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WRINKLES AND WAVES IN SOFT DIELECTRIC PLATES

A thesis submitted by

Hannah Conroy Broderick

to the

School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland, Galway

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Supervisor: Professor Michel Destrade

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Finally, I’d like to thank the Irish Research Council for their financial support.
Declaration

I declare that the work in this thesis is my own, or, in the case of Chapters 2-4, joint work with my co-authors. I have not obtained a degree in this University, or elsewhere, on the basis of this work. My contributions to each chapter are as follows:

Chapter 2: I derived the Stroh formalism and electroacoustic moduli components. I analysed the Stroh formalism and dispersion equations with my co-authors. I wrote the first draft of the appendix and later drafts of the main text.

Chapter 3: I conducted the initial Hessian stability analysis, later expanded on by co-authors, and all geometric stability analysis. I generated all plots relating to the analytical work. I wrote the first draft and edited subsequent drafts.

Chapter 4: I extended the Stroh formalism to include dynamics and conducted the theoretical analysis, later verified by my collaborators. I wrote the first and later drafts.

__________________________________________________________

Hannah Conroy Broderick
Abstract

This article-based thesis comprises a collection of three articles, each of which constitutes a separate chapter, written and formatted in pre-print manuscript form. The general aim of the thesis is to model instabilities and waves in soft dielectric elastomer plates, with a particular focus on wrinkle formation and wave propagation modes.

Soft dielectric materials are smart materials that deform in the presence of an electric field. They have potential promising applications in devices such as artificial muscles and soft robotics, where there is great demand for materials that can undergo repeated large deformations.

In principle, large deformations can be obtained by exploiting the so-called snap-through instability. However, this phenomenon is difficult to achieve and control in practice, as the material often fails due to electric breakdown, or due to wrinkles appearing on the surface of the material. Here we study in turn the stability of voltage and charge-controlled soft dielectric plates. We investigate Hessian and geometric instability modes. We find that voltage-controlled dielectrics can wrinkle in compression and extension, whereas charge-controlled dielectrics can only wrinkle in compression. We find that charge-controlled actuation is more stable than voltage-controlled actuation.

Studies on waves in dielectric materials suggest the possibility of controlling the wave velocity by applying an appropriate electric field. This paves the way for applying acoustic non-destructive evaluation techniques to dielectric plates, a technique already used in purely elastic materials. Here we study Lamb wave propagation in dielectric plates subject to electrical and mechanical loadings. We look at the effects of the pre-stress, the electric field and the strain-stiffening on the wave characteristics.

This work relies on theoretical and numerical treatments, using the multi-
physics theory of nonlinear electro-elasticity, the incremental theory of small deformations and motions superposed on a large actuation, the Stroh formalism, the numerical resolution of boundary-value problems, and Finite Element simulations.
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Chapter 1

Introduction

1.1 Dielectric Elastomers

Dielectric elastomers are soft active materials that deform under the influence of an electric field. Typically the dielectric material is coated with soft electrodes such as carbon grease and an electric field is applied across the thickness. As the electric field increases, the thickness decreases and the material expands in the planar directions, resulting in an increase in surface area (Suo, 2010). These materials have promising applications as actuators in a variety of devices (Lu et al., 2020) including artificial muscles (Ashley, 2003; Pelrine et al., 2002), soft robots (Gu et al., 2017, 2018; Li et al., 2017), energy generators (Chiang Foo et al., 2012; Huang et al., 2013; Shian et al., 2014), sensors (Sun et al., 2014) and tactile Braille displays (Carpi et al., 2011; Chakraborti et al., 2012; Frediani et al., 2018).

The electromechanical response of soft materials was first discovered by Röntgen (1880). He applied charges of opposite signs onto the faces of a rubber strip, and found that its length increased “by several centimetres”. He also noted that the deformation could be reversed by removing the charges (Keplinger et al., 2010). Around the same time, Quincke (1880) conducted a similar experiment and found that the expansion due to an applied charge in rubber was 10 times that of glass. These early results showed that elastic materials could undergo reversible deformations when subject to an applied electrical charge. However, little attention was afforded to this phenomenon in following years.

Dielectric elastomers gained increased attention again following the results of
Introduction

Figure 1.1: Examples of applications of dielectric elastomers. (a) Soft robotic fish from Li et al. (2017), (b) Location and pressure sensor from Sun et al. (2014) and (c) Robotic gripper from Kofod et al. (2007).

Pelrine et al. (2000), who achieved an areal actuation strain of more than 100% using the acrylic elastomer VHB 4910. This large strain was achieved by pre-straining the dielectric film, resulting in a stiffening in one direction. The film expanded in the preferred, less stiff direction, causing a large expansion. This large deformation opened the door for many applications of dielectric elastomers, in particular as actuators for soft robotics and artificial muscles (Pelrine et al., 2002). In recent years, this large expansion has continued, with much attention focused on increasing the actuation performance of dielectric elastomer actuators (An et al., 2015; Keplinger et al., 2012). The current record for actuation strain is an areal actuation strain of 2200% achieved by An et al. (2015) by exploiting the snap-through instability in an inflated dielectric elastomer tube (Lu et al., 2020).

Typically, large deformations are achieved in dielectric elastomers via the
Snap-through instability (An et al., 2015; Keplinger et al., 2012; Li et al., 2013; Rudykh et al., 2012; Zhao and Suo, 2010). In principle, when the voltage reaches a critical value the stretch increases rapidly, causing a large deformation. However, this phenomenon is difficult to achieve in practice as the material often fails due to electric breakdown (Blok and LeGrand, 1969; Huang et al., 2012; Zurlo et al., 2017). Electric breakdown can be avoided by, for instance, pre-stretching the dielectric elastomer before it is electrically actuated (Jiang et al., 2016; Li et al., 2011) or by designing elastomers with shorter polymer chains (Koh et al., 2011). Snap-through may also fail due to other instabilities such as wrinkling (Liu et al., 2016; Plante and Dubowsky, 2006), compression failure (De Tommasi et al., 2011) or band localisation (Gei et al., 2014), among many others.

Of particular interest are wrinkling instabilities, which may be unstable and cause electric breakdown (Plante and Dubowsky, 2006) or may be stable and exploited for applications including actuation (Conn and Rossiter, 2012) or dynamic patterning (Wang et al., 2012). Understanding instabilities in dielectric elastomers is vital for applications. Theoretical models allow us to predict these instabilities so that they may be avoided or exploited, as needed.


1.2 Electroelasticity

The theory of nonlinear electroelasticity was first developed by Toupin (1956), who combined the theories of continuum mechanics and electrostatics to develop a unified theory for elastic dielectrics based on constitutive equations. The theory was later simplified and expanded to include dynamics and thermal and other effects by a variety of authors (Dorfmann and Ogden, 2017; Eringen, 1963; Eringen and Maugin, 1990; McMeeking and Landis, 2004; Tiersten, 1971).

The advent of new materials that could undergo large deformations sparked renewed interest in the theory of dielectric elastomers, leading to the development of the Lagrangian formulation of electroelasticity by Dorfmann and Ogden (2005, 2014a, 2017). In that formulation, the analysis is based on a total energy density function depending on the deformation gradient and the Lagrangian (or nominal) form of the electric field or displacement, leading to constitutive equations in terms of a total Cauchy stress. The Lagrangian formulation is now widely used to solve boundary value problems in a variety of geometries, including plates, tubes and shells (Dorfmann and Ogden, 2017).

The formulation is based on the notion of Lagrangian or nominal field variables, i.e. the total (mechanical and Maxwell) stress, electric field and electric displacement. Consider a dielectric elastomer in the absence of stress and electric field with reference configuration $B_r$. After the application of mechanical loads and an electric field the elastomer has current configuration $B$. In this configuration, the total stress, electric field and electric displacement are denoted $\tau$, $E$ and $D$, respectively. The Lagrangian fields $(T, E_L, D_L)$ are found by conducting the following pull-back operations from $B$ to $B_r$ (Dorfmann and Ogden, 2005; Suo, 2010)

\[ T = JF^{-1}\tau, \quad E_L = F^T E, \quad D_L = JF^{-1}D, \] (1.1)

where $F$ is the deformation gradient from $B_r$ to $B$, and $J$ its determinant.

Consider the total energy density functions $\Omega = \Omega(F, E_L)$ and $\Omega^* = \Omega^*(F, D_L)$, with the connection (Dorfmann and Ogden, 2006)

\[ \Omega^*(F, D_L) = \Omega(F, E_L) + E_L \cdot D_L. \] (1.2)

For energy densities of the form $\Omega$, the constitutive equations are given by

\[ T = \frac{\partial\Omega}{\partial F^*}, \quad D_L = -\frac{\partial\Omega}{\partial E_L}, \] (1.3)
whereas energy densities of the form $\Omega^*$ give constitutive equations

$$
\begin{align*}
T &= \frac{\partial \Omega^*}{\partial F}, \\
E_L &= \frac{\partial \Omega^*}{\partial D_L}.
\end{align*}
$$

(1.4)

Note that an energy function with dependence on the polarisation may also be chosen, for example see Yang et al. (2017).

The energy density function is typically composed of two parts, the purely elastic part, for example the neo-Hookean or Gent material models, and the electromechanical part, which models the coupling between the mechanical and electric fields. The electromechanical coupling is usually modelled using the ideal dielectric assumption $D = \varepsilon E$, where $\varepsilon$ is the permittivity of the material (Suo, 2010). The energy density function is then of the form

$$
\Omega^* = W(F) + \frac{1}{2\varepsilon} D \cdot D,
$$

(1.5)

where $W(F)$ is the purely elastic energy function. Although this linear relationship is idealised, it has been verified for low to moderate values of the electric field (Zurlo et al., 2018). However, in general, the permittivity $\varepsilon$ may depend on the deformation (Wissler and Mazza, 2007).

Modelling and predicting instabilities are important aspects of the theory of dielectric elastomers. In particular, models for the pull-in instability and electric breakdown are valuable for predicting failure. The Hessian method is often used to model electric breakdown (Norris, 2008; Zhao and Suo, 2007). In this method, the material is said to experience electric breakdown when the Hessian matrix associated with the second variation of the energy ceases to be positive definite (Zhao and Suo, 2007). The Hessian method provides a good estimate for the breakdown voltage and stretch, corresponding to experimental results (Norris, 2008); however, it is limited to homogeneous deformations in thin materials, as it does not take the thickness into account (Dorfmann and Ogden, 2019). In spite of this limitation, it can also be used to predict the onset of necking, which is not predicted by other theories (Fu et al., 2018). Pull-in and wrinkling instabilities have also been modelled using energy minimisation (Greaney et al., 2019; Zurlo et al., 2017), tension field theory (De Tommasi et al., 2011) and many other methods (Dorfmann and Ogden, 2019).

The incremental formulation of electroelasticity was developed by Dorfmann and Ogden (2010a) and applied to investigate the stability of an electroelastic
Large deformation
Incremental

Figure 1.3: The incremental formulation. The material first undergoes a large deformation. An incremental deformation is then superposed on top of the large deformation.

half-space. The incremental theory is a “small-on-large” theory where a linearised incremental deformation is superposed onto a large finite deformation, see Figure 1.3. The advantages of the incremental theory are that it can be used to model homogeneous and inhomogeneous deformations, and that it includes the effect of the thickness on the instabilities. This theory has been used to model instabilities in many systems, for example, dielectric plates (Dorfmann and Ogden, 2014b; Yang et al., 2017), tubes (Bortot and Shmuel, 2018; Su et al., 2016) and composites (Bertoldi and Gei, 2011; Rudykh et al., 2014). The incremental theory has also been extended to the dynamic case by Dorfmann and Ogden (2010b) to model electroelastic waves in a dielectric half-space. Additionally, this theory has been applied to model waves in dielectric plates (Shmuel et al., 2012) and cylinders (Wu et al., 2017).

1.3 Outline of the thesis

Using the incremental theory of electroelasticity, we model wrinkling instabilities in dielectric plates in Chapter 2. We introduce the main parts of the incremental theory and derive the Stroh formalism in the electrostatic voltage-controlled case. Using this formalism, we model two-dimensional wrinkles on a voltage-controlled
dielectric plate. We consider both the ideal dielectric and a material that can undergo the snap-through instability, i.e. the Gent dielectric. We find that the voltage-controlled dielectric plate can wrinkle in extension as well as in compression. This Chapter was published as an article in the *Journal of the Mechanics and Physics of Solids* (Su et al., 2018).

In Chapter 3, we investigate the stability of a charge-controlled dielectric plate. We introduce the notion of Hessian stability, often used to determine the breakdown of dielectric elastomers, and derive its version in charge-controlled actuation. We transform the Stroh formalism of Chapter 2 to the charge-controlled case and investigate the geometric stability. We find that charge-controlled dielectric plates can only wrinkle in compression, and are therefore more stable than voltage-controlled plates. We verify our analytical results with Finite Element simulations and find very good agreement. The corresponding article was published in the *International Journal of Engineering Science* (Conroy Broderick et al., 2020b).

Chapter 4 focuses on wave propagation and we consider electro-elastic Lamb waves in dielectric plates. We introduce the relevant theory of acoustic electroelasticity and extend the Stroh formalism derived in Chapter 2 to include wave propagation. We consider the effect of the electric field, pre-stress and strain stiffening on the wave velocity. This Chapter was published as an article in *Extreme Mechanics Letters* (Conroy Broderick et al., 2020a).

Finally, Chapter 5 contains some concluding remarks.

**Bibliography**


Introduction


Chapter 2

Wrinkles in soft dielectric plates

Yipin Su$^{1,2}$, Hannah Conroy Broderick$^1$, Weiqiu Chen$^2$, Michel Destrade$^{1,2}$

$^1$ School of Mathematics, Statistics and Applied Mathematics, NUI Galway
$^2$ Department of Engineering Mechanics, Zhejiang University

Abstract

We show that a smooth giant voltage actuation of soft dielectric plates is not easily obtained in practice. In principle one can exploit, through pre-deformation, the snap-through behaviour of their loading curve to deliver a large stretch prior to electric breakdown. However, we demonstrate here that even in this favourable scenario, the soft dielectric is likely to first encounter the plate wrinkling phenomenon, as modelled by the onset of small-amplitude sinusoidal perturbations on its faces. We provide an explicit treatment of this incremental boundary value problem. We also derive closed-form expressions for the two limit cases of very thin membranes (with vanishing thickness) and of thick plates (with thickness comparable to or greater than the wavelength of the perturbation). We treat explicitly examples of ideal dielectric free energy functions (where the mechanical part is of the neo-Hookean, Mooney-Rivlin or Gent form) and of dielectrics exhibiting polarisation saturation. In addition to the expected buckling mode coming from the purely elastic case, we discover a second mode occurring at large voltages in extension. We find that plates always wrinkle anti-symmetrically, be-
fore the symmetric modes can be reached. Finally we make the link with the classical results of the Hessian electro-mechanical instability criterion and of Euler buckling for an elastic column.

2.1 Introduction

When a soft dielectric plate is put under a large voltage applied to its faces, it expands in its plane. At first, the expansion increases slowly and almost linearly with the voltage until, typically, a local maximum is reached. Then in theory, the voltage drops suddenly, until it reaches a local minimum, rises again, to reach the same level it had at the earlier maximum, and then continues to rise. In practice the voltage doesn’t drop: it stays at the level of the first maximum while the plate expands rapidly, until it starts increasing again with the stretch. The membrane is said to experience a snap-through expansion (An et al., 2015; Dorfmann and Ogden, 2014a; Li et al., 2017; Rudykh et al., 2012; Zhao and Suo, 2010). This large and almost instantaneous extension is highly desirable in experiments but is rarely achieved because during the snap-through the elastomer fails due to electric breakdown (Blok and LeGrand, 1969; Huang et al., 2012b; Koh et al., 2011). Graphically, the curve of the electric breakdown crosses the voltage-stretch curve before the snap-through portion is completed.

This undesirable outcome can be avoided in a number of ways, in principle (Jiang et al., 2015, 2016; Koh et al., 2011; Li et al., 2011a). We could for instance try to design a dielectric material with a free energy density such that the snap-through sequence is completed prior to electrical breakdown. But it seems that such a material has not been synthesised yet. We could pre-stretch the membrane so that the snap-through path is shifted below that of the un-stretched membrane. But in that scenario the snap-through actuation gain is greatly reduced. Moreover, with larger pre-stretch, the corresponding path might become increasingly monotonic and the snap-through possibility will then disappear altogether.

These possible events are summarised in Figure 2.1, where we take a Gent ideal dielectric with $J_m = 97.2$ (Dorfmann and Ogden, 2014a; Gent, 1996) as a representative stiffening parameter of elastomers (a different value stretches or shrinks the plots, but the essential results remain the same). The critical dimensionless
Figure 2.1: Principle of the snap-through giant actuation. Solid lines are the voltage-stretch curves for homogeneous loading at different levels of pre-stress ($\bar{s} = 0, 0.8, ..., 4.5$), when the plate is modelled by the Gent ideal dielectric (here $J_m = 97.2$ (Dorfmann and Ogden, 2014a; Gent, 1996)). The dotted line corresponds to the onset of snap-through instability. The dashed parts of the voltage-stretch curves are the theoretical response of the elastomer after the snap-through instability is triggered, which will not happen in practice. The blue dash-dotted lines are hypothetical Electrical Breakdown curves. The situation described by $\bar{E}_{BD}^{(3)}$ is the most favourable, allowing the initially unstrained material to expand and experience a large snap-through from A to B. For $\bar{E}_{BD}^{(2)}$, this will not be allowed, but a certain level of pre-stress (here $\bar{s} = 0.8, 1.5$) will give a (smaller) snap-through transition (from C to D, from E to F, respectively). As the hypothetical $\bar{E}_{BD}$ curve slides down further, this possibility will vanish eventually (see $\bar{s} = 2.5, 4.5$ curves). For $\bar{E}_{BD}^{(1)}$, no snap-through is possible.

Electric field of the plate is $\bar{E}_{BD} = V_{BD}/(h\sqrt{\mu/\varepsilon})$, where $V_{BD}$ is the voltage causing electrical breakdown of the elastomer and $h$ is its current thickness. This material constant $\bar{E}_{BD}$ is also known as the dielectric strength (Pelrine et al., 2000). For an equal-biaxially stretched dielectric elastomer, $\bar{E}_{0BD} = \lambda^{-2}\bar{E}_{BD}$, where $\bar{E}_{0BD}$ is the nominal measure of $\bar{E}_{BD}$ and $\lambda$ is the in-plane stretch. This relation is displayed by the blue dash-dotted lines in Figure 2.1, for hypothetical values of the dielectric strength ($\bar{E}_{BD} = 0.5, 20, 40$).

In this chapter we show that in any case, the snap-through scenario is derailed
Wrinkles in soft dielectric plates

because the loading curve crosses that of *wrinkle formation*. Indeed, several experiments (Jiang et al., 2015, 2016; Liu et al., 2016; Plante and Dubowsky, 2006) have shown that sinusoidal wrinkles appear in soft dielectric plates under high voltage, see examples in Figure 2.2. Here we model and predict how they will form.

![Figure 2.2: Experimental evidence of electro-mechanical wrinkling instability: (a) collapse of a thin film of the rubber-like material VHB 4905/4910 put under a large voltage (Plante and Dubowsky, 2006); (b) wrinkling of a VHB 4910 membrane under high voltage (Liu et al., 2016); (c) Electric activation of acrylic elastomers (Pelrine et al., 2000). We estimate that the ratio of the initial plate thickness to the wrinkle wavelength is \( H/L \simeq 0.17, 0.35 \) in Cases (a) and (b), respectively.](image)

In Section 2.2 we begin by recalling the equations governing the large deformation of a dielectric plate subject to pre-stretch and voltage. We also recall the Hessian stability criterion of Zhao and Suo (2007) in Section 2.3 and make the connection between energy minimisation and the snap-through instability.

We then rely on the theory of incremental deformations superposed on large actuation (Bertoldi and Gei, 2011; Bortot and Shmuel, 2018; Dorfmann and Ogden, 2010a,b; Gei et al., 2012; Rudykh and deBotton, 2011; Rudykh et al., 2014; Su et al., 2016) to solve the boundary value problem of small-amplitude sinusoidal wrinkles appearing on the mechanically-free faces of the plate (Section 2.4).

This problem was treated earlier by Dorfmann and Ogden (2014b,c) and more recently, by Yang et al. (2017b) and Díaz-Calleja et al. (2017), but not in a fully analytical manner as here. Here we present a general framework to solve the boundary-value problem for a general free energy density. We manage to obtain
analytical results in the case of the Gent ideal dielectric, a model which exhibits the typical non-monotonic snap-through loading curve, see Figure 2.1, and also in the cases of neo-Hookean and Mooney-Rivlin ideal dielectrics. Thanks to the Stroh formulation and the surface impedance method (Destrade, 2015), we obtain closed-form expressions for the dispersion equation. We are also able to separate the symmetric and antisymmetric modes of buckling and to solve the dispersion equations in a numerically robust manner.

Then in Section 2.5 we derive the explicit equations giving the thin-plate and the short-wave limits. Plotting the two corresponding curves gives a narrow region where all physical plate dimensions and wrinkle wavelengths are located. We find that it crosses all loading curves before the snap-through can be completed.

In Section 2.6 we present further results, for a dielectric exhibiting polarisation saturation, and for the specialisation of our analytical formulas to known results in classical Euler buckling theory for elastic columns. We also make the link with, and extend the Hessian criterion of instability (Zhao and Suo, 2007) for electro-elastic dielectrics of a certain thickness.

Finally, Section 2.7 recapitulates the results and puts them into a wider context, thanks to the extension of the analysis to a tri-axial pre-stretch conducted in the Appendix.

### 2.2 Large actuation

We write the free energy density for the dielectric plate as \( \Omega = \Omega(F, E_L) \), where \( F \) is the deformation gradient and \( E_L \) is the Lagrangian form of the electric field \( E \): \( E_L = F^T E \). We introduce the following complete set of invariants for an isotropic incompressible dielectric (Dorfmann and Ogden, 2005, 2006; Goshkoledia and Rudykh, 2017; Rudykh et al., 2014),

\[
\begin{align*}
I_1 &= \text{tr} \ c, & I_2 &= \text{tr} (c^{-1}), \\
I_4 &= E_L \cdot E_L, & I_5 &= E_L \cdot c^{-1} E_L, & I_6 &= E_L \cdot c^{-2} E_L,
\end{align*}
\]  

(2.1)

where \( c = F^T F \) is the right Cauchy-Green deformation tensor.

In the appendix we present results for general materials, where in all generality \( \Omega \) can be written as \( \Omega = \Omega(I_1, I_2, I_4, I_5, I_6) \). In the main text we specialise the results to the Gent ideal dielectric (Gent, 1996; Huang et al., 2012a), which
Wrinkles in soft dielectric plates

exhibits the snap-through response. Its free energy is

\[ \Omega_G = -\frac{\mu J_m}{2} \ln \left(1 - \frac{I_1 - 3}{J_m}\right) - \frac{\varepsilon}{2} I_5, \]  

(2.2)

where \( \mu \) is the initial shear modulus in the absence of electric field (in Pa), \( J_m \) is the stiffening parameter (dimensionless) and \( \varepsilon \) is the permittivity (in F/m). When \( J_m \to \infty \), the neo-Hookean ideal dielectric (Zhao and Suo, 2007) is recovered,

\[ \Omega_{\text{NH}} = \frac{\mu}{2} (I_1 - 3) - \frac{\varepsilon}{2} I_5, \]  

(2.3)

but that model does not provide snap-through loading behaviour.

We call \( H \) the initial thickness of the plate. We apply a voltage \( V \) on the faces of the plate. Then the only non-zero component of the Lagrangian electric field is \( E_{L2} = V/H \), which we call \( E_0 \). We call \( x_1, x_3 \) the in-plane Eulerian principal axes and \( x_2 \) the transverse axis, so that we have the following principal stretches and electric field components,

\[ \lambda_1 = \lambda_3 = \lambda, \quad \lambda_2 = \lambda^{-2}, \quad E_1 = E_3 = 0, \quad E_2 = \lambda^2 E_0. \]  

(2.4)

Note that in the appendix we give results for a bi-axially stretched plate, when \( \lambda_1 \) is not necessarily equal to \( \lambda_3 \).

Introducing the function \( \omega = \omega(\lambda, E_0) \) as

\[ \omega = \Omega(2\lambda^2 + \lambda^{-4}, 2\lambda^{-2} + \lambda^4, E_0^2, \lambda^4 E_0^2, \lambda^8 E_0^2), \]  

(2.5)

for a plate with no mechanical traction applied on the faces \( x_2 = \pm h/2 \), where \( h \) is the thickness of the deformed plate, we find the following compact expression for the equi-biaxial nominal stress component required to maintain the deformation (Dorfmann and Ogden, 2014b):

\[ s = \frac{1}{2} \frac{\partial \omega}{\partial \lambda}, \]  

(2.6)

where \( s \) is the nominal stress applied along the in-plane directions. For example, for the Gent ideal dielectric we have (Lu et al., 2012)

\[ s_G = \mu \left( \frac{\lambda - \lambda^{-5}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} - \frac{\varepsilon \lambda^3 E_0^2}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} \right), \]  

(2.7)

which we can easily invert to find the voltage \( V_G = E_0 H \). Here, a non-dimensional measure of \( V_G \) is

\[ \bar{E}_0 = \frac{V_G}{H \sqrt{\mu/\varepsilon}} = \frac{E_0}{\sqrt{\mu/\varepsilon}} = \sqrt{\frac{\lambda^{-2} - \lambda^{-8}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} - \lambda^{-3} \bar{s}}, \]  

(2.8)
where $\bar{s} = s_G/\mu$ is a non-dimensional measure of stress. We use this formula to plot the non-dimensional loading curves of Figure 2.1, as well as the curve for the onset of snap-through, corresponding to $d\bar{E}_0/d\lambda = 0$.

Note that the electric displacement vector $\mathbf{D} = (0, D, 0)$, with $D$ being the only non-zero component, is related to the electric field through the formula

$$D = -\lambda_1^{-1}\lambda_3^{-1}\frac{\partial \omega}{\partial E_0}.$$ (2.9)

In this chapter we study the possibility of homogeneously deformed plates buckling inhomogeneously as sketched in Figure 2.3. Typically, we find that the onset of these buckling modes is governed by a dispersion equation relating the critical stretch $\lambda_{cr}$ to the ratio of the plate thickness $H$ by the wavelength of the wrinkles $L$. In effect, we find that it can be factorised into the product of a dispersion equation for antisymmetric wrinkles and one for symmetric wrinkles, see Figures 2.3(c) and (d).

Figure 2.3: When put under voltage $V$ and/or stress $\sigma$, a rectangular plate made of a soft dielectric and with faces covered by compliant electrodes deforms homogeneously (a) and (b). It can even deform so severely as to lose its stability and buckle into antisymmetric (c) or symmetric (d) modes of wrinkles.
2.3 Hessian stability

We first examine the stability of the large actuation using the Hessian criterion (Zhao and Suo, 2007). Here we consider an energy density function of the form \( \Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_L) \), where \( \mathbf{D}_L = \mathbf{F}^{-1} \mathbf{D} \) is the Lagrangian form of the electric displacement \( \mathbf{D} \), and \( \Omega \) and \( \Omega^* \) are connected by the Legendre transform (Dorfmann and Ogden, 2006)

\[
\Omega^*(\mathbf{F}, \mathbf{D}_L) = \Omega(\mathbf{F}, \mathbf{E}_L) + \mathbf{E}_L \cdot \mathbf{D}_L.
\]  

(2.10)

We let \( \mathbf{D}_0 = \mathbf{D}_{L,2} = \lambda^{-2} \mathbf{D} \) be the only non-zero component of \( \mathbf{D}_L \), and introduce the function \( \omega^* = \omega^*(\lambda, D_0) \) which from (2.5) and the connection (2.10), is related to \( \omega = \omega(\lambda, E_0) \) via

\[
\omega^*(\lambda, D_0) = \omega(\lambda, E_0) + E_0 D_0.
\]

(2.11)

To investigate the stability, we consider the non-dimensional free energy of the system

\[
\psi = \bar{\omega}^*(\lambda, D_0) - 2\lambda \bar{s} - \bar{E}_0 \bar{D}_0,
\]

(2.12)

where \( \bar{\omega}^* = \omega^*/\mu \) and \( \bar{D}_0 = D_0/\sqrt{\mu\varepsilon} \); and use the Hessian method of Zhao and Suo (2007) to find where the system is energetically minimal. Note that the free energy of the system cannot be constructed using the energy density function \( \omega(\lambda, E_0) \). For a more detailed discussion on free energies, their construction and stability, see Chapter 3.

Following Zhao and Suo (2007), the free energy of the system is minimised when the Hessian matrix of the second variations is positive definite, or equivalently the system is no longer stable when the Hessian matrix ceases to be positive definite, i.e. when its determinant vanishes. For the Gent ideal dielectric

\[
\bar{\omega}^* = -\frac{J_m}{2} \ln \left( 1 - \frac{2\lambda^2 + \lambda^{-4} - 3}{J_m} \right) + \frac{1}{2}\lambda^{-4} \bar{D}_0^2,
\]

(2.13)

the determinant of the Hessian vanishes when

\[
\bar{E}_0^2 = \frac{1}{3} \left( \frac{\lambda^{-2} + 5\lambda^{-8}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} \right) + \frac{4}{3J_m} \left( \frac{1 - \lambda^{-6}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} \right)^2,
\]

(2.14)

where we have made use of the connection \( \bar{E}_0 = \partial \bar{\omega}^*/\partial D_0 \), or equivalently of Equation (2.9).

This condition is exactly the onset of snap-through curve in Figure 2.1, i.e. the onset of snap-through curve (or the Hessian) delineates the boundary between
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stable and unstable regions in $\bar{E}_0 - \lambda$ space. In particular, it shows that by undergoing snap-through, the material minimises its energy.

Take, for example, the loading curve in the absence of pre-stress, when $\bar{s} = 0$. As the electric field $\bar{E}_0$ is increased, the stretch increases along the stable branch towards the maximum (Point A). The loading curve then snaps from Point A to Point B, thereby minimising the energy. The stretch then continues to increase with increased electric field. If we now decreased the electric field from Point B, the material would follow this new branch to the local minimum, i.e. the next point where the loading curve intersects the Hessian. At this minimum, the material then snaps back to the initial branch and continues to decrease to the initial stretch for decreasing electric field.

Similar behaviour is seen for all values of pre-stress that give loading curves exhibiting snap-through, i.e. those with local maxima. Whereas, for higher pre-stresses, the loading curve is monotonic and always below the onset curve and therefore always energetically minimal.

2.4 Small-amplitude wrinkles

We now linearise the governing equations and boundary conditions in the neighbourhood of the large electro-elastic deformation. We adopt the point of view that the existence of an incremental solution at an equilibrium state implies that the second variation of the energy is not positive definite, and signals the onset of instability, at least in the linearised sense. This connection has been established by several researchers over the years, see for instance the reviews by Sawyers (1996) or Ogden (2000), or the more recent investigation by Chen et al. (2018).

We introduce the following fields: $u$, the small-amplitude mechanical displacement; $\dot{T}_{2i}$, the incremental mechanical traction on the planes $x_2 = \text{const.}$; and $\dot{D}_L$, $\dot{E}_L$, the incremental electric displacement and electric field, respectively. These fields are functions of $x$, the position vector in the current (actuated) configuration (Dorfmann and Ogden, 2014c).

We focus on two-dimensional wrinkles, and thus take the fields to be functions of $x_1, x_2$ only. This leads to (not shown here) $u_3 = \dot{T}_{3i} = \dot{E}_{L3} = \dot{D}_{L3} = 0$. Because the updated, incremental version of the Maxwell equation $\text{Curl}\, \bar{E}_L = 0$ is $\text{curl}\, \dot{E}_L = 0$ (Dorfmann and Ogden, 2010b), we can introduce the electric
potential \( \varphi \) by

\[
\dot{E}_{L1} = -\partial \varphi / \partial x_1, \quad \dot{E}_{L2} = -\partial \varphi / \partial x_2. \tag{2.15}
\]

We then seek solutions with sinusoidal shape along \( x_1 \) and amplitude variations along \( x_2 \), in the form

\[
\{ u_1, u_2, \dot{D}_{L2}, \dot{T}_{21}, \dot{T}_{22}, \varphi \} = \Re \left\{ [k^{-1}U_1, k^{-1}U_2, i \Delta, i \Sigma_{21}, i \Sigma_{22}, k^{-1}\Phi] e^{ikx_1} \right\}, \tag{2.16}
\]

where \( U_1, U_2, \Delta, \Sigma_{21}, \Sigma_{22}, \Phi \) are functions of \( kx_2 \) only and \( k = 2\pi / \mathcal{L} \) is the wavenumber.

Our main result is that the governing equations can be put in the form

\[
\eta' = i N \eta, \tag{2.17}
\]

where

\[
\eta = \begin{bmatrix} U_1 & U_2 & \Delta & \Sigma_{21} & \Sigma_{22} & \Phi \end{bmatrix}^T = \begin{bmatrix} U & S \end{bmatrix}^T, \tag{2.18}
\]

is the Stroh vector and the prime denotes differentiation with respect to \( kx_2 \).

Here \( U = \begin{bmatrix} U_1 & U_2 & \Delta \end{bmatrix}^T, \ S = \begin{bmatrix} \Sigma_{21} & \Sigma_{22} & \Phi \end{bmatrix}^T \) are the generalised displacement and traction vectors, respectively, and \( N \) is the Stroh matrix. In the appendix we show that \( N \) has the following block structure

\[
N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1 \end{bmatrix}, \tag{2.19}
\]

where the \( N_i \) \( (i = 1, 2, 3) \) are real symmetric. We find that these \( 3 \times 3 \) submatrices are

\[
N_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} \frac{1}{c} & 0 & \frac{d}{c} \\ 0 & 0 & 0 \\ \frac{d}{c} & 0 & \frac{d^2}{c} - f \end{bmatrix},
\]

\[
N_3 = \begin{bmatrix} \frac{e^2}{g} - 2(b + c) & 0 & -\frac{e}{g} \\ 0 & c - a & 0 \\ -\frac{e}{g} & 0 & \frac{1}{g} \end{bmatrix}. \tag{2.20}
\]
where $a$, $b$, $c$, $d$, $e$, $f$, and $g$ are electro-elastic moduli. Their general expressions in terms of the free energy density $\Omega$ are given in the appendix. For equi-biaxial deformations, they read

\[
a = 2 \left[ \lambda^2 (\Omega_1 + \lambda^2 \Omega_2) + \lambda^4 (\Omega_5 + (2\lambda^4 + \lambda^{-2})\Omega_6) \right] E_0^2,
\]

\[
b = 2 \left\{ (\lambda^{-4} - \lambda^2) [(\lambda^{-4} - \lambda^2)(\Omega_{11} + 2\lambda^2\Omega_{12} + \lambda^4\Omega_{22}) \\
- 2\lambda^4(\Omega_{15} + 2\lambda^4\Omega_{16} + \lambda^2\Omega_{25} + 2\lambda^6\Omega_{26})E_0^2] + \lambda^8(\Omega_{55} + 4\lambda^4\Omega_{56} + 4\lambda^8\Omega_{66})E_0^4 \right\}

+ (\lambda^2 + \lambda^{-4})(\Omega_1 + \lambda^2\Omega_2) + \lambda^4 \left[ \Omega_5 + 2(3\lambda^4 - \lambda^{-2})\Omega_6 \right] E_0^2,
\]

\[
c = 2\lambda^{-2} \left[ \lambda^{-2}(\Omega_1 + \lambda^2\Omega_2) + \lambda^4\Omega_6 E_0^2 \right],
\]

\[
d = -2\lambda^2 \left[ \Omega_5 + (\lambda^4 + \lambda^{-2})\Omega_6 \right] E_0,
\]

\[
e = 4\lambda^2 \left[ (\lambda^{-4} - \lambda^2)(\lambda^{-4}\Omega_{44} + \Omega_{15} + \lambda^4\Omega_{16} + \lambda^{-2}\Omega_{24} + \lambda^2\Omega_{25} + \lambda^6\Omega_{26}) \\
- \lambda^4(\lambda^{-4}\Omega_{45} + 2\Omega_{46} + \Omega_{55} + 3\lambda^4\Omega_{56} + 2\lambda^6\Omega_{66})E_0^2 - (\Omega_5 + 2\lambda^4\Omega_6) \right] E_0,
\]

\[
f = 2(\lambda^2\Omega_4 + \Omega_5 + \lambda^{-2}\Omega_6),
\]

\[
g = 4\lambda^4 \left[ \lambda^{-8}\Omega_{44} + 2\lambda^{-4}\Omega_{45} + 2\Omega_{46} + \Omega_{55} + 2\lambda^4\Omega_{56} + \lambda^8\Omega_{66} \right] E_0^2

+ 2(\lambda^{-4}\Omega_4 + \Omega_5 + \lambda^4\Omega_6),
\]

\[\text{(2.21)}\]

where $\Omega_i = \partial\Omega/\partial I_i$ and $\Omega_{ij} = \partial^2\Omega/\partial I_i \partial I_j$ for $i = 1, 2, 4, 5, 6$.

Specialising to the Gent ideal dielectric, we find

\[
a = \mu(2\lambda^2\dddot{W}' - \lambda^4\dddot{E}_0), \quad c = 2\mu\lambda^{-4}\dddot{W}'', \quad 2b = 4\mu(\lambda^{-4} - \lambda^2)^2\dddot{W}'' + a + c, \quad \text{(2.22)}
\]

\[
d = \sqrt{\mu\varepsilon\lambda^2\dddot{E}_0}, \quad e = 2d, \quad f = g = -\varepsilon, \quad \text{(2.23)}
\]

where

\[
\dddot{W}' = \frac{1}{2 \left[ 1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m \right]^2},
\]

\[
\dddot{W}'' = \frac{1}{2J_m \left[ 1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m \right]^2}. \quad \text{(2.24)}
\]

The Stroh equation must be solved subject to the incremental boundary conditions on the faces of the plate of no mechanical traction and no electrical field, i.e.,

\[
S(-kh/2) = 0, \quad S(kh/2) = 0. \quad \text{(2.25)}
\]
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Now because the Stroh matrix is constant, the resolution of (2.17) is straightforward. It reduces to an eigenvalue problem, yielding a complete set of six linearly independent eigensolutions with exponential variations in $x_2$. Then the boundary conditions give a linear $6 \times 6$ homogeneous system of equations for the six unknowns $U_1(\pm h/2), U_2(\pm h/2), \Delta(\pm h/2)$, for which the determinant must be zero: this is the dispersion equation.

Using the usual matrix manipulations of plate acoustics (see Nayfeh (1995) for instance), the six exponential solutions can be decoupled into two sets of three (hyperbolic) trigonometric solutions, one corresponding to antisymmetric modes, the other to symmetric modes, see sketches in Figure 2.3. The dispersion equation itself factorises into two corresponding equations. In the appendix we give those equations for an arbitrary tri-axial pre-stretch, for materials with free energies of the forms

$$
\Omega = W(I_1) - \frac{\varepsilon}{2} I_5, \quad \Omega = \frac{\mu(1 - \beta)}{2} (I_1 - 3) + \frac{\mu \beta}{2} (I_2 - 3) - F(I_5),
$$

(2.26)

where $W$ and $F$ are arbitrary functions (note that the Gent ideal dielectric belongs to the first type), and $0 \leq \beta \leq 1$ is a constant.

For the Gent ideal dielectric under an equi-biaxial pre-stretch (2.4), we find the following explicit dispersion equation for the anti-symmetric wrinkles:

$$
2\tilde{W} \left[ p_1(1 + p_2^2)^2 \tanh(\pi p_1 \lambda^{-2} H/L) - p_2(1 + p_1^2)^2 \tanh(\pi p_2 \lambda^{-2} H/L) \right]
= (p_2^2 - p_1^2) \lambda^6 \tilde{E}_0^2 \tanh(\pi \lambda^{-2} H/L),
$$

(2.27)

where

$$
p_{1,2} = \frac{\lambda^3}{2} \pm \frac{1}{2} \sqrt{1 + 2(\lambda - \lambda^{-2})^2 \frac{W''}{W'}} \pm \frac{\lambda^3 - 1}{2} \sqrt{1 + 2(\lambda + \lambda^{-2})^2 \frac{W''}{W'}},
$$

(2.28)

For symmetric wrinkles, the dispersion equation is the same except that tanh is replaced with coth everywhere.

In the case of the neo-Hookean ideal dielectric (2.3), we take $J_m \to \infty$ and have $W' = 1/2, W'' = 0, p_1 = \lambda^3, p_2 = 1$, so that the dispersion equations simplify to

$$
\left[ \frac{\tanh(\lambda \pi H/L)}{\tanh(\lambda^{-2} \pi H/L)} \right]^{\pm 1} = \frac{(1 + \lambda^6)^2}{4 \lambda^3} + \frac{\lambda^5 (1 - \lambda^6)}{4} \tilde{E}_0^2,
$$

(2.29)
where the $+1$ ($-1$) exponent corresponds to anti-symmetric (symmetric) wrinkles. Note that these equations recover the purely elastic buckling criterion when $\bar{E}_0 = 0$ (Ogden and Roxburgh, 1993).

We now plot the dispersion curves for the Gent ideal dielectric as the plate is loaded homogeneously by an increasing voltage.

When $\bar{E}_0 = 0$, see Figure 2.4(a), we recover the purely mechanical case. The lower/dashed (upper/full) curve corresponds to symmetric (anti-symmetric) buckling. We see that in extension ($\lambda > 1$), the plate is always stable, whereas in contraction ($\lambda < 1$), it buckles antisymmetrically, with $\lambda_{cr} \simeq 1$ when $H/L$ is small (thin plate, long wavelength) and $\lambda_{cr} \simeq 0.661$ when $H/L$ is large (thick plate, short wavelength). Note that here “large” and “thick” simply mean that the initial plate thickness $H$ is of the order of the wavelength $L$.

When $\bar{E}_0 = 0.2$, see Figure 2.4(b), the landscape is the same, with the curves slightly shifted upwards. The plate only buckles in contraction.

When $\bar{E}_0 = 0.4$, see Figure 2.4(c), we see that, in addition to contractile buckling, the possibility of wrinkling in extension has now emerged. The plate buckles anti-symmetrically when $\lambda$ reaches a critical value $\lambda_{cr}$ between 2.65 (thin-plate limit) and 2.81 (short-wavelength limit), depending on the ratio $H/L$.

Similarly, when $\bar{E}_0 = 0.6$, Figure 2.4(d) shows that the plate wrinkles anti-symmetrically in extension when $\lambda$ reaches a critical value $\lambda_{cr}$ between 1.65 (thin-plate limit) and 1.78 (short-wave limit), depending on the ratio $H/L$.

From this rapid analysis, we conclude that it is unnecessary to study the dispersion equation in detail for the Gent ideal dielectric, and that the thin-plate and short-wave limits suffice to find global, wavelength-independent critical stretches of wrinkling in extension.

### 2.5 Thin-plate and short-wave instabilities

In the previous section we saw that put under a sufficiently large voltage, a dielectric plate wrinkles anti-symmetrically in extension. Depending on the ratio $H/L$ of thickness to wavelength, the plate wrinkles at a critical stretch located in between a lower bound, corresponding to the limit for thin plates $H/L \to 0$ and an upper bound, the limit for short wavelengths $H/L \to \infty$. In this section we present explicit expressions for these two limits.
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Figure 2.4: Dispersion curves for electrically loaded \( E_0 = 0, 0.2, 0.4, 0.6 \) dielectric plates: critical stretch ratio \( \lambda_{cr} \) of compression (lower curves) and of extension (upper curves) against the initial thickness to wavelength ratio \( H/L \). (a) and (b): for low voltages, the plate can wrinkle only in compression. (c) and (d): For higher voltages, the dielectric plate can wrinkle in extension, see the thick upper line, corresponding to anti-symmetric wrinkles. Then, the critical stretch is located between its limit values in the thin-plate \( (H/L \to 0) \) and short-wave \( (H/L \to \infty) \) limits.

First, the thin-plate limit can be found with the asymptotic behaviour of \( \tanh \) as its argument is small in the dispersion equation (2.27) for the Gent ideal dielectric. However, we can in fact give the thin plate limit in the most general case. Using the results of Shuvalov (2000), it is easy to show that the buckling condition when \( H/L \to 0 \) is simply

\[
\det N_3 = 0,
\]

where \( N_3 \) is the Stroh sub-matrix given in Equation (2.20). It factorises to give

\[(a - c)(b + c) = 0,\]

and the anti-symmetric mode corresponds to \( a - c = 0 \). Combining Equations (2.5) and (2.6), we find that

\[a - c = (\lambda/2)\frac{\partial \omega}{\partial \lambda},\]

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anti-symmetric buckling is equivalent to
\[
\frac{\partial \omega}{\partial \lambda} = 0. \quad (2.31)
\]
Comparing with Equation (2.6), we see that in general, the loading curve \( E_0 - \lambda \) with no pre-stress (\( \bar{s} = 0 \)) is in fact the buckling limit for plates of vanishing thickness. This makes sense from a mechanical point of view: a purely elastic membrane with vanishing thickness buckles as soon as it is contracted (\( \lambda_{cr} = 1.0 \)); a dielectric membrane with vanishing thickness can first be stretched by applying a voltage, to \( \lambda = \lambda_0 \), say, and then it will buckle as soon as it is contracted, so that \( \lambda_{cr} = \lambda_0 \) now. Here the equation reads
\[
\frac{\lambda^{-2} - \lambda^{-8}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} = \bar{E}_0^2. \quad (2.32)
\]
Next the short-wave limit is found from Equation (2.27) by replacing \( \tanh \) with 1, its value as \( H/L \to \infty \). After some re-arrangement, we find that it reads
\[
2\lambda(\lambda^9 + \lambda^6 + 3\lambda^3 - 1)\bar{W}' + 4(\lambda^6 - 1)^2\bar{W}'' = \lambda^8(1 + \lambda^3)\bar{E}_0^2 \sqrt{1 + 2(\lambda - \lambda^{-2})^2}\bar{W}'. \quad (2.33)
\]
For example, when \( \bar{E}_0 = 0.6 \) (and \( J_m = 97.2 \)) we find that the roots to this equation are \( \lambda_{cr} = 0.665 \) in contraction and \( \lambda_{cr} = 1.78 \) in extension, as reported on Figure 2.4(d). Note that the purely elastic case (\( \bar{E}_0 = 0 \)) is consistent with the surface stability criterion of an elastic Gent material (Destrade and Scott, 2004), giving \( \lambda_{cr} = 0.661 \) here. For the neo-Hookean ideal dielectric of Equation (2.3), the equation simplifies to
\[
\lambda^9 + \lambda^6 + 3\lambda^3 - 1 = \lambda^8(1 + \lambda^3)\bar{E}_0^2, \quad (2.34)
\]
Note that Dorfmann and Ogden (2014b,c) studied the surface instability of an ideal neo-Hookean dielectric, but it was charge-controlled instead of voltage-controlled as here.

In the purely elastic case (\( \bar{E}_0 = 0 \)), the equation above recovers the critical stretch ratio of surface instability for a half-space under equi-biaxial strain as \( \lambda_{cr} = 0.666 \), a result which can be traced back to the works of Green and Zerna (1954), Flavin (1963), and Biot (1963a).

Now the two equations (2.32) and (2.33) delineate a region in the \( \bar{E}_0 - \lambda \) landscape of Figure 2.1 where a given plate is going to buckle in contraction.
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when subject to compressive forces and/or wrinkle in extension when subject to a sufficiently large voltage $\tilde{E}_0$. The precise value of the corresponding $\lambda_c$, depends on $H/L$ but the region between the two curves is narrow enough to draw general conclusions.

In Figure 2.5 we plot the loading curves for the same Gent ideal dielectric ($J_m = 97.2$) used to generate the plots of Figure 2.1, together with the curves for the thin-plate and short-wave limits. We see that the snap-through scenario is not going to unfold completely for the $\bar{s} = 0$ curve: as soon as $\tilde{E}_0$ reaches its maximum (A) and the plate starts expanding with voltage remaining fixed at that value, we enter the wrinkling zone between the thin-plate and the short-wave limits. The same is true for the $\bar{s} = 0.8$ pre-stressed plate, as again, the snap-through from C to D hits the wrinkling zone and cannot be completed. Only the $\bar{s} = 1.5$ pre-stressed plate might be able to achieve a snap-through from E to F without wrinkling.

For plates subject to sufficiently large pre-stress ($\bar{s} = 2.5, 4.5$), the snap-through, thin-plate and short-wave instabilities are avoided, and the plates can deform until they fail by electrical breakdown.

In Figure 2.4, we saw that plates buckle in contraction only when $\tilde{E}_0$ is small (e.g., $\tilde{E}_0 = 0.0, 0.2$, see Figures 2.4(a) and (b)), and then in contraction and in extension for sufficiently large voltage (e.g., $\tilde{E}_0 = 0.4, 0.6$, see Figures 2.4(c) and (d)). The critical voltage where the wrinkling in extension is first expressed is determined by

$$\frac{\partial \tilde{E}_0}{\partial \lambda} = 0, \quad \text{where} \quad \tilde{E}_0 = \sqrt{\frac{\lambda^{-2} - \lambda^{-8}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m}},$$

(2.35)

which corresponds to the $\tilde{E}_0 - \lambda$ loading curve with no pre-stress ($\bar{s} = 0$).

In our calculation ($J_m = 97.2$), Equation (2.35) has two real roots $\tilde{E}_0 = 0.69$ and $\tilde{E}_0 = 0.28$. The former corresponds to the voltage at point A (onset of snap-through at the local maximum of the curve) and the latter to the voltage at the local minimum of the curve, see Figure 2.5.

Finally, we inserted experimental results in Figure 2.5 on actual voltage-stretch curves due to Huang et al. (2012b). They do show that plates can be stretched by voltage a little bit further than the onset of snap-through suggests, by going beyond the maximum of the curve, and that pre-stretch allows for further absolute stretch of the plate by voltage, although the relative gain is affected
Figure 2.5: How the snap-through actuation is counter-acted by plate instabilities. Solid lines are the voltage-stretch curves for homogeneous loading at different levels of pre-stress ($\bar{s} = 0, 0.8, ..., 4.5$), when the plate is modelled by the Gent ideal dielectric (here $J_m = 97.2$, (Dorfmann and Ogden, 2014a; Gent, 1996)). Their intersection with the black dashed line shows where a snap-through should start. However, the non-prestressed plate ($\bar{s} = 0$) will meet the buckling zone found between the thin-plate and the short-wave limit curves on its way from A to B. Similarly for the $\bar{s} = 0.8$ pre-stressed plate. Only the $\bar{s} = 1.5$ pre-stressed plate might be able to experience a snap-through transition from E to F. Inset: Experimental voltage (kV)–stretch data digitised from Huang et al. (2012b).

by the pre-stretch, see Huang et al. (2012b) for a more detailed discussion.

2.6 Further results

2.6.1 Plate wrinkling for dielectrics with polarisation saturation

Many dielectrics exhibit the phenomenon of polarisation saturation, in the sense that the electric displacement $D$ increases monotonically with the electric field
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$E$ but with an asymptotic upper bound $D_s$, say (Li et al., 2011b, 2012; Liu et al., 2012). This characteristic can be captured by the following form of energy density,

$$\Omega = \frac{\mu(1 - \beta)}{2} (I_1 - 3) + \frac{\mu\beta}{2} (I_2 - 3) - \frac{D_s^2}{\varepsilon} \ln \left( \cosh \left( \frac{\sqrt{I_5}}{D_s} \right) \right), \tag{2.36}$$

where $D_s > 0$ and $0 \leq \beta \leq 1$ are constants. Then the electric displacement $D$ is related to the electric field $E$ through (see Equation (2.9) and Figure 2.6),

$$D = D_s \tanh(\varepsilon E/D_s). \tag{2.37}$$

The nominal stress required to effect an equi-biaxial stretch with a transverse electrical field is found from Equation (2.6) as

$$s = \mu(1 - \beta)(\lambda - \lambda^{-5}) + \mu\beta(\lambda^3 - \lambda^{-3}) - \lambda D_s E_0 \tanh (\lambda^2 \varepsilon E_0/D_s). \tag{2.38}$$

We introduce the following quantities

$$\bar{s} = s/\mu, \quad \bar{E}_0 = E_0 \sqrt{\varepsilon/\mu}, \quad \bar{D}_s = D_s/\sqrt{\mu \varepsilon}; \tag{2.39}$$

to obtain the non-dimensional version of this equation as

$$\bar{s} = (1 - \beta)(\lambda - \lambda^{-5}) + \beta(\lambda^3 - \lambda^{-3}) - \lambda \bar{D}_s \bar{E}_0 \tanh (\lambda^2 \bar{E}_0/\bar{D}_s). \tag{2.40}$$

For a given level of pre-stress $\bar{s}$, it gives an implicit relationship between the voltage and the stretch, which we solve to plot the $\bar{E}_0 - \lambda$ curves of Figure 2.6. For these plots we took $\beta = 0.2$ and $\bar{D}_s = 4\sqrt{5}$ (corresponding to $k = 1/4$ and $D_s/\sqrt{C_1\varepsilon} = 10$ in the paper by Liu et al. (2012)). We also plot the curve for the onset of snap-through, found by differentiating implicitly Equation (2.40) and taking $d\bar{E}_0/d\lambda = 0$.

We note that, as outlined above, the thin-plate limit corresponds to this equation in the absence of pre-stress, i.e., when $\bar{s} = 0$. Since the energy density (2.36) is of the form $(2.26)_2$, we can make use of the results found using this choice of energy (see appendix, (2.111)), to obtain the short-wave limit as

$$p_3 \left[ \lambda^{-2}(1 - \beta) + \beta \right] (\lambda^3 + 1 + 3\lambda^{-3} - \lambda^{-6}) - (\lambda + \lambda^{-2}) \bar{E}_0 \bar{D}_s \tanh (\lambda^2 \bar{E}_0/\bar{D}_s) = 0, \tag{2.41}$$

where

$$p_3 = \sqrt{\frac{\bar{D}_s}{2\lambda^2 \bar{E}_0}} \sinh \left( \frac{2\lambda^2 \bar{E}_0}{\bar{D}_s} \right). \tag{2.42}$$
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Figure 2.6: A dielectric with polarisation saturation: solid lines are the voltage-stretch curves for homogeneous loading at different levels of pre-stress ($\bar{s} = 0, 0.6, 1.0, 1.5$). The non-prestressed plate ($\bar{s} = 0$) will meet the buckling zone found between the thin-plate and the short-wave limit curves on its way from A to B. The pre-stressed plates may be able to avoid buckling, depending on their thickness. Inset: Polarisation saturation of the electric displacement with the electric field for the corresponding model (Liu et al., 2012).

is the imaginary part of the third eigenvalue of the corresponding Stroh matrix.

We plot the thin-plate limit and short-wave limit curves in Figure 2.6. We can see that when the material is not under mechanical pre-stress ($\bar{s} = 0$), snap-through does not occur, as the wrinkling zone between the thin-plate and short-wave instabilities is reached before it can be completed. However, we also see that the plate may avoid buckling when it is pre-stressed, and potentially achieve snap-through, depending on the thickness of the material.

2.6.2 Correction to the thin-plate buckling equation

Finally, we can exploit the exact dispersion equations to establish approximations to the dispersion equations when the plate is thin. For this exercise, we specialise the analysis to the Mooney-Rivlin ideal dielectric model, with free energy density

$$
\Omega = \frac{\mu (1 - \beta)}{2} (I_1 - 3) + \frac{\mu \beta}{2} (I_2 - 3) - \frac{\varepsilon}{2} I_5,
$$

(2.43)
where $\beta$ ($0 \leq \beta \leq 1$) is a constant. Note that the neo-Hookean ideal dielectric (2.3) corresponds to $\beta = 0$.

In this case, the dispersion equation for anti-symmetric wrinkles (the first to appear) reads

$$\frac{\tanh(\lambda \pi H/L)}{\tanh(\lambda^{-2} \pi H/L)} = \frac{(\lambda^2 \beta + 1 - \beta)(1 + \lambda^6)^2 + \bar{E}_0^2 \lambda^2 (1 - \lambda^6)}{4 \lambda^3 [1 + \beta (\lambda^2 - 1)]}, \quad (2.44)$$

which reduces to Equation (2.29) for $\beta = 0$. At the zero-th order in $H/L$, we have the thin-plate equation (2.31), here:

$$\bar{E}_0^2 = (1 - \beta + \beta \lambda^2)(\lambda^{-2} - \lambda^{-8}). \quad (2.45)$$

This equation has one root (low values of $\bar{E}_0$) or two roots (higher $\bar{E}_0$) for $\lambda$, which we call $\lambda_0$.

The next order in $H/L$ is order two. With some manipulations of Equation (2.44), we find the following correction,

$$\lambda = \lambda_0 - 2 \left[ \frac{\lambda_0^3 (1 - \beta + \beta \lambda_0^2)}{3 \beta \lambda_0^2 + (1 - \beta)(4 - \lambda_0^6)} \right] (\pi H/L)^2. \quad (2.46)$$

This expression is valid for both the smallest root of the thin-plate equation (corresponding to buckling in contraction) and the largest root (wrinkling in extension under a large voltage), when it exists. That latter case is a departure from the purely elastic case ($\lambda_0 \equiv 1$), where there is no loss of stability in extension (Beatty and Pan, 1998). For the neo-Hookean ideal dielectric (2.3), we take $\beta = 0$ and the expression reduces to

$$\lambda = \lambda_0 - 2 \left[ \frac{\lambda_0^3}{4 - \lambda_0^6} \right] (\pi H/L)^2. \quad (2.47)$$

Finally, $\lambda_0 = 1$ in the purely elastic case, and from Equation (2.46) we recover the Euler solution for the buckling of a slender plate under equi-biaxial load: $\lambda = 1 - (2/9)(\pi H/L)^2$, see Biot (1963b), Sawyers (1996), or Beatty and Pan (1998) for the connection with the classical formula of the corresponding critical end thrust (see also Yang et al. (2017b)).

This type of expansion in $H/L$ can be performed for any free energy by using the Stroh matrix, see Shuvalov (2000) for details. It allows us to link our stability analysis to that based on the Hessian criterion. That stability criterion is based on minimising the free energy once it has been expanded in terms of the plate
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Figure 2.7: Critical stretch $\lambda_{cr}$ versus the initial thickness to wavelength ratio $H/L$ for the anti-symmetric instability modes of a neo-Hookean ideal dielectric plate (left: $\bar{E}_0 = 0.6$, right: $\bar{E}_0 = \sqrt{3}/2^{4/3} \approx 0.69$). Black: exact solutions from the analytical dispersion equation (2.29) and Red: Euler buckling approximations from Equation (2.47) (left) and Equation (2.49) (right).

thickness up to the first power (Zurlo et al., 2017). It corresponds to the onset of snap-through $d\bar{E}_0/d\lambda = 0$, and of compression-induced failure (De Tommasi et al., 2011). For instance, take the neo-Hookean ideal dielectric (2.3): in the absence of a pre-stress, the onset of snap-through/Hessian criterion occurs when (Zhao and Suo, 2007)

$$\lambda = 2^{1/3} \approx 1.26, \quad \bar{E}_0 = \frac{\sqrt{3}}{2^{4/3}} \approx 0.69. \quad (2.48)$$

However, Equation (2.47) will not work here because the numerator of the correction would be zero. By carefully re-doing the expansion in this special case, we find the first correction to this criterion in terms of the plate thickness to be of order one for the stretch (and the next term is of order three) and of order two for the voltage (and the next term is of order three). Explicitly,

$$\lambda = 2^{1/3} \pm \frac{2^{1/6}}{3} (\pi H/L) \approx 1.26 \pm 1.18 (H/L),$$

$$\bar{E}_0 = \frac{\sqrt{3}}{2^{4/3}} - \frac{2^{1/3}}{3^{3/2}} (\pi H/L)^2 \approx 0.69 - 2.39 (H/L)^2. \quad (2.49)$$

Figures 2.7 show the dispersion curves obtained by the exact equation (2.29) and the Euler column buckling approximations (2.47) and (2.49) of a neo-Hookean plate. When $\bar{E}_0 < 0.69$ (left figure), Equation (2.46) has two real roots,
and the $\lambda_{cr} - H/L$ curves for the thin plate are approximated quadratically. For the case of the Hessian instability criterion ($\bar{E}_0 = \sqrt{3}/2^{4/3}$), Equation (2.46) has a single real root, and the $\lambda_{cr} - H/L$ curves are approximated linearly for small thicknesses.

The figures confirm the special character of the Hessian criterion to study stability. Prior to reaching the maximum in the voltage-stretch loading curve (for example when $\bar{E}_0 = 0.6$, as in Figure 2.7a), a thin membrane will buckle when subject to a small amount of contraction ($\lambda_{cr}$ slightly smaller than $\lambda_0 = 1.1$ in Figure 2.7a), but can be stretched by a large amount before it wrinkles in extension (the stretch has to reach a value slightly larger than $\lambda_0 = 1.62$). At the Hessian criterion ($\bar{E}_0 = 0.69$), a thin membrane can only be contracted or stretched by a small amount before it buckles or wrinkles ($\lambda_{cr}$ is close to $\lambda_0 = 2^{1/3}$ for both buckling and wrinkling).

2.7 Conclusions

2.7.1 Results of the current chapter

Motivated by experimental evidence, we presented a theoretical investigation on the wrinkling modes of instability of a dielectric plate subject to the combined action of electrical voltage in thickness and in-plane equi-biaxial pre-stresses, based on nonlinear electroelasticity and the associated incremental theory. We derived the dispersion equation and decoupled it into explicit antisymmetric and symmetric modes. We recovered classical elastic results of plate buckling when the voltage is absent. For specific energy functions (Gent and neo-Hookean ideal dielectrics, Mooney-Rivlin dielectric with polarisation saturation), we obtained the thin-plate and short-wave limits analytically. Our numerical calculations show that plates subject to small-to-moderate voltage require contractile loadings to buckle. For larger applied voltage, the plates can buckle/wrinkle in both contraction and extension, with the critical stretch confined between the thin-plate and short-wave limits. In either case, we found that plates bifurcate anti-symmetrically first and that the symmetric mode is never attained. We determined the threshold value of the voltage where the buckling in extension is first encountered. For thin plates, we extended the Hessian criterion (Zhao and Suo, 2007) to account for thickness
2.7.2 Lateral boundary conditions: wrinkling, buckling

So far we have not specified the lateral boundary conditions that can be applied on the end faces of the plate, located at $x_1 = 0, \ell_1$ and $x_3 = 0, \ell_3$, say. Instead we wrote our general solution in terms of the wrinkles’ wavelength $L$, because it can handily be specialised to several scenarios.

For instance we may assume that the lateral dimensions of the plate are large enough to ignore the actual boundary conditions on the end faces, and that the only relevant boundary conditions are those applying on the upper and lower faces. This assumption can be used to model loaded plates with a high number of wrinkles, see experimental examples in Figure 2.2.

We may also assume that the plate has finite lateral dimensions, say it is a square of initial side length $L$.

A given low-to-moderate voltage $\bar{E}_0$ corresponds to a single actuation stretch $\lambda_0$, say. Here the plate buckles in contraction, when $\lambda = \lambda_{cr} \leq \lambda_0$. In practical terms, it means that if we place frictionless walls at $x_1 = 0, \lambda_0 L$, and at $x_3 = 0, \lambda_0 L$, we can push them with equal force against the plate until its side length is $\lambda_{cr} L$, where it buckles. The corresponding wrinkles produce no incremental displacement nor traction in the $x_3$ direction, so that sliding contact is ensured with the walls at $x_3 = 0, \lambda_0 L$. A short analysis of the incremental fields of the solution (not reproduced here) reveals that both the normal displacement $u_1$ and the shear traction $\dot{T}_{12}$ are proportional to $\sin(kx_1)$. It follows that the incremental boundary condition of sliding contact with the walls at $x_1 = 0, \lambda_{cr} L$ is fully satisfied when $k\lambda_{cr} L = n\pi$, where $n$ is an integer. We may then replace the $H/L$ term in the dispersion equations with $n\lambda_{cr}^{-1} H/(2L)$. Hence we used the plate slenderness ratio $H/L$ as a variable to plot Figure 2.8, where we solved the dispersion equations (2.29) for a neo-Hookean ideal dielectric for $n = 1, 2, 3$ in turn. We see that the $n = 1$ anti-symmetric mode is always the first one met, in contraction and in extension, so that the plate buckles in a half-period pattern, see sketch 4(a) by Yang et al. (2017b).

For higher voltages, a second, larger activation stretch occurs, at $\lambda = \hat{\lambda}_0$, say. Under those voltages, the plate can buckle in contraction ($\lambda_{cr} \leq \lambda_0$) as above, and also in extension ($\lambda_{cr} \geq \hat{\lambda}_0$). At $x_1 = 0, \lambda_{cr} L$, we can apply sliding conditions
Dispersion critical curves for an equi-biaxially stretched neo-Hookean ideal dielectric square plate with initial side $L$ and thickness $H$, in terms of its slenderness $H/L$, for two levels of voltage: $E_0 = 0.6$ (left) and $E_0 = \sqrt{3}/2^{4/3} \simeq 0.69$ (right), the value given by the Hessian criterion. The integer $n$ gives the number of half-periods across the plate. We see that the plate can buckle/wrinkle in contraction ($\lambda_{cr} \leq \lambda_0$) and in extension ($\lambda_{cr} \geq \hat{\lambda}_0$). The plate always buckles/wrinkles anti-symmetrically with one half-period only ($n = 1$, thick black curves). The other modes ($n > 1$ and symmetric modes) are never reached.

again, or alternatively, incremental dead-load conditions: $\dot{T}_{11} = \dot{T}_{12} = 0$ (by adjusting the incremental Lagrange multiplier $\dot{p}$). We cannot have $u_1 = 0$ and $u_2 = 0$ simultaneously there, but we can have $u_1 = 0$ and $u_2(0) = u_2(\lambda_{cr}L)$. In that case, the end faces both rise or dip during wrinkling by the same amount, which can be absorbed in the analysis by a superposed translation. Again, we find that the plate wrinkles anti-symmetrically in a half-period pattern.

For the Hessian voltage, we have $\lambda_0 = \hat{\lambda}_0 = 2^{1/3}$ and the plate can buckle in the neighbourhood of $\lambda_{cr}$ both in contraction and in extension with a half-period pattern ($n = 1$). We find from Equation (2.49) that the critical stretch is then approximated as

$$\lambda_{cr} = 2^{1/3} \pm \frac{5}{12} (\pi H/L) \simeq 1.26 \pm 0.466(H/L),$$

$$E_{ocr} = \frac{\sqrt{3}}{24/3} - \frac{1}{27/33/2} (\pi H/L)^2 \simeq 0.69 - 0.377(H/L)^2.$$  \hspace{1cm} (2.50)

2.7.3 Connections with existing results

In the main part of the chapter, we presented theoretical and numerical calculations with respect to equi-biaxially deformed plates, but our strategy is readily extended to general tri-axial deformations, see the Appendix. In what follows, we
specialise the general results to specific deformations and materials to compare and contrast our results with those of others.

Dorfmann and Ogden (2010a) derived the incremental equations of electro-elasticity and used them to obtain the criterion of surface instability for a class of dielectric models which includes the neo-Hookean model (by taking $\alpha = 0$ in their equation (109)). They assumed the dielectric half-space to be immersed in a uniform external electric field and took plane strain conditions. In a later paper (Dorfmann and Ogden, 2014b), they extended the analysis to plates with finite thickness under equi-biaxial stretch, as here, and considered both the cases of plates immersed in a uniform external field and of electrode-covered plates. In those two investigations, the authors took the Lagrangian electric displacement to be a constant, in contrast to our study, where we take the Lagrangian electric field $E_0$ to be constant. It follows that we cannot make a direct connection with their results and only give the counterpart explicit dispersion equations below.

Hence, for a neo-Hookean ideal dielectric plate stretched equi-biaxially, we found the explicit dispersion equations and surface wrinkling bifurcation criterion as Equations (2.29) and (2.34), respectively, which we recall below as

\[
\left[ \frac{\tanh(\pi \lambda H/L)}{\tanh(\pi \lambda^{-1} H/L)} \right]^{\pm1} = \frac{(1 + \lambda^6)^2}{4\lambda^3} + \frac{\lambda^8(1 - \lambda^6)}{4} \bar{E}_0^2, \tag{2.51}
\]

(where the $+1/-1$ exponent corresponds to anti-symmetric/symmetric wrinkles), and

\[
\lambda^6 + \lambda^4 + 3\lambda^2 - 1 = \lambda^8(1 + \lambda^3)\bar{E}_0^2. \tag{2.52}
\]

As shown with the dispersion curves of Figure 2.4, these equations are robust and easy to solve numerically, while the determinantal equation of Dorfmann and Ogden (2014b) is probably ill-conditioned, as it presents instability and a merging of symmetric and anti-symmetric modes.

For a neo-Hookean ideal dielectric plate in plane strain, we take $\lambda_1 = \lambda$, $\lambda_2 = 1/\lambda$, $\lambda_3 = 1$ in the Appendix equation (2.102), to find the dispersion equations as

\[
\left[ \frac{\tanh(\pi \lambda H/L)}{\tanh(\pi \lambda^{-1} H/L)} \right]^{\pm1} = \frac{(1 + \lambda^4)^2}{4\lambda^2} + \frac{\lambda^2(1 - \lambda^4)}{4} \bar{E}_0^2, \tag{2.53}
\]

and the surface instability equation ($H/L \to \infty$) as

\[
\lambda^6 + \lambda^4 + 3\lambda^2 - 1 = \lambda^4(1 + \lambda^2)\bar{E}_0^2. \tag{2.54}
\]
Here the main difference with the equi-biaxial case is that the plane-strain voltage-stretch curve is monotone:

\[ E_0 = \sqrt{1 - \lambda^{-4} - \lambda^{-1}}. \]  

(2.55)

Because this equation corresponds to the thin-plate limit, it follows that there is now only one possibility of buckling, as to a given voltage corresponds a single actuation stretch \( \lambda_0 \). Here the plate can only buckle in contraction, when \( \lambda = \lambda_{cr} \leq \lambda_0 \).

Díaz-Calleja et al. (2017) took the same set-up as Dorfmann and Ogden (2014b) (in equi-biaxial strain), and also took the electric displacement as constant, so that again, we cannot compare our results to theirs directly. They used an extended Mooney-Rivlin model and an Ogden model in turn, and obtained the dispersion equation as a \( 6 \times 6 \) determinant, not decoupled into symmetric and anti-symmetric modes. Their numerical results do not seem to reveal an extensional mode of wrinkling, and seem to present numerical instabilities.

Fu et al. (2018) recently studied the localized necking of ideal neo-Hookean dielectric plates in plane strain. They expanded the bifurcation criterion of Dorfmann and Ogden (2014b) in terms of \( kH \), similar to our analysis in Section 2.6.2. In an effort to model the initiation of necking in tension, they focused on the symmetric mode of wrinkling. We can also provide such an expression by expanding in a Maclaurin series the bifurcation criterion (2.53) with the \(-1\) exponent, but there is little value in pursuing this avenue, simply because, as we saw, the symmetric modes are never arrived at, as they are always preceded by anti-symmetric modes.

Finally, Yang et al. (2017b) looked at the buckling of ideal dielectric plates in plane strain. However, when they specialised their results to the neo-Hookean case, they chose for simplicity to not consider the increments in the electrical fields, only in the mechanical fields. As a result they obtained the following dispersion relation, see their Equations (69)-(70),

\[ \left[ \frac{\tanh(\pi \lambda H/L)}{\tanh(\pi \lambda^{-1} H/L)} \right]^{\pm 1} = \frac{(1 + \lambda^4)^2 + \lambda^4(1 + \lambda^4)E_0^2}{4\lambda^2 + 2\lambda^6 E_0^2}. \]  

(2.56)

The numerical resolution of this equation for the anti-symmetric case shows that it brings little difference compared to the dispersion curve of the exact equation (2.53) for plates subject to small voltages (up to \( E_0 = 0.5 \), say). However,
for larger applied voltage, the difference is quite dramatic, as the equation above under-estimates the critical stretch by at least 15% for thin plates. Also, their equation has roots when $\tilde{E}_0 > 1$, which is not allowed according to (2.55), see also Yang et al. (2017a). The conclusion is that we must keep all increments (mechanical, electrical, and coupled) to solve the boundary value problem.

2.7.4 Limitations

Our study is restricted in several ways and can be improved upon.

For example we did not consider viscosity (Hong, 2011; Park and Nguyen, 2013) in any way, although there is strong evidence to support its role in the problem of electro-mechanical instability.

We also discarded the possibility of the wrinkles wavefront being oriented at a different angle instead of being normal to $x_1$. It could very well be that the first wrinkles to develop are oblique (Carfagna et al., 2017). Further, it could also be the case that wrinkles with a wavefront normal to $x_2$ develop. They might even be compatible with our wrinkles and combine to create a two-dimensional pattern (Yin et al., 2018).

We should also note that our analysis is restricted to the onset of linearised instability and thus we cannot provide any insight into the post-buckling behaviour, which is beyond the scope of this work. However, the wrinkling analysis conducted in this chapter is often a necessary precursor to such an analysis, for example in order to predict creases (Sigaeva et al., 2018).

Finally, the present incremental theory framework does not seem appropriate to initiate necking (Fu et al., 2018) of a thin plate under large voltage. Our analysis predicts that the $n = 2$ symmetric mode in extension (which is the perfect precursor to necking), can never be attained, as the plate has already reached the bifurcation criterion for the $n = 1$ anti-symmetric mode, where the linearised theory breaks down. Clearly a nonlinear treatment is required to capture necking.

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2.A Appendix

Here we give the general expressions for the electro-acoustic moduli and the derivation of the Stroh matrix. The appendix is self-contained and some equations from the main text are repeated.

2.A.1 Electro-acoustic moduli

We use the push-forward versions of the incremental constitutive equations when the free energy is of the form $\Omega = \Omega(F, E_L)$ (Dorfmann and Ogden, 2014c):

$$
\dot{T} = \mathcal{A}_0 L + \Gamma_0 \dot{E}_L + p L - \dot{p} I, \quad \dot{D}_L = -\Gamma_0^T L - K_0 \dot{E}_L,
$$

where $\dot{T}$ is the push-forward version of the incremental mechanical traction, $\dot{E}_L$ and $\dot{D}_L$ are the push-forward versions of the incremental electric field and electric displacement, respectively, $L = \text{grad } u$ is the gradient of the small-amplitude mechanical displacement $u$, $p$ is the Lagrange multiplier due to the incompressibility condition, with $\dot{p}$ its increment, and $\mathcal{A}_0$, $\Gamma_0$ and $K_0$ are fourth-, third- and second-order tensors, respectively, the so-called electro-acoustic moduli.

The electro-acoustic moduli tensors are given in terms of the first and second derivatives of the energy density function $\Omega$ with respect to the deformation gradient $F$ and the Lagrangian form of the electric field $E_L = F^T E$, where $E$ is the electric field. Using the invariants given in the main sections of the chapter,
we obtain the following expressions for the components of the moduli tensors,

\[ A_{ijk} = 4 \{ \Omega_{14} b_{ij} b_{kl} + \Omega_{22} (I_1 b - b^2)_{ij} (I_1 b - b^2)_{kl} \\
+ \Omega_{12} [b_{ij} (I_1 b - b^2)_{ij} + b_{kl} (I_1 b - b^2)_{ij}] - \Omega_{15} (b_{ij} E_k E_l + b_{kl} E_i E_j) \\
- \Omega_{25} [E_i E_j (I_1 b - b^2)_{ij} + E_k E_i (I_1 b - b^2)_{ij}] - \Omega_{26} [(I_1 b - b^2)_{ij} (E_k (b^{-1} E)_l + (b^{-1} E)_l) \\
+(b^{-1} E)_k E_l + (I_1 b - b^2)_{kl} (E_k (b^{-1} E)_l + (b^{-1} E)_l)] + \Omega_{55} E_i E_j E_k E_l \\
+ \Omega_{56} [E_i E_j (E_k (b^{-1} E)_l + (b^{-1} E)_l) E_i E_l + E_k E_l (E_i (b^{-1} E)_l + (b^{-1} E)_l)] \\
+ \Omega_{66} (E_i (b^{-1} E)_l + (b^{-1} E)_l) E_j (E_k (b^{-1} E)_l + (b^{-1} E)_l)] \\
+ 2 \{ \Omega_{14} \delta_{ik} b_{jl} + \Omega_{15} (2 b_{ij} b_{kl} - b_{ij} b_{jl} - b_{ik} b_{jl} + \delta_{ik} (I_1 b - b^2)_{jl}) \\
+ \Omega_{5} (\delta_{ij} E_k E_l + \delta_{ik} E_j E_l + \delta_{il} E_j E_k) \\
+ \Omega_{6} [b_{ik} E_j E_l + b_{il} E_j E_k + b_{ij} E_l E_k + \delta_{ij} (E_i (b^{-1} E)_l + (b^{-1} E)_l)] \\
+ \delta_{il} (E_j (b^{-1} E)_k + (b^{-1} E)_j E_k) + \delta_{jl} (E_i (b^{-1} E)_k + (b^{-1} E)_l)] \}, \]

(2.58)

\[ \Gamma_{ijk} = 4 \{ b_{ij} [\Omega_{14} (b E)_k + \Omega_{15} E_k + \Omega_{16} (b^{-1} E)_k] \\
+ (I_1 b - b^2)_{ij} [\Omega_{24} (b E)_k + \Omega_{25} E_k + \Omega_{26} (b^{-1} E)_k] \\
- E_i E_j [\Omega_{45} (b E)_k + \Omega_{55} E_k + \Omega_{56} (b^{-1} E)_k] \\
- (E_i (b^{-1} E)_j + (b^{-1} E)_j) [\Omega_{46} (b E)_k + \Omega_{56} E_k + \Omega_{66} (b^{-1} E)_k] \} \\
- 2 [\Omega_{5} (\delta_{ik} E_j + \delta_{ik} E_j) + \Omega_{6} (\delta_{ik} (b^{-1} E)_j + \delta_{ik} (b^{-1} E)_j + b_{ik} E_i)] \], \]

(2.59)

\[ K_{ij} = 4 \{ \Omega_{45} (b E)_i (b E)_j + \Omega_{55} E_i E_j + \Omega_{66} (b^{-1} E)_i (b^{-1} E)_j \\
+ \Omega_{45} [(b E)_i E_j + E_i (b E)_j] + \Omega_{46} [(b E)_i (b^{-1} E)_j + (b^{-1} E)_j] \} \\
+ \Omega_{56} [(E_i (b^{-1} E)_j + (b^{-1} E)_j E_j)] + 2 (\Omega_{4} b_{ij} + \Omega_{5} \delta_{ij} + \Omega_{6} b_{ij}^{-1}) \}, \]

(2.60)

where \( b = FF^T \) is the left Cauchy-Green deformation tensor, \( \Omega_{ij} = \partial^2 \Omega / \partial I_i \partial I_j \) and \( \Omega_i = \partial \Omega / \partial I_i \), for \( i, j = 1, 2, 4, 5, 6 \).

These expressions are derived from the derivatives of the invariants with respect to the deformation gradient \( F \) and to the Lagrangian electric field \( E_L \). The non-zero first derivatives are as follows,
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\[
\frac{\partial I_1}{\partial F_{ia}} = 2F_{ia}, \quad \frac{\partial I_2}{\partial F_{ia}} = 2(I_1 F_{ia} - c_{a\gamma} F_{i\gamma}),
\]

\[
\frac{\partial I_5}{\partial F_{ia}} = -2c_{a\gamma}^{-1} E_{L\gamma} F_{\delta i}^{-1} E_{L\delta}, \quad \frac{\partial I_6}{\partial F_{ia}} = -2 \left(c_{a\gamma}^{-1} E_{L\gamma} F_{\delta i}^{-1} E_{L\delta} + c_{a\gamma}^{-1} E_{L\gamma} c_{\delta p}^{-1} F_{p\delta}^{-1} E_{L\delta}\right),
\]

\[
\frac{\partial I_4}{\partial E_{La}} = 2E_{La}, \quad \frac{\partial I_5}{\partial E_{La}} = 2c_{a\gamma}^{-1} E_{L\gamma}, \quad \frac{\partial I_6}{\partial E_{La}} = 2c_{a\gamma}^{-2} E_{L\gamma}, \tag{2.61}
\]

the non-zero second derivatives with respect to \(F\) are

\[
\frac{\partial^2 I_1}{\partial F_{ia} \partial F_{k\beta}} = 2\delta_{ik}\delta_{a\beta}, \quad \frac{\partial^2 I_2}{\partial F_{ia} \partial F_{k\beta}} = 2 \left(2F_{ia} F_{k\beta} - F_{i\beta} F_{ka} + \delta_{ik}(I_1 \delta_{a\beta} - c_{a\beta}) - b_{ik}\delta_{a\beta}\right),
\]

\[
\frac{\partial^2 I_5}{\partial F_{ia} \partial F_{k\beta}} = 2E_{L\gamma} E_{L\delta} \left(c_{a\beta}^{-1} F_{\gamma k}^{-1} F_{\delta i}^{-1} + c_{\beta\gamma}^{-1} F_{\alpha k}^{-1} F_{\delta i}^{-1} + c_{a\gamma}^{-1} F_{\delta k}^{-1} F_{\delta i}^{-1}\right),
\]

\[
\frac{\partial^2 I_6}{\partial F_{ia} \partial F_{k\beta}} = 2 \left[c_{a\beta}^{-2} F_{\gamma k}^{-1} F_{\delta i}^{-1} + c_{\beta\gamma}^{-2} F_{\alpha k}^{-1} F_{\delta i}^{-1} + c_{a\gamma}^{-2} F_{\delta k}^{-1} F_{\delta i}^{-1} + c_{a\beta}^{-1} \left(c_{\alpha q}^{-1} F_{qk}^{-1} F_{\delta i}^{-1} - c_{\delta q}^{-1} F_{q k}^{-1} F_{\delta i}^{-1} + c_{\delta q}^{-1} F_{q k}^{-1} F_{\delta i}^{-1}\right)\right] E_{L\gamma} E_{L\delta}, \tag{2.62}
\]

the non-zero second derivatives with respect to \(E_L\) are

\[
\frac{\partial^2 I_4}{\partial E_{La} \partial E_{L\beta}} = 2\delta_{a\beta}, \quad \frac{\partial^2 I_5}{\partial E_{La} \partial E_{L\beta}} = 2c_{a\beta}^{-1}, \quad \frac{\partial^2 I_6}{\partial E_{La} \partial E_{L\beta}} = 2c_{a\beta}^{-2}, \tag{2.63}
\]

and the mixed second derivatives are

\[
\frac{\partial^2 I_5}{\partial F_{ia} \partial E_{L\beta}} = -2 \left(c_{a\beta}^{-1} F_{\gamma i}^{-1} + c_{\alpha\gamma}^{-1} F_{\beta i}^{-1}\right) E_{L\gamma},
\]

\[
\frac{\partial^2 I_6}{\partial F_{ia} \partial E_{L\beta}} = -2 \left[c_{a\gamma}^{-2} F_{\beta i}^{-1} + c_{a\beta}^{-2} F_{\gamma i}^{-1} + F_{p\delta}^{-1} \left(c_{a\gamma}^{-1} c_{\beta p}^{-1} + c_{a\beta}^{-1} c_{\gamma p}^{-1}\right)\right] E_{L\gamma}. \tag{2.64}
\]

Note that some of these derivatives were first derived by Rudykh et al. (2014).

2.A.2 Two-dimensional wrinkles in transverse electric field

We look for two-dimensional solutions to the incremental equations so that \(u = u(x_1, x_2)\) only. Then \(\dot{p}, \dot{D}_L\) and \(\dot{E}_L\) are also functions of \(x_1, x_2\) only.
Although we do not show it here, we find that this leads to $u_3 = 0$, $\dot{D}_{L3} = 0$ and $\dot{E}_{L3} = 0$ for our problem of principal wrinkles in a transverse electrical field. Because $\text{curl} \, \dot{E}_L = 0$, we can introduce the electric potential $\varphi$ and write that

$$\dot{E}_{L1} = -\varphi, \quad \dot{E}_{L2} = -\varphi,.$$  \hfill (2.65)

The push-forward versions of the incremental constitutive equations then have the following non-zero entries,

$$\dot{T}_{11} = (A_{0111} + p)u_{1,1} + A_{0122}u_{2,2} - \Gamma_{0112}\varphi, - \dot{p},$$

$$\dot{T}_{12} = (A_{0121} + p)u_{1,2} + A_{0122}u_{2,1} - \Gamma_{0112}\varphi, 1,$$

$$\dot{T}_{21} = (A_{0121} + p)u_{2,1} + A_{0122}u_{1,2} - \Gamma_{0112}\varphi, 1,$$

$$\dot{T}_{22} = (A_{0222} + p)u_{2,2} + A_{0122}u_{1,1} - \Gamma_{0222}\varphi, 2 - \dot{p},$$ \hfill (2.66)

and

$$\dot{D}_{L1} = -\Gamma_{0211}(u_{1,2} + u_{2,1}) + K_{011}\varphi, 1,$$

$$\dot{D}_{L2} = -\Gamma_{0112}u_{1,1} - \Gamma_{0222}u_{2,2} + K_{022}\varphi, 2,$$ \hfill (2.67)

because all other components of the electro-elastic moduli are zero. Here $A_{0ijk}, \Gamma_{0ijk}$ and $K_{0ij}$ are given by Equations (2.58), (2.59) and (2.60) respectively.

The equilibrium equations in the incremental case, $\text{div} \, \dot{T} = 0$ and $\text{div} \, \dot{D}_L = 0$, are then as follows,

$$\dot{T}_{11,1} + \dot{T}_{21,2} = 0, \quad \dot{T}_{12,1} + \dot{T}_{22,2} = 0, \quad \dot{D}_{L1,1} + \dot{D}_{L2,2} = 0,$$  \hfill (2.68)

which together with the incompressibility condition,

$$\text{div} \, \mathbf{u} = u_{1,1} + u_{2,2} = 0, \hfill (2.69)$$

fully describe the incremental motion.

### 2.A.3 Stroh formulation

We look for solutions that are harmonic in the $x_1$-direction, i.e., solutions of the form

$$\{u_1, u_2, \dot{D}_{L2}, \dot{T}_{21}, \dot{T}_{22}, \varphi\} = \Re\{[k^{-1}U_1, k^{-1}U_2, i\Delta, i\Sigma_{21}, i\Sigma_{22}, k^{-1}\Phi] e^{j\kappa x_1}\}. \hfill (2.70)$$
where \( U_1, U_2, \Delta, \Sigma_{21}, \Sigma_{22} \) and \( \Phi \) are functions of \( kx_2 \) only, and \( k = 2\pi/\mathcal{L} \) is the wavenumber. We can then rewrite the full problem in Stroh form, i.e., as

\[
\eta' = iN\eta, \tag{2.71}
\]

where

\[
\eta = \begin{bmatrix} U_1 & U_2 & \Delta & \Sigma_{21} & \Sigma_{22} & \Phi \end{bmatrix}^T = \begin{bmatrix} U & S \end{bmatrix}^T, \tag{2.72}
\]
is the Stroh vector, the prime denotes differentiation with respect to \( kx_2 \), and \( N \) is the Stroh matrix, which can be partitioned as

\[
N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^\dagger \end{bmatrix}, \tag{2.73}
\]

where \( \dagger \) denotes the Hermitian operator. We derived the Stroh matrix \( N \) as follows.

First, substituting \( u_1 \) and \( u_2 \) into the incompressibility condition, (2.69), gives

\[
U_2' = -iU_1, \tag{2.74}
\]
the second line of the Stroh equation. We then substitute the expression for \( \dot{D}_{I2} \) into equation (2.67) and using (2.74) we get the following expression for \( \Phi' \),

\[
\Phi' = i \left[ \frac{\Gamma_{0112} - \Gamma_{0222}}{K_{022}} U_1 + \frac{1}{K_{022}} \Delta \right], \tag{2.75}
\]
i.e., the last line of the Stroh equation. Similarly, we can get an expression for \( U_1' \) by using \( \dot{T}_{21} \) in Equation (2.66), so that

\[
U_1' = i \left[ \frac{- \left( A_{01221} + p \right)}{A_{02121}} U_2 + \frac{1}{A_{02121}} \Sigma_{21} + \frac{\Gamma_{0211}}{A_{02121}} \Phi \right], \tag{2.76}
\]
which is the first line of the Stroh equation.

In order to get the remaining three equations, we must use the equilibrium equations (2.68). We first find an expression for \( \dot{p} \) by rearranging the expression (2.66) for \( \dot{T}_{22} \). We then substitute this into (2.66) and use (2.75) and (2.68) to find the fourth line of the Stroh equation as follows,

\[
\Sigma_{21}' = -i \left\{ A_{01111} + A_{02222} - 2A_{01122} + 2p - \frac{(\Gamma_{0112} - \Gamma_{0222})^2}{K_{022}} \right\} U_1 \tag{2.77}
\]

\[
+ \Sigma_{22} - \frac{(\Gamma_{0112} - \Gamma_{0222})}{K_{022}} \Delta \right\}. \]
Similarly, we use \((2.66)_2\), \((2.76)\) and \((2.68)_2\) to find the fifth Stroh equation as
\[
\Sigma'_{22} = i \left\{ \left[ \frac{(A_{01221} + p)^2}{A_{02121}} - A_{01212} \right] U_2 - \frac{(A_{01221} + p)}{A_{02121}} \Sigma_{21} \right. \\
\left. - \Gamma_{0211} \left( \frac{A_{01221} + p}{A_{02121}} - 1 \right) \Phi \right\}. 
\] (2.78)

Finally, to get an equation for \(\Delta'\), we use \((2.67)_1\), \((2.76)\) and \((2.68)_3\) so that,
\[
\Delta' = i \left\{ \Gamma_{0211} \left[ 1 - \frac{(A_{01221} + p)}{A_{02121}} \right] U_2 + \frac{\Gamma_{0211}}{A_{02121}} \Sigma_{21} + \left( \frac{(\Gamma_{0211})^2}{A_{02121}} - K_{011} \right) \Phi \right\}. 
\] (2.79)

We can then write these six equations in the Stroh matrix form. Adopting the following shorthand notation,
\[
a = A_{01212}, \quad c = A_{02121}, \quad 2b = A_{01111} + A_{02222} - 2A_{01122} - 2A_{01221}, \\
d = \Gamma_{0211}, \quad e = \Gamma_{0222} - \Gamma_{0112}, \quad f = K_{011}, \quad g = K_{022}, 
\] (2.80)
we find the partitions of the Stroh matrix, \(N_1\), \(N_2\) and \(N_3\), as follows,
\[
N_1 = \begin{bmatrix} 0 & -1 + \tau_{22}/c & 0 \\ -1 & 0 & 0 \\ 0 & d\tau_{22}/c & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1/c & 0 & d/c \\ 0 & 0 & 0 \\ d/c & 0 & d^2/c - f \end{bmatrix}, \\
N_3 = \begin{bmatrix} -2(b + c - \tau_{22}) + e^2/g & 0 & -e/g \\ 0 & -a + (c - \tau_{22})^2/c & 0 \\ -e/g & 0 & 1/g \end{bmatrix}, 
\] (2.81)
where we have also made use of the connection \(A_{01221} + p = A_{02121} - \tau_{22}\) (see Chadwick (1997) or Shams et al. (2011)).

In particular, in this chapter, we solve a problem where there is no electric field external to the plate, and so \(\tau_{22} = 0\) in the expressions above.
In general, the expressions (2.80) read as follows,

\[ a = 2\left[\lambda_i^2 \Omega_1 + \lambda_i^2 \lambda_3^2 \Omega_2 + \lambda_i^2 \lambda_3^2 E_0^2 \left(\Omega_5 + (\lambda_i^{-2} + 2\lambda_i^2 \lambda_3^2)\Omega_6\right)\right], \]

\[ 2b = 4 \left\{ \left[\lambda_i^2 - \lambda_1^{-2} \lambda_3^2\right] \left(\Omega_{11} + 2\lambda_i^2 \lambda_3^2 \Omega_{12} + \lambda_i^2 \lambda_3^2 \Omega_{22}\right) + 2\lambda_i^2 \lambda_3^2 E_0^2 \left(\Omega_{15} + 2\lambda_i^2 \lambda_3^2 \Omega_{16} + \lambda_i^2 \lambda_3^2 \Omega_{25} + 2\lambda_i^2 \lambda_3^2 \Omega_{26}\right) \right\} + 2 \left\{ \left(\lambda_i^2 + \lambda_1^{-2} \lambda_3^2\right) \left(\Omega_1 + \lambda_3^2 \Omega_2\right) + \lambda_i^2 \lambda_3^2 E_0^2 \left[\Omega_5 + 2(3\lambda_i^2 \lambda_3^2 - \lambda_1^{-2})\Omega_6\right] \right\}, \]

\[ c = 2 \left[\lambda_i^{-2} \lambda_3^{-2} \Omega_1 + \lambda_i^{-2} \Omega_2 + \lambda_i^2 E_0^2 \Omega_6\right], \]

\[ d = -2\lambda_1 \lambda_3 \left[\Omega_5 + (\lambda_1^{-2} + \lambda_i^2 \lambda_3^2)\Omega_6\right] E_0, \]

\[ e = 4\lambda_1 \lambda_3 \left[\left(\lambda_i^{-2} \lambda_3^{-2} - \lambda_1^{-2} \lambda_3^{-2}\right)\Omega_{14} + \Omega_{15} + \lambda_i^2 \lambda_3^2 \Omega_{16} + \lambda_i^{-2} \Omega_{24} + \lambda_i^2 \lambda_3^2 \Omega_{25} + \lambda_i^2 \lambda_3^2 \Omega_{26}\right] -\lambda_i^2 \lambda_3^2 E_0^2 \left[\lambda_i^{-2} \lambda_3^{-2} \Omega_{45} + 2\Omega_{46} + \Omega_{55} + 3\lambda_i^2 \lambda_3^2 \Omega_{56} + 2\lambda_i^4 \lambda_3^4 \Omega_{66}\right] - \left(\Omega_5 + 2\lambda_i^2 \lambda_3^2 \Omega_6\right) E_0, \]

\[ f = 2(\lambda_i^2 \Omega_4 + \Omega_5 + \lambda_i^{-2} \Omega_6), \]

\[ g = 4 \left[\lambda_i^4 \lambda_3^{-4} \Omega_{44} + 2\lambda_i^{-2} \lambda_3^{-2} \Omega_{45} + 2\Omega_{46} + \Omega_{55} + 2\lambda_i^2 \lambda_3^2 \Omega_{56} + \lambda_i^4 \lambda_3^4 \Omega_{66}\right] \lambda_i^2 \lambda_3^2 E_0^2 + 2(\lambda_i^{-2} \lambda_3^{-2} \Omega_4 + \Omega_5 + \lambda_i^2 \lambda_3^2 \Omega_6). \]  
(2.82)

We can non-dimensionalise both the Stroh constants and entries of \(\eta\) by introducing the following dimensionless moduli,

\[ \bar{a} = a/\mu, \quad \bar{b} = b/\mu, \quad \bar{c} = c/\mu, \quad \bar{\tau}_{22} = \tau_{22}/\mu, \]

\[ \bar{d} = d/\sqrt{\mu\varepsilon}, \quad \bar{e} = e/\sqrt{\mu\varepsilon}, \quad \bar{f} = f/\varepsilon, \quad \bar{g} = g/\varepsilon, \]  
(2.83)

and dimensionless fields,

\[ \bar{U}_i = U_i, \quad \bar{\Sigma}_{2i} = \Sigma_{2i}/\mu, \quad \bar{\Delta} = \Delta/\sqrt{\mu\varepsilon}, \quad \bar{\Phi} = \Phi/\sqrt{\varepsilon/\mu}, \]  
(2.84)

for \(i = 1, 2\), where \(\bar{X}\) denotes a dimensionless measure of \(X\), and \(\mu\) and \(\varepsilon\) are the initial shear modulus and initial permittivity of the dielectric material,

\[ \mu = 2(\Omega_1 + \Omega_2)|_{I_1=I_2=3,I_4=I_5=I_6=0}; \quad \varepsilon = -2(\Omega_4 + \Omega_5 + \Omega_6)|_{I_1=I_2=3,I_4=I_5=I_6=0}. \]  
(2.85)

The finite fields can also be non-dimensionalised by introducing

\[ \bar{E}_0 = E_0/\sqrt{\varepsilon/\mu}, \quad \bar{D} = D/\sqrt{\mu\varepsilon}, \quad \bar{I}_a = (\varepsilon/\mu)I_a \quad (\alpha = 4, 5, 6). \]  
(2.86)

Once \(a\) is replaced by \(\mu\bar{a}\), \(b\) by \(\mu\bar{b}\), etc., the equations of equilibrium can be re-written in their non-dimensional form \(\bar{\eta}' = i\bar{N}\bar{\eta}'\). For the rest of the appendix, the overline notation is understood everywhere, and all quantities are non-dimensional.
2.A.4 Method of resolution for plates

Since \( N \) has constant entries, we look for solutions to (2.71) of the form,

\[
\eta(kx_2) = \eta^0 e^{iqkx_2},
\]

which results in an eigen-problem for the eigenvalues \( q \) and eigenvectors \( \eta^0 \) of the matrix \( N \),

\[
(N - qI) \eta^0 = 0. \tag{2.88}
\]

The characteristic equation associated with this eigen-problem is

\[
 cgq^6 + [2bg + cf - (d - e)^2]q^4 + [2bf + ag + 2d(d - e)]q^2 + af - d^2 = 0. \tag{2.89}
\]

This equation is bi-cubic in \( q \) and does not depend on the Cauchy stress \( \tau_{22} \) for any choice of energy density function.

After calculating the eigenvalues \( q_j \) and eigenvectors \( \eta^{(j)} \), \( j = 1,2,\ldots,6 \), for the Stroh matrix \( N \), we can construct the solution to (2.71) for a plate of electroelastic material,

\[
\eta(kx_2) = \begin{bmatrix} U(kx_2) \\ S(kx_2) \end{bmatrix} = \sum_{j=1}^{6} c_j \eta^{(j)} e^{jqkx_2}, \tag{2.90}
\]

where \( c_j \) for \( j = 1,2,\ldots,6 \) are arbitrary constants to be determined from the boundary conditions. The eigenvalues come in conjugate pairs because the bicubic has real coefficients. We specialise the analysis to free energies for which the \( q_j \) are pure imaginary, and so we write them as \( q_j = ip_j \) and \( q_{j+3} = -ip_j \) for \( j = 1,2,3 \), where \( p_1, p_2, p_3 \) are real. Then the eigenvectors are also conjugate pairs, \( \eta^{(j)} = \eta^{(j+3)} \) for \( j = 1,2,3 \).

The incremental equations must be solved subject to the boundary conditions of no incremental mechanical tractions and no incremental electric field on the faces of the plate, i.e., \( S(kh/2) = S(\text{kh}/2) = 0 \). Using this boundary condition and (2.90), we can write the following matrix equation,

\[
\begin{bmatrix} S(kh/2) \\ S(\text{kh}/2) \end{bmatrix} = \begin{bmatrix} F_1E_1^- & F_2E_2^- & F_3E_3^- & F_4E_4^+ & F_5E_5^+ & F_6E_6^+ \\ G_1E_1^- & G_2E_2^- & G_3E_3^- & G_4E_4^+ & G_5E_5^+ & G_6E_6^+ \\ H_1E_1^- & H_2E_2^- & H_3E_3^- & H_4E_4^+ & H_5E_5^+ & H_6E_6^+ \\ F_1E_1^+ & F_2E_2^+ & F_3E_3^+ & F_4E_4^- & F_5E_5^- & F_6E_6^- \\ G_1E_1^+ & G_2E_2^+ & G_3E_3^+ & G_4E_4^- & G_5E_5^- & G_6E_6^- \\ H_1E_1^+ & H_2E_2^+ & H_3E_3^+ & H_4E_4^- & H_5E_5^- & H_6E_6^- \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = 0, \tag{2.91}
\]
where $F_j = \eta_4^{(j)}$, is the fourth component, $G_j = \eta_5^{(j)}$ is the fifth component, and $H_j = \eta_6^{(j)}$ is the sixth component of the eigenvector $\eta^{(j)}$, and $E_j^\pm = e^{\pm p_j kh/2}$, for $j = 1, 2, \ldots, 6$. We also note that since $q_j + 3 = -q_j$, we have $E_j^\pm = E_j^\mp$, for $j = 1, 2, 3$.

For our choices of free energy densities, we find that $F_{j+3} = F_j$, $G_{j+3} = -G_j$, $H_{j+3} = H_j$, for $j = 1, 2, 3$. Then some simple linear manipulations (Nayfeh, 1995) of the matrix result in two $3 \times 3$ blocks, the antisymmetric and symmetric modes, and its determinant factorises as follows,

$$\begin{vmatrix} F_1 C_1 & F_2 C_2 & F_3 C_3 \\ G_1 S_1 & G_2 S_2 & G_3 S_3 \\ H_1 C_1 & H_2 C_2 & H_3 C_3 \end{vmatrix} \times \begin{vmatrix} F_1 S_1 & F_2 S_2 & F_3 S_3 \\ G_1 C_1 & G_2 C_2 & G_3 C_3 \\ H_1 S_1 & H_2 S_2 & H_3 S_3 \end{vmatrix} = 0,$$  \hspace{1cm} \text{(2.92)}

where $C_j = \cosh(p_j kh/2)$ and $S_j = \sinh(p_j kh/2)$. Here the antisymmetric mode is described by the determinant on the left, and the symmetric mode by the one on the right. We then get the following expressions for the dispersion equations in general,

$$G_1 (F_3 H_2 - F_2 H_3) \tanh(p_1 kh/2) + G_2 (F_1 H_3 - F_3 H_1) \tanh(p_2 kh/2) + G_3 (F_2 H_1 - F_1 H_2) \tanh(p_3 kh/2) = 0,$$  \hspace{1cm} \text{(2.93)}

for the antisymmetric mode and,

$$G_1 (F_3 H_2 - F_2 H_3) \coth(p_1 kh/2) + G_2 (F_1 H_3 - F_3 H_1) \coth(p_2 kh/2) + G_3 (F_2 H_1 - F_1 H_2) \coth(p_3 kh/2) = 0,$$  \hspace{1cm} \text{(2.94)}

for the symmetric mode. The quantity $kh$ can be expressed as $kh = 2\pi \lambda_1^{-1} \lambda_3^{-1} H/L$, where $H$ is the initial thickness of the plate and $L$ is the wavelength of the wrinkles, two quantities that are easy to measure experimentally.

2.A.5 Examples

Of course, solving the bicubic (2.89) is quite cumbersome in the general case, but for some special forms of the free energy density, it simplifies quite a lot. Hence, for generalised neo-Hookean ideal dielectrics, which are such that

$$\Omega = W(I_1) - \frac{\varepsilon}{2} I_5,$$  \hspace{1cm} \text{(2.95)}

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where \( W \) is an arbitrary function of \( I_1 \) only, we find that it factorises as

\[
(q^2 + 1) \left\{ q^4 + \left[ 1 + \lambda_1^4 \lambda_3^2 + 2(\lambda_1^3 \lambda_3 - \lambda_1^{-1} \lambda_3^{-1})^2 \frac{W''}{W'} \right] \right\} q^2 + \lambda_1^4 \lambda_3^2 = 0. \tag{2.96}
\]

Here we call \( q_1 \) and \( q_2 \) the two roots of the factorised biquadratic with positive imaginary part.

Now we define all six eigenvalues and the three real numbers \( p_j \) by

\[
q_1 = -q_4 = ip_1, \quad q_2 = -q_5 = ip_2, \quad q_3 = -q_6 = i \quad (p_3 = 1). \tag{2.97}
\]

The real quantities \( p_1, p_2 \) are such that

\[
p_1^2 p_2^2 = \lambda_1^4 \lambda_3^2, \quad p_1^2 + p_2^2 = 1 + \lambda_1^4 \lambda_3^2 + 2(\lambda_1^3 \lambda_3 - \lambda_1^{-1} \lambda_3^{-1})^2 \frac{W''}{W'} \tag{2.98}
\]

Solving for real positive \( p_1, p_2 \) gives

\[
p_{1,2} = \frac{\lambda_1^2 \lambda_3 + 1}{2} \sqrt{1 + 2(\lambda_1 - \lambda_1^{-1} \lambda_3^{-1})^2 \frac{W''}{W'}} \pm \frac{\lambda_1^2 \lambda_3 - 1}{2} \sqrt{1 + 2(\lambda_1 + \lambda_1^{-1} \lambda_3^{-1})^2 \frac{W''}{W'}} \tag{2.99}
\]

The six eigenvectors are

\[
\eta^{(1)} = \begin{bmatrix}
-ip_1 \\
1 \\
-ip_1 \lambda_1 \lambda_3 E_0 \\
2(1 + p_1^2) \frac{W''}{\lambda_1^2 \lambda_3^2} + \lambda_1^2 \lambda_3^2 E_0^2 \\
2i (\lambda_1 \lambda_3^2 + p_1^2) W' \\
-\lambda_1 \lambda_3 E_0
\end{bmatrix}, \quad \eta^{(2)} = \begin{bmatrix}
-ip_2 \\
1 \\
-ip_2 \lambda_1 \lambda_3 E_0 \\
2(1 + p_2^2) \frac{W''}{\lambda_1^2 \lambda_3^2} + \lambda_1^2 \lambda_3^2 E_0^2 \\
2i (\lambda_1^4 \lambda_3^2 + p_2^2) W' \\
-\lambda_1 \lambda_3 E_0
\end{bmatrix}, \quad \eta^{(3)} = \begin{bmatrix}
0 \\
0 \\
i \lambda_1 \lambda_3 E_0 \\
i \lambda_1 \lambda_3 E_0 \\
-1
\end{bmatrix}, \quad \eta^{(4)} = \overline{\eta^{(1)}}, \quad \eta^{(5)} = \overline{\eta^{(2)}}, \quad \eta^{(6)} = \overline{\eta^{(3)}}. \tag{2.100}
\]
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From these expressions we deduce the dispersion equation for anti-symmetric buckling (2.93) in the form

\[ 2W' \left[ p_1(1 + p_2)^2 \tanh \left( \pi p_1 \lambda_1^{-1} \lambda_3^{-1} H/L \right) - p_2(1 + p_1)^2 \tanh \left( \pi p_2 \lambda_1^{-1} \lambda_3^{-1} H/L \right) \right] = (p_2 - p_1)^2 \lambda_1^4 \lambda_3^3 E_0^2 \tanh \left( \pi \lambda_1^{-1} \lambda_3^{-1} H/L \right). \]  

(2.101)

Here we can take \( H/L \to 0 \) and \( H/L \to \infty \) to establish explicit expressions for the thin-plate and the short-wave limits, as in the main text. The equation is valid for any plate made of a generalised neo-Hookean ideal dielectric (2.95), subject to a bi-axial pre-stretch \( \lambda_1, \lambda_3 \).

For example, for a neo-Hookean ideal dielectric, we have \( W' = 1/2, W'' = 0 \), \( p_1 = 1, p_2 = \lambda_1^2 \lambda_3 \), and the dispersion equations simplify to

\[ \left[ \frac{\tanh(\pi \lambda_1 H/L)}{\tanh(\pi \lambda_1^{-1} \lambda_3^{-1} H/L)} \right]^{\pm 1} = \frac{(1 + \lambda_1^4 \lambda_3^2)^2}{4 \lambda_1^2 \lambda_3} + \frac{(1 - \lambda_1^4 \lambda_3^2)}{4} \lambda_1^2 \lambda_3^3 E_0^2. \]  

(2.102)

In the short wavelength limit \( (H/L \to \infty) \), we find that it can be rearranged as

\[ (\lambda_1^2 \lambda_3)^3 + (\lambda_1^2 \lambda_3)^2 + 3 (\lambda_1^2 \lambda_3) - 1 = (1 + \lambda_1^2 \lambda_3) \lambda_1 \lambda_3^3 E_0^2; \]  

(2.103)

and we check that the left hand-side is the cubic established by Flavin (1963) for the surface stability of a purely elastic half-space.

Another example of a free energy density for which it is possible to make good progress is defined by the following class,

\[ \Omega = \frac{\mu(1 - \beta)}{2} (I_1 - 3) + \frac{\mu \beta}{2} (I_2 - 3) - F(I_5), \]

(2.104)

where \( F \) is an arbitrary function of \( I_5 \) only and \( 0 \leq \beta \leq 1 \). Then we find that the characteristic equation (2.89) factorises fully, as

\[ (q^2 + 1)(q^2 + \lambda_1^4 \lambda_3^2) \left[ (2\lambda_1^2 \lambda_3^2 E_0^2 F'' + F')q^2 + F' \right] = 0. \]  

(2.105)

The six eigenvalues can again be written in terms of three real numbers \( p_j \),

\[ q_1 = -q_4 = ip_1, \quad q_2 = -q_5 = ip_2, \quad q_3 = -q_6 = ip_3, \]

(2.106)

where

\[ p_1 = 1, \quad p_2 = \lambda_1^2 \lambda_3, \quad p_3 = \sqrt{\frac{F'}{2\lambda_1^2 \lambda_3^2 E_0^2 F'' + F'}}. \]  

(2.107)
We find that the corresponding eigenvectors are

\[
\eta^{(1)} = \begin{bmatrix}
i \\ -1 \\ 2i\lambda_1\lambda_3 F'^2 E_0 \\ -2\lambda_1^2\lambda_3^2 F'^2 E_0^2 - 2\lambda_1^{-2}\kappa \\ -i\lambda_3^2(\lambda_1^2 + \lambda_1^{-2}\lambda_3^{-2})\kappa \\ \lambda_1\lambda_3 E_0 \\
\end{bmatrix},
\eta^{(2)} = \begin{bmatrix}
i\lambda_1 \\ -\lambda_1^{-1}\lambda_3^{-1} \\ 2i\lambda_1^2\lambda_3 F'^2 E_0 \\ -2\lambda_1\lambda_3 F'^2 E_0^2 - \lambda_1^{-1}\lambda_3(\lambda_1^2 + \lambda_1^{-2}\lambda_3^{-2})\kappa \\ -2i\lambda_1^{-1}\kappa \\ E_0 \\
\end{bmatrix},
\eta^{(3)} = \begin{bmatrix}
0 \\ 0 \\ -2iF' \\ -2p_3\lambda_1\lambda_3 F'^2 E_0 \\ -2i\lambda_1\lambda_3 F'^2 E_0 \\ p_3 \\
\end{bmatrix},
\eta^{(4)} = \eta^{(1)},
\eta^{(5)} = \eta^{(2)},
\eta^{(6)} = \eta^{(3)}.
\]

(2.108)

where

\[
\kappa = \lambda_3^{-2} + \beta(1 - \lambda_3^{-2}).
\]

(2.109)

We then find that the antisymmetric mode for the thin-plate limit \((H/L \to 0)\), \(a - c = 0\), in this case reads

\[
2\lambda_1^2 F'^2 E_0^2 + \kappa(\lambda_1^2 - \lambda_1^{-2}\lambda_3^{-2}) = 0,
\]

(2.110)

and that the short-wave limit, \(H/L \to \infty\), is as follows,

\[
p_3\kappa(\lambda_1^2 + \lambda_1\lambda_3^{-1} + 3\lambda_1^{-1}\lambda_3^{-2} - \lambda_1^{-3}\lambda_3^{-3}) - 2\lambda_1^2(\lambda_1 + \lambda_1^{-1}\lambda_3^{-1}) F'^2 E_0^2 = 0.
\]

(2.111)

**Bibliography**


Wrinkles in soft dielectric plates


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Wrinkles in soft dielectric plates


Wrinkles in soft dielectric plates


Chapter 3

Stability analysis of charge-controlled soft dielectric plates

Hannah Conroy Broderick\textsuperscript{1}, Michele Righi\textsuperscript{2}, Michel Destrade\textsuperscript{1}, Ray W. Ogden\textsuperscript{3}

\textsuperscript{1} School of Mathematics, Statistics and Applied Mathematics, NUI Galway
\textsuperscript{2} TeCip Istituto, Scuola Superiore Sant’Anna
\textsuperscript{3} School of Mathematics and Statistics, University of Glasgow

Abstract

We examine the stability of a soft dielectric plate deformed by the coupled effects of a mechanical pre-stress applied on its lateral faces and an electric field applied through its thickness under charge control. The electric field is created by spraying charges on the major faces of the plate: although in practice this mode of actuation is harder to achieve than a voltage-driven deformation, here we find that it turns out to be much more stable in theory and in simulations.

First we show that the electromechanical instability based on the Hessian criterion associated with the free energy of the system does not occur at all for charge-driven dielectrics for which the electric displacement is linear in the electric field. Then we show that the geometric instability associated with the formation
of small-amplitude wrinkles on the faces of the plate that arises under voltage control does not occur either under charge control. This is in complete contrast to voltage-controlled actuation, where Hessian and wrinkling instabilities can occur once certain critical voltages are reached.

For the mechanical pre-stresses, two modes that can be implemented in practice are used: equi-biaxial and uni-axial. We confirm the analytical and numerical stability results of homogeneous deformation modes with Finite Element simulations of real actuations, where inhomogeneous fields may develop. We find complete agreement in the equi-biaxial case, and very close agreement in the uni-axial case, when the pre-stress is due to a dead-load weight. In the latter case, the simulations show that small inhomogeneous effects develop near the clamps, and eventually a compressive lateral stress emerges, leading to a breakdown of the numerics.

3.1 Introduction

Soft dielectric materials can undergo large actuation stretches when a potential difference is induced in the material. Typically, compliant electrodes such as carbon grease are smeared onto the faces of a soft dielectric elastomer plate and a voltage is applied across the thickness of the material. As the voltage increases the material gradually expands in area until a maximum voltage is reached, at which point a rapid large deformation known as snap-through occurs (Zhao and Suo, 2007). The large actuation achieved due to the snap-through behaviour is desirable for many applications but is difficult to achieve in practice. Snap-through is often prevented by electric breakdown (Zhao and Suo, 2007) or by instabilities such as inhomogeneities (Bertoldi and Gei, 2011), compression failure (De Tommasi et al., 2011), band localisation (Gei et al., 2014), wrinkles (Bortot and Shmuel, 2018; Liu et al., 2016; Pelrine et al., 2000; Plante and Dubowsky, 2006; Su et al., 2018a; Yang et al., 2017), membrane wrinkling (Greaney et al., 2019), etc.

Various methods have been proposed for avoiding electric breakdown without sacrificing the large actuation. For example, if the material is pre-stretched before the voltage is applied, electric breakdown may be avoided, but the stretch
gain achieved might be reduced (Su et al., 2018b). Another method proposed is charge-controlled actuation, as shown experimentally by Keplinger et al. (2010) and theoretically by Li et al. (2011). In charge-controlled actuation, charges of opposite signs are sprayed on opposite planar surfaces of a dielectric plate, inducing a potential difference, and hence an electric field in the dielectric, thereby inducing a deformation. In principle this method of actuation annihilates the possibility of snap-through because the theoretical charge-stretch loading curves are monotonic (Li et al., 2011). In this chapter, we investigate the stability of a charge-driven dielectric plate, which has not been considered previously; the results of which are significantly different from those for voltage control.

We first focus on equi-biaxial loading and show that charge-controlled actuation is stable since the Hessian criterion—or rather, its version for this problem—for onset of instability is never met (Section 3.2.2). This result is far from straightforward to obtain, because the Hessian determinant of the energy density is always negative, from which it could erroneously be concluded that the actuation is unstable. In fact, we show that the second variation of the free energy of the whole system is always positive, which ensures stability throughout. This is in sharp contrast to the corresponding situation for voltage-controlled actuation, which is well-known to become unstable once a critical voltage is reached (Zhao and Suo, 2007).

We then highlight another new, and complementary, feature of charge control by showing that charge-controlled actuation is also stable with respect to geometric instability because, provided the material is pre-stretched, small-amplitude inhomogeneous wrinkled solutions superposed on the large homogeneous actuation do not develop (Section 3.3). Again, this contrasts with the situation for voltage-controlled actuation, for which dielectric plates eventually wrinkle under sufficiently large voltages (Dorfmann and Ogden, 2014, 2019; Su et al., 2018a).

In Section 3.4 we model the experiments of Keplinger et al. (2010), where a plate was pre-stretched by a weight prior to charge-controlled actuation. We thus study the stability of a homogeneously deforming plate under uni-axial tension and charge-actuation and again we find Hessian-based and geometric stability in this case, again contrary to the corresponding situation for voltage-controlled actuation.

Finally in Section 3.5 we use Finite Element simulations to account for the
finite dimensions of plates. We find that in the equi-biaxial case there are no
differences between the results of the homogeneous loading analytical modelling
and those of the Finite Element method, because the plate is free to stretch lat-
erally and the loading curves are indeed monotonic (no snap-through). However,
for the uni-axial case we find that the clamping of the plate required to apply the
weight leads to non-homogeneous deformations with local variations of stresses
and strains compared to the homogeneous solution, and that these effects build
up and eventually lead to a breakdown of the simulation. We identify the point
of breakdown as corresponding to the appearance of compressive stresses in the
plate.

3.2 Equations of electroelasticity

Consider the stress-free reference configuration \( B_r \) of an electroelastic material
in the absence of an electric field and applied mechanical loads. Points in \( B_r \)
are labelled by the position vector \( X \). When subject to loads and an electric
field under static conditions the material occupies the configuration \( B \), with the
material point \( X \) now at \( x \). Let \( F = \text{Grad} x \) denote the deformation gradient
from \( B_r \) to \( B \), where Grad is the gradient operator with respect to \( X \). We denote
by \( E \) and \( D \), respectively, the electric field and electric displacement vectors in
\( B \), and by \( \tau \) the Cauchy stress tensor (which in general depends on \( F \) and either
\( E \) or \( D \)).

It has been found advantageous (Dorffmann and Ogden, 2005; Zhao and Suo,
2007) to formulate constitutive equations in terms of the Lagrangian field vari-
bles, denoted \( E_L, D_L \), and the nominal stress tensor \( T \), which are related to \( E, \)
\( D \) and \( \tau \) by the following pull-back operations (from \( B \) to \( B_r \))

\[
E_L = F^T E, \quad D_L = J F^{-1} D, \quad T = J F^{-1} \tau, \quad (3.1)
\]

where \( J = \text{det} F \).

The constitutive equations are based on the use of so-called ‘total’ energy
functions, depending either on \( F \) and \( E_L \), denoted \( \Omega \), or on \( F \) and \( D_L \), denoted
\( \Omega^* \), with the (partial) Legendre transform connection

\[
\Omega^*(F, D_L) = \Omega(F, E_L) + D_L \cdot E_L. \quad (3.2)
\]
Henceforth, we confine attention to incompressible materials, so that the constraint $J \equiv 1$ is in force. Then we have the constitutive relations
\begin{align}
T &= \frac{\partial \Omega}{\partial F} - p F^{-1}, \\
T^* &= \frac{\partial \Omega^*}{\partial F} - p^* F^{-1},
\end{align}
where $p$ and $p^*$ are Lagrange multipliers associated with the constraint, in general with $p^* \neq p$, and
\begin{align}
D_L &= -\frac{\partial \Omega}{\partial E_L}, \\
E_L &= \frac{\partial \Omega^*}{\partial D_L}.
\end{align}

The governing equations are
\begin{align}
\text{Div } T &= 0, \\
\text{Curl } E_L &= 0, \\
\text{Div } D_L &= 0,
\end{align}
where Div and Curl are the divergence and curl operators with respect to $X$.

We shall consider the situation in which there is no external field, so that on the boundary $\partial B_r$ of $B_r$ the standard electric boundary conditions associated with the equations (3.5) are simply
\begin{align}
T^T N &= t_A, \\
N \times E_L &= 0, \\
N \cdot D_L &= -\sigma_F \text{ on } \partial B_r,
\end{align}
where $N$ is the unit outward normal on $\partial B_r$, $t_A$ is the applied mechanical traction per unit area of $\partial B_r$, and $\sigma_F$ is the surface charge density per unit area of $\partial B_r$.

In considering applications to dielectric elastomers, which are isotropic electroelastic materials, the functional dependence of $\Omega$ and $\Omega^*$ can be expressed in terms of five invariants. First of all, the isotropic purely kinematic invariants defined by
\begin{align}
I_1 &= \text{tr } e, \\
I_2 &= \frac{1}{2} I_1^2 - \text{tr } (e^2),
\end{align}
where $e = F^T F$ is the right Cauchy–Green deformation tensor. Secondly, invariants associated with $E_L$, which typically are taken to be
\begin{align}
I_4 &= E_L \cdot E_L, \\
I_5 &= E_L \cdot (e^{-1} E_L), \\
I_6 &= E_L \cdot (e^{-2} E_L),
\end{align}
as in Dorfmann and Ogden (2005), and, thirdly, invariants associated with $D_L$, here defined by
\begin{align}
I_4^* &= D_L \cdot D_L, \\
I_5^* &= D_L \cdot (c D_L), \\
I_6^* &= D_L \cdot (c^2 D_L),
\end{align}
as used in Dorfmann and Ogden (2005) in different notation.
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The expanded forms of the constitutive relations (3.3), when converted to Eulerian form using (3.1) with $J=1$, are

$$\tau = 2\Omega_1 \mathbf{b} + 2\Omega_2 (I_1 \mathbf{b} - \mathbf{b}^2) - p\mathbf{I} - 2\Omega_5 \mathbf{E} \otimes \mathbf{E}$$
$$+ 2\Omega_6 (\mathbf{b}^{-1} \mathbf{E} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{b}^{-1} \mathbf{E}),$$  \hfill (3.10)

$$\tau = 2\Omega_1^* \mathbf{b} + 2\Omega_2^* (I_1 \mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} + 2\Omega_5^* \mathbf{D} \otimes \mathbf{D}$$
$$+ 2\Omega_6^* (\mathbf{bD} \otimes \mathbf{D} + \mathbf{D} \otimes \mathbf{bD}),$$  \hfill (3.11)

$$\mathbf{D} = -2(\Omega_4 \mathbf{b} + \Omega_5 \mathbf{I} + \Omega_6 \mathbf{b}^{-1}) \mathbf{E},$$  \hfill (3.12)

$$\mathbf{E} = 2(\Omega_4^* \mathbf{b}^{-1} + \Omega_5^* \mathbf{I} + \Omega_6^* \mathbf{b}) \mathbf{D},$$  \hfill (3.13)

where $\mathbf{I}$ is the identity tensor, $\mathbf{b} = \mathbf{FF}^T$ is the left Cauchy–Green deformation tensor, $\Omega_i = \partial \Omega / \partial I_i$, $i = 1, 2, 4, 5, 6$, $\Omega_i^* = \partial \Omega^* / \partial I_i$, $i = 1, 2$, and $\Omega_i^* = \partial \Omega^* / \partial I_i^*$, $i = 4, 5, 6$.

### 3.2.1 Specialisation to biaxial deformations of a plate

We now consider the application of the above theory to the biaxial deformation of a rectangular plate. The plate has sides of lengths $L_1, L_2, L_3$ in the reference configuration $B_r$, where $L_2 = H$ is the thickness of the plate, which is small compared with its lateral dimensions. Mechanical loads are applied in the 1 and 3 directions; also, a potential difference, say $V$, exists between the major surfaces of the plate and the associated charges on the surfaces are denoted $\pm Q$. As a result, the plate is stretched homogeneously with stretches $\lambda_1$ and $\lambda_3$ parallel to the major surfaces, and, by incompressibility, a stretch $\lambda_2 = \lambda_1^{-1}\lambda_3^{-1}$ normal to the major surfaces. The potential difference generates an electric field with a single component $E = E_2$, associated with an electric displacement component $D = D_2$. The corresponding components of the Lagrangian fields are $E_L = \lambda_2 E$ and $D_L = \lambda_2^{-1} D$.

In terms of the potential difference $V$ and the associated charges $\pm Q$ on the surfaces, we have the simple connections

$$E_L = -V/H, \quad D_L = -\sigma_F = -Q/L_1L_3.$$  \hfill (3.14)

Thus, for a fixed potential, $E_L$ is fixed, while fixed charge $Q$ corresponds to fixed $D_L$.  

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For this combination of deformation and electric field, Ω and Ω* specialise accordingly. The invariants are now given in terms of the independent stretches \( \lambda_1, \lambda_3 \) and \( E_L \) and \( D_L \) by

\[
I_1 = \lambda_1^2 + \lambda_3^2 + \lambda_1^{-2}\lambda_3^{-2}, \quad I_2 = \lambda_1^{-2} + \lambda_3^{-2} + \lambda_1^2\lambda_3^2, \tag{3.15}
\]

\[
I_4 = E_L^2, \quad I_5 = \lambda_1^2\lambda_3^2 E_L^2, \quad I_6 = \lambda_1^4\lambda_3^4 E_L^2, \tag{3.16}
\]

\[
I_4^* = D_L^2, \quad I_5^* = \lambda_1^{-2}\lambda_3^{-2}D_L^2, \quad I_6^* = \lambda_1^{-4}\lambda_3^{-4}D_L^2. \tag{3.17}
\]

We denote the specialisations of Ω and Ω* by \( \omega \) and \( \omega^* \), respectively, and the independent variables by \( (\lambda_1, \lambda_3, E_L) \) and \( (\lambda_1, \lambda_3, D_L) \), respectively, with, from the connection (3.2),

\[
\omega^*(\lambda_1, \lambda_3, D_L) = \omega(\lambda_1, \lambda_3, E_L) + D_L E_L. \tag{3.18}
\]

Since the resulting deformation is purely biaxial the corresponding nominal stress is coaxial with the edges of the plate; we denote its components by \( t_1, t_2, t_3 \).

We now assume that there is no mechanical traction on the major faces of the plate so that the boundary condition (3.6)\(_1\) yields \( t_2 = 0 \). Then, on elimination of the hydrostatic stress from (3.10) and (3.11), we obtain the simple formulas

\[
t_1 = \frac{\partial \omega}{\partial \lambda_1} = \frac{\partial \omega^*}{\partial \lambda_1}, \quad t_3 = \frac{\partial \omega}{\partial \lambda_3} = \frac{\partial \omega^*}{\partial \lambda_3}, \tag{3.19}
\]

and from (3.12) and (3.13)

\[
D_L = -\frac{\partial \omega}{\partial E_L}, \quad E_L = \frac{\partial \omega^*}{\partial D_L}. \tag{3.20}
\]

The particular case of equi-biaxial deformations is of special interest, for then, with \( \lambda_1 = \lambda_3 = \lambda \), and incompressibility giving \( \lambda_2 = \lambda^{-2} \), we may introduce the following further specialisations of the total energy functions,

\[
\tilde{\omega}(\lambda, E_L) = \omega(\lambda, \lambda, E_L), \quad \tilde{\omega}^*(\lambda, D_L) = \omega^*(\lambda, \lambda, D_L). \tag{3.21}
\]

We also have \( t_1 = t_3 = t \), say, so that

\[
t = \frac{1}{2} \frac{\partial \tilde{\omega}}{\partial \lambda} = \frac{1}{2} \frac{\partial \tilde{\omega}^*}{\partial \lambda}, \quad D_L = -\frac{\partial \tilde{\omega}}{\partial E_L}, \quad E_L = \frac{\partial \tilde{\omega}^*}{\partial D_L}. \tag{3.22}
\]

For illustration, we now consider models for which \( \mathbf{D} = \varepsilon \mathbf{E} \), where \( \varepsilon \), the material permittivity, is taken to be a constant. These are “ideal” dielectrics in
the terminology of Suo (2010). Note that this linear relationship has recently been verified by Zurlo et al. (2018) using experimental data for low to moderate values of the electric field for the acrylic dielectric elastomer VHB 4905. In general, \( \varepsilon \) may depend on the deformation, as has been shown in Wissler and Mazza (2007), for example, for the acrylic dielectric elastomer VHB 4910, but for our present purposes we consider it to be a material constant.

Then, \( \Omega \) and \( \Omega^* \) have the forms

\[
\Omega = W(I_1, I_2) - \frac{\varepsilon}{2} I_5 = W(I_1, I_2) - \frac{\varepsilon}{2} E_L \cdot (c^{-1} E_L),
\]

(3.23)

\[
\Omega^* = W(I_1, I_2) + \frac{1}{2\varepsilon} I_5^* = W(I_1, I_2) + \frac{1}{2\varepsilon} D_L \cdot (c D_L),
\]

(3.24)

and, for the biaxial deformations of a plate considered above,

\[
\omega = w(\lambda_1, \lambda_3) - \frac{\varepsilon}{2} \lambda_1^2 \lambda_3^2 E_L^2,
\]

(3.25)

where \( w(\lambda_1, \lambda_3) = W(I_1, I_2) \) with \( I_1 \) and \( I_2 \) given by (3.15) and \( D_L = \lambda_1^2 \lambda_3^2 E_L \).

For equi-biaxial deformations, we have

\[
\bar{\omega} = \tilde{\omega}(\lambda) - \frac{\varepsilon}{2} \lambda^4 E_L^2, \quad \bar{\omega}^* = \tilde{\omega}(\lambda) + \frac{1}{2\varepsilon} \lambda^{-4} D_L^2,
\]

(3.26)

where \( \tilde{\omega}(\lambda) = w(\lambda, \lambda) \) and \( D_L = \varepsilon \lambda^4 E_L \).

For our subsequent applications we consider two representative energy density functions, a \textit{neo-Hookean} dielectric model and a \textit{Gent} dielectric model, defined, in the two representations, by

\[
\Omega_{nH} = \frac{\mu}{2} (I_1 - 3) - \frac{\varepsilon}{2} I_5, \quad \Omega_G = -\frac{\mu J_m}{2} \ln \left(1 - \frac{I_1 - 3}{J_m}\right) - \frac{\varepsilon}{2} I_5,
\]

(3.27)

\[
\Omega^*_{nH} = \frac{\mu}{2} (I_1 - 3) + \frac{1}{2\varepsilon} I_5^*, \quad \Omega^*_G = -\frac{\mu J_m}{2} \ln \left(1 - \frac{I_1 - 3}{J_m}\right) + \frac{1}{2\varepsilon} I_5^*,
\]

(3.28)

where \( \mu \) is the shear modulus in the absence of an electric field and \( J_m \) is a stiffening parameter. Notice that \( \Omega_G \) recovers \( \Omega_{nH} \) and \( \Omega^*_G \) recovers \( \Omega^*_{nH} \) in the limit \( J_m \to \infty \).

We now express the equations in dimensionless form by defining the following quantities

\[
\bar{\omega} = \omega / \mu, \quad \bar{\omega}^* = \omega^* / \mu, \quad \tilde{\omega} = \tilde{\omega} / \mu, \quad \tilde{\omega}^* = \tilde{\omega}^* / \mu,
\]

\[
D_0 = D_L / \sqrt{\mu \varepsilon}, \quad E_0 = E_L \sqrt{\varepsilon / \mu}, \quad s = t / \mu,
\]

(3.29)
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Figure 3.1: Plots of (a) $D_0$ versus $\lambda$ and (b) $E_0$ versus $\lambda$ based on equation (3.31) and the connection $E_0 = \lambda^{-4} D_0$ for the neo-Hookean dielectric in equi-biaxial deformation, for values of non-dimensional pre-stress $s = 0, 1, 2, 3$ (continuous curves). In (a) we also display the (dashed) curve of $D_0 = \sqrt{(\lambda^6 + 5)/3}$, the intersections of which with the continuous curves correspond to the maxima in (b).

so that $D_0 = \lambda^4 E_0$ in the equi-biaxial case, and

$$s = \frac{1}{2} \frac{\partial \hat{\omega}}{\partial \lambda} = \frac{1}{2} \frac{\partial \hat{\omega}^*}{\partial \lambda}, \quad D_0 = -\frac{\partial \hat{\omega}}{\partial E_0}, \quad E_0 = \frac{\partial \hat{\omega}^*}{\partial D_0}. \quad (3.30)$$

Based on either $\hat{\omega}$ or $\hat{\omega}^*$, we now obtain the expression for $D_0$ in terms of $\lambda$ and $s$ for the neo-Hookean and Gent dielectric models as

$$D_0 = \sqrt{\lambda^6 - 1 - \lambda^5 s}, \quad D_0 = \sqrt{\frac{\lambda^6 - 1}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} - \lambda^5 s}, \quad (3.31)$$

respectively, and note that the latter reduces to the former when $J_m \to \infty$.

Figures 3.1(a) and 3.2(a) show plots of these curves with $D_0$ versus $\lambda$ for several fixed values of $s$, and Figures 3.1(b) and 3.2(b) display the corresponding plots of $E_0$ versus $\lambda$ based on the connection $E_0 = \lambda^{-4} D_0$. The value $J_m = 97.2$ given by Gent (1996) has been used here. Also shown in Figure 3.1(a) is the curve $D_0 = \sqrt{(\lambda^6 + 5)/3}$, which cuts the fixed $s$ curves at points where $E_0$ is a maximum in Figure 3.1(b), which also shows the corresponding dashed curve. Similarly for the Gent dielectric in Figures 3.2(a) and 3.2(b), although, for the larger values of $s$, there is no maximum in (b) and no corresponding intersection.

We now turn to the analysis of the stability of the plate based on the Hessian criterion.
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Figure 3.2: Plots of (a) $D_0$ versus $\lambda$ and (b) $E_0$ versus $\lambda$ based on equation (3.31) and $E_0 = \lambda^{-4}D_0$ for the Gent model, for fixed values of $s = 0, 1, 2, 3, 3$ (continuous curves) and in (a) the (dashed) curve of $D_0$ versus $\lambda$, the intersections of which with the continuous curves correspond to the maxima in (b). Note that for larger values of $s$ there is no intersection. Note also that the value $s = 2.3$ has been used here instead of $s = 2$ to enable the dashed curve to be distinguished from the continuous curve at larger values of $\lambda$ in (a).

3.2.2 Analysis of the Hessian stability criterion

Electro-mechanical instability is often considered to occur when the Hessian matrix associated with the second variation of the free energy for the whole system ceases to be positive definite (Zhao and Suo, 2007). The rationale of this criterion is that equilibrium corresponds to an extremum of the free energy (and thus its first variation is zero), and that the equilibrium is stable when it corresponds to a minimum of the free energy (and then its second variation is positive).

In different notation and in dimensionless form, the free energy of the whole system, here denoted $\psi^*$, considered in Zhao and Suo (2007) has the form

$$\psi^*(\lambda_1, \lambda_3, D_0) = \bar{\omega}^*(\lambda_1, \lambda_3, D_0) - s_1 \lambda_1 - s_3 \lambda_3 - D_0 E_0,$$

and vanishing of its first variation (for fixed $s_1, s_3, E_0)$, with $s_1 = t_1/\mu$, $s_3 = t_3/\mu$, yields the dimensionless versions of the constitutive relations involving $\omega^*$ in (3.19) and (3.20).

If, instead, we use $E_0$ as the independent electric variable, then the corresponding ‘energy’, denoted $\psi$, vanishing of the first variation of which yields the constitutive relations in terms of $\omega$ in (3.19) and (3.20), is given by

$$\psi(\lambda_1, \lambda_3, E_0) = \bar{\omega}(\lambda_1, \lambda_3, E_0) - s_1 \lambda_1 - s_3 \lambda_3 + D_0 E_0.$$  

(3.33)
Note that, on use of (3.18) in dimensionless form, we have \( \psi^* = \psi - D_0E_0 \), so that \( \psi \) is the Legendre transform of \( \psi^* \) with respect to the conjugate variables \( E_0 \) and \( D_0 \) related by (3.30).

For the free energy \( \psi^* \) of the whole system to be at a minimum, its second variation must be positive, i.e. the associated Hessian matrix must be positive definite, at a point of equilibrium. The second variations of \( \psi^* \) and \( \psi \) are written compactly as

\[
\delta^2 \psi^* = \delta a^* \cdot (\mathbf{H}^* \delta a^*), \quad \delta^2 \psi = \delta a \cdot (\mathbf{H} \delta a),
\]

respectively, with first variations \( \delta a^* = [\delta \lambda_1, \delta \lambda_3, \delta D_0]^T \), \( \delta a = [\delta \lambda_1, \delta \lambda_3, \delta E_0]^T \), where \( \mathbf{H}^* \) and \( \mathbf{H} \) are the corresponding Hessian matrices, which are given by

\[
\mathbf{H}^* = \begin{pmatrix}
\bar{\omega}^*_{11} & \bar{\omega}^*_{13} & \bar{\omega}^*_{1D_0} \\
\bar{\omega}^*_{13} & \bar{\omega}^*_{33} & \bar{\omega}^*_{3D_0} \\
\bar{\omega}^*_{1D_0} & \bar{\omega}^*_{3D_0} & \bar{\omega}^*_{D_0D_0}
\end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix}
\bar{\omega}_{11} & \bar{\omega}_{13} & \bar{\omega}_{1E_0} \\
\bar{\omega}_{13} & \bar{\omega}_{33} & \bar{\omega}_{3E_0} \\
\bar{\omega}_{1E_0} & \bar{\omega}_{3E_0} & \bar{\omega}_{E_0E_0}
\end{pmatrix},
\]

with the subscripts representing partial derivatives.

For the equi-biaxial case these become \( 2 \times 2 \) matrices, given by

\[
\mathbf{H}^* = \begin{pmatrix}
\bar{\omega}^*_{\lambda\lambda} & \bar{\omega}^*_{\lambda D_0} \\
\bar{\omega}^*_{\lambda D_0} & \bar{\omega}^*_{D_0D_0}
\end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix}
\bar{\omega}_{\lambda\lambda} & \bar{\omega}_{\lambda E_0} \\
\bar{\omega}_{\lambda E_0} & \bar{\omega}_{E_0E_0}
\end{pmatrix},
\]

and we now focus on this case for illustration.

It is straightforward to show that \( \bar{\omega}^*_{D_0D_0} = -1/\bar{\omega}_{E_0E_0} \) by using the formulas (3.30)\textsubscript{3,4}. Now the determinants of the Hessians above are given by

\[
\det \mathbf{H}^* = \bar{\omega}^*_{\lambda\lambda} \bar{\omega}^*_{D_0D_0} - \bar{\omega}^2_{\lambda D_0}, \quad \det \mathbf{H} = \bar{\omega}_{\lambda\lambda} \bar{\omega}_{E_0E_0} - \bar{\omega}^2_{\lambda E_0},
\]

and on specialising (3.18) we have \( \bar{\omega}^* \left( \lambda, D_0 \right) = \bar{\omega} \left( \lambda, E_0 \right) + D_0 E_0 \), from which the following connections, given in Dorfmann and Ogden (2019) in dimensional form, can be obtained:

\[
\det \mathbf{H}^* = \bar{\omega}^*_{\lambda\lambda} \bar{\omega}^*_{D_0D_0} = -\bar{\omega}_{\lambda\lambda}/\bar{\omega}_{E_0E_0}, \quad \det \mathbf{H} = \bar{\omega}^*_{\lambda\lambda} \bar{\omega}_{E_0E_0}.
\]

These equations are independent of the specific forms of \( \bar{\omega}^* \) and \( \bar{\omega} \), and so are valid for any choice of (equi-biaxial) energy density function. They have some interesting interpretations, which we now discuss in respect of the neo-Hookean dielectric, for which

\[
\bar{\omega}^* = \frac{1}{2}(2\lambda^2 + \lambda^{-4} - 3) + \frac{1}{2}\lambda^{-4}D_0^2, \quad \bar{\omega} = \frac{1}{2}(2\lambda^2 + \lambda^{-4} - 3) - \frac{1}{2}\lambda^4E_0^2.
\]
and hence
\[ s = \frac{1}{2} \dot{\omega}_\lambda = \lambda - \lambda^{-5} - \lambda^3 E_0^2 = \lambda - \lambda^{-5} - \lambda^{-5} D_0^2 = \frac{1}{2} \dot{\omega}_\lambda^*, \tag{3.40} \]
and
\[ \dot{\omega}_{\lambda\lambda} = 2 (1 + 5 \lambda^{-6} + 5 \lambda^{-6} D_0^2), \quad \dot{\omega}_{\lambda\lambda} = 2 (1 + 5 \lambda^{-6} - 3 \lambda^2 E_0^2), \tag{3.41} \]
\[ \dot{\omega}_{\lambda D_0} = -4 \lambda^{-5} D_0, \quad \dot{\omega}_{D_0 D_0} = \lambda^{-4}, \quad \dot{\omega}_{\lambda E_0} = -4 \lambda^3 E_0, \quad \dot{\omega}_{E_0 E_0} = -\lambda^4. \tag{3.42} \]

Thus here,
\[ \det \mathbf{H}^* = 2 \lambda^{-10} (\lambda^6 + 5 - 3 D_0^2), \quad \det \mathbf{H} = -2 \lambda^4 (1 + 5 \lambda^{-6} + 5 \lambda^2 E_0^2). \tag{3.43} \]

Note that the maxima of \( E_0 \) in Figure 3.1(b) correspond to \( \det \mathbf{H}^* = 0 \), equivalently \( \dot{\omega}_{\lambda\lambda} = 0 \), which also corresponds to a maximum of \( s \) at fixed \( E_0 \). Note that \( \mathbf{H}^* \) is positive definite up to the maxima as \( E_0 \) is increased from 0. By contrast, \( s \) is monotonic with respect to \( \lambda \) at fixed \( D_0 \) and \( \dot{\omega}_{\lambda\lambda}^* > 0 \), while \( \det \mathbf{H} < 0 \) and \( \mathbf{H} \) is indefinite, thus defining a saddle point of \( \dot{\omega} \). Note that it would be incorrect to conclude here that the charge-controlled actuation is unstable, as we now show.

Consider the connection
\[ \dot{\omega}^*(\lambda, D_0) = \dot{\omega}(\lambda, E_0) + E_0 D_0, \tag{3.44} \]
the first variation of which yields
\[ \dot{\omega}_\lambda^* \delta \lambda + \dot{\omega}_{D_0}^* \delta D_0 = \dot{\omega}_\lambda \delta \lambda + \dot{\omega}_{E_0} \delta E_0 + E_0 \delta D_0 + D_0 \delta E_0, \tag{3.45} \]
leading to
\[ \dot{\omega}_\lambda^* = \dot{\omega}_\lambda, \quad E_0 = \dot{\omega}_{D_0}^*, \quad D_0 = -\dot{\omega}_{E_0}. \tag{3.46} \]
If we now take the second variation then the terms involving \( \delta^2 \lambda, \delta^2 E_0, \delta^2 D_0 \) cancel and we are left with the quadratic connection
\[ \dot{\omega}_{\lambda\lambda}^*(\delta \lambda)^2 + 2 \dot{\omega}_{\lambda D_0}^* \delta \lambda \delta D_0 + \dot{\omega}_{D_0 D_0}^*(\delta D_0)^2 = \dot{\omega}_{\lambda\lambda}(\delta \lambda)^2 + 2 \dot{\omega}_{\lambda E_0} \delta \lambda \delta E_0 \\
+ \dot{\omega}_{E_0 E_0}(\delta E_0)^2 + 2 \delta E_0 \delta D_0. \tag{3.47} \]
From (3.46), we obtain
\[ \delta D_0 = -(\dot{\omega}_{\lambda E_0} \delta \lambda + \dot{\omega}_{E_0 E_0} \delta E_0), \tag{3.48} \]

and hence, by substituting for \( \delta D_0 \) on the right-hand side of (3.47), we obtain
\[
\tilde{\omega}_{\lambda\lambda}^*(\delta \lambda)^2 + 2\tilde{\omega}_{\lambda\lambda D_0}^* \delta \lambda \delta D_0 + \tilde{\omega}_{D_0D_0}^*(\delta D_0)^2 = \tilde{\omega}_{\lambda\lambda}^*(\delta \lambda)^2 - \tilde{\omega}_{E_0E_0}^*(\delta E_0)^2. \tag{3.49}
\]

For stability we require the left-hand side to be positive since this is the second variation of the actual free energy \( \psi^* \) (so the free energy is minimised), whether we have voltage-control of the deformation (when \( \lambda \) and \( D_0 \) are free to vary) or charge-control of the deformation (when \( \lambda \) and \( E_0 \) are free to vary).

For fixed \( E_0 \), in a voltage-controlled experiment, this reduces simply to \( \tilde{\omega}_{\lambda\lambda} > 0 \), and this fails where \( E_0 \) is a maximum. For the neo-Hookean dielectric, it reads \( \lambda^{-2} + 5\lambda^{-8} - 3E_0^2 > 0 \), and \( E_0 = \sqrt{(\lambda^{-2} + 5\lambda^{-8})/3} \) is the plot going through the maxima of each loading curve for different values of the pre-load \( s \), as shown by the dashed curve in Figure 3.1(b).

For fixed \( D_0 \), in a charge-controlled experiment, the left-hand side is positive if \( \tilde{\omega}_{\lambda\lambda} > 0 \), and for the neo-Hookean dielectric, this reads \( 1 + 5\lambda^{-6}(1 + D_0^2) > 0 \), which holds true for all \( D_0 \). In the case of a perfect dielectric, we have \( D_0 = \lambda^4E_0 \), and hence \( 0 = 4\lambda^4\lambda E_0 + \lambda^4\delta E_0 \), so for the right-hand side of (3.49) to be positive we have
\[
\tilde{\omega}_{\lambda\lambda} - 16\tilde{\omega}_{E_0E_0} E_0^2 \lambda^{-2} = 2(1 + 5\lambda^{-6} + 5\lambda^2E_0^2) > 0, \tag{3.50}
\]
which confirms the result \( \tilde{\omega}_{\lambda\lambda}^* > 0 \) for the neo-Hookean dielectric, and thus, that the second variation of the free energy is always positive.

We can therefore conclude that under charge control, equi-biaxial activation is stable according to the Hessian criterion since we have \( \tilde{\omega}_{\lambda\lambda}^* > 0 \) for the considered neo-Hookean model. On the other hand, activation under voltage control can become unstable in the Hessian criterion sense, as is well known, since the inequality \( \tilde{\omega}_{\lambda\lambda} > 0 \) can fail. The results for the Gent model (not developed here) follow the same pattern.

### 3.3 Incremental stability analysis

To investigate the possibility of geometric instabilities, namely the formation of small-amplitude wrinkles on the faces of the plate, we linearise the governing equations and boundary conditions in the neighbourhood of a large deformation and initial electric field.
We introduce the incremental mechanical displacement \( \mathbf{u} \), the incremental nominal stress tensor \( \mathbf{T} \) and the incremental Lagrangian electric field and displacement, \( \mathbf{E}_L \) and \( \mathbf{D}_L \), respectively, all of which are functions of the deformed position \( \mathbf{x} \) (Dorfmann and Ogden, 2014). Let \( \mathbf{T}_0 \), \( \mathbf{E}_{L0} \) and \( \mathbf{D}_{L0} \) denote their push-forward forms from the reference to the deformed configuration, as defined by \( \mathbf{T}_0 = F \mathbf{T}, \mathbf{E}_{L0} = F^{-T} \mathbf{E}_L, \mathbf{D}_{L0} = F \mathbf{D}_L \). These satisfy the governing equations

\[
\text{div} \mathbf{T}_0 = 0, \quad \text{curl} \mathbf{E}_{L0} = 0, \quad \text{div} \mathbf{D}_{L0} = 0, \tag{3.51}
\]

and the relevant incremental constitutive equations are

\[
\dot{\mathbf{T}}_0 = \mathbf{A}_0 \mathbf{L} + p \mathbf{L} - p \mathbf{I} + \mathbf{A}_0 \mathbf{E}_{L0}, \quad \dot{\mathbf{D}}_{L0} = -\mathbf{A}_0^T \mathbf{L} - \mathbf{A}_0 \dot{\mathbf{E}}_{L0}, \tag{3.52}
\]

where \( \mathbf{A}_0, \mathbf{A}_0 \) and \( \mathbf{A}_0 \) are, respectively, fourth-, third- and second-order electroelastic moduli tensors (see Su et al. (2018a) for their general expressions), and \( \mathbf{L} \) is the displacement gradient \( \text{grad} \mathbf{u} \), \( \mathbf{u} \) being the incremental displacement, which, by incompressibility, satisfies \( \text{tr} \mathbf{L} \equiv \text{div} \mathbf{u} = 0 \).

Attention is now focused on two-dimensional wrinkles (Su et al., 2018a) so that the fields are functions of the components \( x_1, x_2 \) of \( \mathbf{x} \) only, and \( u_3 = \dot{E}_{L03} = \dot{D}_{L03} = 0 \). The governing equations then reduce to

\[
\dot{T}_{011,1} + \dot{T}_{021,2} = 0, \quad \dot{T}_{012,1} + \dot{T}_{022,2} = 0, \\
\dot{E}_{L01,2} - \dot{E}_{L02,1} = 0, \quad \dot{D}_{L01,1} + \dot{D}_{L02,2} = 0, \tag{3.53}
\]

where subscripts 1 and 2 following a comma signify differentiation with respect to \( x_1 \) and \( x_2 \), respectively.

From (3.53)_3 we can introduce the scalar electric potential \( \varphi \) such that

\[
\dot{E}_{L01} = -\varphi, \quad \dot{E}_{L02} = -\varphi. \tag{3.54}
\]

We now focus on models of the form

\[
\Omega(I_1, I_5) = W(I_1) - \frac{1}{2} \varepsilon I_5, \tag{3.55}
\]
for which the relevant components of the moduli tensors reduce to

\[
\begin{align*}
A_{0111} &= 4W_{11}\lambda_1^4 + 2W_1\lambda_2^2, \\
A_{0122} &= 4W_{11}\lambda_1^2\lambda_2^2 - \varepsilon E_2^2, \\
A_{01212} &= 2W_1\lambda_2^2 - \varepsilon E_2, \\
A_{012|1} &= A_{021|1} = \varepsilon E_2, \\
A_{011|1} &= A_{022|1} = A_{011|2} = 0, \\
A_{012|2} &= A_{021|2} = 0, \\
A_{011} &= A_{022} = -\varepsilon, \\
A_{012} &= 0.
\end{align*}
\]

(3.56)

Note that we used the connection \( p = A_{02121} \), which is a special case of a general formula given in, for example, equation (9.88) of Dorfmann and Ogden (2014). The vertical bar between the components of \( A_0 \) is used to distinguish the single index (associated with a vector) from the pair of indices associated with a second-order tensor.

Now, for brevity, we introduce the notations

\[
\begin{align*}
a &= A_{01212}, \\
b &= A_{01111} + A_{02222} - 2A_{01122}, \\
c &= A_{02121}, \\
d &= A_{0121|1}.
\end{align*}
\]

(3.57)

Then, on elimination of \( \dot{p} \) and use of the incompressibility equation \( u_{1,1} + u_{2,2} = 0 \), the required incremental constitutive equations can be written compactly in the form

\[
\begin{align*}
\dot{T}_{011} &= \dot{T}_{022} + 2(b + c)u_{1,1} + 2d\varphi, \\
\dot{T}_{012} &= au_{2,1} + cu_{1,2} - d\varphi, \\
\dot{T}_{021} &= c(u_{1,2} + u_{2,1}) - d\varphi, \\
\dot{D}_{L01} &= -d(u_{1,2} + u_{2,1}) - \varepsilon\varphi, \\
\dot{D}_{L02} &= 2du_{1,1} - \varepsilon\varphi.
\end{align*}
\]

(3.58)

We now convert the system of equations to a first-order system with six variables based on the Stroh approach. For this purpose we choose the variables \( u_1, u_2, \varphi, \dot{T}_{021}, \dot{T}_{022}, \dot{D}_{L02} \) and consider increments that are sinusoidal in the \( x_1 \) direction, i.e. solutions of the form

\[
\begin{align*}
\{ u_1, u_2, \varphi, \dot{T}_{021}, \dot{T}_{022}, \dot{D}_{L02} \} &= \Re\{e^{ikx_1} [U_1, U_2, \Phi, ik\Sigma_{21}, ik\Sigma_{22}, ik\Delta] \},
\end{align*}
\]

(3.59)

where \( U_1, U_2, \Phi, \Sigma_{21}, \Sigma_{22} \) and \( \Delta \) are all functions of \( kx_2 \), \( k = 2\pi/\mathcal{L} \) is the wave number and \( \mathcal{L} \) is the wavelength of the wrinkles.
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We now arrange the variables so that they all have the same dimensions by defining a Stroh vector $\eta$ as

$$\eta = (U, S) = [U_1, U_2, \sqrt{\varepsilon/\mu} \Phi, \Sigma_{21}/\mu, \Sigma_{22}/\mu, \Delta/\sqrt{\mu \varepsilon}], \quad (3.60)$$

where $U$ is the ‘displacement’ vector and $S$ is the ‘traction’ vector. After a little manipulation, the equations (3.53) and (3.58) are cast in the form

$$\eta'_1 = i(-\eta_2 + d\bar{c}^{-1}\eta_3 + \bar{c}^{-1}\eta_4),$$
$$\eta'_2 = -i\eta_1,$$
$$\eta'_3 = i(2\bar{d}\eta_1 - \eta_6),$$
$$\eta'_4 = i[-(2\bar{b} + 2\bar{c} + 4\bar{d}^2)\eta_1 - \eta_5 + 2\bar{d}\eta_6],$$
$$\eta'_5 = i[(\bar{c} - \bar{a})\eta_2 - \eta_4],$$
$$\eta'_6 = i[(\bar{d}^2\bar{c}^{-1} + 1)\eta_3 + \bar{d}\bar{c}^{-1}\eta_4], \quad (3.61)$$

where $\bar{a} = a/\mu$, $\bar{b} = b/\mu$, $\bar{c} = c/\mu$ and $\bar{d} = d/\sqrt{\mu \varepsilon}$, and a prime denotes differentiation with respect to $kx_2$. Similarly to Su et al. (2018a) we can thus write the equations in Stroh form, i.e. as

$$\eta' = iN\eta, \quad (3.62)$$

where $N$ is the Stroh matrix and $\eta$ is the Stroh vector, defined in (3.60). Note that the vector $\eta$ is different from its counterpart in the voltage-controlled case (Su et al., 2018a), due to the different electric boundary conditions and scalings. In the voltage-controlled case, the incremental electric boundary condition is in terms of the electric potential $\Phi$ (which must be zero on the faces), whereas in the charge-controlled case, the incremental electric boundary condition is in terms of the electric displacement $\Delta$. In the present situation the Stroh matrix has the dimensionless form

$$N = \begin{bmatrix}
N_1 & N_2 \\
N_3 & N_1^T
\end{bmatrix}, \quad (3.63)$$
where

\[ N_1 = \begin{bmatrix} 0 & -1 & \bar{d}/\bar{c} \\ -1 & 0 & 0 \\ 2\bar{d} & 0 & 0 \end{bmatrix} , \quad N_2 = \begin{bmatrix} 1/\bar{c} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} , \]

\[ N_3 = \begin{bmatrix} -2(\bar{b} + \bar{c}) - 4\bar{d}^2 & 0 & 0 \\ 0 & \bar{c} - \bar{a} & 0 \\ 0 & 0 & \bar{d}^2/\bar{c} + 1 \end{bmatrix} . \quad (3.64) \]

For the models (3.23) and (3.24) for which \( W(I_1, I_2) \) depends on only \( I_1 \), i.e. \( W = W(I_1) \), including the Gent dielectric model (3.27), in equi-biaxial activation we have

\[ \bar{a} = 2\lambda^2 \bar{W}' - \lambda^{-4} D_0^2 , \quad \bar{c} = 2\lambda^{-4} \bar{W}' , \]

\[ 2\bar{b} = 4(\lambda^{-4} - \lambda^2)\bar{W}'' + \bar{a} + \bar{c} , \quad \bar{d} = \lambda^{-2} D_0 , \quad (3.65) \]

where \( \bar{W}(I_1) = W(I_1)/\mu \), and henceforth we restrict attention to this specialisation. For the Gent dielectric,

\[ \bar{W}' = \frac{1}{2[1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m]} , \]

\[ \bar{W}'' = \frac{1}{2J_m [1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m]^2} , \quad (3.66) \]

and we recall that \( E_0 = \lambda^{-4} D_0 \). For the neo-Hookean dielectric, the expressions simplify considerably as: \( \bar{W}' = 1/2 \) and \( \bar{W}'' = 0 \).

To investigate the conditions for wrinkling, it is sufficient to calculate the thin-plate and thick-plate limits of the dispersion equation, as the behaviour of a plate with finite thickness lies in between the two (Su et al., 2018a).

The thin-plate limit is calculated from the Stroh matrix as (Shuvalov, 2000; Su et al., 2018a)

\[ \text{det} \, N_3 = 0 , \quad (3.67) \]

which simplifies here to

\[ (\bar{a} - \bar{c})(\bar{b} + \bar{c} + 2\bar{d}^2)(\bar{d}^2 + \bar{c}) = 0 . \quad (3.68) \]

As in the voltage-controlled case, the thin-plate limit can be separated into symmetric and anti-symmetric modes. Anti-symmetric modes are governed by the
equation $\bar{a} - \bar{c} = 0$, as in the voltage-controlled case. For the neo-Hookean and the Gent dielectric models this yields

$$D_0 = \sqrt{\lambda^6 - 1}, \quad D_0 = \sqrt{\frac{\lambda^6 - 1}{1 - (2\lambda^2 + \lambda^4 - 3)/J_m}}, \quad (3.69)$$

respectively, which is the same as (3.31) in the absence of pre-stress ($s = 0$). No symmetric modes are possible as they are governed by the equation $\bar{b} + \bar{c} + 2\bar{d}^2 = 0$, which has no real solutions in $(\lambda, D_0)$. Likewise, the third factor in (3.68) yields no solutions.

To calculate the thick-plate limit, we first construct a matrix with the eigenvectors $\eta^{(j)}$, $j = 1, 2, 3$, of $N$ with corresponding eigenvalues with positive imaginary part, stacked as the columns as

$$
\begin{bmatrix}
A \\
B
\end{bmatrix} = 
\begin{bmatrix}
| & | & | \\
\eta^{(1)} & \eta^{(2)} & \eta^{(3)} \\
| & | & |
\end{bmatrix}, \quad (3.70)
$$

where $A$ and $B$ defined above are $3 \times 3$ matrices and explicit expressions for the components of $\eta^{(j)}$, $j = 1, 2, 3$, are

$$
\eta^{(1)} = 
\begin{bmatrix}
0 \\
0 \\
1 \\
-\lambda^2D_0 \\
-i\lambda^2D_0 \\
i
\end{bmatrix}, \quad \eta^{(j)} = 
\begin{bmatrix}
i\lambda^8p_j \\
-\lambda^8 \\
\lambda^6D_0 \\
-2\lambda^4\bar{W}''(p_j^2 + 1) - \lambda^4D_0^2 \\
-2i\bar{W}'p_j^{-1}\lambda^4(\lambda^6 + p_j^2) \\
i\lambda^6p_jD_0
\end{bmatrix}, \quad (3.71)
$$

for $j = 2, 3$ and where $p_{2,3}$ and $\bar{W}'$ are given by (3.74) below and (3.66)$_1$, respectively.

Then the thick-plate limit is given by

$$
\det (iBA^{-1}) = 0. \quad (3.72)
$$

Based on the analysis of Stroh (see Ting, 1996 or Shuvalov, 2000, for instance), we recall that $iBA^{-1}$ is Hermitian and the above equation is a single real equation, as distinct from $\det B = 0$, which is a complex equation, although its real and imaginary parts are in proportion.
For models with $W = W(I_1)$, including that of Gent, equation (3.72) is a quadratic in $D_0^2$, explicitly

$$D_0^4 - 2W' \left[ \lambda^3(p_2 + p_3) + 2(\lambda^3 - 1) \right] D_0^2 - 4W''^2 \left[ \lambda^3(p_2 + p_3)^2 - (\lambda^3 - 1)^2 \right] = 0,$$

(3.73)

where

$$p_{2,3} = \frac{\lambda^3 + 1}{2} \sqrt{1 + 2(\lambda - \lambda^{-2})^2 W'' W'} \mp \frac{\lambda^3 - 1}{2} \sqrt{1 + 2(\lambda + \lambda^{-2})^2 W'' W'},$$

(3.74)

where $(-)$ and $(+)$ correspond to $p_2$ and $p_3$, respectively. Note that for the neo-Hookean specialisation, since $W' = 1/2$, $W'' = 0$, we obtain $p_2 = 1$ and $p_3 = \lambda^3$ and the thick-plate limit becomes

$$D_0^4 - (\lambda^6 + 3\lambda^3 - 2)D_0^2 - (\lambda^9 + \lambda^6 + 3\lambda^3 - 1) = 0.$$

(3.75)

In the absence of charge ($D_0 = 0$), this reduces to the classical elastic case and recovers the critical stretch for surface instability under equi-biaxial stretch of Green and Zerna (1954), specifically $\lambda = 0.666$.

We plot the thick- and thin-plate limits, along with the loading curves (3.31$_1$) for the neo-Hookean dielectric model for different values of pre-stress in Figure 3.3. The loading curves are monotonic, and so the material will not experience the snap-through phenomenon of voltage-controlled actuation (Li et al., 2011). As shown in the previous section, this is connected to the sign of the second variation of the free energy being always positive.

These theoretical predictions are compared with Finite Element simulations (see Section 3.5), the results of which are represented by dots in the figure, which also exhibit the stability.

The region between the thick-plate and thin-plate limits is where wrinkling could occur. However, the pre-stretched loading curves do not cross this region, so there is no wrinkling. Charge-controlled dielectric plates are therefore geometrically stable, and will not exhibit wrinkling (provided $s > 0$). This situation again contrasts with voltage-controlled plates, which can wrinkle in compression, as here, but also in extension (Dorfmann and Ogden, 2014, 2019; Su et al., 2018a), which is not possible here.
3.3 Winkles are not expressed in equi-biaxially pre-stretched charge-controlled plates. Here the solid curves are the loading curves for the neo-Hookean dielectric model with pre-stresses $s = 0, 0.8, 1.5, 2.5, 4.5$. The dashed curve is the thick-plate limit (3.75). None of the pre-stretched curves cross the greyed zone where wrinkling occurs, between the thick-plate (dashed curve) and thin-plate ($s = 0$ loading curve) limits, so wrinkling does not take place. The dots are the result of Finite Element calculations using COMSOL Multiphysics® (Section 3.5), which turn out to be very stable numerically. We conducted the same calculations for the Gent dielectric with $J_m = 97.2$ and found almost identical plots (not shown here).

3.4 Activation under uni-axial dead load

In order to model the experiments of Keplinger et al. (2010), we now consider a plate that is pre-stretched by a uni-axial dead load. A weight is applied in the $x_1$-direction and charges are deposited on the lateral faces of the dielectric so that an electric field is induced in the $x_2$-direction.

In dimensionless form, the loading curves relating the uni-axial stress $s$, the electric displacement component $D_0$ and the electric field $E_0$ to the stretches $\lambda_1$ and $\lambda_3$ for the neo-Hookean model (3.27)_1 are given by

$$s = \lambda_1 - \lambda_1^{-1}\lambda_3^2, \quad D_0^2 = \lambda_1^2\lambda_3^4 - 1, \quad E_0^2 = \lambda_1^{-2} - \lambda_1^{-4}\lambda_3^{-4},$$

which lead to expressions for $D_0-\lambda_1$ and $E_0-\lambda_1$ in terms of $s$ (see, for example,
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Huang and Suo (2012); Lu et al. (2012) for details in the voltage-controlled case), namely

\[ D_0 = \sqrt{\lambda_1^4(\lambda_1 - s)^2 - 1}, \quad E_0 = \lambda_1^{-1}\sqrt{1 - \lambda_1^{-4}(\lambda_1 - s)^{-2}}. \] (3.77)

Plots of \( D_0 \) and \( E_0 \) versus \( \lambda_1 \) based on (3.77) for several fixed values of \( s \) are shown in Figures 3.4(a) and 3.4(b), respectively, as the continuous curves. Notice, in particular, that \( D_0 \) is monotonic in \( \lambda_1 \), while \( E_0 \) exhibits maxima, these behaviours being associated with loss of Hessian stability, as we elaborate on below.

\[ \begin{align*}
\bar{\omega}_{11}^* \delta \lambda_1^2 + 2\bar{\omega}_{13}^* \delta \lambda_1 \delta \lambda_3 + \bar{\omega}_{33}^* \delta \lambda_3^2 + 2\bar{\omega}_{1D_0}^* \delta \lambda_1 \delta D_0 + 2\bar{\omega}_{3D_0}^* \delta \lambda_3 \delta D_0 + \bar{\omega}_{D_0D_0}^* \delta D_0^2 \\
= \bar{\omega}_{11} \delta \lambda_1^2 + 2\bar{\omega}_{13} \delta \lambda_1 \delta \lambda_3 + \bar{\omega}_{33} \delta \lambda_3^2 + 2\bar{\omega}_{1E_0} \delta \lambda_1 \delta E_0 \\
+ 2\bar{\omega}_{3E_0} \delta \lambda_3 \delta E_0 + \bar{\omega}_{E_0E_0} \delta E_0^2 \quad + 2\delta E_0 \delta D_0 \\
= \bar{\omega}_{11} \delta \lambda_1^2 + 2\bar{\omega}_{13} \delta \lambda_1 \delta \lambda_3 + \bar{\omega}_{33} \delta \lambda_3^2 - \bar{\omega}_{E_0E_0} \delta E_0^2. \quad (3.78)
\end{align*} \]

For the second variations of the free energy of the whole system \( \psi^* \) to be positive, the \( 3 \times 3 \) Hessian matrix \( \mathbf{H}^* \) must be positive definite (recall (3.34)).
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According to the equality above, this is equivalent under voltage control (when \( \lambda_1, \lambda_3 \) and \( D_0 \) are free to vary and \( E_0 \) is fixed) to

\[
\bar{\omega}_{11} \delta \lambda_1^2 + 2\bar{\omega}_{13} \delta \lambda_1 \delta \lambda_3 + \bar{\omega}_{33} \delta \lambda_3^2 > 0, \tag{3.79}
\]

for non-zero \( \delta \lambda_1 \) and/or \( \delta \lambda_3 \), i.e. it is equivalent to the leading \( 2 \times 2 \) minor in \( H \) being positive definite.

On the other hand, under charge control (when \( \lambda_1, \lambda_3 \) and \( E_0 \) are free to vary and \( D_0 \) is fixed), the left hand side of the equality (3.78) tells us that leading \( 2 \times 2 \) minor of \( H^* \) should be positive definite for stability, i.e.

\[
\bar{\omega}_{11}^* \delta \lambda_1^2 + 2\bar{\omega}_{13}^* \delta \lambda_1 \delta \lambda_3 + \bar{\omega}_{33}^* \delta \lambda_3^2 > 0, \tag{3.80}
\]

for non-zero \( \delta \lambda_1 \) and/or \( \delta \lambda_3 \).

The latter inequality always holds for the neo-Hookean dielectric model, since \( \bar{\omega}_{11}^* > 0 \) and the leading \( 2 \times 2 \) minor of \( H^* \) is positive definite, with determinant

\[
1 + 3\lambda_1^{-4} \lambda_3^{-4}(\lambda_1^2 + \lambda_3^2)(1 + D_0^2) + 5\lambda_1^{-6} \lambda_3^{-6}(1 + D_0^2)^2, \tag{3.81}
\]

which factorises in the equi-biaxial case, with \( \lambda_1 = \lambda_3 = \lambda \), as

\[
[1 + 5\lambda^{-6}(1 + D_0^2)][1 + \lambda^{-6}(1 + D_0^2)], \tag{3.82}
\]

the first factor coinciding with the corresponding result in the purely equi-biaxial case.

Also, using (3.76)\(_3\), we find

\[
\det H^* = 4\lambda_1^{-6} \lambda_3^{-6}(3\lambda_3^2 + \lambda_1^2 - \lambda_1^2 \lambda_3^6), \tag{3.83}
\]

which corresponds to

\[
\bar{\omega}_{11} \delta \lambda_1^2 + 2\bar{\omega}_{13} \delta \lambda_1 \delta \lambda_3 + \bar{\omega}_{33} \delta \lambda_3^2 = 0, \tag{3.84}
\]

for fixed \( E_0 \). This condition means that the leading \( 2 \times 2 \) minor of \( H \) is indefinite, which can hold for fixed \( E_0 \) (at least for the neo-Hookean model), and we also have \( \det H < 0 \).

Plots of \( E_0 \) versus \( \lambda_1 \) for the uni-axial case are shown in Figure 3.4(b) for \( s = 0, 1, 2, 3 \), and the connection between \( E_0 \) and \( \lambda_1 \) where \( \det H^* = 0 \) is also shown as the dashed curve that passes through the maximum points of \( E_0 \).
Figure 3.4(a) are shown corresponding plots of $D_0$ (for $s = 0, 1, 2, 3, 4$) versus $\lambda_1$, the dashed curve corresponding to where $\det \mathcal{H}^* = 0$.

In conclusion, for a neo-Hookean dielectric subject to a uni-axial dead load, activation with voltage control can become unstable, but charge-controlled activation is always stable in the sense of the Hessian free energy criterion.

For the study of the geometric stability, we again refer to the limit cases. First, the thin-plate limit, again $\det N_3 = 0$, reduces to

$$D_0^2 = \lambda_1^2 \lambda_3^2 - 1. \quad (3.85)$$

The thick-plate limit, is a quadratic in $D_0^2$ given by

$$D_0^4 - (\lambda_1^4 \lambda_3^2 + 3 \lambda_1^2 \lambda_3 - 2) D_0^2 - (\lambda_1^4 \lambda_3^2 + \lambda_1^2 \lambda_3^2 + 3 \lambda_1^2 \lambda_3 - 1) = 0. \quad (3.86)$$

Note that these two equations apply for all $\lambda_1 (> 0)$ and, in particular, they recover the conditions for the equi-biaxial case (3.69) and (3.75) when $\lambda_1 = \lambda_3 = \lambda$.

The limit conditions above relate to wrinkles aligned with the direction of the uni-axial load. In Figure 3.5 we plot the corresponding $D_0-\lambda_1$ curves by solving each condition (3.85) and (3.86) together with (3.76)$_2$. The loading curves (3.77)$_1$ are also plotted, for different values of uni-axial pre-stress $s$.

As in the equi-biaxial case, the thin-plate limit is equivalent to the loading curve in the absence of pre-stress ($s = 0$). The wrinkling zone between the thin- and thick-plate limits is not reached by any of the curves corresponding to a pre-stress ($s > 0$), and so the uni-axially pre-stretched plate will not wrinkle in the direction of the load. Note that in the absence of charge ($D_0 = 0$), we again recover the purely elastic case, where $\lambda_2 = \lambda_1^{-1/2}$, and the critical stretch for uni-axial surface instability is the Biot value, $\lambda_1 = 0.444$ (Biot, 1963).

We also investigated wrinkles perpendicular to the direction of the load using the same method. There we looked for wrinkles in the $(x_2, x_3)$-plane, and constructed the Stroh formulation for the variables

$$\{ u_3, u_2, \varphi, T_{023}, T_{022}, D_{L02} \}. \quad (3.87)$$

We then found that the thin-plate condition is identical to the loading curve equation (3.76)$_2$ for $s = 0$, and that the thick-plate limit is

$$D_0^4 - (\lambda_1^2 \lambda_3^2 + 3 \lambda_1 \lambda_3^2 - 2) D_0^2 - (\lambda_1^2 \lambda_3^2 + \lambda_1 \lambda_3^2 + 3 \lambda_1 \lambda_3^2 - 1) = 0. \quad (3.88)$$
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Figure 3.5: Wrinkles are not expressed for uni-axially-loaded, charge-driven dielectric plates. The solid curves are the loading curves for the neo-Hookean dielectric (3.27) with pre-stress $s = \alpha mg/\mu A$, where $\alpha = 0.05, 0.3, 0.7, 1.0, 1.5$, and the other characteristics taken from the Keplinger et al. (2010) membrane ($m = 150$ g, $\mu = 9833.07$ Pa, $A = 50$ mm$^2$). The left-most dashed (blue) curve is the thick-plate limit curve (3.86) and the other dashed (black) curve is the thin-plate limit (3.85) curve, equivalent to the hypothetical no-weight curve ($s = 0$). The shaded region between the thick and thin-plate limits represents values of $D_0$ and $\lambda_1$ for which wrinkling could occur. Because the loading curves for the pre-stressed plate ($s > 0$) are all monotonic, they will not cross into the wrinkling region, provided the material is pre-stretched, and so wrinkling will not occur in the direction of the uni-axial load. The dots result from Finite Element computations, and follow the theoretical curves closely, although the clamping of the plate creates local, non-homogeneous fields. The main difference with the theoretical predictions is that the simulations eventually breakdown numerically, as indicated by red crosses.

On solving this condition together with (3.76)$_2$, no real solutions are found, and so there are no wrinkles perpendicular to the uni-axial load.

In the next section we see that the Hessian and geometric stabilities found from the homogeneous deformation fields can be contradicted by local inhomogeneous effects, as shown in numerical simulations.
3.5 Finite Element simulations

To complement the results of the theory, we developed electroelastic Finite Element (FE) models of the equi-biaxial and the uni-axial experiments using the commercial software COMSOL Multiphysics® (COMSOL, 2016), and coupled the elasticity and electrostatics in two different ways.

In the fully coupled model, COMSOL® uses the second Piola-Kirchhoff stress tensor, denoted $P$, and implements incompressibility via a volumetric energy function in the form $\kappa (\det F - 1)^2/2$, where $\kappa$ is the initial bulk modulus, taken to be orders of magnitude larger than the shear modulus.

The second way to solve the coupled problem is by considering the effect of the Maxwell stress tensor as a fictitious mechanical boundary condition in the purely elastic problem. Since there are no charges within the volume of a dielectric, it is possible to consider the Maxwell stress as a pressure applied on the external faces of the volume. This adds a boundary traction $\tau_m n$ to the mechanical problem, where $n$ is the outward normal to the deformed surface of the specimen and $\tau_m$ is the Maxwell stress tensor

$$\tau_m = E \otimes D - \frac{1}{2}(E \cdot D)I.$$  \hfill (3.89)

We found that both methods lead to the same results, although we noted that imposing the Maxwell stress tensor as a pressure boundary condition seemed to be a slightly more stable method in the uni-axial case.

In the equi-biaxial case, we found no difference between the predictions of the analytical model and those of the FE model, which also displayed stability and could be performed at any level of charge control, see Figure 3.3.

By contrast, a major difference between the analytical model and the FE model arises in the uni-axial case, because the simulations for the latter eventually breakdown. Note that this breakdown is not due to a failure of the Hessian criterion because, as we have seen previously, it is always stable in charge-controlled actuation.

We identified the reason for this numerical breakdown to be due to the boundary conditions in the areas close to the clamping playing an initially small but eventually significant role. In the analytical model the strain is homogeneous and the material is free to deform in the transverse $x_3$-direction. In the real-world experiments (Keplinger et al., 2010) and in the FE numerical model, the top
and bottom parts of the material are clamped and the strain is inhomogeneous in these neighbourhoods, see Figure 3.6(b). This behaviour is local, however, and the stretch in the direction of the uni-axial tension due to the weight (the $x_1$-direction) is almost completely homogeneous, as can be seen in Figure 3.6(a).

When the stretched plate is electrically activated it expands in area and its thickness reduces. While it is free to expand in the direction of the dead load (the $x_1$-direction), the situation in the transverse $x_3$-direction is different. There is a central zone where the influence of the clamping is weak, so that the normal stress component $P_{33}$ in the $x_3$-direction remains close to zero, as in the homogeneous case. On the other hand, the portions of material closer to the clamping areas suffer from the fixed displacement in the $x_3$-direction imposed by the clamps. There the application of the uni-axial tension due to the weight increases $P_{33}$, as is clearly visible in Figure 3.7.

When the plate is progressively activated with an increasing uniform charge distribution on its faces, the stress component $P_{33}$ near the clamping zone is progressively reduced until a critical value just below zero is reached: in this configuration, the plate undergoes a lateral compression that makes it buckle in the $x_3$-direction (De Tommasi et al., 2011). At that point the FE computation
Figure 3.7: Finite Element simulations of a charge-driven plate subject to a dead-load. The total second Piola-Kirchhoff stress lateral component $P_{33}$ (normalised with respect to $\mu$) along the centre line of the material in the $x_1$-direction (direction of uni-axial tension, in cm). The plate’s characteristics are the same as for Figure 3.6. The uppermost curve corresponds to the static dead-load condition ($D_0 = 0$); then an increasing charge activation is performed until the simulation reaches the instability point, where the stress is slightly compressive throughout (lowest curve) and the computation crashes. The colour coding for the levels of $P_{33}$ in the simulations goes from about 1 kPa in blue to about 0 kPa in dark orange. The stiffness matrix breaks down, presumably because the stiffness matrix stops being positive definite and the solver has trouble converging. This phenomenon does not occur in the analytical model and in the equi-biaxial case, as $\lambda_3$ is homogeneous then and $P_{33}$ is imposed from the boundary condition and is identically equal to zero everywhere.

For larger weights, the levels of the $P_{33}$ stress component before activation are higher, making it possible to activate the dielectric plate with a larger value of the electric charge before the instability condition is reached, as can be seen from the dots in Figure 3.5.

Despite the significant difference in the transverse behaviour between the analytical and numerical model, due to the different boundary condition imposed, there is very good agreement in the results, as can be seen in Figure 3.5. As long as the FE model stays below the point of negative $P_{33}$, the $D_0-\lambda$ curves follow those of the homogeneously deformed analytical model very closely.
3.6 Conclusion

In conclusion, we found that both equi-biaxial and uni-axial modes in the charge-control actuation of a dielectric plate are stable, whether the stability analysis is based on a Hessian criterion for the free energy of the whole system, or on the formation of small-amplitude inhomogeneous wrinkles.

By comparing the different Hessian criteria that result from the voltage- and charge-controlled situations, we found that charge-controlled actuation is always stable with respect to the Hessian criterion, in complete contrast to voltage-controlled actuation, which, according to the Hessian criterion, can become unstable.

We also investigated the possibility of small-amplitude wrinkles and found that the wrinkling conditions in the limiting cases of thin and thick plates occur only in compression, whereas it has been shown that wrinkles can exist in extension in the voltage-controlled case (Su et al., 2018a). As a result, charge-controlled actuation, which always occurs in extension, is also geometrically stable, again in contrast to voltage-control actuation.

To account for the difference between a theoretical homogeneous uni-axial deformation and the local inhomogeneous fields created by clamps in practice, we also conducted Finite Element simulations to verify our analytical results. We found complete agreement in the equi-biaxial case and very close agreement in the uni-axial case. So the assumption of homogeneous deformation is well justified for modelling the behaviour of charge-controlled activation of a dielectric plate in equi-biaxial stretch, and in uni-axial stretch when the aspect ratio of the specimen is high.

In the uni-axial case, Finite Element simulations reveal that the fringe effects are localised in a portion of area near the clamping zone and that they do not significantly affect the homogeneous loading curves of the system, although they have a strong effect on the eventual instability of the setup, a possibility that the homogeneous solution cannot capture. We may also argue that the emergence of compressive lateral stresses inside the plate seen in the simulations has a real-world counterpart, and that an equi-biaxial pre-stress leads to larger actuations than a uni-axial pre-stress in practice.

Of course, the plate may become unstable due to other causes than free en-
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energy instability or inhomogeneous small-amplitude wrinkles. Other mechanisms include for instance charge localisation (Lu et al., 2014) or thickness effects (Fu et al., 2018; Zurlo et al., 2017).

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Bibliography


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Stability analysis of charge-controlled soft dielectric plates


Chapter 4

Electro-elastic Lamb waves in dielectric plates

Hannah Conroy Broderick¹, Luis Dorfmann², Michel Destrade¹

¹ School of Mathematics, Statistics and Applied Mathematics, NUI Galway
² Department of Civil and Environmental Engineering, Tufts University

Abstract

We study the propagation of Lamb waves in soft dielectric plates subject to mechanical and electrical loadings and find the explicit expressions for the dispersion equations in the cases of neo-Hookean and Gent dielectrics. We elucidate the effects of the electric field, of the thickness-to-wavelength ratio, of pre-stress and of strain-stiffening on the wave characteristics.

4.1 Introduction

Wave propagation in soft dielectric materials has been shown to depend on both the underlying deformation and the applied electric field for a variety of deformations and geometries.

Dorfmann and Ogden (2010) derived the incremental formulation and showed the dependence of the surface wave velocity on the electric field in an electro-
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elastic half-space (voltage-controlled case). The case of a dielectric plate under plane strain was investigated by Shmuel et al. (2012), who showed the effects of pre-stretch and applied electric displacement on the wave velocity (charge-controlled case). Ziser and Shmuel (2017) investigated experimentally pre-stretched voltage-controlled films and found that the velocity of the fundamental flexural mode decreases with increasing voltage under plane strain. In a recent study, Wang et al. (2019) considered the vibration of multi-layered compressible plates and highlighted interesting phenomena such as frequency jumping in plates undergoing snap-through.

There has been significant work on cylinders and tubes (Chen and Dai, 2012; Dorfmann and Ogden, 2020; Shmuel and deBotton, 2013; Su et al., 2016; Wu et al., 2017), showing that the velocity of the wave depends on the electric field and the direction in which it is applied, as well as the direction of propagation relative to the underlying electro-mechanical deformation. In particular, Wu et al. (2017) highlighted the possibility of using wave propagation to detect defects or cracks in the material based on their analysis of circumferential waves.

The dependence of the wave velocity on the applied electric field suggests the possibility of controlling the velocity of the propagating wave by applying an appropriate electric field (Shmuel et al., 2012; Wang et al., 2019). It also paves the way for applying acoustic non-destructive evaluation techniques to dielectric plates.

Here, we first recall the equations governing the large deformation of a soft dielectric plate and then the subsequent propagation of small-amplitude Lamb waves, using incremental theory (Section 4.2). We write the equations of motion in the Stroh form, and then solve them for the neo-Hookean dielectric model (Section 4.3). We separate symmetric from antisymmetric modes of propagation to obtain explicit expressions for the corresponding dispersion equations, and for their limits in the long wavelength-thin plate and short wavelength-thick plate regimes. In Section 4.4, we solve the dispersion equations numerically to highlight the effects of loading and geometry on the wave propagation. We also use the Gent dielectric model to look at the effects of strain-stiffening and snap-through on the plate’s acoustics.
4.2 Equations of motion

We use the incremental theory of electro-elasticity (Dorfmann and Ogden, 2010) to analyse wave propagation in a finitely deformed electro-active plate, within the context of the electro-acoustic approximation (Ogden, 2009). For completeness of presentation we first summarise the main parts of the theory.

The incremental (small-amplitude) mechanical displacement is denoted $u$ and the increments of the total nominal stress and of the Lagrangian forms of the electric displacement and the electric field are denoted by $\dot{T}$, $\dot{D}_L$, $\dot{E}_L$, respectively. For an incompressible electro-elastic material the corresponding push-forward measures are obtained as

$$\dot{T}_0 = F \dot{T}, \quad \dot{D}_{L0} = F \dot{D}_L, \quad \dot{E}_{L0} = F^{-T} \dot{E}_L,$$

where $F$ is the deformation gradient. The incremental quantities satisfy the Lagrangian form of the governing equations

$$\text{Div} \dot{T} = \rho u_{tt}, \quad \text{Div} \dot{D}_L = 0, \quad \text{Curl} \dot{E}_L = 0,$$

and, equivalently, their updated (Eulerian) forms

$$\text{div} \dot{T}_0 = \rho u_{tt}, \quad \text{div} \dot{D}_{L0} = 0, \quad \text{curl} \dot{E}_{L0} = 0,$$

where $\rho$ is the (constant) mass density per unit volume and $,t$ denotes the time derivative.

With no external field and no applied mechanical traction the incremental boundary conditions have the simple forms

$$\dot{T}^T N = 0, \quad \dot{E}_L \times N = 0, \quad \dot{D}_L \cdot N = \dot{\sigma}_F,$$

where $\dot{\sigma}_F$ identifies an increment in the referential charge density on the electrodes attached to the major surfaces of the plate.

Superposed on the current configuration we consider an incremental motion, tracked by the incremental deformation gradient $\dot{F}$, combined with an increment in the electric field $\dot{E}_L$. This results in increments of the total nominal stress and of the Lagrangian electric displacement field as specified by the incremental forms of the constitutive equations (Dorfmann and Ogden, 2019b).
In particular, for an incompressible material we have \( \det F = 1 \) at all times. Then, \( \dot{T} \) and \( \dot{D}_L \) have the forms
\[
\dot{T} = \mathcal{A} \dot{F} + \mathcal{A}_0 \dot{E}_L - \dot{p} F^{-1} + p F^{-1} \dot{F} F^{-1}, \tag{4.5}
\]
and
\[
\dot{D}_L = -\mathcal{A}^T \dot{F} - \mathcal{A} \dot{E}_L, \tag{4.6}
\]
where \( \mathcal{A}, \mathcal{A}_0, \mathcal{A} \) are, respectively, fourth-, third- and second-order electro-elastic moduli tensors, and \( p \) is a Lagrange multiplier due to incompressibility (and \( \dot{p} \) is its increment). The use of (4.1) gives the updated forms of the incremental constitutive equations
\[
\dot{T}_0 = \mathcal{A}_0 L + \mathcal{A}_0 \dot{E}_L 0 + p L - \dot{p} I, \tag{4.7}
\]
and
\[
\dot{D}_{L0} = -\mathcal{A}_0^T L - \mathcal{A}_0 \dot{E}_{L0}, \tag{4.8}
\]
where \( I \) is the identity tensor and \( L \) the gradient of the Eulerian version of the incremental displacement vector \( u \), see Dorfmann and Ogden (2014b) for details. The latter satisfies the incremental incompressibility condition
\[
\text{tr } L = \text{div } u = 0. \tag{4.9}
\]
From (4.4) we find the updated incremental boundary conditions
\[
\dot{T}_0^T n = 0, \quad \dot{E}_{L0} \times n = 0, \quad \dot{D}_{L0} \cdot n = \dot{\sigma}_{F0}, \tag{4.10}
\]
where \( \dot{\sigma}_{F0} \) is the Eulerian form of the charge density increment.

In what follows we focus attention on isotropic and incompressible electro-elastic materials with properties dependent on just two invariants, denoted \( I_1 \) and \( I_5 \), and defined as
\[
I_1 = \text{tr } C, \quad I_5 = E_L \cdot C^{-1} E_L, \tag{4.11}
\]
where \( C = F^T F \) is the right Cauchy-Green deformation tensor.

We denote the lateral dimensions and the total thickness of an electroelastic plate in the undeformed configuration by \( 2L \) and \( 2H \), respectively. The deformed
configuration is defined using the Cartesian coordinates \((x_1, x_2, x_3)\) with \(x_2\) oriented normal to the major surfaces. We focus on equi-biaxial deformations with principal stretches
\[
\lambda_1 = \lambda_3 = \lambda, \quad \lambda_2 = \lambda^{-2}.
\] (4.12)

The deformed plate is then in the region
\[
-\ell \leq x_1 \leq \ell, \quad -h \leq x_2 \leq h, \quad -\ell \leq x_3 \leq \ell,
\] (4.13)
where \(2h = 2\lambda^{-2}H\) and \(2\ell = 2\lambda L\) are the plate’s dimensions in the deformed configuration.

In addition, a potential difference (voltage) is applied between the compliant electrodes attached at the top and bottom surfaces. The in-plane dimensions are much larger than the thickness and the edge effects can therefore be neglected. It follows that the accompanying electric and electric displacement fields have a single component each, denoted \(E_2\) and \(D_2\), respectively, along the normal to the major surfaces of the plate. Equation (4.1), specialised to the underlying configuration gives the Lagrangian counterparts as
\[
E_{L2} = \lambda^{-2}E_2, \quad D_{L2} = \lambda^2 D_2,
\] (4.14)
while the invariants (4.11) have the simple forms
\[
I_1 = 2\lambda^2 + \lambda^{-4}, \quad I_5 = \lambda^4 E_{L2}^2.
\] (4.15)

Superimposed on the underlying configuration we consider two-dimensional electro-elastic increments with components \(u_1, u_2, \dot{E}_{L01}, \dot{E}_{L02}\), which depend on \(x_1, x_2\) and \(t\) only.

Equation (4.3) suggests the introduction of a scalar function \(\varphi = \varphi(x_1, x_2, t)\) such that
\[
\dot{E}_{L01} = -\varphi_{,1}, \quad \dot{E}_{L02} = -\varphi_{,2},
\] (4.16)
where the subscripts 1 and 2 following a comma denote partial derivatives with respect to \(x_1\) and \(x_2\), respectively. Using (4.7) we find the non-zero incremental stress components \(\dot{T}_{011}, \dot{T}_{012}, \dot{T}_{021},\) and \(\dot{T}_{022}\) and the non-zero incremental electric displacement components \(\dot{D}_{L01}\) and \(\dot{D}_{L02}\) from (4.8) (Dorfmann and Ogden, 2010,
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2014b; Su et al., 2018). Equation (4.3) and (4.3) then specialise to

\[ \dot{T}_{011,1} + \dot{T}_{021,2} = \rho u_{1,tt}, \]
\[ \dot{T}_{012,1} + \dot{T}_{022,2} = \rho u_{2,tt}, \]  \hspace{1cm} (4.17)
\[ \dot{D}_{L01,1} + \dot{D}_{L02,2} = 0, \]  \hspace{1cm} (4.18)

together with the incompressibility condition (4.9)

\[ u_{1,1} + u_{2,2} = 0. \]  \hspace{1cm} (4.19)

It remains to specify the incremental boundary conditions (4.10) on the major surfaces \( x_2 = \pm h \), as

\[ \dot{T}_{021} = \dot{T}_{022} = 0, \quad \dot{E}_{L01} = 0, \]  \hspace{1cm} (4.20)

while the boundary condition (4.10) is not used, as we are considering a voltage-controlled plate.

Specifically, we seek solutions with sinusoidal dependence in the \( x_1 \) direction for the variables \( u_1, u_2, \dot{D}_{L02}, \dot{T}_{021}, \dot{T}_{022}, \varphi \) in the form

\[ \left\{ u_1, u_2, \dot{D}_{L02}, \dot{T}_{021}, \dot{T}_{022}, \varphi \right\} = \Re \left\{ [U_1, U_2, ik\Delta, ik\Sigma_{21}, ik\Sigma_{22}, \Phi] e^{ik(x_1-\nu t)} \right\}, \]  \hspace{1cm} (4.21)

where the constant \( k \) is the wave number, \( \nu \) the wave speed and the amplitude functions \( U_1, U_2, ik\Delta, ik\Sigma_{21}, ik\Sigma_{22}, \Phi \) are functions of \( kx_2 \) only.

In what follows, it is convenient to consider the variables \( U_1, U_2, \Delta \) as the components of a generalised displacement vector \( U \), and the variables \( \Sigma_{21}, \Sigma_{22}, \Phi \) as the components of a generalised traction vector \( S \). We find that the governing equations can then be arranged in the Stroh form

\[ \eta' = iN\eta, \]  \hspace{1cm} (4.22)

where \( \eta = (U, S)^T \) is the Stroh vector and the prime denotes differentiation with respect to \( kx_2 \). The \( 6 \times 6 \) matrix \( N \) has the form

\[ N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{bmatrix}, \]  \hspace{1cm} (4.23)

where the \( N_i \) are \( 3 \times 3 \) sub-matrices and \( N_2, N_3 \) are symmetric (Dorfmann and Ogden, 2019a; Shuvalov, 2000; Su et al., 2018).
Next, we introduce shorthand notations for the coefficients in (4.7) and (4.8). For an energy function $\Omega$ that depends linearly on the invariants $I_1$ and $I_5$, we use

\begin{align*}
a &= A_{01212} = 2(\lambda^2 \Omega_1 + \lambda^4 E_{L2}^2 \Omega_5), \\
c &= A_{0211} = 2\lambda^{-4} \Omega_1, \\
2b &= A_{0111} + A_{0222} - 2A_{0122} - 2A_{0121} \\
&= 4(\lambda^2 - \lambda^{-4}) \Omega_{11} + a + c, \\
d &= A_{0211} = -2\lambda^2 E_{L2} \Omega_5, \\
e &= A_{0222} - A_{0112} = 2d, \\
f &= A_{011} = A_{022} = 2\Omega_5,
\end{align*}

where $\Omega_j = \partial \Omega / \partial I_j$ and $\Omega_{ij} = \partial^2 \Omega / \partial I_i \partial I_j$ for $i, j = 1, 5$. We also recall the connections (Dorfmann and Ogden, 2019b)

\[ A_{0jkl} - A_{0ijkl} = (\tau_{jl} + p\delta_{jl}) \delta_{ik} - (\tau_{il} + p\delta_{il}) \delta_{jk}, \]

which for $\tau_{22} = 0$ results in $p = c$.

We introduce dimensionless versions of (4.24) as follows

\begin{align*}
\bar{a} &= a/\mu, \\
\bar{b} &= b/\mu, \\
\bar{c} &= c/\mu, \\
\bar{d} &= d/\sqrt{\mu \varepsilon}, \\
\bar{e} &= e/\sqrt{\mu \varepsilon}, \\
\bar{f} &= f/\varepsilon,
\end{align*}

where $\mu$ is the initial shear modulus associated with purely elastic deformations and $\varepsilon$ is the (constant) electric permittivity. Together with the dimensionless field measures

\begin{align*}
\bar{E}_0 &= E_{L2} \sqrt{\varepsilon/\mu}, \\
\bar{D}_0 &= D_{L2} / \sqrt{\mu \varepsilon},
\end{align*}

and the non-dimensional components of the Stroh vector $\eta$,

\begin{align*}
\bar{U}_i &= U_i, \\
\bar{\Sigma}_{2i} &= \Sigma_{2i} / \mu, \\
\bar{\Delta} &= \Delta / \sqrt{\mu \varepsilon}, \\
\bar{\Phi} &= \Phi \sqrt{\varepsilon / \mu},
\end{align*}

we arrive at a non-dimensional version of the Stroh formulation (4.22) (Su et al., 2018).

To derive the Stroh equations, we follow the procedure in Su et al. (2018). As the only change in the governing equations from Su et al. (2018) is to the
equilibrium equation for the stress, i.e. \((4.3)_1\) or equivalently \((4.17)\), the only entries that change are those related to \(\Sigma_{21}\) and \(\Sigma_{22}\). As a result, \(N_1\) and \(N_2\) are identical to the forms derived in Su et al. (2018), and \(N_3\) becomes

\[
N_3 = \begin{bmatrix}
\frac{\bar{e}^2}{f} - 2(\bar{b} + \bar{c}) + \bar{v}^2 & 0 & -\bar{e} \\
0 & \bar{c} - \bar{a} + \bar{v}^2 & 0 \\
-\bar{e} & 0 & \frac{1}{f}
\end{bmatrix}
\]

(4.28)

where \(\bar{v}^2 = \rho v^2/\mu\) is a non-dimensional version of \(v^2\).

### 4.3 Resolution for the neo-Hookean dielectric plate

We consider solutions of the form \(\eta = \eta_0 e^{-pkz^2}\), which reduce \((4.22)\) to the eigen-problem \(N\eta_0 = ip\eta_0\). Hence, the eigenvalues and eigenvectors \(p_j, \eta^{(j)}\), \(j = 1, \ldots, 6\) are determined by solving the characteristic equation

\[
det(N - ipI) = 0,
\]

(4.29)

where \(I\) is the \(6 \times 6\) identity matrix. Note that the form of the solution used here is equivalent to the one in Su et al. (2018) using \(p = iq\). It follows that the general solution for \(\eta\) has the form

\[
\eta = \sum_{j=1}^{6} c_j \eta^{(j)} e^{-p_j k z^2},
\]

(4.30)

where \(c_j, j = 1, \ldots, 6\) are constants to be determined by the boundary conditions \((4.20)\).

To illustrate the solution, we now specialise the constitutive model \(\Omega(I_1, I_5)\) to the neo-Hookean electroelastic form (Zhao and Suo, 2007)

\[
\Omega_{\text{NH}} = \frac{\mu}{2}(I_1 - 3) - \frac{\varepsilon}{2} I_5.
\]

(4.31)

Then we find, using a Computer Algebra System, that solving the characteristic equation \((4.29)\) results in

\[
p_1 = -p_4 = 1, \quad p_2 = -p_5 = 1, \\
p_3 = -p_6 = \lambda^2 \sqrt{\lambda^2 - \bar{v}^2},
\]

(4.32)
with corresponding eigenvectors

\[
\eta^{(1)} = \begin{bmatrix}
\lambda^4 \\
-i\lambda^4 \\
2\lambda^6 \bar{E}_0 \\
2i \\
\lambda^4 (\bar{v}^2 - \lambda^2 + \lambda^4 \bar{E}_0) - 1 \\
0
\end{bmatrix},
\]

\[
\eta^{(2)} = \begin{bmatrix}
i\lambda^6 \bar{E}_0 \\
-\lambda^6 \bar{E}_0 \\
i(\lambda^6 - \lambda^4 \bar{v}^2 + \lambda^8 \bar{E}_0^2 + 1) \\
\lambda^2 \bar{E}_0 (\lambda^6 - \lambda^4 \bar{v}^2 - \lambda^8 \bar{E}_0^2 - 1) \\
\lambda^4 (\bar{v}^2 - \lambda^2 + \lambda^4 \bar{E}_0^2) - 1
\end{bmatrix},
\]

\[
\eta^{(3)} = \begin{bmatrix}
i\lambda^6 \sqrt{\lambda^2 - \bar{v}^2} \\
-\lambda^4 \\
i\lambda^8 \bar{E}_0 \sqrt{\lambda^2 - \bar{v}^2} \\
\lambda^4 (\bar{v}^2 - \lambda^2 - \lambda^4 \bar{E}_0^2) - 1 \\
-2i\lambda^2 \sqrt{\lambda^2 - \bar{v}^2} \\
\lambda^6 \bar{E}_0
\end{bmatrix},
\]

and \(\eta^{(4)}\), \(\eta^{(5)}\), \(\eta^{(6)}\) are the respective complex conjugates of these three vectors.

The boundary conditions (4.20) on the surfaces \(x_2 = \pm h\), in terms of the generalised traction vector \(S\) require

\[
S(\pm kh) = 0.
\]

These boundary conditions constitute six homogeneous equations that are conveniently represented as a 6 × 6 matrix equation. For a non-trivial solution the determinant of the matrix must vanish. The resulting equation can be factorised to give two independent equations, which identify the configurations in which antisymmetric and symmetric propagating waves may occur (Dorfmann and Ogden, 2019a; Su et al., 2018). Specifically, for subsonic waves (\(v < \lambda \sqrt{\mu/\rho}\), so that \(\bar{v}^2 < \lambda^2\)), we find the following explicit dispersion equations,

\[
\frac{(\lambda^6 - \lambda^4 \bar{v}^2 + 1)^2 - \lambda^8(\lambda^6 - \lambda^4 \bar{v}^2 - 1) \bar{E}_0^2}{4\lambda^2 \sqrt{\lambda^2 - \bar{v}^2}} = \left[\frac{\tanh(kH \sqrt{\lambda^2 - \bar{v}^2})}{\tanh(kH \lambda^2)}\right]^\pm_1,
\]

(4.35)
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where the exponents ±1 correspond to antisymmetric and symmetric modes, respectively.

To evaluate the response in the short wavelength-thick plate and long wavelength-thin plate limits, we note that \( kH = 2\pi H/L \), where \( L \) denotes the wavelength. Therefore, for short wavelengths-thick plates (Rayleigh surface waves), \( kH \rightarrow \infty \) and equation (4.35) specialises to

\[
\frac{(\lambda^6 - \lambda^4\bar{v}^2 + 1)^2 - \lambda^8(\lambda^6 - \lambda^4\bar{v}^2 - 1)\bar{E}_0^2}{4\lambda^2\sqrt{\lambda^2 - \bar{v}^2}} = 1.
\]

(4.36)

In the long wavelength-thin plate limit, \( kH \rightarrow 0 \) and the antisymmetric mode simplifies to

\[
\lambda^8\bar{E}_0^2 - \lambda^4(\lambda^2 - \bar{v}^2) + 1 = 0,
\]

(4.37)

while symmetric incremental modes occur when

\[
\lambda^8\bar{E}_0^2 - \lambda^4(\lambda^2 - \bar{v}^2) - 3 = 0.
\]

(4.38)

As expected, the above equations recover the purely elastic case (Ogden and Roxburgh, 1993) when \( \bar{E}_0 = 0 \) and the static electro-elastic case (Su et al., 2018) when \( \bar{v} = 0 \).

When \( \bar{v}^2 > \lambda^2 \), using \( \tanh(ix) = i\tan(x) \), we obtain

\[
\frac{(\lambda^6 - \lambda^4\bar{v}^2 + 1)^2 - \lambda^8(\lambda^6 - \lambda^4\bar{v}^2 - 1)\bar{E}_0^2}{4\lambda^2\sqrt{\bar{v}^2 - \lambda^2}} = \mp \left[ \frac{\tan(kH\sqrt{\bar{v}^2 - \lambda^2})}{\tanh(kH\lambda^{-2})} \right]^{\pm1},
\]

(4.39)

where the upper and lower signs correspond to antisymmetric and symmetric modes, respectively. When \( kH \ll 1 \), Eq. (4.39) reduces to (4.37) for antisymmetric modes and to (4.38) for symmetric modes. When \( kH \rightarrow \infty \), the limit is indeterminate, as the limit of \( \tan(kH\sqrt{\bar{v}^2 - \lambda^2}) \) is then undefined. This is expected because supersonic Rayleigh waves do not exist here; as seen in Figure 4.4, \( \bar{v}^2 \) is always less than \( \lambda^2 \) for thick plates.

### 4.4 Numerical results and discussion

#### 4.4.1 Neo-Hookean dielectric plate

On specialising to the constitutive model (4.31) and using the boundary condition \( \tau_{22} = 0 \), the in-plane nominal stress \( T = \tau_{11}/\lambda = \tau_{33}/\lambda \) is obtained as

\[
T = \mu(\lambda - \lambda^{-5}) - \varepsilon \lambda^3 E_{12}^2,
\]

(4.40)
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Figure 4.1: The loading curve $\bar{E}_0$ against the in-plane stretch $\lambda$ for a neo-Hookean electroelastic plate with no pre-stress ($\bar{T} = 0.0$, upper curve) and with pre-stress $\bar{T} = 0.8$ (lower curve). The field $\bar{E}_0$ increases initially with increasing values of $\lambda$, obtains its maximum $\bar{E}_{\text{max}}$ (marked by ‘•’ and decreases afterwards. As the pre-stress increases, $\bar{E}_{\text{max}}$ decreases, see Su et al. (2018) for a detailed discussion.

or, equivalently

$$\bar{E}_0 = \sqrt{\lambda^{-2} - \lambda^{-8} - \lambda^{-3}\bar{T}}, \quad (4.41)$$

where $\bar{T} = T/\mu$ is a dimensionless measure of the nominal stress. Relation (4.41) is illustrated by the loading curve $\bar{E}_0$ against the in-plane stretch $\lambda$, with pre-stresses $\bar{T} = 0$ and $\bar{T} = 0.8$ in Figure 4.1. In the absence of pre-stress, the maximum value, $\bar{E}_{\text{max}} = \sqrt{3}/2^{4/3} \approx 0.69$, that the plate can support occurs at the critical stretch $\lambda = 2^{1/3} \approx 1.26$, as shown by Zhao and Suo (2007). Loading curves for increasing amounts of pre-stress show a continuous reduction in the value of $\bar{E}_{\text{max}}$, see for example Dorfmann and Ogden (2019b); Su et al. (2018); Zhao and Suo (2007).

We first consider a plate in the absence of pre-stress, i.e. we take $\bar{T} = 0.0$. To evaluate the effect of an electric field on the wave speeds of symmetric and antisymmetric modes, we must consider values of $\bar{E}_0$ and $\lambda$ on the loading curve (4.41). Results show that for increasing values of $\bar{E}_0$ the overall trend of the curves is maintained as illustrated in Figure 4.2. In particular, for $\bar{E}_0 = 0$ we recover the elastic results of an un-stretched plate (Ogden and Roxburgh, 1993).

A notable feature of the effect of $\bar{E}_0$ on $\bar{v}$ is the reduction of the speed of
Figure 4.2: The dimensionless Lamb wave speed $\bar{v} = v \sqrt{\rho/\mu}$ against $kH$ of antisymmetric and symmetric modes shown by solid and dashed curves, respectively, for a neo-Hookean electroelastic plate with $\bar{T} = 0$. (a) Purely elastic un-stretched case $\bar{E}_0 = 0, \lambda = 1$, (b) Moderate electrical loading $\bar{E}_0 = 0.4, \lambda \approx 1.03$, and (c) Maximal electrical loading $\bar{E}_0 = \bar{E}_{\text{max}} = \sqrt{3}/2^{4/3}, \lambda = 2^{1/3}$.
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Figure 4.3: The dimensionless wave speed $\bar{v} = v\sqrt{\rho/\mu}$ of the fundamental symmetric and antisymmetric modes against the stretch $\lambda$ in the long wavelength-thin plate limit ($kH \to 0$) for a neo-Hookean dielectric plate with no pre-stress ($\bar{T} = 0$) and with pre-stress $\bar{T} = 0.8$. The wave speed when $\bar{E}_0 = \bar{E}_{\text{max}}$, is marked by 'o' for each case. Beyond $\bar{E}_{\text{max}}$ the plate undergoes the pull-in instability, as indicated by the dashed portion of the curves.

The corresponding Rayleigh wave speed is obtained from (4.36) using (4.41) to express the stretch $\lambda$ in terms of the dimensionless field $\bar{E}_0$. This is illustrated in Figure 4.4 where $\bar{E}_{\text{max}}$ is again indicated by 'o'. For $\bar{E}_0 = 0$, we recover Lord Rayleigh’s result (Rayleigh, 1885) of the purely elastic surface wave, $\tilde{v}^2 = 0.9126$. As $\bar{E}_0$ increases with increasing $\lambda$, the Rayleigh wave speed increases as well until it reaches its maximum at $\bar{E}_0 \approx 0.6329, \lambda \approx 1.1251$, before $\bar{E}_{\text{max}}$ is reached. The speed then begins to decrease and past $\bar{E}_{\text{max}}$, when the electric field decreases at increased stretch, it decreases rapidly to zero.

These trends are in contrast to those of Shmuel et al. (2012), who found that both the Rayleigh wave speed and the thin-plate limit of the fundamental symmetric mode increase monotonically with increased electric displacement. However, they considered a charge-controlled dielectric plate under the influence
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Figure 4.4: The dimensionless squared Rayleigh wave speed \( \bar{v}^2 = \rho v^2/\mu \) against \( \bar{E}_0 \) for a neo-Hookean electroelastic plate with \( \bar{T} = 0.0 \) and \( \bar{T} = 0.8 \), and \( kH \to \infty \) (short wavelength-thick plate limit). The wave speed \( \bar{v}^2 \) when \( \bar{E}_0 = \bar{E}_{\text{max}} \) for each case is again marked by ‘●’.

of a plane strain deformation, and so we cannot perform a direct comparison. Instead we can compare their results with the case of a voltage-controlled plate under plane strain. In that case we also find that the fundamental symmetric mode in the long-wavelength limit decreases monotonically. We can again see this by substituting the loading curve into the thin-plate limit giving \( \bar{v} = 2\lambda^{-1} \). The Rayleigh wave speed follows a similar trend to the equi-biaxial case, increasing towards a maximum as the electric field increases. The speed then decreases asymptotically, corresponding to the asymptote of the loading curve.

4.4.2 Pre-stressed dielectric plate

We now investigate the effect of pre-stress on wave propagation in a neo-Hookean dielectric plate. We use a pre-stress of \( \bar{T} = 0.8 \) as a representative example. In that case the corresponding pre-stretch is \( \lambda \simeq 1.2 \) in the absence of an electric field, and the loading curve reaches its maximum at \( \bar{E}_0 = \bar{E}_{\text{max}} \simeq 0.4148 \), \( \lambda \simeq 1.5916 \), as seen in Figure 4.1.

We again consider values of \( \bar{E}_0 \) and \( \lambda \) along the loading curve (4.41), and plot the dispersion curves (4.35) when \( \bar{T} = 0.8 \) in Figure 4.5. When \( \bar{E}_0 = 0 \), the dispersion curves recover the corresponding curves in the elastic case (see, for
example, Ogden and Roxburgh (1993) for similar plots at different values of pre-stress). As before, as $\bar{E}_0$ increases towards $\bar{E}_{\text{max}}$, the overall trend of the $\bar{E}_0 = 0$ curve is maintained. The symmetric long-wave limit decreases and the thick-plate limit increases as $\bar{E}_0$ is increased towards $\bar{E}_{\text{max}}$, following similar trends to the case without pre-stress (Figs. 4.3, 4.4).

The major difference between the pre-stressed and non-pre-stressed cases is the behaviour in the long wavelength-thin plate regime: once a pre-stress is introduced, the speed of the antisymmetric modes in this limit become non-zero. As the stretch is increased, the speed of the fundamental antisymmetric mode in that limit also increases (Figure 4.3), which can be seen explicitly by substituting the loading curve (4.41) into Eq. (4.37), giving $\bar{v} = \sqrt{\lambda \bar{T}}$, a monotonically increasing function of $\lambda$ for any positive $\bar{T}$. We note that in the plane strain case, the antisymmetric long wavelength limit becomes $\bar{v} = \sqrt{\lambda^{-1} \bar{T}}$, a monotonically decreasing function for any positive $\bar{T}$, corresponding to the slowing found experimentally by Ziser and Shmuel (2017).

By substituting (4.41) into (4.38) we see that the limit for the symmetric modes becomes

$$\bar{v} = \sqrt{4\lambda^{-4} + \lambda \bar{T}},$$

(4.42)

in the pre-stressed case. As $\lambda$ increases beyond $\lambda \approx 1.5916$, the $\lambda \bar{T}$ term dominates and the speed in the long wavelength-thin plate limit begins to increase, unlike in the case without pre-stress where it decreases monotonically. As a result, for large $\lambda$ the symmetric and antisymmetric thin-plate/long wavelength limits converge to the same value ($\bar{v} = \sqrt{\lambda \bar{T}}$), as seen in Figure 4.3.

The short wavelength-thick plate limit ($kH \gg 1$) also deviates from the trend in the case with no pre-stress (Figure 4.4). Initially, the wave speed $\bar{v}^2$ increases as the electric field increases, until it reaches the maximum $\bar{E}_0 \approx 0.4148$, as before. At this point, the curve changes suddenly and the speed begins to increase, as the electric field decreases with increased stretch. At lower values of pre-stress (not shown here), the wave speed initially follows a similar trend as in Figure 4.4, and begins to decrease, after which it begins to increase, as in the $\bar{T} = 0.8$ case.
Figure 4.5: The dimensionless wave speed $\bar{v} = v\sqrt{\rho/\mu}$ against $kH$ of antisymmetric (solid) and symmetric (dashed) modes for a pre-stressed neo-Hookean electroelastic plate, with $\bar{T} = 0.8$. (a) The elastic case with $\bar{E}_0 = 0, \lambda \simeq 1.2$, (b) Moderate electric field $\bar{E}_0 = 0.4, \lambda \simeq 1.4387$, and (c) The maximal electric field $\bar{E}_0 \simeq 0.4148, \lambda \simeq 1.5916$. 

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4.4.3 Gent dielectric plate

In this section we investigate wave propagation characteristics in a plate made of an electro-elastic, strain-stiffening Gent dielectric, with energy function

$$\Omega_G = -\frac{\mu J_m}{2} \log \left[ 1 - \frac{(I_1 - 3)}{J_m} \right] - \frac{\varepsilon I_5}{2},$$  \hspace{1cm} (4.43)

where the invariants $I_1, I_5$ are given in (4.15) and $J_m$ is a material constant, capturing strain-stiffening effects at large stretches. Following the work of Dorfmann and Ogden (2014a, 2019b) we use the specific value $J_m = 97.2$, which is typical for soft rubber.

The in-plane nominal stress is now connected to the stretch $\lambda$ and the Lagrangian electric field by

$$T = \mu \frac{\lambda - \lambda^{-5}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m} - \varepsilon \lambda^3 E_{Ez}^2,$$  \hspace{1cm} (4.44)

leading to the following non-dimensional form of the loading curve

$$\bar{E}_0 = \sqrt{\frac{\lambda^{-2} - \lambda^{-8}}{1 - (2\lambda^2 + \lambda^{-4} - 3)/J_m}} - \lambda^{-3} \bar{T}.$$  \hspace{1cm} (4.45)

Figure 4.6 illustrates relation (4.45) for the special case when $\bar{T} = 0$, highlighting the snap-through instability. The field $\bar{E}_0$ increases with increasing $\lambda$ and reaches a local maximum, $\bar{E}_{\text{max}} \approx 0.689$ when $\lambda \approx 1.264$. The stretch then increases with constant electric field until it reaches $\lambda \approx 6.926$, after which the stretch and electric field increase together, i.e. the snap-through instability, see (Dorfmann and Ogden, 2019b; Su et al., 2018) for detailed discussions.

The explicit expression of the Stroh matrix $N$ for the Gent dielectric is obtained by following the steps identified in Section 4.3 with the derivatives in (4.24) evaluated using the energy function (4.43). Then, the solution of (4.29) gives the eigenvalues

$$p_1 = 1,$$  \hspace{1cm} (4.46)

$$p_{2,3} = \frac{1}{2} \sqrt{(1 + \kappa)^2 + 2(\lambda^4 - \lambda^{-2})^2 \frac{\Omega_{111}}{\Omega_1}} \pm \frac{1}{2} \sqrt{(1 - \kappa)^2 + 2(\lambda^4 - \lambda^{-2})^2 \frac{\Omega_{111}}{\Omega_1}},$$
Figure 4.6: The loading curve $\bar{E}_0$ against the in-plane stretch $\lambda$ for a Gent electro-elastic plate with $J_m = 97.2$ and $\bar{T} = 0$.

with

$$p_4 = -p_1, \quad p_5 = -p_2, \quad p_6 = -p_3,$$  \hfill (4.47)

where

$$\kappa = \sqrt{\lambda^6 - \frac{\lambda^4 v^2}{2\bar{\Omega}_1}},$$  \hfill (4.48)

and

$$\Omega_1 = \frac{1}{2} \frac{J_m}{3 + J_m - I_1}, \quad \Omega_{11} = \frac{1}{2} \frac{J_m}{(3 + J_m - I_1)^2},$$  \hfill (4.49)

are the (non-dimensional) derivatives of (4.43) with respect to $I_1$. In addition,
the solution of (4.29) gives the corresponding eigenvectors

\[
\eta^{(1)} = \begin{bmatrix}
0 \\
0 \\
-i \\
-\lambda^2 \bar{E}_0 \\
-i \lambda^2 \bar{E}_0 \\
1
\end{bmatrix},
\]

\[
\eta^{(2)} = \begin{bmatrix}
i \lambda^4 p_2 \\
-\lambda^4 \\
i \lambda^6 \bar{E}_0 p_2 \\
-(2\bar{\Omega}_1 (p_2^2 + 1) + \lambda^8 \bar{E}_0^2) \\
i (2\bar{\Omega}_1 (\lambda^6 + p_3^2) - \lambda^4 \bar{v}^2) / p_2 \\
\lambda^6 \bar{E}_0
\end{bmatrix},
\]

\[
\eta^{(3)} = \begin{bmatrix}
i \lambda^4 p_3 \\
-\lambda^4 \\
i \lambda^6 \bar{E}_0 p_3 \\
-(2\bar{\Omega}_1 (p_3^2 + 1) + \lambda^8 \bar{E}_0^2) \\
i (2\bar{\Omega}_1 (\lambda^6 + p_3^2) - \lambda^4 \bar{v}^2) / p_3 \\
\lambda^6 \bar{E}_0
\end{bmatrix},
\] (4.50)

with \(\eta^{(4)}, \eta^{(5)}, \eta^{(6)}\) their complex conjugates.

For antisymmetric wave modes and \(\bar{v}^2/(2\bar{\Omega}_1) < \lambda^2\) we find the following dispersion equation

\[
\kappa (p_2^2 - p_3^2) \lambda^8 \bar{E}_0^2 \tanh (\lambda^{-2} k H) + p_3 (p_2^2 + 1) \left[ 2\bar{\Omega}_1 (\lambda^6 + p_3^2) - \lambda^4 \bar{v}^2 \right] \tanh (\lambda^{-2} p_2 k H)
\]

\[
- p_2 (p_3^2 + 1) \left[ 2\bar{\Omega}_1 (\lambda^6 + p_3^2) - \lambda^4 \bar{v}^2 \right] \tanh (\lambda^{-2} p_3 k H) = 0, \quad (4.51)
\]

where the in-plane stretch \(\lambda\) and the field \(\bar{E}_0\) are connected by (4.45). The corresponding equation for symmetric modes is obtained from (4.51) by replacing the hyperbolic function \(\tanh\) with \(\coth\). Note that for \(\bar{v}^2/(2\bar{\Omega}_1) > \lambda^2\) the eigenvalue \(p_2\) is imaginary and, therefore, to obtain real solutions the dispersion equation for antisymmetric and symmetric modes must be multiplied by the imaginary unit \(i\).

Again the dispersion relation (4.51) recovers the electrostatic case when \(\bar{v} = 0\) and the elastic case when \(\bar{E}_0 = 0\).
When \( kH \gg 1 \) (Rayleigh waves), Equation (4.51) and its symmetric counterpart tend to

\[
\lambda^8 \kappa (p_2 + p_3) \bar{E}_0^2 + \lambda^4 \bar{v}^2 (p_2^2 + p_3^2 + 1 + \kappa) - 2 \left[ \lambda^6 (p_2 + p_3)^2 + (1 - \kappa)(\lambda^6 - \kappa) \right] \bar{\Omega}_1 = 0.
\]

(4.52)

The Rayleigh wave speed \( \bar{v}^2 \) in (4.52) is evaluated using (4.45) and depicted in Figure 4.7 against the field \( \bar{E}_0 \). For the purely elastic case with no pre-stress (\( \bar{E}_0 = 0, \lambda = 1 \)) we again recover the Rayleigh wave speed of a linear incompressible solid (Rayleigh, 1885), \( \bar{v}^2 = 0.9126 \) (see Destrade and Ogden (2005); Destrade and Scott (2004) for Rayleigh waves in a deformed, purely elastic, Gent material). Initially, the wave speed follows a similar trend as in the neo-Hookean case (Figure 4.4), but shortly after the snap-through point (\( \bar{E}_0 \simeq 0.689 \)) the speed begins to increase substantially with decreasing electric field, due to the exponential stiffening of the material with increasing stretch after the snap-through instability.

When the plate undergoes the snap-through instability, the Rayleigh wave speed is increased by more than a thousandfold, with a jump to the value \( \bar{v}^2 \simeq 1.045 \), and then increases further with the electric field. Wang et al. (2019) found a similar phenomenon, with natural frequency jumping occurring during snap-through. However, they considered a pre-stretched plate and a compressible energy density function, which results in a smaller increase in stretch due to the snap-through instability than when incompressibility is enforced and when there is no pre-stretch. For the Gent dielectric considered here, the increase in stretch is very large (about 600\%, see Figure 4.6), which explains the very large jump in speed (or frequency).

In the long wavelength-thin plate limit \( kH \to 0 \), the dispersion equation for antisymmetric incremental modes simplifies to

\[
\lambda^8 \bar{E}_0^2 - 2 \bar{\Omega}_1 (\lambda^6 - 1) + \lambda^4 \bar{v}^2 = 0,
\]

(4.53)

while the wave speed for symmetric modes is governed by

\[
\lambda^{12} \bar{E}_0^2 + \lambda^8 \bar{v}^2 - 4 \bar{\Omega}_{11} (\lambda^6 - 1)^2 - 2 \lambda^4 \bar{\Omega}_1 (\lambda^6 + 3) = 0.
\]

(4.54)

It is easy to check that Eq. (4.53) gives the trivial solution \( \bar{v} = 0 \) for \( \bar{E}_0 \) and \( \lambda \) taken on the loading curve (4.45), while Eq. (4.54) simplifies to

\[
8 \lambda^4 \bar{\Omega}_1 + 4 \bar{\Omega}_{11} (\lambda^6 - 1)^2 - \lambda^8 \bar{v}^2 = 0.
\]

(4.55)
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Figure 4.7: The dimensionless Rayleigh wave speed $\bar{v}^2$ against $\bar{E}_0$ for a Gent electro-elastic plate with no pre-stress ($\bar{T} = 0$) in the short wavelength-thick plate limit. (a) The jump in velocity due to snap-through. (b) The Rayleigh wave speed before the snap-through instability. The values of $\bar{v}^2$ corresponding to $\bar{E}_0 = \bar{E}_{\text{max}} = 0.689$ are denoted by $\bullet$.

The corresponding non-dimensionalised velocity $\bar{v}$ of the symmetric fundamental mode in the limit $kH \to 0$ against the in-plane stretch $\lambda$ is shown in Figure 4.8. As $\lambda$ increases towards the snap-through point ($\lambda \simeq 1.264$), the speed of the symmetric fundamental mode decreases, as in the neo-Hookean case (Figure 4.3). When $\lambda \simeq 1.9$, the speed begins to increase as the stretch increases, and continues to increase monotonically beyond the past snap-through point ($\lambda \simeq 6.926$). This
Figure 4.8: The dimensionless velocity $\bar{v}$ of the fundamental symmetric mode in the long wavelength-thin plate limit ($kH \to 0$) against the stretch $\lambda$ for a Gent electro-elastic plate. The wave speed marked by '•' occurs when $\lambda = 1.264$ and $\bar{E}_0 = \bar{E}_{\text{max}} = 0.689$.

effect is due to the stiffening after the snap-through instability.

To illustrate the effect of the snap-through instability on the wave velocity, we plot the dispersion relation (4.51) at the snap-through point ($\bar{E}_0 = \bar{E}_{\text{max}} \simeq 0.689$, $\lambda \simeq 1.264$) and the past snap-through point where $\lambda \simeq 6.926$ in Figure 4.9. At the snap-through point ($\lambda \simeq 1.264$), the curves follow the same trend as those of the neo-Hookean energy density function, see Figure 4.2(c). After the snap-through has taken place ($\lambda \simeq 6.926$), the velocity increases dramatically. The speeds of the symmetric modes in the $kH \ll 1$ and $kH \gg 1$ regimes have both increased, but the overall trend of the fundamental modes remains the same. The dramatic increase in the values of the speed is due to the very large stretch at the past snap-through point, where the plate is under large strain and is much stiffened.

The first antisymmetric and symmetric modes also converge to the short-wave limit much slower than in all other cases. This is again due to the large stretch, as the competition between $kH$ and $\lambda^{-2}$ is greater for large $\lambda$. 
Figure 4.9: The non-dimensionalised Lamb wave speed \( \bar{v} = v \sqrt{\rho/\mu} \) against \( kH \) of antisymmetric and symmetric modes shown by solid and dashed curves, respectively, for a Gent electroelastic plate with no pre-stress \( (\bar{T} = 0.0) \). (a) The response at the snap-through point where \( \bar{E}_{\text{max}} = 0.689 \) and \( \lambda = 1.264 \), and (b) The past snap-through point where \( \bar{E}_{\text{max}} = 0.689 \) and \( \lambda = 6.926 \), see Figure 4.6.

### 4.5 Concluding remarks

In this chapter, we investigated Lamb wave characteristics in the presence of an electric field generated by a potential difference between two flexible electrodes mounted on the major surfaces of a dielectric plate. The incremental governing equations and the electrical and mechanical boundary conditions are given in Stroh form and solved numerically for neo-Hookean and Gent dielectric models. The dispersion equation is factorised to give two independent equations,
which identify the configurations where antisymmetric and symmetric propagating waves may occur. Explicit expressions of the dispersion equations for the short wavelength-thick plate and long wavelength-thin plate limits are obtained. In particular, we investigated the effects of plate thickness, of the electric field, of an in-plane pre-stress and of strain-stiffening on the wave characteristics.

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Bibliography


Electro-elastic Lamb waves in dielectric plates


Chapter 5

Conclusions and Perspectives

In Chapter 2, we modelled wrinkles in dielectric plates via small-amplitude perturbations superposed on the surface of the plate. We derived the Stroh formalism and applied it to two-dimensional wrinkles oriented along the planar axis of a dielectric plate. Dispersion equations were derived and decoupled into symmetric and antisymmetric modes. The dispersion equations are consistent with the classic elastic results of Biot (1963) and Green and Zerna (1954), in the absence of an electric field. We found that for low-to-moderate electric fields, dielectric plates can wrinkle only in compression, whereas dielectric plates under high electric fields can wrinkle both in extension and compression.

The plates first wrinkle antisymmetrically and symmetric modes are never attained. The linearised model is therefore not appropriate to model symmetric bifurcation modes such as necking (Fu et al., 2018). By extending the model to include higher order perturbations, we may be able to predict symmetric modes.

We then considered a charge-controlled dielectric plate in Chapter 3. We found the charge-controlled Hessian criterion and showed that it is never met. We also investigated geometric stability and found that charge-controlled dielectric plates can only wrinkle in compression, in contrast to the voltage-controlled case of Chapter 2. As a result, charge-controlled actuation is more stable than voltage-controlled actuation, with respect to both the Hessian criterion and wrinkling. We verified our analytical results with Finite Element simulations, to account for the difference between the homogeneous deformation of the analytical model and the inhomogeneous deformation due to clamps in an experiment. The FE simulations have very close agreement, but eventually breakdown due to
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inhomogeneities in the deformation at the clamping site. So the homogeneous deformation assumption is only appropriate for materials with high aspect ratios.

Despite charge-controlled actuation being more stable than voltage-control in theory, it is more difficult to implement in practice. Charge-controlled actuation is typically achieved by spraying charges onto the surface of the plate, which is impractical for many applications. A possible solution to this is charge-localisation, as proposed by Lu et al. (2014). This form of actuation is more practical for applications, but may be susceptible to wrinkling.

Lastly, in Chapter 4, we investigated the propagation of electro-elastic Lamb waves in dielectric plates, by extending the Stroh formalism to the dynamic case. We derived the decoupled dispersion relations and showed that they were consistent with both the purely elastic case (Ogden and Roxburgh, 1993) and the electrostatic case (Chapter 2). We investigated the effect of the electric field, thickness, pre-stress and strain-stiffening on the wave propagation. The influence of these effects indicates that the wave velocity can be tuned by controlling each of these parameters. Further analysis of the interaction between pre-stress and strain-stiffening is needed, in particular how this interaction affects the frequency jumping phenomenon seen during snap-through (Wang et al., 2019).

One possible avenue of further research is viscoelasticity. Many dielectric elastomers, including the widely used VHB 4905/4910, exhibit viscoelastic behaviour (Hong, 2011; Zhang et al., 2014). Viscoelasticity effects the electromechanical instabilities in dielectric elastomers including pull-in, wrinkling and electrocreasing (Kollosche et al., 2015; Park and Nguyen, 2013; Wang et al., 2016). In particular, it can delay or eliminate instabilities under certain loading conditions (Wang et al., 2016) and cause transitions between wrinkling modes (Kollosche et al., 2015).

Wrinkles in dielectric elastomers may also propagate in more than one direction and interact to form complex patterns, see for example Godaba et al. (2019); Plante and Dubowsky (2006). This interaction can not be predicted using a two-dimensional model, a full three-dimensional model of wrinkling in dielectric elastomers is needed to completely capture this phenomenon. Theoretical models of patterning have been widely studied in elastic materials (e.g. Chen and Hutchinson (2004); Yin et al. (2018)), however there exist few similar models in electroelastic materials. Models for patterning in dielectric elastomers
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will benefit many applications, including dynamic patterning (Wang et al., 2012).

Bibliography


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