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Author(s)	Duddy, Conal; Piggins, Ashley
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A Foundation for Pareto Optimality*

Conal Duddy†and Ashley Piggins‡

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Abstract

Can an axiomatic justification be given for the requirement that society picks all and only Pareto optimal alternatives at each profile of individual preferences? Using the framework of fixed-agenda social choice theory, we present a characterization of the Pareto optimal social choice correspondence. We introduce a new independence condition, P-independence. When combined with three natural assumptions, P-independence leads to the conclusion that the social choice set and the Pareto optimal set are the same.

Keywords: Pareto optimal social choice correspondence; Fixed agenda; Oligarchy; P-oligarchy; P-independence

1 Introduction

The Pareto optimal social choice correspondence (POSCC) selects the Pareto optimal (or Pareto efficient) alternatives at each profile of individual preferences. It is a benchmark social choice correspondence. Any correspondence

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†Department of Economics, Cork University Business School, University College Cork, Cork, Ireland. Email: conal.duddy@ucc.ie

‡School of Business and Economics, and Whitaker Institute, National University of Ireland Galway, Ireland. Email: ashley.piggins@nuigalway.ie

that selects Pareto inefficient alternatives at any preference profile would surely be rejected on normative grounds. In the formal literature on voting rules, Pareto optimality is taken to be a basic axiom (Zwicker 2016). Economists view Pareto optimality as a necessary, but not sufficient, condition for social choice. However, does a formal argument exist to justify sufficiency (as well as necessity)? We provide such an argument, an axiomatic characterization of the POSCC.

The framework assumes a fixed set of at least three alternative social states X with the requirement that only X itself is available and not strict subsets of X.[1] There is a finite population of individuals and each individual has preferences over the alternatives in X. Individual preferences are represented by orderings, i.e. reflexive, transitive and complete binary relations. A list of individual preferences is called a profile. A social choice correspondence (SCC) is a correspondence mapping profiles into non-empty subsets of X. If the profile is p and the SCC is C, then $C(p)$ is the social choice set where $\emptyset \neq C(p) \subseteq X$. The POSCC selects the Pareto optimal alternatives at each profile. Since individual preferences are orderings, the POSCC can never output the empty set at any profile and so is a genuine SCC.

A counterpart of the POSCC in the traditional social choice framework is the Pareto extension rule, and this rule has been characterized by Sen (1969, 1970). The traditional framework considers collective choice rules rather than SCCs, i.e. functions mapping profiles of individual preferences into a single social preference relation. To establish a complete social preference relation, the Pareto extension rule follows the standard Pareto dominance relation but completes it by declaring all Pareto-incomparable pairs of alternatives to be socially indifferent. The social preference relation that emerges is complete but quasi-transitive, i.e. only the strict part of the social preference rela-

[1] This is known as the fixed-agenda approach to social choice. A pioneering early paper is Hansson (1969). A survey of the field is given by Deb (2011). Related papers that use the framework include, among others, Peris and Sánchez (2001) and Sánchez and Peris (2006).

tion is guaranteed to be transitive. Sen proves that if the co-domain of the collective choice rule is equal to the set of all reflexive, complete and quasi-transitive binary relations on X, then the Pareto extension rule is the unique rule satisfying the axioms of unrestricted domain, independence of irrelevant alternatives, strong Pareto and anonymity.[2]

Some intuition for Sen's result can be provided by Gibbard's (1969) oligarchy theorem.[3] Gibbard proves that for co-domains of the kind considered by Sen, if the collective choice rule satisfies the axioms of unrestricted domain, independence of irrelevant alternatives and a weakening of strong Pareto (weak Pareto), then there exists an oligarchy. A set of individuals G is an oligarchy if and only if (i) the unanimous strict preference of all members of G for x over y implies that society strictly prefers x to y and (ii) the strict preference of any member of G for x over y implies that society does not strictly prefer y to x. Property (i) says that the oligarchs are decisive when they act in concert with one another, and (ii) says that each oligarch has veto power. Under the assumption of completeness, property (ii) is equivalent to stating that society regards x to be at least as good as y whenever any member of G strictly prefers x to y. Therefore, if two members of G have opposing strict preferences over a pair of alternatives, then society is indifferent between them. By assuming anonymity, Sen is making the whole society the oligarchy. From this, a simple strengthening of weak Pareto to strong Pareto quickly establishes the characterization.

The characterization presented here complements Sen's but deals with SCCs rather than collective choice rules.[4] On the face of it, a proof strategy similar to the one described above should work for SCCs as oligarchy results

[2]Weymark (1984) axiomatically characterizes the Pareto rule under the assumption that social preferences are quasi-orderings, i.e. reflexive, transitive, but not necessarily complete, binary relations. The Pareto rule is different from the Pareto extension rule in that Pareto-incomparable alternatives are not ranked by society.

[3]Gibbard's original manuscript has been published as Gibbard (2014a, 2014b).

[4]Campbell and Nagahisa (1994) provide a foundation for Pareto aggregation in a model of classical welfare economics.

for SCCs have been known since Denicolò (1987, 1993). However, the definition of oligarchy in this framework is not sufficiently strong enough to allow us to characterize the POSCC. In the framework of SCCs, the definition of oligarchy mirrors the one above. The counterpart of property (i) says that the unanimous strict preference of all members of G for x over y implies that y is not in the social choice set, and the counterpart of property (ii) says that the strict preference of any member of G for x over y implies that the social choice set cannot be a singleton containing y.

Consider the following SCC. First, at every profile, we identify for each individual the set of alternatives that are maximal for that individual. These are the alternatives that, intuitively, are at the "top" of an individual's preference ordering. Next, take the union of these maximal sets of alternatives across all of the individuals. Finally, eliminate any alternatives in this set that are Pareto dominated by any of the other alternatives in the set. This determines the social choice set at the original profile. This SCC is oligarchical (as defined above), anonymous and satisfies strong Pareto optimality (as defined for SCCs), but it is not the POSCC. Suppose that there are three alternatives x, y and z, and two individuals. Individual 1 strictly prefers x to y and y to z, and individual 2 strictly prefers z to y and y to x. The maximal-alternatives SCC, C^{MA}, has $C^{MA}(p) = \{x, z\}$ whereas the POSCC, C^{PO}, has $C^{PO}(p) = \{x, y, z\}$.

To achieve a characterization of the POSCC, the original definition of oligarchy for SCCs needs to be strengthened by adding an additional property. We call this property (property (iii)), *forcing*. To explain this condition, consider alternative y in the previous example. One of the oligarchs (individual 1) strictly prefers y to z and another (individual 2) strictly prefers y to x. Therefore, we can find among the oligarchs, a strict preference for y over every other alternative. The forcing property says that if this is the case, then the social choice set must contain y. Under the property, an individual oligarch can force an alternative into the social choice set by placing it uniquely at the top of their preference ordering. Note that C^{MA} violates forcing whereas C^{PO} does not.

To accomplish the characterization, we introduce a new fixed-agenda independence condition, P-independence (P refers to Pareto). This is intermediate in logical strength between two other conditions in the literature, Denicolò's (1985, 1987, 1993) independence and weak independence conditions. When combined with weak Pareto optimality for SCCs, independence leads to dictatorial social choice correspondences and weak independence leads to the original kind of oligarchical correspondences. We show that P-independence and weak Pareto optimality lead to the stronger kind of oligarchical correspondences. Further, if an SCC satisfies P-independence, strong Pareto optimality (a strengthening of weak Pareto optimality), anonymity, and an axiom dealing with the case in which everyone is indifferent between a pair of alternatives, then the SCC must be the POSCC. It is straightforward to see that the POSCC satisfies these axioms.

Of all these axioms, only the first and last are unfamiliar. The last axiom says that if everyone is indifferent between x and y, then either both x and y are in the social choice set, or neither are. If we think of an SCC as partitioning X into two parts, a "winning" part $(C(p))$ and a "losing" part $(X - C(p))$, the axiom requires that a pair of alternatives over which everyone is indifferent must be in the same part of the partition. This axiom is present for technical reasons.

The key axiom is P-independence. The difference between P-independence and the other independence conditions in the literature can be explained intuitively using the idea of "blocking".[5] We denote alternatives by x, y, preference profiles by p, p' and the restriction of p to the pair $\{x, y\}$ by $p\{x, y\}$. We say that y blocks x at p if and only if: for all profiles p' (including p), if $p\{x, y\} = p'\{x, y\}$ then $x \notin C(p')$. If y blocks x at p then x is not in the social choice set at p and, in addition, x is not in the social choice set at ev-

[5]We thank Bill Zwicker for this suggestion.

ery profile in which each individual's pairwise ranking of x and y is the same as at p. A social choice correspondence C satisfies Denicolò's independence condition if and only if: $x \notin C(p)$ implies that *each* $y \in C(p)$ blocks x at p.[6] In contrast, a social choice correspondence satisfies P-independence if and only if: $x \notin C(p)$ implies that *some* alternative $y \in C(p)$ blocks x at p. Clearly P-independence is logically weaker than independence.

A social choice correspondence C satisfies Denicolò's weak independence condition if and only if: $x \notin C(p)$ and $\{y\} = C(p)$ implies that y blocks x at p. P-independence is logically stronger than weak independence and the two concepts coincide when the social choice correspondence always selects a single alternative at each preference profile.

In Duddy and Piggins (2019) the following S-independence condition is also proposed. A social choice correspondence satisfies S-independence if and only if: $x \notin C(p)$ implies that some alternative y (not necessarily in $C(p)$) blocks x at p. S-independence is weaker than P-independence and is logically independent of weak independence. Duddy and Piggins (2019) characterize a family of social choice correspondences called S-correspondences using this property.

To the best of our knowledge, this paper provides the first characterization of the POSCC using the framework of fixed-agenda social choice theory. This is surprising given the central role Pareto optimality plays in economics and the practical importance of the fixed agenda model.

The following section provides our basic definitions. Section 3 contains the proof of the characterization theorem.

[6] This independence condition is violated by the POSCC, see Campbell and Kelly (2002, pp.77-78).

2 Preliminaries

There is a finite set of alternatives X containing at least three elements, that is, $|X| \geq 3$ where $|X|$ denotes the cardinality of X. The population is $N = \{1, ..., n\}$ with $n \in \mathbb{N} - \{1\}$, where $\mathbb{N}$ denotes the set of natural numbers (excluding zero). Let $R \subseteq X \times X$ be a (binary) relation. We write xRy instead of $(x, y) \in R$ and $\neg xRy$ instead of $(x, y) \notin R$. The primitive R is a weak preference relation and xRy means that x is at least as good as y. The asymmetric factor P of R is defined by, for all $x, y \in X$, xPy if and only if [xRy and $\neg yRx$]. The symmetric factor I of R is defined by, for all $x, y \in X$, xIy if and only if [xRy and yRx]. P and I are interpreted as the strict preference relation and the indifference relation corresponding to R.

A relation R is reflexive if and only if, for all $x \in X$, xRx. R is complete if and only if, for all $x, y \in X$, such that $x \neq y$, xRy or yRx. R is transitive if and only if, for all $x, y, z \in X$, [xRy and yRz] implies xRz.

An ordering is a reflexive, complete and transitive relation. The set of all orderings on X is denoted by $\mathcal{R}$ and its $|N|$-fold Cartesian product is $\mathcal{R}^{|N|}$. A preference profile is an $|N|$-tuple $(R_1, ..., R_{|N|}) \in \mathcal{R}^{|N|}$. We write p for $(R_1, ..., R_{|N|})$ and p' for $(R'_1, ..., R'_{|N|})$ and so on. For convenience, we write $p\{x, y\} = p'\{x, y\}$ to mean that profiles p and p' are identical when restricted to alternatives x and y. Formally, $p\{x, y\} = p'\{x, y\}$ means that, for every individual $i \in N$, [xR_iy if and only if xR'_iy] and [yR_ix if and only if yR'_ix].

A social choice correspondence (SCC) on X is a correspondence C which to any profile $p \in \mathcal{R}^{|N|}$ assigns a non-empty subset of X, $C(p)$. Since $C(p)$ is defined for any profile $p \in \mathcal{R}^{|N|}$ we are assuming an unrestricted domain.

We present axioms on social choice correspondences. The following formalize the intuition that Pareto inferior alternatives should not be selected. Weak Pareto optimality appears in Denicolò (1985, 1993).

Weak Pareto optimality. For all $x, y \in X$ and for all $p \in \mathcal{R}^{|N|}$, if xP_iy

for all $i \in N$ then $y \notin C(p)$.

Strong Pareto optimality. For all $x, y \in X$ and for all $p \in \mathcal{R}^{|N|}$, if $[xR_iy$ for all $i \in N$ and $\exists j \in N$ such that $xP_jy]$ then $y \notin C(p)$.

Strong Pareto optimality implies weak Pareto optimality. Strong Pareto optimality says that if everyone weakly prefers x to y and someone strictly prefers x to y, then y is not in the social choice set. This condition is the counterpart to part (ii) of Weymark's (1984, p.238) definition of the strong Pareto principle which says that if everyone weakly prefers x to y and someone strictly prefers x to y, then society strictly prefers x to y. Weymark's condition is stated in the relational framework, whereas ours is stated in the choice correspondence framework.

However, we need an additional axiom to play the role of counterpart to part (i) of Weymark's definition. Weymark's part (i) says that if everyone is indifferent between x and y, then society is indifferent between x and y. Our counterpart to this axiom in the choice correspondence framework is called "twinning".

Twinning. For all $x, y \in X$ and for all $p \in \mathcal{R}^{|N|}$, if xI_iy for all $i \in N$ then $[x \in C(p)$ and $y \in C(p)]$ or $[x \notin C(p)$ and $y \notin C(p)]$.

Twinning says that if everyone is indifferent between x and y, then either both x and y are in the social choice set, or else neither are. The SCC must treat x and y as "twins" and not separate them. As will be seen, twinning plays a technical role in the proof and seems a natural translation of Weymark's indifference condition.

The statements of the following independence conditions are taken from Denicolò (1993).

Weak independence. For all $x, y \in X$ and for all $p, p' \in \mathcal{R}^{|N|}$, the following implication holds: if $\{x\} = C(p)$ and $p\{x, y\} = p'\{x, y\}$, then $y \notin C(p')$.

Independence. For all $x, y \in X$ and for all $p, p' \in \mathcal{R}^{|N|}$, the following implication holds: if $x \in C(p)$, $y \notin C(p)$, and $p\{x, y\} = p'\{x, y\}$, then $y \notin C(p')$.

Independence implies weak independence.

An undesirable property of a social choice correspondence is dictatorship.

Dictatorship. There exists $d \in N$ such that, for all $x, y \in X$ and for all $p \in \mathcal{R}^{|N|}$, if xP_dy then $y \notin C(p)$.

Denicolò (1985 Theorem 1, 1993 Theorem 3) proves that if a social choice correspondence satisfies weak Pareto optimality and independence then it is dictatorial. Dictatorship can be weakened to oligarchy, and the following definition of oligarchy is from Denicolò (1987, 1993).

Oligarchy. (i) There exists $L \subseteq N$ such that, for all $x, y \in X$ and for all $p \in \mathcal{R}^{|N|}$, if xP_iy for all $i \in L$ then $y \notin C(p)$. (ii) Further, for all $i \in L$ and for all $p \in \mathcal{R}^{|N|}$, if xP_iy then $\{y\} \neq C(p)$.

Property (i) is the statement that the oligarchs are decisive and property (ii) is the statement that each oligarch exercises veto power. Denicolò (1987 Theorem 6, 1993 Theorem 4) proves that if a social choice correspondence satisfies weak Pareto optimality and weak independence, then it is oligarchic.

Our stronger definition of oligarchy retains (i) and (ii), and adds the requirement that the oligarchs can force an alternative into the social choice set. We call this stronger oligarchy a P-oligarchy.

***P*-oligarchy**. Properties (i) and (ii) from the definition of oligarchy still apply. (iii) Further, for all $x \in X$ and for all $p \in \mathcal{R}^{|N|}$, if for every $y \in X - \{x\}$ there exists some $i \in L$ such that xP_iy then $x \in C(p)$.

Under weak independence (the weakest independence condition considered here) property (iii) implies property (ii) but the converse does not hold. To see that (iii) implies (ii), assume that (iii) holds but (ii) does not. If (ii) is false then $\exists p \in \mathcal{R}^{|N|}$ and $\exists i \in L$ such that xP_iy and $\{y\} = C(p)$. By weak independence, for all $p' \in \mathcal{R}^{|N|}$ such that $p'\{x,y\} = p\{x,y\}$ we must have $x \notin C(p')$. However, consider a profile p' in which individual i has x uniquely at the top of his or her ordering, and in which every other individual keeps their $\{x,y\}$ pairwise ranking the same as at p. By (iii) we have $x \in C(p')$ which contradicts the requirement that $x \notin C(p')$.

We introduce the following independence condition.

***P*-independence**. For all $x \in X$ and for all $p \in \mathcal{R}^{|N|}$, if $x \notin C(p)$ then $\exists y \in C(p)$ such that, for all $p' \in \mathcal{R}^{|N|}$ with $p\{x,y\} = p'\{x,y\}$, we have $x \notin C(p')$.

As noted earlier, P-independence is intermediate in logical strength between independence and weak independence. The POSCC satisfies P-independence. If an alternative, say x, is not in the social choice set, then it must be Pareto dominated by some other alternative, say y. If y itself is not in the social choice set, then it must be Pareto dominated by z, which, by the transitivity of individual preferences, must also Pareto dominate x. Therefore, something in the social choice set will Pareto dominate x. At every profile where each individual's pairwise ranking of x and this dominating alternative remains the same, x is still Pareto dominated and, hence, x will remain unchosen.

The P-independence condition can be justified by combining a revealed preference argument with one based on independence of irrelevant alternatives. Suppose we accept that $x \notin C(p)$ implies there exists $y \in C(p)$ such that yPx, where P is the (strict) social preference relation at p. Further, at any profile p, if there exists $y \in X$ such that yPx then $x \notin C(p)$. This is the revealed preference argument. Second, if yPx and $p'\{x,y\} = p\{x,y\}$ then $yP'x$. This is the argument based on independence of irrelevant alternatives. P-independence can be derived from these two arguments.[7]

A common social choice axiom is anonymity.

[7]As a referee points out, the first part of this revealed preference argument bears some similarity to the argument for Axiom 3 in Bandyopadhyay and Sengupta (1991, p.207).

Anonymity. For all bijections $\rho : N \to N$ and for all $p, p' \in \mathcal{R}^{|N|}$, if $P_i = P'_{\rho(i)}$ for all $i \in N$ then $C(p) = C(p')$.

Anonymity says that swapping individual preferences has no effect on the social choice set. This reflects the idea that what matters for social choice is preferences and not who holds them.

Finally, an SCC C^{PO} is the POSCC if and only if, for all $p \in \mathcal{R}^{|N|}$, $C^{PO}(p) = \{x \in X \mid \nexists y \in X \text{ such that } [yR_ix \text{ for all } i \in N \text{ and } \exists j \in N \text{ such that } yP_jx]\}$.

3 A characterization

We say that a set of individuals $V \subseteq N$ is decisive over some pair of alternatives $\{x, y\}$ if it is the case that, for all $p \in \mathcal{R}^{|N|}$, xP_iy for all $i \in V$ implies that $y \notin C(p)$. Lemma 1 is based on Sen's (2017, p. 332) Spread of Decisiveness lemma which, in the original, refers to collective choice rules and not social choice correspondences.

Lemma 1. *For any SCC satisfying P-independence and weak Pareto optimality, if there exists a set of individuals who are decisive over a pair of alternatives $\{x, y\}$, then this set of individuals is decisive over all pairs of alternatives.*

Proof. Take any other pair $\{a, b\}$ and assume that x, y, a, b are distinct alternatives.[8] Assume that everyone in V strictly prefers a to x, x to y, and y to b. Assume that everyone not in V strictly prefers a to x, and y to b. For everyone in society, these four alternatives are strictly preferred to all of the other alternatives in $X - \{x, y, a, b\}$. By weak Pareto optimality,

Bandyopadhyay and Sengupta use their axiom to characterize a quasi-transitive rational choice function. Note, however, that Bandyopadhyay and Sengupta are considering a model where the agenda of alternatives can vary whereas we treat this as fixed.

[8] The reasoning is similar when two of these alternatives are the same.

$C(p) \subseteq \{a, y\}$, and the decisiveness of V implies that $y \notin C(p)$. Therefore, $\{a\} = C(p)$. Note that only the individuals in V have had their preferences over $\{a, b\}$ specified, and so by P-independence V is decisive over $\{a, b\}$. $\square$

Lemma 2 proves that for every social choice correspondence satisfying P-independence and weak Pareto optimality there is a P-oligarchy.

Lemma 2. *For any SCC satisfying P-independence and weak Pareto optimality, there is a P-oligarchy.*

Proof. Consider the set D of all decisive sets of individuals. By weak Pareto optimality this set must contain N and so is non-empty. Note that $\emptyset \notin D$ since this would contradict weak Pareto optimality. Since N is finite, there exists a set (or sets) in D containing the fewest individuals. Suppose that there exists more than one such set, and so we can label any two of them G and G'. If these two sets are disjoint, and if those in G strictly prefer x to every other alternative and those in G' strictly prefer every alternative to x, then the decisiveness of G implies that $\{x\} = C(p)$ and the decisiveness of G' implies $x \notin C(p)$. This is a contradiction.

If they have at least one member in common, let those in $G - \{G \cap G'\}$ strictly prefer x to y and x to z (leaving the y vs. z ranking unspecified), those in $G \cap G'$ strictly prefer z to x and x to y, and those in $G' - \{G \cap G'\}$ strictly prefer y to x and z to x (leaving the y vs. z ranking unspecified). All individuals in N rank every other alternative in $X - \{x, y, z\}$ as strictly inferior to x, y and z. Weak Pareto optimality implies that $C(p) \subseteq \{x, y, z\}$. The decisiveness of G implies that $y \notin C(p)$ and the decisiveness of G' implies that $x \notin C(p)$. It follows that $\{z\} = C(p)$. Note, however, that only the individuals in $G \cap G'$ have had their preferences over $\{z, y\}$ specified, and so P-independence implies that $G \cap G'$ is decisive over $\{z, y\}$ and, by Lemma 1, $G \cap G'$ is globally decisive. This contradicts the minimality of G and G'. Therefore, the smallest decisive set, which we label G^*, must be unique.

We know that part (i) of the definition of oligarchy applies to this group G^*. But it remains for us to prove that part (iii) of the definition of P-oligarchy also applies to G^* (we have already seen that part (iii) implies part (ii) and so do not need to prove part (ii)). Suppose that, at some profile p', for every alternative that is distinct from x there is at least one individual in G^* who strictly prefers x to that alternative. We need to prove that $x \in C(p')$. If G^* contains only one individual and the property is satisfied, then this follows directly from part (i) of the definition of oligarchy. So assume that $|G^*| > 1$. Let us suppose, by way of contradiction, that $x \notin C(p')$. By P-independence $\exists y \in C(p')$ such that, for all $p \in \mathcal{R}^{|N|}$ with $p\{x, y\} = p'\{x, y\}$, we have $x \notin C(p)$. In other words, y is the alternative that blocks x at p'. By assumption, we have $xP'_i y$ for some individual $i \in G^*$. Let G_i denote the set of all individuals in G^* who strictly prefer x to y at p'. Given that G^* is the smallest *decisive* set, there must exist a non-empty set $G_j = G^* - G_i$ such that either $yI'_j x$ or $yP'_j x$ for each $j \in G_j$. If G_j were empty, then $G^* = G_i$ and by the decisiveness of G^* we would have $y \notin C(p')$, a contradiction.

Now consider a profile where $zP^*{}_i y$, $xP^*{}_i y$ for all $i \in G_i$ (leaving the ranking of z vs. x for each individual in G_i unspecified), and $zP^*_j x$, $zP^*_j y$ for all $j \in G_j$ with each individual in G_j holding the same preference over x and y that he or she held at profile p'. Again, at this profile, all individuals (both those in G^* and those in $N - G^*$) strictly prefer x, y and z to every other alternative in $X - \{x, y, z\}$. Weak Pareto optimality implies that $C(p^*) \subseteq \{x, y, z\}$. Decisiveness of G^* implies that $y \notin C(p^*)$ and P-independence implies that $x \notin C(p^*)$. Therefore, $\{z\} = C(p^*)$. However, only the individuals in $G_j \subset G^*$ have had their preferences over $\{z, x\}$ specified. By P-independence they are, therefore, decisive over $\{z, x\}$ and by Lemma 1, globally decisive. This contradicts the minimality of G^*. □

We can now state our characterization theorem.

Theorem 3. *An SCC satisfies strong Pareto optimality, P-independence,* *anonymity and twinning if and only if it is the POSCC.*

Proof. It is straightforward to see that the POSCC satisfies the axioms. To prove the converse, note that an SCC satisfying strong Pareto optimality satisfies weak Pareto optimality. Lemma 2 implies that there is a P-oligarchy G^*. By anonymity, this oligarchy is N.

If $x \in C(p)$ and $x \notin C^{PO}(p)$ then we have a contradiction with the assumption of strong Pareto optimality. Suppose that $x \notin C(p)$ and $x \in C^{PO}(p)$. By P-independence, $\exists y \in C(p)$ such that, for all $p' \in \mathcal{R}^{|N|}$ with $p\{x,y\} = p'\{x,y\}$, we have $x \notin C(p')$. This y blocks x at p. Note, however, that since $x \in C^{PO}(p)$ it cannot be the case that $[yR_ix$ for all $i \in N$ and $\exists j \in N$ such that $yP_jx]$. Since individual preferences are orderings, then either (a) $\exists j \in N$ such that xP_jy, or (b) xI_iy for all $i \in N$. If (a) then the forcing power of j implies that y cannot block x at p, a contradiction. If (b) then the fact that $x \notin C(p)$ and $y \in C(p)$ violates twinning. □

The axioms used in the characterization are logically independent. The SCC $C(p) = X$ for all $p \in \mathcal{R}^{|N|}$ satisfies all of the axioms except strong Pareto optimality. Consider a dictatorial SCC in which the social choice set is always equal to the alternatives at the top of the dictator's ordering, provided that they are not Pareto dominated. This SCC satisfies all of the axioms except anonymity. The well-known Borda SCC (Suzumura 2002, p.3) satisfies all of the axioms except P-independence. Finally, consider the following SCC. Take two alternatives, x and y. The correspondence chooses all of the Pareto optimal alternatives at each profile. However, at a profile in which everyone is indifferent between alternatives x and y *and* both x and y are Pareto optimal, then x is not included in the social choice set (but y is). This SCC satisfies all of the axioms except twinning.

To conclude, we present a simple proof of Denicolò's original dictatorship result based on the analysis above. Of course, this is a version of Arrow's (1951) theorem for social choice correspondences when there is a fixed agenda of social states.[9]

Theorem 4. *For any SCC satisfying independence and weak Pareto optimality there is a dictator.*

Proof. Lemma 2 still holds as independence implies P-independence. Let G^* be the P-oligarchy and assume that $|G^*| > 1$. Partition G^* into two non-empty parts, G_i and G_j. Let those in G_i hold the preferences xP_izP_iy and those in G_j hold the preferences yP_jxP_jz. Assume that all individuals (both those in G^* and those in $N - G^*$) strictly prefer x, y and z to all other alternatives in $X - \{x, y, z\}$. Weak Pareto optimality implies that $C(p) \subseteq \{x, y, z\}$. Decisiveness of G implies that $z \notin C(p)$. The veto-power of those in G_i implies that $\{y\} \neq C(p)$ and similarly for G_j we have $\{x\} \neq C(p)$. Therefore, $\{x, y\} = C(p)$.

Consider now a profile p' in which those in G_i hold the preferences $zP'_ixP'_iy$ and those in G_j hold the preferences $yP'_jzP'_jx$, again with those three alternatives being strictly preferred to all others. Assume that all other individuals hold exactly the same preferences in p' as in p. As before, weak Pareto optimality implies that $C(p') \subseteq \{x, y, z\}$. Independence implies that $z \notin C(p')$. Veto-power implies that $\{y\} \neq C(p')$ and $\{x\} \neq C(p')$. Therefore, $\{x, y\} = C(p')$. However, this contradicts the decisiveness of G^* since everyone in G^* strictly prefers z to x. Therefore, the assumption that $|G^*| > 1$ is false and so there must be one P-oligarch. A dictator is a P-oligarchy containing just one individual. □

4 Conclusion

We have characterized the POSCC using the axioms strong Pareto optimality, P-independence, anonymity and twinning. The key condition is P-independence. P-independence is a logical strengthening of Denicolò's (1987,

[9] Theorem 4 follows the proof of Arrow's theorem in Piggins (2017).

1993) weak independence condition, but a weakening of his original independence condition (Denicolò 1985). The reason for the strengthening is that any SCC satisfying weak Pareto optimality and weak independence is oligarchical. However, this definition of oligarchy is too weak to allow us to characterize the POSCC which is a natural oligarchical correspondence. If an SCC satisfies both weak Pareto optimality and P-independence, then a stronger form of oligarchy exists that we call a P-oligarchy.

Under a P-oligarchy the oligarchs can force an alternative x into the social choice set if, for every other alternative, at least one oligarch strictly prefers x to that alternative. An implication of this is that an individual oligarch can force x into the social choice set by placing it uniquely at the top of their preference ordering. Once we assume anonymity, all of the individuals in society have this power. This gives some intuition for our result. If x is Pareto optimal at profile p, but is not among the chosen alternatives, then, by P-independence, something that is chosen, say y, must block x at p. However, since x is Pareto optimal, we can expect some individual to strictly prefer x to y at p. But this individual's forcing power means that y cannot block x at p, which establishes a contradiction.

We conclude with some directions for future research.[10] First of all, as we have seen, P-independence is useful for characterizing the POSCC. But is it useful in any other social choice context? Secondly, we have assumed an unrestricted domain of preferences but in economic environments preferences are often restricted.[11] Do the fixed-agenda impossibility theorems still hold on restricted domains? A natural place to start would be to consider single-peaked preferences. This might lead to a fixed-agenda characterization of the SCC that selects the Condorcet winner, for example.

[10]These were suggested to us by a referee.

[11]See Kalai, Muller and Satterthwaite (1979) and the survey by Le Breton and Weymark (2011).

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