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2-Uniform Covering Groups of Elementary Abelian 2-Groups

PhD Thesis

by

Dana Saleh

Supervisor: Dr. Rachel Quinlan

School of Mathematical and Statistical Sciences

University of Galway

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UNIVERSITY OF GALWAY

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Declaration

I declare that this thesis represents my own work, and I have not obtained a degree at the University of Galway, or elsewhere, on the basis of the work described in the thesis. The collaborative contributions have been indicated clearly and acknowledged. Due references have been provided on all supporting kinds of literature and resources.

Abstract

A covering group of an elementary abelian group of order $p^{(n)}$ is a group G of order $p^{n+\binom{n}{2}}$ consisting of the following data:

- G has generators $x_1, \ldots, x_n$.
- The commutator subgroup of G is equal to the centre and is an elementary abelian group of order $p^{\binom{n}{2}}$ or rank $\binom{n}{2}$ generated by $\binom{n}{2}$ simple commutators $[x_i, x_j]$.
- $G/Z(G)$ is an elementary abelian group of order $p^{(n)}$, generated by $\bar{x}_1, \ldots, \bar{x}_n$, where $\bar{x}$ denotes the coset $xZ(G)$ of $Z(G)$ in G.

In general, an elementary abelian group has many non-isomorphic covering groups whose enumeration and/or classification is a difficult problem. Different covering groups are determined by specifying the pth powers of the generators $\bar{x}_i$ as elements of the elementary abelian group G'. For an odd prime p, the problem can be expressed purely in terms of linear algebra, because the mapping from G to G′ that takes every element to its p^{th} power is a linear transformation of F_p-vector spaces, from G/G' to G'. For $p = 2$, this is not the case, and the subject has more of a combinatorial flavour. An invariant of covering groups of C_2^n is the minimum number k of distinct squares of elements in a generating set. If $k = 1$, the corresponding covering groups are called uniform and it is known that their isomorphism types are in bijective correspondence with the isomorphism types of simple undirected graphs on n vertices. The goal of this thesis is to extend this graph correspondence to the case $k = 2$, which is called 2-uniform. Graphs that encode 2-uniform covering groups are equipped with vertex and edge colourings, both with two colours. We again obtain a correspondence between group and graph isomorphism types. Theorem 3.12 presents a class of graphs that includes at least one representative of every isomorphism type of covering groups. Many groups are represented by a single graph in this class, and the exceptions are explored in Chapters 4 through 8. For a 2-uniform covering group G of C_2^n, the *uniform rank* $\rho(G)$ of G is defined as the maximum number of elements with the same square in a minimal generating set, and the *uniform corank* is $n - \rho(G)$. Theorem 3.8 establishes

that covering groups whose uniform corank is at least 4 are almost always represented by exactly one graph in the class described in Theorem 3.12. The exceptions to this are investigated in Chapter 4, and the main results are documented in Theorems 4.6, 4.8 and 4.10. Further failures of bijectivity in the correspondence between group isomorphism types and the graphs of Theorem 3.12 occur for all covering groups of uniform corank 1, some of uniform corank 2 or 3, and some whose uniform rank is at most 3. These cases are explored in Chapter 5 (on groups of corank 3), Chapter 6 (on corank 2), and Chapter 7 (on corank 1). Chapter 7 presents a refinement of the correspondence of Theorem 3.12, for the special case of covering groups of uniform corank 1. Finally, groups whose uniform rank is at most 3 are considered in Chapter 8.

Chapter 1

Introduction

For a finite group G, a *Schur cover* or *covering group* or *stem cover* of G is a finite group H with a normal subgroup $N \subseteq Z(H) \cap H'$ with $H/N \cong G$, that has maximal order amongst all groups with this property. A pair of groups (H, N) with $N \subseteq Z(H) \cap H'$ and $H/N \cong G$ is referred to as a *stem extension* of G. Thus a covering group corresponds to a stem extension with groups of maximal possible order. A group may have multiple non-isomorphic covering groups, but in all cases the normal subgroup N is isomorphic to the Schur Multiplier $M(G)$ of G. If $G \cong F/R$ for a free group F of finite rank, and if (H, N) is a stem extension of R, then H is the image of F under a homomorphism whose kernel contains $[F, R]$, and which maps $F' \cap R$ onto N. In this situation, the Schur Multiplier of G is given by

$$M(G) \cong (F' \cap R)/[F, R]. \tag{1.1}$$

Furthermore, $(F' \cap R)/[F, R]$ is the torsion subgroup of the abelian group $R/[F, R]$, and every covering group of G may be realized as $(F/[R, F])/C$, where C is a torsion-free complement of $(F' \cap R)/[F, R]$ in $R/[F, R]$. We refer to Chapter 7 of [4] for an account of these points, and of the general theory of covering groups. The study of covering groups was initiated by Schur in the early years of the 20th century, in his development of the projective representation theory of finite groups [5, 6].

If H is a covering group of G, with $H/N \cong G$, where $N \subseteq Z(H) \cap H'$, then the Schur multiplier of G captures the connection between expressions for the identity element of G as products of commutators, and the analogous expressions involving representatives of the corresponding cosets of N in H. A non-identity element of $(F' \cap R)/[F,R]$ corresponds to a product of commutators in F that maps to a non-identity element of N in a mapping from $F/[F,R]$ onto H.

Example 1.1. If G is a finite cyclic group of order k, then we may take F to be an infinite cyclic group generated by X, with $R = \langle X^k \rangle$. Then F' is trivial, $M(G)$ is trivial, and (G, id) is the only stem extension of G. This is consistent with the observation that if H/A is cyclic for a central subgroup A of any finite group H, then H is abelian and H' is trivial.

Example 1.2. Suppose that $G \cong C_2 \times C_2$ and let (H, N) be a stem extension of G. Let x and y be elements of H with $H/N = \langle xN, yN \rangle$. Then

$$[x,y] = x^{-1}y^{-1}xy \in N,$$

and since $N \subseteq Z(H)$, we have $[x,y] = yx^{-1}y^{-1}x$ also. Moreover, $y^2 \in N$, $x^2 \in N$, and

$$\begin{aligned} [x,y]^2 &= (x^{-1}y^{-1}xy)(yx^{-1}y^{-1}x) \\ &= x^{-1}y^{-1}xy^2x^{-1}y^{-1}x \\ &= y^2x^{-1}y^{-1}xx^{-1}y^{-1}x \\ &= y^2x^{-1}y^{-2}x \\ &= \mathrm{id}. \end{aligned}$$

Since no other non-identity commutators can occur in H, it follows that $|N| = 1$ or 2, and H has order 4 or 8. The two non-abelian groups of order 8 are covering groups of $C_2 \times C_2$.

The theme of this thesis is the classification, up to isomorphism, of covering groups of elementary abelian 2-groups. For a prime p and positive integer n, the elementary abelian p-group of order p^n is the direct product of n copies of the cyclic group C_p of order p. Written additively, it is the vector space of dimension n over the field $\mathbb{F}_p$ of p elements. Elementary abelian groups possess a particular abundance of distinct covering groups.

In [7], Ursula Martin Webb investigates the number $A(p, n)$ of all isomorphism types of covering groups of C_p^n for odd p, and shows that it is bounded below by

$$\frac{p^{n\binom{n}{2}}}{|GL(n,p)|}\left(p^{-3n^2/2+9n/2-4}(p^n-1)(p+p^{n-1}-1)(p-1)+1\right).$$

This result alone shows that the elementary abelian group of order 81 has at least 12555 distinct covering groups.

In [3], it is shown that the isomorphism classes of covering groups of C_2^n that possess a generating set consisting of n elements all having the same square, are in bijective correspondence with the isomorphism types of simple undirected graphs on n vertices. These *uniform* examples represent a tiny proportion of all covering groups of C_2^n.

Our aim in this thesis is not to attempt an complete enumeration of all isomorphism types, but to extend this connection between groups and graphs. We study another class of covering groups of elementary abelian 2-groups, namely those with a generating set including elements with exactly two distinct squares. We refer to such groups as 2-uniform, and they are the main object of attention.

The thesis is organised as follows. Chapter 2 contains some essential ideas from group theory that are used throughout the thesis, and explains the main themes. The detailed study of 2-uniform covering groups commences in Chapter 3, where the main theoretical machinery is developed, including a method of representing these groups with 2-coloured graphs. The remainder of the thesis is devoted to the analysis of various special cases, involving groups that admit multiple non-isomorphic graph representations according to our scheme.

Chapter 2

Background

In this chapter we present some group theory definitions and properties that are essential to our study.

Definition 2.1. *The centre of a group* G *is the set of elements that commute with every element of* G*. It is denoted by* $Z(G)$*.*

$$Z(G) = \{z \in G \mid \forall g \in G, zg = gz\}.$$

Definition 2.2. *For elements* x*,* y *of a group* G*, the commutator of* x *and* y *is the element* $[x, y]$*, defined by*

$$[x, y] = x^{-1}y^{-1}xy.$$

We observe that $[x, y]^{-1} = [y, x]$, for all group elements x, y.

Definition 2.3. *The commutator subgroup of a group* G *is the subgroup generated by the commutators of its elements, and is denoted by* G'*.*

$$G' = [G, G] = \langle x^{-1}y^{-1}xy \mid x, y \in G\rangle.$$

The commutator subgroup G' is the unique smallest normal subgroup of G for which G/G' is abelian. It can range from the identity subgroup to the whole group (in the case of a perfect group).

We refer to any element of G' that has the form $[x, y]$ for some $x, y \in G$, as a *simple commutator*. Every element of G' can be written as a product of simple commutators.

The length of a commutator is the least number of simple commutators required to write the element.

Definition 2.4. *A group* G *is* nilpotent of class 2 *if* G *is not abelian and* $G' \subseteq Z(G)$. *This means that every element of* G *of the form* $x^{-1}y^{-1}xy$ *belongs to the center.*

For any elements x and y of any group G, the elements xy and yx are related by the equation

$$yx = xy(y^{-1}x^{-1}yx) = xy[y, x].$$

This means that we can rearrange any appearance yx to $xy[y, x]$ and this introduces the commutator $[y, x]$ to the expression. Working with commutators is generally a complicated matter, but in a group that is *nilpotent* group of class 2, all commutators belong to the centre of G, so they commute with all elements of the group.

We will list here some properties of the commutator algebra in a nilpotent group of class 2, that will be used throughout this thesis.

Lemma 2.5. *Let* G *be a nilpotent group of class 2. For any integer* k

$$[x, y]^k = [x^k, y] = [x, y^k], \ \forall x, y \in G.$$

Proof. This can be proved by induction on k, using the fact that all commutators belong to the center.

1. Base case $k = 1 : [x, y]^1 = [x^1, y] = [x, y^1]$.

2. Induction hypothesis: Suppose that the result holds for $[x, y]^{k-1}$: $[x, y]^{k-1} = [x^{k-1}, y] = [x, y^{k-1}]$.

3. We now consider $[x, y]^k$.

$$\begin{aligned}
[x^k, y] &= x^{-k}y^{-1}x^k y \\
&= x^{-k}y^{-1}x^{k-1}xy \\
&= = x^{-k}y^{-1}x^{k-1}yx[x, y] \\
&= x^{-1}x^{-(k-1)}y^{-1}x^{k-1}yx[x, y] \\
&= x^{-1}[x^{k-1}, y]x[x, y] \\
&= x^{-1}x[x, y]^{k-1}[x, y] \\
&= [x, y]^k.
\end{aligned}$$

A similar argument shows that $[x, y^k] = [x, y]^k$. □

The following lemma is an extension of Lemma 2.5.

Lemma 2.6. *Let* G *be a nilpotent group of class 2 and let* x, y, z *be elements of* G. *Then*

1. $[x, z][y, z] = [xy, z]$

2. $[z, x][z, y] = [z, xy]$

Proof. For 1.,

$$\begin{aligned}
[xy, z] &= (xy)^{-1}z^{-1}xyz \\
&= y^{-1}x^{-1}z^{-1}xyz \\
&= y^{-1}x^{-1}z^{-1}xzy[y, z] \\
&= y^{-1}[x, z]y[y, z] \\
&= y^{-1}y[x, z][y, z] \\
&= [x, z][y, z]
\end{aligned}$$

We can prove 2. in a similar way. □

We now consider powers of products in nilpotent groups of class 2.

Lemma 2.7. *Let* G *be a nilpotent group of class 2. For all* $x, y \in G$ *and all positive integers* k*:*

$$(xy)^k = x^k y^k [y, x]^{\frac{k(k-1)}{2}}.$$

Proof. We prove the lemma by induction on k.

1. Base case $k = 2$: $(xy)^2 = xyxy = xxy[y, x]y = x^2 yy[y, x] = x^2 y^2 [y, x]$.

2. Induction hypothesis: We assume that

$$(xy)^{k-1} = x^{k-1} y^{k-1} [y, x]^{\frac{(k-1)(k-2)}{2}}.$$

3. Now we consider $(xy)^k$:

$$\begin{aligned}
(xy)^k &= (xy)^{k-1}(xy) = x^{k-1}y^{k-1}[y,x]^{\frac{(k-1)(k-2)}{2}}(xy) \\
&= x^{k-1}y^{k-1}(xy)[y,x]^{\frac{(k-1)(k-2)}{2}} \\
&= x^{k-1}xy^{k-1}[y^{k-1},x]y[y,x]^{\frac{(k-1)(k-2)}{2}} \\
&= x^k y^k [y^{k-1},x][y,x]^{\frac{(k-1)(k-2)}{2}} \\
&= x^k y^k [y,x]^{(k-1)}[y,x]^{\frac{(k-1)(k-2)}{2}} \\
&= x^k y^k [y,x]^{\frac{k(k-1)}{2}}.
\end{aligned}$$

□

Next we introduce some objects that have a central role in our study.

Definition 2.8. *A group* G *is a* $p-$group *if every element of* G*, except the identity, has order equal to a power of the same prime* p*.*

Definition 2.9. *The elementary abelian group of order* p^n*, for a prime* p*, is the direct product of* n *cyclic groups of order* p*.*

$$C_p^n \cong C_p \times C_p \times \cdots \times C_p.$$

If the i^{th} copy of C_p is generated by x_i, then the elements of the elementary abelian group have the form: $x_1^{i_1} x_2^{i_2} .... x_n^{i_n}; 0 \leqslant i_k < p$.

The group operation is commutative and is given by:

$$(x_1^{i_1}x_2^{i_2}\dots x_n^{i_n})\cdot(x_1^{j_1}\,x_2^{j_2}\dots x_n^{j_n}) = x_1^{i_1+j_1}x_2^{i_2+j_2}\dots x_n^{i_n+j_n}\ ;\ i_k+j_k \text{ is modulo } p.$$

If the group operation is written additively, C_p^n is just the vector space of dimension n over the field $\mathbb{F}_p$ of p elements.

Example 2.10. For $p = 3$ and $n = 2$, the elementary abelian group of order 3^2 is $< x_1 > \times < x_2 >$ with the elements:

$$Id,\ x_1,\ x_2,\ x_1^2,\ x_2^2,\ x_1x_2,\ x_1^2x_2,\ x_1x_2^2,\ x_1^2x_2^2.$$

Recalling the general description of a covering group of finite groups in Chapter 1, we now give a description of covering groups of elementary abelian groups. Covering groups of elementary abelian 2-groups are the focus of attention in this study.

Definition 2.11. *A covering group of an elementary abelian group of order* p^n *is a group* G *of order* $p^{n+\binom{n}{2}}$ *consisting of the following data:*

- *G has generators* $x_1, \dots, x_n$.
- *The commutator subgroup of G is equal to the centre and is an elementary abelian group of order* $p^{\binom{n}{2}}$ *or rank* $\binom{n}{2}$ *generated by* $\binom{n}{2}$ *simple commutators* $[x_i, x_j]$.
- $G/Z(G)$ *is an elementary abelian group of order* p^n, *generated by* $\bar{x}_1, \dots, \bar{x}_n$, *where* $\bar{x}$ *denotes the coset* xZ *of* Z *in* G.

The below example shows that an elementary abelian group can have multiple non-isomorphic covering groups.

Example 2.12. The dihedral group D_8 is a non-abelian group of order 8 and it is a covering group of $C_2 \times C_2$, since:

- D_8 has a generating set $\{x, y\}$, where $x^4 = y^2 = 1$.
- The commutator subgroup of D_8 is $G' = \{1, x^2\} \cong Z(D_8)$, where $[x, y] = x^{-1}y^{-1}xy = x^3yxy = x^3x^{-1}yy = x^2$. G' is an elementary abelian group of order 2.
- $D_8/Z(D_8)$ is an elementary abelian group of order 4.

Also, the quaternion group Q_8 is a non-abelian group of order 8 and it is a covering group of $C_2 \times C_2$, since:

- Q_8 has a generating set $\{x, y\}$, where $x^4 = y^4 = 1$.
- The commutator subgroup of Q_8 is $G' = \{1, x^2\} \cong Z(Q_8)$, where $[x, y] = x^{-1}y^{-1}xy = x^3y^3xy = x^3xyy = y^2 = x^2$. G' is elementary abelian group of order 2.
- $Q_8/Z(Q_8)$ is an elementary abelian group of order 4

Since any covering group of $C_2 \times C_2$ is a non-abelian group of order $2^3 = 8$, we conclude that the group $C_2 \times C_2$ has two distinct covering groups D_8 and Q_8. These are the two non-isomorphic non abelian groups of order 8.

Now, we introduce a description for the elements of G, where G is a covering group of an elementary abelian $p - group$ generated by $\{x_1, x_2, \ldots, x_n\}$. The commutator subgroup of G is equal to the center and is an elementary abelian group of order $p^{\binom{n}{2}}$, generated by $\binom{n}{2}$ simple commutators $[x_i, x_j]$. This means that for every element of G expressed as a word of the generators, we can rearrange it into the form

$$x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}z; 0 \leqslant i_j \leqslant p-1, \tag{2.1}$$

where z is an element of the commutator subgroup of G. Since $G/Z(G)$ has exponent p, then x_i^p belongs to the commutator subgroup for every i. This means that in the expression (2.1), we can adjust so that each i_j is in the range 0 to $p - 1$ and reach the conclusion that every element of G has a unique expression of the form of (2.1).

We can describe the binary operation of multiplication in G as follows: for elements $(x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}z_1)$ and $(x_1^{j_1}x_2^{j_2}\ldots x_n^{j_n}z_2)$ of G,

$$(x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}z_1)\cdot(x_1^{j_1}x_2^{j_2}\ldots x_n^{j_n}z_2) = x_1^{[i_1+j_1]_p}x_2^{[i_2+j_2]_p}\ldots x_n^{[i_n+J_n]_p}z_3, \tag{2.2}$$

Where:

- $[i_k + j_k]_p$ denotes the remainder on dividing $i_k + j_k$ by p.
- z_3 is not the product of z_1 and z_2, it involves z_1 and z_2, various commutators of the x_i, and any p^{th} powers of x_i that are arising from cases which $[i_k + j_k]_p$ exceeds p.

We now let G be a covering group of C_p^n. We will refer to any minimal generating set of G as a *basis* of G. We say that a subset of G is *independent* if its elements are linearly independent in G/G', regarded as a vector space over $\mathbb{F}_p$. Thus a basis is a maximal independent set.

Let F be a free group of rank n, with generators $X_1, \dots, X_n$. Let G be a covering group of C_p^n with basis $\{x_1, \dots, x_n\}$. Then there is an epimorphism $\phi : F \to G$ with $\phi(X_i) = x_i$ for each i. Since G has exponent p^2 or p, G' has exponent p, and all commutators and squares are central in G, the kernel R of ϕ contains the subgroup H of F generated by all elements of the forms

$$X^{p^2},\ [X^p, Y],\ [X, Y]^p, [[X, Y], Z],\ \text{for } X, Y, Z \in F.$$

We note that the above forms are preserved under conjugation, so H is normal in F. We write $\bar{X}_i$ for the element of F/H represented by X_i. The centre of the group F/H is an elementary abelian of order $p^{\binom{n}{2}+n}$, generated by $\bar{X}_1^p, \dots, \bar{X}_n^p$ and the $\binom{n}{2}$ simple commutators $[\bar{X}_i, \bar{X}_j]_{i<j}$. See [2] for a discussion of this point. Since the centre of F/H strictly contains the commutator subgroup, F/H is not a covering group of C_p^n. However, every covering group of C_p^n may be realized as a quotient of F/H, modulo a subgroup C of order p^n that is a complement of $(F/H)'$ in $Z(F/H)$. Such a subgroup is elementary abelian, generated by elements of the form $X_1^p c_1, \dots X_n^p c_n$, where each c_i belongs to $(F/H)'$ and has a unique expression as a product of the $[\bar{X}_i, \bar{X}_j]$. If $c_i = \theta_i(\bar{X}_i, \dots, \bar{X}_n)$ and $G = (F/H)/C$, then $x_i^p = \theta_i(x_1, \dots, x_n)$ in G. Choosing a complement C of $(F/H)'$ in $Z(F/H)$ amounts to designating the pth power of each of the generators $x_1, \dots, x_n$ of G as a product of the basic simple commutators $[x_i, x_j]$. This can be done freely and independently for each x_i, with different choices corresponding to different choices of C. Different choices for the p^{th} power map on generators may lead to isomorphic covering groups, and determining when this occurs is a difficult problem in general.

The basic arithmetic of (2.2) is the same for all covering groups, and we need to specify how the x_i^p is written as an element of G'. If G is a covering group of elementary abelian group C_p^n, generated by $\{x_1, x_2, \dots, x_n\}$, then $Z(G)$ is generated by the $\binom{n}{2}$ simple commutators $[x_i, x_j]; 1 \leqslant i < j \leqslant n$. Designating a particular covering group of

$C_p^{(n)}$ amounts to writing the p^{th} power of the generator x_k for $(k = 1, \ldots, n)$ explicitly in the form

$$x_k^p = \prod_{i<j} [x_i, x_j]^{e_{ijk}}; 0 \leqslant e_{ijk} < p$$

Since $|G'| = p^{\binom{n}{2}}$, for each i we have $p^{\binom{n}{2}}$ choices for how to assign x_i^p and since the x_i are independent, we can make these choices independently for each i. The number of ways to write down the p^{th} power mapping on the generators $\{x_1, x_2, \ldots, x_n\}$ is $p^{n\binom{n}{2}}$. However the $p^{n\binom{n}{2}}$ choices do not all determine non-isomorphic covering groups, and the key problem is to determine when they do.

This choice on generators determines the "p^{th} power map", defined by:

$$\phi : G \to G', \;\; x \to x^p.$$

If p is odd, and x, y are elements of a covering group G of C_p^n, then from lemma 2.7 and the fact that $[x, y]^p = \mathrm{id}$, we can write

$$(xy)^p = x^p y^p [x, y]^{\frac{p(p-1)}{2}} = x^p y^p.$$

This means that for a covering group G of C_p^n for an odd prime p, the mapping from G to G' that takes x to x^p is a group homomorphism. The kernel of this mapping contains G', and the image is a subgroup of G' which is elementary abelian group of order p^k for some k in the range 0 to n. We call k the *rank* of the covering group.

The p^{th} power mapping on G may be regarded as a group homomorphism from G/G' to G'. Since G/G' is an elementary abelian group of order p^n, so G/G' may be considered as a vector space of dimension n over the field $\mathbb{F}_p$. Moreover, G' is an elementary abelian group of order $p^{\binom{n}{2}}$ and may be considered as a vector space of dimension $\binom{n}{2}$ over the field $\mathbb{F}_p$. Therefore, the p^{th} power mapping on G is a linear transformation from $\mathbb{F}_p^n$ to $\mathbb{F}_p^{\binom{n}{2}}$ and its rank is defined as the dimension of its image.

Definition 2.13. *For odd* p*, the* rank *of a covering group* G *of* C_p^n *is the rank of* ϕ*, considered as a linear transformation from* $\mathbb{F}_p^n$ *to* $\mathbb{F}_p^{\binom{n}{2}}$*.*

When the rank is 0 then $x_i^p = \mathrm{id}$ for all x, which means that the group G has exponent p. There is exactly one covering group of C_p^n of exponent p when p is odd , corresponding to the choice of the trivial homomorphism as the p^{th} power mapping.

The classification of the covering groups of rank 1 of C_p^n up to isomorphism was discussed in the article [1]. If G is a covering group of C_p^n of rank 1, we can choose a generating set $\{x_1, x_2, \ldots, x_n\}$ for G with the property that $x_i^p = 1$ for $i \geqslant 2$, so the only element of the generating set that is not in the kernel of the p^{th} power map is x_1. If we write $x_1^p = r$, then r is a non-identity element in the group of p^{th} powers in G. This means that r must be a generator of this group, since it is cyclic of a prime order. Since $r \in G'$, we can write r as a product of expressions of the form $[x_i, x_j]^{k_{ij}}$ where $i < j$ and $0 \leqslant k_{ij} < p$. The following theorem appears in [1].

Theorem 2.14. *The group C_p^n for odd p has $n-1$ isomorphism types of covering groups of rank 1.*

This theorem was proved by showing that a covering group G of C_p^n for an odd p has a basis $\{x_1, x_2, \ldots, x_n\}$ where $\langle x_2, x_3, \ldots, x_n \rangle$ is the kernel of the p^{th} power map and x_1^p has one of the following forms:

1. $x_1^p = [x_2, x_3][x_4, x_5] \ldots [x_{2k}, x_{2k+1}]$

2. $x_1^p = [x_1, x_2][x_3, x_4] \ldots [x_{2k-1}, x_{2k}]$

The two conditions determine non-isomorphic covering groups. For a given k and n, with $2k \leqslant n$, there are at most two non-isomorphic covering groups of C_p^n of rank 1, in which non-identity p^{th} powers have commutator length k. If $n = 2k$, there is only one. This accounts for the $n-1$ in the statement of Theorem 2.14.

In this study, our main theme is to develop an analogous theory for case of covering groups of elementary abelian 2-groups, where the squaring map is not a homomorphism. If x and y are non-commuting elements in a covering group G of an elementary abelian 2-group, then $(xy)^2 = x^2y^2[x, y] \neq x^2y^2$. Thus the squaring map in a covering group of C_2^n does not induce a linear transformation. This means that the linear algebra methods that apply to the case of odd p do not extend to $p = 2$, and the subject has more of a combinatorial flavour. Nevertheless, we aim to adapt the concept of rank to the case of 2-groups, and consider analogues of Theorem 2.14.

Definition 2.15. *A covering group* G *of* C_2^n *is* uniform *if it has a generating set consisting of* n *elements all having the same square. Such a generating set is called a* uniform basis*.*

Uniform covering groups were defined and discussed in [3]. We present some important concepts and theorems relating to the uniform case. It is apparent that a uniform covering group has many non-uniform bases. Given a covering group of C_2^n in terms of a generating set, there is no quick way to decide if this group is uniform, unless the generating set that we are given happens to have the uniform property. In this respect the concept of "uniform" for $p = 2$ differs from the concept of "rank 1" for odd p.

Example 2.16. A covering group G of $C_2 \times C_2$ has generating set $\{x, y\}$, and commutator subgroup of order 2, generated by $[x, y]$. As noted in Example 2.12, choosing $x^2 = y^2 = [x, y]$ gives Q_8; other three options all give D_8.

1. $x^2 = y^2 = [x, y]$
   In this case $x^4 = y^4 = \mathrm{Id}$ and $yx = x^3y$. This is a non abelian group of order 8 that is generated by two elements both of order 4. It is the *quaternion* group of order 8.

   $$Q_8 = \{\mathrm{Id}, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

   Since $x^2 = y^2 = [x, y]$, we observe that $\{x, y\}$ is a uniform basis and that Q_8 is a uniform covering group of C_2^2.

2. $x^2 = y^2 = \mathrm{id}$
   In this case $x = x^{-1}$ and $y = y^{-1}$, then $(xy)^2 = xyxy = x^{-1}y^{-1}xy = [x, y]$. This is a non-abelian group of order 8 generated by a pair of elements both of order 2. It is the *Dihedral* group of order 8.

   $$D_8 = \{\mathrm{Id}, x, xy, (xy)^2, (xy)^3, (xy)x, (xy)^2x, (xy)^3x\}$$

Since $x^2 = y^2 = \mathrm{Id}$, we observe $\{x, y\}$ is a uniform basis and D_8 is uniform.

Thus both Q_8 and D_8 are *uniform* covering groups of $C_2^{(2)}$. The quaternion group Q_8 is generated by two elements whose square is the unique non-identity commutator and the dihedral group D_8 is generated by two elements whose square is the identity. For the other two choices of the squaring map, the related basis for the dihedral group D_8 are not uniform. On the other hand, every basis of Q_8 is uniform.

We now let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be a basis of G, and introduce a set V of n vertices labelled by the elements of $\mathcal{B}$. For $1 \leqslant i < j \leqslant n$, the *basic simple commutator* $[x_i, x_j]$ is represented by the edge comprising the two vertices labelled by x_i and x_j. Every element of G' has a unique expression as a product of basic simple commutators, and is represented by the graph on V whose edges correspond to the commutators that occur in this expression. Subject to the choice of a basis of G, we have a one-to-one correspondence between the set of all graphs on n labelled vertices and the elements of G'. Also, the uniform covering group of C_2^n with a uniform basis can be associated with a graph as follows.

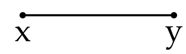
Let $\mathcal{B} = \{x_1, \ldots . x_n\}$ be a uniform basis of a covering group G of C_2^n, where $x_i^2 = r$ for $1 \leqslant i \leqslant n$. Then r has a unique expression as a product of some of the $\binom{n}{2}$ commutators $[xi, xj]$. We use this fact to define a graph $\Gamma_{\mathcal{B}}(G)$ of order n , as follows. The *order* of a graph is its number of vertices.

- The vertex set of $\Gamma_{\mathcal{B}}(G)$ consists of n vertices $X_1, \ldots, X_n$ (corresponding to the basis elements $x_1, \ldots, x_n$).
- A pair of vertices is adjacent via an edge if the commutator of their corresponding elements appears in the expression for r as a product of commutators of elements of $\mathcal{B}$.

Therefore, if $r = 1$, then $\Gamma_{\mathcal{B}}(G)$ is the null graph on n vertices.

Returning to Example 2.16 for the case of covering groups of $C_2 \times C_2$, we introduce a pair of vertices corresponding to the elements x and y of a uniform basis. The quaternion group Q_8 is generated by two elements whose square is the unique non-identity commutator and the dihedral group D_8 is generated by two elements whose square is the identity. Therefore we can associate the following two graphs to these two uniform covering groups as follows.

1. For Q_8, $x^2 = y^2 = [x, y]$, and a uniform basis for Q_8 corresponds to the following graph.



2. For D_8, $x^2 = y^2 = \mathrm{id}$, for any uniform basis $\{x, y\}$. The corresponding graph is the null graph on two vertices.

Also, it in shown in [3] that the isomorphism type of $\Gamma_{\mathcal{B}}(G)$ does not depend on the choice of uniform basis $\mathcal{B}$. This means that if a covering group G has a uniform basis $\mathcal{B}$ that corresponds to a graph $\Gamma_{\mathcal{B}}(G)$, then every uniform basis of G corresponds to a graph isomorphic to $\Gamma_{\mathcal{B}}(G)$. In view of this, we may refer to the graph of a uniform covering group without reference to a particular basis, and write $\Gamma(G)$ instead of $\Gamma_B(G)$. The following theorem is one of the main results of [3].

Theorem 2.17. *Let G_1 and G_2 be uniform covering groups for C_2^n. Then $G_1 \cong G_2$ if and only if $\Gamma(G_1)$ and $\Gamma(G_2)$ are isomorphic graphs. The number of isomorphism types of uniform covering groups for C_2^n is equal to the number of isomorphism types of graphs of order n.*

Given a simple graph Γ with vertex set $\{v_1, \dots, v_n\}$, a uniform covering group G of C_2^n corresponding to Γ is determined as follows. For generators $\{x_1, \dots, x_n\}$, we define the common square of all the x_i to be the element $\prod_{i<j}[x_i, x_j]^{e_{ij}}$, where e_{ij} is 1 or 0 according as the vertices v_i and v_j are adjacent in Γ, or not. The simple commutators $[x_i, x_j]$ are central in G, and $Z(G)$ is generated by the $\binom{n}{2}$ simple commutators $[x_i, x_j]; 1 \leqslant i < j \leqslant n$, and the common square of all the x_i is an element of $Z(G)$.

Comparing Theorem 2.17 and Theorem 2.14, we note that the number of isomorphism types of uniform covering groups of C_2^n greatly exceeds the number of covering groups of rank 1 of C_p^n, where p is odd.

Example 2.18. If $n = 6$ and p is odd, then from Theorem 2.14, the group C_p^6 has $n-1 = 5$ isomorphism types of rank 1 covering groups. The group C_2^6 has 156 isomorphism types of uniform covering groups, corresponding to the 156 isomorphism classes of simple undirected graphs on six vertices.

Subject to the choice of a basis $\mathcal{B}$ for a covering group G of C_2^n, any element of G' may be described, as outlined above, by a graph on a set of n vertices labelled by the elements of $\mathcal{B}$. The distinguishing feature of uniform covering groups is that a single such graph is sufficient to fully specify the group. Our theme in this thesis is to explore the case of covering groups that are not uniform but possess a basis whose elements have only two distinct squares. Such groups will be called 2-*uniform*, and they can be described using graphs with a 2-colouring of both their vertex and edge sets.

One invariant that can distinguish non-isomorphic covering groups of C_2^n is the minimum number of distinct squares of elements of a generating set. In the next chapter we will define the concept of an *2-uniform covering group* of an elementary abelian 2-group and establish a correspondence between 2-uniform covering groups of C_2^n and a certain class of graphs of order n with two edge colours.

Chapter 3

2-uniform groups

In this chapter, we discuss an extension of the graph representation of uniform covering groups, to the case of non-uniform covering groups possessing bases whose elements have two distinct squares instead of one, which we call *2-uniform*. This is the core subject of the thesis. The main results of this chapter are

- Theorem 3.8, which establishes that in many cases, there is only one possibility for the two squares of the elements in such a basis;
- Theorem 3.12, which presents a class of graphs encoding all 2-uniform covering groups.

We show in this chapter that a 2-coloured graph with at least three blue vertices, and at least as many blue as red, is 2-uniform if and only if it satisfies the conditions of Theorem 3.12.

Definition 3.1. *We will say that a covering group* G *of* C_2^n *is* 2-uniform *if it is not uniform, and it has a basis* $\mathcal{B}$ *with the property that*

$$|\{x^2 : x \in \mathcal{B}\}| = 2.$$

We will refer to a basis of the type described in Definition 3.1 as a *2-square basis* of G. Any covering group that possesses a 2-square basis is either 2-uniform or uniform. We may use a 2-square basis to associate a graph to G, by extending the graph interpretation of a uniform basis as used in Chapter 2. We use vertex colours to distinguish the elements of a 2-square basis according to their two distinct squares, and corresponding edge-colours to distinguish their respective squares. By a 2-coloured graph, we mean a loopless undirected graph in which every vertex is coloured either blue or red, and every edge is coloured either blue or red. A pair of vertices may be adjacent via a blue edge and a red edge, but multiple edges of the same colour cannot occur. We say that two 2-coloured graphs are *isomorphic* if there is a bijection between their vertex sets that preserves adjacency and non-adjacency, and either preserves the colours of both vertices and edges, or switches the colours of all vertices and all edges.

Definition 3.2. *Let* G *be a covering group of* C_2^n *with a basis* $\mathcal{B}$*, and let* $c \in G'$*. We define* $\Gamma_{\mathcal{B}}(c)$ *to be the graph whose vertices represent the elements of* $\mathcal{B}$*, in which two vertices are adjacent if and only if the commutator of the corresponding elements of* $\mathcal{B}$ *occurs in the expression for* c *as a product of commutators of elements of* $\mathcal{B}$*.*

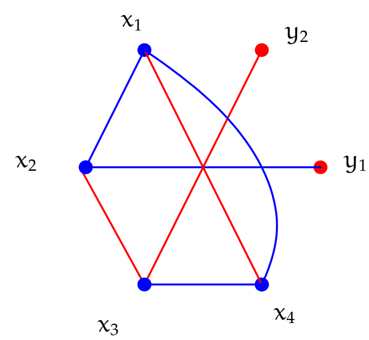
Let $\mathcal{B} = \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\}$ be a 2-square basis of a covering group G of C_2^n, where $x_i^2 = r$ for $i \leqslant k$, $y_j^2 = s$ for $j > k$, and r and s are distinct elements of G'. We define the 2-coloured graph of G with respect to the basis $\mathcal{B}$, denoted $\Gamma_{\mathcal{B}}(G)$, as follows.

- The vertex set of $\Gamma_{\mathcal{B}}(G)$ consists of k blue vertices, corresponding to the basis elements $x_1, \ldots, x_k$ and $n - k$ red vertices, corresponding to the basis elements $y_{k+1}, \ldots, y_n$;
- The blue edges of $\Gamma_{\mathcal{B}}(G)$ comprise the edge set of the graph $\Gamma_{\mathcal{B}}(r)$.
- The red edges of $\Gamma_{\mathcal{B}}(G)$ comprise the edge set of the graph $\Gamma_{\mathcal{B}}(s)$.

Example 3.3. Let $G = \langle x_1, x_2, x_3, x_4, y_1, y_2 \rangle$, where

- $x_1^2 = x_2^2 = x_3^2 = x_4^2 = r = [x_1, x_2][x_1, x_4][x_2, y_1][x_3, x_4]$,
- $y_1^2 = y_2^2 = s = [x_2, x_3][x_1, x_4][x_3, y_2]$.

The graph that represents the group G is below:



On the other hand, if Γ is a 2-coloured graph on n vertices, we may associate a covering group G to Γ as follows. A generating set of G consists of n elements $X_1, \ldots, X_n$, corresponding to the vertices of Γ. The following set of relators specifies G (see [2]):

- The square of each generator that corresponds to a blue vertex is the element of G' represented by the blue edges.
- The square of each generator that corresponds to a red vertex is the element of G' represented by the red edges.
- All elements of the forms X_i^4, $[X_i^2, X_j]$, $[X_i, X_j]^2$ and $[[X_i, X_j], X_k]$ are equal to the identity element in G.

We say that 2-coloured graphs are *isomorphic* if there is a bijection between their vertex sets that preserves adjacency and non-adjacency, and either preserves the colours of both vertices and edges, or switches the colours of all vertices and all edges.

Definition 3.4. *Suppose that Γ_1 and Γ_2 are 2-coloured graphs of order n. Then Γ_1 and Γ_2 are* isomorphic *if there is a bijection $\phi : V(\Gamma_1) \to V(\Gamma_2)$ with the following properties:*

1. *$\phi(u)$ and $\phi(v)$ have the same colour in Γ_2 if and only if u and v have the same colour in Γ_1. This means that ϕ either maps all blue vertices of Γ_1 to blue vertices of Γ_2 (and red to red) or it maps all blues to reds and all reds to blues. The latter case can only occur if n is even and exactly half of the vertices in each graph are blue and half red.*
2. *If ϕ preserves blue vertices, then it also preserves blue edges. This means that $\phi(u)\phi(v)$ is a blue edge in Γ_2 if and only if uv is a blue edge in Γ_1. Same for red edges.*

3. *In the special case where ϕ sends blue vertices to red vertices, it also sends blue edges to red edges (and red to blue). In this case $\phi(u)\phi(v)$ is a blue edge of Γ_2 if and only if uv is a red edge in Γ_1. Also $\phi(u')\phi(v')$ is a red edge in Γ_2 if and only if $u'v'$ is a blue edge in Γ_1.*

Also, a 2-uniform group may have multiple 2-square bases, and may be represented by non-isomorphic 2-coloured graphs, as the following example shows.

Example 3.5. Let G be the 2-uniform covering group of C_2^4 with 2-square basis $\{x_1, x_2, y_3, y_4\}$, where $x_1^2 = x_2^2 = [x_1, x_2][y_3, y_4]$, and $y_3^2 = y_4^2 = [x_1, y_3]$. Then $(x_1y_3)^2 = x_1^2y_3^2[x_1, y_3] = x_1^2$. It follows that $\{x_1, x_2, x_1y_3, y_4\}$ is another 2-square basis of G, in which

$$x_1^2 = x_2^2 = (x_1y_3)^2 = [x_1, x_2][x_1y_3, y_4][x_1, y_4],$$

and $y_4^2 = [x_1, x_1y_3]$. Thus the following nonisomorphic 2-coloured graphs both represent this 2-uniform covering group G of C_2^4.

Example 3.5 shows that, even for bases consisting of elements with the same pair of squares, some variation is possible in the numbers of blue and red vertices in the corresponding graphs. This difficulty will be resolved by refining the concept of a 2-square basis to that of a *2-uniform basis*, which is one which maximizes the number of elements having a single square.

Definition 3.6. *For any covering group G of C_2^n, the* uniform rank *of G, denoted $\rho(G)$, is the maximum k with the property that k independent elements of G have the same square. The* uniform corank *of G, denoted $\Phi(G)$, is defined as $n - \rho(G)$.*

In a 2-uniform covering group of C_2^n, the uniform rank is at least $\lfloor \frac{n}{2} \rfloor$ and at most $n - 1$.

Definition 3.7. *Let* G *be a 2-uniform covering group of* C_2^n*. A* 2-uniform basis *of* G *is a generating set* $\{x_1, \ldots, x_n\}$ *with the following properties:*

- $x_1, \ldots, x_k$ *have the same square* r.
- $x_{k+1}, \ldots, x_n$ *have the same square* s*, where* $s \neq r$.
- k *is the uniform rank of* G.

We now establish that every 2-uniform covering group of an elementary abelian 2-group possesses a 2-uniform basis. This allows us to restrict our attention to 2-coloured graphs that arise from 2-uniform bases. We will refer to such graphs as *2-uniform graphs*, and give a descriptive characterisation of them in terms of their graph-theoretic properties. We also establish conditions for the existence of a unique 2-uniform basis in a covering group. This step identifies a large class of covering groups that are represented by a unique 2-uniform graph. The exceptions to this situation will be categorised in this chapter, and analysed later. Theorem 3.8 is the main technical ingredient required to establish that every covering group possesses a 2-uniform basis. The following notation, which will be used throughout the paper, occurs in the proof.

If X is a subset of a covering group G of C_2^n, we write $C(X)$ for the element of G' that is given by the product of the commutators $[x, y]$, over all unordered pairs $\{x, y\}$ of distinct elements of X. If $X = \{x_1, x_2, \ldots, x_t\}$ we write $C(x_1, \ldots, x_t)$ for $C(X)$. If the elements of X are independent in G and are included in a basis $\mathcal{B}$, then $\Gamma_{\mathcal{B}}(C(X))$ consists of a clique on those vertices representing the elements of X, with remaining vertices isolated.

Theorem 3.8. *Let* G *be a 2-uniform covering group of* C_2^n*, where* $n \geqslant 4$*, and let* $\mathcal{B} = \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\}$ *be a generating set of* G*, where* $x_i^2 = r$ *for* $i = 1, \ldots, k$*, and* $y_j^2 = s$ *for* $j = k+1, \ldots, n$*, and where* $k \geqslant n - k$*. Then no element of* $G' \backslash \{r, s\}$ *is the square of elements of four independent cosets of* G' *in* G.

Proof. Let $t \in G' \backslash \{r, s\}$, and suppose that t is the square of elements from four independent cosets of G' in G. Each of these cosets has a unique representative that is a product of elements of the ordered basis $\mathcal{B}$, appearing in the same order as they do in $\mathcal{B}$. Let these four elements be z_1, z_2, z_3, z_4, and for $i = 1, \ldots, 4$ let X_i and Y_i respectively denote the sets of elements of $\{x_1, \ldots, x_k\}$ and $\{y_{k+1}, \ldots, y_n\}$ that occur in z_i. We note that $|X_i \cup Y_i| \geqslant 2$ in each case, since $z_i^2 \notin \{r, s\}$. Then

$$r^{|X_1|}s^{|Y_1|}C(X_1 \cup Y_1) = r^{|X_2|}s^{|Y_2|}C(X_2 \cup Y_2) = r^{|X_3|}s^{|Y_3|}C(X_3 \cup Y_3) = r^{|X_4|}s^{|Y_4|}C(X_4 \cup Y_4).$$

In each case, the expression $r^{|X_i|}s^{|Y_i|}$ is either equal to id, r, s or rs. No two of these can coincide, since the four elements $C(X_i \cup Y_i)$ of G' are distinct. After relabelling if necessary we write

$$C(X_1 \cup Y_1) = rC(X_2 \cup Y_2) = sC(X_3 \cup Y_3) = rsC(X_4 \cup Y_4), \tag{3.1}$$

where $|X_1|$, $|Y_1|$, $|Y_2|$ and $|X_3|$ are even, and $|X_2|$, $|Y_3|$, $|X_4|$ and $|Y_4|$ are odd. Multiplying the expressions in (3.1) together, we obtain

$$C(X_1 \cup Y_1)C(X_4 \cup Y_4) = C(X_2 \cup Y_2)C(X_3 \cup Y_3). \tag{3.2}$$

Let V be a set of vertices corresponding to the elements of $\mathcal{B}$, and for $i = 1, \ldots, 4$, let V_i be the subset of V corresponding to $X_i \cup Y_i$. Let Γ_i be the graph on vertex set V, whose edges form a complete graph on V_i. The sets $V_1, \ldots, V_4$ are distinct, and each has at least two elements since $t \notin \{r, s\}$. The statement (3.2) translates to the following equality involving edge sets.

$$E(\Gamma_1) \triangle E(\Gamma_4) = E(\Gamma_2) \triangle E(\Gamma_3).$$

We write Γ for the graph induced by the edges in $E(\Gamma_1) \triangle E(\Gamma_4)$. We note that Γ has at least three vertices, and that Γ is not a complete graph.

Let u and v be a pair of non-adjacent vertices in Γ. Then either u and v both belong to $V_2 \cap V_3$, or one of these vertices belongs to $V_2 \backslash V_3$ and the other to $V_3 \backslash V_2$.

Suppose the former case. Then u and v have the same set of neighbours in Γ, and this set is $V_2 \triangle V_3$. The subgraph of Γ induced on $V_2 \triangle V_3$ is complete (if $V_2 \supseteq V_3$ or $V_3 \supseteq V_2$) or consists of two complete components, on the sets $V_2 \backslash V_3$ and $V_3 \backslash V_2$. The set consisting of u, v and their non-neighbours in Γ is $V_2 \cap V_3$. Thus the sets V_2 and V_3

are determined by the non-adjacent pair $\{u, v\}$ and the hypothesis that $\{u, v\} \subseteq V_2 \cap V_3$. If, in addition, $\{u, v\} \subseteq V_1 \cap V_4$, then the same reasoning leads to the contradiction that $\{V_1, V_4\} = \{V_2, V_3\}$. Thus if $\{u, v\} \subseteq V_2 \cap V_3$, then we may assume that $u \in V_1 \backslash V_4$ and $v \in V_4 \backslash V_1$.

Similar reasoning leads from the hypothesis $u \in V_2 \backslash V_3$ and $v \in V_3 \backslash V_2$ to the conclusion that $\{u, v\} \in V_1 \cap V_4$. In this case V_2 consists of u and its neighbours in Γ, and if $u \notin V_1 \cap V_4$ then either $V_1 = V_2$ or $V_4 = V_2$.

We proceed with $u \in V_1 \backslash V_4$, $v \in V_4 \backslash V_1$, and $\{u, v\} \subseteq V_2 \cap V_3$. The vertices u and v have the same set of neighbours in Γ, which is $V_2 \triangle V_3$. It follows that $V_1 \backslash V_4 = \{u\}$ (since any other vertex in $V_1 \backslash V_4$ would be adjacent to u but not v in Γ) and that $V_4 \backslash V_1 = \{v\}$.

If any vertex of Γ belongs to all four of the Γ_i, then its neighbour set is simultaneously equal to $V_1 \triangle V_4$ and $V_2 \triangle V_3$. Since these two sets are different, it follows that $V_1 \cap V_2 \cap V_3 \cap V_4$ is empty, and $V_2 \cap V_3 \subseteq V_1 \triangle V_4 = \{u, v\}$. Hence $V_2 \cap V_3 = \{u, v\}$. Moreover, $V_1 \cap V_4 = V_2 \triangle V_3$. We may assume that $V_2 \backslash V_3$ includes an element x, since $V_2 \triangle V_3$ is not empty. Then $x \in V_1 \cap V_4$, and $N(x) = \{u, v\}$. It follows that $V_2 \backslash V_3 = \{x\}$. Similarly $V_3 \backslash V_2$ has at most one element. We have two possibilities.

1. $V_1 = \{u, x\}$, $V_4 = \{v, x\}$, $V_2 = \{u, v, x\}$, $V_3 = \{u, v\}$. In this case Γ is a path on three vertices, with edges ux and vx.

2. There is a single vertex y in $V_3 \backslash V_2$. In this case $V_1 = \{u, x, y\}$, $V_4 = \{v, x, y\}$, $V_2 = \{u, v, x\}$, $V_3 = \{u, v, y\}$. The graph Γ is a cycle of length 4, and it has two different representations as the symmetric difference of two copies of K_3.

Neither of these solutions satisfies the parity requirements in (3.2), and we conclude that no element of $G' \backslash \{r, s\}$ can occur as the square of elements from more than three independent cosets of G' in G. □

Suppose now that G is a 2-uniform covering group of C_2^n whose uniform rank is at least 4. Suppose that $\{x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\}$ is a generating set of G, $x_i^2 = r$ for $i \in \{1, \ldots, m\}$, and $y_j^2 = s \neq r$, for $j \in \{m+1, \ldots, n\}$. If $\rho(G) \in \{m, n-m\}$, then this generating set is a 2-uniform basis of G. If not, let S be a set of $\rho(G)$ independent elements of G all having the same square. By Theorem 3.8, this common square must either be r or s, and after relabelling the elements of the generating set if necessary, we may assume that it is r. Then we may extend the set $\{x_1, \dots, x_m\}$ to a set $\{x_1, \dots, x_{\rho(G)}\}$ of independent elements of G with square r, discarding an element y_i from the original generating set for each of the newly introduced elements $x_{m+1}, \dots, x_{\rho(G)}$. The result is a 2-uniform basis of G.

It remains to consider the case where G is a 2-uniform covering group of $C_2^{(n)}$ with $\rho(G) \leqslant 3$. In this case $n \leqslant 6$. Both covering groups of C_2^2 are uniform, so the cases of interest occur when $n \in \{3, 4, 5, 6\}$. We first observe that if $\rho(G) = n - 1$, then any set of $n - 1$ independent elements with the same square can be extended to a 2-uniform basis by adding one further element. On the other hand if $\rho(G) = \lceil \frac{n}{2} \rceil$, then every minimal generating set of G, whose elements have two distinct squares, must have $\rho(G)$ elements with one square and $n - \rho(G)$ elements with the other. Such a set is therefore a 2-uniform basis. This observation accounts for the remaining cases, which occur when $(\rho(G), n) \in \{(2, 4), (3, 5), (3, 6)\}$. We have proved the following statement.

Theorem 3.9. *If n is a positive integer and G is a 2-uniform covering group of C_2^n, then G possesses a 2-uniform basis.*

If G is a 2-uniform covering group of C_2^n, then the graph that represents G with respect to a 2-uniform basis has $\rho(G)$ vertices of one colour, and $n - \rho(G)$ of the other. We will adopt the convention that the colour blue is used for $\rho(G)$ vertices representing basis elements with the same square, and red for the remainder. From now on, we will only consider graphs that are written with respect to 2-uniform bases, and thus only graphs that have at least as many blue as red vertices.

Definition 3.10. *A 2-uniform graph is a 2-coloured graph that represents a 2-uniform covering group with respect to a 2-uniform basis.*

The remainder of this chapter discusses how to recognize a 2-uniform graph. We consider the question of how a 2-coloured graph with at least $\frac{n}{2}$ blue vertices, could fail to be 2-uniform. Suppose that $\mathcal{B} = \{x_1, \dots, x_k, y_{k+1}, \dots, y_n\}$ is a 2-square basis of a covering group G of C_2^n, where $k \geqslant \frac{n}{2}$, $x_i^2 = r$ for each x_i, $y_j^2 = s$ for each y_j, and $s \neq r$. We write $\mathcal{X}$ for $\{x_1, \dots, x_k\}$ and $\mathcal{Y}$ for $\{y_{k+1}, \dots, y_n\}$. If $\mathcal{B}$ is not a 2-uniform basis of G, then

$k < \rho(G)$. If $\rho(G) \geqslant 4$, then it follows from Theorem 3.8 that a 2-uniform basis of G possesses $\rho(G)$ elements with square r, or $\rho(G)$ elements with square s. This means either that $g^2 = r$ for some $g \in G \backslash \langle x_1, \dots x_k \rangle$, or that $h^2 = s$ for some $h \in G \backslash \langle y_{k+1}, \dots, y_n \rangle$, and in the latter case that G contains enough independent elements h of this type to extend $\{y_{k+1}, \dots, y_n\}$ to a set of $\rho(G)$ elements. Our next lemma establishes the circumstances under which such adjustments are possible.

Lemma 3.11. *Let* G *be a 2-uniform covering group of* C_2^n*, with a 2-square basis* $\mathcal{B}$ *as above. If none of the following conditions holds, then the maximum number of independent elements of* G *having square* r *is* k*. If exactly one of them holds, this number is* $k+1$*. If* (b) *and* (c) *hold with* $S_b \cap \mathcal{Y} = S_c \cap \mathcal{Y}$*, it is* $k+1$*. In other cases where two of the three conditions hold, it is* $k+2$*.*

(a) $r = C(S_a)$ *for some subset* S_a *of* $\mathcal{B}$*, with* $|S_a \cap \mathcal{X}|$ *and* $|S_a \cap \mathcal{Y}| > 0$ *both even.*

(b) $s = C(S_b)$*, for some subset* S_b *of* $\mathcal{B}$*, with* $|S_b \cap \mathcal{X}|$ *and* $|S_b \cap \mathcal{Y}|$ *both odd.*

(c) $rs = C(S_c)$*, for some subset* S_c *of* $\mathcal{B}$*, where* $|S_c \cap \mathcal{X}|$ *is even and positive, and* $|S_c \cap \mathcal{Y}|$ *is odd.*

Proof. The maximum number of independent elements of G that have square r is the dimension of the vector subspace of G/G' spanned by all cosets consisting of elements with square r. Since the set of cosets represented by elements of $\mathcal{X}$ extends to a basis of this space, it is sufficient to consider whether G can include elements with square r that do not belong to the subgroup generated by $\mathcal{X}$ and G'.

If such an element x exists, we may assume that $x = s_1 s_1 \dots s_m$; where the s_i are elements of $\mathcal{B}$ and we write $S = \{s_1, \dots, s_m\}$. Then

$$r = x^2 = r^e s^f C(S),$$

where $e = |S \cap \mathcal{X}|$, $f = |S \cap \mathcal{Y}|$, and $f \geqslant 1$ since $x \notin \langle \mathcal{X}, G' \rangle$. This equation is satisfied if and only if one of the following occurs

(a) e and f are both even and $r = C(S)$;

(b) e and f are both odd and $r = rsC(S)$, so $s = C(S)$;

(c) e is even, f is odd and $r = sC(S)$, so $rs = C(S)$.

We now show that at most two of these three conditions can hold simultaneously. Suppose that the first two both hold, for subsets S_a and S_b of $\mathcal{B}$ in place of S, each having at least two elements. If $|S_a \cap S_b| \geqslant 2$, let x and y be elements of $S_a \cap S_b$ and let $z \in S_a \triangle S_b$. Then $[x, z]$ and $[y, z]$ occur in rs, but $[x, y]$ does not, so rs cannot be represented by a clique as in (c). If $S_a \cap S_b = \{x\}$, then $S_a \backslash S_b$ and $S_b \backslash S_a$ are non-empty, with respective elements y and z. Then $[x, y]$ and $[x, z]$ occur in rs but $[y, z]$ does not, which is again inconsistent with (c). Finally if $S_a \cap S_b = \emptyset$, let $x, y \in S_a$ and $z, w \in S_b$. Then $[x, y]$ and $[z, w]$ occur in rs but $[x, z]$ does not, so rs does not have the form described in (c).

Each of the three conditions that holds in G yields an element of square r that is independent of the $x_1, \ldots, x_k$. If both (b) and (c) hold with $S_b \cap \mathcal{Y} = S_c \cap \mathcal{Y}$ then the process yields only $k + 1$ independent elements that can occur together in a basis. Otherwise, if two of the three conditions hold, we obtain $k + 2$ independent elements.

□

Applying Lemma 3.11 to the element s instead of r, we note the maximum number of independent elements of G whose square is s is at most $n - k + 2$, and the value of this number is determined by the conditions (a), (b), (c) in the statement of the lemma, with the roles of $\mathcal{X}$ and $\mathcal{Y}$ reversed. The conditions of Lemma 3.11 may be expressed as properties of the graph $\Gamma_{\mathcal{B}}(G)$ and used to characterize 2-uniform graphs. Before proceeding with this description, we introduce some notation that will apply to 2-coloured graphs in general.

For a 2-coloured graph Γ, we write Γ^B and Γ^R respectively for the subgraphs induced by the sets of blue and red edge in Γ. We write $\Gamma^{B \triangle R}$ for the subgraph of Γ induced by the edge set $E(\Gamma^B) \triangle E(\Gamma^R)$, with each edge retaining its colour in Γ. We write $\Gamma^\star$ for the *colour opposite* of Γ, which is obtained from Γ by switching the colour of every vertex and every edge, from blue to red or from red to blue. It is clear that Γ and $\Gamma^\star$ represent the same group G, with respect to the same two-square basis.

The following statement of properties of 2-uniform graphs is a consequence of Lemma 3.11.

Theorem 3.12. *Let Γ be a 2-coloured graph with at least as many blue vertices as red. If Γ is a 2-uniform graph, then the following conditions hold.*

(a) Γ^{B} *is not a clique on an even number of blue vertices and a positive even number of red vertices;*

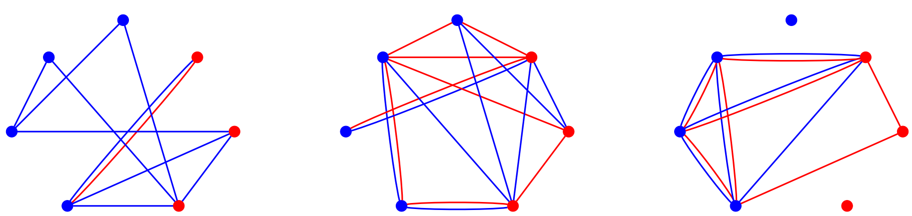
(b) Γ^{R} *is not a clique on an odd number of blue vertices and an odd number of red vertices;*

(c) $\Gamma^{B \triangle R}$ *is not a clique on an even number of blue vertices and an odd number of red vertices;*

(d) If the numbers of blue and red vertices in Γ *are equal, then items (a),(b) and (c) above apply to the colour opposite* $\Gamma^{\star}$ *of* Γ.

(e) If the number of blue and red vertices in Γ *differ by 1, then the colour opposite* $\Gamma^{\star}$ *fails at most one of conditions (a),(b),(c), or fails both (b) and (c) with cliques involving the same set of red vertices.*

Example 3.13. These three 2-coloured graphs, each having more blue than red vertices, all fail to be 2-uniform graphs, respectively on the basis of items (b), (c) and (e) of Theorem 3.12.



We now consider when the necessary conditions listed in Theorem 3.11 are sufficient to guarantee that a graph is 2-uniform.

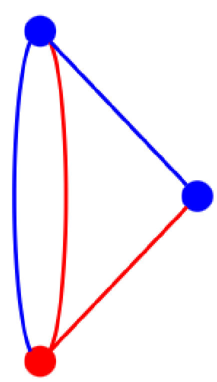
Let Γ be a 2-coloured graph on n vertices, with $t \geqslant 3$ blue vertices, where $t \geqslant n-t$. Then Γ determines a covering group G of C_2^n, and Γ is a 2-uniform graph if and only if t is equal to the uniform rank k of G. Let $\mathcal{B}$ be a 2-square basis of G whose elements correspond to the vertices of Γ. Let r and s be the elements of G' determined respectively by blue and red edges of Γ. If the uniform rank of G exceeds t, then it follows from Theorem 3.8 that either r or s is the square of four independent elements of G. In this case at least one of the conditions of Theorem 3.12 does not hold for Γ.

We conclude that a 2-coloured graph with at least three blue vertices, and at least as many blue as red, is 2-uniform if and only if it satisfies the conditions of Theorem 3.12. In particular, Theorem 3.12 fully characterizes 2-uniform graphs on five or more vertices.

If $n \leqslant 4$, it is possible for a 2-uniform covering group G to have a 2-square basis $\mathcal{B}$ comprising elements with squares r and s, both occurring with multiplicity strictly less than the uniform rank of G, and neither occurring as the square of an element of any 2-uniform basis. The following example shows a graph which satisfies all the conditions of Theorem 3.12 but is not 2-uniform.

Example 3.14. Let G be the covering group of C_2^3 determined by the graph Γ shown below. Let $\{x_1, x_2, y\}$ be the 2-square basis of G determined by the vertices of Γ, where x_1 and x_2 correspond to the blue vertices with blue degrees of 2 and 1 respectively. The squares of the basis elements are

$$x_1^2 = x_2^2 = [x_1, x_2][x_1, y], y^2 = [x_1, y][x_2, y].$$



Also,

$$\begin{aligned}(x_1x_2)^2 &= [x_1, x_2]\\(x_1x_2y)^2 &= [x_1, x_2]\\(x_2y)^2 &= [x_1, x_2].\end{aligned}$$

We notice that this graph represents a uniform covering group with the uniform basis $\{x_1x_2, x_1x_2y, x_2y\}$. Therefore, it is not a 2-uniform graph.

Chapter 4

Exchange operations on 2-uniform graphs

Our ambition is to construct a bijective correspondence between isomorphism classes of 2-uniform covering groups of C_2^n, and an appropriate collection of 2-coloured graphs of order n. A graph is constructed not intrinsically from a group, but from a 2-square basis. As Example 3.5 indicates, a covering group of C_2^n may have multiple 2-square bases, possibly even corresponding to graphs whose vertex colourings partition n differently.

Theorem 3.12 gives a full description of 2-uniform graphs of order 5 or greater. We now consider the question of when non-isomorphic 2-uniform graphs describe isomorphic groups. This requires that the number of blue (resp. red) vertices in each graph are equal, since the number of blue vertices is the uniform rank, an invariant of the group. The remainder of the thesis is devoted to the question of when a 2-uniform covering group of an elementary abelian 2-group has multiple 2-uniform bases, determining non-isomorphic 2-uniform graphs. We remark that this always occurs in the case of a 2-uniform graph of C_2^n of uniform corank 1, since a set of $n-1$ independent elements can be extended to a 2-uniform basis by the addition of any element from outside their span. The special case of corank 1 will be discussed in Chapter 7; in the meantime we restrict attention to 2-uniform covering groups whose uniform corank is at least 2.

The theme of Chapter 4 is the possibility that a covering group could have multiple 2-uniform bases involving elements with the same two squares. The main results of this chapter are Theorem 4.6, Theorem 4.8 and Theorem 4.10, which detail the conditions under which multiple bases of this type exist, and the relationships between their corresponding 2-uniform graphs.

Let G be a covering group of C_2^n, with a basis $\mathcal{B}$. Before considering multiple 2-uniform bases, we discuss the effect on the graph of a single element of G' of changing one or two elements of $\mathcal{B}$. These details will be applied to our analysis of 2-uniform bases later in this chapter.

For an element c of G', the graph of c with respect to $\mathcal{B}$ will be denoted by $\Gamma_{\mathcal{B}}(c)$. Its vertices are labelled by the elements of $\mathcal{B}$, and its edges are those pairs of basis elements that appear as commutators in the unique expression for c as a product of basic simple commutators from $\mathcal{B}$. One may consider the relationships between the graphs that represent c with respect to different bases of G. The case of a pair of bases that differ only in one or two elements will be of particular interest, and we conclude this section by noting the graph transformations that correspond to basis changes of this nature. We now let $\mathcal{B}'$ be a basis of G that differs from $\mathcal{B}$ in either exactly one or exactly two elements. We assume that $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}'}(c)$ have the same vertex set, with the relevant vertex or pair of vertices relabelled in the transition from one graph to the other.

If the element c is a non-identity commutator in G, then $c = [p, q]$. Since c depends only on the cosets pG' and qG', we may assume that each of p and q are products of elements of $\mathcal{B}$. Let P and Q respectively denote the sets of vertices of $\Gamma_{\mathcal{B}}(c)$ that represent the basis elements that occur in p and q. Expanding the expression $[p, q]$ in terms of the basis elements, we observe that the edges of $\Gamma_{\mathcal{B}}(c)$ and their incident vertices comprise a complete tripartite graph with parts $P \backslash Q$, $Q \backslash P$ and $P \cap Q$, or a complete bipartite graph if one of these three sets is empty. It follows that a graph represents a simple commutator (i.e. an element of G' of the form $[p, q]$) if and only if it has a connected component that is complete tripartite or complete bipartite, with remaining vertices isolated. This situation will arise frequently in our analysis, so we introduce the following notation for the set of edges that represents the commutator of a pair of elements from specified cosets of G' in G.

Definition 4.1. *For sets of vertices* P *and* Q*, we denote by* $E(P, Q)$ *the set of edges of the complete tripartite (or bipartite or null) graph whose parts are* $P \backslash Q$*,* $Q \backslash P$ *and* $Q \cap P$*.*

In general, we write $E(\Gamma)$ for the edge set of a graph Γ. For a pair of sets A and B, $A \triangle B$ denotes the symmetric difference of A and B.

Theorem 4.2. *Suppose that* $\mathcal{B}$ *and* $\mathcal{B}' = \mathcal{B}\backslash\{x\} \cup \{y\}$ *are bases of* G*, and let* $c \in G'$*. Let* v *be the vertex that represents* x *in* $\Gamma_{\mathcal{B}}(c)$ *and* y *in* $\Gamma_{\mathcal{B}'}(c)$*. Let* P *be the set of neighbours of* v *in* $\Gamma_{\mathcal{B}}(c)$*, and let* Q *be the set of vertices representing elements of* $\mathcal{B}\backslash\{x\}$ *that occur in the expression for* y *as a product of elements of* $\mathcal{B}$ *(modulo* G'*). Then*

$$E(\Gamma_{\mathcal{B}'}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P, Q).$$

Proof. Let q and p respectively denote the products (in some specified order) of the elements of $\mathcal{B}$ represented by the vertices of P and of Q. Then

$$c = [x, p]c' = [yq, p]c' = [q, p][y, p]c',$$

where c' is a product of simple commutators involving the elements of $\mathcal{B} \cap \mathcal{B}'$. Since c' is represented by the same set of edges in both graphs, and the edges that represent $[y, p]$ with respect to $\mathcal{B}'$ coincide with those that represent $[x, p]$ with respect to $\mathcal{B}$, it follows that the graph $\Gamma_{\mathcal{B}'}(c)$ is obtained from $\Gamma_{\mathcal{B}}(c)$ by switching the status of all edges that represent commutators that occur in the expansion of $[q, p]$ in terms of elements of $\mathcal{B}\backslash\{x\}$. These edges are exactly those of the set $E(P, Q)$. □

If the sets P and Q coincide in the situation of Theorem 4.2, then the graphs $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}'}(c)$ also coincide. We note the following special case of this situation, which will arise in our analysis.

Corollary 4.3. *Let* c *be an element of* G' *whose graph with respect to the basis* $\mathcal{B}$ *consists of a clique on* $k \geqslant 2$ *vertices, with any remaining vertices isolated. Let* x *be the product in* G*, in some order, of those basis elements* $x_1, \dots, x_k$ *that are represented by non-isolated vertices. Then the graphs* $\Gamma_{\mathcal{B}}(c)$ *and* $\Gamma_{\mathcal{B}'}(c)$ *coincide, where* $\mathcal{B}'$ *is a basis obtained from* $\mathcal{B}$ *by replacing any of* $x_1, \dots, x_k$ *with* x*.*

We now consider the relationship between $\Gamma_{\mathcal{B}}(c)$ and $\Gamma_{\mathcal{B}''}(c)$, where $c \in G'$ and the basis $\mathcal{B}''$ is obtained from $\mathcal{B}$ by replacing two elements x_1 and x_2 with y_1 and y_2. Since $\mathcal{B}$ and $\mathcal{B}''$ are both generating sets of G, we may assume that the expression for y_1 as a product of elements of $\mathcal{B}$ (modulo G') involves x_1 but not x_2, and that the corresponding expression for y_2 involves x_2. We write P_1 and P_2 respectively for the sets of neighbours of the vertices representing x_1 and x_2 in $\Gamma_{\mathcal{B}}(c)$. We write Q_1 and Q_2 for the respective sets of vertices representing elements of $\mathcal{B}\backslash\{x_1\}$ and $\mathcal{B}\backslash\{x_2\}$ that appear in the expressions for y_1 and y_2 as products of elements of $\mathcal{B}$. We will prove the following theorem through two applications of Theorem 4.2.

Theorem 4.4. *The edge set of $\Gamma_{\mathcal{B}''(c)}$ depends on c and y_2 as follows:*

1. *If the vertices representing x_1 and x_2 are not adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ does not include x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2, Q_2).$$

2. *If the vertices representing x_1 and x_2 are adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ does not include x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2 \triangle Q1, Q_2).$$

3. *If the vertices representing x_1 and x_2 are not adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ includes x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2, Q_2 \triangle Q_1).$$

4. *If the vertices representing x_1 and x_2 are adjacent in $\Gamma_{\mathcal{B}}(c)$, and the expression for y_2 as a product of elements of $\mathcal{B}$ includes x_1, then*

$$E(\Gamma_{\mathcal{B}''}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1) \triangle E(P_2 \triangle Q1, Q_2 \triangle Q_1).$$

Proof. We write $\mathcal{B}'$ for the basis of G that results from replacing x_1 with y_1 in $\mathcal{B}$. From a direct application of Theorem 4.2,

$$E(\Gamma_{\mathcal{B}'}(c)) = E(\Gamma_{\mathcal{B}}(c)) \triangle E(P_1, Q_1).$$

We now write P_2' for the set of neighbours of the vertex representing x_2 in $\Gamma_{\mathcal{B}'}(c)$, and Q_2' for the set of vertices representing elements of $\mathcal{B}'\backslash\{x_2\}$ that occur in the expression for y_2 as a product of elements of the basis $\mathcal{B}'$. By applying Theorem 4.2 again, we may describe the edge set of $\Gamma_{\mathcal{B}''}$ in terms of the sets P_1, Q_1, P_2' and Q_2'. To describe it in terms of the original data pertaining to $\mathcal{B}$, we need to consider how P_2' and Q_2' depend of P_1, P_2, Q_1, Q_2 and the edges of $\Gamma_{\mathcal{B}}(c)$.

If the commutator $[x_1, x_2]$ occurs in the description of c in terms of simple commutators involving elements of $\mathcal{B}$, then the vertex representing x_2 belongs to $P_1\backslash Q_1$, and $P_2' = P_2 \triangle Q_1$. Otherwise $P_2' = P_2$.

If x_1 is involved in the expression for y_2 as a product of elements of $\mathcal{B}$, then the vertex representing x_1 belongs to Q_2, and $Q_2' = Q_2 \triangle Q_1$. Otherwise $Q_2' = Q_2$. □

We now return to considering the possibility that a group has multiple distinct 2-uniform bases involving elements with the same pair of squares r and s.

Lemma 4.5. *Let G be a covering group, with 2-square basis $\mathcal{B} = \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\}$, where $x_i^2 = r$, $y_j^2 = s \neq r$, and suppose that k and $n-k$ are both at least 2. Suppose that another 2-square basis $\mathcal{B}'$ can be obtained by including the element $z = x_1 \ldots x_p y_{k+1} \ldots y_{k+q}$, where $z^2 \in \{r, s\}$, and eliminating some element w of $S = \{x_1, \ldots, x_p, y_{k+1}, \ldots, y_{k+q}\}$. Then*

$$z^2 = r^p s^q C(S),$$

z^2 is either equal to r or s, and one of the following occurs.

1. *If p and q are both even, then $C(S)$ is either equal to r or s. This occurs if either $\Gamma_{\mathcal{B}}^B$ or $\Gamma_{\mathcal{B}}^R$ is a clique on an even number of blue and and even number of red vertices.*

2. *If p and q are both odd, then $C(S)$ is either equal to r or s. This occurs if either $\Gamma_{\mathcal{B}}^B$ or $\Gamma_{\mathcal{B}}^R$ is a clique on an odd number of blue and odd number of red vertices.*

3. *If p is even and q is odd, then $sC(S)$ is either equal to r or s. Since $sC(S) = s$ is impossible, this occurs if $C(S) = rs$, which means that $\Gamma_{\mathcal{B}}^{B \triangle R}$ is a clique on an even number of blue and odd number of red vertices.*

4. *If* p *is odd and* q *is even, then* $rC(S)$ *is either equal to* r *or* s*. Since* $rC(S) = r$ *is impossible, this occurs if* $C(S) = rs$*, which means that* $\Gamma_{\mathcal{B}}^{B\triangle R}$ *is a clique on an odd number of blue and an even number of red vertices.*

In each of these four cases, any element w of S can be eliminated from $\mathcal{B} \cup \{z\}$ to form the alternative 2-square basis $\mathcal{B}'$. If $\mathcal{B}' = (\mathcal{B} \cup \{z\}) \backslash \{w\}$, a description of the relationship between the edge sets of the graphs $\Gamma_{\mathcal{B}'}$ and $\Gamma_{\mathcal{B}}$ is provided by a direct application of Theorem 4.2. The vertex sets may differ by the colour of a single vertex, if the elements w and z have different squares. These general considerations may be applied to all 2-square bases. Our interest however is in the case of 2-uniform graphs, in which the number of blue vertices coincides with the uniform rank of the associated group, and is thus maximal among all 2-coloured graphs representing that group. If $\Gamma_{\mathcal{B}}$ is a 2-uniform graph, a basis change of the type described above cannot replace a red vertex with a blue one; graphs that admit this possibility are excluded by Theorem 3.12. Basis changes that replace a blue vertex with a red one do not preserve the 2-uniform property and are thus not of interest (except in the case where the numbers of blue and red vertices differ by 1, which is considered below).

For a 2-uniform graph of uniform corank at least 2, we refer to the operation of adjusting one 2-uniform basis to another, by replacing a single element, as an *exchange* operation. We refer to the transition between their corresponding graphs as an exchange operation of graphs, where we assume that both graphs have the same vertex set, with a single vertex relabelled in the transition. In Theorem 4.6, we give a graph-theoretic description of the exchange operations on 2-uniform graphs that preserve the colour of the relabelled vertex (and hence preserve the 2-uniform property). We refer to exchanges of this type as *simple exchanges*.

Before stating Theorem 4.6, which describes the effect of a simple exchange on a 2-uniform graph, we introduce some notation for the neighbour set of a vertex, via coloured or uncoloured edges.

For a vertex v of a 2-coloured graph Γ, we write $E(v)$ for the set $E(N^B(v), N^R(v))$, where $N^B(v)$ and $N^R(v)$ respectively denote the sets of neighbours of v in Γ, via blue and red edges. If $N^{B\backslash R}(v)$ denotes the set of vertices of Γ that are adjacent to v via blue edges only, and $N^{R\setminus B}(v)$ and $N^{R\cap B}(v)$ are similarly defined, then $E(v)$ is the edge set of the complete tripartite (or bipartite or null) graph with parts $N^{B\setminus R}(v)$, $N^{R\setminus B}(v)$ and $N^{B\cap R}(v)$. We consider $E(v)$ itself to be a set of uncoloured edges, and write $E^R(v)$ and $E^B(v)$ respectively for the same set of edges, all coloured red or all blue.

In the statement of Theorem 4.6 below, we consider that the graphs Γ' and Γ have the same vertex set, with a single vertex relabelled in the transition from one graph to the other.

Theorem 4.6. *Let Γ be a 2-uniform graph of order n, with at least two red vertices, describing a 2-uniform covering group G of C_2^n, with respect to a basis $\mathcal{B}$. If Γ' is a 2-uniform graph describing G with respect to a basis obtained from $\mathcal{B}$ by a simple exchange operation, then at least one of the following occurs.*

1. *(Type 1) Γ^B is a clique on an even number of blue vertices, and $E(\Gamma') = E(\Gamma) \triangle E^R(v)$, for some vertex v of the clique.*

2. *(Type 2) Γ^R is a clique on an even number of blue vertices and a positive even number of red vertices, and $E(\Gamma') = E(\Gamma) \triangle E^B(v)$, for some red vertex v of the clique.*

3. *(Type 3) Γ^B is a clique on an odd number of blue and an odd number of red vertices, and $E(\Gamma') = E(\Gamma) \triangle E^R(v)$, for some red vertex v of the clique.*

4. *(Type 4) $\Gamma^{B\triangle R}$ is a clique on an odd number of blue and an even number of red vertices, and $E(\Gamma') = E(\Gamma) \triangle E^R(v) \triangle E^B(v)$, for some red vertex v of the clique.*

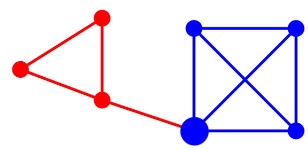
Proof. We consider how each of the exchanges described in Lemma 4.5 affects a 2-uniform graph. By Theorem 3.12, only certain cases of items 1., 2. and 4. of Lemma 4.5 can occur in a 2-uniform graph. Item 1. occurs if Γ^B is a clique on an even number of vertices(Type 1), or Γ^R a clique on even number of blue and a positive even number of red vertices (Type 2). Item 2. occurs if Γ^B a clique on an odd number of blue and an odd number of red vertices (Type 3). Item 4. occurs if $\Gamma^{B\triangle R}$ a clique on an odd number of blue vertices and an even number of red vertices (Type 4).

In each case, the effect on the graph of the basis adjustment described in Lemma 4.5 follows from application of Theorem 4.2 and Corollary 4.3. For example, in an exchange of Type 1 above, it follows from Corollary 4.3 that Γ and Γ' have the same set of blue edges. By Theorem 4.2, the set of red edges of Γ' differs from that of Γ by the set $E(P,Q)$, where P and Q are the sets of neighbours of the exchanged vertex v in Γ^R and Γ^B respectively. Thus $E(P,Q) = E(v)$, and $E^R(\Gamma') = E^R(\Gamma) \triangle E^R(v)$. Since the blue edges are identical in Γ and Γ', it follows that $E(\Gamma') = E(\Gamma) \triangle E^R(v)$.

The statements for Types 2, 3 and 4, are obtained by similar reasoning. □

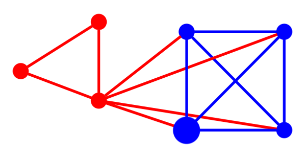
Example 4.7. We present here examples of simple exchange operations of each of the types in Theorem 4.6. We show their alternative graphs.

1. Let Γ_1 be the following 2-uniform graph.



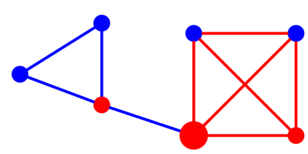
Γ_1

Since Γ_1^B is a clique on four blue vertices, by Theorem 4.6, we may apply a type 1 exchange and Γ_1' is given by $E(\Gamma_1) \triangle E^R(X_1)$ for the vertex X_1 of the clique that is distinguished by size.



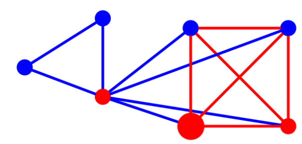
Γ_1'

2. Let Γ_2 be the following 2-uniform graph.



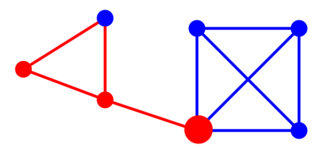
Γ_2

Since Γ_2^R is a clique on two blue vertices and two red vertices, by Theorem 4.6, we may apply type 2 exchange and Γ_2' may be given by $E(\Gamma_2) \triangle E^B(Y_1)$, for the red vertex Y_1 of the clique that is distinguished by size.



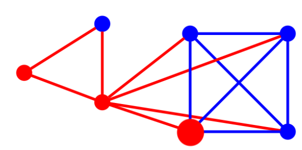
Γ_2'

3. Let Γ_3 be the following 2-uniform graph.



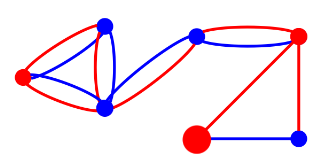
Γ_3

Since Γ_3^B is a clique on one red and three blue vertices, then Γ_3' may be given by $E(\Gamma_3) \triangle E^R(Y_1)$, for the red vertex Y_1 of the clique that is distinguished by size.



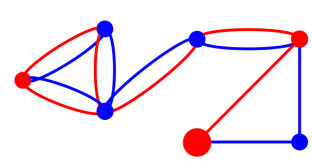
Γ_3'

4. Let Γ_4 the 2-uniform graph that represent the group G.



Γ_4

Since $\Gamma_4^{B \triangle R}$ is a clique on one blue and two red vertices, then Γ_4' may be given by $E(\Gamma_4) \triangle E^R(Y_1) \triangle E^B(Y_1)$, for the red vertex Y_1 of the clique that is distinguished by size.



Γ_4'

It remains to consider exchange operations that switch the vertex colours. If the uniform rank k of G exceeds the uniform corank $n-k$ by only 1 or 2, then Lemma 3.11 gives conditions under which G may possess a 2-uniform basis having k elements of square s and $n-k$ of square r. We conclude this chapter by giving a description of the corresponding graph operations in such cases.

Theorem 4.8. *Let Γ be a 2-uniform graph of order $n = 2k-1$, with k blue vertices and $k-1$ red vertices, describing a 2-uniform covering group G of C_2^n with respect to a basis $\mathcal{B}$ consisting of k elements with square r and $k-1$ elemnents with square s. Suppose that Γ' is a 2-uniform graph that describes G with respect to a basis obtained from $\mathcal{B}$ by replacing an element of square r with an element of square s. Then at leat one of the following occurs.*

1. *Γ^R is a clique on a positive even number of blue vertices and an even number of red vertices, and Γ' is the colour opposite of the graph Γ_1 defined as follows for some blue vertex v of this clique. The vertex v is coloured red in Γ_1, and $E(\Gamma_1) = E(\Gamma) \triangle E^B(v)$.*

2. *Γ^B is a clique on a odd number of blue vertices and an odd number of red vertices, and Γ' is the colour opposite of the graph Γ_1 defined as follows for some blue vertex v of this clique. The vertex v is coloured red in Γ_1, and $E(\Gamma_1) = E(\Gamma) \triangle E^R(v)$.*

3. *$\Gamma^{B\triangle R}$ is a clique on an odd number of blue vertices and an even number of red vertices, and Γ' is the colour opposite of the graph Γ_1 defined as follows for some blue vertex v of this clique. The vertex v is coloured red in Γ_1, and $E(\Gamma_1) = E(\Gamma \triangle E^R(v) \triangle E^B(v)$.*

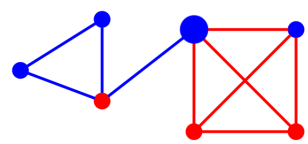
Proof. As noted in the proof of Theorem 4.6, it follows from Theorem 3.12, that only items 1., 2. and 4. of Lemma 4.5 can arise in the 2-uniform graph Γ. By Item 1. of Lemma 4.5, an additional red vertex can replace a blue vertex if Γ^R is a clique on an even number of red vertices and a positive even number of blue vertices. Item 2. occurs if Γ^B is a clique on an odd number of blue and an odd number of red vertices; in this case a blue vertex of this clique may be replaced with a red one. Item 4. occurs if $\Gamma^{B\triangle R}$ is a clique on an odd number of blue and and even number of red vertices; again a blue vertex of this clique can be replaced with a red one in this case.

Each of these exchanges produces a graph with k red vertices and $k-1$ blue vertices, whose edge set is described by application of Theorem 4.2 and Corollary 4.3, as in the proof of Theorem 4.6. The colour opposite of such a graph is an alternative 2-uniform graph representing the same group. □

Theorem 4.8 is proved by direct application of Theorem 4.2 and Corollary 4.3.

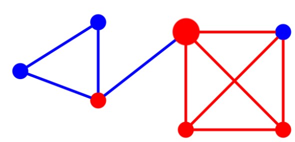
Example 4.9. We will present here examples of 2-uniform graphs that satisfies the conditions of the Theorem 4.8 and show their alternative graphs.

1. Let Γ_1 be the following 2-uniform graph.

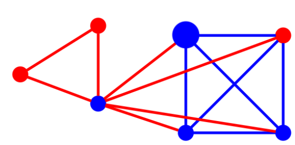


Γ_1

Since Γ_1^R is a clique on a two blue and two red vertices, we may choose the blue vertex X_1 of this clique that is distinguished by size, transform Γ_1 to Γ_1' by switching the colour of X_1 from blue to red, and then define Γ_1'' to be the colour opposite of the graph with edge set $E(\Gamma_1')\triangle E^B(X_1)$.

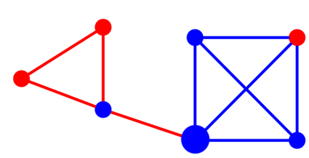


Γ_1'



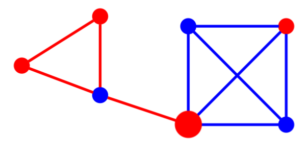
Γ_1''

2. Let Γ_2 be the following 2-uniform graph.

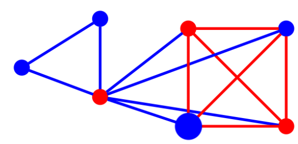


Γ_2

Since Γ_2^B is a clique on one red and three blue vertices, we may choose the blue vertex X_1 of this clique,that is distinguished by size, transform Γ_2 to Γ_2' by switching the colour of X_1 from blue to red, and then define Γ_2'' to be the colour opposite of the graph with edge set $E(\Gamma_2')\triangle E^R(X_1)$.

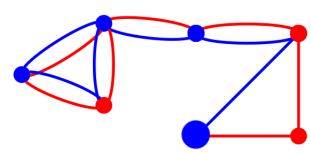


Γ_2'



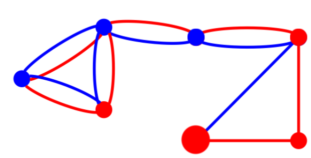
Γ_2''

3. Let Γ be the following 2-uniform graph.

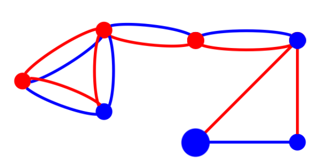


Γ_3

Since $\Gamma_3^{B\triangle R}$ is a clique on one blue and two red vertices, we may choose the blue vertex X_1 of this clique, that is distinguished by size, transform Γ_3 to Γ_3' by switching the colour of X_1 from blue to red, and then define Γ_3'' to be the colour opposite of the graph with edge set $E(\Gamma_3')\triangle E^R(X_1)\triangle E^B(X_1)$.



Γ_3'



Γ_3''

Finally, if $k = (n-k)+2$, and exactly two of the three conditions of Theorem 4.8 hold in Γ (involving different sets of blue vertices), we can increase the number of independent elements of square s by 2, to obtain a 2-uniform basis in which the number of elements of square s is the uniform rank k. We refer to a change of basis of this nature as a *double exchange*. Let the 2-uniform graph Γ, with vertex set V, corresponding to a 2-uniform basis of a covering group G, with k elements of square r represented by the blue vertices, and $k-2$ vertices of square s represented by the red vertices. Let Γ_1 and Γ_2, with vertex sets V_1 and V_2 respectively, be the edge-induced subgraphs of Γ that respectively satisfy two of the three conditions in Theorem 4.8, and let c_1 and c_2 be the elements of G' represented by the edge sets of the cliques Γ_1 and Γ_2. Then $\{c_1, c_2\} \subset \{r, s, rs\}$.

A double exchange operation from Γ to Γ' begins with the selection of a blue vertex v_1 of the clique Γ_1, and a blue vertex v_2 of the clique Γ_2, representing elements x_1 and x_2 of a basis $\mathcal{B}$. In the alternative basis $\mathcal{B}'$, x_1 and x_2 are respectively replaced by z_1 and z_2, which are the products of the elements of $\mathcal{B}$ represented respectively by the vertices of Γ_1 and Γ_2. A necessary condition for $\mathcal{B}'$ to generate the group is that the vertices v_1 and v_2 do not both belong to both Γ_1 and Γ_2. We may assume that Γ_1 includes the vertex v_1 and not v_2.

Since v_2 is incident with no edge of Γ_1, it follows from Corollary 4.3 that the set of edges representing c_1 is the same for both bases. We apply Theorem 4.4 to c_2. The sets P_2 Q_2 coincide, both are equal to $V_2 \backslash \{v_2\}$. If v_1 is incident with no edge of Γ_2, then P_2 is empty and c_2 is described by the same set of edges with respect to both bases, by item 1. of Theorem 4.4.

If the vertex v_1 belongs to the clique Γ_2, then Item 4 of Theorem 4.4 applies, and (since $P_2 = Q_2$), it asserts that edge sets that represent c_2 with respect to the two bases differ by $E(P_1, Q_1) = E(v_1)$, where P_1 and Q_1 are respectively the sets of neighbours of v_1 in Γ_1 and Γ_2. The colour(s) of the adjusted edges depends on whether c_2 coincides with the element r, s or rs.

The following statement summarizes the double exchange operation on graphs.

Theorem 4.10. *Let Γ be a 2-uniform graph satisfying exactly two of the three conditions of Theorem 4.8, on cliques Γ_1 and Γ_2, with vertex sets V_1 and V_2 respectively, involving different sets of red vertices. Let v_1 and v_2 be blue vertices of Γ_1 and Γ_2 respectively, where v_2 does not belong to Γ_1. Let Φ be the graph obtained from Γ by recolouring the vertices v_1 and v_2 from blue to red, and adjusting the edge set as follows:*

1. *If v_1 does not belong to Γ_2, then $E(\Phi) = E(\Gamma)$.*
2. *If v_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^R$, then $E(\Phi) = E(\Gamma) \triangle E^R(v_1)$.*
3. *If v_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^B$, then $E(\Phi) = E(\Gamma) \triangle E^B(v_1)$.*
4. *If v_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^{B \triangle R}$, then $E(\Phi) = E(\Gamma) \triangle E^R(v_1) \triangle E^B(v_1)$.*

Then the colour opposite Γ' of Φ is a 2-uniform graph representing the same covering group as Γ.

Example 4.11. Let Γ be a 2-uniform graph satisfying first two conditions of Theorem 4.8, on cliques Γ_1 and Γ_2, with vertex sets X_1 and X_2 respectively, involving different sets of red vertices. X_1 and X_2 are the blue vertices of Γ_1 and Γ_2 respectively, that they are distinguished by size and where X_2 does not belong to Γ_1.

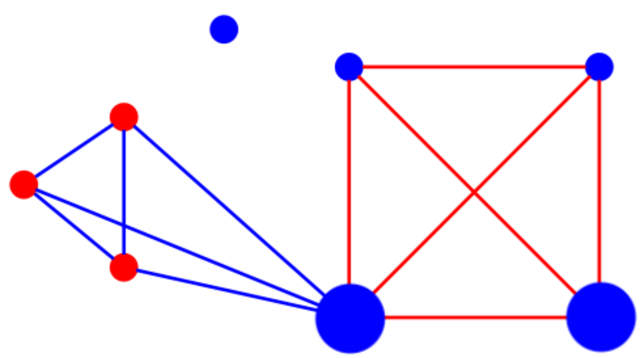


Figure 4.1: Γ

By Theorem 4.10, the graph Φ can be obtained from Γ by recolouring the vertices X_1 and X_2 from blue to red, and since X_1 belongs to Γ_2 and $\Gamma_2 = \Gamma^R$, the edge set of Φ is $E(\Phi) = E(\Gamma) \triangle E^R(X_1)$.

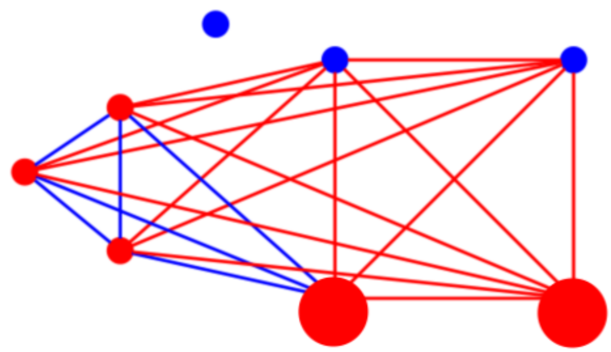


Figure 4.2: Φ

Therefore the colour opposite of Φ is a 2-uniform graph Γ' that represents the same covering group as Γ.

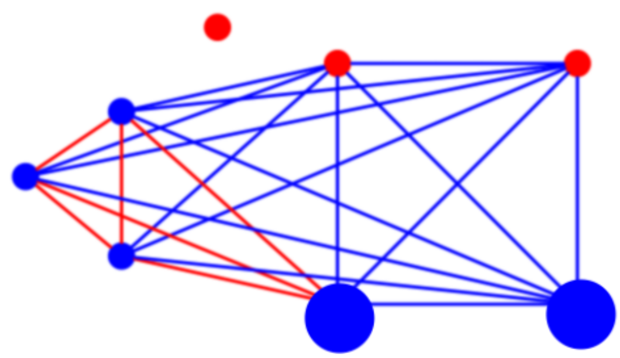


Figure 4.3: Γ'

Chapter 5

Groups of uniform corank 3

Chapter 4 gives an account of those 2-uniform covering groups of C_2^n that admit multiple 2-uniform bases consisting of elements with the same pair of squares. By Theorem 3.8, if r and s are the squares of the elements of a 2-uniform basis of a covering group G of corank at least 4, then *every* 2-uniform basis of G consists of elements with squares r and s, and may be obtained from $\mathcal{B}$ through a sequence of exchange operations of the types described in Chapter 4. In the case of a 2-uniform covering group of C_2^n whose uniform rank k is at least $n-3$, a 2-uniform basis consists of k elements with square r and up to three elements with a different square s. If $k \geqslant 4$, Theorem 3.8 asserts that only one choice is available for element r, but multiple choices may exist for s (and certainly do in the case $k = n-1$). In this section we consider this possibility in the case of uniform corank 3. Our analysis is presented with the assumption that $n \geqslant 7$, but can easily be extended to the case of groups whose uniform rank and corank are both equal to 3. In this case all considerations apply to the colour opposite of all graphs in question, as well to the graphs themselves.

The main result of this chapter is Theorem 5.1, which establishes necessary conditions for a covering group of uniform corank 3, and uniform rank at least 4, to possess multiple 2-uniform bases involving elements with different squares.

Let G be a 2-uniform covering group of C_2^n of uniform corank 3, where $n \geqslant 7$. Let $\mathcal{B} = \{x_1, \dots, x_k, y_1, y_2, y_3\}$ be a 2-uniform basis of G, where $x_i^2 = r$ and $y_i^2 = s$, $r \neq s$. We write $\mathcal{X}$ and $\mathcal{Y}$ for the subsets $\{x_1, \dots, x_k\}$ and $\{y_1, y_2, y_3\}$ of $\mathcal{B}$. By Theorem 3.8, no element of G', except r and possibly s, is the square of more than three independent elements of G'. We now establish the conditions under which $\mathcal{B}$ may be adjusted to a new 2-uniform basis $\mathcal{B}'$, by replacing y_1, y_2, y_3 with independent elements z_1, z_2, z_3 having the same square s', where $s' \neq s$.

Suppose that z_1, z_2, z_3 are elements of G with these properties. Since the squaring map in G is constant on cosets of G', we may assume that each of z_1, z_2, z_3 is the product of some elements of $\mathcal{B}$. We write $\mathcal{Z}_i$ for the set of elements of $\mathcal{B}$ that occur in z_i. For each i, we may write

$$s' = z_i^2 = r^{|\mathcal{Z}_i \cap \mathcal{X}|} s^{|\mathcal{Z}_i \cap \mathcal{Y}|} C_i, \tag{5.1}$$

where $C_i = C(\mathcal{Z}_i)$. Since C_1, C_2 and C_3 are distinct elements of G', their prefixes $r^{|\mathcal{Z}_i \cap \mathcal{X}|} s^{|\mathcal{Z}_i \cap \mathcal{Y}|}$ must also be distinct for $i = 1, 2, 3$.

After relabelling if necessary, we may assume that $|\mathcal{Z}_1 \cap \mathcal{Y}|$ and $|\mathcal{Z}_2 \cap \mathcal{Y}|$ have the same parity. Then $|\mathcal{Z}_1 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$ have opposite parity. Comparing the descriptions of z_1^2 and z_2^2 in (5.1), we find that $r = C_1C_2$, where C_1 and C_2 are elements of G' whose graphs with respect to $\mathcal{B}$ are nontrivial cliques, whose numbers of blue vertices have opposite parity, and whose numbers of red vertices have the same parity. Now $|\mathcal{Z}_3 \cap \mathcal{X}|$ has the same parity as exactly one of $|\mathcal{Z}_1 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$; we may assume this to be $|\mathcal{Z}_1 \cap \mathcal{X}|$, after relabelling again if necessary. Then $|\mathcal{Z}_3 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$ have opposite parity. Comparing the expressions for z_2^2 and z_3^2 in (5.1) gives $s = C_2C_3$, where $C_3 \in G'$ is represented on the vertex set of $\Gamma_{\mathcal{B}}$ by a clique whose numbers of blue and red vertices are respectively of the same and opposite parity to those of the graph representing C_2.

The following Theorem notes the meaning of these observations in terms of a 2-uniform graph representing G. We note that a graph satisfying the conditions of Theorem 5.1 cannot also satisfy the conditions in any of Theorem 4.6, Theorem 4.8 or Theorem 4.10. If a 2-uniform covering group of corank 3 that has multiple 2-uniform bases related by exchange operations of the types described in chapter 4, the same group cannot have multiple 2-uniform bases related by the considerations in this section.

Theorem 5.1. *Let Γ be a 2-uniform graph of order $n \geqslant 7$, with three red vertices. Let G be the 2-uniform covering group of C_2^n with basis $\mathcal{B}$ determined by Γ. Then G contains elements*

z_1, z_2, z_3 representing different cosets of G' and all having the same square s', with $s' \notin \{r, s\}$ if and only if the following conditions hold in Γ.

1. *$E(\Gamma^B) = E(\Phi_1) \triangle E(\Phi_2)$, where Φ_1 and Φ_2 are nontrivial cliques whose numbers of blue vertices have opposite parity and whose numbers of red vertices have the same parity, and;*

2. *$E(\Gamma^R) = E(\Phi_2) \triangle E(\Phi_3)$, where Φ_3 is a nontrivial clique whose numbers of blue and red vertices respectively have the same and opposite parity to the corresponding numbers in Φ_2.*

If these conditions are satisfied, let z_i be the product in G of the basis elements represented by the vertices of Φ_i (in any order). Then $z_1^2 = z_2^2 = z_3^2$.

For a graph Γ satisfying the conditions of Theorem 5.1, it is not automatic that the elements z_1, z_2, z_3 are independent of the $n-3$ basis elements represented by blue vertices in Γ. This requires a linear independence condition which we express in matrix terms as follows. Let v_1, v_2, v_3 be labels on the red vertices of Γ. Define a 3×3 matrix $B \in M_3(\mathbb{F}_2)$ whose (i, j) entry is 1 if the vertex v_j occurs in the clique Φ_i, and 0 otherwise. Then $\{z_1, z_2, z_3\}$ extends the set of elements of $\mathcal{B}$ represented by blue vertices in Γ to a 2-uniform basis $\mathcal{B}'$ of G, if and only if B is nonsingular in $M_3(\mathbb{F}_2)$.

Our theme for the remainder of this section is a description of the relationship between the graphs determined by the 2-uniform bases $\mathcal{B}$ and $\mathcal{B}'$ of G, when the matrix B is nonsingular.

We begin with some remarks on the uniqueness of Φ_1, Φ_2 and Φ_3, under the conditions of Theorem 5.1. This involves the application of Theorem 3.8 and its proof. It was shown there that the edge set of any graph has at most one expression as the symmetric difference of the edge sets of two cliques, with the two exceptions of the path P_3 on 3 vertices, and the cycle C_4 on four vertices. Each of these has two expressions as the symmetric difference of a pair of cliques. Under the conditions of Theorem 5.1, the question of alternative possibilities for the Φ_i (and hence the z_i) arises if Γ^B or Γ^R is a copy of P_3 or C_4. For both P_3 and C_4, it is routine to check that there is no colouring of the vertices that yields a decomposition satisfying both the parity conditions of Theorem 5.1 and the requirement that the 3×3 matrix B is nonsingular. We conclude that if $\mathcal{B} = \{x_1, \dots, x_{n-3}, y_1, y_2, y_3\}$ is a 2-uniform basis of a covering group G of C_2^n of corank 3, with $x_i^2 = r$ and $y_i^2 = s \neq r$, then there is at most choice for a set $\{z_1G', z_2G', z_3G'\}$, with the property that $\mathcal{B}' = \{x_1, \dots, x_{n-3}, z_1, z_2, z_3\}$ is an alternative 2-uniform basis of G', where $z_i^2 = s' \neq s$.

We now assume that G is a covering group of corank 3 of C_2^n, possessing 2-uniform bases $\mathcal{B}$ and $\mathcal{B}'$ as above. We write P for the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$, whose jth column records the $\mathcal{B}$-coordinates of the jth element of $\mathcal{B}'$. The first $n-3$ columns of P coincide with those of the identity matrix, and the last three columns respectively correspond to z_1, z_2, z_3, which we assume to be ordered according to the description in Theorem 5.1. Thus P has the following form, where v_1, v_2, v_3 are vectors in $\mathbb{F}_2^{n-3}$, with the property that the numbers of entries equal to 1 in v_1 and v_2 have opposite parity, and the numbers of entries equal to 1 in v_2 and v_3 have the same parity. The lower right block B is a non-singular matrix in $M_3(\mathbb{F}_2)$, with the property that the numbers of entries equal to 1 in its first two columns have the same parity, and the number of entries equal to 1 in its third column has the opposite parity to these.

$$P = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1 & v_2 & v_3 \\ & | & | & | \\ \hline 0_{(n-3)\times 3} & & B_{3\times 3} & \end{array}\right]. \tag{5.2}$$

The graph of $\Gamma_{\mathcal{B}}(G)$ can be constructed from P as follows. For $i = 1, 2, 3$, we write E_i for the edge set of the clique on the set of vertices representing those elements of $\mathcal{B}$ where a 1 occurs in column $(n-3)+i$ of P; i.e. those elements of $\mathcal{B}$ that occur in z_i. The set of blue edges in $\Gamma_{\mathcal{B}}(G)$ is $E_1 \triangle E_2$, and the set of red edges is $E_2 \triangle E_3$. The change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is the inverse of P in $M_n(\mathbb{F}_2)$, given by

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1B^{-1} & v_2B^{-1} & v_3B^{-1} \\ & | & | & | \\ \hline & & & \\ 0_{(n-3)\times 3} & & B^{-1}_{3\times 3} & \\ & & & \end{array}\right]. \tag{5.3}$$

The graph $\Gamma_{\mathcal{B}'}$ that represents G with respect to $\mathcal{B}'$ can be constructed from P^{-1} as $\Gamma_{\mathcal{B}}$ is from P. The edge-induced subgraph $\Gamma'^{B}_{\mathcal{B}}$ comprising its blue edges has the form $\Psi_1 \triangle \Psi_2$, where Ψ_1 and Ψ_2 are cliques whose numbers of red vertices have the same parity and whose numbers of blue vertices have opposite parity. If we assume $\Gamma_{\mathcal{B}}$ and $\Gamma'_{\mathcal{B}}$ to have the same vertex set (with the red vertices labelled differently), the vertices of the cliques Ψ_1 and Ψ_2 are written in some pair of the last three columns of P^{-1}; these are the two columns in which the numbers of 1s among the last three entries have the same parity. Similarly, $\Gamma'^{R}_{\mathcal{B}} = \Psi_2 \triangle \Psi_3$, where the clique Ψ_3 is described by the remaining columns of P^{-1}, which also contains the information to distinguish Ψ_1 from Ψ_2, on the basis that that the numbers of blue vertices in Ψ_2 and Ψ_3 have the same parity.

We now detail the transformations from $\Gamma_{\mathcal{B}}$ to $\Gamma'_{\mathcal{B}}$ corresponding to the distinct possibilities for the matrix B in the lower right 3×3 block of the matrix P. In the following analysis of these cases, we write S, T, U respectively for the vertex sets of the cliques Φ_1, Φ_2 and Φ_3, and use the superscripts B and R to denote their sets of blue and red vertices.We may reorder the elements y_1, y_2, y_3 in $\mathcal{B}$ as necessary, to ensure that the 3×3 matrix B in the lower right block of P has one of the following standard forms. Each of these forms occurs in two versions, depending on whether $|S^B|$, which is the number of 1s in v_1, is even or odd. We have a total of 16 cases, some pairs of which are equivalent under the the transition between the two bases. Distinguishing the cases on the basis of graph $\Gamma_{\mathcal{B}}$ generally requires the expression for the sets of blue and red edges as symmetric differences of cliques.

1. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Case 1.(a): $|S^B|$ is odd. Case 1.(b): $|S^B|$ is even.

2. $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Case 2.(a): $|S^B|$ is odd. Case 2.(b): $|S^B|$ is even.

3. $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Case 3.(a): $|S^B|$ is odd. Case 3.(b): $|S^B|$ is even.

4. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Case 4.(a): $|S^B|$ is odd. Case 4.(b): $|S^B|$ is even.

5. $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Case 5.(a): $|S^B|$ is odd. Case 5.(b): $|S^B|$ is even.

6. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Case 6.(a): $|S^B|$ is odd. Case 6.(b): $|S^B|$ is even.

7. $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Case 7.(a): $|S^B|$ is odd. Case 7.(b): $|S^B|$ is even.

8. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Case 8.(a): $|S^B|$ is odd. Case 8.(b): $|S^B|$ is even.

We now analyse the transformation between $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$ in all cases.

1. In Case 1, we write P as in (5.2) and observe

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_3 & v_1+v_3 & v_1+v_2+v_3 \\ & | & | & | \\ \hline & 0 & 1 & 1 \\ 0_{(n-3)\times 3} & 0 & 0 & 1 \\ & 1 & 1 & 1 \end{array}\right].$$

After reordering the last three columns and last three rows to obtain a standard form as above, we have the following descriptions of the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ respectively for Cases 1(a) and 1(b).

$$1.(a)\ \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_3 & v_1+v_2+v_3 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 0 & 1 & 0 \\ & 0 & 1 & 1 \end{array}\right] \quad 1.(b)\ \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_2+v_3 & v_3 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 1 & 0 & 0 \\ & 1 & 0 & 1 \end{array}\right]$$

The matrices above are of types 7(b) and 8(b) respectively, and we conclude that Cases 1(a) and 1(b) are respectively equivalent to 7(b) and 8(b), in terms of the covering groups that they describe.

2. In Case 2,

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_3 & v_3 & v_2+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 0 & 0 & 1 \\ & 1 & 1 & 1 \end{array}\right].$$

After reordering the last three columns and last three rows to obtain a standard form as above, we have the following descriptions of the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ respectively for Cases 2(a) and 2(b).

$$2.(a) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_3 & v_2+v_3 & v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 0 \end{array}\right] \quad 2.(b) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2+v_3 & v_1+v_3 & v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 0 \end{array}\right]$$

The matrices above are again of types 2(a) and 2(b) respectively, for these cases the graphs with respect to both $\mathcal{B}$ and $\mathcal{B}'$ are of the same type, 2(a) or 2(b). In these cases, the graphs $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$ are related in Case 2(a) by

$$V(\Psi_1) = V(\Phi_1) \triangle V^B(\Phi_3), V(\Psi_2) = V(\Phi_2) \triangle V^B(\Phi_3),\ V(\Psi_3) = V(\Phi_3)$$

and in Case 2(b) by

$$V(\Psi_1) = V(\Phi_2) \triangle V^B(\Phi_3), V(\Psi_2) = V(\Phi_1) \triangle V^B(\Phi_3),\ V(\Psi_3) = V(\Phi_3)$$

3. In Case 3,

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_2+v_3 & v_2+v_3 & v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 1 & 1 & 0 \\ & 1 & 1 & 1 \end{array}\right].$$

In Cases 3(a) and 3(b), this may be adjusted to the following standard forms

$$3.(a) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_2+v_3 & v_3 & v_2+v_3 \\ & | & | & | \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 1 & 0 & 0 \\ & 1 & 0 & 1 \end{array}\right] \quad 3.(b) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_3 & v_1+v_2+v_3 & v_2+v_3 \\ & | & | & | \\ \hline & 1 & 1 & 1 \\ 0_{(n-3)\times 3} & 0 & 1 & 0 \\ & 0 & 1 & 1 \end{array}\right]$$

The matrices above are of types 8(a) and 7(a) respectively, and we conclude that Cases 3(a) and 3(b) are respectively equivalent to 8(a) and 7(a), in terms of the covering groups that they describe.

4. In Case 4,

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2+v_3 & v_1+v_2+v_3 & v_1+v_3 \\ & | & | & | \\ \hline & 0 & 1 & 1 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 1 & 0 & 1 \end{array}\right].$$

In Cases 4(a) and 4(b), this may be adjusted to the following standard forms

$$4.(a) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2+v_3 & v_1+v_3 & v_1+v_2+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 1 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 1 \end{array}\right]$$

$$4.(b) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1+v_3 & v_2+v_3 & v_1+v_2+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 1 \\ 0_{(n-3)\times 3} & 1 & 1 & 1 \\ & 0 & 1 & 1 \end{array}\right]$$

The matrices above are again of types 4(b) and 4(a) respectively; the graphs that represent 4(a) and 4(b) are equivalent.

5. In Case 5,

$$P^{-1} = \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_1 & v_2 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 1 \\ 0_{(n-3)\times 3} & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right].$$

In Cases 5(a) and 5(b), this may be adjusted to the following standard forms

$$5.(a) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2 & v_1 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 0 & 1 & 1 \\ & 0 & 0 & 1 \end{array}\right] \quad 5.(b) \left[\begin{array}{c|ccc} & | & | & | \\ I_{n-3} & v_2 & v_1 & v_1+v_3 \\ & | & | & | \\ \hline & 1 & 0 & 0 \\ 0_{(n-3)\times 3} & 0 & 1 & 1 \\ & 0 & 0 & 1 \end{array}\right]$$

The matrices above are again of types 6(b) and 6(a) respectively, and we conclude that Cases 5(a) and 5(b) are respectively equivalent to 6(b) and 6(a), in terms of the covering groups that they describe.

If $n \geqslant 7$, a 2-uniform covering group of corank 3 of C_2^n that satisfies the conditions of Theorem 5.1 possesses exactly two 2-uniform bases $\mathcal{B}$ and $\mathcal{B}'$, up to coset representatives modulo G'. The graphs corresponding to the two bases are encoded by the change of basis matrices P and P^{-1}, and are typically non-isomorphic. The conclusion of this section is that in order to list all isomorphism types of such groups, it is sufficient to consider matrices of types 1(a), 1(b), 2(a), 2(b), 3(a), 3(b), 4(a), 5(a) and 5(b). The associated graphs capture every group isomorphism type once, except for those encoded by matrices of types 2(a) and 2(b), which are generally represented by two different graphs. Since the three columns in the upper right $(n-3) \times 3$ region can be chosen independently, the number of matrices of each of these types is $(2^{n-4})^3$. Most isomorphism types of groups of types 2 are counted twice by this count of distinct matrices, but on all other cases, the distinct matrices correspond bijectively with the isomorphism classes of groups. The number of isomorphism types of 2-uniform covering groups of C_2^n and uniform corank 3, that admit two different choices for the common square of exactly three elements of a 2-uniform basis, is bounded above by

$$8 \times (2^{n-4})^3 = 2^{3n-9}.$$

Example 5.2. Let Γ be the 2-uniform graph in Figure 5.1. Let G be the 2-uniform covering group of C_2^7 detemined by Γ, with 2-uniform basis $\mathcal{B} = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$ corresponding to the vetices of Γ. The graoh Γ satisfies the following conditions.

1. $E(\Gamma^B) = E(\Phi_1) \triangle E(\Phi_2)$, where Φ_1 is a clique on one blue and three red vertices and Φ_2 is a clique on two blue and one red vertices, and;
2. $E(\Gamma^R) = E(\Phi_2) \triangle E(\Phi_3)$, where Φ_3 is a nontrivial clique one two blue and two red vertices.

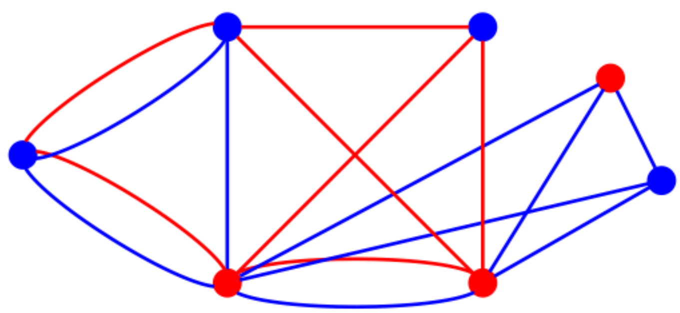


Figure 5.1: Γ

Therefore, by Theorem 5.1, G contains elements z_1, z_2, z_3 where z_i are product in G of the basis elements of $\mathcal{B}$ represented by the vertices of Φ_i and $z_1^2 = z_2^2 = z_3^2 = s'; s' \notin \{r, s\}$. The alternative 2-uniform basis is $\mathcal{B}' = \{x_1, x_2, x_3, x_4, z_1, z_2, z_3\}$, and the change of basis matrix P from $\mathcal{B}'$ to $\mathcal{B}$ is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is the inverse of P in $M_7(\mathbb{F}_2)$, given by

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Hence, the alternative 2-uniform graph of G that corresponds to $\mathcal{B}'$ is given in Figure 5.2.

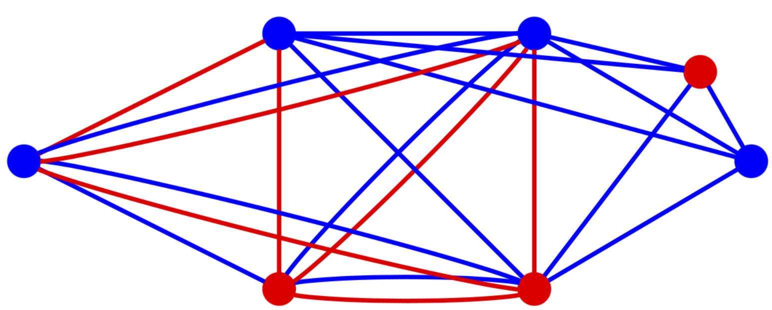


Figure 5.2: Γ'

Moreover, Γ is of Case 8(a) which is equivalent to the case 3(a).

Chapter 6

Groups of uniform corank 2

Let G be a 2-uniform covering group of C_2^n of uniform corank 2, where $n \geqslant 6$. Let $\mathcal{B} = \{x_1, \ldots, x_k, y_1, y_2\}$ be a 2-uniform basis of G, where $x_i^2 = r$ and $y_i^2 = s$, $r \neq s$. We write $\mathcal{X}$ and $\mathcal{Y}$ for the subsets $\{x_1, \ldots, x_k\}$ and $\{y_1, y_2\}$ of $\mathcal{B}$. By Theorem 3.8, no element of G' apart from r and possibly s is the square of more than three independent elements of G', but it is possible that y_1 and y_2 can be replaced in $\mathcal{B}$ by elements z_1 and z_2, to form an alternative 2-uniform basis $\mathcal{B}'$. In this situation, $\mathcal{B}' = \{x_1, \ldots, x_k, z_1, z_2\}$, where $z_1^2 = z_2^2 = s'$ and $s' \notin \{r, s\}$. In this chapter, we consider the conditions on $\Gamma_{\mathcal{B}}(G)$ which admit this possibility. As in Chapter 5, we consider the change of basis matrix P from $\mathcal{B}'$ to $\mathcal{B}$, whose columns list the coordinates of the elements of $\mathcal{B}'$ with respect to $\mathcal{B}$. Unlike the case of uniform corank 3, this matrix does not fully determine the group. We discuss the relationship between the graphs $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$.

We assume that G contains elements z_1 and z_2 as described above, and as in Chapter 5 we write $\mathcal{X}$ and $\mathcal{Y}$ for the subsets $\{x_1, \ldots, x_k\}$ and $\{y_1, y_2\}$ of $\mathcal{B}$. We may assume that each of z_1 and z_2 is a product of elements of $\mathcal{B}$, and we write $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively for the sets of elements of $\mathcal{B}$ that occur in z_1 and z_2. We note that each of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ has at least two elements. That $\mathcal{X} \cup \{z_1, z_2\}$ generates G requires that the sets $\mathcal{Z}_1 \cap \mathcal{Y}$ and $\mathcal{Z}_2 \cap \mathcal{Y}$ are distinct and non-empty. Comparing the expressions for z_1^2 and z_2^2 in terms of the elements of $\mathcal{B}$, we observe that $z_1^2 = z_2^2$ if and only if one of the following conditions holds.

Case 1 $r = C_1 C_2$, where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$

by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. This occurs if $|\mathcal{X} \cap \mathcal{Z}_1|$ and $|\mathcal{X} \cap \mathcal{Z}_2|$ have opposite parity, and $|\mathcal{Y} \cap \mathcal{Z}_1|$ and $|\mathcal{Y} \cap \mathcal{Z}_2|$ have the same parity (which must be odd). After relabelling, we may interpret this last condition as saying that $y_1 \in \mathcal{Z}_1 \backslash \mathcal{Z}_2$, $y_2 \in \mathcal{Z}_2 \backslash \mathcal{Z}_1$, $|\mathcal{Z}_1|$ is odd and $|\mathcal{Z}_2|$ is even.

Case 2 $s = C_1 C_2$, where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$ by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. This occurs if $|\mathcal{X} \cap \mathcal{Z}_1|$ and $|\mathcal{X} \cap \mathcal{Z}_2|$ have the same parity, and $|\mathcal{Y} \cap \mathcal{Z}_1|$ and $|\mathcal{Y} \cap \mathcal{Z}_2|$ have opposite parity. After relabelling, we may infer from this last condition that $y_1 \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, and $y_2 \in \mathcal{Z}_2 \backslash \mathcal{Z}_1$. We distinguish the following subcases:

Case 2(a) $|\mathcal{X} \cap \mathcal{Z}_1|$ and $|\mathcal{X} \cap \mathcal{Z}_2|$ are odd.

Case 2(b) $|\mathcal{X} \cap \mathcal{Z}_1|$ and $|\mathcal{X} \cap \mathcal{Z}_2|$ are even.

Case 3 $rs = C_1 C_2$, where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$ by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively. This occurs if $|\mathcal{X} \cap \mathcal{Z}_1|$ and $|\mathcal{X} \cap \mathcal{Z}_2|$ have opposite parity, and $|\mathcal{Y} \cap \mathcal{Z}_1|$ and $|\mathcal{Y} \cap \mathcal{Z}_2|$ have opposite parity. As in the second case above, we may assume in this situation that both y_1 and y_2 occur in $\mathcal{Z}_2$ and that only y_1 occurs in $\mathcal{Z}_1$. Again we consider two subcases, depending on the numbers of blue vertices in the cliques describing C_1 and C_2.

Case 3(a) $|\mathcal{X} \cap \mathcal{Z}_1|$ is odd and $|\mathcal{X} \cap \mathcal{Z}_2|$ is even.

Case 3(b) $|\mathcal{X} \cap \mathcal{Z}_1|$ is even and $|\mathcal{X} \cap \mathcal{Z}_2|$ is odd.

It is possible for more than one of Cases 1, 2 and 3 to occur simultaneously, so that there may be multiple choices for the pair of elements $\{z_1, z_2\}$. It is even possible, with Case 3(b), that the same graph may admit two different choices for C_1 and C_2, in a case where rs is represented by a 4-cycle that has two different descriptions as the symmetric difference of two copies of the complete graph K_3. In all other cases, it follows from Theorem 3.8 and the parity restrictions that there is only one possible choice for the pair (C_1, C_2) corresponding to the description of r, s or rs as a product of two elements represented by complete graphs.

In each of the three cases, we write $\mathcal{B}'$ for the basis obtained from $\mathcal{B}$ by replacing y_1 and y_2 by z_1 and z_2, and consider the relationship between the graphs $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}'}$. We consider these two graphs to have the same vertex set, where the red vertices that represent y_1 and y_2 in $\Gamma_{\mathcal{B}}$ respectively represent z_1 and z_2 in $\Gamma_{\mathcal{B}'}$. In all cases, Theorem 4.4 provides a template for the description of the relationship between the two graphs.

As in Chapter 5, we may consider the change of basis matrix P from $\mathcal{B}'$ to $\mathcal{B}$, whose columns list the coordinates of the elements of $\mathcal{B}'$ with respect to $\mathcal{B}$. Unlike the case of uniform corank 3, this matrix does not fully describe the group, but only one of the three elements r, s and rs. The matrix P, and its inverse, have the following forms.

$$P = \left[\begin{array}{c|cc} & | & | \\ I_{n-2} & v_1 & v_2 \\ & | & | \\ \hline & 1 & e \\ 0_{(n-2)\times 2} & 0 & 1 \end{array}\right], \qquad P^{-1} = \left[\begin{array}{c|cc} & | & | \\ I_{n-2} & v_1 & ev_1 + v_2 \\ & | & | \\ \hline & 1 & e \\ 0_{(n-2)\times 2} & 0 & 1 \end{array}\right],$$

where $e = 0$ or 1, and v_1 and v_2 are columns with entries in $\mathbb{F}_2$. We write $n(v)$ for the number of non-zero entries in the column vector v of the top right block of the matrix P. If $e = 0$, then $n(v_1)$ is even and $n(v_2)$ is odd. The condition that $z_1^2 = z_2^2$ means that the above cases and subcases are encoded in the matrix P as in the following table.

	e	$n(v_1)$	$n(v_2)$
Case 1	0	even	odd
Case 2(a)	1	odd	odd
Case 2(b)	1	even	even
Case 3(a)	1	odd	even
Case 3(b)	1	even	odd

From the description of P^{-1} in terms of P, we note that if P describes an instance of Case 2(a), then P^{-1} describes one of Case 3(a), and vice versa. Hence every 2-uniform graph that satisfies condition 2(a) is equivalent to one that satisfies condition 3(a), and it is sufficient to consider one of these conditions in a description of graphs that describe 2-uniform covering groups of uniform corank 2, up to isomorphism.

In all other rows of the table above, the matrices P and P^{-1} correspond to the same row of the table. In these cases, the relationship between the 2-uniform graphs $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$ is described by Theorem 4.4.

1. In Case 1, we have $s' = z_1^2 = sC_1$. By Corollary 4.3, the graphs representing C_1 and C_2, and hence r, are the same with respect to both bases, so $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$ have the same sets of blue edges. We write Q_1 and Q_2 for the respective sets of blue vertices in the cliques representing the elements C_1 and C_2 with respect to $\mathcal{B}$, and we write P_1 and P_2 for the sets of neighbours of y_1 and y_2 in $\Gamma_{\mathcal{B}}(s)$. Then the set of red edges of $\Gamma_{\mathcal{B}'}(s')$, hence of $\Gamma_{\mathcal{B}'}(G)$, is given by $E(\Gamma_{\mathcal{B}'}(s))\triangle E(C_1)$, and from Theorem 4.4 we have

$$E(\Gamma_{\mathcal{B}'}(s')) = \begin{cases} E(\Gamma_{\mathcal{B}}(s))\triangle E(P_1, Q_1)\triangle E(P_2\triangle Q_1, Q_2) & \text{if the red vertices of } \Gamma_{\mathcal{B}}(G) \text{ are} \\ & \text{adjacent via a red edge} \\ E(\Gamma_{\mathcal{B}}(s))\triangle E(P_1, Q_1)\triangle E(P_2, Q_2) & \text{otherwise} \end{cases}$$

2. In Case 2(b), $s' = z_2^2 = C_2$. By inspecting the entries of the last two columns of P^{-1} (or by applying Theorem 4.4 to the graph of C_2 with respect to $\mathcal{B}$), we observe that the set of red edges in $\Gamma_{\mathcal{B}'}(G)$ is the symmetric difference of the edge sets of the cliques on the sets of vertices representing $\mathcal{Z}_1$ and $(\mathcal{Z}_1\triangle\mathcal{Z}_2)\cup\{z_1\}$. The blue edges of $\Gamma_{\mathcal{B}}(G)$ are independent of the red edges and of condition 2(a), and Theorem 4.4 describes how they change under the change of basis. We write P_1 and P_2 for the sets of neighbours in $\Gamma_{\mathcal{B}}(r)$ of the vertices representing y_1 and y_2 respectively, and Q_1 and Q_2 for the sets of vertices respectively representing $\mathcal{Z}_1\backslash\{y_1\}$ and $\mathcal{Z}_2\backslash\{y_2\}$. Then the set of blue edges of $\Gamma_{\mathcal{B}'}(G)$ is given by

$$E(\Gamma_{\mathcal{B}'}(r)) = \begin{cases} E(\Gamma_{\mathcal{B}}(r))\triangle E(P_1, Q_1)\triangle E(P_2\triangle Q_1, Q_2) & \text{if the red vertices of } \Gamma_{\mathcal{B}}(G) \text{ are} \\ & \text{adjacent via a blue edge} \\ E(\Gamma_{\mathcal{B}}(r))\triangle E(P_1, Q_1)\triangle E(P_2, Q_2) & \text{otherwise} \end{cases}$$

3. In case 3(b), $s' = z_2^2 = sC_1$. From the matrix P^{-1} we note that $\Gamma_{\mathcal{B}'}(rs')$ is the symmetric difference of the cliques on the sets of vertices representing $\mathcal{Z}_1$ and

$(\mathcal{Z}_1 \triangle \mathcal{Z}_2) \cup \{z_1\}$, which respectively involve an even and odd number of blue vertices. The set of blue edges in $\Gamma_{\mathcal{B}'}(G)$ is given, as in Case 2(b), by

$$E(\Gamma_{\mathcal{B}'}(r)) = \begin{cases} E(\Gamma_{\mathcal{B}}(r)) \triangle E(P_1, Q_1) \triangle E(P_2 \triangle Q_1, Q_2) & \text{if the red vertices of } \Gamma_{\mathcal{B}}(G) \text{ are adjacent via a blue edge} \\ E(\Gamma_{\mathcal{B}}(r)) \triangle E(P_1, Q_1) \triangle E(P_2, Q_2) & \text{otherwise} \end{cases}$$

where P_1, P_2, Q_1, Q_2 have the same definitions as in Case 2(b). Finally, the set of red edges in $\Gamma_{\mathcal{B}'}(G)$ is the symmetric difference of the edges sets of the graphs representing rs' and r.

Example 6.1. Let G be a 2-uniform covering group of C_2^6 of uniform corank 2. Let $\mathcal{B} = \{x_1, x_2, x_3, x_4, y_1, y_2\}$ be a 2-uniform basis of G, where $x_i^2 = r$ and $y_i^2 = s$, $r \neq s$. Let $r = C_1C_2$ where C_1 and C_2 are elements of G' represented with respect to $\mathcal{B}$ by cliques on the sets of vertices corresponding to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ respectively

$$\begin{aligned} z_1 &= x_1x_2x_4y_1 \\ z_2 &= x_3x_4y_2 \end{aligned}$$

The 2-uniform graph that represents G and corresponds to $\mathcal{B}$ is

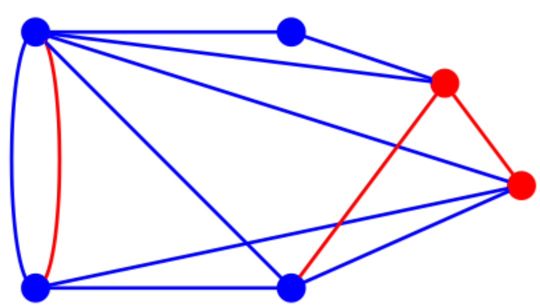


Figure 6.1: Γ

The alternative basis $\mathcal{B}'$ of the group G is obtained from $\mathcal{B}$ by replacing y_1 and y_2 by z_1 and z_2. The change of basis matrix P from $\mathcal{B}'$ to $\mathcal{B}$, whose columns list the coordinates of the elements of $\mathcal{B}'$ with respect to $\mathcal{B}$ and the inverse matrix P^{-1} are equal

$$P = P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have $s' = z_2^2 = sC_2$ and r, are the same with respect to both bases, so $\Gamma_{\mathcal{B}}(G)$ and $\Gamma_{\mathcal{B}'}(G)$ have the same sets of blue edges. We write $Q_1 = \{x_1, x_2, x_4\}$ and $Q_2 = \{x_3, x_4\}$ for the respective sets of blue vertices in the cliques representing the elements C_1 and C_2 with respect to $\mathcal{B}$, and we write $P_1 = \{y_2\}$ and $P_2 = \{x_2, y_1\}$ for the sets of neighbours of y_1 and y_2 in $\Gamma_{\mathcal{B}}(s)$. Then the set of the red edges is $E(P_1, Q_1) \triangle E(P_2 \triangle Q_1, Q_2)$, and the the alternative 2-uniform graph of G that corresponds to $\mathcal{B}'$ is

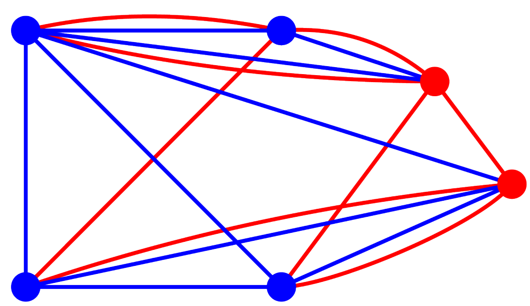


Figure 6.2: Γ'

Chapter 7

Groups of uniform corank 1

In this chapter, we suppose that G is an 2-uniform covering group of C_2^n, with $\rho(G) = n-1$ where $n \geqslant 5$. Let $x_1, \ldots, x_{n-1}$ be independent elements of G all with square r. Then $\{x_1, \ldots, x_{n-1}\}$ may be extended to a 2-uniform basis of G by the addition of *any* element y of G that does not belong to the subgroup $X = \langle x_1, \ldots, x_{n-1}\rangle$, and every uniform basis includes $n-1$ elements with square r, by Theorem 3.8. Having chosen y, we write $\Gamma(y)$ for the graph of G with respect to the basis $\{x_1, \ldots, x_{n-1}, y\}$, which has $n-1$ blue vertices representing $x_1, \ldots, x_{n-1}$, and a single red vertex representing y.

We show in Lemma 7.2 and Lemma 7.3 that it is possible to choose y so that the graph $\Gamma(y)$ has a particular form, referred to as *standard form*. In the special case of corank 1, a 2-coloured graph in standard form may be taken as a refinement of the concept of a 2-uniform graph. The remainder of the chapter is devoted to the question of when non-isomorphic graphs in standard form represent isomorphic groups.

Lemma 7.1. *The neighbours in* $\Gamma(y)$ *of the red vertex, via blue edges, do not depend on the choice of* y.

Proof. Suppose that y and y' are different elements of $G \backslash X$. Then $y \in y'xG'$ for some $x \in X$. After relabelling the elements of $\mathcal{B}$ we may suppose that

$$r = [y, x_1 \ldots x_p]c,$$

where $c \in X'$. Then

$$r = [y'x, x_1 \ldots x_p]c = [y', x_1 \ldots x_p]c',$$

where $c' \in X'$. Hence the neighbours of the red vertex in the blue parts of both $\Gamma(y)$ and $\Gamma(y')$ are the vertices representing $x_{i_1}, \dots, x_{i_p}$. $\square$

We continue to write $\{x_1, \dots, x_p\}$ for the set of neighbours of the red vertex via blue edges, in a 2-uniform graph representing G.

Lemma 7.2. *If* p *is even, then for every subset* S *of* $\{x_1, \dots, x_{n-1}\}$*, there is exactly one choice of* y *for which the red vertex is adjacent via red edges in* $\Gamma(y)$ *precisely to those vertices representing elements of* S*. In particular there is exactly one choice of* y *for which the red vertex is incident with no red edge in* $\Gamma(y)$*.*

Proof. We assume that p is even, and choose $z \in G \backslash X$. If $x_{j_1}, \dots, x_{j_q}$ are the basis elements representing the neighbours of the red vertex via red edges in $\Gamma(z)$, we may write

$$z^2 = [z, x_{j_1} \dots x_{j_q}]\, c,$$

where $c \in X'$. Define y by

$$y = \begin{cases} z x_{j_1} \dots x_{j_q} & \text{if } q \text{ is even} \\ z x_{j_1} \dots x_{j_q} x_1 \dots x_p & \text{if } q \text{ is odd} \end{cases}$$

For even q, $y^2 = z^2 r^q C(z, x_{j_1}, \dots, x_{j_q}) \in [z, x_{j_1} \dots x_{j_q}]^2 X'$, and the red vertex in $\Gamma(y)$ is incident with no red edge.

For odd q, $y^2 = z^2 r^q [z, x_{j_1} \dots x_{j_q} x_1 \dots x_p] C(\{x_{j_1} \dots, x_{j_q}\} \triangle \{x_1, \dots, x_p\})$. Since $r^q = r \in [z, x_1 \dots x_p]X'$, again in this case we have $y^2 \in X'$, and the red vertex in $\Gamma(y)$ is incident with no red edge.

For any subset $S = \{x_{i_1}, \dots, x_{i_t}\}$ of $\{x_1, \dots, x_{n-1}\}$, we may define y_S by

$$y_S = \begin{cases} y x_{i_1} \dots x_{i_t} & \text{if } t \text{ is even} \\ y x_{i_1} \dots x_{i_t} x_1 \dots x_p & \text{if } t \text{ is odd} \end{cases}$$

Then it is easily confirmed that the neighbours via red edges of the red vertex in $\Gamma(y_S)$ are exactly those blue vertices that represent elements of S. Moreover every possible neighbour set occurs for exactly one choice of an element of G/G' that completes $\{x_1 G', \dots, x_{n-1} G'\}$ to a basis of G/G'. $\square$

The following lemma deals with the alternative case, where the red vertex is adjacent via blue edges to an odd number of blue vertices.

Lemma 7.3. *If p is odd, then the red degree of the red vertex is either even for every choice of y or odd for every choice of y. Furthermore,*

1. *If this degree is even for all y, then for every subset S of even cardinality of $\{x_1, \ldots, x_{n-1}\}$, there are exactly two choices of yG' for which the neighbours via red edges of the red vertex in $\Gamma(y)$ are precisely those vertices representing elements of S. These two choices of y differ from each other (modulo G') by the element $x_1 \ldots x_p$, the product of the basis elements represented by the neighbours of the red vertex via blue edges. In particular, in this case there are two choices of yG' for which the red vertex is incident with no red edge in $\Gamma(y)$.*

2. *If this degree is odd for all y, then for every subset S of odd cardinality of $\{x_1, \ldots, x_{n-1}\}$ there are exactly two choices of yG' for which the neighbours via red edges of the red vertex in $\Gamma(y)$ are precisely those vertices representing elements of S. These two choices of y differ from each other (modulo G') by the element $x_1 \ldots x_p$. In particular, there are two choices of yG' for which the red vertex in $\Gamma(y)$ has the same neighbour set via red and blue edges.*

Proof. We assume that p is odd and choose $z \in G \backslash X$. We write

$$z' = zx_1 \ldots x_p.$$

Then

$$\begin{aligned} (z')^2 &= z^2 r^p C(z, x_1, \ldots, x_p) \\ &= [z, x_1 \ldots x_p] r [z, x_1 \ldots x_p] c, \\ &= rc. \end{aligned}$$

where $c \in X'$. Thus the red vertex has the same set of neighbours in $\Gamma(z)$ and $\Gamma(z')$, whenever z' and z are related by $z' \in zx_1 \ldots x_p G'$.

Now let S be any subset of $x_1, \ldots, x_{n-1}$ and let x be the product of the elements of S (in some order). Choose $y \in G \backslash \langle X \rangle$, and let N_y be the set of neighbours of the red vertex via red edges in $\Gamma(y)$. Then

$$(yx)^2 = y^2 r^{|S|}[yx, x].$$

Thus the set of neighbours via red edges of the red vertex in $\Gamma(yx)$ is

- $N_y \triangle S$, if $|S|$ is even;
- $N_y \triangle S \triangle \{x_1, \ldots, x_p\}$, if $|S|$ is odd.

Since p is odd, the red degree of the red vertex has the same parity in $\Gamma(y)$ and $\Gamma(yx)$, for all choices of x. Since the symmetric difference is a group operation on the power set of $\{x_1, \ldots, x_{n-1}\}$, every subset whose cardinality has the same parity as N_y occurs (as the neighbour set via red edges of the red vertex) for two choices of S, one with odd and one with even cardinality.

In particular, if $|N_y|$ is even, then $\{x_1, \ldots, x_{n-1}\}$ may be extended (in two ways) to a 2-uniform basis of G whose graph has the property that its red vertex is incident with no red edge. If $|N_y|$ is odd, the $\{x_1, \ldots, x_{n-1}\}$ may be extended (in two ways) to a 2-uniform basis whose graph has the property that the neighbours of the red vertex via red edges coincide with those via blue edges. $\square$

It remains to consider the relationship between the two 2-uniform graphs representing G, and having the properties described in Lemma 7.3, in the case that p is odd. Suppose that G is a group satisfying the hypotheses of Lemma 7.3, and that the element y of $G \backslash \langle x_1, \ldots, x_{n-1} \rangle$ has been chosen so that the red vertex in graph $\Gamma(y)$ is either incident with no red edge, or has the same set of n_B neighbours via both blue and red edges.

Theorem 7.4. *If $x_1, \ldots, x_p$ are the basis elements represented by the neighbours of the red vertex in $\Gamma(y)$, where p is odd, let $y' = x_1 \ldots x_p y$. Then the graph $\Gamma(y')$ that represents G with respect to the basis $\{x_1, \ldots x_{n-1}, y'\}$ is related to $\Gamma(y)$ as follows:*

- *The two graphs are considered to have the same vertex set, where the red vertex represents y in $\Gamma(y)$ and y' in $\Gamma(y')$;*
- *$\Gamma(y)$ and $\Gamma(y')$ have the same set of blue edges;*

- *The set of red edges in* $\Gamma(y')$ *is given by* $E^R(\Gamma(y) \triangle S \triangle T$*, where* S *and* T *respectively denote the set of blue edges amongst the blue vertices of* $\Gamma(y)$ *and the edge set of the complete graph on the vertices representing* $x_1, \dots, x_p$.

Proof. That the sets of blue vertices coincide in $\Gamma(y)$ and $\Gamma(y')$ follows from the fact that

$$r = [y, x_1 \dots x_p]C = [y', x_1 \dots x_p]C,$$

where C is a product of commutators involving the elements $x_1, \dots, x_{n-1}$, which is represented by the same set of edges in both graphs.

That the sets of red edges are related as described above follows from the observation that

$$\begin{aligned} (y')^2 &= (x_1 \dots x_p y)^2 \\ &= r^p s[x_1 \dots x_p, y] \prod_{1 \leqslant j < k \leqslant p} [x_{i_j}, x_{i_k}] \\ &= rs[x_1 \dots x_p, y'] \prod_{1 \leqslant j < k \leqslant p} [x_{i_j}, x_{i_k}]. \end{aligned}$$

The red edges of $\Gamma(y')$ are those that represent commutators that occur in the element $s' = (y')^2$ of G', with respect to the basis $\{x_1, \dots, x_{n-1}, y'\}$. For $1 \leqslant j \leqslant p$, the commutators $[y', x_{i_j}]$ all occur in r, and either all or none of them occur in s. Hence they occur in s' if and only if they occur in s, and the sets of red edges incident with the red vertex coincide in $\Gamma(y)$ and $\Gamma(y')$. For basis elements x_i and x_j represented by blue vertices, the commutator $[x_i, x_j]$ occurs in s' if and only if it occurs in exactly one of s, r and $\prod_{1 \leqslant j < k \leqslant p} [x_{i_j}, x_{i_k}]$ or in all three of them, hence the conclusion. □

In order to classify 2-uniform covering groups of C_2^n uniform corank 1 with 2-uniform graphs, it is sufficient to consider 2-uniform graphs with a single red vertex, which is either incident with no red edge, or has the same set of neighbours, of odd cardinality, via both red and blue edges. We refer to such graphs as being in *standard form*. Such a graph could admit a simple exchange operation of Type 1 in Theorem 4.6, if its blue edges form a clique on an even number of blue vertices. This can occur only if If $n \geqslant 5$, no other graph transformations can arise, that preserve the property of being in standard form and the isomorphism type of the group.

Assume that $n \geqslant 5$ and let G be a covering group of C_2^n of uniform corank 1. If G is represented by a 2-uniform graph in standard form, in which the red vertex is incident with a positive even number of blue edges, then this is the only example in standard form that represents G. If G is represented by a 2-uniform graph in standard form where the red vertex is isolated, then this is the only graph in standard form that represents G, unless it admits exchange operations as mentioned above. If the red vertex is incident with an odd number of blue edges in a standard 2-uniform graph representing Γ, then it follows from Theorem 4.6, that no exchange operations preserving the property of being in standard form are possible. However, each group of this type is represented by two generally non-isomorphic graphs in standard form, as described in Theorem 7.4. The collection of all standard 2-uniform graphs in which the red vertex is incident with an odd number of blue edges has two graphs representing each of the groups that occur, with exceptions only in cases where the two graphs described in Theorem 7.4 are isomorphic. Graphs in this collection have a natural occurrence in pairs; corresponding to each example in which the red vertex is incident with no red edge, is one in which the red vertex has the same neighbours via red edges as blue. Those graphs in which the red vertex is incident with no red edge account for half of all graphs in this collection, and their number approximates (and slightly overestimates) the number of isomorphism types of covering groups involved. We conclude that for $n \geqslant 5$, the number of isomorphism types of covering groups of uniform corank 1 of C_2^n is closely approximated by the number of 2-uniform graphs of standard form on n vertices, in which the red vertex is incident with no red edge.

Chapter 8

Small special cases

Let G be a 2-uniform covering group of C_2^n, having a 2-uniform basis consisting of elements with squares r and s in G'. Theorem 3.8 asserts that no element of G', except possibly r and s, can be the square of four independent elements of G. Consequently, if the uniform rank $\rho(G)$ of G exceeds 3, the square of the "blue" elements of a 2-uniform basis is almost fully determined by G. Unless the uniform rank and corank coincide or almost coincide, it is the unique element of G' that occurs as the square of $\rho(G)$ independent elements. In any case, it occurs as the square corresponding to either the blue or red vertices in the graph determined by every uniform basis.

It remains to consider 2-uniform covering groups of uniform rank at most three, where more choices potentially exist for a 2-uniform basis. That is the theme of this chapter. We consider 2-uniform covering groups of uniform rank 2, with uniform corank 1 or 2, and groups of uniform rank 3, with uniform corank 1, and 2.

8.1 Uniform rank 2, corank 1

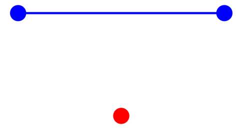
The goal of this section is to enumerate the isomorphism types of 2-uniform covering groups of C_2^3 of uniform rank 2. Let G be a 2-uniform covering group of C_2^3 of uniform rank 2. We consider the problem of distinguishing the possible isomorphism types of G via 2-uniform graphs.

Since not every 2-coloured graph with one red and two blue vertices is a 2-uniform graph, by Theorem 3.12 we need only consider 2-coloured graphs Γ that satisfy the following conditions.

(a) Γ^R is not a clique on one blue vertex and one red vertex.

(b) $\Gamma^B \triangle \Gamma^R$ is not a clique on two blue vertices and one red vertex.

Moreover, by Lemma 7.2 and Lemma 7.3, we may assume that the red vertex is isolated in the red part of the graph, or that in both the red and blue parts the red vertex has the same single blue neighbour. There are only 9 graphs on three vertices satisfying all of these properties. They are presented below.



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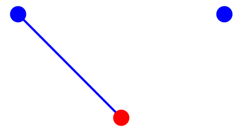

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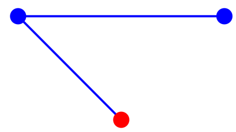

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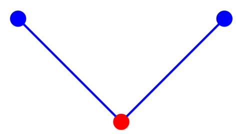

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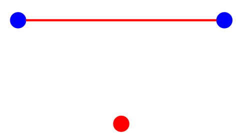

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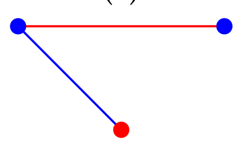

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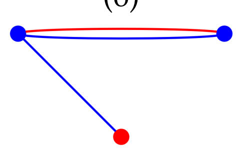

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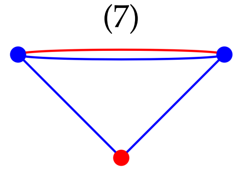

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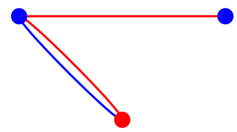


(9)

Table 8.1

Every 2-uniform covering group of C_2^3 of uniform rank 2 is represented by one of these graphs. Now the question is whether any pairs of these graphs represent the same group. In this case and since $n = 3$, it is feasible to analyze each graph explicitly and investigate the alternative 2-uniform bases that represent the same group.
In the analysis below we assume in each case that the two blue vertices represent elements x_1 and x_2 and that the red vertex represents an element called y.

1. Graph (1) represents a group that has

$$x_1^2 = x_2^2 = [x_1, x_2],\ y^2 = 1.$$

Then

$$\begin{aligned} (x_1y)^2 &= [x_1, x_2y] \\ (x_2y)^2 &= [x_2, x_1y] \\ (x_1x_2)^2 &= [x_1, x_2] \\ (x_1x_2y)^2 &= [x_1x_2, x_2y] \end{aligned}$$

The only alternative 2-uniform basis (for which the red vertex is incident with no red edges) comes from replacing either x_1 or x_2 with x_1x_2. This does not change the graph. So no other of these 9 graphs represents the same group as the first one.

2. Graph (2) represents a group that has

$$x_1^2 = x_2^2 = [x_1, y], y^2 = 1.$$

Then

$$\begin{aligned}
(x_1y)^2 &= 1 \\
(x_2y)^2 &= [x_1x_2, y] \\
(x_1x_2)^2 &= [x_1, x_2] \\
(x_1x_2y)^2 &= [x_1x_2, x_2y]
\end{aligned}$$

We may replace x_1, x_2, and y by x_1y, y, and x_1x_2 respectively, this results in a representation of the same group as graph (5). So graphs (2) and (5) represent the same group.

3. Graph (3) represents a group that has

$$x_1^2 = x_2^2 = [x_1, x_2y], y^2 = 1.$$

Then

$$\begin{aligned}
(x_1y)^2 &= [x_1, x_2] \\
(x_2y)^2 &= [x_1y, x_2y] \\
(x_1x_2)^2 &= [x_1, x_2] \\
(x_1x_2y)^2 &= [x_1y, x_2y]
\end{aligned}$$

We may replace y in the 2-uniform basis with the element x_1y, this results in a representation of the same group as graph (7). So graphs (3) and (7) represent the same group.

We may also notice that G has other 2-uniform bases such as $\{x_1x_2, x_1y, y\}$, but the graph with respect to each of these is isomorphic to (3) or (7).

4. Graph(4) represents a group that has

$$x_1^2 = x_2^2 = [x_1x_2, y], y^2 = 1.$$

Then

$$\begin{aligned}(x_1y)^2 &= [x_2, y] \\ (x_2y)^2 &= [x_1, y] \\ (x_1x_2)^2 &= [x_1, x_2] \\ (x_1x_2y)^2 &= [x_1y, x_2y]\end{aligned}$$

We notice that there is no adjustment to the basis can give the same group that graph (4) represents.

5. As noted earlier, the group represented by graph (5) is isomorphic to that represented by graph (2).

6. Graph (6) represents a group that has

$$x_1^2 = x_2^2 = [x_1, y], y^2 = [x_1, x_2].$$

Then

$$\begin{aligned}(x_1y)^2 &= [x_1, x_2] \\ (x_2y)^2 &= [x_1y, x_2y] \\ (x_1x_2)^2 &= [x_1, x_2] \\ (x_1x_2y)^2 &= [x_1x_2, y]\end{aligned}$$

We notice that graph (6) represents a uniform covering group with the uniform basis $\{x_1x_2, x_1y, y\}$. Also, we may replace x_1, x_2 and y with the element x_1x_2, x_1y, and x_1x_2y respectively this results in a representation of the same group as graph (9).

7. As noted earlier, the group represented by graph (7) is isomorphic to that represented by graph (3).

8. Graph (8) represents a group that has

$$x_1^2 = x_2^2 = [x_1y, x_2y], y^2 = [x_1, x_2].$$

Then

$$\begin{aligned} (x_1y)^2 &= [x_1, x_2y] \\ (x_2y)^2 &= [x_1y, x_2] \\ (x_1x_2)^2 &= [x_1, x_2] \\ (x_1x_2y)^2 &= [x_2, y] \end{aligned}$$

We notice that there is no alternative 2-uniform basis in this case.

9. Graph (9) represents a group that has

$$x_1^2 = x_2^2 = [x_1, x_2y], y^2 = [x_1, y].$$

Then

$$\begin{aligned} (x_1y)^2 &= [x_1, x_2y] \\ (x_2y)^2 &= [x_1, y] \\ (x_1x_2)^2 &= [x_1, x_2] \\ (x_1x_2y)^2 &= [x_1x_2, y] \end{aligned}$$

We notice that graph (9) represents a uniform covering group with the uniform basis $\{x_1, x_2, x_1y\}$. As noted earlier, the group represented by graph (9) is isomorphic to that represented by graph (6).

We conclude that there are 5 different isomorphism types of 2-uniform covering groups of C_2^3 of uniform rank 2.

8.2 Uniform rank 3, corank 1

We now consider 2-uniform covering group of C_2^4 of uniform rank 3. Every such group is represented by a 2-uniform graph with three blue vertices and one red, in standard form in the sense of Chapter 7. In view of the number of graphs involved, it is not feasible to conduct an exhaustive manual search of alternative 2-uniform bases within each group as in Section 8.1. This section presents an account an account of all 2-uniform graphs in standard form, on three blue vertices and one red vertex. We note pairs of such graphs that represent the same group, according to the equivalence described in Theorem 7.4, or under the exchange operations in Chapter 4. The outcome is an upper bound on the number of isomorphism types of 2-uniform covering groups of C_2^4 of uniform rank 3.

As in Section 8.1, it follows from Theorem 3.12 that we need only consider graphs Γ satisfying the following conditions.

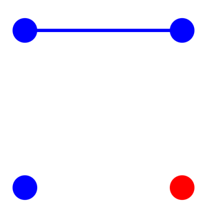
(a) Γ^R is not a clique on one blue and one red vertices.

(b) $\Gamma^B \triangle \Gamma^R$ is not a clique on two blue vertices and one red vertex.

By Lemma 7.2 and Lemma 7.3, we may assume that the red vertex is isolated in the red part of the graph, or that in both the red and blue parts the red vertex has the same single blue neighbour or three blue neighbors.

Also we need to apply Theorem 7.4 on the graphs that already satisfy the above mentioned Lemmas in order to figure out the graphs that represent the same group. It is obvious that Theorem 7.4 can be applied on the graphs where the red vertex has one or three blue neighbors via red and blue edges.

We notice that after adding one isolated blue vertex to the graphs that represent the uniform rank 2 and corank 1 case, these graphs will represent some of the uniform rank 3 and corank 1 case graphs. Those 5 graphs are listed in the table below.



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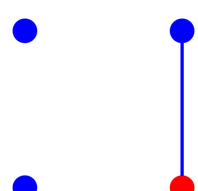

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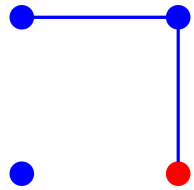
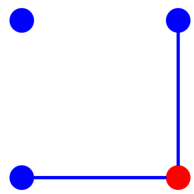
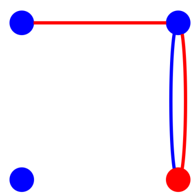


(3)



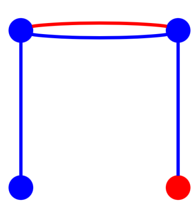
(4)



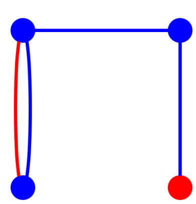
(5)

Table 8.2

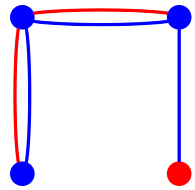
Now, in the table below, we present pairs of graphs that represent the same group according to Theorem 7.4.



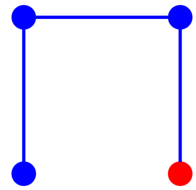
(1-a)



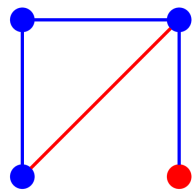
(1-b)



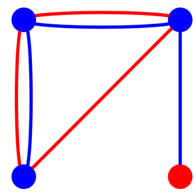
(2-a)



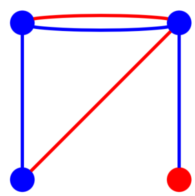
(2-b)



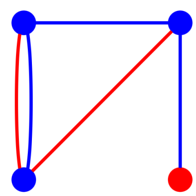
(3-a)



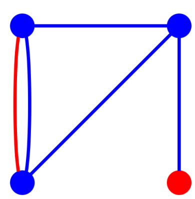
(3-b)



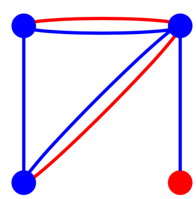
(4-a)



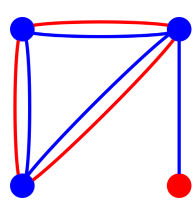
(4-b)



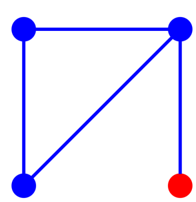
(5-a)



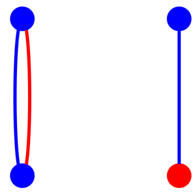
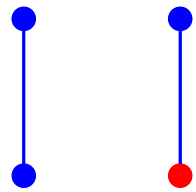
(5-b)



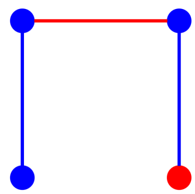
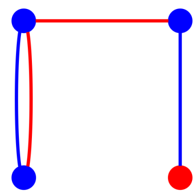
(6-a)



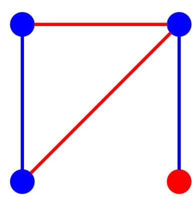
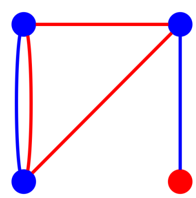
(6-b)

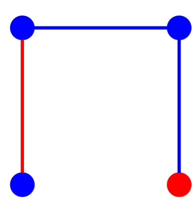
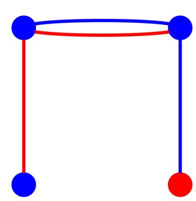
(7-a) (7-b)

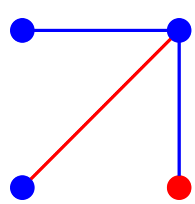
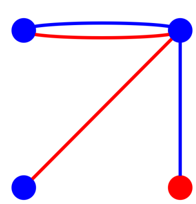
(8-a) (8-b)

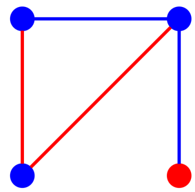
(9-a) (9-b)

(10-a) (10-b)

(11-a) (11-b)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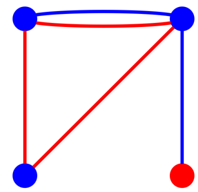
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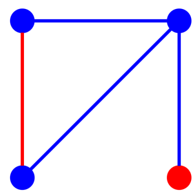
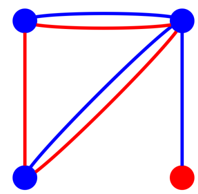

(13-a)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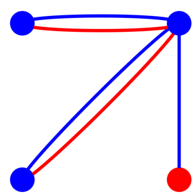
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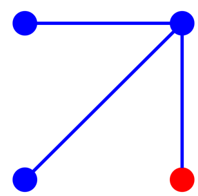
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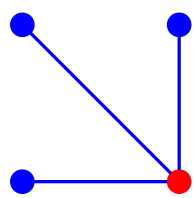
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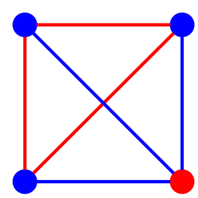
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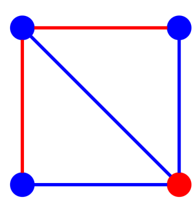
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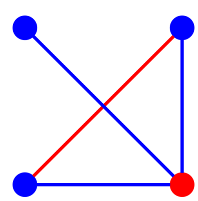
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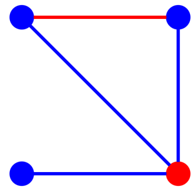

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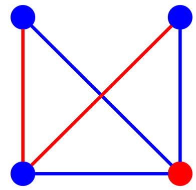

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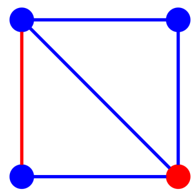

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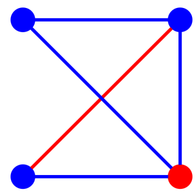

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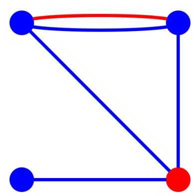

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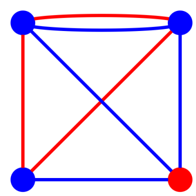
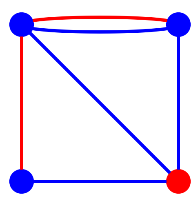
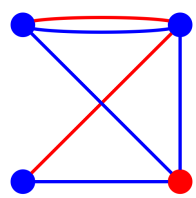


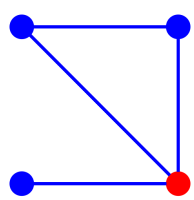
(19-b)



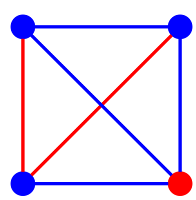
(20-a)



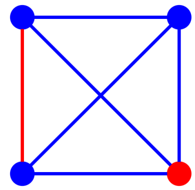
(20-b)



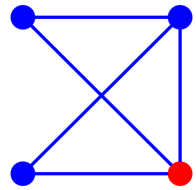
(21-a)



(21-b)



(22-a)



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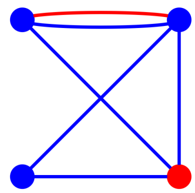
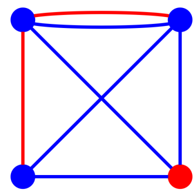
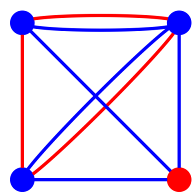


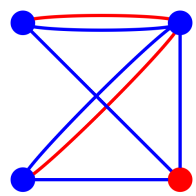
(23-a)



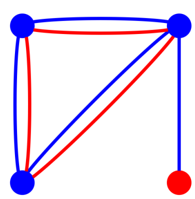
(23-b)



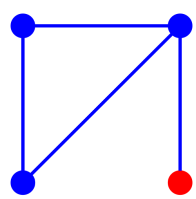
(24-a)



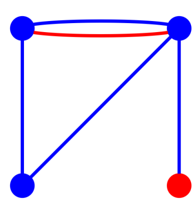
(24-b)



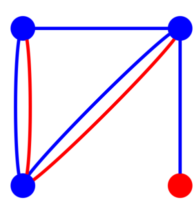
(25-a)



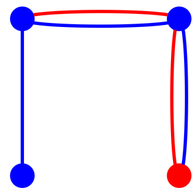
(25-b)

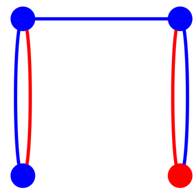


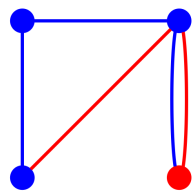
(26-a)

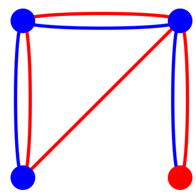


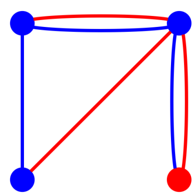
(26-b)


(27-a)


(27-b)


(28-a)


(28-b)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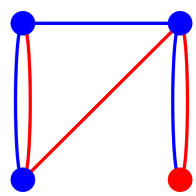
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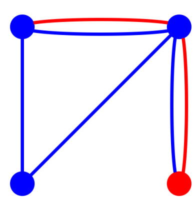
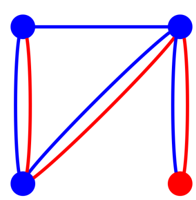
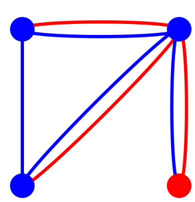
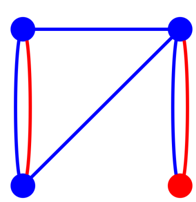

(30-a)


(30-b)


(31-a)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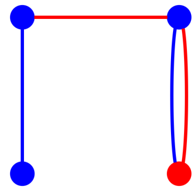
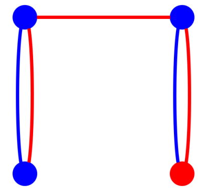
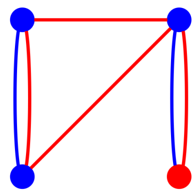


(32-a)



(32-b)



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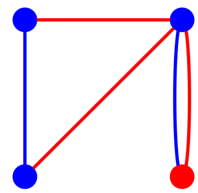
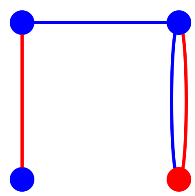
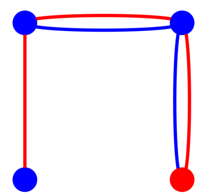


(33-b)



(34-a)



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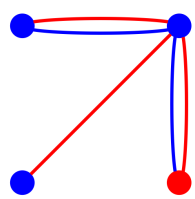

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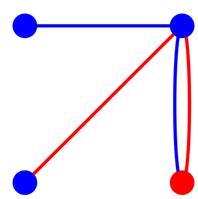
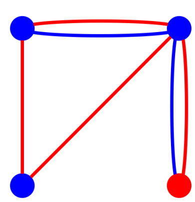
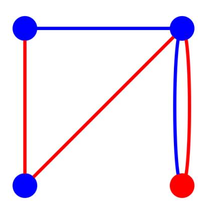


(35-b)



(36-a)



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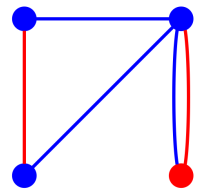
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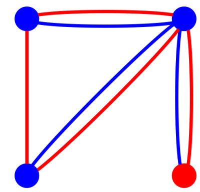
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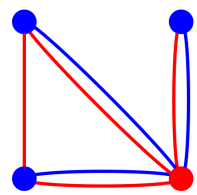
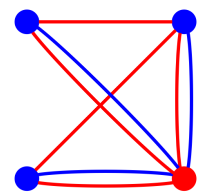

(38-a)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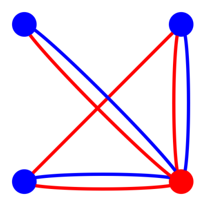
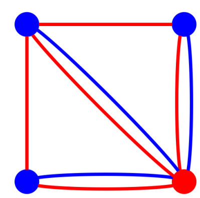

(39-a)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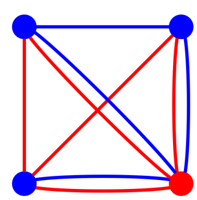
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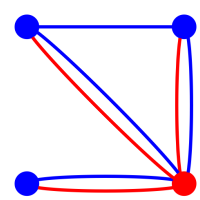
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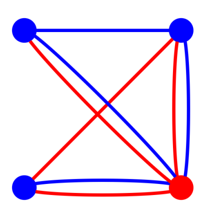
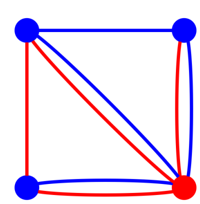

(41-a)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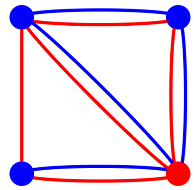
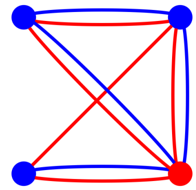
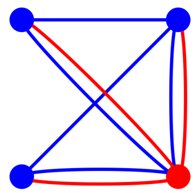
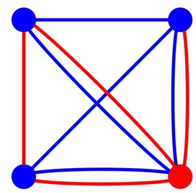
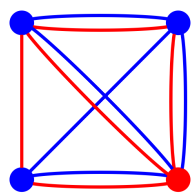
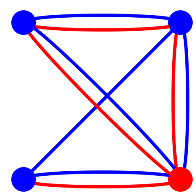
 

(42-a) (42-b)

(43-a) (43-b)

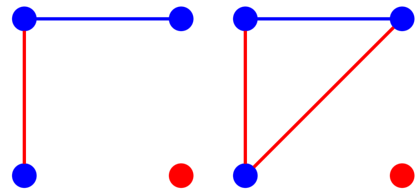
 

(44-a) (44-b)

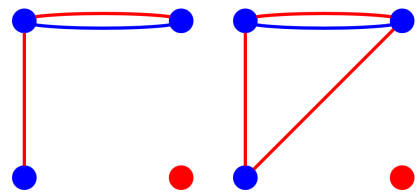
Table 8.3

We can apply exchange operations of Types 1 and 4 in Theorem 4.6, when Γ^{B} is a clique on an even number of blue vertices and when $\Gamma^{B \triangle R}$ is a clique on three blue vertices and zero red respectively. It is obvious that types 2 and 3 do not arise in the uniform rank 3 and corank 1 case, since we only have one red vertex, which is either incident with no red edge, or has the same neighbours via both red and blue edges, in which case the number of such neighbours is odd.

In the table below, we present the pairs of graphs that are equivalent under exchange operations of Types 1 and 4, and represent the same group.



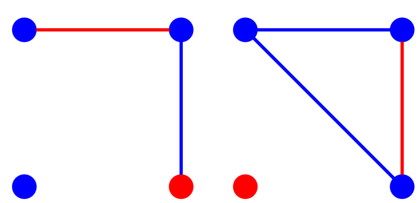
(1-a) (1-b)



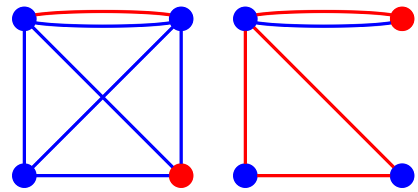
(2-a) (2-b)

Table 8.4

Since the uniform rank and corank differ by 2, the double exchange operation can be applied on the graphs that satisfy two of three conditions of Theorem 4.8. We only have two such graphs. We list below the two pairs of graphs that we obtained by applying Theorem 4.10. We notice that the graphs (1-a) and (1-b) in the below table represent a uniform covering group.



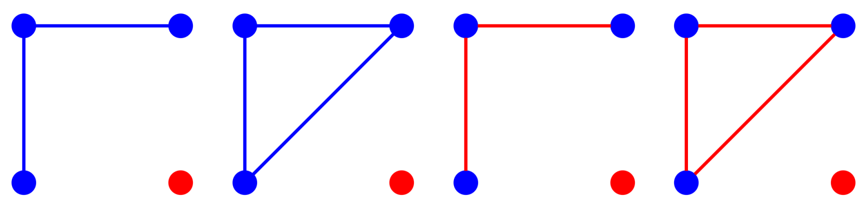
(1-a) (1-b)



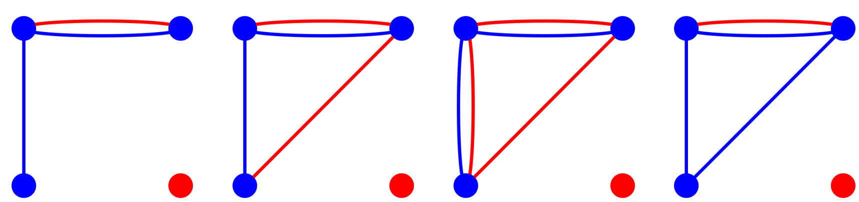
(2-a) (2-b)

Table 8.5

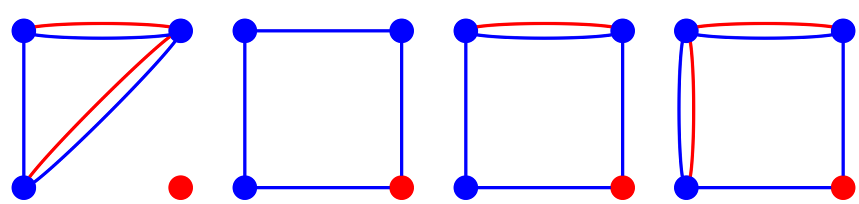
Finally, we list below table the remaining 2-uniform graphs on three blue vertices and one red, each of which is the unique 2-uniform graph representing the corresponding group.



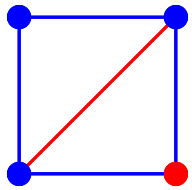
(1) (2) (3) (4)



(5) (6) (7) (8)



(9) (10) (11) (12)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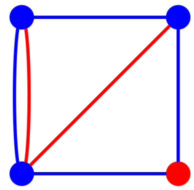
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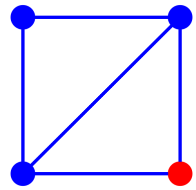
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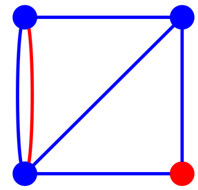
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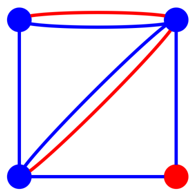
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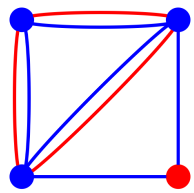
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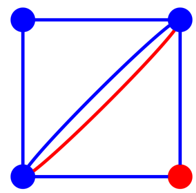
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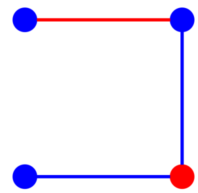
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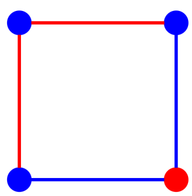
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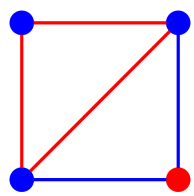
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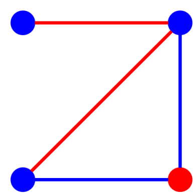
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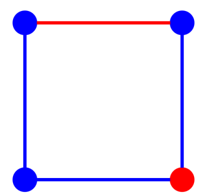
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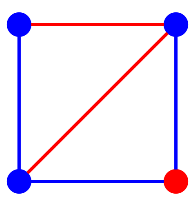
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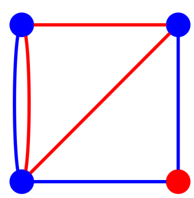
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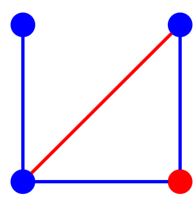
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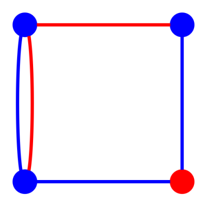
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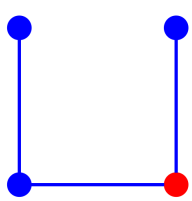
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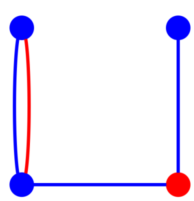
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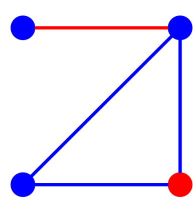
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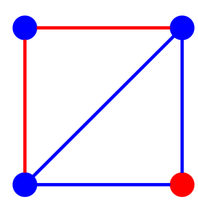
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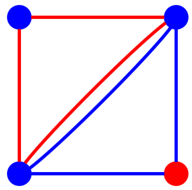
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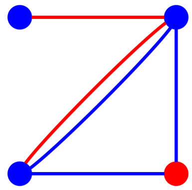
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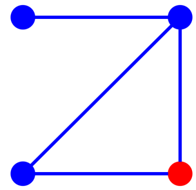
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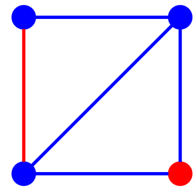
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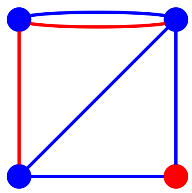
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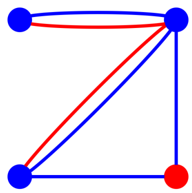
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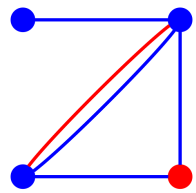
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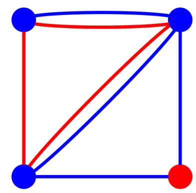
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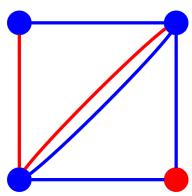
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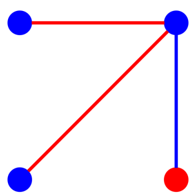
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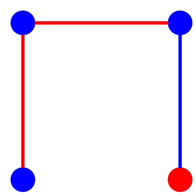
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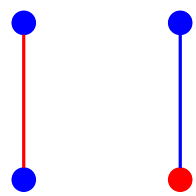
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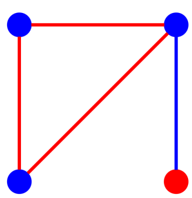
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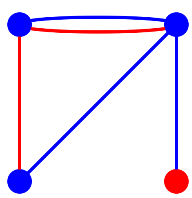
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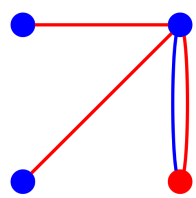
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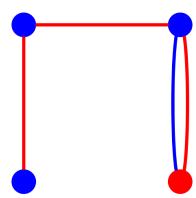
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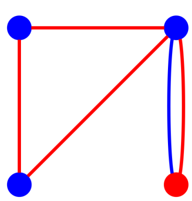
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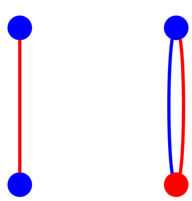
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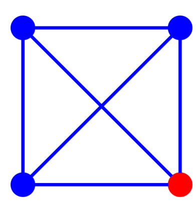
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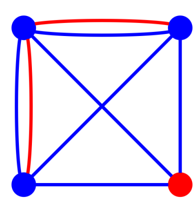
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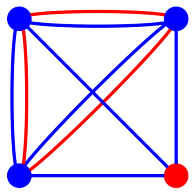
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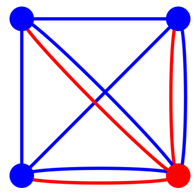
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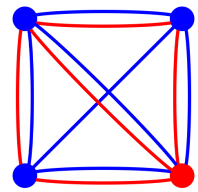
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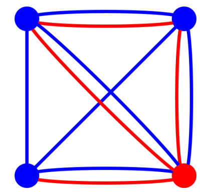
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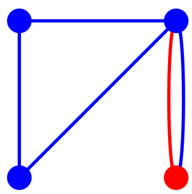
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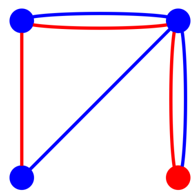
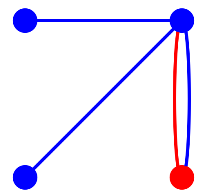

(58)


(59)

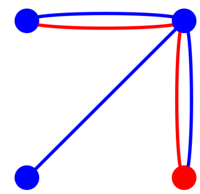

(60)

Table 8.6

We conclude that there are at most 113 different isomorphism types of 2-uniform covering groups of C_2^4 of uniform rank 3.

8.3 Uniform rank 2, corank 2

Let G be a 2-uniform covering group of C_2^4 of uniform corank 2. Let $\mathcal{B} = \{x_1, x_2, y_1, y_2\}$ be a 2-uniform basis of G, where $x_i^2 = r$ and $y_i^2 = s$, $r \neq s$. We write $\mathcal{X}$ and $\mathcal{Y}$ for the subsets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ of $\mathcal{B}$. Our concern is the 2-uniform graphs that represent the covering group G of C_2^4. Since not every 2-coloured graph with two red and two blue vertices is a 2-uniform graph, therefore we need to exclude the graphs that are not 2-uniform. First we exclude the graphs that are not satisfying the conditions in Theorem 3.12, so every graph that represents the covering groups of C_2^4 must satisfy the following conditions:

(a) Γ^B is not a clique on two blue vertices and two red vertices.

(b) Γ^R is not a clique on an one blue vertex and one red vertex.

(c) $\Gamma^B \triangle \Gamma^R$ is not a clique on two blue vertices and one red vertex.

(d) since $n_B(\Gamma) = n_R(\Gamma) = 2$, therefore Γ^R is not a clique on two blue vertices and two red vertices, Γ^B is not a clique on an one blue vertex and one red vertex, and $\Gamma^B \triangle \Gamma^R$, is not a clique on two red vertices and one blue vertex.

Now from the graphs that satisfy the above conditions, we need also to exclude the graphs that are not uniform because they have a uniform rank 3 not 2, these cases arise from the fact that we could have some element $t \in G'$, other than r and s, that is the square of three independent elements of G. Suppose G contains independent elements z_1, z_2, and z_3, where $t = z_1^2 = z_2^2 = z_3^2$. We may assume that each of z_1, z_2, z_3 is the product of some elements of $\mathcal{B}$.

We write $\mathcal{Z}_i$ for the set of elements of $\mathcal{B}$ that occur in z_i. For each i, we may write

$$t = z_i^2 = r^{|\mathcal{Z}_i \cap \mathcal{X}|} s^{|\mathcal{Z}_i \cap \mathcal{Y}|} C_i, \tag{8.1}$$

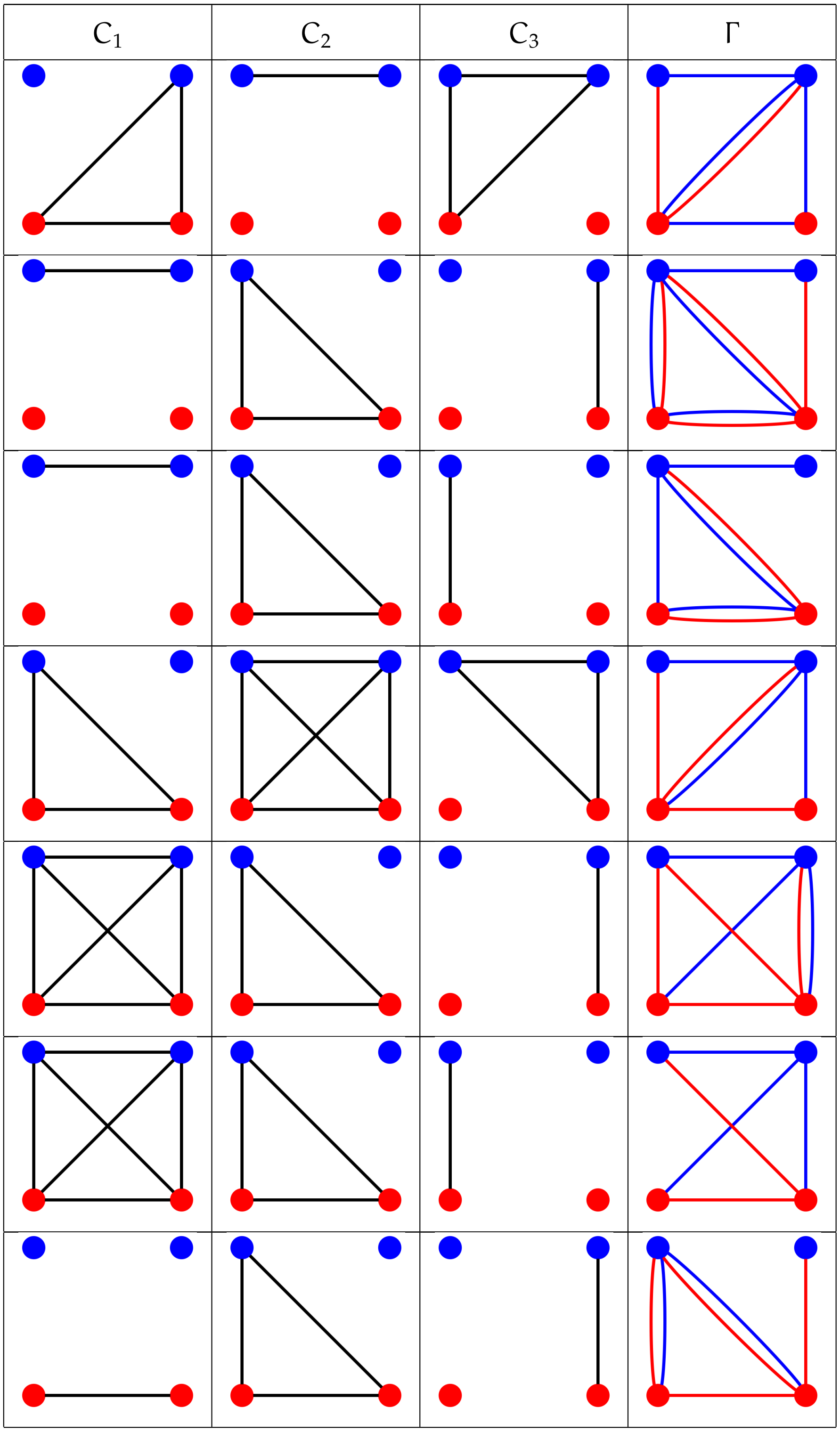
where C_i is the element of G' whose graph with respect to $\mathcal{B}$ is the clique on those vertices that correspond to elements of $\mathcal{Z}_i$. Since C_1, C_2 and C_3 are distinct elements of G', their prefixes $r^{|\mathcal{Z}_i \cap \mathcal{X}|} s^{|\mathcal{Z}_i \cap \mathcal{Y}|}$ must also be distinct for $i = 1, 2, 3$.

After relabelling if necessary, we obtain the following descriptions for r, s, rs:

- $|\mathcal{Z}_1 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$ have the opposite parity, and $|\mathcal{Z}_1 \cap \mathcal{Y}|$ and $|\mathcal{Z}_2 \cap \mathcal{Y}|$ have the same parity, then $r = C_1C_2$.

- $|\mathcal{Z}_1 \cap \mathcal{X}|$ and $|\mathcal{Z}_2 \cap \mathcal{X}|$ have the same parity, and $|\mathcal{Z}_1 \cap \mathcal{Y}|$ and $|\mathcal{Z}_2 \cap \mathcal{Y}|$ have the opposite parity, then $s = C_2C_3$.

where $C_3 \in G'$ is represented on the vertex set of $\Gamma_{\mathcal{B}}$ by a clique whose numbers of blue and red vertices are respectively of the same and opposite parity to those of the graph representing C_2. We notice that for the above description $rs = C_1C_3$. In the following table we are presenting all possibilities for C_1, C_2, and C_3 that satisfying the parity conditions mentioned above. The edges color in these graphs is black in the first three columns, where the fourth column contains the graphs that represent the group where the blue graph is the symmetric difference of the graphs of C_1 and C_2, and the red graph is the symmetric difference of the graphs of C_2 and C_3, these graphs represent uniform rank 3 groups. Therefore we will omit every graph Γ that has any of the descriptions below from the 2-uniform graph collection describing groups of uniform rank 2 and uniform corank 2.


C_1
C_2
C_3
Γ

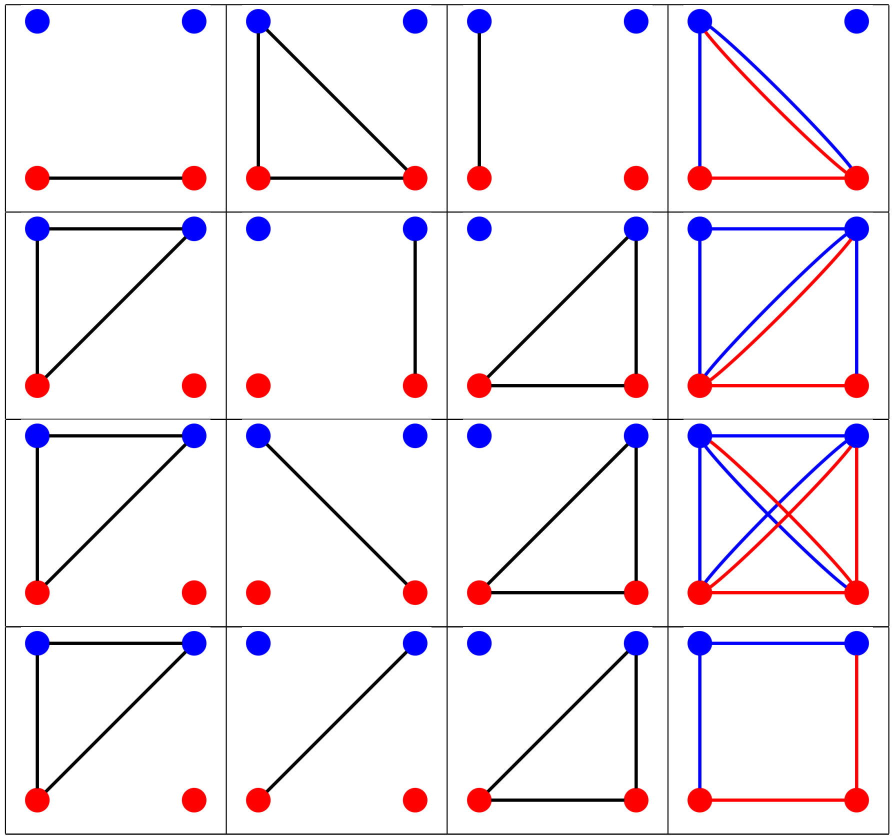


Table 8.7

We now consider graphs that are equivalent under the exchange operations of Theorem 4.6. By Theorem 4.6, alternative 2-uniform graphs Γ and Γ' corresponding to exchange operations arise in the following ways.

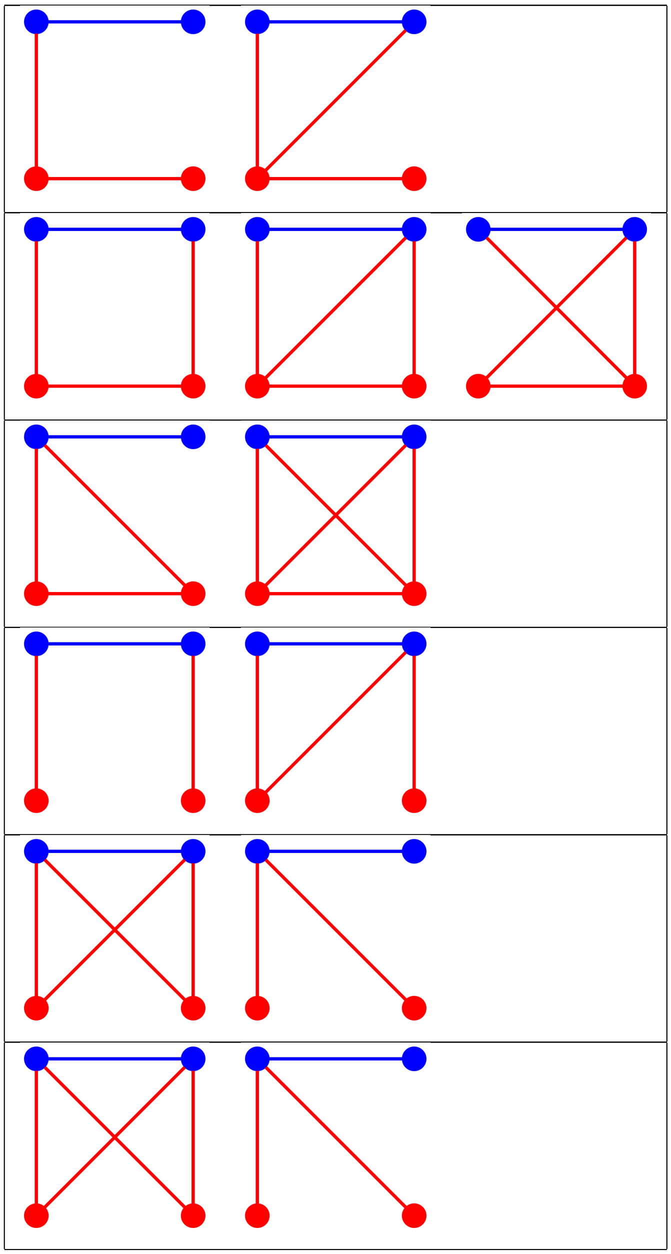
1. (Type 1) If Γ^B is a clique on an two blue vertices, then Γ' may be given by $\Gamma \triangle K^R(W)$ for any vertex W of the clique.

2. (Type 2) If Γ^R is a clique on either two blue vertices and a two red vertices or a clique on two red vertices , then Γ' may be given by $\Gamma \triangle K^B(W)$, for any red vertex W of the clique.

3. (Type 3) If $\Gamma^B_{\mathcal{B}}$ is a clique on an one blue and one red vertices, then Γ' may be given by $\Gamma \triangle K^R(W)$, for any red vertex W of the clique.

4. (Type 4) If $\Gamma^{B\triangle R}$ is a clique on one blue and an two red vertices, then Γ' may be given by $\Gamma \triangle K^R(W) \triangle K^B(W)$, for any red vertex W of the clique.

Listed below are pairs and triples of graphs representing the same group. We note that (since the uniform rank and corank coincide), the colour opposite of each graph in this section represents the same group as the graph itself.



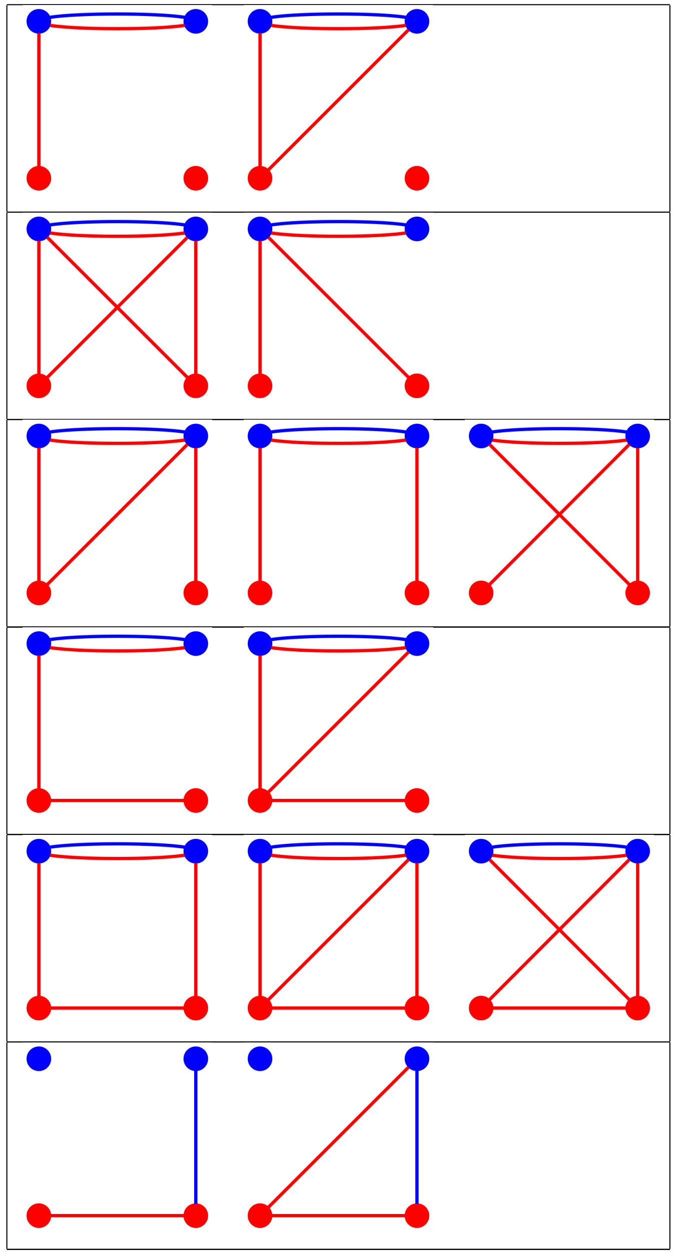


Table 8.8

8.4 Uniform rank 3, corank 2

Let G be a 2-uniform covering group of C_2^5 of uniform corank 2. Let $\mathcal{B} = \{x_1, x_2, x_3, y_1, y_2\}$ be a 2-uniform basis of G, where $x_i^2 = r$ and $y_i^2 = s$, $r \neq s$. By Theorem 3.8, it is possible that y_1 and y_2 can be replaced in $\mathcal{B}$ by elements z_1 and z_2, to form an alternative 2-uniform basis $\mathcal{B}'$. In this situation, $\mathcal{B}' = \{x_1, \ldots, x_k, z_1, z_2\}$, where $z_1^2 = z_2^2 = s'$ and $s' \notin \{r, s\}$. Also, it is possible that x_1, x_2 and x_3 can be replaced in $\mathcal{B}$ by elements w_1, w_2 and w_3, to form an alternative 2-uniform bases $\mathcal{B}''$. In this situation, $\mathcal{B}' = \{w_1, w_2, w_3, y_1, y_2\}$, where $w_1^2 = w_2^2 = w_3^2 = r'$ and $r' \notin \{r, s\}$. Moreover, other combinations with x_1, y_j, and w_k can form alternative bases for the same group.

In addition, alternative graphs to the $\Gamma_{\mathcal{B}}$ that represent the same group can potentially be obtained by applying exchange operations as outlined in Theorems 4.6. We observe that the exchange operations of Chapter 4 cannot be applied to groups that also possess multiple 2-uniform bases involving elements with different squares.

We presenting the examples below, to show how multiple graphs representing the same group can arise in different ways in the case of 2-uniform graphs of uniform rank 3 and corank 2. An enumeration of the groups of this type is not within reach at present.

Example 8.1. Let G be the 2-uniform covering group of C_2^5 with 2-uniform basis $\{x_1, x_2, x_3, y_1, y_2\}$, where $x_1^2 = x_2^2 = x_3^2 = r$, and $y_1^2 = y_2^2 = s$, $r \neq s$. Let $\Gamma_{\mathcal{B}}$ be the 2-uniform graph that represents the group G.

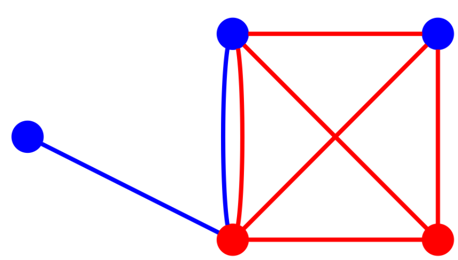


Figure 8.1: $\Gamma_{\mathcal{B}}$

Since $\Gamma_{\mathcal{B}}^R$ is a clique on two blue and two red vertices, by Theorem 4.6, we may apply a type 2 exchange, for example to obtain the graph $\Gamma_{\mathcal{B}_1}$ below. This graph is given by $\Gamma \triangle K^B(Y_1)$ for the red vertex Y_1 of the clique that has a blue degree of 2.

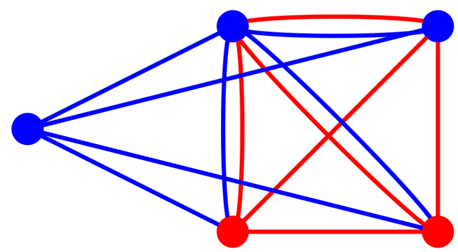


Figure 8.2: $\Gamma_{\mathcal{B}_1}$

Alternatively, we can apply a type 2 exchange to the other red vertex of the clique, to obtain the graph $\Gamma_{\mathcal{B}_2}$ as follows.

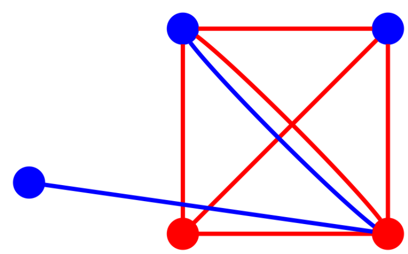


Figure 8.3: $\Gamma_{\mathcal{B}_2}$

By Theorem 4.8, and since $\Gamma^R_{\mathcal{B}}$ is a clique on two blue and two red vertices, we can apply the exchange operation separately on the two blue vertices of $\Gamma^R_{\mathcal{B}}$. We write X_1 for the vertex that has a blue degree of one, and X_2 for the other blue vertex. For X_1, we transform Γ to Γ_1 by switching the colour of X_1 from blue to red, and then define $\Gamma_{\mathcal{B}_3}$ to be the colour opposite of $\Gamma_1 \triangle K^B(X_1)$. A similar exercise for X_2 yields the graph $\Gamma_{\mathcal{B}_4}$ below.

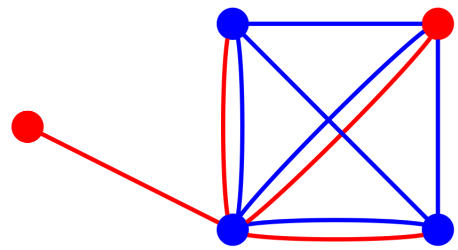


Figure 8.4: $\Gamma_{\mathcal{B}_3}$

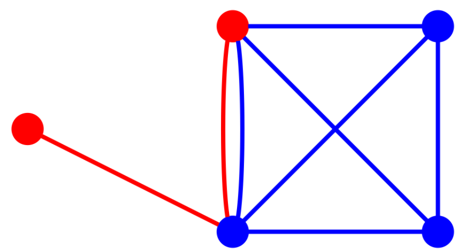


Figure 8.5: $\Gamma_{\mathcal{B}_4}$

The graphs $\Gamma_{\mathcal{B}}, \Gamma_{\mathcal{B}_1}, \Gamma_{\mathcal{B}_2}, \Gamma_{\mathcal{B}_3}, \Gamma_{\mathcal{B}_4}$ all represent the same group G.

The second example describes a case that admits changes of basis of the type described in Chapter 5.

Example 8.2. Let G be the the 2-uniform covering group of C_2^5 with 2-uniform basis $\mathcal{B} = \{x_1, x_2, x_3, y_1, y_2\}$, where $x_1^2 = x_2^2 = x_3^2 = r$, and $y_1^2 = y_2^2 = s$, $r \neq s$. Let $\Gamma_{\mathcal{B}}$ the 2-uniform graph that represents the group G where

1. $\Gamma^B = \Phi_1 \triangle \Phi_2$, where ϕ_1 is a clique on one two red and two blue vertices and Φ_2 is a clique on two blue vertices and one red vertex and;

2. $\Gamma^R = \Phi_2 \triangle \Phi_3$, where Φ_3 is a clique on one blue and one red vertices.

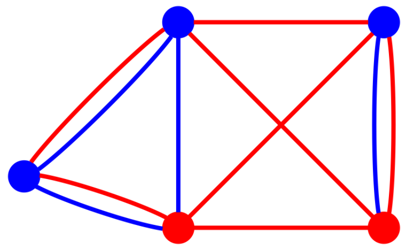


Figure 8.6: $\Gamma_{\mathcal{B}}$

Let z_i be the product in G of the basis elements represented by the vertices of Φ_i as follows:

$$\begin{aligned} z_1 &= x_1x_2y_1y_2 \\ z_2 &= x_2x_3y_1 \\ z_3 &= x_1y_2 \end{aligned}$$

where $z_1^2 = z_2^2 = z_3^2 = s'$. Listing the basis elements of $\mathcal{B}_1$ in the order y_1, y_2, z_1, z_2, z_3, the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}$ is

$$P = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Therefore, the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}_1$ is the inverse of P, given by

$$P^{-1} = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right].$$

The graph $\Gamma_{\mathcal{B}_1}$ is

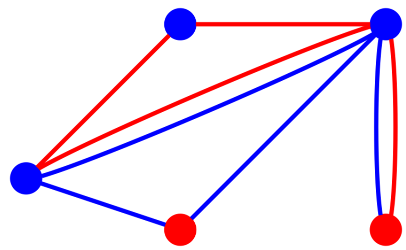


Figure 8.7: $\Gamma_{\mathcal{B}_1}$

Now, we use the matrix description of change of 2-uniform basis, as described in Chapter 5 to present the alternative graphs that correspond to the possible alternative 2-uniform bases. Let A^B and A^R be the matrices in $M_n(\mathbb{F}_2)$ whose (i,j)-entry is 1 if the commutator of the ith and the jth elements appears in the unique expression for r and s respectively. Consequently the adjacency matrices A^B and A^R for the blue and red graphs in $\Gamma_{\mathcal{B}}$ that correspond to r and s respectively are

$$A^B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$A^R = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

For the basis $\mathcal{B}_2 = \{x_1, x_2, x_3, z_1, z_2\}$, where $x_1^2 = x_2^2 = x_3^2 = r$ and $z_1^2 = z_2^2 = s'$, the change basis matrices from $\mathcal{B}$ to $\mathcal{B}_2$ is

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Therefore, the matrix for $\mathcal{B}_2$ that correspond to r is given by

$$Q^T A^B Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Since $z_2 = x_2 x_3 y_1$, then $s' = sC_2$ where $C_2 = [x_2, x_3][x_2, y_1][x_3, y_1]$. The matrix of C_2 with respect to $\mathcal{B}_2$ is

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Therefore, the matrix of s' with respect to $\mathcal{B}_2$ is given by

$$
\begin{aligned}
Q^T A^R Q + K_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}.
\end{aligned}
$$

The graph $\Gamma_{\mathcal{B}_2}$ is

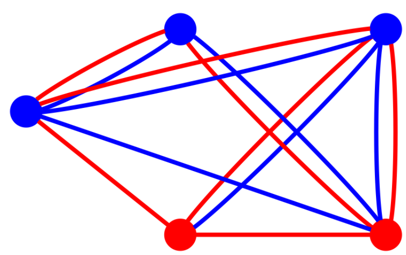


Figure 8.8: $\Gamma_{\mathcal{B}_2}$

We apply the same method for the bases $\mathcal{B}_3 = \{x_1, x_2, x_3, z_2, z_3\}$ and $\mathcal{B}_4 = \{x_1, x_2, x_3, z_1, z_3\}$, where $x_1^2 = x_2^2 = x_3^2 = r$ and $z_1^2 = z_2^2 = z_3^2 = s'$, and find the corresponding graphs $\Gamma_{\mathcal{B}_3}$ and $\Gamma_{\mathcal{B}_4}$ respectively, as shown below.

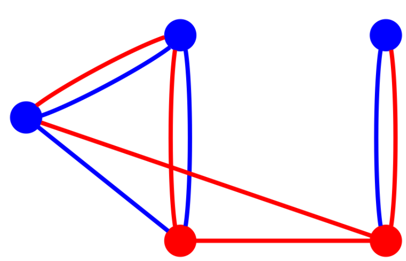


Figure 8.9: $\Gamma_{\mathcal{B}_3}$

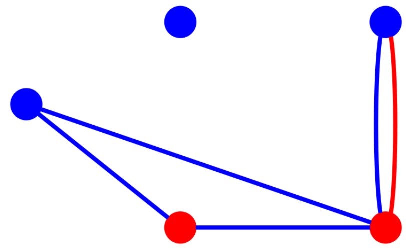


Figure 8.10: $\Gamma_{\mathcal{B}_4}$

Finally, the graph $\Gamma_{\mathcal{B}_5}$ corresponds to the basis $\mathcal{B}_5 = \{z_1, z_2, z_3, x_1, x_2\}$, where $z_1^2 = z_2^2 = z_3^2 = r'$ and $x_1^2 = x_2^2 = s$.

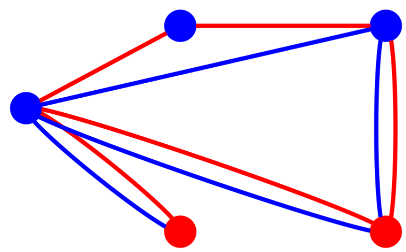


Figure 8.11: $\Gamma_{\mathcal{B}_5}$

The graphs $\Gamma_{\mathcal{B}}$, $\Gamma_{\mathcal{B}_1}$, $\Gamma_{\mathcal{B}_2}$, $\Gamma_{\mathcal{B}_3}$, $\Gamma_{\mathcal{B}_4}$, and $\Gamma_{\mathcal{B}_5}$ all represent the same group G.

This concludes our analysis for covering groups of uniform rank at most 3. The case of groups of uniform rank 3 and uniform corank 3 differs from the general corank 3 considerations of Chapter 5 only by the possibility of colour switches, so we do not include a detailed analysis for this case.

Chapter 9

Conclusion

Our goal was to construct a bijective correspondence between isomorphism classes of 2-uniform covering groups of C_2^n, and an appropriate collection of 2-coloured graphs of order n. We now report on the extent to which this goal has been achieved, and mention some possible avenues for further investigation.

Lemma 3.11 presents a description of 2-uniform graphs of order 5 or greater, which include at least one graph representing every isomorphism type of group. Groups of uniform corank 4 or greater are represented by a unique 2-uniform graph, with a few exceptions that are considered in Chapter 4. The collection of 2-uniform graphs with at least four red vertices provides a correspondence of the desired type, with a few cases where the bijectivity fails, which have been documented in detail.

The case of covering groups of uniform corank 1, 2 or 3 is more complicated, due to Theorem 3.8, which is the essential driver of our analysis. In these cases, the more extensive possibilities for a 2-uniform basis mean that more mechanisms occur for a group to have multiple non-isomorphic representations by 2-uniform graphs. Nevertheless, in the case of coranks 2 and 3 (and uniform rank at least 4), most 2-uniform group are represented by a unique 2-uniform graph. The exceptions are discussed in detail in Chapters 5 and 6.

The case of corank 1 (and rank 4 or greater) is particularly interesting, because of the much greater number of choices for a 2-uniform basis. In this case, we were able to refine the concept of a 2-uniform graph by specializing to graphs in standard form.

Graphs of standard form can admit at most one type of the exchange operations of Chapter 4. Thus, it is arguably in the case of groups of uniform corank 1 that we come closest to achieving the goal of exhibiting a bijective correspondence between isomorphism types of groups and a collection of 2-coloured graphs. With a few exceptions, groups of corank 1 are represented either by exactly one or exactly two graphs in standard form.

In Chapter 8, we discussed isomorphism types of 2-uniform covering groups of uniform rank at most three, where more choices potentially exist for a 2-uniform basis. We gave a complete enumeration in the case of uniform rank 2 or 3 and corank 1. In the case of uniform rank and corank both equal to 2, we gave a complete account of those groups with multiple graph representations. A complete enumeration for groups of uniform rank 3 and corank 2 or 3 remains out of our reach at present.

We have not discovered an analogue of Theorem 2.17 for 2-uniform covering groups, that can be stated in a succinct way. We suspect that no such neat statement exists, given the complexity of 2-uniform bases.

It would be of interest to determine the number of isomorphism types of covering groups of rank 2 of an elementary abelian group of specified odd order. As noted in Chapter 2, this is a problem of linear algebra that would involve different techniques from those employed here. Another direction of possible further investigation would be to apply our methods to other families of 2-groups of nilpotency class 2.

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