}$$

and the planes of constant amplitude are

$$(\sin\phi) m\hat{\mathbf{m}} \cdot \mathbf{x} + (\cos\phi)\hat{\mathbf{n}} \cdot \mathbf{x} = \text{constant}. \tag{2.8}$$

The particle paths are ellipses with principal axes along $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$. We note that

$$\begin{aligned}&\mathbf{u}(\mathbf{0}, t) = |\delta| T[m\hat{\mathbf{m}} \cos(\psi + \phi - \omega t) - \hat{\mathbf{n}} \sin(\psi + \phi - \omega t)], \\ &\mathbf{u}(\mathbf{0}, \omega t = \psi + \phi) = |\delta| T m\hat{\mathbf{m}}, \\ &\mathbf{u}(\mathbf{0}, \omega t = \psi + \phi + \pi/2) = |\delta| T\hat{\mathbf{n}}. \end{aligned} \tag{2.9}$$

Hence the sense of description for the motion of the particle initially at $\mathbf{x} = \mathbf{0}$ and now on the ellipse (2.6) with center at $\mathbf{x} = \mathbf{0}$, is from the tip of the principal axis along $\hat{\mathbf{m}}$ towards the tip of the principal axis along $\hat{\mathbf{n}}$. This is clearly the case for the motions of all other particles also.

For circularly polarized waves we take $m = 1$ so that the planes of constant phase are orthogonal to the planes of constant amplitude. Also,

$$\mathbf{u} = |\delta| T \exp -\{\omega T[(\sin\phi)\hat{\mathbf{m}} + (\cos\phi)\hat{\mathbf{n}}]\cdot\mathbf{x}\} \\ \times [\hat{\mathbf{m}}\cos(\kappa - \omega t) - \hat{\mathbf{n}}\sin(\kappa - \omega t)], \quad (2.10)$$

where

$$\kappa = \omega T[(\cos\phi)\hat{\mathbf{m}} - (\sin\phi)\hat{\mathbf{n}}]\cdot\mathbf{x} + \psi + \phi. \quad (2.11)$$

The radius of the circle of polarization for the particle initially at $\mathbf{x}$ is

$$|\epsilon\delta| T \exp -\{\omega T[(\sin\phi)\hat{\mathbf{m}} + (\cos\phi)\hat{\mathbf{n}}]\cdot\mathbf{x}\}. \quad (2.12)$$

The displacement $\epsilon\mathbf{u}$ where $\mathbf{u}$ is given by (2.10) corresponds to an infinite train of circularly polarized longitudinal inhomogeneous plane waves.

3 Basic equations.

3.1 Hyperelastic materials.

We consider a body made of homogeneous isotropic elastic material, initially in a state of rest $\mathcal{B}_0$. When the body is deformed to a state $\mathcal{B}$ (say), a material particle which was at $\mathbf{X}$ in $\mathcal{B}_0$ moves to the position $\mathbf{x}$, at time t, where

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \quad (3.1)$$

The left Cauchy-Green strain tensor $\mathbb{B}$ is defined by

$$\mathbb{B}_{ij} = (\partial x_i/\partial X_A)(\partial x_j/\partial X_A), \quad (3.2)$$

in a rectangular Cartesian coordinate system, fixed in $\mathcal{B}$.

Three principal invariants of $\mathbb{B}$ are $\mathrm{I}, \mathrm{II}, \mathrm{III}$ defined by

$$\mathrm{I} = \text{ tr } \mathbb{B}, \quad 2\mathrm{II} = \mathrm{I}^2 - \text{ tr } (\mathbb{B}^2), \quad \mathrm{III} = \text{ det } \mathbb{B}. \quad (3.3)$$

The constitutive equation for the material, relating the Cauchy stress tensor $\mathbb{T}$ with the strain tensor $\mathbb{B}$ is [6]

$$\mathbb{T} = N_0\mathbf{1} + N_1\mathbb{B} - N_{-1}\mathbb{B}^{-1}, \quad (3.4)$$

where the material coefficients N_Γ ($\Gamma = 0, \pm 1$) are functions of $\mathrm{I}, \mathrm{II}, \mathrm{III}$. In the case of a hyperelastic material, the strain energy density function W, measured per unit mass, is characteristic of the material and a function of $\mathrm{I}, \mathrm{II}, \mathrm{III}$:

$$W = W(\mathrm{I}, \mathrm{II}, \mathrm{III}), \tag{3.5}$$

and then N_0, N_1, N_{-1} are given by [6]

$$\begin{aligned} N_0 &= 2[\mathrm{II}\,\mathrm{III}^{-\frac{1}{2}}\frac{\partial W}{\partial \mathrm{II}} + \mathrm{III}^{\frac{1}{2}}\frac{\partial W}{\partial \mathrm{III}}], \\ N_1 &= 2\mathrm{III}^{-\frac{1}{2}}\frac{\partial W}{\partial \mathrm{I}}, \quad N_{-1} = 2\mathrm{III}^{\frac{1}{2}}\frac{\partial W}{\partial \mathrm{II}}. \end{aligned} \tag{3.6}$$

3.2 Finite static pure homogeneous deformation.

We assume that the body is first subjected to a finite triaxial static homogeneous deformation, as it is deformed from $\mathcal{B}_0$ to $\mathcal{B}$. Then, the corresponding three principal axes of deformation constitute a fixed rectangular Cartesian coordinate system $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ say, in which the deformation (3.1) is written as

$$\mathbf{x} = \lambda_1 X \mathbf{i} + \lambda_2 Y \mathbf{j} + \lambda_3 Z \mathbf{k}, \tag{3.7}$$

where the λ's are the principal stretch ratios, assumed, without loss of generality, to be ordered as

$$\lambda_1 > \lambda_2 > \lambda_3. \tag{3.8}$$

In this case, the left Cauchy–Green tensor reduces to

$$\mathbb{B} = \lambda_1^2 \mathbf{i} \otimes \mathbf{i} + \lambda_2^2 \mathbf{j} \otimes \mathbf{j} + \lambda_3^2 \mathbf{k} \otimes \mathbf{k}, \tag{3.9}$$

with corresponding invariants

$$\mathrm{I} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad \mathrm{II} = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad \mathrm{III} = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{3.10}$$

The Cauchy stress necessary to maintain the basic finite static homogeneous deformation (3.7) has components

$$\begin{aligned} T_{\alpha\beta} &= 0, \quad \alpha \neq \beta \\ T_{\alpha\alpha} &= N_0 + N_1 \lambda_\alpha^2 - N_{-1}\lambda_\alpha^{-2}, \end{aligned} \tag{3.11}$$

where the N_Γ ($\Gamma = -1, 0, 1$) are defined as in (3.6), at values of $\mathrm{I}, \mathrm{II}, \mathrm{III}$ given by (3.10).

3.3 Superposed 'longitudinal' inhomogeneous plane waves.

Let the body be subjected to a further deformation bringing it from the state $\mathcal{B}$ to the state $\overline{\mathcal{B}}$, corresponding to the propagation of an infinitesimal *'longitudinal' inhomogeneous plane wave* of complex exponential type in the deformed material, such that the current position $\overline{\mathbf{x}}$ in $\overline{\mathcal{B}}$, of a particle at $\mathbf{x}$ given by (3.7) in $\mathcal{B}$, and at $\mathbf{X}$ in $\mathcal{B}_0$, is given by

$$\overline{\mathbf{x}} = \mathbf{x} + \epsilon\{\mathbf{S}e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\}^+. \tag{3.12}$$

Here, ϵ is a small parameter, such that terms of order higher than ϵ may be neglected in comparison with first order terms.

Now, corresponding to the superposition of the small-amplitude motion (3.12) upon the large deformation (3.7) are the quantities $\overline{\mathbb{B}}$, $\overline{\mathbb{B}}^{-1}$, $\overline{\mathrm{I}}$, $\overline{\mathrm{II}}$, $\overline{\mathrm{III}}$, $\overline{N}_{-1}$, $\overline{N}_0$, $\overline{N}_1$, and $\overline{\mathbb{T}}$, that is, respectively, the left Cauchy-Green tensor and its inverse, the first three invariants, the three material constitutive parameters, and the Cauchy stress tensor. Each of these quantities may be expanded up to the first order in ϵ, and written in the form,

$$\begin{aligned}
&\overline{\mathbb{B}} = \mathbb{B} + \epsilon\{i\omega\mathbb{B}^* e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\}^+, \dots \\
&\overline{\mathrm{I}} = \mathrm{tr}\ \overline{\mathbb{B}} = \mathrm{I} + \epsilon\{i\omega\mathrm{I}^* e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\}^+, \dots \\
&\overline{\mathbb{T}} = \mathbb{T} + \epsilon\{i\omega\mathbb{T}^* e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\}^+.
\end{aligned} \tag{3.13}$$

Here, we have,

$$\begin{aligned}
&\mathbb{B}^* = \mathbf{S}\otimes\mathbb{B}\mathbf{S} + \mathbb{B}\mathbf{S}\otimes\mathbf{S}, \\
&(\mathbb{B}^{-1})^* = -\mathbf{S}\otimes\mathbb{B}^{-1}\mathbf{S} - \mathbb{B}^{-1}\mathbf{S}\otimes\mathbf{S}, \\
&\mathrm{I}^* = 2(\mathbf{S}\cdot\mathbb{B}\mathbf{S}), \\
&\mathrm{II}^* = 2[\mathrm{II}(\mathbf{S}\cdot\mathbf{S}) - \mathrm{III}(\mathbf{S}\cdot\mathbb{B}^{-1}\mathbf{S})], \\
&\mathrm{III}^* = 2\mathrm{III}(\mathbf{S}\cdot\mathbf{S}), \\
&N_\Gamma^* = \mathrm{I}^* N_{\Gamma,\mathrm{I}} + \mathrm{II}^* N_{\Gamma,\mathrm{II}} + \mathrm{III}^* N_{\Gamma,\mathrm{III}}, \quad (\Gamma = -1, 0, 1), \\
&\mathbb{T}^* = N_0^*\mathbf{1} + N_1^*\mathbb{B} - N_{-1}^*\mathbb{B}^{-1} + N_1\mathbb{B}^* - N_{-1}(\mathbb{B}^{-1})^*.
\end{aligned} \tag{3.14}$$

Finally, the mass density $\overline{\rho}$ in the current configuration is

$$\overline{\rho} = \mathrm{III}^{-\frac{1}{2}}\rho(1 - \epsilon\{i\omega(\mathbf{S}\cdot\mathbf{S})e^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\}^+). \tag{3.15}$$

3.4 Equations of motion.

In the dynamic state $\overline{\mathcal{B}}$, the equations of motion, written in the absence of body forces, read

$$\frac{\partial \overline{T}_{ij}}{\partial \overline{x}_j} = \overline{\rho} \frac{\partial^2 \overline{x}_i}{\partial t^2}. \tag{3.16}$$

Up to the first order in ϵ, they are equivalent to $\mathbb{T}^* \cdot \mathbf{S} = \rho \mathbf{S}$, or

$$\Theta \mathbf{S} + \Phi \mathbb{B} \mathbf{S} + \Gamma \mathbb{B}^{-1} \mathbf{S} = \mathbf{0}, \tag{3.17}$$

where

$$\begin{aligned}
&\Theta := N_0^* + N_1(\mathbf{S} \cdot \mathbb{B} \mathbf{S}) + N_{-1}(\mathbf{S} \cdot \mathbb{B}^{-1} \mathbf{S}) - \rho, \\
&\Phi := N_1^* + N_1(\mathbf{S} \cdot \mathbf{S}), \\
&\Gamma := N_{-1}^* - N_{-1}(\mathbf{S} \cdot \mathbf{S}).
\end{aligned} \tag{3.18}$$

When referred to axes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the principal axes of $\mathbb{B}$, Eqs. (3.17) may be written

$$(\Theta + \Phi \lambda_\alpha^2 + \Gamma \lambda_\alpha^{-2}) C_\alpha = 0, \quad \alpha = 1, 2, 3, \text{ no sum}, \tag{3.19}$$

where $\mathbf{C} = C_1 \mathbf{i} + C_2 \mathbf{j} + C_3 \mathbf{k}$. These equations have to be satisfied for all choices of C_1, C_2, C_3, and for all positive choices of λ_1, λ_2, λ_3 satisfying (3.8). Noting that

$$\begin{vmatrix} 1 & \lambda_1^2 & \lambda_1^{-2} \\ 1 & \lambda_2^2 & \lambda_2^{-2} \\ 1 & \lambda_3^2 & \lambda_3^{-2} \end{vmatrix} = (\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)/\mathrm{III} \neq 0, \tag{3.20}$$

it follows that we must have

$$\Theta = \Phi = \Gamma = 0. \tag{3.21}$$

Thus, writing $\mathbf{S} = N\mathbf{C}$, we have

$$\begin{aligned}
\rho N^{-2} = (2N_{0,\mathrm{I}} + N_1) \mathbf{C} \cdot \mathbb{B} \mathbf{C} + (-2\mathrm{III} N_{0,\mathrm{II}} + N_{-1}) \mathbf{C} \cdot \mathbb{B}^{-1} \mathbf{C} \\
+ (2\mathrm{II} N_{0,\mathrm{II}} + 2\mathrm{III} N_{0,\mathrm{III}}) \mathbf{C} \cdot \mathbf{C},
\end{aligned} \tag{3.22}$$

and also

$$\begin{aligned}
&2N_{1,\mathrm{I}}(\mathbf{C} \cdot \mathbb{B} \mathbf{C}) - 2\mathrm{III} N_{1,\mathrm{II}}(\mathbf{C} \cdot \mathbb{B}^{-1} \mathbf{C}) \\
&\qquad + [N_1 + 2\mathrm{II} N_{1,\mathrm{II}} + 2\mathrm{III} N_{1,\mathrm{III}}](\mathbf{C} \cdot \mathbf{C}) = 0, \qquad (3.23) \\
&2N_{-1,\mathrm{I}}(\mathbf{C} \cdot \mathbb{B} \mathbf{C}) - 2\mathrm{III} N_{-1,\mathrm{II}}(\mathbf{C} \cdot \mathbb{B}^{-1} \mathbf{C}) \\
&\qquad + [-N_{-1} + 2\mathrm{II} N_{-1,\mathrm{II}} + 2\mathrm{III} N_{-1,\mathrm{III}}](\mathbf{C} \cdot \mathbf{C}) = 0. \qquad (3.24)
\end{aligned}$$

Here $\mathbf{C}$ is assumed prescribed. Then, $\mathbf{S} = N\mathbf{C}$ is determined from (3.22). The response functions N_1, N_{-1} must be such that (3.23) and (3.24) are satisfied for all choices of $\mathbf{C}$ and all positive choices of λ_1, λ_2, λ_3.

4 Propagation of 'longitudinal' inhomogeneous plane waves for any directional bivector C.

Here we find the most general form of the stored energy density for which infinitesimal longitudinal inhomogeneous plane waves may propagate in the finitely deformed material for any choice of directional bivector $\mathbf{C}$.

4.1 'Longitudinal' waves of general polarization.

Equations (3.23) and (3.24) must be satisfied for any $\mathbf{C}$. In particular, choose $\mathbf{C} \neq \mathbf{0}$ such that $\mathbf{C} \cdot \mathbf{C} = 0$, $\mathbf{C}{\cdot}\mathbb{B}^{-1}\mathbf{C} = 0$. In that case, $\mathbf{C}{\cdot}\mathbb{B}\mathbf{C} \neq 0$ and it follows from (3.23), (3.24) that

$$N_{1,\mathrm{I}} = N_{-1,\mathrm{I}} = 0. \tag{4.1}$$

Next choose $\mathbf{C} \neq \mathbf{0}$ such that $\mathbf{C} \cdot \mathbf{C} = 0$, $\mathbf{C}{\cdot}\mathbb{B}\mathbf{C} = 0$. In that case, $\mathbf{C}{\cdot}\mathbb{B}^{-1}\mathbf{C} \neq 0$ and it follows that

$$N_{1,\mathrm{II}} = N_{-1,\mathrm{II}} = 0. \tag{4.2}$$

Thus

$$N_\Gamma = N_\Gamma(\mathrm{III}), \quad \Gamma = -1, 1. \tag{4.3}$$

The N_Γ are functions of III alone, and equations (3.23) and (3.24) reduce to

$$[N_1 + 2\mathrm{III}N_1'](\mathbf{C} \cdot \mathbf{C}) = 0, \quad [-N_{-1} + 2\mathrm{III}N_{-1}'](\mathbf{C} \cdot \mathbf{C}) = 0. \tag{4.4}$$

These equations must be satisfied with any $\mathbf{C}$, in particular with $\mathbf{C}$ such that $\mathbf{C} \cdot \mathbf{C} \neq 0$ (e.g. $\mathbf{C} = \mathbf{i} + 2i\mathbf{j}$). So,

$$N_1 + 2\mathrm{III}N_1' = 0, \quad -N_{-1} + 2\mathrm{III}N_{-1}' = 0, \tag{4.5}$$

and by integration,

$$N_1 = 2\nu\mathrm{III}^{-\frac{1}{2}}, \quad N_{-1} = 2\mu\mathrm{III}^{\frac{1}{2}}, \tag{4.6}$$

where ν and μ are two constant parameters, independent of the strain invariants. The corresponding hyperelastic material has the following constitutive equation, by (3.4),

$$\mathbb{T} = N_0(\mathrm{I}, \mathrm{II}, \mathrm{III})\mathbf{1} + 2\nu\mathrm{III}^{-\frac{1}{2}}\mathbb{B} - 2\mu\mathrm{III}^{\frac{1}{2}}\mathbb{B}^{-1}. \tag{4.7}$$

In order that this expression should correspond to that of an hyperelastic material, it is seen that N_0 has to be independent of I and linear in II, thus

$$N_0 = 2\left(\mathrm{II}\ \mathrm{III}^{-\frac{1}{2}}\mu + \mathrm{III}^{\frac{1}{2}}\frac{\mathrm{d}h}{\mathrm{d\ III}}\right), \tag{4.8}$$

so that the strain energy density is given by

$$W = \nu\mathrm{I} + \mu\mathrm{II} + h(\mathrm{III}), \tag{4.9}$$

characteristic of the Hadamard material [2]. Here h is an arbitrary function of III. We conclude:

> The most general material in which 'longitudinal' inhomogeneous plane waves may propagate for any choice of the directional bivector $\mathbf{C}$, when it is held in any state of finite static pure homogeneous deformation, is the Hadamard material.

This same conclusion was reached by John [2] for the propagation of finite amplitude *homogeneous* plane waves.

Using (3.22) and (4.6), we note that the slowness $\mathbf{S}$ corresponding to the directional bivector $\mathbf{C}$ is given by $\mathbf{S} = N\mathbf{C}$, where

$$\begin{aligned}\rho N^{-2} = 2(N_{0,\mathrm{I}} + \nu\mathrm{III}^{-\frac{1}{2}})\mathbf{C}\cdot\mathbb{B}\mathbf{C} + 2(-\mathrm{III}N_{0,\mathrm{II}} + \mu\mathrm{III}^{\frac{1}{2}})\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}\\ + 2(\mathrm{II}N_{0,\mathrm{II}} + \mathrm{III}N_{0,\mathrm{III}})\mathbf{C}\cdot\mathbf{C}.\end{aligned} \tag{4.10}$$

In the particular case of a hyperelastic Hadamard material, on using (4.8) and (4.9), equation (4.10) becomes

$$\begin{aligned}\rho N^{-2} = 2\nu\mathrm{III}^{-\frac{1}{2}}\mathbf{C}\cdot\mathbb{B}\mathbf{C} - 2\mu\mathrm{III}^{\frac{1}{2}}\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}\\ + 4[\mu\mathrm{II}\ \mathrm{III}^{-\frac{1}{2}} + \mathrm{III}(\mathrm{III}^{\frac{1}{2}}h')']\mathbf{C}\cdot\mathbf{C}.\end{aligned} \tag{4.11}$$

For any choice of the directional bivector $\mathbf{C}$, whether it is taken to be linear, elliptic, or circular, the corresponding wave train is linearly, elliptically, or circularly polarized, respectively, the displacement field being given by

$$\overline{\mathbf{x}} = \mathbf{x} + \epsilon\{N\mathbf{C}e^{i\omega(N\mathbf{C}\cdot\mathbf{x}-t)}\}^{+}. \tag{4.12}$$

wherein N is given by (4.10).

Example

As an example we may choose $\mathbf{C}$ such that

$$\frac{C_1^2}{C_3^2} = \left(\frac{\lambda_2^4 - \lambda_3^4}{\lambda_1^4 - \lambda_2^4}\right)\left(\frac{\lambda_1^2}{\lambda_3^2}\right), \quad \frac{C_2^2}{C_3^2} = -\left(\frac{\lambda_1^4 - \lambda_3^4}{\lambda_1^4 - \lambda_2^4}\right)\left(\frac{\lambda_2^2}{\lambda_3^2}\right). \tag{4.13}$$

Then

$$\mathbf{C}\cdot\mathbb{B}\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} = 0, \tag{4.14}$$

and (4.11) gives

$$\rho N^{-2} = -4[\mu \mathrm{II}\ \mathrm{III}^{-\frac{1}{2}} + \mathrm{III}(\mathrm{III}^{\frac{1}{2}}h')']\frac{(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)}{\lambda_3^2(\lambda_1^2 + \lambda_2^2)}C_3^2. \tag{4.15}$$

Thus NC_3 is determined in terms of λ_α, h', and μ, and from (4.13), NC_1 and NC_2 are also determined. Thus the displacement field (4.12) may be written explicitly in terms of the basic deformation stretches λ_α, and μ, h'.

We note in passing that (4.14) may be interpreted [5] in terms of the two special central planes for which the sections of the ellipsoids $\mathbf{x}\cdot\mathbb{B}\mathbf{x} = 1$, $\mathbf{x}\cdot\mathbb{B}^{-1}\mathbf{x} = 1$, are a pair of similar and similarly situated ellipses. The plane of $\mathbf{C}$ must coincide with either of the two special central planes and also the ellipse of $\mathbf{C}$ is similar and similarly situated to the elliptical sections of the $\mathbb{B}$ and $\mathbb{B}^{-1}$ ellipsoids by the plane of $\mathbf{C}$.

4.2 'Longitudinal' waves of circular polarization.

Now we determine the most general material for which infinitesimal 'longitudinal' inhomogeneous plane waves of *circular* polarization may propagate for any choice of isotropic directional bivector $\mathbf{C}$, when the material is held in an arbitrary state of finite static pure homogeneous deformation. Of course, as we have seen already, such waves may propagate in a deformed Hadamard material.

The condition that such waves may propagate is that equations (3.23) and (3.24) be satisfied for all isotropic $\mathbf{C}$: $\mathbf{C}\cdot\mathbf{C} = 0$ and for all positive choice of λ_1, λ_2, λ_3. When $\mathbf{C}\cdot\mathbf{C} = 0$, equations (3.23) and (3.24) reduce to

$$N_{\Gamma,\mathrm{I}}(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) - \mathrm{III}N_{\Gamma,\mathrm{II}}(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}) = 0, \quad (\Gamma = -1, 1). \tag{4.16}$$

Because λ_1, λ_2, λ_3 are arbitrary, it follows that

$$N_{\Gamma,\mathrm{I}} = N_{\Gamma,\mathrm{II}} = 0, \quad (\Gamma = -1, 1), \tag{4.17}$$

so that the corresponding constitutive equation is

$$\mathbb{T} = N_0(\mathrm{I}, \mathrm{II}, \mathrm{III})\mathbf{1} + N_1(\mathrm{III})\mathbb{B} - N_{-1}(\mathrm{III})\mathbb{B}^{-1}. \tag{4.18}$$

The corresponding materials are called 'Generalized Hadamard materials'. The class encompasses Hadamard materials.

For any choice of isotropic $\mathbf{C}$, the wave train is circularly polarized, with the plane of $\mathbf{C}$ being the plane of polarization, and the corresponding slowness $\mathbf{S} = N\mathbf{C}$ being given through

$$\rho N^{-2} = (2N_{0,\mathrm{I}} + N_1)\mathbf{C}\cdot\mathbb{B}\mathbf{C} + (N_{-1} - 2\mathrm{III}N_{0,\mathrm{II}})\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}, \tag{4.19}$$

on using (3.22).

If the Generalized Hadamard material is to be hyperelastic, then it may be shown that N_0 must have the form

$$N_0 = \mathrm{I}\ \mathrm{III}^{\frac{1}{2}}(\mathrm{III}^{\frac{1}{2}}N_1)' - \mathrm{II}\ \mathrm{III}^{-\frac{1}{2}}(\mathrm{III}^{\frac{1}{2}}N_{-1})' + \mathrm{III}^{\frac{1}{2}}h'(\mathrm{III}), \tag{4.20}$$

and the corresponding form of the strain energy density W is

$$2W = \mathrm{I}\ \mathrm{III}^{\frac{1}{2}}N_1(\mathrm{III}) + \mathrm{II}\ \mathrm{III}^{-\frac{1}{2}}N_{-1}(\mathrm{III}) + h(\mathrm{III}), \tag{4.21}$$

where N_1, N_{-1}, and h are arbitrary functions of III.

Remark: A universal relation.

Using equation (4.19) we derive a universal relation among the wave slownesses corresponding to three choices of the directional bivector $\mathbf{C}$. We write $\mathbf{C_1} = \mathbf{i} + i\mathbf{j}$, $\mathbf{C_2} = \mathbf{j} + i\mathbf{k}$, $\mathbf{C_3} = \mathbf{k} + i\mathbf{j}$. Then $\mathbf{C}_\alpha \cdot \mathbf{C}_\alpha = 0$, and

$$\begin{aligned}
\mathbf{C_1}\cdot\mathbb{B}\mathbf{C_1} &= \lambda_1^2 - \lambda_2^2, &\quad \mathbf{C_1}\cdot\mathbb{B}^{-1}\mathbf{C_1} &= \lambda_1^{-2} - \lambda_2^{-2},\\
\mathbf{C_2}\cdot\mathbb{B}\mathbf{C_2} &= \lambda_2^2 - \lambda_3^2, &\quad \mathbf{C_2}\cdot\mathbb{B}^{-1}\mathbf{C_2} &= \lambda_2^{-2} - \lambda_3^{-2},\\
\mathbf{C_3}\cdot\mathbb{B}\mathbf{C_3} &= \lambda_3^2 - \lambda_1^2, &\quad \mathbf{C_3}\cdot\mathbb{B}^{-1}\mathbf{C_3} &= \lambda_3^{-2} - \lambda_2^{-2}.
\end{aligned} \tag{4.22}$$

Let the value of N corresponding to $\mathbf{C}_\alpha$ be $N(\mathbf{C}_\alpha)$. Then

$$\rho N^{-2}(\mathbf{C_1}) = (2N_{0,\mathrm{I}} + N_1)(\lambda_1^2 - \lambda_2^2) + (N_{-1} - 2\mathrm{III}N_{0,\mathrm{II}})(\lambda_1^{-2} - \lambda_2^{-2}), \tag{4.23}$$

etc., and we have the universal relation

$$N^{-2}(\mathbf{C_1}) + N^{-2}(\mathbf{C_2}) + N^{-2}(\mathbf{C_3}) = 0. \tag{4.24}$$

5 Propagation of 'longitudinal' inhomogeneous plane waves, circularly polarized in special planes

Now we seek elastic isotropic materials which allow the propagation of two very special circularly polarized 'longitudinal' inhomogeneous plane waves, irrespective of the basic static homogeneous deformation. The circle of polarization of the waves is to be one or the other of the two central sections of the ellipsoid $\mathbf{x}\cdot\mathbb{B}^n\mathbf{x} = 1$ (the $\mathbb{B}^n$-ellipsoid), where $\mathbb{B}$ is the left Cauchy-Green strain tensor (3.9) corresponding to the basic static deformation.

We first consider the cases where $n = 1$ and where $n = -1$, and then treat the general case.

5.1 Circle of polarization lies in a plane of central circular section of the $\mathbb{B}$-ellipsoid or of the $\mathbb{B}^{-1}$-ellipsoid.

The $\mathbb{B}$-ellipsoid is the surface described by $\mathbf{x}\cdot\mathbb{B}\mathbf{x} = 1$. Because $\mathbb{B}$ is a positive definite tensor, there exist two central planes which cut the ellipsoid in circular sections. Boulanger and Hayes [5] have proved that the circles are described by the bivector $\mathbf{A}$ (say) such that $\mathbf{A}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbb{B}\mathbf{A} = 0$. Hence, the wave (3.12), with a bivector $\mathbf{C}$ satisfying

$$\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}\mathbf{C} = 0, \tag{5.1}$$

is a 'longitudinal' inhomogeneous plane wave, circularly polarized, with a central circular section of the $\mathbb{B}$-ellipsoid as circle of polarization. When (5.1) is satisfied, it follows that $\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} \neq 0$ unless $\mathbf{C} \equiv \mathbf{0}$, so that (3.23), (3.24) reduce to

$$N_{1,\mathrm{II}} = N_{-1,\mathrm{II}} = 0. \tag{5.2}$$

These must hold for all possible positive λ_1, λ_2, λ_3. Hence $N_1 = N_1(\mathrm{I},\mathrm{III})$, $N_{-1} = N_{-1}(\mathrm{I},\mathrm{III})$ and the corresponding constitutive equation is

$$\mathbb{T} = N_0(\mathrm{I},\mathrm{II},\mathrm{III})\mathbf{1} + N_1(\mathrm{I},\mathrm{III})\mathbb{B} - N_{-1}(\mathrm{I},\mathrm{III})\mathbb{B}^{-1}. \tag{5.3}$$

This is, of course, inclusive of the Generalized Hadamard materials class (4.18). When (5.1) holds, then, from (3.22),

$$\rho N^{-2} = (-2\mathrm{III}N_{0,\mathrm{II}} + N_{-1})\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}. \tag{5.4}$$

To determine the possible $\mathbf{C}$, we recall that the Hamiltonian decomposition of the $\mathbb{B}$ tensor is

$$\mathbb{B} = \lambda_2^2\mathbf{1} + \tfrac{1}{2}(\lambda_1^2 - \lambda_3^2)[\mathbf{h}^+\otimes\mathbf{h}^- + \mathbf{h}^-\otimes\mathbf{h}^+], \tag{5.5}$$

where

$$\mathbf{h}^\pm = \delta\mathbf{i} \pm \phi\mathbf{k}, \quad \delta = \sqrt{\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2-\lambda_3^2}}, \quad \phi = \sqrt{\frac{\lambda_2^2-\lambda_3^2}{\lambda_1^2-\lambda_3^2}}, \quad \delta^2+\phi^2 = 1, \tag{5.6}$$

$$\delta^2\lambda_3^2 + \phi^2\lambda_1^2 = \lambda_2^2, \quad \delta^2\lambda_3^{-2} + \phi^2\lambda_1^{-2} - \lambda_2^{-2} = \delta^2\phi^2(\lambda_1^2-\lambda_3^2)^2/\mathrm{III}. \tag{5.7}$$

By (5.1) it follows, using (5.5), that $(\mathbf{C}\cdot\mathbf{h}^+)(\mathbf{C}\cdot\mathbf{h}^-) = 0$. Hence either $\mathbf{C}\cdot\mathbf{h}^+ = 0$ or $\mathbf{C}\cdot\mathbf{h}^- = 0$, so that the only possible $\mathbf{C}$ are

$$\mathbf{C} = \phi\mathbf{i} \pm i\mathbf{j} - \delta\mathbf{k}, \quad \text{or} \quad \mathbf{C} = \phi\mathbf{i} \pm i\mathbf{j} + \delta\mathbf{k}. \tag{5.8}$$

For the first pair of possible $\mathbf{C}$, $\mathbf{C} = \phi\mathbf{i} + i\mathbf{j} - \delta\mathbf{k}$ and $\mathbf{C} = \phi\mathbf{i} - i\mathbf{j} - \delta\mathbf{k}$, the polarization circles lie in the same plane (with normal $\mathbf{h}^+$) of central circular section of the $\mathbb{B}$-ellipsoid. However the two circles are described in opposite senses. A similar comment applies to the second pair of possible $\mathbf{C}$, $\mathbf{C} = \phi\mathbf{i} \pm i\mathbf{j} + \delta\mathbf{k}$. The reason why there are four possibilities for $\mathbf{C}$ is that if $\mathbf{C}$ is a solution of (5.1) then so also is its complex conjugate $\overline{\mathbf{C}}$, because $\mathbb{B}$ is real and there are just two central circular sections of the $\mathbb{B}$-ellipsoid.

Now, for all these four possible choices of $\mathbf{C}$, $\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} = \delta^2\lambda_3^{-2} + \phi^2\lambda_1^{-2} - \lambda_2^{-2} = \delta^2\phi^2(\lambda_1^2 - \lambda_3^2)^2/\mathrm{III}$ and so N is given by

$$\rho N^{-2} = \delta^2\phi^2(\lambda_1^2 - \lambda_3^2)^2\mathrm{III}^{-1}(N_{-1} - 2\mathrm{III}N_{0,\mathrm{II}}). \tag{5.9}$$

We note that $\mathbf{C}$ and N are determined completely by the basic deformation. The planes of constant phase and the planes of constant amplitude are orthogonal. From (5.9), N is either purely real or purely imaginary. When N is real (imaginary), the planes of constant phase (amplitude) are $\phi x \mp \delta z =$ const. and the planes of constant amplitude (phase) are $y =$ const. Of course, any scalar multiple of $\mathbf{S}$ is a possible amplitude bivector. The displacements corresponding to $(5.8)_1$ are

$$\overline{\mathbf{x}} = \mathbf{x} + \epsilon e^{-\omega N y}\{(\phi\mathbf{i} \pm i\mathbf{j} - \delta\mathbf{k})e^{i\omega[N(\phi x - \delta z) - t]}\}^+. \tag{5.10}$$

Essentially, there are two possible circularly polarized longitudinal inhomogeneous plane waves for which $\mathbf{C}\cdot\mathbb{B}\mathbf{C} = 0$.

Also, we note that for (5.3) to represent a hyperelastic material, the response function $N_{-1}(\mathrm{I},\mathrm{III})$ must be independent of I. The response functions are then of the form

$$\begin{aligned} &N_{-1} = G(\mathrm{III}), \quad N_0 = \mathrm{II}G'(\mathrm{III}) + [\mathrm{II}/(2\mathrm{III})]G(\mathrm{III}) + K(\mathrm{I},\mathrm{III}),\\ &N_1 = \mathrm{III}^{-\frac{1}{2}}L(\mathrm{I}) + \mathrm{III}^{-\frac{1}{2}}\textstyle\int \partial K(\mathrm{I},\mathrm{III})/\partial\mathrm{I}\ \mathrm{III}^{-\frac{1}{2}}\mathrm{dIII}, \end{aligned} \tag{5.11}$$

with corresponding strain energy density W given by

$$2W = \textstyle\int L(\mathrm{I})\mathrm{dI} + \int K(\mathrm{I},\mathrm{III})\mathrm{III}^{-\frac{1}{2}}\mathrm{dIII} + \mathrm{II}\ \mathrm{III}^{-\frac{1}{2}}G(\mathrm{III}), \tag{5.12}$$

where G, K, L are arbitrary functions of their arguments.

Similarly, the wave (3.12), with a bivector $\mathbf{C}$ satisfying

$$\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} = 0, \tag{5.13}$$

is a 'longitudinal' inhomogeneous plane wave, circularly polarized, with a central circular section of the $\mathbb{B}^{-1}$-ellipsoid as circle of polarization. When (5.13) is satisfied, then (3.23) and (3.24) reduce to

$$N_{1,\mathrm{I}} = N_{-1,\mathrm{I}} = 0, \tag{5.14}$$

which are to be satisfied for all positive λ_1, λ_2, λ_3. The corresponding constitutive equation is

$$\mathbb{T} = N_0(\mathrm{I},\mathrm{II},\mathrm{III})\mathbf{1} + N_1(\mathrm{II},\mathrm{III})\mathbb{B} - N_{-1}(\mathrm{II},\mathrm{III})\mathbb{B}^{-1}. \tag{5.15}$$

Again, the corresponding class of materials is inclusive of the Generalized Hadamard materials class. When (5.13) holds, then

$$\rho N^{-2} = (N_1 + 2N_{0,\mathrm{I}})\mathbf{C}\cdot\mathbb{B}\mathbf{C}. \tag{5.16}$$

To determine those $\mathbf{C}$ for which $\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} = 0$, we use the Hamiltonian decomposition of the $\mathbb{B}^{-1}$ tensor:

$$\mathbb{B}^{-1} = \lambda_2^{-2}\mathbf{1} + \tfrac{1}{2}(\lambda_1^{-2} - \lambda_3^{-2})[\mathbf{l}^+\otimes\mathbf{l}^- + \mathbf{l}^-\otimes\mathbf{l}^+], \tag{5.17}$$

where

$$\mathbf{l}^\pm = \hat{\delta}\mathbf{i} \pm \hat{\phi}\mathbf{k}, \quad \hat{\delta} = \sqrt{\frac{\lambda_2^{-2} - \lambda_1^{-2}}{\lambda_3^{-2} - \lambda_1^{-2}}}, \quad \hat{\phi} = \sqrt{\frac{\lambda_3^{-2} - \lambda_2^{-2}}{\lambda_3^{-2} - \lambda_1^{-2}}}, \quad \hat{\delta}^2 + \hat{\phi}^2 = 1,$$
$$\hat{\delta}^2\lambda_3^{-2} + \hat{\phi}^2\lambda_1^{-2} = \lambda_2^{-2}, \quad \hat{\delta}^2\lambda_3^2 + \hat{\phi}^2\lambda_1^2 - \lambda_2^2 = \hat{\delta}^2\hat{\phi}^2(\lambda_3^{-2} - \lambda_1^{-2})^2\mathrm{III}. \tag{5.18}$$

Here the only possible $\mathbf{C}$ are those for which $\mathbf{C}\cdot\mathbf{C} = 0$, namely $\mathbf{C}\cdot\mathbf{l}^+ = 0$ or $\mathbf{C}\cdot\mathbf{l}^- = 0$. Thus, suitable $\mathbf{C}$ are

$$\mathbf{C} = \hat{\phi}\mathbf{i} \pm i\mathbf{j} - \hat{\delta}\mathbf{k}, \quad \text{or} \quad \mathbf{C} = \hat{\phi}\mathbf{i} \pm i\mathbf{j} + \hat{\delta}\mathbf{k}. \tag{5.19}$$

For all these four possible choices of $\mathbf{C}$, the corresponding complex scalar slowness N is given by

$$\rho N^{-2} = \hat{\delta}^2\hat{\phi}^2(\lambda_3^{-2} - \lambda_1^{-2})^2\mathrm{III}(N_1 + 2\mathrm{III}N_{0,\mathrm{I}}). \tag{5.20}$$

In order that the constitutive equation (5.15) represent that of a hyperelastic material, the response functions N_0, N_1, N_{-1} must satisfy certain compatibility equations [7, §86]. Using those compatibility equations it is found that for (5.15) the response function $N_1(\mathrm{II}, \mathrm{III})$ must be independent of II. Then integrating the compatibility equations it is found that

$$N_1 = R(\mathrm{III}), \quad N_0 = \mathrm{I}\ \mathrm{III}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{dIII}}[R(\mathrm{III})\mathrm{III}^{\frac{1}{2}}] + S(\mathrm{II}, \mathrm{III}), \tag{5.21}$$

where R and S are arbitrary functions of their arguments and N_{-1} is a solution of

$$\tfrac{1}{2}N_{-1} + \mathrm{III}\frac{\partial N_{-1}}{\partial \mathrm{III}} + \mathrm{II}\frac{\partial N_{-1}}{\partial \mathrm{II}} + \mathrm{III}\frac{\partial S(\mathrm{I}, \mathrm{III})}{\partial \mathrm{II}} = 0. \tag{5.22}$$

The characteristics of this equation are

$$\frac{\mathrm{dIII}}{\mathrm{III}} = \frac{\mathrm{dII}}{\mathrm{II}} = \frac{\mathrm{d}N_{-1}}{\frac{1}{2}N_{-1} + \mathrm{III}\partial S/\partial \mathrm{II}}, \tag{5.23}$$

the solutions of which depend upon S. To make progress we assume

$$S = M(\mathrm{III})\mathrm{II} + T(\mathrm{III}), \tag{5.24}$$

where M and T are arbitrary functions of III. In this case the general solution of (5.23) is

$$f(\mathrm{II}/\mathrm{III},\ \mathrm{III}^{\frac{1}{2}}N_{-1} - \textstyle\int \mathrm{III}^{\frac{1}{2}}M(\mathrm{III})\mathrm{dIII}) = 0, \tag{5.25}$$

where f is an arbitrary function, or

$$N_{-1} = -\mathrm{III}^{-\frac{1}{2}}\textstyle\int \mathrm{III}^{\frac{1}{2}}M(\mathrm{III})\mathrm{dIII} + \mathrm{III}^{-\frac{1}{2}}h(\mathrm{II}/\mathrm{III}), \tag{5.26}$$

where h is an arbitrary function. It is found that h must be zero for a hyperelastic material. We obtain

$$2W = \mathrm{I}\ \mathrm{III}^{\frac{1}{2}}R(\mathrm{III}) - \mathrm{II}\ \mathrm{III}^{-1}\textstyle\int \mathrm{III}^{\frac{1}{2}}M(\mathrm{III})\mathrm{dIII} + \textstyle\int \mathrm{III}^{-\frac{1}{2}}T(\mathrm{III})\mathrm{dIII}, \tag{5.27}$$

and

$$\begin{aligned} &N_1 = R(\mathrm{III}), \quad N_{-1} = -\mathrm{III}^{-\frac{1}{2}}\textstyle\int \mathrm{III}^{\frac{1}{2}}M(\mathrm{III})\mathrm{dIII}, \\ &N_0 = \tfrac{1}{2}\mathrm{I}R(\mathrm{III}) + \mathrm{I}\ \mathrm{III}R'(\mathrm{III}) + \mathrm{II}M(\mathrm{III}) + T(\mathrm{III}), \end{aligned} \tag{5.28}$$

where R, M, T are arbitrary functions of III. Of course alternative choices of S will lead to alternative forms for W.

Finally, we recap for the constitutive model (5.3) (or (5.15)) that amongst the waves which may propagate in the material when it is held in an arbitrary state of finite static homogeneous deformation are two infinitesimal circularly polarized longitudinal plane waves whose circles of polarization lie in the planes of the central circular sections of the $\mathbb{B}$-ellipsoid (or the $\mathbb{B}^{-1}$-ellipsoid) corresponding to the finite static deformation.

5.2 Circle of polarization lies in a plane of central circular section of the $\mathbb{B}^n$-ellipsoid.

Here we determine the most general form of the response functions N_0, N_1, N_{-1} such that the circle of polarization lies in a plane of central circular section of the $\mathbb{B}^n$-ellipsoid, $\mathbf{x}\cdot\mathbb{B}^n\mathbf{x} = 1$, where $\mathbb{B}$ is the left Cauchy-Green strain tensor corresponding to the finite static homogeneous deformation. As in the previous cases ($n = \pm 1$), it is found that the response functions N_1, N_{-1} depend just upon two invariants of $\mathbb{B}$. They take the forms $N_\Gamma = N_\Gamma(\mathrm{III}, \mathrm{tr}\ \mathbb{B}^n)$. Up to a sign there are four directional bivectors $\mathbf{C}$ such that two infinitesimal circularly polarized longitudinal plane waves may propagate and whose circles of polarization lie in the planes of central circular section of the $\mathbb{B}^n$-ellipsoid.

Now we have

$$\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^n\mathbf{C} = 0, \tag{5.29}$$

and equations (3.22), (3.23), (3.24) become

$$\rho N^{-2} = (2N_{0,\mathrm{I}} + N_1)\mathbf{C}\cdot\mathbb{B}\mathbf{C} + (-2\mathrm{III}N_{0,\mathrm{II}} + N_{-1})\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}, \tag{5.30}$$

and

$$\begin{aligned} N_{1,\mathrm{I}}(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) &= \mathrm{III}N_{1,\mathrm{II}}(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}),\\ N_{-1,\mathrm{I}}(\mathbf{C}\cdot\mathbb{B}\mathbf{C}) &= \mathrm{III}N_{-1,\mathrm{II}}(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}). \end{aligned} \tag{5.31}$$

Now it may be shown (Appendix) that when (5.29) hold, then

$$(\mathbf{C}\cdot\mathbb{B}\mathbf{C})/(\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}) = \mathrm{III}(\partial\mathrm{tr}\mathbb{B}^n/\partial\mathrm{II})/(\partial\mathrm{tr}\mathbb{B}^n/\partial\mathrm{I}), \tag{5.32}$$

so that (5.31) become

$$N_{\Gamma,\mathrm{I}}\frac{\partial\mathrm{tr}\mathbb{B}^n}{\partial\mathrm{II}} - N_{\Gamma,\mathrm{II}}\frac{\partial\mathrm{tr}\mathbb{B}^n}{\partial\mathrm{I}} = 0, \quad \Gamma = \pm 1. \tag{5.33}$$

This suggests that we write $N_\Gamma(\mathrm{I},\mathrm{II},\mathrm{III})$ in terms of the set of invariants $\mathrm{I},\mathrm{III},\mathrm{tr}\mathbb{B}^n$ ($n \neq 1$) which is equivalent to the set $(\mathrm{I},\mathrm{II},\mathrm{III})$. We write

$$N_\Gamma = \hat{N}_\Gamma(\mathrm{I},\mathrm{III},\mathrm{tr}\mathbb{B}^n), \quad \Gamma = \pm 1, \tag{5.34}$$

and

$$\begin{aligned} \frac{\partial N_\Gamma}{\partial\mathrm{I}} &= \frac{\partial\hat{N}_\Gamma}{\partial\mathrm{I}} + \frac{\partial\hat{N}_\Gamma}{\partial\mathrm{tr}\mathbb{B}^n}\frac{\partial\mathrm{tr}\mathbb{B}^n}{\partial\mathrm{I}},\\ \frac{\partial N_\Gamma}{\partial\mathrm{II}} &= \frac{\partial\hat{N}_\Gamma}{\partial\mathrm{II}} + \frac{\partial\hat{N}_\Gamma}{\partial\mathrm{tr}\mathbb{B}^n}\frac{\partial\mathrm{tr}\mathbb{B}^n}{\partial\mathrm{II}}, \end{aligned} \tag{5.35}$$

so that (5.33) becomes

$$\frac{\partial \hat{N}_\Gamma}{\partial \mathrm{I}} = 0, \quad \Gamma = \pm 1. \tag{5.36}$$

and thus, because these are to be valid for all $\lambda_1, \lambda_2, \lambda_3 > 0$, we have

$$\hat{N}_\Gamma = \hat{N}_\Gamma(\mathrm{III}, \mathrm{tr}\mathbb{B}^n), \quad \Gamma = \pm 1. \tag{5.37}$$

Hence, those materials such that when they are in any state of finite static homogeneous deformation, two infinitesimal 'longitudinal' inhomogeneous circularly polarized plane waves may propagate, the circle of polarization being in the planes of central circular sections of the $\mathbb{B}^n$-ellipsoid, where $\mathbb{B}$ is the strain tensor associated with the finite static homogeneous deformation, have constitutive equation

$$\mathbb{T} = \hat{N}_0(\mathrm{I}, \mathrm{III}, \mathrm{tr}\mathbb{B}^n)\mathbf{1} + \hat{N}_1(\mathrm{III}, \mathrm{tr}\mathbb{B}^n)\mathbb{B} - \hat{N}_{-1}(\mathrm{III}, \mathrm{tr}\mathbb{B}^n)\mathbb{B}^{-1}. \tag{5.38}$$

This result is in accord with what we found previously when $n = \pm 1$. Indeed, for $n = 1$, $\mathrm{tr}\mathbb{B}^n = \mathrm{I}$ and (5.38) becomes (5.3), whilst for $n = -1$, $\mathrm{tr}\mathbb{B}^n = \mathrm{II}/\mathrm{III}$ and (5.38) may be written in the form of (5.15).

Returning to the general case, upon using (5.30), (5.32), and (5.35), we find

$$\rho N^{-2} = (2\frac{\partial \hat{N}_0}{\partial \mathrm{I}} + \hat{N}_1)\mathbf{C}\cdot\mathbb{B}\mathbf{C} + \hat{N}_{-1}\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}, \tag{5.39}$$

Thus, if $\mathbf{C}$ is chosen so that $\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^n\mathbf{C} = 0$, then a circularly polarized plane wave with slowness $\mathbf{S} = N\mathbf{C}$ (N given by (5.39)) may propagate in the material with constitutive equation (5.38) when it is held in an arbitray state of finite static homogeneous deformation. The possible choices of $\mathbf{C}$ satisfying $\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^n\mathbf{C} = 0$ are

$$\mathbf{C} = \phi_n\mathbf{i} \pm i\mathbf{j} - \delta_n\mathbf{k}, \quad \mathbf{C} = \phi_n\mathbf{i} \pm i\mathbf{j} + \delta_n\mathbf{k}, \tag{5.40}$$

where

$$\delta_n = \sqrt{\frac{\lambda_1^{2n} - \lambda_2^{2n}}{\lambda_1^{2n} - \lambda_3^{2n}}}, \quad \phi_n = \sqrt{\frac{\lambda_2^{2n} - \lambda_3^{2n}}{\lambda_1^{2n} - \lambda_3^{2n}}}, \quad \delta_n^2 + \phi_n^2 = 1. \tag{5.41}$$

We note that the terms occuring in (5.39) for all four possible choices of $\mathbf{C}$ are given by

$$\begin{aligned} \mathbf{C}\cdot\mathbb{B}\mathbf{C} &= (\lambda_1^2 - \lambda_2^2)\phi_n^2 - (\lambda_2^2 - \lambda_3^2)\delta_n^2, \\ \mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C} &= [\lambda_1^2(\lambda_2^2 - \lambda_3^2)\delta_n^2 - \lambda_3^2(\lambda_1^2 - \lambda_2^2)\phi_n^2]/\mathrm{III}. \end{aligned} \tag{5.42}$$

Special Case $n = \frac{1}{2}$

The expressions simplify greatly in the case when $n = \frac{1}{2}$, that is, when we assume that the circles of polarization lie in the planes of central circular section of the $\mathbb{B}^{\frac{1}{2}}$-ellipsoid. Thus $\mathbf{C}$ is such that $\mathbf{C} \cdot \mathbf{C} = 0$, $\mathbf{C} \cdot \mathbb{B}^{\frac{1}{2}} \mathbf{C} = 0$, so that the possible $\mathbf{C}$ are given by

$$\begin{aligned} \sqrt{\lambda_1 - \lambda_3}\mathbf{C} &= \sqrt{\lambda_2 - \lambda_3}\mathbf{i} \pm i\mathbf{j} + \sqrt{\lambda_1 - \lambda_2}\mathbf{k}, \\ \sqrt{\lambda_1 - \lambda_3}\mathbf{C} &= \sqrt{\lambda_2 - \lambda_3}\mathbf{i} \pm i\mathbf{j} - \sqrt{\lambda_1 - \lambda_2}\mathbf{k}. \end{aligned} \tag{5.43}$$

The corresponding constitutive equation is

$$\mathbb{T} = \hat{N}_0(\mathrm{I}, \mathrm{II}, \mathrm{III})\mathbf{1} + \hat{N}_1(\mathrm{III}, \mathrm{tr}\mathbb{B}^{\frac{1}{2}})\mathbb{B} - \hat{N}_{-1}(\mathrm{III}, \mathrm{tr}\mathbb{B}^{\frac{1}{2}})\mathbb{B}^{-1}. \tag{5.44}$$

Using (5.39), the slownesses $\mathbf{S} = N\mathbf{C}$ of all four possible circularly polarized waves with directional bivectors $\mathbf{C}$ given by (5.43) are such that

$$\rho N^{-2} = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)[2\frac{\partial \hat{N}_0}{\partial \mathrm{I}} + \hat{N}_1 + \frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{(\lambda_1\lambda_2\lambda_3)^2}\hat{N}_{-1}]. \tag{5.45}$$

Appendix

Here we present a proof of (5.32).

For convenience we write

$$\alpha = \lambda_1^2, \quad \beta = \lambda_2^2, \quad \gamma = \lambda_3^2, \tag{E.46}$$

so that

$$\begin{aligned} &\mathbb{B} = \mathrm{diag}\,(\alpha, \beta, \gamma), \quad \mathrm{I} = \alpha + \beta + \gamma, \quad \mathrm{III} = \alpha\beta\gamma, \\ &\mathrm{II} = \alpha\beta + \beta\gamma + \gamma\alpha, \quad \mathrm{tr}\mathbb{B}^n = \alpha^n + \beta^n + \gamma^n. \end{aligned} \tag{E.47}$$

It may be checked that

$$\frac{\partial \alpha}{\partial \mathrm{I}} = \frac{\alpha(\beta - \gamma)\mathrm{III}}{\beta\gamma(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}, \quad \frac{\partial \alpha}{\partial \mathrm{II}} = \frac{-(\beta - \gamma)\mathrm{III}}{\beta\gamma(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}, \tag{E.48}$$

and that

$$\begin{aligned} \frac{\partial \mathrm{tr}\mathbb{B}^n}{\partial \mathrm{I}} &= \frac{n[(\beta - \gamma)\alpha^{n+1} + (\gamma - \alpha)\beta^{n+1} + (\alpha - \beta)\gamma^{n+1}]}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}, \\ \frac{\partial \mathrm{tr}\mathbb{B}^n}{\partial \mathrm{II}} &= \frac{-n[(\beta - \gamma)\alpha^n + (\gamma - \alpha)\beta^n + (\alpha - \beta)\gamma^n]}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}. \end{aligned} \tag{E.49}$$

Thus, we have the identity

$$n\frac{\partial \mathrm{tr}\mathbb{B}^{n+1}}{\partial \mathrm{II}} = -(n+1)\frac{\partial \mathrm{tr}\mathbb{B}^{n}}{\partial \mathrm{I}}. \tag{E.50}$$

If $\mathbf{C}$ is such that $\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^n\mathbf{C} = 0$, i.e.

$$C_1^2 + C_2^2 + C_3^2 = 0, \quad C_1^2\alpha^n + C_2^2\beta^n + C_3^2\gamma^n = 0, \tag{E.51}$$

then

$$\frac{C_1^2}{C_3^2} = \frac{\beta^n - \gamma^n}{\alpha^n - \beta^n}, \quad \frac{C_2^2}{C_3^2} = \frac{\gamma^n - \alpha^n}{\alpha^n - \beta^n}, \tag{E.52}$$

so that

$$\begin{aligned}\frac{\mathbf{C}\cdot\mathbb{B}\mathbf{C}}{\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}} &= \frac{C_1^2\alpha + C_2^2\beta + C_3^2\gamma}{C_1^2/\alpha + C_2^2/\beta + C_3^2/\gamma}\\ &= -\mathrm{III}\frac{(\beta-\gamma)\alpha^n + (\gamma-\alpha)\beta^n + (\alpha-\beta)\gamma^n}{(\beta-\gamma)\alpha^{n+1} + (\gamma-\alpha)\beta^{n+1} + (\alpha-\beta)\gamma^{n+1}}.\end{aligned} \tag{E.53}$$

Thus, when $\mathbf{C}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbb{B}^n\mathbf{C} = 0$, we have the result

$$\frac{\mathbf{C}\cdot\mathbb{B}\mathbf{C}}{\mathbf{C}\cdot\mathbb{B}^{-1}\mathbf{C}} = \mathrm{III}\frac{\partial \mathrm{tr}\mathbb{B}^n/\partial \mathrm{II}}{\partial \mathrm{tr}\mathbb{B}^n/\partial \mathrm{I}}. \tag{E.54}$$

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