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On the application of robust numerical methods to a complete-flow wave-current model

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Abstract

We aim to show how parameter-robust numerical method may be employed to solve equations arising in the modeling of wave-current interactions.

We present two such models: a complete flow model for wave-current interaction in the presence of weakly turbulent flow leading to an Orr-Sommerfeld type problem, and a system of two singularly perturbed reaction-diffusion equations from a k-ϵ turbulence model. The numerical results are compared with experimental data.

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1. Introduction

Parameter-robust numerical methods are of significant interest in modern numerical analysis: they yield accurate, layer-resolved, computed solutions to singularly perturbed differential equations. Importantly, their accuracy is independent of the singular perturbation parameter, and thus the width of the boundary layers.

Many of these methods are *mesh based*. That is, they use the same discretizations as one would use for a "classical" problem whose solution does not exhibit layers. Instead of modifying the scheme to stabilize it, a mesh tailored to the specific problem is used. In this study we employ the *a priori* fitted piecewise-uniform meshes of Shishkin, see, e.g., [9].
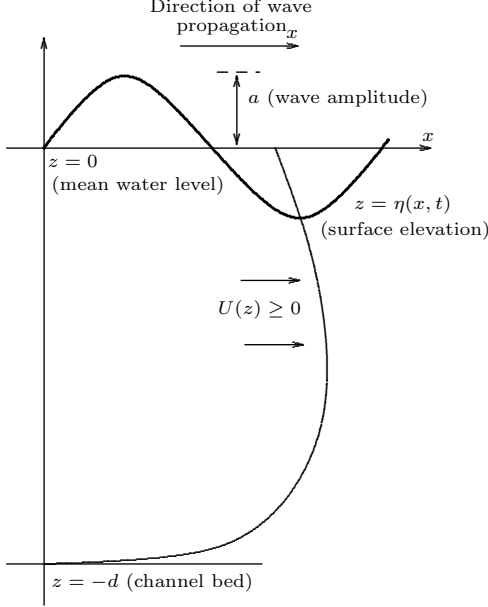
A complete flow model for the interaction of waves and currents leads to a variant on the fourth-order Orr-Sommerfeld equation, and is described in §2. A crucial component of the model is a depth-varying eddy viscosity distribution. For a given current profile, this is computed using a two equation turbulence model described in §4.

Since the models generate the function that establishes the width of the boundary layers, it is appropriate to use a numerical method whos accuracy does not depend on the layer width.

2. A complete flow model

The physical system under consideration consists of waves propagating in the positive x-direction that are of amplitude a, length λ, and frequency ω. These are present in a channel of depth d. We compute velocity components that vary only with depth and so use the depth z as a real-valued independent variable. Boundary conditions are evaluated at the *mean water level*, $z = 0$, and at the channel bed $z = -d$. The wave amplitude and frequency are taken as known and are used as inputs to the model. The wavelength λ, or equivalently the wavenumber k, can be measured experimentally and are also

outputs of the model. Here k is a complex-valued quantity. The real part of k is the number of waves per unit length and the imaginary part of k determines the spatial wave decay due to dissipation.

Direction of wave propagation x

a (wave amplitude)

x

$z = 0$ (mean water level)

$z = \eta(x, t)$ (surface elevation)

$U(z) \geq 0$

$z = -d$ (channel bed)

Also present is a steady current $U(z)$, where $-d \leq z \leq 0$. This represents the horizontal velocity of a fluid particle in the absence of waves or, if waves are present, the horizontal velocity averaged over a wave cycle. Hence it is referred to as the *mean* current. In the diagram above, $U(z)$ is acting in the same direction as the waves, and so is called a *wave-following current*. Alternatively, we may consider a *wave-opposing current*.

The model presented here assumes that $U(z)$ is known and so is said to be of *reactive* type [7]. Other models that attempt to predict $U(z)$ are said to be *generative* (see, e.g., [1, 2]).

The main assumptions of the model are that quantities are functions of x, z and t (i.e., none vary in the y-direction); the bed is locally horizontal; boundary conditions can be evaluated at the mean water level; the fluid temperature, and hence its viscosity, is constant over depth; the slope of the waves is small.

A key aspect of the model we present here is the use of an *eddy viscosity distribution*. The Boussinesq eddy viscosity concept assumes that there is an analogy between turbulence stress and viscous stresses in laminar flow. For our purposes it can be taken to mean that there is a depth-varying nonnegative quantity, the (kinematic) eddy viscosity $\nu_t(z)$, which mimics the behaviour of the kinematic molecular viscosity ν which is constant over depth at a fixed temperature. Therefore the model (1–5) below is formulated so that the term ν is augmented by $\nu_t(z)$.

Owing to the interaction of waves with the steady current, both the horizontal and vertical velocities of a particle, $u_T(x, z, t)$ and $w_T(x, z, t)$, do indeed vary with time. It is these quantities, along with the wavenumber, that we want to predict.

Considering first the horizontal component, the velocity is expressed as the sum of the steady mean current $U(z)$ and a wave-like term written as a harmonic

$$u_T(x, z, t) = U(z) + u(x, z, t). \qquad \text{where} \qquad u(x, z, t) = \sum_{n=1}^{\infty} u_n(z) \cos n(kx - \omega t).$$

The vertical velocity has zero mean component and so it is assumed that

$$w_T(x, z, t) = w(x, z, t) = \sum_{n=1}^{\infty} w_n(z) \sin n(kx - \omega t).$$

We intend to compute approximations to the first order coefficient $u_1(z)$ and the wavenumber k. From these one can compute first harmonic $w_1(z)$ of the vertical component of the orbital velocity using the continuity equation: $u(z) = w'(z)/k$.

In this model, the stream-function $\Psi(x, z, t)$ is introduced and is defined as

$$u_1(z) = \frac{\partial \Psi}{\partial z}, \qquad w_1(z) = -\frac{\partial \Psi}{\partial x},$$

which is then given the form

$$\Psi(x, z, t) = \Re\left\{\psi(z) \exp(i\theta)\right\},$$

where $\theta = kx - \omega t$ is the phase function and $\Re\{\cdot\}$ denotes the real part of a complex number.

The differential equation that must be solved is the fourth-order problem

$$-\frac{i\nu}{\omega - kU}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^2\psi + \frac{\mathrm{d}\psi^2}{\mathrm{d}z^2} - \left(k^2 - \frac{k}{\omega - kU}\frac{\mathrm{d}^2 U}{\mathrm{d}z^2}\right)\psi =$$

$$\frac{i}{\omega - kU}\left\{\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2\right)\left\{\nu_t\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2\right)\psi\right\} - 4k^2\frac{\mathrm{d}}{\mathrm{d}z}\left(\nu_t\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)\right\} \qquad \text{on } \Omega := (-d, 0), \quad (1)$$

subject to the bottom boundary conditions

$$\psi = 0, \quad \frac{\mathrm{d}\psi}{\mathrm{d}z} = 0, \qquad \text{on } z = -d, \tag{2}$$

and, at the free surface conditions $(z = 0)$,

$$\psi = a\left(\frac{\omega}{k} - U\right), \tag{3}$$

$$(\nu + \nu_t)\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} = -a\left((\nu + \nu_t)\frac{\mathrm{d}^2 U}{\mathrm{d}z^2} + \frac{\mathrm{d}\nu_t}{\mathrm{d}z}\frac{\mathrm{d}U}{\mathrm{d}z} + k(\nu + \nu_t)(\omega - kU)\right), \tag{4}$$

and

$$i(\nu + \nu_t)\frac{\mathrm{d}^3\psi}{\mathrm{d}z^3} + i\frac{\mathrm{d}\nu_t}{\mathrm{d}z}\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} - \left\{\omega - kU + 3ik^2(\nu + \nu_t)\right\}\frac{\mathrm{d}\psi}{\mathrm{d}z} = a\left((\omega - kU)\left\{\frac{\mathrm{d}U}{\mathrm{d}z} - ik\frac{\mathrm{d}\nu_t}{\mathrm{d}z}\right\} - gk\right). \tag{5}$$

3. Numerical Solution of the Orr-Sommerfeld Equations

The equation (1) is singularly perturbed because the coefficient ν and function $\nu_t(z)$, multiplying the highest derivatives are both small in magnitude. As a consequence the solution exhibits layers at the boundaries and particularly at the lower boundary.

A similar problem, though real-valued and linear, was studied by Sun and Stynes [12]. They showed how to construct a piecewise uniform mesh on which one can apply a finite element method based. The solution to the discrete problem converges uniformly to the continuous one independently of a perturbation parameters.

Our finite element formulation is based of the sesquelinear form

$$B_{(\nu+\nu_t)}(u, v) := ((\nu + \nu_t)u'', v'') + k^2\left((\nu + \nu_t)u, v''\right) + 4k^2\left((\nu + \nu_t)u', v'\right)$$

$$+ ([i(\omega - kU) + k^2(\nu + \nu_t)]u'', v) + \left(\left[ikU'' - ik^2(\omega - kU) + k^4(\nu + \nu_t)\right]u, v\right), \quad (6)$$

for all $u, v \in H^2_\star(\Omega)$, where $H^2_\star(\Omega)$ is a subspace of $H^2(\Omega)$ chosen so that functions belonging to it satisfy the boundary conditions (2)–(3).

We seek not only an approximation of $\psi(z)$ but also an approximation of $\psi'(z)$ from which to predict the vertical component $w(z)$ of the flow velocity. Thus it is natural to choose piecewise cubic Hermite basis functions, which lie in $C^1(\bar{\Omega})$:

$$v_h(z) = \sum_{i=1}^{N} v_i\phi_i^0(z) + \sum_{i=1}^{N} v_i'\phi_i^1(z),$$

where $\phi_i^0(z)$ and $\phi_i^1(z)$ are the usual basic functions for Hermite interpolation.

An iterative scheme is then formulated based on boundary condition (5). Now the problem is to find approximations $\psi(x)$, $\psi'(x)$ and k such that the variational formulation of (1)–(4) and the extra condition (5) are satisfied. For further details of the numerical method, see [6] and [7].

4. The Turbulence Model

To calculate a suitable eddy viscosity distribution $\nu_t(z)$ for a given current profile, we employ a model due to Thomas [13]. This is a so-called "two equation model" based on the well-known k-ϵ model, where k is the kinematic energy per unit mass of the turbulent motion, and ϵ is the rate of viscous dissipation (see, e.g., [10]). In the study of waves and currents, the notations k and ϵ are usually reserved for the wavenumber and wave-slope respectively. Therefore we follow Thomas's notation and use E for the turbulent kinetic energy and D for the rate of dissipation.

The model leads to a system of two coupled singularly perturbed nonlinear reaction-diffusion equations that are solved for $E(z)$ and $D(z)$. Then the Kolmogorov-Prandtl expression relates these to $\nu_t(z)$:

$$\nu_t(z)D(z) = C_\mu f_\mu(z)E(z)^2 \quad \text{for } z \in [-d, 0]. \tag{7}$$

Here C_μ is an empirical closure constant, and $f_\mu(z)$ is a damping function used in order to obtain correct behaviour near the bed, and taken to depend on the local turbulence Reynolds number.

The approach of Thomas [13] is to expand $\nu_t(z)$, $E(z)$ and $D(z)$ in terms of the wave slope ϵ. Here we concentrate only on the zero-order equations. Our unknowns, then, will be the functions $E_0(z)$ and $D_0(z)$, which are the zero-order terms in the expansions of $E(z)$ and $D(z)$, and satisfy and then develop the zero-order E-D model as

$$\nu_t^0(z)D_0(z) = C_\mu f_\mu(z)E_0^2(z) \qquad \text{for } z \in \bar{\Omega}, \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\nu + \frac{\nu_t^0}{\sigma_E}\right)\frac{\mathrm{d}E_0}{\mathrm{d}z}\right] + \nu_t^0(z)\left[\frac{\mathrm{d}U}{\mathrm{d}z}\right]^2 = D_0(z) \qquad \text{for } z \in \Omega, \tag{8b}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\nu + \frac{\nu_t^0}{\sigma_D}\right)\frac{\mathrm{d}D_0}{\mathrm{d}z}\right] + C_{1D}f_1(z)C_\mu f_\mu(z)E_0\left[\frac{\mathrm{d}U}{\mathrm{d}z}\right]^2 = C_{2D}f_2(z)\frac{D_0^2}{E_0} \qquad \text{for } z \in \Omega, \tag{8c}$$

where $\Omega := (-d, 0)$. The boundary conditions are

$$E_0(-d) = E'(0) = D_0'(-d) = 0, \quad D_0(0) = (5.87/d)E_0^{\frac{3}{2}}(0). \tag{9}$$

The three functions $f_\mu(z)$, $f_1(z)$ and $f_2(z)$ in (8) are "wall function", and are present in the formulation to ensure correct behaviour in the near-wall region. In general their value is close to unity over all of Ω, except near the bottom boundary where they change rapidly. For details of the these functions and the values of the empirically established terms C_{1D}, C_{2D} (closure constants) and σ_E and σ_D (diffusion constants) we refer the reader to [4].

We solve numerically the E-D equations by applying a finite element method on a piecewise uniform mesh. Such meshes for systems of reaction-diffusion equations have been analysed by Shishkin and collaborators, see e.g., [11]. Solutions to the system that we study here exhibit two distinct, interacting layers near each boundary. It has been shown that a piecewise uniform mesh can yield a parameter-robust approximation with finite difference [8] and finite element [5] methods.

To resolve the interacting layers at the bed, we construct a piecewise uniform mesh with two transition parameters τ_1 and τ_2 chosen according to the formula

$$\tau_1 = -d + C_1\nu \ln N \quad \text{and} \quad \tau_2 = -d + C_2\sqrt{\nu}\ln N,$$

where C_1 and C_2 are user chosen values.

We divide $[-d, 0]$ into the three subintervals $[-d, \tau_1]$, $[\tau_1, \tau_2]$, and $[\tau_2, 0]$. We place a uniform mesh on each subinterval in such a way that there are $N/4$ evenly spaces mesh points on each of the subintervals $[-d, \tau_1]$, $[\tau_1, \tau_2]$, and $N/2$ mesh points in the remainder of the region.

5. Numerical Results

We now compare numeric results with data from physical experiments [3]. Waves of amplitude $a = 0.05987$m and frequency $\omega = 0.9844$ Hz were propagated in a flume with water of depth 0.5m.

The current distribution $U(z)$ for the wave-following case is shown opposite, and is shown on the left of Figure 1. The data is fitted with a continuous function which is then used as an input for the E–D model. The resulting $\nu_t^0(z)$ is shown in the centre of Figure 1. A detail of the $\nu_t^0(z)$ close to the channel bed is shown on the right.
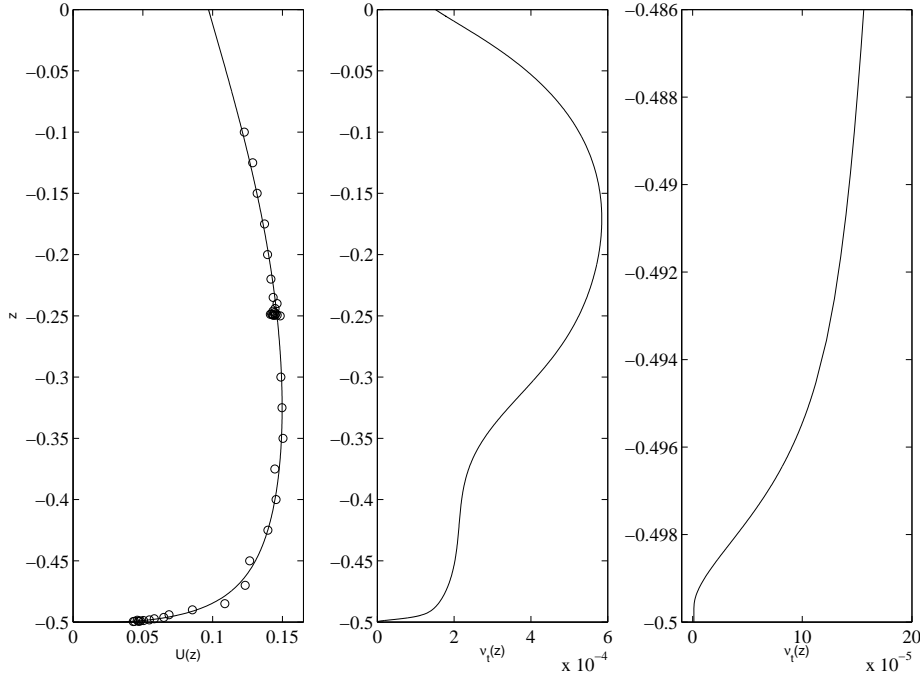

Figure 1: A current profile $U(z)$ and computed $\nu_t^0(z)$

Figure 2 below show the predicted horizontal component of the orbital velocity. We give both the results obtained when $\nu_t^0(z) \equiv 0$ (i.e., we neglect the effects of turbulence) and with $\nu_t^0(z)$ as given in Figure 1. From the picture on the left we see that both approaches yield predictions that agree very well (over most of the depth) with the experimentally obtained data. In the picture on the right we show the results obtained in the region closest to the channel bed. Here the model for pure viscous flow clearly under estimates the width of the boundary layer. When the eddy viscosity term in included, it seems that the layer is excessively dissipated. The computed wave numbers were $k = 2.149 + 4.509 \times 10^{-4}i$ for $\nu_t^0 \equiv 0$, and $k = 2.149 + 3.230 \times 10^{-3}i$ when the turbulence term in included.

6. Conclusion

We have shown that piecewise uniform meshes can be successfully applied to the study of wave-current interactions.

Within the region closest to the bed, the agreement between experimental data and the numerical simulation of the model is not entirely satisfactory. However, careful selection of the problem parameters, data parameterization, and choice of wave-functions is an area of on-going research.

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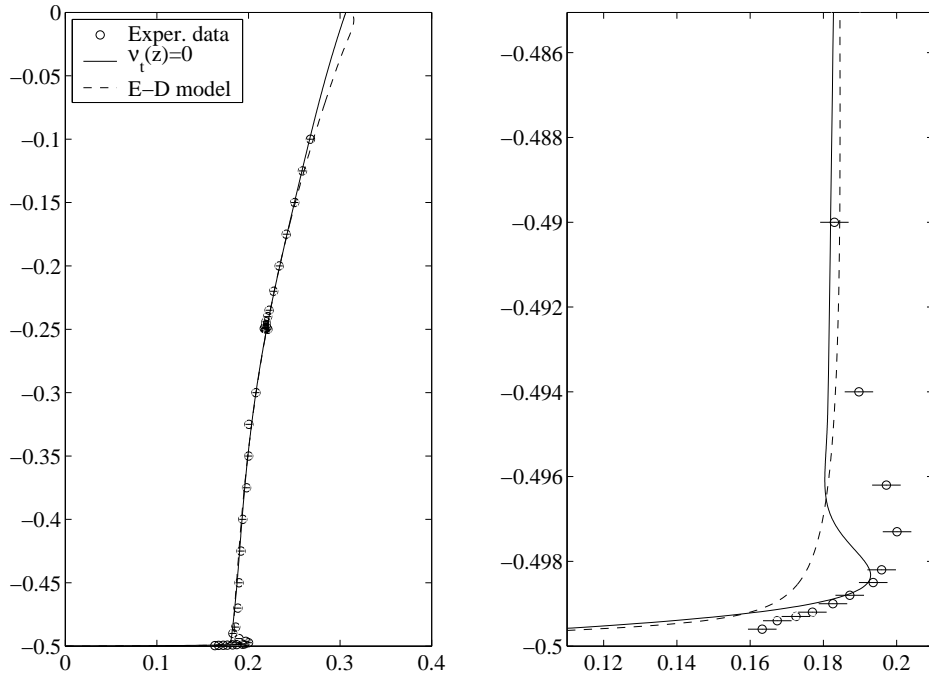
Figure 2: Prediction of the horizontal component of the vertical orbital velocity

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