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Author(s)	Duddy, Conal; Piggins, Ashley
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Aggregating Partitions

Conal Duddy and Ashley Piggins

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Department of Economics
National University of Ireland, Galway

http://www.economics.nuigalway.ie

Aggregating partitions*

Conal Duddy[†] and Ashley Piggins[‡]

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Abstract

Consider the following social choice problem. A group of individuals seek to partition a finite set X into two subsets. The individuals may disagree over the partition and an aggregation rule is applied to determine a compromise outcome. We permit collective indifference and so the outcome is a pair of disjoint subsets of X which may or may not partition X. Critically, neither subset can contain all of the elements in X. We present four normatively desirable properties that identify one aggregation rule uniquely. These properties are similar to those Young (J. Econ. Theory 9 (1974) 43-52) used in his characterization of the Borda rule.

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[†]Government of Ireland Scholar, J.E. Cairnes School of Business and Economics, National University of Ireland Galway, University Road, Galway, Ireland. Email: conal.duddy@gmail.com

[‡]J.E. Cairnes School of Business and Economics, National University of Ireland Galway, University Road, Galway, Ireland. Email: ashley.piggins@nuigalway.ie

1 Introduction

Consider the following social choice problem. A group of individuals seek to partition a finite set X into two subsets. The individuals may disagree over the partition and so an aggregation rule is applied to determine a compromise outcome. This outcome is a pair of disjoint subsets of X which may or may not partition X. This allows for collective indifference. Critically, neither of these subsets can contain all of the elements in X. Each individual's opinion can be described by a single non-empty, strict subset of X corresponding to one of the two parts of the proposed partition, with its complement corresponding to the other part.[1]

The following examples illustrate some applications of this problem. The first example is taken from List [5].

Example 1. Which worlds are possible?

In logic propositions are sometimes modelled as sets of possible worlds as opposed to sentences in a formal language. Let X denote a finite set containing at least two possible worlds. A proposition is a tautology if it is true in every possible world. A proposition is a contradiction if there is no world in which it is possible. Non-contradictory and non-tautological propositions are, therefore, non-empty, strict subsets of the set of possible worlds. Imagine that each member of a group accepts a single non-contradictory and non-tautological proposition, represented by a set of possible worlds (these are the worlds for which the individual believes the proposition to be true). The group then seeks to aggregate these views in order to form a collective judgment (these are the worlds for which the group believes the proposition to be true). How should this collective judgment be determined?

Our second motivating example is a variation on Kasher and Rubinstein [4].[2]

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[1] Our approach is analogous to models of social choice in which individuals have strict preferences, but social preference is weak.

[2] A literature on this topic has developed following Kasher and Rubinstein. Miller [6]

Example 2. Who is a J?

Consider the following "group identification problem". Each member of a group makes a judgment as to which members of that group have a certain property. This property could be a religious affiliation, for instance. Each individual believes that at least one member of the group has the property, but not all. The group then seeks to aggregate these judgments on who has the given property into a collective judgment. How should this collective judgment be determined?

Example 3. Work allocation.

A firm is opening a new, second office. Rather than hire new staff, a group of managers at the firm will send a subset of the staff at the first office over to the second office. All agree that it would not be appropriate to leave either of the two offices without any staff. How should the group decide who to send?

Example 4. Job candidates.

Imagine that a number of candidates apply for a job. Each member of the appointment panel must look at each application and decide whether or not the candidate should be invited to interview. All of the panel members agree that it would not be practical to reject all of the candidates. They also agree that they should be at least minimally discerning and not invite every candidate to interview. How should the panel decide who to invite?

Example 5. Location of public goods.

Public goods are to be provided to a region containing two cities. An assembly is to decide on how the goods should be allocated to the two cities, with the understanding that duplication of any one good would be prohibitively expensive. It would be politically unacceptable for all of the public goods to be assigned to just one of the two cities. How should the assembly decide where to locate the goods?

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is a recent example. Kasher and Rubinstein's work draws from Rubinstein and Fishburn [8].

In what follows we refer to the elements of X simply as alternatives. And we distinguish between the two disjoint subsets of X by referring to them as the set of alternatives that are "approved of" and the set of alternatives that are "disapproved of". The subset that is submitted by an individual is taken to correspond to the former, with the complement of that subset corresponding to the latter.

1.1 Aggregation rules

Imagine that X contains five alternatives, and that there are five individuals in the group. We can create a 5×5 matrix where the rows represent the individuals and the columns represent the alternatives. The elements of the matrix are either 0 or 1. The element 1 in the (i, j) position of the matrix indicates that individual i's subset contains the alternative represented by the j^{th} column. Conversely, the element 0 in the (i, j) position of the matrix indicates that individual i's subset does not contain the alternative represented by the j^{th} column. This framework goes back to Wilson [9].[3]

Suppose that $X = \{v, w, x, y, z\}$. Consider the following profile.

v	w	x	y	z
1	0	0	0	1
0	1	1	1	1
1	1	1	1	0
1	1	0	1	0
1	0	1	1	1

Table 1: A profile.

Which of the alternatives should be approved of and which disapproved of? It might be tempting to think that an alternative should be approved of if and only if the number of individuals who approve of it exceeds the number of

[3]A recent application of this framework is Dokow and Holzman [2].

individuals who disapprove of it. However, this would imply in our example that v, w, x, y and z all receive approval. This contradicts the requirement that not every alternative be approved of. This means that majority voting is not a legitimate aggregation rule for this aggregation problem.

However, consider the following rule. First, assign each alternative a number that is equal to the number of individuals who include that alternative in their subset. Second, add up these numbers and divide by the cardinality of X. For obvious reasons, we call this number "the mean". The rule then says that an alternative is approved of if and only if its number is greater than the mean, and it is disapproved of if and only if its number is less than the mean. If its number equals the mean then it is neither approved nor disapproved (one possible interpretation is that the group is indifferent as to whether it should be approved or disapproved). We call this the "Mean Rule".

As we can see from the example, the mean is $\frac{17}{5}$. So v and y are approved of and w, x and z are disapproved of.

Here is an alternative way of describing the Mean Rule. Consider again the profile in Table 1. The individual in the first row has two alternatives in her subset, and leaves three alternatives out of her subset. This means that we can assign the score of $+3$ to each alternative in her subset, and assign the score of -2 to each alternative left out of her subset. In other words, individual 1 said "no" three times, and so each "yes" is worth three points. Conversely, she said "yes" twice, and so each "no" is worth minus two points. Individual 2 said "no" just once, and so each "yes" is worth one point. She said "yes" four times, and so her "no" is worth minus four points.

This technique enables us to convert the profile into a matrix of scores.

v	w	x	y	z
$+3$	-2	-2	-2	$+3$
-4	$+1$	$+1$	$+1$	$+1$
$+1$	$+1$	$+1$	$+1$	-4
$+2$	$+2$	-3	$+2$	-3
$+1$	-4	$+1$	$+1$	$+1$

Table 2: Score matrix.

The sum of scores for alternative v is 3. Since this is greater than zero, alternative v is approved of. The sum of scores for alternative w is -2. Since this is less than zero, alternative w is disapproved of. If an alternative receives a score of zero then it is neither approved nor disapproved.

It is interesting to note that this is just another way of describing the Mean Rule. The outcome is always identical. However, what we learn through this alternative description is that the more times a person says "yes" the less each "yes" is worth. Similarly, the more times a person says "no" the less each "no" is worth. The more your "yes" is worth, the less your "no" is worth, and vice versa.

This property appears to be reasonable. Imagine that a group of managers are assessing a set of job applicants and are trying to decide who among them should go through to the next round of the recruitment process. If one of the managers approves of nearly all of the candidates, while another manager approves of only a few, we might naturally infer that the first manager is applying a lower standard than the second one is. It might make sense then to place less weight on the approvals given by the first manager, and a greater weight on the approvals given by the second manager. As we have seen, this is what the Mean Rule accomplishes.

The purpose of this paper is to show that the Mean Rule is the only rule that satisfies four normatively desirable axioms. The formal description of these axioms appears in the next section, along with the statement of the model itself. We demonstrate that the axioms are independent at the end of the paper.

2 Model

X is a finite set containing at least two alternatives. A *ballot* is a non-empty, strict subset of X. Each individual's ballot indicates the alternatives that this individual approves of. The complement of the ballot indicates the alternatives that this individual disapproves of. $\mathcal{B}$ is the set of all ballots.

$S_{\mathbb{N}}$ is the set of all finite subsets of the set of natural numbers, including the empty set. These numbers represent the individuals. A *profile* is a function $\pi : \mathcal{B} \to S_{\mathbb{N}}$ such that $\pi(B) \cap \pi(B') = \emptyset$ for all $B, B' \in \mathcal{B}$. The set $\pi(B)$ is interpreted as the set of individuals who cast ballot B. Π is the set of all possible profiles. The set of participating voters is $n(\pi) = \bigcup_{B \in \mathcal{B}} \pi(B)$.

F is the set of all functions from Π to $\mathcal{B} \cup \{\emptyset\}$. An *aggregation rule* is a pair of such functions in the set $\{(f, g) \in F \times F : f(\pi) \cap g(\pi) = \emptyset \text{ for all } \pi \in \Pi\}$. Given an aggregation rule (f, g) and a profile π, $f(\pi)$ is the set of alternatives that are approved of while $g(\pi)$ is the set of alternatives that are disapproved of. These two sets are required to be disjoint, but their union need not be X (as mentioned earlier, this can be interpreted as allowing for indifference at the collective level).

We write $|A|$ to denote the cardinality of A. Given $x \in X$ and $\pi \in \Pi$, denote by $v(x, \pi)$ the sum of $|\pi(B)|$ over all $B \in \mathcal{B}$ such that $x \in B$. In other words, $v(x, \pi)$ is the number of voters who approve of x at profile π. Additionally, let $v(\pi) = \sum_{x \in X} v(x, \pi)$.

We now make some notational remarks. First, given $\pi, \pi' \in \Pi$, let $\pi \cup \pi'$ denote a new profile that we can interpret as joining together the two profiles π and π'. That is, for all $B \in \mathcal{B}$, the set of individuals who cast ballot B at profile $\pi \cup \pi'$ is equal to $\pi(B) \cup \pi'(B)$. Also, if two profiles π and π' share no participating individuals in common, that is, $n(\pi) \cap n(\pi') = \emptyset$, then they are said to be *disjoint profiles*.

3 Axioms

The following axioms are similar to those Young [10] used in his characterization of the Borda rule.[4] A permutation σ of the set of alternatives X induces in a natural way an operation $\hat{\sigma} : \Pi \to \Pi$ with $\pi(\sigma(B)) = \hat{\sigma}\pi(B)$ for all $\pi \in \Pi$ and all $B \in \mathcal{B}$.

An aggregation rule (f, g) may satisfy the following axioms.

Neutral. For all $\pi \in \Pi$ and every permutation σ of X, $\sigma(f(\pi)) = f(\hat{\sigma}\pi)$ and $\sigma(g(\pi)) = g(\hat{\sigma}\pi)$.

This condition says that the criterion for determining whether an alternative is in the approved set, or in the disapproved set, is the same for all alternatives.

Anonymous. For all $\pi, \pi' \in \Pi$, if $|\pi(B)| = |\pi'(B)|$ for every $B \in \mathcal{B}$ then $f(\pi) = f(\pi')$ and $g(\pi) = g(\pi')$.

An aggregation rule is said to be anonymous if the identities of the individuals do not matter.

Faithful. For all $B \in \mathcal{B}$ and all $\pi \in \Pi$ such that $|\pi(B)| = 1$ and $|n(\pi)| = 1$, $f(\pi) = B$ and $g(\pi) = X - B$.

This condition says that when society consists of one individual, the approved set is simply this individual's ballot. Similarly, the disapproved set is the complement of this ballot.

Consistent. For all disjoint profiles $\pi, \pi' \in \Pi$ and all $x \in X$, $x \in f(\pi)$ and $x \notin g(\pi')$ implies $x \in f(\pi \cup \pi')$, while $x \notin f(\pi)$ and $x \in g(\pi')$ implies $x \in g(\pi \cup \pi')$.

This condition says the following. Imagine that two disjoint groups are merged together. If an alternative is approved by one subgroup, and the other

[4] See also Nitzan and Rubinstein [7]. It is worth noting that in a different context these axioms can be used to characterize Approval Voting. See Fishburn [3] and Alós-Ferrer [1].

subgroup either approves of it or is indifferent, then it must be approved by their union. Conversely, if an alternative is disapproved by one subgroup, and the other subgroup either disapproves of it or is indifferent, then it must be disapproved by their union. This is an extension of the familiar Pareto principle.

It is also possible to explain this condition in terms of the matrix representation of profiles described in the earlier section. Moreover, this way of thinking about the consistency condition also helps when working through the proof. Imagine that you take any arbitrary $n \times m$ matrix where $n \geq 1$ and $m \geq 2$. Like before, the rows correspond to the individuals and the columns correspond to the alternatives. Each row contains at least one 1 and at least one 0. Imagine that x is approved of at this profile. Now take another arbitrary $n' \times m$ matrix with $n' \geq 1$ that meets the row constraints described above. Note that for this matrix it need not be the case that $n' = n$. Imagine that x is approved of or regarded as indifferent at this profile (and so it is either in $f(\cdot)$ or $X - f(\cdot) - g(\cdot)$). Now, imagine that these two matrices are concatenated into a single $(n+n') \times m$ dimensional matrix. Consistency says that x must be approved of at this new profile.

Cancellative. For all $\pi \in \Pi$, if $v(x, \pi) = \frac{1}{2}|n(\pi)|$ for every $x \in X$ then $f(\pi) = g(\pi) = \emptyset$.

This condition says that if every alternative is approved by half of the individuals (i.e. it is in their ballots) and disapproved by the other half, then both the approved set and the disapproved set are empty.

4 Theorem

Define a particular aggregation rule (f^*, g^*) as follows. For all $\pi \in \Pi$, let $\bar{v}(\pi)$ denote $v(\pi)$ divided by $|X|$. In other words, $\bar{v}(\pi)$ is the mean number of votes received by the alternatives. For all $\pi \in \Pi$, $f^*(\pi) = \{x \in X \mid v(x, \pi) > \bar{v}(\pi)\}$ and $g^*(\pi) = \{x \in X \mid v(x, \pi) < \bar{v}(\pi)\}$. This rule says that an alternative is approved of if it receives a number of votes greater than the mean, and it is disapproved of if this number is less than the mean. If an alternative receives a number of votes equal to the mean then there is a tie, and it is neither approved nor disapproved.

Our theorem is the following.

Theorem. *The aggregation rule* (f^*, g^*) *is the only one that is neutral, faithful, consistent and cancellative.*

Lemma 1. *If an aggregation rule is consistent and cancellative then it is anonymous.*

Proof. Let (f, g) be an aggregation rule that is consistent and cancellative. Take any three disjoint profiles $\pi, \pi', \pi^* \in \Pi$ such that $|\pi(B)| = |\pi'(B)|$ and $|\pi(B)| = |\pi^*(X - B)|$ for all $B \in \mathcal{B}$. Suppose that there exists $x \in X$ such that $x \in f(\pi)$ and $x \notin f(\pi')$. Cancellation implies both that $f(\pi \cup \pi^*) = \emptyset$ and that $f(\pi' \cup \pi^*) = \emptyset$. However, if $x \notin g(\pi^*)$ then consistency would imply $x \in f(\pi \cup \pi^*)$, while on the other hand if $x \in g(\pi^*)$ then consistency would imply $x \in f(\pi' \cup \pi^*)$. If we suppose instead that $x \in g(\pi)$ and $x \notin g(\pi')$ we come to a similar contradiction by simply interchanging f and g in the above argument. So it must be that $x \in f(\pi) \leftrightarrow x \in f(\pi')$ and $x \in g(\pi) \leftrightarrow x \in g(\pi')$. $\square$

Given any two profiles $\pi, \pi' \in \Pi$, let $\pi + \pi'$ denote an arbitrary profile such that the number of individuals who cast ballot B at profile $\pi + \pi'$ is equal to $|\pi(B)| + |\pi'(B)|$ for all $B \in \mathcal{B}$. For anonymous aggregation rules the number of individuals who have cast each possible ballot is the only information that is relevant. A consequence of Lemma 1 is that a cancellative aggregation rule (f, g) is consistent if and only if it has the following property. For all profiles $\pi, \pi' \in \Pi$ and all $x \in X$, $x \in f(\pi)$ and $x \notin g(\pi')$ implies $x \in f(\pi + \pi')$, while $x \notin f(\pi)$ and $x \in g(\pi')$ implies $x \in g(\pi + \pi')$. In what follows we take this to be an immediate consequence of consistency for aggregation rules that are cancellative.

For every $\pi \in \Pi$, denote by $\ominus\pi$ the profile satisfying $\pi(B) = \ominus\pi(X - B)$ for all $B \in \mathcal{B}$.

Lemma 2. *If (f, g) is an aggregation rule that is consistent and cancellative then, for all $\pi \in \Pi$, $f(\pi) = g(\ominus\pi)$.*

Proof. Let (f, g) be an aggregation rule that is consistent and cancellative. Take any profile $\pi \in \Pi$. Suppose that $x \in f(\pi)$ and $x \notin g(\ominus\pi')$. By consistency, we have $x \in f(\pi + \ominus\pi')$. However cancellation implies that $f(\pi + \ominus\pi') = \emptyset$. Similarly, $x \notin f(\pi)$ and $x \in g(\ominus\pi)$ implies, by consistency, that $x \in g(\pi + \ominus\pi)$ while cancellation implies that $g(\pi + \ominus\pi) = \emptyset$. □

Lemma 3. *If (f, g) is an aggregation rule that is consistent and cancellative then, for all $\pi, \pi' \in \Pi$ and all $x \in X$, $x \in f(\pi + \pi') \cup g(\pi + \pi')$ implies $x \in f(\pi) \cup f(\pi') \cup g(\pi) \cup g(\pi')$.*

Lemma 3 says the following. Consistency and cancellation together imply that if x is regarded as indifferent at profile π and profile π' then it must also be regarded as indifferent at profile $\pi + \pi'$.

Proof. Let (f, g) be an aggregation rule that is consistent and cancellative. Assume there exists $x \in X$ and $\pi, \pi' \in \Pi$ such that $x \in f(\pi + \pi')$ and $x \notin f(\pi) \cup g(\pi) \cup f(\pi') \cup g(\pi')$. Since $x \notin f(\pi)$ and $x \notin f(\pi')$, Lemma 2 implies that $x \notin g(\ominus\pi)$ and $x \notin g(\ominus\pi')$. Consistency implies then that $x \in f(\ominus\pi + \ominus\pi' + \pi + \pi')$. However, cancellation implies that $f(\ominus\pi + \ominus\pi' + \pi + \pi') = \emptyset$. If we suppose instead that $x \in g(\pi + \pi')$ we come to a similar contradiction. □

Lemma 4. *If (f, g) is an aggregation rule that is consistent, cancellative and neutral then the following is true. For all $\pi \in \Pi$, $v(x, \pi) = v(y, \pi)$ for every $x, y \in X$ implies $f(\pi) = g(\pi) = \emptyset$.*

Lemma 4 says the following. Consistency, cancellation and neutrality together imply that if every alternative receives the same number of votes then no alternative is approved nor disapproved.

Proof. Let (f, g) be an aggregation rule that is consistent, cancellative and neutral. Take any profile $\pi \in \Pi$ such that $v(x, \pi) = v(y, \pi)$ for all $x, y \in X$. Suppose that there exists $z \in f(\pi)$. We know, by definition, that $f(\pi)$ cannot contain all of the alternatives in X so there must exist $w \in X - f(\pi)$. We also know, by Lemma 2, that $z \in g(\ominus\pi)$. Let σ be a permutation of X that inverts z and w. Note that at the resulting profile $\hat{\sigma}\pi$ each alternative receives the same numbers of votes as it does at profile π. So, just as cancellation implies $g(\pi + \ominus\pi) = \emptyset$, it also implies $g(\hat{\sigma}\pi + \ominus\pi) = \emptyset$. However, neutrality implies that $z \notin f(\hat{\sigma}\pi)$ and so consistency implies $z \in g(\hat{\sigma}\pi + \ominus\pi)$, a contradiction.

If we suppose instead that $z \in g(\pi)$ we come to a similar contradiction by interchanging everywhere f and g in the above argument. □

For every $\pi \in \Pi$, denote by $\otimes\pi$ an arbitrary profile such that $\otimes\pi(B) = \emptyset$ for every non-singleton $B \in \mathcal{B}$ and $v(x, \pi) = v(x, \otimes\pi)$ for all $x \in X$.

Lemma 5. *If (f, g) is an aggregation rule that is consistent, cancellative and neutral then, for all $\pi, \pi' \in \Pi$, $v(x, \pi) = v(x, \pi')$ for all $x \in X$ implies $f(\pi) = f(\pi')$ and $g(\pi) = g(\pi')$.*

Lemma 5 says the following. Consistency, cancellation and neutrality together imply that any profile π produces the same pair of social subsets as any other profile π' provided that each alternative receives the same number of votes in π' as it does in π.

Proof. Let (f, g) be an aggregation rule that is consistent, cancellative and neutral. This lemma is directly implied by Lemma 1 if $|X| = 2$, so we assume here that $|X| \geq 3$. Take any $B_1, B_2, B_3 \in \mathcal{B}$ with $B_1 = B_2 \cup B_3$ and $B_2 \cap B_3 = \emptyset$. Let $\pi_1, \pi_2 \in \Pi$ be two profiles such that at π_1 there is just one individual and that individual casts ballot B_1, while at π_2 there are two individuals and they cast ballots B_2 and B_3.

Cancellation implies that $f(\pi_1 + \ominus\pi_1) = g(\pi_1 + \ominus\pi_1) = \emptyset$. Lemma 4 implies that $f(\pi_2 + \ominus\pi_1) = g(\pi_2 + \ominus\pi_1) = \emptyset$.

Take any profile $\pi \in \Pi$. By consistency, we know that if $x \in f(\pi + \pi_1)$ then $x \in f(\pi + \pi_1 + \pi_2 + \ominus\pi_1)$. We also know by consistency that if x is in $f(\pi + \pi_1 + \pi_2 + \ominus\pi_1)$ then it cannot be in $g(\pi + \pi_1)$, since this would imply (given that $f(\pi_2 + \ominus\pi_1) = \emptyset$) that $x \in g(\pi + \pi_1 + \pi_2 + \ominus\pi_1)$ which contradicts the requirement that $f(\cdot)$ and $g(\cdot)$ are disjoint. In other words, if x is in $f(\pi + \pi_1 + \pi_2 + \ominus\pi_1)$ then either it is in $f(\pi + \pi_1)$ or it is in neither $f(\pi + \pi_1)$ nor $g(\pi + \pi_1)$. Lemma 3 allows us to rule out the second of these two possibilities. This leaves us with

$$f(\pi + \pi_1) = f(\pi + \pi_1 + \pi_2 + \ominus\pi_1).$$

In a similar way, given that $f(\pi_1 + \ominus\pi_1) = g(\pi_1 + \ominus\pi_1) = \emptyset$, consistency and Lemma 3 also imply

$$f(\pi + \pi_2) = f(\pi + \pi_2 + \pi_1 + \ominus\pi_1).$$

Since the right-hand side is the same in both of these equations, we have $f(\pi + \pi_1) = f(\pi + \pi_2)$. What this means, very loosely speaking, is that we can take from the ballot box any ballot that contains more than one alternative and break it up into two ballots without affecting the outcome. Iteration of this allows us to see that a logical consequence is that $f(\pi) = f(\otimes\pi)$.

Take any profile π' such that $v(x, \pi) = v(x, \pi')$ for all $x \in X$. We know that $f(\pi) = f(\otimes\pi)$ and $f(\pi') = f(\otimes\pi')$. Lemma 1 implies $f(\otimes\pi) = f(\otimes\pi')$ and so $f(\pi) = f(\pi')$.

To see that $g(\pi) = g(\pi')$ simply interchange everywhere f and g. □

Lemma 6. *If (f, g) is an aggregation rule that is consistent, cancellative and neutral then for all $x \in X$ and all $\pi, \pi' \in \Pi$ the following is true. If $v(x, \pi) = v(x, \pi')$ and $v(\pi) = v(\pi')$ then $x \in f(\pi) \leftrightarrow x \in f(\pi')$ and $x \in g(\pi) \leftrightarrow x \in g(\pi')$.*

Lemma 6 says the following. Suppose that we take any alternative x and any pair of profiles π, π'. Imagine that (i) the number of votes received by x

at both π and π' is the same, and (ii) the total number of votes received by all of the alternatives together is the same at both π and π'. Then consistency, cancellation and neutrality jointly imply that either x is approved at both profiles, or x is disapproved at both profiles, or x is neither approved nor disapproved at both profiles.

Proof. If $|X| = 2$ then Lemma 6 is implied by Lemma 1 and so we assume that $|X| > 2$ in what follows. Let (f, g) be an aggregation rule that is consistent, cancellative and neutral. Take any $x \in X$ and any two profiles $\pi, \pi' \in \Pi$ such that $v(x, \pi) = v(x, \pi')$ and $v(\pi) = v(\pi')$. Assume that $x \in f(\pi)$ and $x \notin f(\pi')$. Lemma 5 implies that $x \in f(\otimes\pi)$ and $x \notin f(\otimes\pi')$.

Consider a sequence of profiles beginning with $\otimes\pi$ and ending with $\otimes\pi'$ such that (i) each step in the sequence consists in one individual replacing the single alternative in her ballot with an alternative from the complement of her ballot and (ii) the number of individuals who cast ballot $\{x\}$ remains fixed throughout the sequence. There must exist two profiles π^* and π^{**} that are adjacent to one another in that sequence with $x \in f(\pi^*)$ and $x \notin f(\pi^{**})$.

Lemma 2 implies that $x \notin g(\ominus\pi^{**})$. Let $\hat{\pi} = \pi^* + \ominus\pi^{**}$. Consistency implies that $x \in f(\hat{\pi})$.

Suppose that in the move from π^* to π^{**} an individual replaced alternative y with z. Let σ be the permutation of X that inverts y and z only. Neutrality implies that $x \in f(\hat{\sigma}\hat{\pi})$. Consistency implies that $x \in f(\hat{\pi} + \hat{\sigma}\hat{\pi})$.

We know, however, that $v(w, \hat{\pi} + \hat{\sigma}\hat{\pi}) = \frac{1}{2}\,|n(\hat{\pi} + \hat{\sigma}\hat{\pi})|$ for all $w \in X$. To see this first note that the profiles π^*, π^{**}, $\hat{\sigma}\pi^*$ and $\hat{\sigma}\pi^{**}$ are identical when we restrict attention to $X - \{y, z\}$. It follows from this that $v(u, \hat{\pi} + \hat{\sigma}\hat{\pi}) = \frac{1}{2}\,|n(\hat{\pi} + \hat{\sigma}\hat{\pi})|$ for all $u \in X - \{y, z\}$. Next we consider alternatives y and z. We know that $v(y, \ominus\pi^*) = v(y, \ominus\pi^{**}) - 1$ and that, by definition, $v(y, \pi^*) + v(y, \ominus\pi^*) = |n(\pi^*)|$. Hence $v(y, \pi^*) + v(y, \ominus\pi^{**}) - 1 = |n(\pi^*)|$. In other words, $v(y, \hat{\pi}) - 1 = |n(\pi^*)|$. In a similar way we can see that $v(z, \hat{\pi}) + 1 = |n(\pi^*)|$. The permutation σ simply inverts y and z, and so we have $v(y, \hat{\sigma}\hat{\pi}) + 1 = |n(\pi^*)|$ and $v(z, \hat{\sigma}\hat{\pi}) - 1 = |n(\pi^*)|$. It follows that

$v(y, \hat{\pi}) + v(y, \hat{\sigma}\hat{\pi}) = 2\,|n(\pi^*)|$ and $v(z, \hat{\pi}) + v(z, \hat{\sigma}\hat{\pi}) = 2\,|n(\pi^*)|$. In other words, $v(y, \hat{\pi} + \hat{\sigma}\hat{\pi}) = 2\,|n(\pi^*)|$ and $v(z, \hat{\pi} + \hat{\sigma}\hat{\pi}) = 2\,|n(\pi^*)|$. Given that $2\,|n(\pi^*)| = \frac{1}{2}\,|n(\hat{\pi} + \hat{\sigma}\hat{\pi})|$ we are left with $v(w, \hat{\pi} + \hat{\sigma}\hat{\pi}) = \frac{1}{2}\,|n(\hat{\pi} + \hat{\sigma}\hat{\pi})|$ for all $w \in X$.

Cancellation then implies that $f(\hat{\pi} + \hat{\sigma}\hat{\pi}) = \emptyset$, which contradicts $x \in f(\hat{\pi} + \hat{\sigma}\hat{\pi})$. To see that $x \in g(\pi)$ and $x \notin g(\pi')$ would also have led to a contradiction, simply interchange everywhere f and g. $\square$

Proof of the main theorem. Let (f, g) be an aggregation rule that is consistent, cancellative, neutral and faithful. Take any alternative $x \in X$. Let $\pi \in \Pi$ be any profile such that $v(x, \pi) > \bar{v}(\pi)$. Denote by q the quotient, and by r the remainder, of the division of $v(\pi) - v(x, \pi)$ by $|X| - 1$. Let π^* be the profile with $\pi^*(\{x\}) = n(\pi^*) = \{1, \ldots, q\}$. Construct another profile π^{**} where (i) $|n(\pi^{**})| = v(x, \pi) - q$, (ii) every ballot cast contains x and (iii) the total number of votes received by all of the other alternatives together is r. It is easy to construct such a profile since r is strictly less than the number of alternatives in $X - \{x\}$.

Cancellation implies that $g(\pi^* + \ominus\pi^*) = \emptyset$. And, since at profile π^{**} every ballot contains x, it is clear that faithfulness and consistency imply $x \in f(\pi^{**})$. So consistency implies $x \in f(\pi^* + \ominus\pi^* + \pi^{**})$. Let $\hat{\pi} = \pi^* + \ominus\pi^* + \pi^{**}$. We know that $v(x, \pi^* + \ominus\pi^*) = q$ and $v(x, \pi^{**}) = v(x, \pi) - q$, and so $v(\hat{\pi}, x) = v(\pi, x)$. Also, we know that $v(\pi^* + \ominus\pi^*) - v(x, \pi^*) = q(|X| - 1)$ and $v(\pi^{**}) - v(x, \pi^{**}) = r$, giving us $v(\hat{\pi}) - v(x, \hat{\pi}) = q(|X| - 1) + r$. Recall that, by definition,

$$q + \frac{r}{|X| - 1} = \frac{v(\pi) - v(x, \pi)}{|X| - 1}$$

and so we have $v(\pi) - v(x, \pi) = v(\hat{\pi}) - v(x, \hat{\pi})$. Since $v(x, \hat{\pi}) = v(x, \pi)$ and $v(\hat{\pi}) = v(\pi)$, Lemma 6 implies that $x \in f(\pi)$.

Let $\tilde{\pi}$ be any profile such that $v(x, \tilde{\pi}) < \bar{v}(\tilde{\pi})$. We know that $v(x, \ominus\tilde{\pi}) > \bar{v}(\ominus\tilde{\pi})$. We have seen that this implies $x \in f(\ominus\tilde{\pi})$, and so Lemma 2 implies

$x \in g(\tilde{\pi})$.

Let π' be any profile such that $v(x, \pi') = \bar{v}(\pi')$. Let π'' be the profile with $\pi''(\{x\}) = n(\pi'') = \{1, 2, \ldots, v(x, \pi')\}$. Cancellation implies that $f(\pi'' + \ominus\pi'') = g(\pi'' + \ominus\pi'') = \emptyset$. Since $v(x, \pi'' + \ominus\pi'') = v(x, \pi')$ and $v(\pi'' + \ominus\pi'') = v(\pi')$, Lemma 6 implies that $x \notin f(\pi')$ and $x \notin g(\pi')$.

This completes the proof of the theorem.

Independence of the axioms

If there are just two alternatives in X then any aggregation rule that is faithful, cancellative and consistent must also be neutral. The four properties are logically independent if $|X| \geq 3$ and so we assume this to be the case for the first of the following aggregation rules.

1. Let Y be a non-empty, strict subset of X. Define an aggregation rule (f_1, g_1) as follows. For all $\pi \in \Pi$,

$$(f_1(\pi), g_1(\pi)) = \begin{cases} (Y, \emptyset) & \text{if } f^*(\pi) = \emptyset \text{ and } \bar{v}(\pi) > \frac{1}{2}|n(\pi)| \\ (\emptyset, Y) & \text{if } f^*(\pi) = \emptyset \text{ and } \bar{v}(\pi) < \frac{1}{2}|n(\pi)| \\ (f^*(\pi), g^*(\pi)) & \text{otherwise.} \end{cases}$$

   The aggregation rule (f_1, g_1) is faithful, consistent and cancellative but not neutral.

2. For all $\pi \in \Pi$, $f_2(\pi) = g^*(\pi)$ and $g_2(\pi) = f^*(\pi)$. The aggregation rule (f_2, g_2) is neutral, consistent and cancellative but not faithful.

3. For all $\pi \in \Pi$, $f_3(\pi) = \{x \in X \mid v(x, \pi) = |n(\pi)|\}$ and $g_3(\pi) = \{x \in X \mid v(x, \pi) = 0\}$. An alternative is approved or disapproved only when there is complete consensus among the individuals. The aggregation rule (f_3, g_3) is neutral, faithful and cancellative but not consistent.

4. For all $\pi \in \Pi$, if there exists $B \in \mathcal{B}$ such that $1 \in \pi(B)$ then $f_4(\pi) = B - g^*(\pi)$ and $g_4(\pi) = (X - B) - f^*(\pi)$. Otherwise $f_4(\pi) = f^*(\pi)$ and $g_4(\pi) = g^*(\pi)$. The aggregation rule (f_4, g_4) is neutral, faithful and consistent but not cancellative.

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