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Hochschild (co)homology of Two Families of Complete Intersections

Ph.D. thesis

by

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Declaration

This thesis is presented in fulfillment of the requirements for the degree of Doctor of Philosophy at the School of Mathematics, Statistics and Applied Mathematics, National University of Ireland at Galway, Ireland. I declare that the thesis is all my own work under supervision of Dr. Emil Sköldberg and Dr. Alexander D. Rahm, and that I have not obtained a degree in this University or elsewhere on the basis of any of this work. Where use has been made of the work of other people it has been fully acknowledged and referenced.

Tran Thi Hieu Nghia
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September 2019

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List of Symbols

k	field with unity 1
$\mathrm{char}(k)$	characteristic of the field k
$k[x_1, x_2, \ldots, x_n]$	polynomial ring over the field k
R	commutative ring
$\mathbb{Z}$	the set of all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{N}$	the set of all non-negative integers $\{0, 1, 2, \ldots\}$
$[n]$	the set of all integers from 1 to n $\{1, 2, \ldots, n\}$
$\oplus$	direct sum
$\otimes$	tensor product
$\times$	Cartesian product
$\lvert S \rvert$	cardinality of the set S
$\langle S \rangle$	the ideal generated by the set S
A^{op}	the opposite algebra of A
A^e	$A \otimes A^{\mathrm{op}}$ the enveloping algebra of A
$A^{\otimes n}$	n-fold tensor product $A \otimes \cdots \otimes A$
id_M	the identity map from M to M
$\left(\begin{array}{c} n \\ k \end{array} \right)$	n choose k, $\dfrac{n!}{k!(n-k)!}$
$\gcd(f, g)$	the greatest common divisor of f and g
$\mathrm{lcm}(f, g)$	the least common multiple of f and g
$f \circ g$	the composition of the maps f and g
$\mathrm{Ker}(f)$, $\mathrm{Im}(f)$	the kernel, the image of f respectively

Abstract

The thesis presents the original results on a description of the ring structure in terms of generators and relations of the Hochschild cohomology of the two families of complete intersections: the square-free monomial complete intersections and the numerical semigroup algebras of embedding dimension two. In particular, we use the alternative resolution given by Jorge Guccione and Juan Guccione to describe the Hochschild cohomology. Then we describe the Hochschild cohomology modules via sub-complexes of the Hochschild complex which reduces the computations into smaller and simpler complexes. In the next stage, the cup product is described in terms of the Yoneda product. For more details, we provide an explicit formula of the multiplication on these module structures. Finally, we give a description of the ring structures of the algebras in terms of generators and relations and computed the Hilbert series of these algebras. Based on the ideas for the cohomology version, we give some conjectures on the ring structure of the Hochschild homology of the square-free monomial complete intersections.

Introduction

The theory of cohomology of associative algebras over a field was first introduced by Gerhard Hochschild (1945) [1] and then extended to algebras over more general rings by Henri Cartan and Samuel Eilenberg (1956) [2]. The Hochschild cohomology groups have been investigated for many different classes of algebras. Eilenberg and MacLane defined the cup products for Hochschild cohomology. The Hochschild cohomology of an algebra records substantial information about algebra and has a very rich structure: it is an associative, graded-commutative algebra with respect to the cup product, and it also has a graded Lie bracket of cohomological degree -1; therefore, it has a Gerstenhaber algebra structure [3]. Much progress has been made in describing the ring structure of the former. The explicit computation of these structures is usually very tricky due to the complexity of the multiplicative structure. We can see this in the calculations of the Hochchild cohomology structures of some families of algebras investigated in the works of Cibils and Solotar [4, 5], Holm [6], Siegel and Witherspoon [7], Erdmann and Hellstrøm-Finnsen [8], Chouhy, Herscovich and Solotar [9], etc.

This thesis was initially inspired by the work of Holm [10] in 2000. Holm considered the Hochschild cohomology ring of the k-algebra $k[X]/\langle f\rangle$ where f is a monic element of the polynomial ring $k[X]$ in a single variable and k is a commutative ring. He provided a full description in terms of generators and relations of the ring structure of the Hochschild cohomology of the algebra $A = k[X]/\langle f\rangle$ based on the periodic resolution of A, see [11], as an A^e-module. The contributions of Holm led us to think about the Hochschild cohomology structure of complete intersections with more variables and more generators. In this thesis we provide a concrete method to describe the Hochschild cohomology of the two different complete intersections: the square-free monomial complete intersections and the numerical semigroup algebras of embedding dimension two, see [12, 13]. We use the alternative resolution given by Jorge Guccione and Juan Guccione [14] in place of the bar resolution used by Hochschild and Eilenberg-MacLane to describe the Hochschild cohomology. Denote the algebra that we are considering by A, we interpret the alternative resolution $\mathbf{F}$ for A and then apply the contravariant functor $\mathrm{Hom}_{A^e}(-, A)$ to get a complex that gives rise to the Hochschild cohomology module of the algebra A. We also call this complex the Hochschild complex. For each of the two classes, we give a description of the Hochschild cohomology modules via sub-complexes of the Hochschild complex which reduce the computations into smaller and simpler complexes. In the next stage, we describe the cup product in terms of the Yoneda product. For more details, we provide an explicit formula for the chain map between the shifted resolution of $\mathbf{F}$ and the resolution itself and then infer the formula of the multiplication on these module structures. This multiplication gives the Hochschild module an algebra structure. Finally, we give a description of the ring structures of the algebras in terms of generators and relations and compute the Hilbert series of these algebras. Based on the ideas for the cohomology version, we work out on some conjectures on the Hochschild homology of the square-free monomial complete intersections in the last chapter.

The organisation of the thesis is as follows. Chapter 1 recalls the background material which is necessary throughout the thesis. We also provide some examples and discussions to illustrate some points that we want to highlight in order to support the arguments in the later chapters. Chapter 2 presents the results on computations of the Hochschild cohomology ring of the square-free monomial complete intersections. Chapter 3 presents the results on computations of the Hochschild cohomology ring of the numerical semigroup algebras of embedding dimension two. Finally, Chapter 4 gives some early results and conjectures on the homology version of the Hochschild algebras. In addition, we include in this thesis an appendix of the Macaulay2 code on computing the illustrative examples.

Chapter 1

Preliminaries

1.1 Overview

This chapter presents some essentials in homological algebra which lay the foundation for the Hochschild theory. We also give a quick review of definitions, properties and examples of our central objects, the Hochschild (co)homology of associative algebras. Finally, we include in this chapter the description of the supplementary mathematical tools which are necessary to obtain the results of the thesis. As general references for this chapter we suggest the books of Cartan and Eilenberg [15], MacLane [16], Weibel [17] and Rotman [18].

1.2 Some background on Homological Algebra

For a general background, let us begin by recalling some material on associative algebras, complexes, homology and chain maps. We fix a field k with unity 1 and denote $\otimes := \otimes_k$, the tensor products taken over k, unless otherwise stated. We will also assume the basic knowledge of abelian groups, vector spaces, modules, rings and homomorphisms.

1.2.1 Associative algebras, Bimodules, Complete intersections

Definition 1.1. A set A is said to be an *associative algebra over* k if it has the structure of a k-vector space and a ring in which the multiplication $\mu : A \times A \to A$ (denoted by $\mu(a,b) = ab$ for all $a, b \in A$) is compatible with the scalar multiplication $\nu : k \times A \to A$ (denoted by $\nu(\lambda, a) = \lambda a$ for all $\lambda \in k$, $a \in A$) as follows:

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

for all $a, b \in A$ and $\lambda \in k$.

If, in addition, there is an element 1 such that for all $a \in A$, $1a = a1 = a$, then we say that A is an algebra with identity. We will use the term k-algebra to refer to an associative algebra over k with identity.

Definition 1.2. Let A be a ring. A set M is called an A-*bimodule* if M is both a left and a right A-module, and the two scalar multiplications are related by an associative law:

$$a(mb) = (am)b$$

for all $a, b \in A$ and $m \in M$.

Since the definition of bimodule says that the two possible associations agree, we can write amb with no parentheses if M is an A-bimodule.

Definition 1.3. Let A be a k-algebra with multiplication $\mu : A \times A \to A$. The *opposite algebra* of A, denoted by A^{op}, is exactly A as k-module, but the multiplication $\mu^{\mathrm{op}} : A^{\mathrm{op}} \times A^{\mathrm{op}} \to A^{\mathrm{op}}$ is the opposite of that in A, that is, $\mu^{\mathrm{op}}(a,b) = \mu(b,a)$ for all a, b in A.

For elements a in A^{op}, we write "$a \in A$" where convenient instead of "$a \in A^{\mathrm{op}}$" since the underlying vector spaces are the same. The main feature of the opposite algebra is that a left A^{op}-module M is the same thing as a right A-module via the multiplication $a \cdot m = ma$ for all $a \in A$ and $m \in M$. Similarly, a right A^{op}-module M is the same thing as a left A-module via the multiplication $m \cdot a = am$.

Definition 1.4. Let A be a k-algebra. The *enveloping algebra* of A is $A^e := A \otimes A^{\mathrm{op}}$ where the multiplication is given by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$.

The main feature of the enveloping algebra is that an A-bimodule M can be considered as a left A^e-module via the scalar multiplication

$$(a \otimes b) \cdot m = amb$$

for all $a, b \in A$ and $m \in M$. Also, it is equivalent to a right A^e-module where we define $m \cdot (a \otimes b) = bma$. For simplicity, when we refer to a module we mean a left module unless indicated otherwise.

Let us linger on the following point. Since A can itself be seen as an A-bimodule, we have A as an A^e-module in terms of the above meaning. More generally, the n-fold tensor product $A^{\otimes n} := \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}$ is also an A^e-module with the scalar multiplication given by

$$(a \otimes b) \cdot (c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

Definition 1.5. Let A be a ring. We say that A is an $\mathbb{N}$-*graded ring* (or simply a graded ring without the prefix $\mathbb{N}$) if A is a direct sum of abelian groups $A = \bigoplus_{i \in \mathbb{N}} A_i$ such that $A_i A_j \subseteq A_{i+j}$ for all i, j in $\mathbb{N}$.

Definition 1.6. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}$-graded ring and M an A-module. We say that M is an $\mathbb{N}$-graded module if M is a direct sum of subgroups, $M = \bigoplus_{i \in \mathbb{N}} M_i$, such that $A_i M_j \subseteq M_{i+j}$ for all i, j in $\mathbb{N}$.

A graded k-module that is also a graded ring is called a *graded k-algebra*.

We say that an algebra is *finite dimensional* if the underlying vector space is finite dimensional.

Definition 1.7. Let R be a commutative ring and M an R-module. An element r in R is called a *non-zero divisor* on M if $rm = 0$ implies $m = 0$ for m in M. A sequence $r_1, r_2, \ldots, r_n$ in R is called an *M-regular sequence* if r_i is a non-zero divisor on $M/(r_1, r_2, \ldots, r_{i-1})M$ for all $i = 1, 2, \ldots, n$. An R-regular sequence is called simply a *regular sequence*.

Definition 1.8. A Noetherian local ring R is called a *complete intersection* if its completion is the factor ring of a regular local ring by a regular sequence.

Example 1.9. By the definition, the algebras $k[x_1, x_2, \ldots, x_n]/\langle x_1 x_2 \cdots x_n \rangle$ and $k[s^a, s^b] \cong k[x_1, x_2]/\langle x_1^a - x_2^b \rangle$ where $\gcd(a, b) = 1$ are complete intersections.

In the rest of this subsection, we will focus on a brief review of homological theory.

1.2.2 Complexes, Chain maps, Homotopies, Resolutions

Let R be a ring with multiplicative identity 1. We will take $R = A^e$ or $R = k$ in this thesis.

Definition 1.10. A *chain complex* $(\mathcal{C}_\bullet, d)$ (simply, $\mathcal{C}_\bullet$) over R is a sequence of R-modules and R-homomorphisms (called *differentials*),

$$\mathcal{C}_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

with the composite of adjacent maps being 0: $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. We generally abbreviate $d = d_n$. For each n, elements in the kernel of d_n are called *n-cycles*, elements in the image of d_{n+1} are called *n-boundaries* and the module $\mathrm{H}_n(\mathcal{C}_\bullet) := \dfrac{\mathrm{Ker}(d_n)}{\mathrm{Im}(d_{n+1})}$ is called the *n-th homology* of $\mathcal{C}_\bullet$.

Definition 1.11. A *cochain complex* over R is an analogous sequence

$$\mathcal{C}^\bullet : \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d_n} C^{n+1} \longrightarrow \cdots,$$

where $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$. For each n, the kernel of d^n, written $\mathrm{Ker}(d^n)$, consists of the *n-cocycles*, the image of d^{n-1}, written $\mathrm{Im}(d^{n-1})$, consists of the *n-coboundaries* and the module $\mathrm{H}^n(\mathcal{C}^\bullet) := \dfrac{\mathrm{Ker}(d^n)}{\mathrm{Im}(d^{n-1})}$ is the *n-th cohomology* of the cochain complex $\mathcal{C}^\bullet$.

In practice, we usually require chain complexes to satisfy $C_n = 0$ for $n < 0$ and cochain complexes to satisfy $C^n = 0$ for $n < 0$. Without these conditions, the notions are equivalent. We can leave off the subscript in $\mathcal{C}_\bullet$

and the superscript in $\mathcal{C}^\bullet$, writing $\mathcal{C}$ instead, when it is clear from the context that this notation refers to the whole complex. The following definitions are only given for chain complexes. There are corresponding dual definitions for cochain complexes. In the thesis we use the term *complex* to refer a chain complex unless explicitly stated otherwise.

If a complex can be expressed as a direct sum of sub-complexes, it may reduce the complexity of homology computations by considering the sub-complexes since they are smaller than the original complex.

Definition 1.12. A complex $(\mathcal{A}, \delta)$ is defined to be a *subcomplex* of a complex $(\mathcal{C}, d)$ if A_n is a sub-module of C_n and δ_n is exactly d_n restricted on $A_n \subseteq C_n$ for every $n \in \mathbb{Z}$.

Remark 1.13. If $(\mathcal{C}^{(i)}, d^{(i)})_{i\in I}$ is a family of complexes, then their direct sum is the complex

$$\bigoplus_{i\in I} \mathcal{C}^{(i)} : \cdots \longrightarrow \bigoplus_{i\in I} C^{(i)}_{n+1} \xrightarrow{\bigoplus_{i\in I} d^{(i)}_{n+1}} \bigoplus_{i\in I} C^{(i)}_{n} \xrightarrow{\bigoplus_{i\in I} d^{(i)}_{n}} \bigoplus_{i\in I} C^{(i)}_{n-1} \longrightarrow \cdots,$$

where $\bigoplus\limits_{i\in I} d^{(i)}_n$ acts coordinatewise.
From this, one gets that $\mathrm{H}_n(\bigoplus\limits_{i\in I} \mathcal{C}^{(i)}) \cong \bigoplus\limits_{i\in I} \mathrm{H}_n(\mathcal{C}^{(i)})$.

Definition 1.14. The sequence of R-modules and R-homomorphisms

$$\mathcal{C}_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

is *exact at* C_n if we have the equality $\mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$. We call it an *exact sequence* if it is exact at every C_n, $n \in \mathbb{Z}$.

Every exact sequence is a complex since the equalities $\mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$ imply that $d_n \circ d_{n+1} = 0$. In that case, the homology of the complex vanishes at all n.

Now we define morphisms between chain complexes.

Definition 1.15. Let $(\mathcal{C}, d)$ and $(\mathcal{C}', d')$ be complexes over R. A *chain map* $f : \mathcal{C} \to \mathcal{C}'$ consists of R-module homomorphisms $f_n : C_n \to C'_n$ such that for each n $f_{n-1} \circ d_n = d'_n \circ f_n$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

It can be shown that the chain map $f : \mathcal{C} \to \mathcal{C}'$ induces a map on homology $\mathrm{H}_n(f) : \mathrm{H}_n(\mathcal{C}) \to \mathrm{H}_n(\mathcal{C}')$. Two chain maps $f, g : \mathcal{C} \to \mathcal{C}'$ are *homotopic* (denoted by $f \simeq g$) if there exists a *chain homotopy* s consisting of homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$s_{n-1} \circ d_n + d'_{n+1} \circ s_n = f_n - g_n.$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Chain homotopy is an equivalence relation and chain homotopic maps induce the same homomorphism on (co)homology groups. A chain map $f : \mathcal{C} \to \mathcal{C}'$ is *null-homotopic* if $f \simeq 0$, where 0 is the zero map. A complex $\mathcal{C}$ has a *contracting homotopy* if its identity $\mathrm{id}_{\mathcal{C}}$ is null-homotopic. In that case, $\mathcal{C}$ is exact.

Definition 1.16. A *projective resolution* of a module M is an exact sequence

$$\mathcal{P} : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0,$$

in which each P_i is projective. The complex obtained by deleting M in the above sequence

$$\mathcal{P}_M : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

is called the *truncated projective resolution* of M.

We note that $\mathrm{H}_0(\mathcal{P}_M) \cong M$ and the truncated projective resolution is no longer exact at P_0 unless $M = 0$. If each P_i is a free module, the exact sequence $\mathcal{P}$ is called a *free resolution.* Every module has a free resolution and all free modules are projective. Thus, a module always has a projective resolution. There are many kinds of resolution of a module whose names are based on the features of the component modules in the sequence. In this thesis, we only work with the free resolutions (hence, projective resolutions). So we use the term *resolution* to refer this kind of resolution.

The following theorem is a fundamental result in homological algebra, which implies that the properties of a given module does not depend on the choice of projective resolution.

Theorem 1.17 (Comparison Theorem)**.** *Let M and N be R-modules and $f : M \to N$ an R-module homomorphism.*

$$\begin{tikzcd}
\mathcal{P}: & \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\tilde{f}_2"] & P_1 \arrow[r, "d_i"] \arrow[d, dashed, "\tilde{f}_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed, "\tilde{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\mathcal{Q}: & \cdots \arrow[r] & Q_2 \arrow[r, "d'_2"'] & Q_1 \arrow[r, "d'_1"'] & Q_0 \arrow[r, "\epsilon'"'] & N \arrow[r] & 0,
\end{tikzcd}$$

where the rows are complexes. If each P_i in the top row is projective for each i and the bottom row is exact, then there exists a chain map $\tilde{f} : \mathcal{P}_M \to \mathcal{Q}_N$ (dashed arrows) for which $f \circ \epsilon = \epsilon' \circ \tilde{f}_0$. This chain map is unique up to chain homotopy. We call $\tilde{f}$ a lifting map of f.

Proof. See for instance page 341 in [18]. □

If $\mathcal{P}$, $\mathcal{Q}$ are two projective resolutions of a module M, then the theorem states that there is a chain map $\tilde{f} : \mathcal{P}_M \to \mathcal{Q}_M$ lifting the identity map on M.

We now set up some notations on the essential functor Hom in order to assist the later arguments on cohomology. Let M and N be R-modules over a commutative ring R. We denote $\mathrm{Hom}_R(M, N)$ the set of all R-homomorphisms from M to N. This set forms an abelian group, where the additive identity is the zero map. Moreover, it has a structure of an R-module. Let ${}_R\mathbf{Mod}$ be the category of left modules over R. We define the covariant functor:

$$\begin{aligned} \mathrm{Hom}_R(M, -) :_R \mathbf{Mod} \to_R \mathbf{Mod} \\ N \mapsto \mathrm{Hom}_R(M, N) \end{aligned}$$

and the contravariant functor:

$$\begin{aligned} \mathrm{Hom}_R(-, N) :_R \mathbf{Mod} \to_R \mathbf{Mod} \\ M \mapsto \mathrm{Hom}_R(M, N). \end{aligned}$$

The corresponding definition for the category of right modules is obtained analogously.

Remark 1.18. (i) Let M, N, L be R-modules and $\delta : N \to L$ an R-homomorphism. Then we have a natural morphism

$$\begin{aligned} \delta_* = \mathrm{Hom}_R(M, \delta) : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, L) \\ f \mapsto \delta_*(f) := \delta \circ f. \end{aligned}$$

Similarly, there exists the following morphism corresponding to the R-homomorphism $\partial : M \to N$

$$\begin{aligned}\partial^* = \operatorname{Hom}_R(\partial, L) : \operatorname{Hom}_R(N, L) &\to \operatorname{Hom}_R(M, L)\\ f &\mapsto \partial^*(f) := f \circ \partial.\end{aligned}$$

(ii) Let km and $A^e m$ be free k-modules and A^e-modules respectively generated by the same element m. Then we have the isomorphism:

$$\operatorname{Hom}_{A^e}(A^e m, A) \cong \operatorname{Hom}_k(km, A).$$

The result still holds for any finitely generated free modules.

1.3 Definitions of Hochschild (co)homology

Now we introduce the historical definitions of Hochschild homology and cohomology of algebras, which are based on the construction of the bar complex. More details of definitions can be found in the books of Sarah Witherspoon [19] and Jean-Louis Loday [20].

1.3.1 Bar complex

Let A be a k-algebra. We consider the following sequence:

$$\mathcal{B} : \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \longrightarrow 0,$$

where the components $A^{\otimes n}$ are A^e-modules as mentioned before, the map μ is the multiplication ($\mu(a \otimes b) = ab$) and the maps d_n are given by

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

This is a complex of A^e-modules and A^e-homomorphisms. Moreover, it is exact and a contracting homotopy is given by

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}.$$

The truncation of the above sequence, that is,

$$\mathcal{B}_A : \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \longrightarrow 0 \tag{1.1}$$

is called the *bar complex* of the A^e-module A. In the case that k is a field, (1.1) is a free left A^e-module resolution of A, called the *bar resolution.*

Let K_n be the subspace of $A^{\otimes(n+2)}$ spanned by all elements $1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$ where at least one of the a_i's is in k. We can show that K_n is an A^e-submodule of $A^{\otimes(n+2)}$ and the sequence

$$\cdots \longrightarrow K_2 \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \longrightarrow 0$$

is a subcomplex of (1.1). By this, we have a variant of the bar resolution, which we call the *reduced bar resolution*:

$$\bar{\mathcal{B}}_A : \qquad \cdots \longrightarrow A \otimes \bar{A}^{\otimes 2} \otimes A \longrightarrow A \otimes \bar{A} \otimes A \longrightarrow A \otimes A \longrightarrow 0,$$

where $\bar{A} = A/k$ is a free k-module quotient. This is also a free resolution of A with the differentials are obtained from the differentials in the bar resolution by factoring through $\bar{\mathcal{B}}_A$.

1.3.2 Definitions of Hochschild (co)homology

We now recall the definitions and properties of Hochschild (co)homology, see Witherspoon [19] for more details.

Let M be an A-bimodule. Applying the functor $M \otimes_{A^e} -$ to the bar resolution $\mathcal{B}_A$ of A, we have the complex

$$\cdots \xrightarrow{\mathrm{id}_M \otimes d_2} M \otimes_{A^e} A^{\otimes 3} \xrightarrow{\mathrm{id}_M \otimes d_1} M \otimes_{A^e} A \otimes A \longrightarrow 0, \qquad (1.2)$$

where id_M is the identity map on M.

We have an isomorphism of A^e-modules

$$A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n}$$

given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes (a_1 \otimes \cdots \otimes a_n)$ for all $a_0, \ldots, a_{n+1} \in A$. This together with the isomorphism

$$M \otimes_{A^e} A^e \cong M$$

yields the following isomorphism

$$M \otimes_{A^e} A^{\otimes(n+2)} \cong M \otimes A^{\otimes n}$$

and then the complex

$$M \otimes A^{\otimes \bullet}: \quad \cdots \longrightarrow M \otimes A^{\otimes 2} \xrightarrow{\delta_2} M \otimes A \xrightarrow{\delta_1} M \longrightarrow 0,$$

where δ_n are found by combining the above isomorphisms

$$\begin{aligned}\delta_n(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = & ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}.\end{aligned}$$

Definition 1.19. The *n-th Hochschild homology* $\mathrm{HH}_n(A, M)$ of A with coefficients in an A-bimodule M is the n-th homology of the complex (1.2), equivalently

$$\mathrm{HH}_n(A, M) = \mathrm{H}_n(M \otimes A^{\otimes \bullet}),$$

i.e., $\mathrm{HH}_n(A, M) = \dfrac{\mathrm{Ker}(\delta_n)}{\mathrm{Im}(\delta_{n+1})}$ for all $n \geq 0$, δ_0 is taken to be the zero map and the differentials δ_n are given as above for $n > 0$. Let $\mathrm{HH}_*(A, M) = \bigoplus_{n \geq 0} \mathrm{HH}_n(A, M)$. We call this module the *Hochschild homology* of A with coefficients in the A-bimodule M.

In order to get the Hochschild cohomology version, we apply the functor $\mathrm{Hom}_{A^e}(-, M)$ to the bar complex (1.1). Then we obtain the following complex:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A \otimes A, M) \xrightarrow{d^1} \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{d^2} \cdots, \tag{1.3}$$

where the differentials d^n are given by $d^n(f) := f \circ d_n$ for any element f in $\mathrm{Hom}_{A^e}(A^{\otimes (n+1)}, M)$.

We can show that there is an isomorphism

$$\mathrm{Hom}_{A^e}(A^{\otimes (n+2)}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)$$

and hence, we get the new complex $\mathrm{Hom}_k(A^{\otimes \bullet}, M)$ with the differentials ∂_n identified by combining the isomorphisms naturally.

Definition 1.20. The *n-th Hochschild cohomology* $\mathrm{HH}^n(A, M)$ of A with coefficients in an A-bimodule M is the cohomology of the complex (1.3), equivalently

$$\mathrm{HH}^n(A, M) = \mathrm{H}^n(\mathrm{Hom}_k(A^{\otimes \bullet}, M)),$$

that is, $\mathrm{HH}^n(A, M) = \dfrac{\mathrm{Ker}(\partial^{n+1})}{\mathrm{Im}(\partial^n)}$ for all $n \geq 0$. Let us denote $\mathrm{HH}^*(A, M) = \bigoplus_{n\geq 0} \mathrm{HH}^n(A, M)$. We call this module the *Hochschild cohomology* of A with coefficients in the A-bimodule M.

By definition, Hochschild homology and cohomology are $\mathbb{N}$-graded vector spaces.

Remark 1.21. Cartan and Eilenberg [2] defined the Hochschild homology and cohomology group of A with coefficients in M in terms of the Tor functor and Ext functor by:

$$\mathrm{HH}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A)$$

and

$$\mathrm{HH}^n(A, M) \cong \mathrm{Ext}^n_{A^e}(A, M).$$

We refer the reader to Section 1.2, Chapter 1 [19] to see the interpretation in low degrees of the Hochschild homology and cohomology.
As A is also an A-bimodule, we can consider the case that $M = A$. The resulting Hochschild homology and cohomology of A with coefficients in A are respectively abbreviated by

$$\mathrm{HH}_*(A) = \mathrm{HH}_*(A, A) \text{ and } \mathrm{HH}^*(A) = \mathrm{HH}^*(A, A).$$

Briefly, we call these modules Hochschild homology and cohomology of the algebra A. This is the case of our interest in the thesis.

1.4 An alternative resolution of Guccione et al.

For a given algebra, there can be many options for the resolution of the algebra. The definition of Hochschild homology and cohomology are initially based on the bar resolution. However, in practice we will not use it to compute the Hochschild (co)homology. Instead, we will use a resolution given by Jorge Guccione and Juan Guccione, which is fruitful for explicit computations.

1.4.1 Exterior algebras

We recall the definition of exterior algebras, see [21] or Chapter 3 §5 in [22] for more details.

Definition 1.22. Let V be a vector space over a field k. For any $i \in \mathbb{N}$, we define the *ith-tensor power* of V to be the tensor product of V with itself i times:

$$V^{\otimes i} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{i \text{ times}},$$

with the convention that $V^{\otimes 0} = k$. We call the following direct sum the *tensor algebra*

$$\mathcal{T}(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i},$$

where the multiplication is determined by the canonical isomorphism

$$V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes i+j}$$
$$(v_1 \otimes \cdots v_i) \otimes (w_1 \otimes \cdots w_j) \mapsto v_1 \otimes \cdots v_i \otimes w_1 \otimes \cdots w_j$$

and then extended linearly to all elements of $\mathcal{T}(V)$. The tensor algebra is a graded algebra naturally and also called the free algebra on the vector space V. We consider the quotient defined by

$$\bigwedge V := \mathcal{T}(V)/J,$$

where J is the ideal of $\mathcal{T}(V)$ generated by all elements of the form $v \otimes v$. This structure is called the *exterior algebra* of V and we write $u \wedge v$ for the equivalence class represented by $u \otimes v$ in the quotient $\mathcal{T}(V)/J$.

By the definition, we have that $\bigwedge V$ has the structure of an associative algebra. In particular, $v \wedge v = 0$ for all $v \in V$ and $u \wedge v = -v \wedge u$.

Example 1.23. Let V be the k-vector space generated by n elements $e_1, e_2, \ldots, e_n$. Then the exterior algebra of V is the following graded associative algebra:

$$k \oplus \bigoplus_{i=1}^{n} ke_i \oplus \bigoplus_{\substack{i,j \in [n] \\ i<j}} ke_i \wedge e_j \oplus \cdots,$$

with the convention that $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$.

1.4.2 The resolution of Guccione et al.

We now present the alternative resolution given by Jorge Guccione and Juan Guccione [14], which we will use for the algebras in this thesis. In particular, we recall the alternative resolution for the case of algebra $A = k[x_1, \dots, x_n]/\langle f_1, \dots, f_r\rangle$, where k is an arbitrary commutative ring with unity 1 and $\{f_1, \dots, f_r\}$ is a regular sequence in $k[x_1, \dots, x_n]$.

Let $\mathscr{D}(A)$ be the exterior algebra over A^e of the free A^e-module $A^e e_1 \oplus \cdots \oplus A^e e_n$, see Example 1.23. Let $\mathbf{F}$ be the algebra of divided powers over $\mathscr{D}(A)$ with r variables $t_1, \dots, t_r$, that is, $\mathbf{F}$ is a free module over $\mathscr{D}(A)$ with basis $t_1^{(p_1)} \cdots t_r^{(p_r)}$ ($p_i \in \mathbb{N}$) and the multiplication given by

$$(t_1^{(p_1)} \cdots t_r^{(p_r)}) \cdot (t_1^{(q_1)} \cdots t_r^{(q_r)}) = \prod_{i=1}^{r} \begin{pmatrix} p_i + q_i \\ p_i \end{pmatrix} (t_1^{(p_1+q_1)} \cdots t_r^{(p_r+q_r)}).$$

We assign degree 1 to the elements e_i and degree $2p$ to the elements $t_i^{(p)}$. The algebra $\mathbf{F}$ has basis elements of the form

$$e_{i_1 \cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)},$$

where by $e_{i_1 \cdots i_s}$ we mean $e_{i_1} \wedge \cdots \wedge e_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq n$ and we call the number $s + 2(p_1 + \cdots + p_r)$ the degree of this element. Then $\mathbf{F} = \bigoplus_{m \in \mathbb{N}} F_m$ is a strictly anti-commutative graded A^e-algebra, where F_m is the homogeneous component of degree m, that is the A^e-subspace generated by all elements of degree m.

Let us recall a definition based on the Taylor series development, see [23] for more details.

Definition 1.24. Let $f = \sum x_1^{i_1} \cdots x_n^{i_n}$ be an element in the polynomial ring $P := k[x_1, x_2, \dots, x_n]$. We shall call $T_j(f)$ the sum of the monomials of $T(f)$ which are multiples of $T(x_j)$ and not multiples of $T(x_i)$ for $i < j$, i.e.,

$$T_j(f) = \sum_{\substack{i_j \geq 1 \\ i_{j+1}, \dots, i_n}} \frac{1}{i_j! \cdots i_n!} \cdot \frac{\partial^{\sum i_k} f}{\partial x_j^{i_j} \cdots \partial x_n^{i_n}} \cdot T(x_j)^{i_j} \cdots T(x_n)^{i_n},$$

where T is the Taylor series development from P to P^e given by $T(p) = 1 \otimes p - p \otimes 1$. Sometimes we use $T(p) = p \otimes 1 - 1 \otimes p$ according to our convenience.

Remark 1.25. In [14], the authors stated that the following sequence $\mathbf{F}$ is an A^e-free resolution of the algebra $A = k[x_1, x_2, \ldots, x_n]/\langle f_1, f_2, \ldots, f_r\rangle$, where $f_1, f_2, \ldots, f_r$ is a regular sequence.

$$\mathbf{F} : \cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\mu} A \longrightarrow 0, \tag{1.4}$$

where the map $\mu : F_0 \cong A^e \to A$ is the multiplication and the differentials d_m are given by

$$d_1(e_i) = T(x_i),$$
$$d_2(t_i) = \sum_{m=1}^{n} \frac{T_m(f_i)}{T(x_m)} \cdot e_m,$$

and

$$d_{2p}(t_i^{(p)}) = t_i^{(p-1)} d_2(t_i).$$

The image of the general elements are defined inductively from the above formulas based on the rule:

$$d(xy) = d(x)y + (-1)^{\deg(x)} x d(y),$$

where $\deg(x)$ is the degree of the element x.

We recall in the following remark some technical results that we will use in the later chapters. The full version and proofs of the below results can be found in Section 2 of the work of Guccione, Guccione, Redondo and Villamayor [24].

Remark 1.26. (i) T_j is k-linear. This implies that we can obtain the Taylor series for any polynomial f in $k[x_1, x_2, \ldots, x_n]$ from the monomial components of f.
(ii) [Item (g), Proposition 2.2.4, [24]] Let $f = \sum f_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ be a polynomial in $k[x_1, x_2, \ldots, x_n]$, where k is a field.

$$\frac{T_m(f)}{T(x_m)} = \sum_{i_1, \ldots, i_n} \sum_{l=0}^{i_m - 1} f_{i_1 \cdots i_n} x_1^{i_1} \cdots x_m^{l} \otimes x_m^{i_m - l - 1} \cdot x_{m+1}^{i_{m+1}} \cdots x_n^{i_n}.$$

(iii) By (i), we can reduce (ii) to computations on any monomial $M = x_1^{i_1} \cdots x_n^{i_n}$ of f. And we have the following formula, see Proof of Proposition

2.2.4, [24] for more details.

$$
\begin{aligned}
\frac{T_m(M)}{T(x_m)} =& \frac{(x_1^{i_1}\cdots x_{m-1}^{i_{m-1}}\otimes 1)T_m(x_m^{i_m})(1\otimes x_{m+1}^{i_{m+1}}\cdots x_n^{i_n})}{T(x_m)}\\
=&(x_1^{i_1}\cdots x_{m-1}^{i_{m-1}}\otimes 1)\left(\sum_{l=0}^{i_m-1} x_m^l\otimes x_m^{i_m-l-1}\right)(1\otimes x_{m+1}^{i_{m+1}}\cdots x_n^{i_n})\\
=&\sum_{l=0}^{i_m-1} x_1^{i_1}\cdots x_{m-1}^{i_{m-1}}x_m^l\otimes x_m^{i_m-l-1}x_{m+1}^{i_{m+1}}\cdots x_n^{i_n}
\end{aligned}
$$

since $\dfrac{T_m(x_m^{i_m})}{T(x_m)} = \dfrac{1\otimes x_m^{i_m} - x_m^{i_m}\otimes 1}{1\otimes x_m - x_m\otimes 1} = \sum_{l=0}^{i_m-1} x_m^l \otimes x_m^{i_m-l-1}$.

1.5 Multiplication on Hochschild cohomology

This section will present the multiplication which makes the Hochschild cohomology module into an algebra structure. The cup product is a method of adjoining two cocycles of degree i and j to form a composite cocycle of degree $i+j$. This defines an associative and distributive graded commutative product operation in cohomology, giving the cohomology module the structure of a graded ring, called the cohomology ring. The cup product for the Hochschild cohomology was introduced by Eilenberg and MacLane in 1947 [25]. The definition of the cup product on the Hochschild cohomology will be specified for our case, the Hochschild cohomology of an associative algebra $\mathrm{HH}^*(A)$. There are many equivalent definitions of the associative product on Hochschild cohomology. First we define the cup product at the chain level for functions on the bar complex. Then, we interpret the cup product in terms of the Yoneda product, which will be used in the computations during this thesis. The books [19] by Witherspoon and [26] by Carlson, Townsley, Valero-Elizondo and Zhang may serve as general references on this section.

Definition 1.27. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The *cup product* $f \smile g$ is the element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ defined by

$$(f \smile g)(a_1\otimes\cdots\otimes a_{m+n}) = f(a_1\otimes\cdots\otimes a_m)\cdot g(a_{m+1}\otimes\cdots\otimes a_{m+n}).$$

If $m = 0$, we interpret this formula to be

$$(f \smile g)(a_1 \otimes \cdots \otimes a_n) = f(1) \cdot g(a_1 \otimes \cdots \otimes a_n),$$

and similarly if $n = 0$.

This cup product $\smile$ is associative and induces a well-defined graded associative product on Hochschild cohomology, which is also denoted by the same notation:

$$\smile : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \to \mathrm{HH}^{m+n}(A).$$

In this thesis, we will interpret the cup product in terms of the Yoneda product. Let $\mathbf{F}$ be a free resolution of the A^e-module A and $f : F_i \to A$ an A^e-homomorphism such that $f \circ d_{i+1} = 0$. By the comparison theorem, there is a chain map $\tilde{f}$ consisting of homomorphisms $\tilde{f}_m$, $m \in \mathbb{N}$ that makes the following diagram commute, moreover such a chain map is unique up to chain homotopy.

$$\begin{tikzcd}
\cdots \arrow[r] & F_{i+j} \arrow[r] \arrow[d, dashed, "\tilde{f}_j"] & \cdots \arrow[r] & F_{i+2} \arrow[r, "d_{i+2}"] \arrow[d, dashed, "\tilde{f}_2"] & F_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, dashed, "\tilde{f}_1"] & F_i \arrow[d, dashed, "\tilde{f}_0"] \arrow[dr, "f"] & \\
\cdots \arrow[r] & F_j \arrow[r] \arrow[d, "g"] & \cdots \arrow[r] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, "\epsilon"] & A \\
& A & & & & &
\end{tikzcd} \tag{1.5}$$

Definition 1.28. Let $f \in \mathrm{Hom}_{A^e}(F_i, A)$ and $g = \mathrm{Hom}_{A^e}(F_j, A)$ be cocycles. For any projective resolution $\mathbf{F}$ of the A^e-module A, we extend f to a chain map $\tilde{f} : \mathbf{F} \to \mathbf{F}$ as shown before. We define the map $f \smile g \in \mathrm{Hom}_{A^e}(F_{i+j}, A)$ to be the composition $g \circ \tilde{f}_j$:

$$f \smile g := g \circ \tilde{f}_j.$$

The product $f \smile g$ is again a cocycle because g is a cocycle and $\tilde{f}$ is a chain map. Since $\tilde{f}$ is unique up to homotopy, this induces a well-defined product on cohomology, which is the Yoneda product. We can find in [26] Chapter 4 and in [27] Chapter 1 a detailed proof that the product defined as above gives the Hochschild cohomology an algebra structure and the product does not depend on the choice of the lifting map.

1.6 Multiplication on Hochschild homology

In this section, we shall review the shuffle product, which was introduced by Eilenberg and MacLane. It induces a product on Hochschild homology and yields a result (the so-called Eilenberg-Zilber theorem) which shows that Hochschild homology commutes with tensor product. We suggest the books of MacLane [16] (Chapter 8) and Loday [20] (Chapter 4) as general references.

1.6.1 Permutations

Definition 1.29. Let $[n] := \{1, 2, \ldots, n\}$ be the set of integers from 1 to n. We call a *permutation* of $[n]$ any bijective map of this set onto itself. For a given n, we denote by S_n the set of all the permutations of $[n]$. There exists exactly $n!$ permutations of $[n]$.

Definition 1.30. We call a *transposition* of two elements in $[n]$ any permutation σ for which there exist $i, j \in [n]$ $(i \neq j)$ such that:
We have $\sigma(i) = j$ and $\sigma(j) = i$;
We have $\sigma(k) = k$ for all $k \in [n]$ such that $k \notin \{i, j\}$.

We recall in the following theorem the decomposition of permutations into transpositions.

Theorem 1.31. *Each permutation $\sigma \in S_n$ can be decomposed as a product (in the sense of composition) of transpositions. Such a decomposition is not unique, but the parity of the number of transpositions that decompose a permutation σ depends only on σ itself and not on the considered decomposition.*

Definition 1.32. We define the signature $\epsilon(\sigma)$ of a permutation $\sigma \in S_n$ as:

$$\epsilon(\sigma) := (-1)^{N(\sigma)},$$

where $N(\sigma)$ is a number of transpositions that make it possible to decompose σ.

1.6.2 Shuffle product

Let A be a commutative algebra. We now present a multiplication in Hochschild homology based on the shuffle product, which gives $\mathrm{HH}_*(A)$ a graded commutative algebra structure.

Definition 1.33. Let p, q be non-negative integers. A (p, q)-*shuffle* is a permutation σ in S_{p+q} such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

Let $S_{p,q}$ denote the subset of S_{p+q} consisting of all (p, q)-shuffles.

Definition 1.34. The *shuffle product* on $\mathrm{HH}_*(A)$ is defined at the chain level on the complex (1.2) with $M = A$ by

$$\begin{aligned}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) &\overset{\mathrm{sh}}{\times} (a'_0 \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}) \\ &= \sum_{\sigma \in S_{p,q}} \epsilon(\sigma) a_0 a'_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)}\end{aligned}$$

for all $a'_0, a_0, \ldots a_{p+q} \in A$.

Theorem 1.35. *The shuffle product*

$$\overset{\mathrm{sh}}{\times} : \mathrm{HH}_p(A) \otimes \mathrm{HH}_q(A) \to \mathrm{HH}_{p+q}(A)$$

induces on $\mathrm{HH}_*(A)$ *a structure of graded commutative algebra.*

Remark 1.36. By the definition, we note immediately the two following points:
(i) The elements $a_1, \ldots, a_p$ appear in the same order in the sequence

$$a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(p+q)}$$

and similarly for $a_{p+1}, \ldots, a_{p+q}$.
(ii) For any two given non-negative integers p, q, the number of (p, q)-shuffles in S_{p+q} is $\dfrac{(p+q)!}{p!q!}$.

1.7 Examples

We present in this section some examples in order to have a first view about the Hochschild cohomology of some familiar algebras. We use Chapters 1 and 2 of the book of Witherspoon [19] as general reference.

1.7.1 Example 1

We compute the Hochschild cohomology of the algebra $A = k[x]$.
We consider the sequence of A^e-modules and A^e-homomorphisms

$$0 \longrightarrow k[x] \otimes k[x] \xrightarrow{d} k[x] \otimes k[x] \xrightarrow{\mu} k[x] \longrightarrow 0,$$

where μ is the multiplication and d is the multiplication by the element $x \otimes 1 - 1 \otimes x$. We can prove that this is an exact sequence by a direct calculation.
To construct the cohomology, we apply the functor $\mathrm{Hom}_{A^e}(-, k[x])$ to the truncation of the above complex and use the fact that

$$\mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x]) \cong \mathrm{Hom}_k(k, k[x]) \cong k[x].$$

We get the resulting complex

$$0 \longleftarrow k[x] \xleftarrow{\alpha} k[x] \longleftarrow 0.$$

$$\begin{tikzcd}
\mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x]) \arrow[d, leftrightarrow, "\cong"'] & \mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x]) \arrow[l, "d^*"'] \arrow[d, leftrightarrow, "\cong"] \\
k[x] & k[x] \arrow[l, "\alpha"']
\end{tikzcd}$$

For $a \in k[x]$, a is identified with $f_a \in \mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x])$ where $f_a(1 \otimes 1) = a$. Composing with the differential d, we have that

$$\begin{aligned} f_a((x \otimes 1 - 1 \otimes x) \cdot (1 \otimes 1)) &= (x \otimes 1) \cdot f_a(1 \otimes 1) - (1 \otimes x) \cdot f_a(1 \otimes 1) \\ &= x \cdot a \cdot 1 - 1 \cdot a \cdot x = 0. \end{aligned}$$

So we have $\alpha = 0$:

$$0 \longleftarrow k[x] \xleftarrow{0} k[x] \longleftarrow 0.$$

Hence $\mathrm{HH}^0(k[x]) \cong k[x]$, $\mathrm{HH}^1(k[x]) \cong k[x]$, $\mathrm{HH}^n(k[x]) \cong 0$ for $n \geq 2$.

a	b	$a \smile b$
degree 0	degree 0	multiplication in $k[x]$
degree 0	degree 1	multiplication in $k[x]$
degree 1	degree 1	0
...	...	0

Thus $\mathrm{HH}^*(A) \cong k[x, y]/\langle y^2 \rangle$ where $\deg(x) = 0$ and $\deg(y) = 1$.

1.7.2 Example 2

In this example, we consider the Hochschild cohomology of the algebra $k[x_1, \ldots, x_n]$ based on Example 1 and the following result.

Theorem 1.37. *Let A_1 and A_2 be finite dimensional k-algebras. Then*

$$\mathrm{HH}^*(A_1 \otimes A_2) \cong \mathrm{HH}^*(A_1) \otimes \mathrm{HH}^*(A_2)$$

as algebras, where the algebra on the right side is a graded tensor product algebra.

In fact, the isomorphism in the theorem is an isomorphism of Gerstenhaber algebras. The proof of the theorem can be found in the work of Le and Zhou [28] and in Chapter 2 of the book of Witherspoon [19].

We have that $k[x_1, x_2] \cong k[x_1] \otimes k[x_2]$ and in Example 1 we found that $\mathrm{HH}^*(k[x_i]) \cong k[x_i, y_i]/\langle y_i^2 \rangle$. By the above theorem, we obtain the Hochschild cohomology of $k[x_1, x_2]$ as follows:

$$\begin{aligned} \mathrm{HH}^*(k[x_1, x_2]) &\cong \mathrm{HH}^*(k[x_1]) \otimes \mathrm{HH}^*(k[x_2]) \\ &\cong \frac{k[x_1, y_1]}{\langle y_1^2 \rangle} \otimes \frac{k[x_2, y_2]}{\langle y_2^2 \rangle} \\ &\cong k[x_1, x_2] \otimes \bigwedge(y_1, y_2), \end{aligned}$$

where $\bigwedge(y_1, y_2)$ is the exterior algebra on a vector space with basis y_1, y_2; the degree of x_1, x_2 is 0 and the degree of y_1, y_2 is 1.

Now we get the Hochschild cohomology of $k[x_1, \ldots, x_n]$ by induction on n,

$$\mathrm{HH}^*(k[x_1, \ldots, x_n]) \cong k[x_1, \ldots, x_n] \otimes \bigwedge(y_1, \ldots, y_n).$$

1.7.3 Example 3

We present here the computation of the Hochschild cohomology structure of the algebra $A = k[x]/\langle x^n \rangle$, where $n \geq 2$.

Let us consider the following sequence of A^e-modules:

$$\cdots \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{\mu} A \longrightarrow 0,$$

where $u = x \otimes 1 - 1 \otimes x$, $v = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \cdots + 1 \otimes x^{n-1}$ and μ is the multiplication. This is an exact sequence and also called a *periodic resolution* of A as an A^e-module, see Section 1.3 [11].

Apply the functor $\mathrm{Hom}_{A^e}(-, A)$ to the truncation of the above sequence and identify $\mathrm{Hom}_{A^e}(A^e, A) \cong \mathrm{Hom}_k(k, A)$ with A. We get the resulting sequence:

$$\cdots \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \longleftarrow 0.$$

If n is divisible by the characteristic of k, denoted by $\mathrm{char}(k)$, then $nx^{n-1}\cdot = 0$ and $\mathrm{Ker}(A \xrightarrow{0} A) = A$ everywhere in the sequence. Then $\mathrm{HH}^*(A) \cong A$ for all n. If n is not divisible by $\mathrm{char}(k)$, then we get

$$\mathrm{HH}^i(A) \cong \begin{cases} A & \text{if } i = 0, \\ \langle x \rangle & \text{if } i \text{ odd}, \\ A/\langle x^{n-1} \rangle & \text{if } i \text{ even}. \end{cases}$$

Concerning the cup product of this Hochschild cohomology, we refer the reader to Chapter 1 of [19] for details, to be precise to Example 1.3.11 therein for the case where $\mathrm{char}(k)$ does not divide n and to Example 1.3.12 therein for the case where $\mathrm{char}(k)$ divides n. The multiplication was determined by directly computing compositions of chain maps between the shifted resolution and the resolution itself based on the features of the cohomology. We can find in the work of Holm [10] a general result on the Hochschild cohomology of the algebras $k[x]/\langle f \rangle$ where f is any monic polynomial in $k[x]$.

Discussions. Through the above examples, we can see that the calculation of the Hochschild cohomology structure of an algebra depends significantly on the resolution of the algebra. After identifying the cohomology module, the later computation on the multiplication is based on the features of the cohomology. The work of Holm [10] inspired us to work on families of algebras of the similar flavour, but in more variables and more generators. Depending on our choice of algebra, we will find a suitable method to deal with the complexity of computations, as we have seen in the above examples.

1.8 Algebraic discrete Morse theory

In this section, we will give a brief overview of the algebraic discrete Morse theory as it applies to Chapter 3. We present here the results in the work of

Sköldberg [29], see also Jöllenbeck and Welker [30], which is directly related to this thesis. For more general discrete Morse theory and its applications, we refer the reader to the works of Forman [31] and Kozlov [32]. The idea in discrete Morse theory is to reduce the number of cells in a CW-complex without changing the homotopy type by constructing the new complex via a certain partial matching of the cells. Sköldberg derived an algebraic version of this theory, where the chain complexes of modules with a direct sum decomposition play the role of the CW-complexes.

Definition 1.38. Let R be a ring with unit. A *based complex* of R-modules is a chain complex $\mathbf{F}$ of R-modules together with a direct sum decomposition $F_n = \bigoplus_{\alpha \in I_n} F_\alpha$, where $\{I_n\}$ is a family of mutually disjoint index sets.

Let $d : \bigoplus_n F_n \to \bigoplus_n F_n$ be a graded map. We denote $d_{\beta,\alpha}$ the component of d going from F_α to F_β, that is,

$$d_{\beta,\alpha} : F_\alpha \xrightarrow{\text{inclusion}} F_m \xrightarrow{d} F_n \xrightarrow{\text{projection}} F_\beta \ ,$$

where the order of indices are chosen to agree with the composition of functions.

A *digraph (directed graph)* is a graph that is made up of a set of vertices connected by edges, where the edges have a direction associated with them. For more details and examples, see Bang-Jensen and Gutin [33].

Let $\mathbf{F}$ be a based complex. We construct a digraph $G_{\mathbf{F}}$ with the vertex set $V = \bigcup_n I_n$ and the directed edge set E consisting of $\alpha \to \beta$, where the component $d_{\beta,\alpha}$ is non-zero.

Definition 1.39. A *partial matching* on a digraph $G = (V, E)$ is a subset $\mathcal{M}$ of the edge set E such that no vertex is incident to more than one edge in $\mathcal{M}$.

Let $\mathcal{M}$ be a partial matching on the digraph $G = (V, E)$. We denote $G^{\mathcal{M}} = (V, E^{\mathcal{M}})$ the digraph obtained from G by reversing the direction of each arrow in $\mathcal{M}$ as follows:

$$E^{\mathcal{M}} = (E \smallsetminus \mathcal{M}) \cup \{\beta \to \alpha \mid \alpha \to \beta \in \mathcal{M}\}.$$

Definition 1.40. A *partial order* is a binary relation $\preceq$ over a set P satisfying the following axioms: $a \preceq a$ (reflexivity); if $a \preceq b$ and $b \preceq a$, then

$a = b$ (antisymmetry); and if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity). A partial order on a set P is *well-founded* if there is no strictly descending infinite sequence in P.

Let $\mathcal{M}$ be a partial matching on the digraph $G_{\mathbf{F}}$. For now, we write $\alpha^{(n)}$ to indicate that $\alpha \in I_n$. On each I_n we define a relation $\prec$ such that $\gamma \prec \alpha$ whenever there is a path $\alpha^{(n)} \to \beta \to \gamma^{(n)}$ in $G_{\mathbf{F}}^{\mathcal{M}}$. We call $\mathcal{M}$ a *Morse matching* if for each $\alpha \to \beta$ in $\mathcal{M}$, the corresponding component $d_{\beta,\alpha}$ is an isomorphism and the relation $\prec$ is a well-founded partial order for all n.

Lemma 1.41 (Lemma 1, [29])**.** *Let $\mathbf{F}$ be a based complex such that $G_{\mathbf{F}}$ is a finite directed graph, and let $\mathcal{M}$ be a partial matching on $G_{\mathbf{F}}$ such that $d_{\beta,\alpha}$ is an isomorphism whenever $\alpha \to \beta$ is in $\mathcal{M}$. Then $\mathcal{M}$ is a Morse matching if and only if $G_{\mathbf{F}}^{\mathcal{M}}$ has no directed cycles.*

Let $\mathcal{M}$ be a Morse matching on the based complex $\mathbf{F}$. We now recall the definition of the graded map $\phi : \bigoplus_n F_n \to \bigoplus_n F_n$ of degree 1 (given by Sköldberg [29]) as follows:
If α is minimal with respect to $\prec$ and $x \in F_\alpha$, let

$$\phi(x) = \begin{cases} d_{\alpha,\beta}^{-1}(x) & \text{if } \beta \to \alpha \in \mathcal{M} \text{ for some } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

If α is not minimal with respect to $\prec$ and $x \in F_\alpha$, let

$$\phi(x) = \begin{cases} d_{\alpha,\beta}^{-1}(x) - \sum\limits_{\substack{\beta\to\gamma \\ \gamma\neq\alpha}} \phi d_{\gamma,\beta} d_{\alpha,\beta}^{-1}(x) & \text{if } \beta \to \alpha \in \mathcal{M} \text{ for some } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.42 (Lemma 2, [29])**.** *Let $\mathcal{M}$ be a Morse matching on the based complex $\mathbf{F}$. Then the map ϕ is a splitting homotopy, that is, $\phi^2 = 0$ and $\phi d \phi = \phi$.*

We call $\mathcal{M}$-*critical* the vertex in $G_{\mathbf{F}}^{\mathcal{M}}$ that is unmatched, i.e., not incident to any edge in $\mathcal{M}$. We denote by $\mathcal{M}^0$ the set of $\mathcal{M}$-critical vertices. For each n, we use the notation $\mathcal{M}_n^0$ for the set $\mathcal{M}^0 \cap I_n$. We define the map $\pi : \mathbf{F} \to \mathbf{F}$ by

$$\pi = \mathrm{id} - (\phi d + d\phi).$$

$$\begin{tikzcd}
\cdots \arrow[r] & F_m \arrow[r, "d"] \arrow[d, shift right, "\pi"'] \arrow[d, shift left, "\mathrm{id}"] & F_n \arrow[r, "d"] \arrow[d, shift right, "\pi"'] \arrow[d, shift left, "\mathrm{id}"] \arrow[dl, "\phi"] & F_p \arrow[r] \arrow[d, shift right, "\pi"'] \arrow[d, shift left, "\mathrm{id}"] \arrow[dl, "\phi"] & \cdots \\
\cdots \arrow[r] & F_m \arrow[r, "d"'] & F_n \arrow[r, "d"'] & F_p \arrow[r] & \cdots
\end{tikzcd}$$

Then we have the following result:

Theorem 1.43. *(Theorem 1, [29]) Let $\mathcal{M}$ be a Morse matching on the based complex $\mathbf{F}$. Then the complexes $\mathbf{F}$ and $\pi(\mathbf{F})$ are homotopy equivalent. Furthermore, we have for each n an isomorphism of modules:*

$$\pi(F_n) \cong \bigoplus_{\alpha \in \mathcal{M}_n^0} F_\alpha.$$

Here, we will take note of a special case which we will use to obtain the results in Chapter 3.

Remark 1.44. By Theorem 1.43, in case that $\mathcal{M}^0 = \emptyset$ we get $\pi(F) = 0$. Then by the definition of π, one has that $\phi d + d\phi = \mathrm{id}$, which means that ϕ is a contracting homotopy.

Example 1.45. Let Δ be a cell complex on $\{1, 2, 3, 4\}$, see Figure 1.1. For each i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ and let $\mathbb{Z}^{F_i(\Delta)}$ be a module over $\mathbb{Z}$ whose basis elements are elements in $F_i(\Delta)$.

$$\mathbf{F}:\ 0 \longrightarrow \mathbb{Z} \xrightarrow{\ d_2\ } \mathbb{Z}^4 \xrightarrow{\ d_1\ } \mathbb{Z}^4 \xrightarrow{\ d_0\ } \mathbb{Z} \longrightarrow 0\,,$$

where

$$F_{-1}(\Delta) = \{\emptyset\} = \{u\},$$

$$F_0(\Delta) = \{v_1, v_2, v_3, v_4\},$$

$$F_1(\Delta) = \{e_1, e_2, e_3, e_4\},$$

$$F_2(\Delta) = \{f\};$$

and the differentials are defined by

$$d_0(v_i) = u \text{ for all } i,$$

$$d_1(e_1) = v_2 - v_1,$$

$$d_1(e_2) = v_3 - v_2,$$

$$d_1(e_3) = v_4 - v_3,$$

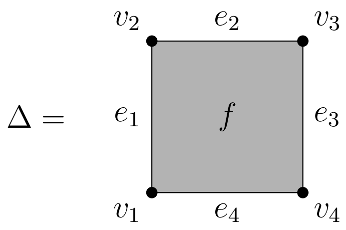

Figure 1.1: Cell complex Δ

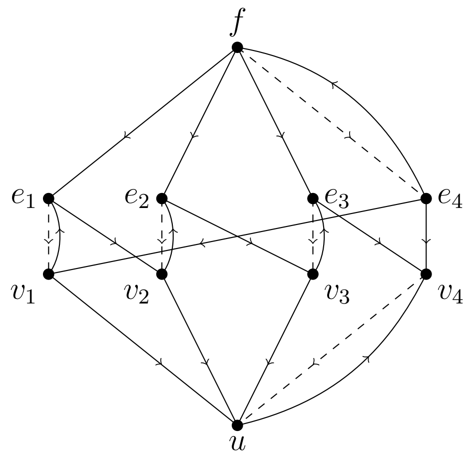

Figure 1.2: A Morse matching $\mathcal{M}$ (dashed arrows) on the digraph $G_{\mathbf{F}}$

$$d_1(e_4) = v_1 - v_4,$$

$$d_2(f) = e_1 + e_2 + e_3 + e_4.$$

We define a matching $\mathcal{M}$ on the digraph $G_{\mathbf{F}}$, see Figure 1.2:

$$M = \{f \to e_4, e_1 \to v_1, e_2 \to v_2, e_3 \to v_3, v_4 \to u\}.$$

We construct the digraph $G_{\mathbf{F}}^{\mathcal{M}}$ as shown before and it is easy to see that there are no directed cycles in $G_{\mathbf{F}}^{\mathcal{M}}$. So $\mathcal{M}$ is a Morse matching on the based complex $\mathbf{F}$. As the Morse matching $\mathcal{M}$ includes all vertices of $G_{\mathbf{F}}$, the map constructed by the formula of ϕ becomes a contracting homotopy of the complex $\mathbf{F}$.

A more detailed example can be found in Chapter 3.

1.9 Hilbert series and Gröbner bases

In commutative algebra, the Hilbert series of a graded finitely generated commutative algebra over a field is used to measure the growth of the dimension of the homogeneous components of the algebra. Here, we recall some notions about Hilbert series and Gröbner bases that we use to describe the graded structure of the Hochschild cohomology rings in the later chapters.

Definition 1.46. Let $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ be an $\mathbb{N}^n$-graded module over a field k, where $M_{\mathbf{a}}$ is the submodule of M generated by elements of degree $\mathbf{a} \in \mathbb{N}^n$. We denote $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. If the vector space dimension $\dim_k(M_{\mathbf{a}})$ is finite for all $\mathbf{a} \in \mathbb{N}^n$, then the formal power series

$$\mathcal{H}(M;\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{N}^n} \dim_k(M_{\mathbf{a}}) \cdot \mathbf{x}^{\mathbf{a}}$$

is the $\mathbb{N}^n$-graded *Hilbert series* of M.

We can generalize the definition of Hilbert series for many other gradings on modules, which in this thesis we will consider the case of $\mathbb{N}\times\mathbb{Z}^n$-graded modules. As general references for this content we suggest the books of Miller and Sturmfels [34], and Villarreal [35].

Example 1.47. Let $K = k[x]$ be the polynomial ring over a field k. Then K is $\mathbb{N}$-graded if we consider $K = \bigoplus_{n\in\mathbb{N}} K_n$, where $K_n := kx^n$, the k-module generated by x^n. We can see that $\dim_k(K_n) = 1$ for all $n \in \mathbb{N}$ and the $\mathbb{N}$-graded Hilbert series of K is the geometric series:

$$\mathcal{H}(K;x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

For more examples and technical aspects, we refer the reader to the book of Wilf [36]. A *total order* on a set P is a partial order $\preceq$ on P (see Definition 1.40) such that for any pair of elements $a, b \in P$, one has either $a \preceq b$ or $b \preceq a$. We denote $S = k[x_1, x_2, \ldots, x_n]$ the polynomial ring in n variables over a field k and $\mathrm{Mon}(S)$ the set of monomials of S. A *monomial order* on S is a total order $\prec$ on $\mathrm{Mon}(S)$ such that: (i) $1 \prec u$ for all $1 \neq u \in \mathrm{Mon}(S)$; (ii) if $u, v \in \mathrm{Mon}(S)$, then $uw \prec vw$ for all $w \in \mathrm{Mon}(S)$.

Example 1.48. We will introduce here an example of monomial order, *pure lexicographic order* induced by the ordering $x_1 \succ x_2 \succ \cdots \succ x_n$ on $S = k[x_1, x_2, \ldots, x_n]$. Let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ and $\mathbf{x}^{\mathbf{b}} = x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ be two elements in S. We define the total order $\prec_{\text{purelex}}$ on $\mathrm{Mon}(S)$ by setting $\mathbf{x}^{\mathbf{a}} \prec_{\text{purelex}} \mathbf{x}^{\mathbf{b}}$ if the leftmost non-zero component of the vector $\mathbf{a} - \mathbf{b}$ is negative.

Now we recall the fundamental material on Gröbner bases, see Herzog and Hibi [37] and Cox, Little and O'Shea [38]. Let us fix a monimial order $\prec$ on S. For any non-zero polynomial $f = \sum\limits_{u \in \mathrm{Mon}(S)} a_u u$ where $a_u \in k$, we define the *initial monomial* of f with respect to $\prec$ (denoted by $\mathrm{in}_{\prec}(f)$) to be the biggest monomial with respect to $\prec$ among the monomials u such that $a_u \neq 0$ in f; and the *leading coefficient* of f to be the coefficient of $\mathrm{in}_{\prec}(f)$ in f.
Let I be a non-zero ideal of S. The *initial ideal* of I with respect to $\prec$, denoted by $\mathrm{in}_{\prec}(I)$, is the monomial ideal of S which is generated by $\{\mathrm{in}_{\prec}(f) \mid 0 \neq f \in I\}$.

Definition 1.49. Let I be a non-zero ideal of S. A finite set of non-zero polynomials $\{g_1, g_2, \ldots, g_s\}$ with each $g_i \in I$ is said to be a *Gröbner basis* of I with respect to $\prec$ if the initial ideal $\mathrm{in}_{\prec}(I)$ of I is generated by the monomials $\mathrm{in}_{\prec}(g_1)$, $\mathrm{in}_{\prec}(g_2)$, ..., $\mathrm{in}_{\prec}(g_s)$.

We have a fact that there always exists a Gröbner basis of any non-zero ideal I with respect to $\prec$; and every Gröbner basis of I is a system of generators of I. Moreover, this set is finite.

Let us now recall Buchberger's algorithm which is used to compute a Gröbner basis of a given ideal I. The details of arguments which support this algorithm can be found in Chapter 2 of [37].

Theorem 1.50. *(The division algorithm) Let $g_1, g_2, \ldots, g_s$ be non-zero polynomials of S. For a given polynomial $f \in S$, there exist polynomials $f_1, f_2, \ldots, f_s$ and f' in S with*

$$f = f_1g_1 + f_2g_2 + \cdots + f_sg_s + f',$$

such that the following conditions satisfied:
(i) if $f' = \sum\limits_{u \in \mathrm{Mon}(S)} a_u u \neq 0$, then no monomial u where $a_u \neq 0$ belongs to

the ideal generated by $\mathrm{in}_\prec(g_1)$*,* $\mathrm{in}_\prec(g_2)$*, …,* $\mathrm{in}_\prec(g_s)$*; and*
(ii) if $f_i \neq 0$*, then* $\mathrm{in}_\prec(f) \succeq \mathrm{in}_\prec(f_i g_i)$*.*

We say that f *reduces* to f' with respect to $g_1, g_2, \ldots, g_s$ and the polynomial f' is said to be a *remainder* of f with respect to $g_1, g_2, \ldots, g_s$. We recall an important property of the initial ideal in the following theorem.

Theorem 1.51. *(Proposition 2.2.5, [37]) Let* I *be a non-zero ideal of* S *and* $\prec$ *a monomial order on* S*. Then the set of monomials which do not belong to* $\mathrm{in}_\prec(I)$ *form a* k*-basis of the quotient ring* S/I*.*

Definition 1.52. Let f and g be polynomials in S. Let c_f, c_g be coefficients of $\mathrm{in}_\prec(f)$ and $\mathrm{in}_\prec(g)$ respectively. We denote $\mathrm{lcm}(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g))$ the least common multiple of $\mathrm{in}_\prec(f)$ and $\mathrm{in}_\prec(g)$. The polynomial

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g))}{c_f \mathrm{in}_\prec(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g))}{c_g \mathrm{in}_\prec(g)} g$$

is called the S-polynomial of f and g.

Theorem 1.53. *(Buchberger's criterion) Let* I *be a non-zero ideal of* S *and* $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$ *a system of generators of* I*. Then* $\mathcal{G}$ *is a Gröbner basis of* I *if and only if for all* $i \neq j$*,* $S(g_i, g_j)$ *reduces to 0 with respect to* $g_1, g_2, \ldots, g_s$*.*

This theorem supplies an algorithm to compute a Gröbner basis from a set of generators of an ideal.

Algorithm 1.54. (Buchberger's algorithm) Let I be an ideal of S with the set of generators $\{g_1, g_2, \ldots, g_s\}$.

- If $S(g_i, g_j)$ reduces to 0 with respect to $g_1, g_2, \ldots, g_s$ for all $i \neq j$, then by Theorem 1.53 $\{g_1, g_2, \ldots, g_s\}$ is a Gröbner basis of I.
- Otherwise, $S(g_i, g_j)$ reduces to some non-zero remainder g_{s+1}, which is in I. We replace the generator set $\{g_1, g_2, \ldots, g_s\}$ by the new set $\{g_1, g_2, \ldots, g_s, g_{s+1}\}$ and compute all S-polynomials for this new set of generators.

After a finite number of steps, the procedure will terminate and a Gröbner basis can be obtained.

The following example is given in order to support the computations in the results in Chapter 3.

Example 1.55. We now apply the above algorithm to find the Gröbner basis of the ideal I generated by $x_1^b - x_2^a$, $x_1^{b-1}t$, $x_2^{a-1}t$, y_2t, y_1^2, y_2^2, y_1y_2, $x_1y_2 - x_2^{a-1}y_1$, $x_2y_2 - x_1^{b-1}y_1$ in the polynomial ring $k[x_1, x_2, y_1, y_2, t]$, where k is a field, a and b are positive integers greater than 1 such that $\gcd(a, b) = 1$. Here we use the pure lexicographic term order ($\prec_{\text{purelex}}$) $x_1 \prec x_2 \prec y_1 \prec y_2 \prec t$.

One has that: $x_1^b = t^0y_2^0y_1^0x_2^0x_1^b$ and $x_2^a = t^0y_2^0y_1^0x_2^ax_1^0$, which implies that

$$(0, 0, 0, 0, b) - (0, 0, 0, a, 0) = (0, 0, 0, -a, b).$$

Hence, $\text{in}_{\prec_{\text{purelex}}}(x_1^b - x_2^a) = x_2^a$. Similarly, $\text{in}_{\prec_{\text{purelex}}}(x_1y_2 - x_2^{a-1}y_1) = x_1y_2$ and $\text{in}_{\prec_{\text{purelex}}}(x_2y_2 - x_1^{b-1}y_1) = x_2y_2$.

Now we compute the S-polynomials for this set of generators consisting of 6 monomials and 3 binomials. It is clear that all S-polynomials of any two monomials are zero. For other pairs of generators, we have:

- $S(x_2^a - x_1^b, x_1^{b-1}t) = x_1^{b-1}t(x_2^a - x_1^b) - x_2^ax_1^{b-1}t = -x_1^{2b-1}t$ (a multiple of $x_1^{b-1}t$), which reduces to 0;
- $S(x_2^a - x_1^b, x_2^{a-1}t) = t(x_2^a - x_1^b) - x_2 \cdot x_2^{a-1}t = -x_1^bt$ reduces to 0 by the same reason;
- $S(x_2^a - x_1^b, y_2t) = y_2t(x_2^a - x_1^b) - x_2^ay_2t = x_1^by_2t$ reduces to 0;
- $S(x_2^a - x_1^b, y_1^2) = y_1^2(x_2^a - x_1^b) - x_2^ay_1^2 = -x_1^by^b$ (a multiple of y^2 since $b \geq 2$) reduces to 0;

and so on.

We can check that every S-polynomial reduces to zero because it is a multiple of some generator. Hence, the given set of generators is a Gröbner basis of I.

Chapter 2

The Hochschild cohomology rings of the square-free monomial complete intersections

In this chapter, we provide the computations on the ring structure of the Hochschild cohomology of the square-free monomial complete intersections. The results in this chapter have been published in Communications in Algebra, see [12].

2.1 Overview

We consider the algebras of the form $k\,[x_1, x_2, \ldots, x_n]\,/\langle m_1, m_2, \ldots, m_r\rangle$, where k is a field and m_i is a square-free monomial such that $\operatorname{supp}(m_i) \cap \operatorname{supp}(m_j) = \emptyset$ if $i \neq j$. Such an algebra is isomorphic to the tensor product of a polynomial ring $k\,[x_1, x_2, \ldots, x_u]$ and some algebras of the form $k\,[x_1, x_2, \ldots, x_v]\,/\langle x_1 x_2 \cdots x_v\rangle$. For example,

$$\frac{k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]}{\langle x_1x_2, x_3x_4x_5\rangle} \cong \frac{k[x_1, x_2]}{\langle x_1x_2\rangle} \otimes \frac{k[x_3, x_4, x_5]}{\langle x_3x_4x_5\rangle} \otimes k[x_6, x_7].$$

Furthermore, since the tensor product is preserved under the action of taking the Hochschild cohomology as stated in Theorem 1.37 and the structure of the Hochschild cohomology of $k\,[x_1, x_2, \ldots, x_u]$ can be found in Section 1.7

(Example 2), it suffices to study the Hochschild cohomology of the algebra $k\left[x_1, x_2, \ldots, x_n\right] / \langle x_1 x_2 \cdots x_n \rangle$, which will be denoted by A in this chapter. Our goal is to give a description in terms of generators and relations for the ring structure of the Hochschild cohomology $\mathrm{HH}^*(A)$ of the algebra A. Since A is a complete intersection as explained in Chapter 1, we can interpret the resolution given by Guccione et al. [14] for our case. The k-module structure will be expressed via sub-modules based on the features of cocycles. In order to describe the cup product, we will give the formula of a lifting map between the shifted resolution and the resolution itself. From this chain map, we can describe the Yoneda product which gives the k-space an algebra structure. In the next step, we use the previous results to describe the generators and relations of the algebra $\mathrm{HH}^*(A)$. In addition, we compute the Hilbert series of $\mathrm{HH}^*(A)$.

2.2 A construction of Hochschild cohomology

For simplicity, we will use the same notation for elements in $k[x_1, x_2, \ldots, x_n]$ and their cosets in the quotient ring $k\left[x_1, x_2, \ldots, x_n\right] / \langle x_1 x_2 \cdots x_n \rangle$ with the convention that $x_1 x_2 \cdots x_n = 0$, if there are no ambiguities.

We give details of the resolution in case of the square-free monomial $f = x_1 x_2 \cdots x_n$ and get the following resolution:

$$F : \cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\mu} A \longrightarrow 0, \tag{2.1}$$

where F_m is the finitely generated free A^e-module with basis elements $e_{i_1 \cdots i_s} \cdot t^{(q)}$ (s, $q \geq 0$ and $s + 2q = m$), where by $e_{i_1 \cdots i_s}$ or e_I with $I = \{i_1, \ldots, i_s\}$ we mean $e_{i_1} \wedge \cdots \wedge e_{i_s}$ ($1 \leq i_1 < \cdots < i_s \leq n$). Then (2.1) is an exact sequence of free A^e-modules F_m with

$$\mu : A^e \to A, \qquad a \otimes b \mapsto ab$$

and the differentials d_m are defined inductively as follows:

$$d_s(e_{i_1\cdots i_s}) = \sum_{j=1}^{s}(-1)^{j-1}(1\otimes x_{i_j} - x_{i_j}\otimes 1)e_{i_1\cdots\widehat{i_j}\cdots i_s};$$
$$d_2(t) = \sum_{j=1}^{n} x_1\cdots x_{j-1}\otimes x_{j+1}\cdots x_n\cdot e_j;$$
$$d_{s+2q}(e_{i_1\cdots i_s}t^{(q)}) = d_s(e_{i_1\cdots i_s})t^{(q)} + d_2(t)\cdot e_{i_1\cdots i_s}\cdot t^{(q-1)}, \text{ if } q\geq 1.$$

For abbreviation, we sometimes write d instead of d_m. In the following, we will present in detail the computations to get the formula of d based on the alternative resolution.

Computations of the differentials d. The general formula of differentials has been recalled in Remark 1.25, Chapter 1. We now refine the formula in the case of monomial $f = x_1x_2\cdots x_n$:

$$d_1(e_i) = T(x_i) = 1\otimes x_i - x_i\otimes 1 \text{ for } i = 1,\ldots,n;$$
$$d_2(t) = \sum_{j=1}^{n}\frac{T_j(f)}{T(x_j)}e_j = \sum_{j=1}^{n} x_1\cdots x_{j-1}\otimes x_{j+1}\cdots x_n\cdot e_j.$$

For the elements of higher degrees, we apply the formula

$$d(xy) = d(x)y + (-1)^{\deg(x)}xd(y), \tag{2.2}$$

where $\deg(x)$ is the degree of x. In our case, we assign degree 1 to the elements e_i and degree 2 to the element t. Thus, the degree of $e_{i_1\cdots i_s}\cdot t^{(q)}$ is $s+2q$. So it is straightforward to obtain the formula of $d_{s+2q}(e_{i_1\cdots i_s}t^{(q)})$. We prove the remaining formula, for $d(e_{i_1\cdots i_s})$, by induction. We have

$$d(e_1e_2) = d(e_1)e_2 - e_1d(e_2) = (1\otimes x_1 - x_1\otimes 1)e_2 - (1\otimes x_2 - x_2\otimes 1)e_1$$

by (2.2). We assume that the formula is true up to any $s-1$. At s, we fix an $j\in[s]$ and use (2.2) as follows:

$$\begin{aligned} d(e_{i_1\cdots i_s}) &= (-1)^{j-1}d(e_{i_j}\wedge e_{i_1\cdots\widehat{i_j}\cdots i_s}) \\ &= (-1)^{j-1}\left(d(e_{i_j})e_{i_1\cdots\widehat{i_j}\cdots i_s} - e_{i_j}d(e_{i_1\cdots\widehat{i_j}\cdots i_s})\right) \\ &= (-1)^{j-1}d(e_{i_j})e_{i_1\cdots\widehat{i_j}\cdots i_s} + \sum_{u\neq j}(-1)^{u-1}d(e_{i_u})e_{i_1\cdots\widehat{i_u}\cdots i_s} \\ &= \sum_{j=1}^{s}(-1)^{j-1}(1\otimes x_{i_j} - x_{i_j}\otimes 1)e_{i_1\cdots\widehat{i_j}\cdots i_s} \end{aligned}$$

since

$$d(e_{i_1\cdots\widehat{i_j}\cdots i_s}) = \sum_{u=1}^{j-1}(-1)^{u-1}d(e_{i_u})e_{i_1\cdots\widehat{i_u}\cdots\widehat{i_j}\cdots i_s} + \sum_{u=j+1}^{s}(-1)^{u-2}d(e_{i_u})e_{i_1\cdots\widehat{i_j}\cdots\widehat{i_u}\cdots i_s}$$

by inductive hypothesis as well as

$$\begin{aligned}(-1)^{j-1}(-e_{i_j}) \wedge (-1)^{u-1}e_{i_1\cdots\widehat{i_u}\cdots\widehat{i_j}\cdots i_s} &= (-1)^{u+j-1}e_{i_j} \wedge e_{i_1\cdots\widehat{i_u}\cdots\widehat{i_j}\cdots i_s}\\ &= (-1)^{u-1}e_{i_1\cdots\widehat{i_u}\cdots i_s}\end{aligned}$$

and similarly

$$(-1)^{j-1}(-e_{i_j}) \wedge (-1)^{u-2}e_{i_1\cdots\widehat{i_j}\cdots\widehat{i_u}\cdots i_s} = (-1)^{u-1}e_{i_1\cdots\widehat{i_u}\cdots i_s}.$$

Applying the contravariant functor $\mathrm{Hom}_{A^e}(-, A)$ to the truncation of the resolution (2.1), we obtain a complex of A^e-modules and A^e-homomorphisms

$$0 \longrightarrow \mathrm{Hom}_{A^e}(F_0, A) \xrightarrow{d^1} \mathrm{Hom}_{A^e}(F_1, A) \xrightarrow{d^2} \mathrm{Hom}_{A^e}(F_2, A) \longrightarrow \cdots$$

with the differentials $d^\bullet$ are canonical maps. From the last complex, by passing to cohomology one gets the Hochschild cohomology $\mathrm{HH}^*(A)$ of A. So far we have constructed the Hochschild cohomology of the algebra A to be an $\mathbb{N}$-graded A^e-module.

2.3 $\mathrm{HH}^*(A)$ as a k-space

In this section, we consider the Hochschild cohomology as a graded k-space and give a description of the structure of this module via simpler complexes. For any $m \in \mathbb{N}$, let $\overline{F}_m$ be the k-space spanned by the same basis elements as F_m. By the definition of F_m, the number of basis elements is finite. There is an isomorphism between the following k-spaces

$$\mathrm{Hom}_{A^e}(F_m, A) \cong \mathrm{Hom}_k(\overline{F}_m, A)$$

for all $m \in \mathbb{N}$. Thus, we get a new complex of k-spaces and k-homomorphisms

$$0 \longrightarrow \mathrm{Hom}_k(\overline{F}_0, A) \xrightarrow{\partial^1} \mathrm{Hom}_k(\overline{F}_1, A) \xrightarrow{\partial^2} \mathrm{Hom}_k(\overline{F}_2, A) \longrightarrow \cdots, \tag{2.3}$$

where the maps ∂^m (for $m \in \mathbb{N}$) will be stated in the subsequent lemma. For abbreviation, we often use ∂ instead of ∂^m when we do not need to specify the index m.

Now let us introduce some notation which will appear in the sequel:

- $[n] := \{1, 2, \dots, n\}$;
- $\operatorname{sgn}(i, I) := (-1)^{|\{j \in I | j < i\}|}$, where $|S|$ is the cardinality of the set S;
- $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where α is the lattice point $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathbb{N}^n$;
- $\operatorname{supp}(\mathbf{x}^\alpha) := \{i \mid \alpha_i > 0\}$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ as above.

Let $e_I t^{(q)}$ be a basis element in $\overline{F}_m$ and $\mathbf{x}^\alpha$ a basis element in A. We denote by $(e_I t^{(q)}, \mathbf{x}^\alpha)$ the k-linear map in $\operatorname{Hom}_k(\overline{F}_m, A)$ which sends $e_I t^{(q)}$ to $\mathbf{x}^\alpha$ and other basis elements to 0, i.e.,

$$(e_I t^{(q)}, \mathbf{x}^\alpha)(e_J t^{(p)}) = \begin{cases} \mathbf{x}^\alpha & \text{if } J = I \text{ and } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

Let us call these basis elements the *standard* elements. Since $\overline{F}_m$ is a finite dimensional space, these standard elements form a basis of $\operatorname{Hom}_k(\overline{F}_m, A)$. We also use the same notation, $(e_I t^{(q)}, \mathbf{x}^\alpha)$, for the residue class in $\operatorname{HH}^*(A)$.

Lemma 2.1. *Let $(e_I t^{(q)}, \mathbf{x}^\alpha)$ be a standard element in $\operatorname{Hom}_k(\overline{F}_m, A)$. We have that*

$$\partial^{m+1}(e_I t^{(q)}, \mathbf{x}^\alpha) = \sum_{i \in I \smallsetminus \operatorname{supp}(\mathbf{x}^\alpha)} \operatorname{sgn}(i, I)(e_{I \smallsetminus \{i\}} t^{(q+1)}, \mathbf{x}^\alpha \cdot x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n).$$

Proof. Let us consider the following diagram

$$\begin{tikzcd} \operatorname{Hom}_{A^e}(F_m, A) \arrow[r, "d^{m+1}"] \arrow[d, leftrightarrow, "\cong"'] & \operatorname{Hom}_{A^e}(F_{m+1}, A) \arrow[d, leftrightarrow, "\cong"'] \\ \operatorname{Hom}_k(\overline{F}_m, A) \arrow[r, "\partial^{m+1}"] & \operatorname{Hom}_k(\overline{F}_{m+1}, A). \end{tikzcd}$$

By combining the isomorphisms and homomorphisms, we can derive ∂^{m+1} from d^{m+1} straightforwardly. As $f = (e_I t^{(q)}, \mathbf{x}^\alpha)$ is a standard element of $\operatorname{Hom}_k(\overline{F}_m, A)$, f is identified with a function in $\operatorname{Hom}_{A^e}(F_m, A)$. A direct calculation shows that it is $(e_I t^{(q)}, \mathbf{x}^\alpha)$, which is also denoted by f by abuse of notation. The canonical homomorphism

$$\begin{array}{rccc} d^{m+1}: & \operatorname{Hom}_{A^e}(F_m, A) & \longrightarrow & \operatorname{Hom}_{A^e}(F_{m+1}, A) \\ & f & \longmapsto & d^{m+1}(f) := f \circ d_{m+1} \end{array}$$

sends f to $d^{m+1}(f)$, an A^e-homomorphism from F_{m+1} to A.

Recall that A is a left and right A^e-module by the scalar multiplication: $(a \otimes b) \cdot c = acb$ and $c \cdot (a \otimes b) = bca$ respectively. Let $e_{i_1 \cdots i_s} t^{(p)}$ (shortly, $e_J t^{(p)}$) be a basis element in F_{m+1}. By the formula of the differential d, one has

$$d^{m+1}(f)(e_J t^{(p)}) = f\left(d_{m+1}(e_J t^{(p)})\right) = K_1 + K_2,$$

where

$$\begin{aligned} K_1 : &= f\left(\sum_{j=1}^{s} (-1)^{j+1} d_1(e_{i_j}) e_{J \setminus \{i_j\}} t^{(p)}\right) \\ &= \sum_{j=1}^{s} (-1)^{j+1} (1 \otimes x_{i_j} - x_{i_j} \otimes 1) \cdot f(e_{J \setminus \{i_j\}} t^{(p)}) \\ &= \sum_{j=1}^{s} (-1)^{j+1} \left(f(e_{J \setminus \{i_j\}} t^{(p)}) \cdot x_{i_j} - x_{i_j} \cdot f(e_{J \setminus \{i_j\}} t^{(p)})\right) = 0; \text{ and} \end{aligned}$$

$$\begin{aligned} K_2 : &= f\left(d_2(t) e_J t^{(p-1)}\right) = f\left(\sum_{i=1}^{n} x_1 \cdots x_{i-1} \otimes x_{i+1} \cdots x_n \cdot e_i \wedge e_J t^{(p-1)}\right) \\ &= \sum_{i \in [n] \setminus J} \operatorname{sgn}(i, J) x_1 \cdots x_{i-1} \cdot f(e_{J \cup \{i\}} \cdot t^{(p-1)}) \cdot x_{i+1} \cdots x_n \\ &= \sum_{i \in [n] \setminus J} \operatorname{sgn}(i, J) x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \cdot f(e_{J \cup \{i\}} \cdot t^{(p-1)}). \end{aligned}$$

Since

$$f(e_{J \cup \{i\}} t^{(p-1)}) = \begin{cases} \mathbf{x}^\alpha & \text{if } J \cup \{i\} = I \text{ and } p - 1 = q, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$K_2 = \begin{cases} \operatorname{sgn}(i, J) x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \cdot \mathbf{x}^\alpha & \text{if } J = I \setminus \{i\} \text{ and } p = q + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where i is an element in I. Note that $\operatorname{sgn}(i, J) = \operatorname{sgn}(i, I)$ when $J = I \setminus \{i\}$ and if $i \in \operatorname{supp}(\mathbf{x}^\alpha)$, then $x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n \cdot \mathbf{x}^\alpha = 0$ in A. Hence, the formula follows. □

From this lemma, we are going to derive some consequences about properties of standard elements in the kernel and the image of ∂.

Corollary 2.2. *Let $(e_I t^{(q)}, \mathbf{x}^\alpha)$ be a standard element. We have that*

$$\partial(e_I t^{(q)}, \mathbf{x}^\alpha) = 0$$

if and only if I is a subset of $\operatorname{supp}(\mathbf{x}^\alpha)$*.*

Corollary 2.3. *The non-zero standard element* $(e_I t^{(q)}, \mathbf{x}^\alpha)$ *occurs as a component of some element in* $\operatorname{Im}(\partial)$ *if and only if the two following conditions hold:*

(i) $q > 0$*; and*

(ii) There exists some index i *in* $[n]\smallsetminus I$ *which satisfies* $\operatorname{supp}(\mathbf{x}^\alpha) = [n]\smallsetminus\{i\}$.

Proof. The first part "⇒" is straightforward by observing the formula of ∂ in Lemma 2.1. Conversely, if $(e_I t^{(q)}, \mathbf{x}^\alpha)$ satisfies the two conditions (i) and (ii), it is a component in the image of the element $\left(e_{I\cup\{i\}} t^{(q-1)}, \dfrac{\mathbf{x}^\alpha}{x_1 \cdots \widehat{x_i} \cdots x_n}\right)$. □

Corollary 2.4. *If* $(e_I t^{(q)}, \mathbf{x}^\alpha)$ *and* $(e_J t^{(p)}, \mathbf{x}^\beta)$ *are standard elements such that their images under* ∂ *have some non-zero component in common, then they are identical.*

Proof. This is an immediate consequence of Corollary 2.3. □

The k-space $\operatorname{Hom}_k(\overline{F}_m, A)$ is generated by standard elements of degree m, i.e.,

$$\operatorname{Hom}_k(\overline{F}_m, A) = \bigoplus_{|I|+2q=m} k(e_I t^{(q)}, \mathbf{x}^\alpha),$$

where $k(e_I t^{(q)}, \mathbf{x}^\alpha)$ is the k-module generated by the element $(e_I t^{(q)}, \mathbf{x}^\alpha)$. Let Γ be the set of standard elements that are not in any component of $\operatorname{Im}(\partial)$. Let us take an element $\gamma = (e_I t^{(q)}, \mathbf{x}^\alpha)$ in Γ. The image of γ under ∂ is given by Lemma 2.1:

$$\partial(\gamma) = \sum_{i \in I\smallsetminus \operatorname{supp}(\mathbf{x}^\alpha)} \operatorname{sgn}(i, I)(e_{I\smallsetminus\{i\}} t^{(q+1)}, \mathbf{x}^\alpha \cdot x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n).$$

From here, we construct a complex M_γ:

$$\begin{aligned} \cdots 0 \longrightarrow & k(e_I t^{(q)}, \mathbf{x}^\alpha) \longrightarrow \\ \longrightarrow & \bigoplus_{i \in I\smallsetminus \operatorname{supp}(\mathbf{x}^\alpha)} k(e_{I\smallsetminus\{i\}} t^{(q+1)}, \mathbf{x}^\alpha \cdot x_1 \cdots x_{i-1} \cdot x_{i+1} \cdots x_n) \longrightarrow 0 \cdots, \end{aligned}$$

where the k-maps are taken to be the k-maps ∂ in (2.3) restricted to the corresponding subspaces. We obtain that each such complex is a subcomplex of (2.3), moreover it is a direct summand of (2.3). By Corollary 2.4, the subcomplexes indexed by elements in Γ have zero intersection. The following theorem shows that the complex (2.3) can be written as a direct sum of subcomplexes indexed by the elements in Γ.

Theorem 2.5. *We have the following direct sum:*

$$\mathrm{Hom}_k(\overline{F}_\bullet, A) = \bigoplus_{\gamma\in\Gamma} M_\gamma.$$

Proof. It is obvious that we have the inclusion $\bigoplus_{\gamma\in\Gamma} M_\gamma \subseteq \mathrm{Hom}_k(\overline{F}_\bullet, A)$ since M_γ is a subcomplex of $\mathrm{Hom}_k(\overline{F}_\bullet, A)$ for all $\gamma \in \Gamma$. For the inverse inclusion, let us consider an arbitrary non-zero basis element $E = (e_I t^{(q)}, \mathbf{x}^\alpha)$ in $\mathrm{Hom}_k(\overline{F}_m, A)$. There are two conceivable cases:
Case 1. If E is a component in the image $\mathrm{Im}(\partial)$, then there exists a unique element γ in $\mathrm{Hom}_k(\overline{F}_{m-1}, A)$, stated in Corollary 2.3, not in the kernel of ∂ such that $\partial(\gamma)$ contains E as a component. It is obvious that $\gamma \in \Gamma$ and the subcomplex M_γ includes E.
Case 2. If E is not any component in $\mathrm{Im}(\partial)$, then E belongs to the subcomplex indexed by E itself, M_E.
Hence, the complex (2.3) can be split into a direct sum of simpler subcomplexes as desired. □

Corollary 2.6. *For each i in $\mathbb{N}$, we have the following isomorphism:*

$$\mathrm{H}^i(\mathrm{Hom}_k(\overline{F}_\bullet, A)) \cong \bigoplus_{\gamma\in\Gamma} \mathrm{H}^i(M_\gamma).$$

Example 2.7. We will see here a slice of a splitting complex for the case $n = 2$, which is also used to illustrate the results throughout this chapter. Let $A = k[x, y]/\langle xy\rangle$, where k is a field. Then some of the subcomplexes in Corollary 2.6 for A are shown below:

$$\cdots$$

$$0 \longrightarrow k(e_1, y^3) \longrightarrow k(t, y^4) \longrightarrow 0$$

$$0 \longrightarrow k(e_1 t, y) \longrightarrow k(t^2, y^2) \longrightarrow 0$$

$$0 \longrightarrow k(t, x^5) \longrightarrow 0$$

$$0 \longrightarrow k(e_1 e_2, 1) \longrightarrow \begin{matrix} k(e_1 t, x) \\ \oplus \\ k(e_2 t, y) \end{matrix} \longrightarrow 0$$

$$\cdots$$

From Corollaries 2.2, 2.3 and Theorem 2.5, we mention here a consequence about the kernel and image of the map ∂.

Remark 2.8. The kernel of ∂ is spanned by the elements $(e_I t^{(q)}, \mathbf{x}^\alpha)$, where I is a subset of $\operatorname{supp}(\mathbf{x}^\alpha)$. The image of ∂ is spanned by $\partial(e_I t^{(q)}, \mathbf{x}^\alpha)$, where I is not a subset of $\operatorname{supp}(\mathbf{x}^\alpha)$.

We have obtained a description of $\mathrm{HH}^*(A)$ as a k-module via simpler complexes. Next we will equip $\mathrm{HH}^*(A)$ with a multiplication which gives this module the structure of a k-algebra.

2.4 An explicit chain map

The goal of this section is to provide an explicit chain map in case of our resolution in order to construct the multiplication on $\mathrm{HH}^*(A)$ in terms of the Yoneda product, which has been recalled in Chapter 1, Section 1.5. This means that we shall find a formula of the chain map $\tilde{f}$, or more precise, formulas of $\tilde{f}_0$, $\tilde{f}_1$, and so on. By direct computing, we obtain formulas for the first homomorphisms, $\tilde{f}_0$, $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$, which make the following diagram commute. Then we generalize the formula for higher index, any $\tilde{f}_j$, which can be found in the subsequent proposition. We prove the proposition by using induction on the index j.

$$\begin{tikzcd}
\cdots \arrow[r] & F_{i+2} \arrow[r, "d_{i+2}"] \arrow[d, "\tilde{f}_2"] & F_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\tilde{f}_1"] & F_i \arrow[d, "\tilde{f}_0"] \arrow[dr, "f"] & \\
\cdots \arrow[r] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, "\mu"] & A
\end{tikzcd} \tag{2.4}$$

We present now some auxiliary results on computations before we state the formula of $\tilde{f}$. The lemma below can be seen as a generalization of Corollary 2.2.

Lemma 2.9. *Let $f : F_i \to A$ be a cocycle and $e_{i_1\cdots i_m} t^{(q)}$ a basis element in F_i. We then have that x_{i_j} is a divisor of $f(e_{i_1\cdots i_m} t^{(q)})$ for all $j \in [m]$ if $f(e_{i_1\cdots i_m} t^{(q)})$ is non-zero.*

Proof. Without loss of generality, we assume that $j = 1$. Let us consider the element $e_{i_2\cdots i_m}t^{(q+1)} \in F_{i+1}$. Applying the differential map, one has that:

$$\begin{aligned} d_{i+1}(e_{i_2\cdots i_m}t^{(q+1)}) = &\sum_{j=2}^{m}(-1)^j(1\otimes x_{i_{j+1}} - x_{i_{j+1}}\otimes 1)e_{i_2\cdots\widehat{i_j}\cdots i_m}t^{(q+1)} \\ &+\sum_{j=1}^{n}(x_1\cdots x_{j-1}\otimes x_{j+1}\cdots x_n)e_j\wedge e_{i_2\cdots i_m}t^{(q)}. \end{aligned}$$

Therefore,

$$f\circ d_{i+1}(e_{i_2\cdots i_m}t^{(q+1)}) = \sum_{j=1}^{n}(x_1\cdots x_{j-1}\cdot x_{j+1}\cdots x_n)f(e_j\wedge e_{i_2\cdots i_m}t^{(q)}).$$

Since $f\circ d_{i+1}(e_{i_2\cdots i_m}t^{(q+1)}) = 0$, we have that:

$$\sum_{j=1}^{n}(x_1\cdots x_{j-1}\cdot x_{j+1}\cdots x_n)f(e_j\wedge e_{i_2\cdots i_m}t^{(q)}) = 0.$$

Multiplying both sides by $x_1\cdots x_{i_1-1}\cdot x_{i_1+1}\cdots x_n$, we obtain that:

$$x_1^2\cdots x_{i_1-1}^2\cdot x_{i_1+1}^2\cdots x_n^2 f(e_{i_1}\wedge e_{i_2\cdots i_m}t^{(q)}) = 0.$$

So x_{i_1} divides $f(e_{i_1\cdots i_m}t^{(q)})$. □

For $z\in[n]$, we set $U_z := \sum_{j=z+1}^{n}(x_1\cdots x_{j-1}\otimes x_{j+1}\cdots x_n)e_j$ with the convention that $U_n = 0$. This notation will be used for the rest of this chapter.

Lemma 2.10. *We have that $d(U_z) = x_1\cdots x_z\otimes x_{z+1}\cdots x_n$.*

Proof. Applying the differential map, one gets

$$\begin{aligned} d(U_z) &= \sum_{j=z+1}^{n}(x_1\cdots x_{j-1}\otimes x_{j+1}\cdots x_n)(1\otimes x_j - x_j\otimes 1) \\ &= \sum_{j=z+1}^{n}(x_1\cdots x_{j-1}\otimes x_j\cdots x_n - x_1\cdots x_j\otimes x_{j+1}\cdots x_n) \\ &= x_1\cdots x_z\otimes x_{z+1}\cdots x_n. \end{aligned}$$

Thus the assertion follows. □

Now we are in the position to obtain the formula of the chain map.

Proposition 2.11. *Let $f : F_i \to A$ be a cocycle in* $\mathrm{Hom}_{A^e}(F_i, A)$. *For a given $j \in \mathbb{N}$, we define an A^e-homomorphism $\tilde{f}_j : F_{i+j} \to F_j$ as follows. Let $x = e_{i_1 \cdots i_m} t^{(q)}$ be a basis element in F_{i+j} and define:*

$$\tilde{f}_j(x) = \sum_{\mathcal{M}} (-1)^{ms+j_1+\cdots+j_r-r} U_{l_s} \cdots U_{l_1} \cdot e_{i_{j_r} \cdots i_{j_1}} t^{(u)} \cdot \\ \cdot \frac{f(e_{i_1 \cdots \widehat{i}_{j_1} \cdots \widehat{i}_{j_r} \cdots i_m} \wedge e_{l_1 \cdots l_s} t^{(q-u-s)})}{x_{l_1} \cdots x_{l_s}} \otimes 1,$$

where the sum is indexed by $\mathcal{M}$ which consists of triples (u, J, L), where $J = \{j_1, \ldots, j_r\}$ and $L = \{l_1, \ldots, l_s\}$ satisfy the following conditions:

$$r + s + 2u = j; 1 \le j_1 < \cdots < j_r \le m; \text{ and } 1 \le l_1 < \cdots < l_s \le n.$$

The chain map $\tilde{f}$ given as above makes the diagram (2.4) *commute.*

The proof of this proposition is given by the combination of the following remarks and lemmas.

Remark 2.12. By Lemma 2.9, we have that $\dfrac{f(e_{i_1 \cdots \widehat{i}_{j_1} \cdots \widehat{i}_{j_r} \cdots i_m} \wedge e_{l_1 \cdots l_s} t^{(q-u-s)})}{x_{l_1} \cdots x_{l_s}}$ is an element in A. Also by this lemma, all elements in the form of a 'fraction' like above are in A throughout this section.

In our first lemma, we can see how the first step, on $\tilde{f}_0$, works with any homomorphism f, not necessarily satisfying $f \circ d_{i+1} = 0$.

Lemma 2.13. *For any homomorphism f in* $\mathrm{Hom}_{A^e}(F_i, A)$, *we have the commutative diagram below:*

$$\begin{tikzcd} F_i \arrow[d, "\tilde{f}_0"'] \arrow[dr, "f"] & \\ F_0 \arrow[r, "\mu"] & A \end{tikzcd}$$

Proof. Indeed, for any basis element $x = e_{i_1 \cdots i_m} t^{(q)}$ in F_i, we have $r+s+2u = 0$. Then, $r = s = u = 0$ is the only option and one gets that

$$\tilde{f}_0(x) = f(x) \otimes 1$$

So we obtain that $\mu \tilde{f}_0 = f$. □

We now turn to the rest of the diagram in the following lemma.

Lemma 2.14. *The homomorphisms defined in Proposition 2.11 make the following diagram commute:*

$$\begin{tikzcd} F_{i+j} \arrow[r, "d_{i+j}"] \arrow[d, "\tilde{f}_j"'] & F_{i+j-1} \arrow[d, "\tilde{f}_{j-1}"] \\ F_j \arrow[r, "d_j"] & F_{j-1} \end{tikzcd}$$

Proof. Let us fix a non-zero standard cocycle $f = (e_K t^{(v)}, \mathbf{x}^\alpha)$. For an arbitrary basis element $x = e_{i_1 \cdots i_m} t^{(q)}$ in F_{i+j}, we show that the diagram commutes by proving that $d_j \circ \tilde{f}_j(x) = \tilde{f}_{j-1} \circ d_{i+j}(x)$.
To simplify the proof, let us introduce some notation. Let M and N be two sets of natural numbers such that $M \cap N = \emptyset$. We denote by $\operatorname{sgn}(M, N)$ the power of -1 such that $e_M \wedge e_N = \operatorname{sgn}(M, N) e_{M \cup N}$. It is simple to show that

$$\operatorname{sgn}(M, N) = \prod_{i \in M} \operatorname{sgn}(i, N).$$

Next, let us consider the summands inside the formula $\tilde{f}_j(x)$ in Proposition 2.11. We shall interpret and simplify the general formula for our case, $f = (e_K t^{(v)}, \mathbf{x}^\alpha)$. It suffices to work on non-zero summands and forget the zero ones. We can see that

$$f(e_{i_1 \cdots \widehat{i}_{j_1} \cdots \widehat{i}_{j_r} \cdots i_m} \wedge e_{l_1 \cdots l_s} t^{(q-u-s)}) = f(e_{(I \smallsetminus J) \cup L} t^{(q-u-s)})$$

up to sign if $(I \smallsetminus J) \cap L = \emptyset$, where $I := \{i_1, \ldots, i_m\}$, $J := \{i_{j_1}, \ldots, i_{j_r}\}$ and $L := \{l_1, \ldots, l_s\}$. Since $f = (e_K t^{(v)}, \mathbf{x}^\alpha)$, we have that $f(e_{(I \smallsetminus J) \cup L} t^{(q-u-s)})$ is non-zero ($= \mathbf{x}^\alpha$) if and only if $(I \smallsetminus J) \cup L = K$ and $q - u - s = v$.
In case of a non-zero summand, we have $(I \smallsetminus J) \cup L = K$ and $(I \smallsetminus J) \cap L = \emptyset$. Hence, we can set $N := I \smallsetminus J = K \smallsetminus L$. We then have that N is a subset of $I \cap K$ and $u = q - v - s = q - v - |K \smallsetminus N|$. As (I, K, q, v) are already fixed, once we know N, we can trace back to (u, J, L) uniquely. All the above observations result that the index (u, J, L) corresponds to a subset $N \subseteq I \cap K$ such that $q - v - |K \smallsetminus N| \geq 0$ (because $u \geq 0$) and the formula for $\tilde{f}_j(x)$ in our case becomes a sum indexed by the subsets N of $I \cap K$ such that (abbreviated by 's.t.') $|N| \geq |K| - q + v$:

$$\tilde{f}_j(x) = \sum_{\substack{N \subseteq I \cap K \\ \text{s.t. } |N| \geq |K| - q + v}} (\dagger) U_{K \smallsetminus N} \cdot e_{I \smallsetminus N} \cdot t^{(q-v-|K \smallsetminus N|)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \smallsetminus N}} \otimes 1,$$

where the sign of the summand corresponding to index N is

$$(\dagger) = (-1)^{m\cdot|K\smallsetminus N|}\cdot \operatorname{sgn}(I\smallsetminus N, N)\cdot \operatorname{sgn}(N, K\smallsetminus N);$$

$\mathbf{x}_M := \prod_{i\in M} x_i$; and whereby U_M (for any $M=\{i_1,\ldots,i_r\}$, $i_1<\cdots<i_r$), we mean $U_{i_r}\wedge\cdots\wedge U_{i_1}$.

Now we will show that $d_j\circ\tilde{f}_j(x)=\tilde{f}_{j-1}\circ d_{i+j}(x)$. All the below computations are to be considered as equations up to sign. The sign of the formula will be considered later.

By applying directly the formula of d to $\tilde{f}_j(x)$, we have that

$$\begin{aligned} d_j\circ\tilde{f}_j(x) = & \sum_{\substack{N\subseteq I\cap K \\ \text{s.t. } |N|\geq|K|-q+v \\ i\in K\smallsetminus N}} d(U_i)\cdot U_{(K\smallsetminus N)\smallsetminus\{i\}}\cdot e_{I\smallsetminus N}t^{(q-v-|K\smallsetminus N|)}\cdot\frac{\mathbf{x}^\alpha}{\mathbf{x}_{K\smallsetminus N}}\otimes 1 \\ & + \sum_{\substack{N\subseteq I\cap K \\ \text{s.t. } |N|\geq|K|-q+v \\ i\in I\smallsetminus N}} d(e_i)\cdot U_{K\smallsetminus N}\cdot e_{(I\smallsetminus N)\smallsetminus\{i\}}t^{(q-v-|K\smallsetminus N|)}\cdot\frac{\mathbf{x}^\alpha}{\mathbf{x}_{K\smallsetminus N}}\otimes 1 \\ & + \sum_{\substack{N\subseteq I\cap K \\ \text{s.t. } |N|>|K|-q+v \\ i\in[n]}} X_i\cdot U_{K\smallsetminus N}\cdot e_{(I\smallsetminus N)\cup\{i\}}t^{(q-v-|K\smallsetminus N|-1)}\cdot\frac{\mathbf{x}^\alpha}{\mathbf{x}_{K\smallsetminus N}}\otimes 1, \end{aligned}$$

where $X_i := x_1\cdots x_{i-1}\otimes x_{i+1}\cdots x_n$. Let us denote these three sums by (L_1), (L_2) and (L_3) respectively.

For the right hand side, we have that

$$d_{i+j}(x)=d_{i+j}(e_It^{(q)})=\sum_{i\in I} d(e_i)\cdot e_{I\smallsetminus\{i\}}t^{(q)}+[q>0]\sum_{i\in[n]\smallsetminus I} X_i\cdot e_{I\cup\{i\}}t^{(q-1)},$$

where $[P]=\begin{cases} 1 & \text{if } P \text{ true,}\\ 0 & \text{if } P \text{ false.}\end{cases}$

We divide the first sum into two smaller parts which correspond to two components of the disjoint union $I=(I\smallsetminus K)\cup(I\cap K)$. We will see the relevance later. Next, we apply the formula of $\tilde{f}$ to the above sum:

$$\tilde{f}_{j-1}\circ d_{i+j}(x)=(R_1)+(R_2)+(R_3),$$

where

$$(R_1) := \sum_{\substack{N \subseteq (I \smallsetminus \{i\}) \cap K \\ \text{s.t. } |N| \geq |K| - q + v \\ i \in I \smallsetminus K}} d(e_i) \cdot U_{K \smallsetminus N} \cdot e_{(I \smallsetminus \{i\}) \smallsetminus N} \cdot t^{(q-v-|K \smallsetminus N|)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \smallsetminus N}} \otimes 1,$$

$$(R_2) := \sum_{\substack{N \subseteq (I \smallsetminus \{i\}) \cap K \\ \text{s.t. } |N| \geq |K| - q + v \\ i \in I \cap K}} d(e_i) \cdot U_{K \smallsetminus N} \cdot e_{(I \smallsetminus \{i\}) \smallsetminus N} \cdot t^{(q-v-|K \smallsetminus N|)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \smallsetminus N}} \otimes 1,$$

$$(R_3) :=$$

$$[q > 0] \sum_{\substack{N \subseteq (I \cup \{i\}) \cap K \\ \text{s.t. } |N| \geq |K| - q + v + 1 \\ i \in [n] \smallsetminus I}} X_i \cdot U_{K \smallsetminus N} \cdot e_{(I \cup \{i\}) \smallsetminus N} \cdot t^{(q-1-v-|K \smallsetminus N|)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \smallsetminus N}} \otimes 1.$$

So far we have got the formula for the left and the right hand sides, which are sums over N and i. To show the equality of the two sides, once again we divide some of the above sums into smaller ones based on i (no change for N) as follows.

- For (L_1), we have $K \smallsetminus N = (K \smallsetminus I) \cup ((K \cap I) \smallsetminus N)$, where the union is disjoint. So (L_1) can be rewritten as a sum of two smaller sums over $K \smallsetminus I$ and $(K \cap I) \smallsetminus N$, denoted by (L_{1A}) and (L_{1B}) respectively, as follows:

$$(L_1) = \sum_{\substack{N \subseteq I \cap K \\ \text{s.t. } |N| \geq |K| - q + v \\ i \in K \smallsetminus I}} d(U_i) \cdot U_{(K \smallsetminus N) \smallsetminus \{i\}} \cdot e_{I \smallsetminus N} t^{(q-v-|K \smallsetminus N|)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \smallsetminus N}} \otimes 1$$
$$+ \sum_{\substack{N \subseteq I \cap K \\ \text{s.t. } |N| \geq |K| - q + v \\ i \in (K \cap I) \smallsetminus N}} d(U_i) \cdot U_{(K \smallsetminus N) \smallsetminus \{i\}} \cdot e_{I \smallsetminus N} t^{(q-v-|K \smallsetminus N|)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \smallsetminus N}} \otimes 1.$$

- For (L_3), if $i \in I \smallsetminus N$, then $e_{(I \smallsetminus N) \cup \{i\}} = 0$. So we only need to consider the cases in which $i \in [n] \smallsetminus (I \smallsetminus N)$. Therefore, we can write (L_3) as a sum of (L_{3A}), (L_{3B}) and (L_{3C}) corresponding to $[n] \smallsetminus (I \cup K)$, N and $K \smallsetminus I$ respectively.

- Similarly, (R_3) is rewritten as a sum of three parts (R_{3A}), (R_{3B}) and (R_{3C}) which correspond to the sums over $[n] \smallsetminus (I \cup K)$, $(K \smallsetminus I) \cap N$ and $(K \smallsetminus I) \smallsetminus N$.

Before comparing the two sides, we look back to (R_1) and (R_2). As $N \subseteq (I \setminus \{i\}) \cap K$ and $i \in I \setminus K$, we can infer that $N \subseteq I \cap K$ in (R_1). Similarly, $N \subseteq (I \setminus \{i\}) \cap K$ and $i \in I \cap K$ in (R_2) yield that $i \notin (I \setminus \{i\}) \cap K$ and hence, $i \notin N$. Thus, we can replace the conditions $N \subseteq (I \setminus \{i\}) \cap K$ and $i \in I \cap K$ by $N \subseteq I \cap K$ and $i \in (I \cap K) \setminus N$.
In order to get

$$(L_1) + (L_2) + (L_3) = (R_1) + (R_2) + (R_3),$$

we prove that

(i) $(L_2) = (R_1) + (R_2)$;

(ii) $(L_{3A}) = (R_{3A})$;

(iii) $(L_{1A}) = (R_{3B})$;

(iv) $(L_{3B}) = (R_{3C})$;

(v) $(L_{1B}) + (L_{3C}) = 0$.

As mentioned before, we will first show that these sums are identical up to sign.

Part (i). Since $i \in I \setminus N = (I \setminus K) \cup ((I \cap K) \setminus N)$, we have $(I \setminus N) \setminus \{i\} = (I \setminus \{i\}) \setminus N$. Thus, $(L_2) = (R_1) + (R_2)$.

Part (ii). We have (ii) because $i \in [n] \setminus (I \cup K)$ implies that $I \cap K = (I \cup \{i\}) \cap K$ and $(I \setminus N) \cup \{i\} = (I \cup \{i\}) \setminus N$.

Part (iii). Let $N' := N \cup \{i\}$ and note that

$$d(U_i) \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \setminus N}} \otimes 1 = X_i \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{(K \setminus N) \setminus \{i\}}} \otimes 1$$

for all $i \in K \setminus N$. Then we get

$$(L_{1A}) = \sum_{\substack{N' \subseteq (I \cup \{i\}) \cap K \\ \text{s.t. } |N'| \geq |K| - q + v + 1 \\ i \in (K \setminus I) \cap N'}} X_i \cdot U_{K \setminus N'} \cdot e_{(I \cup \{i\}) \setminus N'} \cdot t^{(q - v - |K| + |N'| - 1)} \cdot \frac{\mathbf{x}^\alpha}{\mathbf{x}_{K \setminus N'}} \otimes 1,$$

which is exactly the sum (R_{3B}).

Part (iv). $N \subseteq (I \cap K) \cup \{i\}$ becomes $N \subseteq (I \cap K)$ and $(I \cup \{i\}) \smallsetminus N = (I \smallsetminus N) \cup \{i\}$ for all $i \in (K \smallsetminus I) \smallsetminus N$. Then (iv) follows.

Part (v). Let $N' = N \smallsetminus \{i\}$. Then one has

$$(L_{3C}) = \sum_{\substack{N' \subseteq I \cap K \\ \text{s.t. } |N'| \geq |K| - q + v \\ i \in (I \cap K) \smallsetminus N'}} d(U_i) \cdot U_{(K \smallsetminus N') \smallsetminus \{i\}} \cdot e_{I \smallsetminus N'} \cdot t^{(q-v-|K|+|N'|)} \cdot \frac{\mathbf{x}^{\alpha}}{\mathbf{x}_{K \smallsetminus N'}} \otimes 1,$$

which is equal to (L_{1B}).

To complete the proof, we show that the signs of the formulas coincide. We denote by $\operatorname{sign}(T)_{\{i,N\}}$ the sign of the summand corresponding to the pair $\{i, N\}$ of the sum (T). The signs for each of the sums at the index $\{i, N\}$ are calculated as follows:

$$\begin{aligned}\operatorname{sign}(L_1)_{\{i,N\}} =& (-1)^{m \cdot |K \smallsetminus N|} \cdot \operatorname{sgn}(I \smallsetminus N, N) \cdot \operatorname{sgn}(N, K \smallsetminus N) \cdot \\ & \cdot \operatorname{sgn}(i, K \smallsetminus N) \cdot (-1)^{|K \smallsetminus N|-1};\end{aligned}$$

$$\begin{aligned}\operatorname{sign}(L_2)_{\{i,N\}} =& \operatorname{sign}(L_3)_{\{i,N\}} \\ =& (-1)^{m \cdot |K \smallsetminus N|} \cdot \operatorname{sgn}(I \smallsetminus N, N) \cdot \operatorname{sgn}(N, K \smallsetminus N) \cdot \\ & \cdot \operatorname{sgn}(i, I \smallsetminus N) \cdot (-1)^{|K \smallsetminus N|};\end{aligned}$$

$$\begin{aligned}\operatorname{sign}(R_1)_{\{i,N\}} &= \operatorname{sign}(R_2)_{\{i,N\}} \\ &= (-1)^{(m-1) \cdot |K \smallsetminus N|} \cdot \operatorname{sgn}((I \smallsetminus \{i\}) \smallsetminus N, N) \cdot \\ & \cdot \operatorname{sgn}(N, K \smallsetminus N) \cdot \operatorname{sgn}(i, I);\end{aligned}$$

$$\begin{aligned}\operatorname{sign}(R_3)_{\{i,N\}} &= (-1)^{(m+1) \cdot |K \smallsetminus N|} \cdot \operatorname{sgn}((I \cup \{i\}) \smallsetminus N, N) \cdot \\ & \cdot \operatorname{sgn}(N, K \smallsetminus N) \cdot \operatorname{sgn}(i, I).\end{aligned}$$

Now we are in the position to prove that the signs in equations (i) to (v) coincide.

For (i), we need to show that $\operatorname{sign}(L_2)_{\{i,N\}} = \operatorname{sign}(R_1)_{\{i,N\}}$, i.e., we must have that

$$\operatorname{sgn}(I \smallsetminus N, N) \cdot \operatorname{sgn}(i, I \smallsetminus N) = \operatorname{sgn}((I \smallsetminus \{i\}) \smallsetminus N, N) \cdot \operatorname{sgn}(i, I).$$

Indeed, since $N \subseteq I$ and $i \in I \smallsetminus N$, we have

$$\operatorname{sgn}(I \smallsetminus N, N) = \operatorname{sgn}((I \smallsetminus \{i\}) \smallsetminus N, N) \cdot \operatorname{sgn}(i, N)$$

and

$$\operatorname{sgn}(i, N) \cdot \operatorname{sgn}(i, I \smallsetminus N) = \operatorname{sgn}(i, I).$$

Then the result follows.
For (ii), since $i \notin I \cup K$ and $N \subseteq I \cap K$, we have

$$\operatorname{sgn}((I \cup \{i\}) \smallsetminus N, N) = \operatorname{sgn}(I \smallsetminus N, N) \cdot \operatorname{sgn}(i, N)$$

and

$$\operatorname{sgn}(i, N) \cdot \operatorname{sgn}(i, I) = \operatorname{sgn}(i, I \smallsetminus N).$$

Thus,

$$\operatorname{sgn}(I \smallsetminus N, N) \cdot \operatorname{sgn}(i, I \smallsetminus N) = \operatorname{sgn}((I \cup \{i\}) \smallsetminus N, N) \cdot \operatorname{sgn}(i, I),$$

which implies that $\operatorname{sign}(L_{3A})_{\{i,N\}} = \operatorname{sign}(R_{3A})_{\{i,N\}}$.
For (iii), we need to show that $\operatorname{sign}(L_{1A})_{\{i,N'\}} = \operatorname{sign}(R_{3B})_{\{i,N\}}$. First we need to deduce $\operatorname{sign}(L_{1A})_{\{i,N'\}}$ from $\operatorname{sign}(L_{1A})_{\{i,N\}}$, where $N' = N \cup \{i\}$, $i \in K \smallsetminus I$, and $N \subseteq I \cap K$. We have the following identities:

$$|K \smallsetminus N| = |K \smallsetminus N'| + 1,$$

and

$$\begin{aligned}
\operatorname{sgn}((I \cup \{i\}) \smallsetminus N', N') &= \prod_{\substack{j \in (I \cup \{i\}) \smallsetminus N' \\ j<i}} \operatorname{sgn}(j, N') \cdot \prod_{\substack{j \in (I \cup \{i\}) \smallsetminus N' \\ j>i}} \operatorname{sgn}(j, N') \\
&= \prod_{\substack{j \in I \smallsetminus N \\ j<i}} \operatorname{sgn}(j, N) \cdot \prod_{\substack{j \in I \smallsetminus N \\ j>i}} \operatorname{sgn}(j, N) \cdot (-1)^{|\{j \in I \smallsetminus N | j>i\}|} \\
&= \operatorname{sgn}(I \smallsetminus N, N) \cdot (-1)^{m+1-|N'|} \cdot \operatorname{sgn}(i, I \smallsetminus N').
\end{aligned}$$

This implies that

$$\operatorname{sgn}(I \smallsetminus N, N) = (-1)^{m+1-|N'|} \cdot \operatorname{sgn}((I \cup \{i\}) \smallsetminus N', N') \cdot \operatorname{sgn}(i, I \smallsetminus N').$$

By a similar argument, we have

$$\operatorname{sgn}(N, K \smallsetminus N) = (-1)^{|N'|-1} \cdot \operatorname{sgn}(N', K \smallsetminus N') \cdot \operatorname{sgn}(i, K \smallsetminus N') \cdot \operatorname{sgn}(i, N').$$

Together with $\mathrm{sgn}(i, K \setminus N) = \mathrm{sgn}(i, K \setminus N')$, we have

$$\begin{aligned}\mathrm{sign}(L_{1A})_{\{i,N'\}} = (-1)^{(m+1)\cdot|K\setminus N'|} \cdot \mathrm{sgn}((I \cup \{i\}) \setminus N', N') \cdot \cdot \mathrm{sgn}(i, I \setminus N') \\ \cdot \mathrm{sgn}(N', K \setminus N') \cdot \mathrm{sgn}(i, N') \\ = (-1)^{(m+1)\cdot|K\setminus N'|} \cdot \mathrm{sgn}((I \cup \{i\}) \setminus N', N') \cdot \\ \cdot \mathrm{sgn}(N', K \setminus N') \cdot \mathrm{sgn}(i, I),\end{aligned}$$

which is exactly $\mathrm{sign}(R_{3B})_{\{i,N\}}$ when we replace N' by N.
Since

$$\mathrm{sgn}((I \cup \{i\}) \setminus N, N) = \mathrm{sgn}(I \setminus N, N) \cdot \mathrm{sgn}(i, N)$$

and

$$\mathrm{sgn}(i, N) \cdot \mathrm{sgn}(i, I) = \mathrm{sgn}(i, I \setminus N),$$

we have

$$\mathrm{sgn}(I \setminus N, N) \cdot \mathrm{sgn}(i, I \setminus N) = \mathrm{sgn}((I \cup \{i\}) \setminus N, N) \cdot \mathrm{sgn}(i, I).$$

Hence, the sign in (iv) follows.
For the last item, (v), we will show that $\mathrm{sign}(L_{1B})_{\{i,N\}} = -\mathrm{sign}(L_{3C})_{\{i,N'\}}$. Since $i \in N \subseteq (I \cap K)$ and $N' = N \setminus \{i\}$, using the same argument as for item (iii), one has the following observations:

- $|K \setminus N| = |K \setminus N'| - 1$;
- $\mathrm{sgn}(I \setminus N, N) = (-1)^{m-|N'|-1} \cdot \mathrm{sgn}(I \setminus N', N') \cdot \mathrm{sgn}(i, N') \cdot \mathrm{sgn}(i, I \setminus N')$;
- $\mathrm{sgn}(N, K \setminus N) = (-1)^{|N'|} \cdot \mathrm{sgn}(N', K \setminus N') \cdot \mathrm{sgn}(i, K \setminus N') \cdot \mathrm{sgn}(i, N')$; and
- $\mathrm{sgn}(i, K \setminus N) = \mathrm{sgn}(i, K \setminus N')$.

Thus, we get

$$\begin{aligned}\mathrm{sign}(L_{3C})_{\{i,N'\}} = (-1)^{m\cdot|K\setminus N'|} \cdot \mathrm{sgn}(I \setminus N', N') \cdot \mathrm{sgn}(N', K \setminus N') \cdot \\ \cdot \mathrm{sgn}(i, K \setminus N') \cdot (-1)^{|K\setminus N'|},\end{aligned}$$

which is $-\mathrm{sign}(L_{1B})_{\{i,N\}}$ when N' is replaced by N. Hence, we have the equation as desired. □

2.5 The cup product

In this section, we interpret the cup product, which is defined at the chain level of the resolution as a composition of chain maps, see Section 1.5 for the full description. We will now give the product of the two standard cocycles in the following proposition. The formula of the cup product will be computed directly based on the definition of Yoneda product in Section 1.5 and the formula $\tilde{f}$ obtained in previous section.

Proposition 2.15. *Let* $f = \left(e_I t^{(p)}, \mathbf{x}^{\alpha}\right)$ *and* $g = \left(e_J t^{(q)}, \mathbf{x}^{\beta}\right)$ *be cocycles in* $\mathrm{HH}^*(A)$. *Then*

$$f \smile g = \begin{cases} \operatorname{sgn}(I, J) \cdot \left(e_{I\cup J} t^{(p+q)}, \mathbf{x}^{\alpha+\beta}\right) & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, this multiplication is commutative up to sign.

Proof. Let $x = e_K t^{(h)}$ $(K = \{i_1, \dots, i_m\})$ be a basis element of degree $i + j$. By Proposition 2.11, we have

$$g \circ \tilde{f}_j(x) = \sum_{\mathcal{M}} (-1)^{ms+j_1+\cdots+j_r-r} g(U_{l_s} \cdots U_{l_1} e_{i_{j_r}\cdots i_{j_1}} t^{(u)}) \cdot \\ \cdot \frac{f(e_{i_1\cdots \widehat{i}_{j_1} \cdots \widehat{i}_{j_r} \cdots i_m} \wedge e_{l_1\cdots l_s} t^{(h-u-s)})}{x_{l_1} \cdots x_{l_s}}.$$

We then have three conceivable cases as follows.

Case 1. If $h < p+q$, then we must have $u < q$ or $h-u-s < p$, otherwise $s < 0$ which is impossible. Hence,

$$g(U_{l_s} \cdots U_{l_1} e_{i_{j_r}\cdots i_{j_1}} t^{(u)}) = 0$$

or

$$\frac{f(e_{i_1\cdots \widehat{i}_{j_1} \cdots \widehat{i}_{j_r} \cdots i_m} \wedge e_{l_1\cdots l_s} t^{(h-u-s)})}{x_{l_1} \cdots x_{l_s}} = 0$$

and so is their product.

Case 2. In case $h = p + q$,

$$g(U_{l_s} \cdots U_{l_1} \cdot e_{i_{j_r} \cdots i_{j_1}} t^{(u)}) \cdot \frac{f(e_{i_1 \cdots \widehat{i}_{j_1} \cdots \widehat{i}_{j_r} \cdots i_m} \wedge e_{l_1 \cdots l_s} t^{(h-u-s)})}{x_{l_1} \cdots x_{l_s}} \neq 0$$

only if $u = q$ and $h - u - s = p$. This implies that $s = 0$. Then we have

$$\{i_{j_1}, \ldots, i_{j_r}\} = J$$

and

$$K \smallsetminus \{i_{j_1}, \ldots, i_{j_r}\} = I.$$

If $I \cap J = \emptyset$, there is only one such $K = I \cup J$ and one gets that

$$(f \smile g)(e_K t^{(h)}) = \operatorname{sgn}(I, J)\mathbf{x}^{\alpha+\beta}.$$

If $I \cap J \neq \emptyset$, one has $K \smallsetminus J \subsetneq I$ for all K which yields that $(f \smile g)(e_K t^{(h)}) = 0$.

Case 3. If $h > p + q$, then $s > 0$. Indeed, by the argument as in above case, one has $u = q$ and $h - u - s = p$. Therefore, $s = h - p - q > 0$. By the definition of the $U_{l_\bullet}$'s in Lemma 2.10 we can rewrite $U_{l_s} \cdots U_{l_1} \cdot e_{i_{j_r} \cdots i_{j_1}} t^{(u)}$ as a sum of some elements in the form below:

$$(x_1 \cdots x_{z-1} \otimes x_{z+1} \cdots x_n) e_z \cdot a \otimes b \cdot e_R t^{(u)}$$

for some $z \in [n]$ and $a, b \in A$.
Applying the function g on this sum, we get the result in A:

$$x_1 \cdots x_{z-1} \cdot x_{z+1} \cdots x_n \cdot ab \cdot g(e_z \wedge e_R t^{(u)}).$$

Notice that as g is a cocycle, by Lemma 2.9, x_z is a divisor of $g(e_z \wedge e_R t^{(u)})$. Hence, $x_1 \cdots x_{z-1} \cdot x_{z+1} \cdots x_n \cdot g(e_z \wedge e_R t^{(u)})$ is a multiple of $x_1 \cdots x_n$, which is zero in A. Thus the result follows. $\square$

2.6 The ring structure of $\mathrm{HH}^*(A)$

In this section, we are going to give a presentation for the algebra $\mathrm{HH}^*(A)$ by generators and relations.

Theorem 2.16. *Let $k[x_1,\dots,x_n]$ be a polynomial ring over a field k and let A be the quotient ring $k[x_1,\dots,x_n]/\langle x_1\cdots x_n\rangle$. Then we have the isomorphism:*

$$\mathrm{HH}^*(A)\cong k[X_1,\dots,X_n,Y_1,\dots,Y_n,Z]/\mathcal{I},$$

where $k[X_1,\dots,X_n,Y_1,\dots,Y_n,Z]$ is a graded commutative polynomial ring; $\deg X_i=0$, $\deg Y_i=1$ *for all $i\in[n]$ and* $\deg Z=2$*; and the ideal $\mathcal{I}$ is generated by the following relations:*

- $a_1\cdots a_n,$ *where $a_i\in\{X_i,Y_i\}$;*
- Y_i^2 *for all $i\in[n]$;*
- $\dfrac{X_1\cdots X_n}{X_{i_1}\cdots X_{i_m}}\cdot\left(\sum\limits_{j=1}^{m}(-1)^{j+1}Y_{i_1}\cdots\widehat{Y}_{i_j}\cdots Y_{i_m}\right)Z,$
  where $m\in[n]$ and $1\le i_1<i_2<\cdots<i_m\le n$.

Proof. By the construction of Hochschild cohomology, $\mathrm{HH}^m(A)$ consists of the cosets of the cocycles of degree m. From the formula of multiplication, a cocycle of degree m can be factorized into elements of degree 0, 1 and 2. Indeed, assume that $E=\left(e_It^{(p)},\mathbf{x}^\alpha\right)$ is a cocycle, where $I=\{i_1,\dots,i_s\}$, $\alpha=(\alpha_1,\dots,\alpha_n)$. Let us write $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ as $x_{i_1}^{\alpha_{i_1}}\cdots x_{i_s}^{\alpha_{i_s}}\cdot x_{i_{s+1}}^{\alpha_{i_{s+1}}}\cdots x_{i_n}^{\alpha_{i_n}}$. Since $\partial\left(e_It^{(p)},\mathbf{x}^\alpha\right)=0$, it follows by Corollary 2.4 that $I\subseteq\operatorname{supp}(\mathbf{x}^\alpha)=\{i\mid\alpha_i>0\}$, which means that $\alpha_{i_j}>0$ for all $j\in[s]$. Therefore, by the formula of the multiplication for two cochains in Proposition 2.15, we obtain that:

$$\begin{aligned}E&=(e_{i_1}\cdots e_{i_s},x_{i_1}\cdots x_{i_s})\cdot(t,1)^q\cdot\left(1,x_{i_1}^{\alpha_{i_1}-1}\cdots x_{i_s}^{\alpha_{i_s}-1}\cdot x_{i_{s+1}}^{\alpha_{i_{s+1}}}\cdots x_{i_n}^{\alpha_{i_n}}\right)\\&=(e_{i_1},x_{i_1})\cdots(e_{i_s},x_{i_s})\cdot(t,1)^q\cdot(1,x_{i_1})^{\alpha_{i_1}-1}\cdots(1,x_{i_s})^{\alpha_{i_s}-1}\cdot\\&\qquad\qquad\cdot\left(1,x_{i_{s+1}}\right)^{\alpha_{i_{s+1}}}\cdots(1,x_{i_n})^{\alpha_{i_n}}.\end{aligned}$$

Briefly E is factorized into the following elements: $(t,1)$, q times; (e_i,x_i), where $i\in I$; and $(1,x_j)$, $\alpha_j-\beta_j$ times, where $j\in\operatorname{supp}(\mathbf{x}^\alpha)$, $\beta_j=1$ if $j\in I$ and $\beta_j=0$ if $j\notin I$.

For each $i\in[n]$, set X_i to be the coset of the element $(1,x_i)$, Y_i to be the coset of the element (e_i,x_i) and Z to be the coset of the element $(t,1)$. Then $\mathrm{HH}^*(A)$ is generated by X_i, Y_i and Z. As $x_1\cdots x_n=0$, we obtain the relations $a_1\cdots a_n,$ where $a_i\in\{X_i,Y_i\}$. The relations Y_i^2 come from the fact that $e_i\wedge e_i=0$.

Remark 2.17. For any element $\left(e_I t^{(p)}, \mathbf{x}^\alpha\right)$, by Lemma 2.1 we obtain that the image $\partial\left(e_I t^{(p)}, \mathbf{x}^\alpha\right)$ is a multiple of $\partial\left(e_I, 1\right)$. Hence, another relation is $\partial\left(e_I, 1\right)$. Suppose that $I = \{i_1, i_2, \ldots, i_m\}$, where $m \in [n]$ and $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. It follows that

$$\partial\left(e_I, 1\right) = \sum_{j=1}^{m} (-1)^{j+1} \left(e_{i\cdots\widehat{i_j}\cdots i_m} t, \frac{x_1 x_2 \cdots x_n}{x_{i_j}}\right)$$

By relabeling the indices in $x_1 x_2 \cdots x_n$ as $x_{i_1} x_{i_2} \cdots x_{i_m} \cdot x_{i_{m+1}} \cdots x_{i_n}$, we rewrite $\partial\left(e_I, 1\right)$ as a combination of generators as follows:

$$\begin{aligned}
\partial\left(e_I, 1\right) =& \sum_{j=1}^{m} (-1)^{j+1} \left(e_{i\cdots\widehat{i_j}\cdots i_m} t, x_{i_1} \cdots \widehat{x}_{i_j} \cdots x_{i_m} \cdot x_{i_{m+1}} \cdots x_{i_n}\right) \\
=& \left(\sum_{j=1}^{m} (-1)^{j+1} (e_{i_1}, x_{i_1}) \cdots \widehat{(e_{i_j}, x_{i_j})} \cdots (e_{i_m}, x_{i_m})\right) \cdot \\
& \qquad \cdot (t, 1)(1, x_{i_{m+1}}) \cdots (1, x_{i_n}).
\end{aligned}$$

We have shown that $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ generate $\mathrm{HH}^*(A)$ and that they satisfy the relations in $\mathcal{I}$. Now we prove that there exists the isomorphism as in the assertion.

Let $S := k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]$ be the graded commutative polynomial ring over the field k and $\mathcal{J}$ the ideal of S generated by the elements $a_1 \cdots a_n$, where $a_i \in \{X_i, Y_i\}$, and Y_i^2 for all $i \in [n]$. As $\mathcal{J}$ is a monomial ideal, the residue classes of the monomials not belonging to $\mathcal{J}$ form a k-basis of the quotient ring $S/\mathcal{J}$. The monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n} \cdot Y_1^{\beta_1} \cdots Y_n^{\beta_n} \cdot Z^q \in S$ is not in $\mathcal{J}$ if and only if $\beta_i \leq 1$ for all i and $\{i \mid \alpha_i > 0 \text{ or } \beta_i > 0\} \subsetneq [n]$. We can identify a non-zero residue class in $S/\mathcal{J}$ by the k-basis element which represents it. Let us construct the map ψ from $S/\mathcal{J}$ to $\mathrm{Ker}(\partial)$ by sending the k-basis element $X_1^{\alpha_1} \cdots X_n^{\alpha_n} \cdot Y_1^{\beta_1} \cdots Y_n^{\beta_n} \cdot Z^q$ in S to $(e_1^{\beta_1} \cdots e_n^{\beta_n} t^{(q)}, x_1^{\alpha_1+\beta_1} \cdots x_n^{\alpha_n+\beta_n})$ in $\mathrm{Ker}(\partial)$. We can check that ψ is an isomorphism between these algebras.

By Remark 2.17, the image $\mathrm{Im}(\partial)$ is generated by the relations $\partial\left(e_I, 1\right)$ and

$$\psi\left(\frac{X_1 \cdots X_n}{X_{i_1} \cdots X_{i_m}} \cdot \left(\sum_{j=1}^{m} (-1)^{j+1} Y_{i_1} \cdots \widehat{Y}_{i_j} \cdots Y_{i_m}\right) Z\right) = \partial\left(e_I, 1\right),$$

where $I = \{i_1, i_2, \ldots, i_m\}$. Therefore, $\psi^{-1}(\mathrm{Im}(\partial)) = \mathcal{I}$. Hence, $S/\mathcal{I} \cong \dfrac{\mathrm{Ker}(\partial)}{\mathrm{Im}(\partial)} = \mathrm{HH}^*(A)$. □

Example 2.18. Applying the above result to the case $n = 2$, we have that:

$$\mathrm{HH}^*(k[x,y]/\langle xy\rangle) \cong k[x_1, x_2, y_1, y_2, z]/\mathcal{I},$$

where deg $x_i = 0$, deg $y_i = 1$ for $i = 1, 2$, deg $z = 2$ and the ideal $\mathcal{I}$ is generated by x_1x_2, x_1y_2, y_1x_2, y_1y_2, y_1^2, y_2^2, x_1z, x_2z, $(y_1 + y_2)z$.

2.7 The Hilbert series of $\mathrm{HH}^*(A)$

In this final section of the chapter, we apply the previous results to compute the Hilbert series of $\mathrm{HH}^*(A)$.

2.7.1 A decomposition on $\mathrm{HH}^*(A)$

First we introduce a grading on $\mathrm{HH}^*(A)$, which is combined from two different gradings. The first is the $\mathbb{N}$-grading based on the degree of cohomology. In detail, if $(e_I t^{(q)}, \mathbf{x}^\alpha)$ is an element in $\mathrm{Hom}_k(\overline{F}_m, A)$ (which means $|I| + 2q = m$), we let $\mathrm{hdeg}(e_I t^{(q)}, \mathbf{x}^\alpha) = m$. The other one is the $\mathbb{Z}^n$-grading based on the lattice point representation for a variable. Let us set $\mathrm{rdeg}(t, 1) = -(1, 1, \ldots, 1)$ (an n-vector with all 1s), $\mathrm{rdeg}(e_i, 1) = -\mathbf{e}_i$ and $\mathrm{rdeg}(1, x_i) = \mathbf{e}_i$ for all $i \in \{1, 2, \ldots, n\}$, where $\mathbf{e}_i$ is the ith standard basis vector in $\mathbb{N}^n$. The differential ∂ is a 1-homogeneous morphism with respect to the first grading and a 0-homogeneous morphism with respect to the second grading. We call the grading given by combining these gradings the *multidegree* of a standard element, $\mathrm{mdeg}(e_I t^{(q)}, \mathbf{x}^\alpha) := (\mathrm{hdeg}(e_I t^{(q)}, \mathbf{x}^\alpha), \mathrm{rdeg}(e_I t^{(q)}, \mathbf{x}^\alpha))$ in $\mathbb{N} \times \mathbb{Z}^n$. Equivalently, if the element $(e_I t^{(q)}, \mathbf{x}^\alpha)$ whose component e_I is identified by the vector $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, where for any i in $\{1, 2, \ldots, n\}$, ϵ_i is 1 if $i \in I$ and 0 otherwise and $\mathbf{x}^\alpha$ is identified with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then we have

$$\mathrm{mdeg}(e_I t^{(q)}, \mathbf{x}^\alpha) = (|I| + 2q, \alpha_1 - \epsilon_1 - q, \ldots, \alpha_n - \epsilon_n - q).$$

Thus, the element $(e_I t^{(q)}, \mathbf{x}^\alpha)$ contributes the term

$$a_0^{|I|+2q} a_1^{\alpha_1 - \epsilon_1 - q} a_2^{\alpha_2 - \epsilon_2 - q} \cdots a_n^{\alpha_n - \epsilon_n - q}$$

(or briefly, as $\mathbf{a}^\chi$, where $\chi = \mathrm{mdeg}(e_I t^{(q)}, \mathbf{x}^\alpha)$) to the Hilbert series.

Let $\mathbf{H}_\chi$ be the k-module generated by the elements whose multidegree is

$\chi \in \mathbb{N} \times \mathbb{Z}^n$. The Hilbert series of $\mathrm{HH}^*(A) = \oplus_{\chi \in \mathbb{N}\times\mathbb{Z}^n} \mathbf{H}_\chi$ as an $\mathbb{N} \times \mathbb{Z}^n$-graded vector space via the grading above is the formal power series:

$$\mathcal{H}(\mathrm{HH}^*(A); \mathbf{a}) = \sum_{\chi \in \mathbb{N}\times\mathbb{Z}^n} \dim_k(\mathbf{H}_\chi)\mathbf{a}^\chi.$$

2.7.2 Computation of the Hilbert series

Theorem 2.19. *The Hochschild cohomology ring* $\mathrm{HH}^*(A)$ *has the Hilbert series:*

$$\mathcal{H}(\mathrm{HH}^*(A); \mathbf{a}) = \frac{(a_0+1)^{n+1} a_1 \cdots a_n - (a_0+a_1)\cdots(a_0+a_n)(a_0+a_1\cdots a_n)}{(a_1\cdots a_n - a_0^2)\cdot(1-a_1)\cdots(1-a_n)}.$$

Proof. Let us denote H_1 and H_2 the Hilbert series of the cocycles and the coboundaries respectively. As $|I| = \epsilon_1 + \ldots + \epsilon_n$, it follows by a simple computation that:

$$a_0^{|I|+2q} \cdot a_1^{\alpha_1-\epsilon_1-q} \cdots a_n^{\alpha_n-\epsilon_n-q} = (a_0^2(a_1\cdots a_n)^{-1})^q \cdot \prod_{i=1}^n (a_0 a_i^{-1})^{\epsilon_i} a_i^{\alpha_i}.$$

From Corollary 2.2, we recall that the standard element $(e_I t^{(q)}, \mathbf{x}^\alpha)$ is a cocycle if $I \subseteq \mathrm{supp}(\mathbf{x}^\alpha)$, i.e., for any $i \in [n]$ we have $\alpha_i > 0$ if $\epsilon_i = 1$ and $\alpha_i \geq 0$ if $\epsilon_i = 0$. Consequently, we need to eliminate the cases that all $\alpha_i > 0$. Thus we get the first series which counts all cocycles:

$$\mathrm{H}_1 = \frac{1}{1-a_0^2(a_1\cdots a_n)^{-1}} \cdot \left(\prod_{i=1}^n \frac{a_0+1}{1-a_i} - \prod_{i=1}^n \frac{a_0+a_i}{1-a_i}\right).$$

To obtain the series of the coboundaries H_2, we consider the element of the form $(e_I t^{(q)}, \mathbf{x}^\alpha)$ in the same multidegree as above. Then the multidegree of the image $\partial(e_I t^{(q)}, \mathbf{x}^\alpha)$ is

$$(|I| + 2q + 1, \alpha_1 - \epsilon_1 - q, \ldots, \alpha_n - \epsilon_n - q).$$

All cases, $|I \setminus \mathrm{supp}(\mathbf{x}^\alpha)| = m$ for m from 1 to n, are counted, except for $m = 0$ (which means $I \subseteq \mathrm{supp}(\mathbf{x}^\alpha)$ and hence, $\partial(e_I t^{(q)}, \mathbf{x}^\alpha) = 0$). Then we get the series of the coboundaries:

$$\mathrm{H}_2 = \frac{a_0}{1-a_0^2(a_1\cdots a_n)^{-1}} \cdot \left(\prod_{i=1}^n \frac{a_0 a_i^{-1}+1}{1-a_i} - \prod_{i=1}^n \frac{a_0+1}{1-a_i}\right).$$

Now we are able to get the Hilbert series of $\mathrm{HH}^*(A)$, which is $\mathrm{H}_1 - \mathrm{H}_2$. $\square$

2.7.3 Example and Computing with Macaulay2

Example 2.20. We consider the Hilbert series in the case $n = 2$. By Theorem 2.19, we have the Hilbert series of the Hochschild cohomology of the algebra $A = k[x, y]/\langle x, y\rangle$ as follows:

$$\mathcal{H}(\mathrm{HH}^*(A); \mathbf{a}) = \frac{(a_0 + 1)^3 a_1 a_2 - (a_0 + a_1)(a_0 + a_2)\,(a_0 + a_1 a_2)}{(a_1 a_2 - a_0^2)(1 - a_1)(1 - a_2)}.$$

Computing with Macaulay2. We can compute the Hilbert series of $\mathrm{HH}^*(A)$ via the Hilbert series of the algebra isomorphic to it, as shown in Theorem 2.16. For a particular n not too large, we can use Macaulay2 [39, 40] to compute this series. In our case, when we already get the formula of the Hilbert series of $\mathrm{HH}^*(A)$, Macaulay2 can help us check whether or not the Hilbert series for a particular example is computed correctly. We will not use Macaulay2 to obtain the formula of Hilbert series for a general case, i.e., n is undetermined. We can find in Appendix A the Macaulay2 code for some small examples, where $n = 2$ and $n = 3$. All the details of the code for computing the Hilbert series and the code for checking the computations of the Hilbert series are also provided in Appendix A.

Chapter 3

The Hochschild cohomology rings of the numerical semigroup algebras of embedding dimension two

In this chapter, we provide the computation on the ring structure of the Hochschild cohomology of the numerical semigroup algebras of embedding dimension two. The content of this chapter will be published in Journal of Pure and Applied Algebra [13].

3.1 Overview

Let a and b be two coprime positive integers and k an arbitrary field. This chapter presents a description of the Hochschild cohomology ring of the numerical semigroup algebras $k[s^a, s^b] \subseteq k[s]$ of embedding dimension two, which is a class of non-monomial complete intersection in two variables.

Our approach relies on the construction of the free resolution of complete intersections given by Guccione et al. [14, 24]. We then provide a description of the Hochschild cohomology as a k-module by splitting the cochain complex into sub-complexes based on the features of cocycles. For the multiplicative structure, we interpret the cup product in terms of the Yoneda product. In order to compute the formula of the cup product of two elements in the module, starting from a cocycle we construct a chain map between the shifted resolution and the resolution itself. Building on a work of Sköldberg [29] on algebraic discrete Morse theory, we work out an explicit description of a contracting homotopy, which allows us to construct the lifting map by combining the differentials of the complex and the contracting homotopy. The formula of the differentials depends significantly on the relation of the two numbers a, b and the characteristic $\mathrm{char}(k)$ of the field k. This yields that the structure of the Hochschild cohomology of $k[s^a, s^b]$ does the same. Therefore, we will consider this structure in two separate cases. The first case is that neither a nor b is divisible by $\mathrm{char}(k)$; hence the second is that $\mathrm{char}(k)$ is a divisor of a or b, where we assume without loss of generality that $\mathrm{char}(k)$ is a divisor of a. For each of these two cases, we provide a description in terms of generators and relations of the Hochschild cohomology of $k[s^a, s^b]$ and subsequently we calculate the Hilbert series of the Hochschild cohomology ring.

3.2 Some auxiliary results

Let S be the semigroup generated by a and b, that is, $S := \{ua + vb \mid u, v \in \mathbb{N}\}$. In this section, we prove some numerical results related to the semigroup S which will be used throughout the chapter. We denote $F(S) := ab - (a + b)$, the Frobenius number of S. This number was stated by Sylvester [41] and has property that it does not belong to S, which plays a key role in the results of the current chapter.

Lemma 3.1. *For an integer d, $db - a \in S$ if and only if $d \geq a$.*

Proof. "$\Rightarrow$": Let $db - a \in S$ and suppose that $d = a - z$ where $z \in \mathbb{Z},\ z \geq 1$. Thus $S \ni (a-z)b-a = F(S)-(z-1)b \Rightarrow F(S) \in S$ which is a contradiction. "$\Leftarrow$": $db - a = a(b - 1) + b(d - a) \in S$ for any integer $d \geq a$. □

To simplify the notation, we introduce $m_1 = (a - 1)b$ and $m_2 = a(b - 1)$ and define the sets $S_i = \{\alpha \in \mathbb{Z} \mid \alpha - m_i \in S\}$ for $i \in \{1, 2\}$. Notice that $m_1, m_2 \in S$ by Lemma 3.1. The relevance of the next lemma will be seen later.

Lemma 3.2. *The following are true:*

(i) For $i \in \{1,2\}$, S_i is equal to $\{m_i + \gamma \mid \gamma \in S\}$.

(ii) $S_1 \cap S_2$ is equal to $\{m_1 + m_2\} \cup \{ab + \gamma \mid \gamma \in S\}$.

Proof. For (i), let $S' = \{m_i + \gamma \mid \gamma \in S\}$. We have $\alpha \in S_i \iff \alpha - m_i \in S \iff \exists \gamma \in S : \alpha - m_i = \gamma$, i.e., $\alpha = m_i + \gamma$. This is equivalent to $\alpha \in S'$. For (ii), let $S'' = \{m_1 + m_2\} \cup \{ab + \gamma \mid \gamma \in S\}$. Choose $\alpha \in S_1 \cap S_2$. In particular, α satisfies $\alpha - m_2 \in S$ and so we can write $\alpha = m_2 + \beta$ for some $\beta = ua + vb \in S$ where $u, v \in \mathbb{N}$.

- If $u = 0$ then $\alpha - m_1 \in S \iff -a + (v+1)b \in S \iff v \geq a - 1$ by Lemma 3.1. Thus $\alpha = m_1 + m_2$ if $v = a - 1$ and $\alpha = ab + \gamma$ where $\gamma = vb - a \in S$ if $v \geq a$.
- If $u > 0$ then we can write $\beta = \gamma + a$ where $\gamma = (u-1)a + vb \in S$ giving $\alpha = ab + \gamma$.

Thus $S_1 \cap S_2 \subseteq S''$. The other inclusion is clear. $\square$

3.3 A construction of Hochschild cohomology

By setting $x_1 \mapsto s^b$ and $x_2 \mapsto s^a$, we have an isomorphism between algebras, $k[s^a, s^b] \cong \dfrac{k[x_1, x_2]}{\langle x_1^a - x_2^b \rangle}$. We will use both algebras according to our convenience. We now interpret the minimal resolution given by Guccione et al. (see [14] or [24]) for the case of the quotient ring of $k[x_1, x_2]$ modulo $\langle x_1^a - x_2^b \rangle$, the ideal generated by the binomial $x_1^a - x_2^b$. The following complex $\mathbf{F}$ is a free A^e-resolution of A:

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\mu} A \longrightarrow 0, \tag{3.1}$$

where F_m is the finitely generated free A^e-module with basis elements $e_{i_1 \cdots i_r} \cdot t^{(q)}$ ($r, q \geq 0$ and $r + 2q = m$), where by $e_{i_1 \cdots i_r}$ or e_I ($I = \{i_1, \ldots, i_r\} \subseteq \{1,2\}, i_1 < \cdots < i_r$), we mean $e_{i_1} \wedge \cdots \wedge e_{i_r}$. We assign degree 1 to the elements e_1, e_2 and assign degree 2 to the element t. We have that (3.1) is an exact sequence of free A^e-modules with

$$\mu : F_0 \to A, \qquad a \otimes b \mapsto ab$$

and the differentials d_m (briefly d) are defined as follows:

$$d(e_1) = s^b \otimes 1 - 1 \otimes s^b;$$
$$d(e_2) = s^a \otimes 1 - 1 \otimes s^a;$$
$$d(t) = \sum_{i=0}^{a-1} s^{ib} \otimes s^{b(a-1-i)} \cdot e_1 - \sum_{i=0}^{b-1} s^{ia} \otimes s^{a(b-1-i)} \cdot e_2.$$

Also, we write down here a corresponding version with respect to variables x_1, x_2 for our convenience.

$$d(e_1) = x_1 \otimes 1 - 1 \otimes x_1;$$
$$d(e_2) = x_2 \otimes 1 - 1 \otimes x_2;$$
$$d(t) = \sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i} \cdot e_1 - \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_2.$$

For higher degrees, we use the following formula inductively:

$$d(xy) = d(x)y + (-1)^\alpha x d(y), \text{ where } x \in F_\alpha.$$

Alternatively, we can write

$$d(e_I t^{(q)}) = \begin{cases} d(t)t^{(q-1)}, & I = \emptyset, \\ d(e_1)t^{(q)} - e_1 d(t) t^{(q-1)}, & I = \{1\}, \\ d(e_2)t^{(q)} - e_2 d(t) t^{(q-1)}, & I = \{2\}, \\ (d(e_1)e_2 - d(e_2)e_1)\, t^{(q)}, & I = \{1,2\}. \end{cases} \quad \text{(if } q > 0\text{)}$$

Computing the differentials d. We now give a brief proof of the formula of d by using the general formulas recalled in Remark 1.25, Chapter 1. Now we get that:

$$d(e_i) = T(x_i) = x_i \otimes 1 - 1 \otimes x_i \text{ for } i \in \{1, 2\}$$

and

$$\begin{aligned} d(t) =& \frac{T_1(f)}{T(x_1)} e_1 + \frac{T_2(f)}{T(x_2)} e_2 = \frac{T_1(x_1^a)}{T(x_1)} e_1 - \frac{T_2(x_2^b)}{T(x_2)} e_2 \\ =& \sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i} \cdot e_1 - \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_2, \end{aligned}$$

where $f = x_1^a - x_2^b$ in our case.
Applying the contravariant functor $\mathrm{Hom}_{A^e}(-, A)$ to the truncation of the above resolution, we obtain a new complex:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(F_0, A) \xrightarrow{d^1} \mathrm{Hom}_{A^e}(F_1, A) \xrightarrow{d^2} \mathrm{Hom}_{A^e}(F_2, A) \longrightarrow \cdots.$$

The i-th Hochschild cohomology of A is the module

$$\mathrm{HH}^i(A) := \frac{\mathrm{Ker}(d^{i+1})}{\mathrm{Im}(d^i)},$$

where d^0 is taken to be the zero map. Now the Hochschild cohomology module of A is defined to be the direct sum of these components, $\mathrm{HH}^*(A) := \bigoplus_{i\geq 0} \mathrm{HH}^i(A)$.

We can consider $\mathrm{HH}^*(A)$ as a k-module by the following argument. For $m \in \mathbb{N}$, let $\overline{F}_m$ be the free k-module generated by the same basis elements as F_m. Then, there is an isomorphism between the following k-spaces:

$$\mathrm{Hom}_{A^e}(F_m, A) \cong \mathrm{Hom}_k(\overline{F}_m, A).$$

Thus one gets the complex:

$$0 \longrightarrow \mathrm{Hom}_k(\overline{F}_0, A) \xrightarrow{\partial^1} \mathrm{Hom}_k(\overline{F}_1, A) \xrightarrow{\partial^2} \mathrm{Hom}_k(\overline{F}_2, A) \longrightarrow \cdots, \tag{3.2}$$

where the differential ∂ will be described later.

Let $e_I t^{(q)}$ be a basis element in $\overline{F}_m$ and s^α a basis element in A. Let $(e_I t^{(q)}, s^\alpha)$ be the k-linear map in $\mathrm{Hom}_k(\overline{F}_m, A)$ which sends $e_I t^{(q)}$ to s^α and other basis elements to 0, that is,

$$(e_I t^{(q)}, s^\alpha)(e_J t^{(p)}) = \begin{cases} s^\alpha & \text{if } J = I \text{ and } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all such k-linear maps is a k-basis of the module $\mathrm{Hom}_k(\overline{F}_m, A)$. We use the notation $[(e_I t^{(q)}, s^\alpha)]$ to denote the residue class represented by $(e_I t^{(q)}, s^\alpha)$ in $\mathrm{HH}^*(A)$. Now we are in the position to describe the formula of ∂.

Lemma 3.3. *The homomorphism ∂ in (3.2) is given by:*

$$\partial(e_I t^{(q)}, s^\alpha) = \begin{cases} 0 & \text{if } I = \emptyset, \\ a(t^{(q+1)}, s^{\alpha+m_1}) & \text{if } I = \{1\}, \\ -b(t^{(q+1)}, s^{\alpha+m_2}) & \text{if } I = \{2\}, \\ b(e_1 t^{(q+1)}, s^{\alpha+m_2}) + a(e_2 t^{(q+1)}, s^{\alpha+m_1}) & \text{if } I = \{1,2\}. \end{cases}$$

Proof. From the following diagram

$$\begin{tikzcd} \mathrm{Hom}_{A^e}(F_m, A) \arrow[r, "d^{m+1}"] \arrow[d, leftrightarrow, "\cong"'] & \mathrm{Hom}_{A^e}(F_{m+1}, A) \arrow[d, leftrightarrow, "\cong"'] \\ \mathrm{Hom}_k(\overline{F}_m, A) \arrow[r, "\partial^{m+1}"] & \mathrm{Hom}_k(\overline{F}_{m+1}, A) \end{tikzcd}$$

we can derive ∂^{m+1} from d^{m+1} straightforwardly.
As $f = (e_I t^{(q)}, s^\alpha)$ is a basis element of $\mathrm{Hom}_k(\overline{F}_m, A)$, f is identified with a function in $\mathrm{Hom}_{A^e}(F_m, A)$. A direct calculation shows that it is $(e_I t^{(q)}, s^\alpha)$, which is also denoted by f by abuse of notation. We have the homomorphism

$$\begin{array}{rccl} d^{m+1}: & \mathrm{Hom}_{A^e}(F_m, A) & \longrightarrow & \mathrm{Hom}_{A^e}(F_{m+1}, A) \\ & f & \longmapsto & d^{m+1}(f) := f \circ d \end{array}$$

For a basis element $e_J t^{(r)} \in F_{m+1}$, we can compute $(d^{m+1}(f))\,(e_J t^{(r)})$ directly and the result is summarized in the following table, Table 3.1.

	$(d^{m+1}(f))\,(e_J t^{(r)})$
$J = \emptyset$	$as^{\alpha+m_1}$, if $I = \{1\}$ and $r-1=q$ $-bs^{\alpha+m_2}$, if $I = \{2\}$ and $r-1=q$ 0, otherwise
$J = \{1\}$	$bs^{\alpha+m_2}$, if $I = \{1,2\}$ and $r-1=q$ 0, otherwise
$J = \{2\}$	$as^{\alpha+m_1}$, if $I = \{1,2\}$ and $r-1=q$ 0, otherwise
$J = \{1,2\}$	0

Table 3.1: Computations of $(d^{m+1}(f))\,(e_J t^{(r)})$

In other words, we have the formula of ∂ as desired. $\square$

We shall divide the rest of this chapter into two separate parts corresponding to two cases. The first, Case I, is when the characteristic $\mathrm{char}(k)$ of the field k is neither a divisor of a nor of b, and the second, Case II, is when $\mathrm{char}(k)$ divides one of a or b, which we without loss of generality assume to be a. For each case, we show the module structure and then the ring structure of $\mathrm{HH}^*(A)$ in terms of generators and relations.

3.4 The ring structure of $\mathrm{HH}^*(A)$ - Case I

This section presents the results on computations of $\mathrm{HH}^*(A)$ when $\mathrm{char}(k)$ is neither a divisor of a nor of b. We begin with the module structure of $\mathrm{HH}^*(A)$ via smaller modules. Next, we give a classification of cocycles which will be used to describe the generators and relations in the later results.

The multiplication is established by constructing a Morse matching (see Sköldberg [29]) on the basis elements of the module $\mathbf{F} := \bigoplus F_m$, which gives us the formula of a contracting homotopy. As main results, we give a description of generators and relations of the ring structure of $\mathrm{HH}^*(A)$ and finally, we define a decomposition on $\mathrm{HH}^*(A)$ and compute the Hilbert series of $\mathrm{HH}^*(A)$ with respect to this decomposition.

3.4.1 The structure of $\mathrm{HH}^*(A)$ as a k-module

The k-vector space $\mathrm{Hom}_k(\overline{F}_m, A)$ is generated by the basis elements of cohomological degree m:

$$\mathrm{Hom}_k(\overline{F}_m, A) = \bigoplus_{|I|+2q=m} k(e_I t^{(q)}, s^\alpha),$$

where $k(e_I t^{(q)}, s^\alpha)$ is the k-vector space generated by $(e_I t^{(q)}, s^\alpha)$. To simplify the notation we will use the notation k instead of $k(e_I t^{(q)}, s^\alpha)$, justified by the isomorphism $k(e_I t^{(q)}, s^\alpha) \cong k$. In order to describe the Hochschild cohomology, we split the cochain complex (3.2) into sub-complexes. Let $\Gamma := \{(t^{(q)}, s^\alpha) \mid q \in \mathbb{N}, \alpha \in S\}$, the set of all basis elements in the kernel of ∂. For each element $\gamma \in \Gamma$, we construct the complex M_γ which includes γ as the generator of the rightmost non-zero entry $(\cdots \longrightarrow \bullet \longrightarrow 0)$. Each M_γ is a sub-complex of (3.2). Moreover, by Lemma 3.3 there are only four options for such M_γ as follows:

Type 1. $0 \longrightarrow k(t^{(q)}, s^\alpha) \longrightarrow 0;$
Type 2. $0 \longrightarrow k(e_1 t^{(q-1)}, s^{\alpha-m_1}) \longrightarrow k(t^{(q)}, s^\alpha) \longrightarrow 0;$
or $0 \longrightarrow k(e_2 t^{(q-1)}, s^{\alpha-m_2}) \longrightarrow k(t^{(q)}, s^\alpha) \longrightarrow 0;$
Type 3. $0 \longrightarrow k(e_1 t^{(q-1)}, s^{\alpha-m_1}) \oplus k(e_2 t^{(q-1)}, s^{\alpha-m_2}) \longrightarrow k(t^{(q)}, s^\alpha) \longrightarrow 0;$
Type 4. $0 \longrightarrow k(e_1 e_2 t^{(q-2)}, s^{\alpha-m_1-m_2}) \longrightarrow$
$\longrightarrow k(e_1 t^{(q-1)}, s^{\alpha-m_1}) \oplus k(e_2 t^{(q-1)}, s^{\alpha-m_2}) \longrightarrow k(t^{(q)}, s^\alpha) \longrightarrow 0.$

The sub-complexes are classified in Table 3.2 based on the corresponding feature of the element $(t^{(q)}, s^\alpha)$. We can see that they cover all possible options of the arguments q, α in $(t^{(q)}, s^\alpha)$.

q	Type of sub-complex	Condition(s) for α in $(t^{(q)}, s^{\alpha})$
$q=0$	$0 \longrightarrow k \longrightarrow 0$	
$q=1$	$0 \longrightarrow k \longrightarrow 0$	$\begin{cases} \alpha - m_1 \notin S \\ \alpha - m_2 \notin S \end{cases}$
	$0 \longrightarrow k \longrightarrow k \longrightarrow 0$	$\alpha - m_1 \in S$ xor $\alpha - m_2 \in S$
	$0 \longrightarrow k^2 \longrightarrow k \longrightarrow 0$	$\begin{cases} \alpha - m_1 \in S \\ \alpha - m_2 \in S \end{cases}$
$q \geq 2$	$0 \longrightarrow k \longrightarrow 0$	$\begin{cases} \alpha - m_1 \notin S \\ \alpha - m_2 \notin S \end{cases}$
	$0 \longrightarrow k \longrightarrow k \longrightarrow 0$	$\alpha - m_1 \in S$ xor $\alpha - m_2 \in S$
	$0 \longrightarrow k^2 \longrightarrow k \longrightarrow 0$	$\begin{cases} \alpha - m_1 \in S \\ \alpha - m_2 \in S \\ \alpha - m_1 - m_2 \notin S \end{cases}$
	$0 \longrightarrow k \longrightarrow k^2 \longrightarrow k \longrightarrow 0$	$\alpha - m_1 - m_2 \in S$

Table 3.2: Classification of the sub-complexes

Proposition 3.4. *The complex* (3.2) *can be written as the direct sum below:*

$$\operatorname{Hom}_k(\overline{F}_{\bullet}, A) = \bigoplus_{\gamma \in \Gamma} M_{\gamma}.$$

Proof. By Lemma 3.3, M_{γ} is a sub-complex of (3.2). Then we have the first inclusion $\bigoplus_{\gamma \in \Gamma} M_{\gamma} \subseteq \operatorname{Hom}_k(\overline{F}_{\bullet}, A)$. Let us consider an arbitrary non-zero basis element $E = (e_I t^{(q)}, s^{\alpha}) \in \operatorname{Hom}_k(\overline{F}_m, A)$ for some $m \in \mathbb{N}$. If $I = \emptyset$, then $E \in \Gamma$ and M_E is the sub-complex containing E. If $I \neq \emptyset$, we have the following cases:

(i) $I = \{1\}$:

- If $\alpha - m_2 \in S$, then E occurs in the sub-complex

$$0 \longrightarrow k(e_1 e_2 t^{(q-1)}, s^{\alpha - m_2}) \longrightarrow kE \oplus k(e_2 t^{(q)}, s^{\alpha + m_1 - m_2}) \longrightarrow$$
$$\longrightarrow k(t^{(q+1)}, s^{\alpha + m_1}) \longrightarrow 0$$

- If $\alpha - m_2 \notin S$, then E occurs in the sub-complex

$$0 \longrightarrow kE \longrightarrow k(t^{(q+1)}, s^{\alpha + m_1}) \longrightarrow 0, \text{ if } \alpha + m_1 - m_2 \notin S; \text{ or}$$

$$0 \longrightarrow kE \oplus k(e_2 t^{(q)}, s^{\alpha + m_1 - m_2}) \longrightarrow k(t^{(q+1)}, s^{\alpha + m_1}) \longrightarrow 0,$$

if $\alpha + m_1 - m_2 \in S$.

(ii) $I = \{2\}$, similarly.
(iii) If $I = \{1, 2\}$, then E occurs in the sub-complex

$$0 \longrightarrow kE \longrightarrow k(e_1t^{(q+1)}, s^{\alpha+m_2}) \oplus k(e_2t^{(q+1)}, s^{\alpha+m_1}) \longrightarrow \\ \longrightarrow k(t^{(q+2)}, s^{\alpha+m_1+m_2}) \longrightarrow 0.$$

We see that any basis element E is contained in a unique sub-complex which belongs to Type 1 to 4. So the inverse inclusion is obtained and the result follows. □

By the above result, module $\mathrm{HH}^*(A)$ is described via sub-modules in the following consequence.

Corollary 3.5. *For each i in $\mathbb{N}$, we have the following isomorphism:*

$$\mathrm{H}^i(\mathrm{Hom}_k(\overline{F}_\bullet, A)) \cong \bigoplus_{\gamma\in\Gamma} \mathrm{H}^i(M_\gamma).$$

3.4.2 Classification of cocycles

The following is a direct consequence of Lemma 3.3 and Proposition 3.4.

Corollary 3.6. *The k-vector space $\bigoplus_{i\in\mathbb{N}} \mathrm{Ker}(\partial^i)$ is generated by the following elements:*

- *$(t^{(q)}, s^\alpha)$, where $\alpha \in S$;*
- *$b(e_1t^{(q)}, s^{\alpha-m_1}) + a(e_2t^{(q)}, s^{\alpha-m_2})$, where $\alpha \in S$ such that $\alpha - m_1 \in S$ and $\alpha - m_2 \in S$.*

We will call the elements described in the above corollary the *standard* elements. In the following remarks, we will give more details about these elements.

Remark 3.7. We look at the elements $b(e_1t^{(q)}, s^{\alpha-m_1}) + a(e_2t^{(q)}, s^{\alpha-m_2})$, where $\alpha \in S$ such that $\alpha - m_1 \in S$ and $\alpha - m_2 \in S$. By Lemma 3.2, for any $\alpha \in \mathbb{Z}$ such that $\alpha - m_1 \in S$ and $\alpha - m_2 \in S$ we have that

$$\alpha \in \{m_1 + m_2\} \cup \{ab + \gamma \mid \gamma \in S\}.$$

If $\alpha = m_1 + m_2$, we have the cocycles $b(e_1t^{(q)}, s^{m_2}) + a(e_2t^{(q)}, s^{m_1})$.
If $\alpha = ab + \gamma$ for some $\gamma \in S$, then we get the cocycles $b(e_1t^{(q)}, s^{\gamma+b}) +$

$a(e_2t^{(q)}, s^{\gamma+a})$ where $\gamma \in S$, which is distinguished from the cocycles above. It is simple to get these cocycles just by substituting $ab+\gamma$ for α in the original elements. To show that it is distinguished from $b(e_1t^{(q)}, s^{m_2}) + a(e_2t^{(q)}, s^{m_1})$, we suppose on the contrary that there is some $\gamma \in S$ such that $\gamma + b = m_2$ or equivalently $\gamma + a = m_1$. Then $\gamma = m_2 - b = a(b-1) - b = ab - a - b = F(S) \in S$, which is a contradiction.

Remark 3.8. According to Table 3.2 and Corollary 3.6, we are able to identify all standard cocycles that are the representatives of the non-zero cohomology classes in $\mathrm{HH}^*(A)$ as follows:

$$(1, s^\alpha), \text{ where } \alpha \in S \tag{3.3}$$

$$(t^{(q)}, s^\alpha), \text{ where } \begin{cases} q > 0 \\ \alpha \in S \\ \alpha - m_1 \notin S \\ \alpha - m_2 \notin S \end{cases} \tag{3.4}$$

$$b(e_1t^{(q)}, s^{\alpha-m_1}) + a(e_2t^{(q)}, s^{\alpha-m_2}), \text{ where } \begin{cases} \alpha \in S \\ \alpha - m_1 \in S \\ \alpha - m_2 \in S \\ \alpha - m_1 - m_2 \notin S \text{ if } q > 0 \end{cases} \tag{3.5}$$

We can express the cocycles in (3.5) as all the elements of the set

$$\{b(e_1t^{(q)}, s^{\gamma+b}) + a(e_2t^{(q)}, s^{\gamma+a}) \mid \gamma \in S;\ \gamma - F(S) \notin S \text{ if } q > 0\}$$

together with the single cocycle $b(e_1, s^{m_2}) + a(e_2, s^{m_1})$. Indeed, by Remark 3.7, we have $\gamma + b = \alpha - m_1$. Then $\alpha - m_1 - m_2 \notin S$ is equivalent to $\gamma + b - m_2 = \gamma - (ab - a - b) = \gamma - F(S) \notin S$.

In the above remark, we have described a k-basis of $\mathrm{HH}^*(A)$. Next, we will construct a multiplicative structure on $\mathrm{HH}^*(A)$.

3.4.3 Morse matching

Let $\mathbf{F}$ be a free resolution of the A^e-module A and $f : F_i \to A$ be an A^e-homomorphism such that $f \circ d_{i+1} = 0$. Our goal now is to provide an explicit chain map $\tilde{f}$ in case of our resolution that makes the below diagram commute. In more details, we will base ourselves on a work of Sköldberg [29] (which we have recalled in Chapter 1) to construct a contracting homotopy ϕ which consists of maps $\phi_j : F_j \to F_{j+1}$ of degree 1.

$$\begin{tikzcd}
\cdots \arrow[r] & F_{i+2} \arrow[r, "d_{i+2}"] \arrow[d, "\tilde{f}_2"] & F_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\tilde{f}_1"] & F_i \arrow[d, "\tilde{f}_0"] \arrow[dr, "f"] & \\
\cdots \arrow[r, shift right] & F_2 \arrow[l, shift right] \arrow[r, shift right, "d_2"'] & F_1 \arrow[l, shift right, "\phi"'] \arrow[r, shift right, "d_1"'] & F_0 \arrow[l, shift right, "\phi"'] \arrow[r, "\mu"] & A
\end{tikzcd} \tag{3.6}$$

The homomorphism $\tilde{f}$ is given by setting

$$\tilde{f}_0(x) = 1 \otimes f(x)$$

and for any $j > 0$, we define $\tilde{f}_j$ inductively on the A^e-basis elements by

$$\tilde{f}_j := \phi_{j-1} \circ \tilde{f}_{j-1} \circ d_{i+j}$$

and extend linearly for other elements. The chain map $\tilde{f}$ defined as above makes diagram (3.6) commute. In the next steps, we will make this chain map explicit.

To denote the elements in the algebra $k[x_1, x_2]$ and their cosets in the quotient ring $k[x_1, x_2]/\langle x_1^a - x_2^b \rangle$, we use the same notation if there are no ambiguities. Then the k-basis of the algebra $k[x_1, x_2]/\langle x_1^a - x_2^b \rangle$ consists of all elements of the form $x_1^u x_2^v$, where $u \geq 0$ and $0 \leq v < b$. From now on, these are default conditions whenever we mention the elements in $k[x_1, x_2]/\langle x_1^a - x_2^b \rangle$.

We can consider the complex $\mathbf{F}$ as a chain complex of $k \otimes A$-modules together with a direct sum decomposition as follows:

$$F_m = \bigoplus_{\alpha \in \mathfrak{I}_m} F_\alpha,$$

where $\{\mathfrak{I}_m\}_{m \in \mathbb{N}}$ is a family of mutually disjoint index sets given by

$$\mathfrak{I}_m = \{(u, v, I, q) \mid u \geq 0, 0 \leq v < b, 2|I| + q = m\}.$$

Here the index (u, v, I, q) corresponds to the basis element $x_1^u x_2^v \otimes 1 \cdot e_I t^{(q)}$ which generates the $k \otimes A$-module $F_{(u,v,I,q)}$. We write $d_{\beta,\alpha}$ for the component of d going from F_α to F_β. Now $\mathbf{F}$ has the structure of a based complex. Let $G_{\mathbf{F}}$ be the digraph with the vertex set $V = \bigcup_{m \in \mathbb{N}} \mathfrak{I}_m$ and with a directed edge $\alpha \to \beta$ whenever the component $d_{\beta,\alpha}$ is non-zero. Next, we construct a partial matching $\mathcal{M}$ on $\mathbf{F}$ by setting

$$\left.\begin{array}{r} x_1^u x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow x_1^{u-1} x_2^v \otimes 1 \cdot e_1 t^{(q)} \\ x_1^u x_2^v \otimes 1 \cdot e_2 t^{(q)} \longrightarrow x_1^{u-1} x_2^v \otimes 1 \cdot e_1 e_2 t^{(q)} \end{array}\right\} \text{ where } u > 0, 0 \leq v < b$$

$$\begin{array}{r} x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow x_2^{v-1} \otimes 1 \cdot e_2 t^{(q)}, \text{ where } 0 < v < b \\ x_2^{b-1} \otimes 1 \cdot e_2 t^{(q)} \longrightarrow 1 \otimes 1 \cdot t^{(q+1)}. \end{array}$$

We can see that this matching includes all of basis elements of $\mathbf{F}$, which is exactly the situation that we described in Remark 1.44, Chapter 1. That means there are no critical points and the constructed map ϕ becomes a contracting homotopy. We denote by $G_{\mathbf{F}}^{\mathcal{M}}$ the digraph with the same vertex set V and the edge set obtained from $G_{\mathbf{F}}$ by reversing the direction of each arrow $\alpha \to \beta$ whenever $\beta \to \alpha$ in $\mathcal{M}$. For each edge $\alpha \to \beta$ in $\mathcal{M}$, it is clear that the corresponding component of the differential $d_{\beta,\alpha}$ is an isomorphism. Now we only need to check that there are no directed cycles in $G_{\mathbf{F}}^{\mathcal{M}}$ to see that $\mathcal{M}$ is a Morse matching. By observing the formula of the differential d and the matching $\mathcal{M}$, we check the absence of directed cycles as follows:
(i) If we have a path

$$x_2^{b-1} \otimes 1 \cdot e_2 t^{(q)} \longrightarrow 1 \otimes 1 \cdot t^{(q+1)} \longrightarrow x_1^u x_2^v \otimes 1 \cdot e_I t^{(r)}$$

in $G_{\mathbf{F}}^{\mathcal{M}}$ where the two first vertices are matched, then one gets $I = \{1\}$ or ($I = \{2\}$, $u = 0$ and $v < b - 1$), i.e., this path ends here and hence, it cannot form a cycle.
(ii) Similarly, if we have a path

$$x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow x_2^{v-1} \otimes 1 \cdot e_2 t^{(q)} \longrightarrow x_1^m x_2^n \otimes 1 \cdot e_I t^{(r)} (\text{where } 0 < v < b),$$

then $I = \{1, 2\}$ (i.e., the path must end here and there is no cycle formed) or one has $x_1^m x_2^n \otimes 1 \cdot e_I t^{(r)} = x_2^{v-1} \otimes 1 \cdot t^{(q)}$. Thus, the path becomes

$$x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow x_2^{v-1} \otimes 1 \cdot e_2 t^{(q)} \longrightarrow x_2^{v-1} \otimes 1 \cdot t^{(q)} \longrightarrow x_2^{v-2} \otimes 1 \cdot e_2 t^{(q)} \longrightarrow \cdots$$

where the power of x_2 is declining and the path eventually terminates at $1 \otimes 1 \cdot t^{(q)}$. Thus, no cycle is formed by this path.
(iii) Let us consider the path

$$x_1^u x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow x_1^{u-1} x_2^v \otimes 1 \cdot e_1 t^{(q)} \longrightarrow x_1^m x_2^n \otimes 1 \cdot e_I t^{(r)} (\text{where } u > 0).$$

Then we have either $I = \{1,2\}$ (i.e., the path ends here) or $x_1^m x_2^n \otimes 1 \cdot e_I t^{(r)} = x_1^{u-1} x_2^v \otimes 1 \cdot t^{(q)}$ and we can extend this path as follows:

$$x_1^u x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow x_1^{u-1} x_2^v \otimes 1 \cdot e_1 t^{(q)} \longrightarrow x_1^{u-1} x_2^v \otimes 1 \cdot t^{(q)} \longrightarrow \cdots \longrightarrow x_2^v \otimes 1 \cdot t^{(q)}$$

and continue with the path in (ii), i.e., there is no directed cycle.
(iv) For the last one, the path

$$x_1^u x_2^v \otimes 1 \cdot e_2 t^{(q)} \longrightarrow x_1^{u-1} x_2^v \otimes 1 \cdot e_1 e_2 t^{(q)} \longrightarrow x_1^m x_2^n \otimes 1 \cdot e_I t^{(r)} (\text{where } u > 0)$$

gives us either $I = \{1\}$ (which ends the path) or $x_1^m x_2^n \otimes 1 \cdot e_I t^{(r)} = x_1^{u-1} x_2^v \otimes 1 \cdot e_2 t^{(q)}$. By continuing this argument, this path is extended to

$$x_2^v \otimes 1 \cdot e_2 t^{(q)},$$

which ends here if $v < b-1$ and ends at $1 \otimes 1 \cdot t^{(q+1)}$ if $v = b-1$. Hence, there is no directed cycle in $G_{\mathbf{F}}^{\mathcal{M}}$ and $\mathcal{M}$ is a Morse matching as desired. We now give the formula of the contracting homotopy ϕ for our case in the following proposition. The general formula of ϕ was recalled in Section 1.8, Chapter 1.

Proposition 3.9. *Let $x = x_1^u x_2^v \otimes 1 \cdot e_I t^{(q)}$ be a basis element of the $k \otimes A$-complex $\mathbf{F}$. We then have the formula of ϕ as follows:*

- *$I = \{1\}$ or $\{1,2\}$: $\phi(x) = 0$;*
- *$I = \{\emptyset\}$: $\phi(x) = \sum_{i=0}^{u-1} x_1^i x_2^v \otimes x_1^{u-1-i} \cdot e_1 t^{(q)} + \sum_{i=0}^{v-1} x_2^i \otimes x_1^u x_2^{v-1-i} \cdot e_2 t^{(q)}$; and*
- *$I = \{2\}$: $\phi(x) = \sum_{i=0}^{u-1} x_1^i x_2^v \otimes x_1^{u-1-i} \cdot e_1 e_2 t^{(q)} - [v = b-1] 1 \otimes x_1^u \cdot t^{(q+1)}$,*

  where $[P] = \begin{cases} 1 & \textit{if } P \textit{ true,} \\ 0 & \textit{if } P \textit{ false.} \end{cases}$

3.4.4 An explicit chain map

In the following lemmas, we will give the formula of $\tilde{f}$ based on the form of f in Corollary 3.6.

Lemma 3.10. *Let $f : F_i \to A$ be a standard cocycle of the form $(t^{(q)}, x_1^u x_2^v)$ in $\mathrm{Hom}_{A^e}(F_i, A)$. For any $j \in \mathbb{N}$, the A^e-homomorphism $\tilde{f}_j : F_{i+j} \to F_j$ of the chain map $\tilde{f}$ has the formula as follows:*

$$\tilde{f}_j(e_J t^{(r)}) = [q \le r] 1 \otimes f(t^{(q)}) \cdot e_J t^{(r-q)}.$$

Proof. We shall prove this lemma by induction on $j \in \mathbb{N}$.

$j = 0$: As $f(e_J t^{(r)}) = 0$ for all $e_J t^{(r)} \neq t^{(q)}$, we then have

$$\tilde{f}_0(e_J t^{(r)}) = \begin{cases} 1 \otimes f(t^{(q)}) & \text{if } e_J t^{(r)} = t^{(q)}, \\ 0 & \text{otherwise.} \end{cases}$$

$j = 1$: $e_1 t^{(q)}$ and $e_2 t^{(q)}$ are all the basis elements of F_{i+1}.

$$\begin{aligned} \tilde{f}_1(e_1 t^{(q)}) &= \phi \circ \tilde{f}_0 \circ d_{i+1}(e_1 t^{(q)}) \\ &= \phi \circ \tilde{f}_0 \left((x_1 \otimes 1 - 1 \otimes x_1) \cdot t^{(q)} - e_1 t^{(q-1)} \cdot d(t) \right) \\ &= 1 \otimes f(t^{(q)}) \cdot \phi(x_1 \otimes 1 - 1 \otimes x_1) = 1 \otimes f(t^{(q)}) \cdot e_1. \end{aligned}$$

Similarly, we get $\tilde{f}_1(e_2 t^{(q)}) = 1 \otimes f(t^{(q)}) \cdot e_2$.

Suppose that the formula holds up to $j - 1 \geq 0$. We need to show that the formula is true at j. Let $x = e_J t^{(r)}$ be a basis element of degree $i + j$.

If $J = \emptyset$, then $r > q$. Hence, by Proposition 3.9 one gets:

$$\begin{aligned} &\tilde{f}_j(x) = (\phi \circ \tilde{f}_{j-1}) \left(\sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i} \cdot e_1 t^{(r-1)} - \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_2 t^{(r-1)} \right) \\ &= \phi \left(\sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i} \cdot e_1 t^{(r-1-q)} - \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_2 t^{(r-1-q)} \right) \cdot 1 \otimes f(t^{(q)}) \\ &= 1 \otimes f(t^{(q)}) \cdot \phi(-x_2^{b-1} \otimes 1 \cdot e_2 t^{(r-1-q)}) = 1 \otimes f(t^{(q)}) \cdot t^{(r-q)}. \end{aligned}$$

If $J = \{1\}$, we have

$$\begin{aligned} &\tilde{f}_j(x) = \phi \circ \tilde{f}_{j-1} \circ d_{i+1}(e_1 t^{(r)}) \\ &= \phi \circ \tilde{f}_{j-1} \left(d(e_1) t^{(r)} + \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_1 e_2 t^{(r-1)} \right) \\ &= 1 \otimes f(t^{(q)}) \cdot \phi \left((x_1 \otimes 1 - 1 \otimes x_1) \cdot t^{(r-q)} + \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_1 e_2 t^{(r-1-q)} \right) \\ &= 1 \otimes f(t^{(q)}) \cdot \phi(x_1 \otimes 1 \cdot t^{(r-q)}) = 1 \otimes f(t^{(q)}) \cdot e_1 t^{(r-q)}. \end{aligned}$$

We also have $\tilde{f}_j(e_2 t^{(r)}) = 1 \otimes f(t^{(q)}) \cdot e_2 t^{(r-q)}$ and $\tilde{f}_j(e_1 e_2 t^{(r)}) = 1 \otimes f(t^{(q)}) \cdot e_1 e_2 t^{(r-q)}$ similarly. □

Lemma 3.11. *Let $f : F_i \to A$ be a cocycle of the form $b(e_1 t^{(q)}, x_1^{u_1} x_2^{u_2}) + a(e_2 t^{(q)}, x_1^{v_1} x_2^{v_2}))$. For $j \in \mathbb{N}$, the formula of the A^e-homomorphism $\tilde{f}_j$:*

$F_{i+j} \to F_j$ *is given as follows:*

$$\begin{aligned}
\tilde{f}_0(e_1t^{(q)}) &= 1 \otimes bx_1^{u_1}x_2^{u_2}; \tilde{f}_0(e_2t^{(q)}) = 1 \otimes ax_1^{v_1}x_2^{v_2};\\
\tilde{f}_1(t^{(q+1)}) &= 1 \otimes bx_1^{u_1}x_2^{u_2} \cdot \delta_1 e_1 - 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot \delta_2 e_2;\\
\tilde{f}_1(e_1e_2t^{(q)}) &= 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot e_1 - 1 \otimes bx_1^{u_1}x_2^{u_2} \cdot e_2;\\
\tilde{f}_{2j}(e_1t^{(q+j)}) &= 1 \otimes bx_1^{u_1}x_2^{u_2} \cdot t^{(j)} - 1 \otimes ax_1^{v_1}x_2^{v_2}\delta_2 \cdot e_1e_2t^{(j-1)};\\
\tilde{f}_{2j}(e_2t^{(q+j)}) &= 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot t^{(j)} - 1 \otimes bx_1^{u_1}x_2^{u_2} \cdot \delta_1 e_1e_2t^{(j-1)};\\
\tilde{f}_{2j+1}(t^{(q+j+1)}) &= 1 \otimes bx_1^{u_1}x_2^{u_2} \cdot \delta_1 e_1 t^{(j)} - 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot \delta_2 e_2 t^{(j)};\\
\tilde{f}_{2j+1}(e_1e_2t^{(q+j)}) &= 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot e_1t^{(j)} - 1 \otimes bx_1^{u_1}x_2^{u_2} \cdot e_2t^{(j)},
\end{aligned}$$

where $\delta_1 = \sum_{i=0}^{a-2}(i+1)x_1^{a-2-i} \otimes x_1^i$ *and* $\delta_2 = \sum_{i=0}^{b-2}(i+1)x_2^{b-2-i} \otimes x_2^i$.

Proof. The basis of F_i consists of $e_1t^{(q)}$ and $e_2t^{(q)}$. We can see that $\tilde{f}_0(e_1t^{(q)}) = 1 \otimes bx_1^{u_1}x_2^{u_2}$ and $\tilde{f}_0(e_2t^{(q)}) = 1 \otimes ax_1^{v_1}x_2^{v_2}$. In the next degree, the basis of F_{i+1} consists of $t^{(q+1)}$ and $e_1e_2t^{(q)}$. By Proposition 3.9 and the definition of $\tilde{f}$, we have that

$$\begin{aligned}
&\tilde{f}_1(t^{(q+1)}) = \phi \circ \tilde{f}_0 \circ d(t^{(q+1)})\\
=&(\phi \circ \tilde{f}_0)\left(\sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i} \cdot e_1t^{(q)} - \sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i} \cdot e_2t^{(q)}\right)\\
=&1 \otimes bx_1^{u_1}x_2^{u_2} \cdot \phi\left(\sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i}\right) - 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot \phi\left(\sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i}\right)\\
=&1 \otimes bx_1^{u_1}x_2^{u_2} \cdot \delta_1 e_1 - 1 \otimes ax_1^{v_1}x_2^{v_2} \cdot \delta_2 e_2
\end{aligned}$$

since

$$\begin{aligned}
&\phi\left(\sum_{i=0}^{a-1} x_1^i \otimes x_1^{a-1-i}\right) = \phi_1\left(1 \otimes x_1^{a-1} + x_1 \otimes x_1^{a-2} + \cdots + x_1^{a-1} \otimes 1\right)\\
=&\phi\left(1 \otimes x_1^{a-1}\right) + \phi\left(x_1 \otimes x_1^{a-2}\right) + \cdots + \phi\left(x_1^{a-1} \otimes 1\right)\\
=&(0) + \left(1 \otimes x_1^{a-2} \cdot e_1\right) + \left(x_1^2 \otimes x_1^{a-3} \cdot e_1 + 1 \otimes x_1^{a-2} \cdot e_1\right) +\\
&+ \cdots + \left(x_1^{a-2} \otimes 1 \cdot e_1 + x_1^{a-3} \otimes x_1 \cdot e_1 + \cdots + 1 \otimes x_1^{a-2} \cdot e_1\right)\\
=&\underbrace{\left(x_1^{a-2} \otimes 1 + 2x_1^{a-3} \otimes x_1 + \cdots + (a-1)1 \otimes x_1^{a-2}\right)}_{=:\delta_1} e_1
\end{aligned}$$

and similarly, $\phi\left(\sum_{i=0}^{b-1} x_2^i \otimes x_2^{b-1-i}\right) = \delta_2 e_2$.

Let us now consider the remaining basis element, $e_1e_2t^{(q)}$:

$$\begin{aligned}
&\tilde{f}_1(e_1e_2t^{(q)}) = \phi\circ\tilde{f}_0\circ d(e_1e_2t^{(q)})\\
&= \phi_1\circ\tilde{f}_0\left(d(e_1)e_2t^{(q)} - e_1d(e_2)t^{(q)}\right)\\
=&\phi\left((x_1\otimes 1 - 1\otimes x_1)\, 1\otimes ax_1^{v_1}x_2^{v_2} - (x_2\otimes 1 - 1\otimes x_2)\, 1\otimes bx_1^{u_1}x_2^{u_2}\right)\\
=&\phi\left(x_1\otimes ax_1^{v_1}x_2^{v_2}\right) - \phi_1\left(x_2\otimes bx_1^{u_1}x_2^{u_2}\right)\\
=&1\otimes ax_1^{v_1}x_2^{v_2}\cdot e_1 - 1\otimes bx_1^{u_1}x_2^{u_2}\cdot e_2.
\end{aligned}$$

For the higher degrees, we shall use induction on even and odd degrees. Suppose that the formula holds up to $2j$, we need to show the formula holds for $2j+1$. Indeed,

$$\begin{aligned}
&\tilde{f}_{2j+1}(t^{(q+j+1)}) =\\
&(\phi\circ\tilde{f}_{2j})\left(\sum_{i=0}^{a-1} x_1^i\otimes x_1^{a-1-i}\cdot e_1t^{(q+j)} - \sum_{i=0}^{b-1} x_2^i\otimes x_2^{b-1-i}\cdot e_2t^{(q+j)}\right)\\
=&\phi\left(\left(\sum_{i=0}^{a-1} x_1^i\otimes x_1^{a-1-i}\right)\cdot\left(1\otimes bx_1^{u_1}x_2^{u_2}\cdot t^{(j)} - 1\otimes ax_1^{v_1}x_2^{v_2}\cdot\delta_2e_1e_2t^{(j-1)}\right)\right)\\
&-\phi\left(\left(\sum_{i=0}^{b-1} x_2^i\otimes x_2^{b-1-i}\right)\cdot\left(1\otimes ax_1^{v_1}x_2^{v_2}\cdot t^{(j)} - 1\otimes bx_1^{u_1}x_2^{u_2}\cdot\delta_1e_1e_2t^{(j-1)}\right)\right)\\
=&\phi\left(\sum_{i=0}^{a-1} x_1^i\otimes bx_1^{a-1-i}x_1^{u_1}x_2^{u_2}\cdot t^{(j)}\right) - \phi\left(\sum_{i=0}^{b-1} x_2^i\otimes ax_2^{b-1-i}x_1^{v_1}x_2^{v_2}\cdot t^{(j)}\right)\\
=&1\otimes bx_1^{u_1}x_2^{u_2}\cdot\delta_1e_1t^{(j)} - 1\otimes ax_1^{v_1}x_2^{v_2}\cdot\delta_2e_2t^{(j)}.
\end{aligned}$$

Using a similar argument, we get the formula of $\tilde{f}_{2j+1}(e_1e_2t^{(q+j)})$.
Now suppose that the formula holds up to $2j-1$, we prove that the formula at $2j$ holds.

$$\tilde{f}_{2j}(e_1t^{(q+j)}) = (\phi\circ\tilde{f}_{2j-1})\left(d(e_1)t^{(q+j)} + \sum_{i=0}^{b-1} x_2^i\otimes x_2^{b-1-i}\cdot e_1e_2t^{(q+j-1)}\right)$$

$$\begin{aligned}
=&\phi\left((x_1\otimes 1 - 1\otimes x_1)\cdot\left(1\otimes bx_1^{u_1}x_2^{u_2}\cdot\delta_1e_1t^{(j-1)} - 1\otimes ax_1^{v_1}x_2^{v_2}\cdot\delta_2e_2t^{(j-1)}\right)\right)\\
&+\phi\left(\left(\sum_{i=0}^{b-1} x_2^i\otimes x_2^{b-1-i}\right)\cdot\left(1\otimes ax_1^{v_1}x_2^{v_2}\cdot e_1t^{(j-1)} - 1\otimes bx_1^{u_1}x_2^{u_2}\cdot e_2t^{(j-1)}\right)\right)\\
=&-\phi\left(x_1\otimes ax_1^{v_1}x_2^{v_2}\cdot\delta_2e_2t^{(j-1)}\right) - \phi\left(x_2^{b-1}\otimes bx_1^{u_1}x_2^{u_2}\cdot e_2t^{(j-1)}\right)\\
=&-1\otimes ax_1^{v_1}x_2^{v_2}\cdot\delta_2e_1e_2t^{(j-1)} + 1\otimes bx_1^{u_1}x_2^{u_2}\cdot t^{(j)}.
\end{aligned}$$

Similarly we get the formula $\tilde{f}_{2j}(e_2t^{(q+j)})$ as desired. $\square$

So far we have obtained the formula of a chain map $\tilde{f}$ that makes diagram (3.6) commute by using a contracting homotopy based on a Morse matching. Beside this method, we can also use induction in order to construct and then prove the formula of $\tilde{f}$.

3.4.5 The cup product

From the formula of $\tilde{f}$, the cup product can be interpreted in terms of the Yoneda product (see [19] Chapter 1 for more details) on $\mathrm{HH}^*(A)$ as follows. Let f and g be cocycles in $\mathrm{Hom}(F_i, A)$ and $\mathrm{Hom}(F_j, A)$ respectively. Then the product of these cocycles, denoted by $f \smile g$, is given by

$$f \smile g := g \circ \tilde{f}_j,$$

which is again a cocycle of homological degree $i + j$. Since $\tilde{f}$ is unique up to homotopy, the cup product induces a well-defined product by passing to cohomology, i.e., we have a multiplication on $\mathrm{HH}^*(A)$. By Lemmas 3.10 and 3.11, we have the product of two standard residue classes in the consequence below.

Corollary 3.12. *The formula of the cup product between two standard residue classes in* $\mathrm{HH}^*(A)$ *is calculated as follows:*

$$[(t^{(p)}, s^{\alpha})] \smile [(t^{(q)}, s^{\beta})] = [(t^{(p+q)}, s^{\alpha+\beta})];$$

$$\begin{aligned}[(t^{(p)}, s^{\alpha})] \smile \big[b(e_1 t^{(q)}, s^{\beta-m_1}) + a(e_2 t^{(q)}, s^{\beta-m_2})\big] \\ =[b(e_1 t^{(p+q)}, s^{\alpha+\beta-m_1}) + a(e_2 t^{(p+q)}, s^{\alpha+\beta-m_2})];\end{aligned}$$

$$\big[b(e_1 t^{(p)}, s^{\alpha-m_1}) + a(e_2 t^{(p)}, s^{\alpha-m_2})\big] \smile \big[b(e_1 t^{(q)}, s^{\beta-m_1}) + a(e_2 t^{(q)}, s^{\beta-m_2})\big] = 0.$$

Moreover, the multiplication is commutative.

Proof. Let $f := (t^{(p)}, s^{\alpha})$ and $g := (t^{(q)}, s^{\beta})$. We calculate the first product as follows:

$$\begin{aligned}(f \smile g)(e_J t^{(u)}) =& g \circ \tilde{f}(e_J t^{(u)}) \\ =& [p \le u] g(e_J t^{(u-p)}) \cdot 1 \otimes f(t^{(p)}) \\ =& \begin{cases} g(t^{(q)}) \cdot f(t^{(p)}) & \text{if } e_J t^{(u-p)} = t^{(q)} \\ 0 & \text{otherwise} \end{cases} \\ =& \begin{cases} s^{\alpha+\beta} & \text{if } e_J t^{(u)} = t^{(p+q)}, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

For the second one, let $f=(t^{(p)},s^{\alpha})$ and $g=b(e_1t^{(q)},s^{\beta-m_1})+a(e_2t^{(q)},s^{\beta-m_2})$. By a similar computation, we get the result:

$$
\begin{aligned}
(f\smile g)(e_Jt^{(u)}) &= \begin{cases} g(e_1t^{(q)})\cdot f(t^{(p)}) & \text{if } e_Jt^{(u-p)}=e_1t^{(q)},\\ g(e_2t^{(q)})\cdot f(t^{(p)}) & \text{if } e_Jt^{(u-p)}=e_2t^{(q)},\\ 0 & \text{otherwise,}\end{cases}\\
&= \begin{cases} bs^{\alpha+\beta-m_1} & \text{if } e_Jt^{(u)}=e_1t^{(p+q)},\\ as^{\alpha+\beta-m_2} & \text{if } e_Jt^{(u)}=e_2t^{(p+q)},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

Now we take two elements $f=b(e_1t^{(p)},s^{\alpha-m_1})+a(e_2t^{(p)},s^{\alpha-m_2})$ of degree i and $g=b(e_1t^{(q)},s^{\beta-m_1})+a(e_2t^{(q)},s^{\beta-m_2})$ of degree j. The basis of F_{i+j} consists of $e_1e_2t^{(p+q)}$ and $t^{(p+q+1)}$. Apply Lemma 3.11, replace x_1,x_2 by s^b,s^a respectively and notice that $j=2q+1$, we get that:

$$
\begin{aligned}
(f\smile g)(e_1e_2t^{(p+q)}) =& g\left(\tilde{f}_j(e_1e_2t^{(p+q)})\right)\\
=& g(1\otimes as^{\alpha-m_2}\cdot e_1t^{(q)}-1\otimes bs^{\alpha-m_1}\cdot e_2t^{(q)})\\
=& (1\otimes as^{\alpha-m_2})\cdot bs^{\beta-m_1}-(1\otimes bs^{\alpha-m_1})\cdot as^{\beta-m_2}\\
=& 0.
\end{aligned}
$$

Let us consider the remaining basis element:

$$
\begin{aligned}
(f\smile g)(t^{(p+q+1)}) =& g(1\otimes bs^{\alpha-m_1}\delta_1e_1t^{(q)}-1\otimes as^{\alpha-m_2}\delta_2e_2t^{(q)})\\
=& \frac{a(a-1)}{2}b^2s^{\alpha+\beta-2m_1+b(a-2)}-\frac{b(b-1)}{2}a^2s^{\alpha+\beta-2m_2+a(b-2)}\\
=& \frac{ab(a-b)}{2}s^{\alpha+\beta-ab}.
\end{aligned}
$$

Here, we consider δ_1, δ_2 using the variable s. So we have shown that

$$f\smile g=\frac{ab(a-b)}{2}(t^{(p+q+1)},s^{\alpha+\beta-ab}).$$

Now we will state that $(t^{(p+q+1)},s^{\alpha+\beta-ab})$ belongs to the image of ∂, i.e., its residue class in $\mathrm{HH}^*(A)$ is zero. By Remark 3.8, we will show that $\alpha+\beta-ab-m_1\in S$ or $\alpha+\beta-ab-m_2\in S$. From Corollary 3.6, there are two options for $\alpha-m_1$ and $\beta-m_1$, which are m_2 and $\gamma+b$ for some $\gamma\in S$.

- If $\alpha-m_1=\beta-m_1=m_2$, then $\alpha=\beta=m_1+m_2$. Hence, $\alpha+\beta-ab-m_1=m_1+a(b-2)\in S$ and $\alpha+\beta-ab-m_2=m_2+(a-2)b\in S$.

- If $\alpha - m_1 = m_2$ and $\beta - m_1 = \gamma + b$ for some $\gamma \in S$, then $\alpha + \beta - ab - m_1 = \gamma + m_2 \in S$ and $\alpha + \beta - ab - m_2 = \gamma + m_1 \in S$. Similarly for $\alpha - m_1 = \gamma + b$ and $\beta - m_1 = m_2$.
- If $\alpha - m_1 = \gamma + b$ and $\beta - m_1 = \eta + b$ for some $\gamma, \eta \in S$, then $\alpha+\beta-ab-m_1 = \gamma+\eta+b \in S$ and $\alpha+\beta-ab-m_2 = \gamma+\eta+a \in S$. $\square$

By supplementing the module $\mathrm{HH}^*(A)$ with a multiplicative structure, this module becomes a k-algebra. By Corollaries 3.6 and 3.12, we have the description of the generators for the algebra $\mathrm{HH}^*(A)$ as follows.

Remark 3.13. (i) For the basic element of the form $[(t^{(p)}, s^{\alpha})]$ (where $\alpha \in S$) there are $u, v \in \mathbb{N}$ such that $\alpha = ua + vb$. Then we can write $[(t^{(p)}, s^{\alpha})]$ as a product of the elements $[(t,1)]$, $[(1,s^a)]$ and $[(1,s^b)]$.
(ii) Likewise, a basic element of the form $[b(e_1 t^{(q)}, s^{\alpha-m_1}) + a(e_2 t^{(q)}, s^{\alpha-m_2})]$ (where $\alpha \in S$) is written as a product of $[(t,1)]$, $[(1,s^a)]$, $[(1,s^b)]$ and either $[b(e_1, s^{m_2}) + a(e_2, s^{m_1})]$ or $[b(e_1, s^b) + a(e_2, s^a)]$, where the two last elements occur once for such a basic element of this type.

Now we are in the position to give the ring structure of $\mathrm{HH}^*(A)$ in the first case.

3.4.6 The ring structure of $\mathrm{HH}^*(A)$

In the following theorem, we will provide the structure of $\mathrm{HH}^*(A)$ in terms of generators and relations.

Theorem 3.14 ($\mathrm{char}(k) \nmid a, b$)**.** *Let k be a field with characteristic $\mathrm{char}(k)$ and $k[s^a, s^b]$ the numerical semigroup algebra, where a and b are coprime positive integers in which $\mathrm{char}(k)$ is neither a divisor of a nor b. The Hochschild cohomology algebra of $k[s^a, s^b]$ is isomorphic to the quotient ring*

$$k[X_1, X_2, Y_1, Y_2, T]/\mathcal{I},$$

where $k[X_1, X_2, Y_1, Y_2, T]$ is a weighted graded commutative polynomial ring in which $\deg(X_1) = \deg(X_2) = 0$, $\deg(Y_1) = \deg(Y_2) = 1$ and $\deg(T) = 2$; $\mathrm{wt}(X_1) = a$, $\mathrm{wt}(X_2) = b$, $\mathrm{wt}(Y_1) = 0$, $\mathrm{wt}(Y_2) = ab-a-b$ and $\mathrm{wt}(T) = -ab$; and the ideal $\mathcal{I}$ is generated by the following relations: $X_1^b - X_2^a$, $X_1^{b-1}T$, $X_2^{a-1}T$, Y_2T, Y_1^2, Y_2^2, Y_1Y_2, $X_1Y_2 - X_2^{a-1}Y_1$, $X_2Y_2 - X_1^{b-1}Y_1$.

Proof. The Hochschild cohomology module $\mathrm{HH}^*(A)$ consists of the cosets of the cocycles in $\mathrm{Ker}(\partial)$. We set X_1 to be the element $[(1, s^a)]$. Similarly, we have X_2 for $[(1, s^b)]$, Y_1 for $[b(e_1, s^b)+a(e_2, s^a)]$, Y_2 for $[b(e_1, s^{m_2})+a(e_2, s^{m_1})]$ and T for $[(t, 1)]$. Let us introduce a multidegree 'mdeg' combined from an $\mathbb{N}$-grading (on the first argument) and a $\mathbb{Z}$-weight (on the second argument) by setting $\mathrm{mdeg}(e_1, 1) = (1, -b)$, $\mathrm{mdeg}(e_2, 1) = (1, -a)$, $\mathrm{mdeg}(1, s) = (0, 1)$, $\mathrm{mdeg}(t, 1) = (2, -ab)$. Then we consider the decomposition of $\mathrm{HH}^*(A)$ induced by our multidegree. The differential ∂ is a 1-homogeneous morphism with respect to the grading and a 0-homogeneous morphism with respect to the weight. By Remark 3.13, we know that $\mathrm{HH}^*(A)$ is generated by X_1, X_2, Y_1, Y_2 and T. The degree ('deg') and the weight ('wt') of these elements follow from the multidegree. To show that the relations in the theorem are satisfied, we use Corollary 3.12 as follows.

- As $(1, s^a)^b = (1, s^{ab}) = (1, s^b)^a$, we have the first relation, $X_1^b - X_2^a$.
- Using the formula in Corollary 3.12, we have the relation Y_1^2,Y_2^2 and Y_1Y_2.
- By Remark 3.8, the standard cocycles in the image of ∂ consist of:
  $(t^{(q)}, s^\alpha)$, where $q > 0$, $\alpha \in S$, $\alpha - m_1 \in S$;
  $(t^{(q)}, s^\alpha)$, where $q > 0$, $\alpha \in S$, $\alpha - m_2 \in S$; and
  $b(e_1t^{(q)}, s^{\alpha+m_2}) + a(e_2t^{(q)}, s^{\alpha+m_1})$, where $q > 0$, $\alpha \in S$, $\alpha - F(S) \in S$.
  From this, we can deduce the relations $X_1^{b-1}T$, $X_2^{a-1}T$ and Y_2T.

So far, we have obtained all generators and relations displayed in the statement. Now we will prove that there is an isomorphism between the algebras, $\mathrm{HH}^*(A)$ and $k[X_1, X_2, Y_1, Y_2, T]/\mathcal{I}$, by showing that there is a bigraded bijection between a k-basis of each.

We first describe the k-basis of the algebra $k[X_1, X_2, Y_1, Y_2, T]/\mathcal{I}$. By Example 1.55, the Gröbner basis of $\mathcal{I}$ with respect to the pure lexicographic term order $X_1 \prec X_2 \prec Y_1 \prec Y_2 \prec T$ on $k[X_1, X_2, Y_1, Y_2, T]$ is determined as follows: $X_2^a - X_1^b$, $X_1^{b-1}T$, $X_2^{a-1}T$, Y_2T, Y_1^2, Y_2^2, Y_1Y_2, $X_1Y_2 - X_2^{a-1}Y_1$, $X_2Y_2 - X_1^{b-1}Y_1$. The leading terms of this Gröbner base are X_2^a, $X_1^{b-1}T$, $X_2^{a-1}T$, Y_2T, Y_1^2, Y_2^2, Y_1Y_2, X_1Y_2, X_2Y_2. From here, one has a k-basis of the algebra $k[X_1, X_2, Y_1, Y_2, T]/\mathcal{I}$ consisting of the following elements:

- $X_1^uX_2^v$, where $u \geq 0$, $0 \leq v < a$;

- $X_1^u X_2^v T^q$, where $0 \le u < b-1$, $0 \le v < a-1$, $q > 0$;
- $X_1^u X_2^v Y_1$, where $u \ge 0$, $0 \le v < a$;
- $X_1^u X_2^v Y_1 T^q$, where $0 \le u < b-1$, $0 \le v < a-1$, $q > 0$; and
- Y_2.

The k-basis of $\mathrm{HH}^*(A)$ was described in Remark 3.8. It can be easily seen that there is a bigraded one-to-one correspondence between: $X_1^u X_2^v$, where $u \ge 0$, $0 \le v < a$ and $(1, s^\alpha)$, where $\alpha \in S$; Y_2 and $b(e_1, s^{m_2}) + a(e_2, s^{m_1})$; $X_1^u X_2^v Y_1$, where $u \ge 0$, $0 \le v < a$ and $b(e_1, s^{\alpha+b}) + a(e_2, s^{\alpha+a})$, where $\alpha \in S$. Now we will show that the rest of the k-bases of $\mathrm{HH}^*(A)$ and $k[X_1, X_2, Y_1, Y_2, T]/\mathcal{I}$ are corresponding to each other as well.

(i) $X_1^u X_2^v T^q$, where $0 \le u < b-1$, $0 \le v < a-1$, $q > 0$ and $(t^{(q)}, s^\alpha)$, where $q > 0$, $\alpha \in S$, $\alpha - m_1 \notin S$, $\alpha - m_2 \notin S$ are equivalent. Indeed, suppose that $\alpha = ua + vb$ $(u, v \in \mathbb{N})$. We will show that

$$\begin{cases} u < b-1 \\ v < a-1 \end{cases} \Leftrightarrow \begin{cases} ua + vb - m_1 \notin S \\ ua + vb - m_2 \notin S \end{cases}.$$

"$\Leftarrow$" Suppose on the contrary that $u \ge b-1$. Then, $ua + vb - m_2 = ua + vb - a(b-1) = vb + a\left(u - (b-1)\right) \in S$, which is a contradiction. Similar for v. "$\Rightarrow$" By Lemma 3.1, $v < a-1$ implies that $vb - a \notin S$. Since $u < b-1$, $\gamma = ua + vb - m_2 = ua + vb - a(b-1) = vb + a\left(u - (b-1)\right) \notin S$. If not, $\gamma \in S$, so $vb - a = \gamma + a(b-2-u) \in S$, which is inconsequential.

(ii) $X_1^u X_2^v Y_1 T^q$, where $0 \le u < b-1$, $0 \le v < a-1$, $q > 0$ corresponds to $b(e_1 t^{(q)}, s^{\alpha+b}) + a(e_2 t^{(q)}, s^{\alpha+a})$, where $\alpha \in S$, $\alpha - F(S) \notin S$, $q > 0$. Suppose that $\alpha = ua + vb$ $(u, v \in \mathbb{Z}, \ge 0)$. We will show that

$$\begin{cases} u < b-1 \\ v < a-1 \end{cases} \Leftrightarrow ua + bv - F(S) \notin S.$$

"$\Leftarrow$" Suppose on the contrary that $u \ge b-1$ or $v \ge a-1$. Then we have $ua + bv - ab + a + b = (u - b + 1)a + bv + b \in S$ or $ua + bv - ab + a + b = (v - a + 1)b + au + a \in S$, which contradicts $ua + bv - F(S) \notin S$. "$\Rightarrow$" Suppose that $u < b-1$ and $v < a-1$. We need to show that $ua + bv - F(S) \notin S$. If $ua + bv - F(S) \in S$, then $ua + bv - F(S) = -a + (v+1)b + (u - b + 2)a \in S$, where $u - b + 2 \le 0$. This implies that $-a + (v+1)b \in S$ (where $v + 1 < a$) which is impossible by Lemma 3.1.

Hence, we have proved that the Hochschild cohomology ring $\mathrm{HH}^*(A)$ is isomorphic to the quotient ring $k[X_1, X_2, Y_1, Y_2, T]/\mathcal{I}$. $\square$

3.4.7 The Hilbert series

Let $\mathbf{H}_{m,n}$ be the k-module generated by the elements whose degree is $(m, n) \in \mathbb{N} \times \mathbb{Z}$. The Hilbert series of $\mathrm{HH}^*(A) = \bigoplus_{(m,n)\in\mathbb{N}\times\mathbb{Z}} \mathbf{H}_{m,n}$ as an $\mathbb{N} \times \mathbb{Z}$-graded vector space via the grading above is the formal series:

$$\mathcal{H}_{\mathrm{HH}^*(A)}(x, y) = \sum_{(m,n)\in\mathbb{N}\times\mathbb{Z}} \dim_k(\mathbf{H}_{m,n}) x^m y^n.$$

This series is computed based on the Hilbert series of the non-zero cocycles, whose description is listed in Remark 3.8. We will use the decomposition introduced in Proof of Theorem A in computing the Hilbert series.

(i) We have that $(1, s^\alpha)$, where $\alpha \in S$, contributes the series

$$\mathrm{H}_1 = \mathcal{H}_{k[s^a,s^b]}(x, y) = \frac{1 - y^{ab}}{(1 - y^a)(1 - y^b)}.$$

(ii) Now we consider the non-zero cocycles of type (3.4) in Remark 3.8, $(t^{(q)}, s^\alpha)$, where $q > 0$, $\alpha \in S$, $\alpha - m_1 \notin S$ and $\alpha - m_2 \notin S$. The element $(t^{(q)}, s^\alpha)$, where $\alpha \in S$, has degree $(2q, q(-ab) + \alpha)$. This element contributes the term $x^{2q}y^{q(-ab)+\alpha}$, which is equivalent to $(x^2y^{-ab})^q y^\alpha$. We notice that

$$\{\alpha \in S \mid \alpha - m_1 \notin S \text{ and } \alpha - m_2 \notin S\} = S \setminus (S_1 \cup S_2)$$

and, by the principle of inclusion-exclusion, we have that

$$\sum_{\alpha\in S\setminus(S_1\cup S_2)} y^\alpha = \sum_{\alpha\in S} y^\alpha - \left(\sum_{\alpha\in S_1} y^\alpha + \sum_{\alpha\in S_2} y^\alpha \right) + \sum_{\alpha\in S_1\cap S_2} y^\alpha. \tag{3.7}$$

By Lemma 3.2, we already have the detailed description of S_1, S_2 and $S_1 \cap S_2$. Now we are able to calculate the Hilbert series formed by this kind of cohomology classes.

- The series given by all $(t^{(q)}, s^\alpha)$, where $\alpha \in S$ and $q > 0$ is

$$\mathrm{H}_{2A} = \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot \mathrm{H}_1.$$

- The element $(t^{(q)}, s^{\alpha})$, where $\alpha \in S_1$ is written as $(t^{(q)}, s^{\gamma+m_1})$, where $\gamma \in S$. Hence, the corresponding degree is $(2q, q(-ab) + m_1 + \gamma)$, which contributes the term $x^{2q}y^{q(-ab)+m_1+\gamma}$. Then, the series given by all $(t^{(q)}, s^{\gamma+m_1})$, where $\gamma \in S$, $q > 0$, is

$$\mathrm{H}_{2B} = \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot y^{m_1} \cdot \mathrm{H}_1.$$

- Similarly, the series given by all $(t^{(q)}, s^{\alpha})$, where $\alpha \in S_2$, $q > 0$, is

$$\mathrm{H}_{2C} = \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot y^{m_2} \cdot \mathrm{H}_1.$$

- The element $(t^{(q)}, s^{\alpha})$, where $\alpha \in S_1 \cap S_2$, is $(t^{(q)}, s^{m_1+m_2})$ or $(t^{(q)}, s^{ab+\gamma})$, where $\gamma \in S$. Hence, by a similar argument, we find out that the series for these elements is

$$\mathrm{H}_{2D} = \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot \left(y^{m_1+m_2} + y^{ab} \cdot \mathrm{H}_1\right).$$

By (3.7), the Hilbert series for the elements of type (3.4) is

$$\mathrm{H}_2 = \mathrm{H}_{2A} - (\mathrm{H}_{2B} + \mathrm{H}_{2C}) + \mathrm{H}_{2D}.$$

(iii) For the cocycles of type (3.5) in Remark 3.8, we have the single cocycle $b(e_1, s^{m_2}) + a(e_2, s^{m_1})$ and the cocycles $b(e_1t^{(q)}, s^{b+\alpha}) + a(e_2t^{(q)}, s^{a+\alpha})$, where $\alpha \in S$ and if $q > 0$, $\alpha - F(S) \notin S$.

- The element $b(e_1, s^{m_2}) + a(e_2, s^{m_1})$ of degree $(1, ab - a - b)$ and the elements $b(e_1, s^{b+\alpha}) + a(e_2, s^{a+\alpha})$ of degree $(1, \alpha)$ contribute the series

$$\mathrm{H}_{3A} = xy^{ab-a-b} + x \cdot \mathrm{H}_1.$$

- For the remaining elements, we notice that

$$\{\alpha \in S \mid \alpha - F(S) \notin S\} = S \smallsetminus \{\gamma + F(S) \mid \gamma \in S \smallsetminus \{0\}\}.$$

  So we have the series

$$\mathrm{H}_{3B} = x \cdot \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot \mathrm{H}_1,$$

which corresponds to the $b(e_1 t^{(q)}, s^{b+\alpha}) + a(e_2 t^{(q)}, s^{a+\alpha})$, where $q > 0$ and $\alpha \in S$.

And the series

$$\mathrm{H}_{3C} = xy^{ab-a-b} \cdot \frac{x^2 y^{-ab}}{1 - x^2 y^{-ab}} \cdot (\mathrm{H}_1 - 1),$$

corresponds to the elements $b(e_1 t^{(q)}, s^{b+\gamma+F(S)}) + a(e_2 t^{(q)}, s^{a+\gamma+F(S)})$, where $q > 0$ and $\gamma \in S \setminus \{0\}$.

Now we get the Hilbert series for all elements of type (3.5), which is

$$\mathrm{H}_3 = \mathrm{H}_{3A} + \mathrm{H}_{3B} - \mathrm{H}_{3C}.$$

Hence, the Hilbert series for $\mathrm{HH}^*(A)$ in Case I is the series

$$\mathcal{H}_{\mathrm{HH}^*(A)}(x, y) = \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_3.$$

3.4.8 Example

Hilbert series of the Hochschild cohomology of the algebra $k[s^2, s^3]$ where k is a finite field with characteristic 101, which is neither a divisor of $a = 2$ nor $b = 3$. By Theorem 3.14, we have the isomorphism:

$$\mathrm{HH}^*(A) \cong k[x_1, x_2, y_1, y_2, t]/I$$

where the ideal I is generated by

$$\begin{gathered} x_1^3 - x_2^2, \\ x_1^2 t, x_2 t, \\ y_2 t, y_1^2, y_2^2, y_1 y_2, \\ x_1 y_2 - x_2 y_1, x_2 y_2 - x_1^2 y_1. \end{gathered}$$

In Appendix B, we present the Macaulay2 code to compute and to check the Hilbert series of this example.

3.5 The ring structure of $\mathrm{HH}^*(A)$ - Case II

In this section, we will use the same arguments as in Case I to describe the ring structure of $\mathrm{HH}^*(A)$ in the case that $\mathrm{char}(k)$ is a divisor of a. Some of our results shall be stated without proof because the reader can establish them analogously to the previous case.

3.5.1 The formula of the cup product

Since $\mathrm{char}(k)$ is a divisor of a, the formula of ∂ becomes

$$\partial(e_I t^{(q)}, s^\alpha) = \begin{cases} 0 & \text{if } I = \emptyset \text{ or } \{1\}, \\ -b(t^{(q+1)}, s^{\alpha+m_2}) & \text{if } I = \{2\}, \\ b(e_1 t^{(q+1)}, s^{\alpha+m_2}) & \text{if } I = \{1,2\}. \end{cases}$$

Then we have an immediate consequence of the information on the kernel and the image of ∂ as follows.

Corollary 3.15. *(i) The kernel of ∂ is spanned by $(e_I t^{(q)}, s^\alpha)$, where $\alpha \in S$ and $I = \emptyset$ or $I = \{1\}$.*

(ii) The image of ∂ is spanned by $(e_I t^{(q)}, s^\alpha)$, where $\alpha \in S$, $I = \emptyset$ or $\{1\}$, $\alpha - m_2 \in S$ and $q > 0$.

As the explicit chain map was constructed independently from the characteristic $\mathrm{char}(k)$, we can interpret this chain map from Case I for Case II.

Lemma 3.16. *(i) Let $f : F_i \to A$ be a cocycle of the form $(t^{(q)}, s^\alpha)$ in $\mathrm{Hom}_{A^e}(F_i, A)$. For any $j \in \mathbb{N}$, the formula of the A^e-homomorphism $\tilde{f}_j : F_{i+j} \to F_j$ is given by*

$$\tilde{f}_j(e_J t^{(r)}) = [q \leq r] e_J t^{(r-q)} \cdot 1 \otimes f(t^{(q)}).$$

(ii) If $f : F_i \to A$ is a cocycle of the form $(e_1 t^{(q)}, s^\alpha)$ in $\mathrm{Hom}_{A^e}(F_i, A)$, then for any $j \in \mathbb{N}$, the A^e-homomorphism $\tilde{f}_j : F_{i+j} \to F_j$ is given by:

$$\tilde{f}_0(e_1 t^{(q)}) = 1 \otimes s^\alpha; \tilde{f}_0(e_2 t^{(q)}) = 0;$$

$$\tilde{f}_1(t^{(q+1)}) = 1 \otimes s^\alpha \delta_1 e_1; \tilde{f}_1(e_1 e_2 t^{(q)}) = -1 \otimes s^\alpha e_2;$$

$$\tilde{f}_{2j}(e_1 t^{(q+j)}) = 1 \otimes s^\alpha t^{(j)}; \tilde{f}_{2j}(e_2 t^{(q+j)}) = -1 \otimes s^\alpha \delta_1 e_1 e_2 t^{(j-1)};$$

$$\tilde{f}_{2j+1}(t^{(q+j+1)}) = 1 \otimes s^\alpha \delta_1 e_1 t^{(j)}; \tilde{f}_{2j+1}(e_1 e_2 t^{(q+j)}) = -1 \otimes s^\alpha e_2 t^{(j)}.$$

Corollary 3.17. *The formula of the cup product between two standard residue classes in $\mathrm{HH}^*(A)$ is given by:*

$$[(t^{(p)}, s^\alpha)] \smile [(t^{(q)}, s^\beta]) = [(t^{(p+q)}, s^{\alpha+\beta})]$$

$$[(t^{(p)}, s^{\alpha})] \smile [(e_1 t^{(q)}, s^{\beta})] = [(e_1 t^{(p+q)}, s^{\alpha+\beta})]$$

$$[(e_1 t^{(p)}, s^{\alpha})] \smile [(e_1 t^{(q)}, s^{\beta})] =$$

$$\begin{cases} [(t^{(p+q+1)}, s^{\alpha+\beta+b(a-2)})] & \textit{if } \mathrm{char}(k) = 2 \textit{ and } 4 \nmid a, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The two first formulas are obtained by computing directly. For the last formula, we have

$$(e_1 t^{(p)}, s^{\alpha}) \smile (e_1 t^{(q)}, s^{\beta}) = \frac{a(a-1)}{2}(t^{(p+q+1)}, s^{\alpha+\beta+b(a-2)}).$$

Recall that $\mathrm{char}(k)$ is a divisor of a. If $\mathrm{char}(k) \neq 2$ or $\mathrm{char}(k) = 2$ and a is divisible by 4, then $\mathrm{char}(k)$ is a divisor of $\frac{a(a-1)}{2}$. Hence, we have $\frac{a(a-1)}{2} = 0$. If $\mathrm{char}(k) = 2$ and 4 is not a divisor of a, then $a = 2n$ where n is an odd number. Then we get $\frac{a(a-1)}{2} = n(2n-1) \equiv 1$ modulo 2. $\square$

3.5.2 The ring structure

Theorem 3.18 ($\mathrm{char}(k) \mid a$). *Let k be a field with characteristic $\mathrm{char}(k)$ and a, b two coprime integers in which $\mathrm{char}(k)$ is a divisor of a. Then the Hochschild cohomology algebra of $k[s^a, s^b]$ is isomorphic to the quotient ring*

$$k[X_1, X_2, Y, T]/\mathcal{I},$$

where $k[X_1, X_2, Y, T]$ is a weighted graded commutative polynomial ring in which $\deg(X_1) = \deg(X_2) = 0$, $\deg(Y) = 1$ and $\deg(T) = 2$; $\mathrm{wt}(X_1) = a$, $\mathrm{wt}(X_2) = b$, $\mathrm{wt}(Y) = -b$ and $\mathrm{wt}(T) = -ab$; and the ideal $\mathcal{I}$ is generated by the relations:

- $X_1^b - X_2^a$*,* $X_1^{b-1}T$*, and* $Y^2 - X_2^{a-2}T$ *if* $\mathrm{char}(k) = 2$ *and* $4 \nmid a$*; or*
- $X_1^b - X_2^a$*,* $X_1^{b-1}T$*, and* Y^2 *otherwise.*

Proof. All cocycles are combinations of the elements $(t^{(q)}, s^{\beta})$ and $(e_1 t^{(q)}, s^{\beta})$ where $\beta \in S$. By Corollary 3.17, we can see that all basis cocycles are products of $(1, s^a)$, $(1, s^b)$, $(e_1, 1)$ and $(t, 1)$. So we set X_1, X_2, Y and T to be the cosets of the elements $(1, s^a)$, $(1, s^b)$, $(e_1, 1)$ and $(t, 1)$ respectively. Then these are generators of the ring. In addition, we easily obtain all the relations $X_1^b - X_2^a$ (as $(1, s^a)^b = (1, s^b)^a$), $X_1^{b-1}T$ by Corollary 3.15 (ii) and $Y^2 - X_2^{a-2}T$ if $\mathrm{char}(k) = 2$ and $4 \nmid a$ (or Y^2 otherwise) by Corollary 3.17.

Now we will show that there is a bigraded bijection between the k-bases of $k[X_1, X_2, Y, T]/\mathcal{I}$ and $\mathrm{HH}^*(A)$. Let us start with $k[X_1, X_2, Y, T]/\mathcal{I}$. The Gröbner basis of $\mathcal{I}$ with respect to the pure lexicographic term order $Y \succ X_1 \succ X_2 \succ T$ consists of $X_1^b - X_2^a$, $Y^2 - X_2^{a-2}T$, $X_1^{b-1}T$ and X_2^aT in the case that $\mathrm{char}(k) = 2$ and a is not divisible by 4. The other case is very similar. Moreover, the Gröbner basis has the same leading terms with respect to the above order, so we can skip this case. Then, we get the k-basis of $k[X_1, X_2, Y, T]/\mathcal{I}$ as follows:

- $X_1^u X_2^v Y^i$, where $0 \le u < b$, $v \ge 0$ and $i \in \{0, 1\}$;
- $X_1^u X_2^v Y^i T^q$, where $0 \le u < b-1$, $0 \le v < a$, $i \in \{0, 1\}$ and $q > 0$.

By Corollary 3.15, we can infer that the standard elements corresponding to the non-zero elements in $\mathrm{HH}^*(A)$ are $(t^{(q)}, s^\alpha)$ and $(e_1 t^{(q)}, s^\alpha)$, where $\alpha \in S$ and if $q > 0$, $\alpha - m_2 \notin S$. In the following, we will see the correspondence between the k-bases of the two rings:

- $X_1^u X_2^v$ ($0 \le u < b$, $v \ge 0$) corresponds to $(1, s^\alpha)$ where $\alpha \in S$.
- $X_1^u X_2^v Y$ ($0 \le u < b$, $v \ge 0$) corresponds to (e_1, s^α) where $\alpha \in S$.
- To show that $X_1^u X_2^v Y^i T^q$ ($0 \le u < b-1$, $0 \le v < a$, $i \in \{0, 1\}$ and $q > 0$) corresponds to $(e_I t^{(q)}, s^\alpha)$ ($\alpha \in S$, $\alpha - m_2 \notin S$, $I = \emptyset$ or $I = \{1\}$, and $q > 0$), we have to prove that

  $$ua + vb - a(b-1) \notin S \Leftrightarrow \begin{cases} 0 \le u < b-1 \\ 0 \le v < a \end{cases},$$

  where $\alpha = ua + vb$, $u, v \ge 0$.
  Indeed, if $u \ge b-1$ or $v \ge a$, then $ua + vb - a(b-1) \in S$, which is a contradiction. For the other implication, suppose that we have the hypothesis on the right hand side, i.e., we can write $u = b-2-d$ and $v = a-1-e$ for some $d, e \ge 0$. If we have $ua + vb - a(b-1) \in S$, then $(b-2-d)a + (a-1-e)b - a(b-1) = F(S) - ad - be \in S$. This implies that $F(S) \in S$, which is a contradiction again. $\square$

3.5.3 The Hilbert series

We define the formal series as for the previous case and use the same decomposition for grading the Hochschild cohomology ring $\mathrm{HH}^*(A)$ in this case. Using a similar argument to Case I, we now compute the Hilbert series for $\mathrm{HH}^*(A)$ in Case II as follows:
(i) The elements of the form $(1, s^\alpha)$ (where $\alpha \in S$) contribute the series $\mathrm{H}_1 = \mathcal{H}_{k[s^a,s^b]}(x,y)$ as in Case I.
(ii) The elements of the form (e_1, s^α) (where $\alpha \in S$) have multidegree of $(1, -b+\alpha)$. This contributes a series $xy^{-b}\mathrm{H}_1$ into the final result.
(iii) Now we consider the elements $(e_I t^{(q)}, s^\alpha)$ (where $\alpha \in S$, $\alpha - m_2 \notin S$, $I = \emptyset$ or $I = \{1\}$, and $q > 0$). When $I = \emptyset$, the element $(t^{(q)}, s^\alpha)$ (where $\alpha \in S$ and $q > 0$) of multidegree $(2q, (-ab)q + \alpha)$ contribute the series

$$\frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot \mathrm{H}_1.$$

Similarly, when $I = \emptyset$, the element $(t^{(q)}, s^\alpha)$ (where $\alpha - m_2 \in S$ and $q > 0$) is equivalent to $(t^{(q)}, s^\gamma + m_2)$ (where $\gamma \in S$ and $q > 0$) by setting $\gamma = \alpha - m_2$. This element has multidegree $(2q, (-ab)q + \gamma)$, which contributes the series

$$\frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot y^{m_2} \cdot \mathrm{H}_1.$$

Hence, the series of the elements $(t^{(q)}, s^\alpha)$ (where $\alpha \in S$, $\alpha - m_2 \notin S$ and $q > 0$) is the subtraction of the two above series, which is

$$\frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot (1 - y^{m_2}) \cdot \mathrm{H}_1.$$

Analogously we get the series of the elements $(e_1 t^{(q)}, s^\alpha)$ (where $\alpha \in S$, $\alpha - m_2 \notin S$ and $q > 0$) as follows:

$$xy^{-b} \cdot \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot (1 - y^{m_2}) \cdot \mathrm{H}_1.$$

Hence, the series for (iii) is the sum of two sub-cases:

$$(1 + xy^{-b}) \cdot \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot (1 - y^{m_2}) \cdot \mathrm{H}_1.$$

The Hilbert series of $\mathrm{HH}^*(A)$ is obtained by taking the sum of three series (i), (ii) and (iii) above. Reduce this sum, we have:

$$\mathcal{H}_{\mathrm{HH}^*(A)}(x,y) = \left(1 + \frac{x^2y^{-ab}}{1 - x^2y^{-ab}} \cdot (1 - y^{m_2})\right) \cdot (1 + xy^{-b})\mathrm{H}_1.$$

3.5.4 Example

We consider the Hochschild cohomology of the algebra $k[s^2, s^3]$ where k is a finite field with characteristic $\mathrm{char}(k) = 2$, which is divisible by $a = 2$. By Theorem 3.18, we have the isomorphism:

$$\mathrm{HH}^*(A) \cong k[x_1, x_2, y_1, y_2, t]/I,$$

where the ideal I is generated by $x_1^3 - x_2^2, y^2 - t, x_1^2 t$.
In Appendix B, we present the Macaulay2 code in order to compute and check the Hilbert series of this example.

Chapter 4

The Hochschild homology rings of the square-free monomial complete intersections

4.1 Overview

In this last chapter, we will consider the Hochschild homology of the square-free monomial complete intersections. More specific, we examine the structure of the Hochschild homology of the algebra $k[x_1, x_2, \dots, x_n]/\langle x_1 x_2 \cdots x_n \rangle$, which is denoted by A. As doing before, we will again construct the Hochschild homology module by using the alternative resolution of Guccione et al. [24]. Next, we give a description of this module via smaller modules based on the features of the cycles. At the early stage, we will provide some conjectures on the multiplication and construct some illustrative examples to check the conjectures in some simple cases.

4.2 A construction of Hochschild homology module

Let us start with some notation in this chapter. For succinctness, we generally abbreviate $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ when mentioning monomials in $k[x_1, x_2, \dots, x_n]$. We will use the same notation for elements in A with the convention that $x_1 x_2 \cdots x_n = 0$.

Now we interpret the free resolution for A^e-module A given by Guccione et al. [24] for our case, which is exactly the resolution for the cohomology version, see Chapter 2.

$$\mathbf{F}: \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0,$$

or,

$$\cdots \xrightarrow{d_3} \bigoplus_{i<j} A^e e_i \wedge e_j \bigoplus A^e t \xrightarrow{d_2} \bigoplus_{j=1}^{n} A^e e_j \xrightarrow{d_1} A^e \longrightarrow A \longrightarrow 0,$$

where F_m is the finitely generated A^e-module with basis elements $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s} \cdot t^{(q)}$ such that $s + 2q = m$. Here we assign degree 1 to e_i and degree $2q$ to $t^{(q)}$. We abbreviate $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s}$ by e_I, where $I = \{i_1, i_2, \ldots, i_s\}$ in which $1 \leq i_1 < i_2 < \cdots < i_s \leq n$.
The following is computation of differentials d_m (shortly, d):

$$\begin{aligned}
d_s(e_{i_1 \cdots i_s}) &= \sum_{j=1}^{s} (-1)^{j-1}(1 \otimes x_{i_j} - x_{i_j} \otimes 1) e_{i_1 \cdots \widehat{i_j} \cdots i_s};\\
d_2(t) &= \sum_{j=1}^{n} x_1 \cdots x_{j-1} \otimes x_{j+1} \cdots x_n \cdot e_j;\\
d_{s+2q}(e_{i_1 \cdots i_s} t^{(q)}) &= d_s(e_{i_1 \cdots i_s}) t^{(q)} + d_2(t) \cdot e_{i_1 \cdots i_s} t^{(q-1)}, \text{ if } q \geq 1.
\end{aligned}$$

The module F_m can be seen as $A^e \otimes V_m$, where V_m is the k-space generated by the same basis elements of F_m. Applying the functor $(A \otimes_{A^e} -)$ to the truncation of the above resolution, together with the fact that $A \otimes_{A^e} A^e \cong A$ (see Proposition 2.14, Chapter 2, [42]) one gets the new complex of A^e-modules:

$$A \otimes \mathbf{V}: \cdots \xrightarrow{\delta_3} \bigoplus_{i<j} A \otimes k(e_i \wedge e_j) \bigoplus A \otimes kt \xrightarrow{\delta_2} \bigoplus_{j=1}^{n} A \otimes k e_j \xrightarrow{\delta_1}$$
$$\longrightarrow A \longrightarrow 0$$

with the corresponding differentials δ induced from d as follows:

$$\begin{aligned}
\delta(\mathbf{x}^\alpha \otimes e_I) &= 0;\\
\delta(\mathbf{x}^\alpha \otimes e_I t^{(q)}) &= \sum_{i \notin I} \operatorname{sgn}(i, I) \mathbf{x}^\alpha \cdot x_1 \cdots \widehat{x_i} \cdots x_n \otimes e_{I \cup \{i\}} t^{(q-1)}, \text{ if } q \geq 1.
\end{aligned}$$

We will write down here a brief proof of the above formula.

$$\begin{tikzcd}
\mathbf{F}: \cdots \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] \arrow[d, leftrightarrow, "\cong"] & F_1 \arrow[r, "d_1"] \arrow[d, leftrightarrow, "\cong"] & F_0 \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\
A^e \otimes \mathbf{V}: \cdots \arrow[r, "\bar{d}_3"] & A^e \otimes V_2 \arrow[r, "\bar{d}_2"] & A^e \otimes V_1 \arrow[r, "\bar{d}_1"] & A^e \otimes V_0 \arrow[r] & 0
\end{tikzcd}$$

The isomorphisms between the modules of $\mathbf{F}$ and $A^e \otimes \mathbf{V}$ have the nature of the following isomorphism:

$$\begin{aligned}
A^e &\cong A^e \otimes k, \\
a \otimes b &\mapsto (a \otimes b) \otimes 1, \\
m(a \otimes b) &\mapsfrom (a \otimes b) \otimes m,
\end{aligned}$$

where $a, b \in A$ and $m \in k$.

By combining these isomorphisms and the formula of d, we get the formula of $\bar{d}$:

$$\begin{aligned}
\bar{d}\left((1 \otimes 1) \otimes e_I\right) &= \sum_{j=1}^{s} (-1)^{j-1} (1 \otimes x_{i_j} - x_{i_j} \otimes 1) \otimes e_{I \setminus \{i_j\}}; \\
\bar{d}(t) &= \sum_{j=1}^{n} (x_1 \cdots x_{j-1} \otimes x_{j+1} \cdots x_n) \otimes e_j; \\
\bar{d}\left((1 \otimes 1) \otimes e_I t^{(q)}\right) &= \sum_{j=1}^{s} (-1)^{j-1} (1 \otimes x_{i_j} - x_{i_j} \otimes 1) \otimes e_{I \setminus \{i_j\}} t^{(q)} \\
&\quad + \sum_{j=1}^{n} (x_1 \cdots x_{j-1} \otimes x_{j+1} \cdots x_n) \otimes e_j \wedge e_I t^{(q-1)}, \text{ if } q \geq 1.
\end{aligned}$$

In the next step, we apply the functor $(A \otimes_{A^e} -)$ to the above complex:

$$\begin{tikzcd}
A \otimes_{A^e} A^e \otimes \mathbf{V}: \cdots \arrow[r, "\mathrm{id}_A \otimes \bar{d}_2"] & A \otimes_{A^e} A^e \otimes V_1 \arrow[r, "\mathrm{id}_A \otimes \bar{d}_1"] \arrow[d, leftrightarrow, "\cong"] & A \otimes_{A^e} A^e \otimes V_0 \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\
A \otimes \mathbf{V}: \cdots \arrow[r, "\delta_2"] & A \otimes V_1 \arrow[r, "\delta_1"] & A \otimes V_0 \arrow[r] & 0
\end{tikzcd}$$

Similarly, we have the isomorphism between $A \otimes_{A^e} A^e \otimes \mathbf{V}$ and $A \otimes \mathbf{V}$ as follows:

$$\begin{aligned}
A \otimes_{A^e} A^e &\cong A, \\
a \otimes (b \otimes c) &\mapsto abc, \\
a \otimes (1 \otimes 1) &\mapsfrom a,
\end{aligned}$$

where $a, b, c \in A$.

Now we take $\mathbf{x}^\alpha \otimes e_I$ in $A \otimes V_m$. Then $\mathbf{x}^\alpha \otimes e_I$ identifies with $(\mathbf{x}^\alpha \otimes (1 \otimes 1)) \otimes e_I$ via the isomorphism $A \cong A \otimes_{A^e} A^e$. So we get

$$\mathrm{id}_A \otimes \bar{d}\,((\mathbf{x}^\alpha \otimes (1 \otimes 1)) \otimes e_I) = \sum_{j=1}^{s} (-1)^{j-1} \mathbf{x}^\alpha \otimes (1 \otimes x_{i_j} - x_{i_j} \otimes 1) \otimes e_{I \smallsetminus \{i_j\}}.$$

And again, via the isomorphism $A \otimes_{A^e} A^e \cong A$ we have $\delta(\mathbf{x}^\alpha \otimes e_I) = 0$.

By the same argument and noticing that $e_i \wedge e_I = \mathrm{sgn}(i, I) e_{I \cup \{i\}}$, we obtain the remaining formula of δ. By the definition of Hochschild homology (recalled in Chapter 1), for any $n \geq 0$ the *n-th Hochschild homology* $\mathrm{HH}_n(A)$ of A is the n-th homology of the above complex

$$\mathrm{H}_n(A \otimes \mathbf{V}) = \frac{\mathrm{Ker}(\delta_n)}{\mathrm{Im}(\delta_{n+1})},$$

where δ_0 is taken to be the zero map. We call $\mathrm{HH}_*(A) = \bigoplus_{n \geq 0} \mathrm{HH}_n(A)$ the *Hochschild homology* of A and denote it $\mathrm{HH}_*(A)$. Now we shall give some more details of the structure of this $\mathbb{N}$-graded module.

We start with some immediate consequences of the elements in the kernel and the image of the differentials δ in the following remark.

Remark 4.1. For $q > 0$, we have $\delta(\mathbf{x}^\alpha \otimes e_I t^{(q)}) \neq 0$ if and only if there exists some i such that $i \notin I \cup \mathrm{supp}(\mathbf{x}^\alpha)$. This yields that:

(i) The basis element $\mathbf{x}^\alpha \otimes e_I t^{(q)}$ occurs in the kernel of δ if and only if $q = 0$ or ($q > 0$ and $I \cup \mathrm{supp}(\mathbf{x}^\alpha) = [n]$).

(ii) The basis element $\mathbf{x}^\alpha \otimes e_I t^{(q)}$ occurs as a component of some element in the image of δ if and only if there exists some i such that $\mathrm{supp}(\mathbf{x}^\alpha) = [n] \smallsetminus \{i\}$ and $i \in I$.

Let Γ be the set of all elements $\gamma = \mathbf{x}^\alpha \otimes e_I t^{(q)}$ such that γ is not any component in $\mathrm{Im}(\delta)$ and let M_γ be the sub-complex of $A \otimes \mathbf{V}$ constructed by γ based on the formula of δ as follows:

$$0 \longrightarrow k(\mathbf{x}^\alpha \otimes e_I t^{(q)}) \xrightarrow{\ \delta\ } \bigoplus_{i \notin I} \mathrm{sgn}(i, I) k \left(\mathbf{x}^\alpha \cdot x_1 \cdots \widehat{x}_i \cdots x_n \otimes e_{I \cup \{i\}} t^{(q-1)}\right) \longrightarrow 0.$$

There are two options for a such sub-complex.

Type 1: $0 \longrightarrow k \longrightarrow 0$

Type 2: $0 \longrightarrow k \longrightarrow k^m \longrightarrow 0$, $m > 0$.

Here we have identified the one dimensional k-space $k(\mathbf{x}^\alpha \otimes e_I t^{(q)})$ with k and the m dimensional $\bigoplus_{m \text{ folds}} k$ with k^m. Using Remark 4.1, we obtain a complete classification of the above sub-complexes based on the features of the triple $(\operatorname{supp}(\mathbf{x}^\alpha), I, q)$ in the left-most non-zero component $\mathbf{x}^\alpha \otimes e_I t^{(q)}$ of the sub-complexes.

Lemma 4.2. *(Classification of sub-complexes, necessary and sufficient condition)*
(i) The basis element has the corresponding sub-complex Type 1 if it occurs in $\operatorname{Ker}(\delta)$ *and it is not a component of any element in* $\operatorname{Im}(\delta)$. *Such the elements are* $\mathbf{x}^\alpha \otimes e_I t^{(q)}$ *that satisfy one of the following conditions:*

- $q = 0$ *and* $|\operatorname{supp}(\mathbf{x}^\alpha)| < n - 1$*; or*
- $q = 0$ *and there is some* $i \in [n]$ *such that* $\operatorname{supp}(\mathbf{x}^\alpha) = [n] \setminus \{i\}$ *and* $i \notin I$*; or*
- $q > 0$, $|\operatorname{supp}(\mathbf{x}^\alpha)| < n - 1$ *and* $I \cup \operatorname{supp}(\mathbf{x}^\alpha) = [n]$.

(ii) For the sub-complex Type 2, the basis element corresponding to the leftmost non-zero position of the sub-complex is the element $\mathbf{x}^\alpha \otimes e_I t^{(q)}$ *such that* $q > 0$ *and* $|[n] \setminus (I \cup \operatorname{supp}(\mathbf{x}^\alpha))| = m$.
(iii) If there are two elements $\mathbf{x}^\alpha \otimes e_I t^{(q)}$ *and* $\mathbf{x}^\beta \otimes e_J t^{(p)} \in \Gamma$ *such that their images under* δ *have some mutual non-zero component, then they are identical.*

Proof. The results in (i) and (ii) are obtained by observing Remark 4.1. For (iii), suppose that

$$E = \mathbf{x}^\alpha \cdot x_1 \cdots \widehat{x}_i \cdots x_n \otimes e_{I\cup\{i\}} t^{(q-1)} = \mathbf{x}^\beta \cdot x_1 \cdots \widehat{x}_j \cdots x_n \otimes e_{J\cup\{j\}} t^{(p-1)}$$

is some mutual non-zero component of the images of the two given elements under the action of δ. Since E is non-zero, we have $i \notin I \cup \operatorname{supp}(\mathbf{x}^\alpha)$ and $j \notin J \cup \operatorname{supp}(\mathbf{x}^\beta)$. If $i \neq j$, comparing $\operatorname{supp}(\mathbf{x}^\alpha \cdot x_1 \cdots \widehat{x}_i \cdots x_n)$ and $\operatorname{supp}(\mathbf{x}^\beta \cdot x_1 \cdots \widehat{x}_j \cdots x_n)$ gives us $i \in \operatorname{supp}(\mathbf{x}^\alpha)$ and $j \in \operatorname{supp}(\mathbf{x}^\beta)$, which is a contradiction. So $i = j$, $I = J$, $\operatorname{supp}(\mathbf{x}^\alpha) = \operatorname{supp}(\mathbf{x}^\beta)$ and $q = p$. The result follows. □

Theorem 4.3. *We have the isomorphism*

$$A \otimes \mathbf{V} \cong \bigoplus_{\gamma\in\Gamma} M_\gamma.$$

Proof. By the definition of M_γ and the fact (iii) in the above lemma, we get the inclusion $\bigoplus_{\gamma\in\Gamma} M_\gamma \subseteq A\otimes \mathbf{V}$. Now we show the inverse inclusion, i.e., for any non-zero element $E = \mathbf{x}^\alpha \otimes e_I t^{(q)}$, under the action of δ, it belongs to a complex M_γ for some $\gamma \in \Gamma$. If E is not a component in the image of δ, then E is in Γ and belongs to the complex indexed by itself M_E. If E is a component in $\mathrm{Im}(\delta)$, then by Remark 4.1 (ii) there is unique i such that $\mathrm{supp}(\mathbf{x}^\alpha) = [n] \smallsetminus \{i\}$ and $i \in I$. Thus, we can trace back the pre-image of E, $\gamma = \dfrac{\mathbf{x}^\alpha}{x_1\cdots\widehat{x_i}\cdots x_n} \otimes e_{I\smallsetminus\{i\}} t^{(q+1)}$, which is not in the kernel of δ by Remark 4.1. Then, E belongs to M_γ, where $\gamma \in \Gamma$. □

Corollary 4.4. *We have a description of the Hochschild homology module via the homology of the sub-complexes $\{M_\gamma\}_{\gamma\in\Gamma}$:*

$$\mathrm{HH}_i(A) \cong \bigoplus_{\gamma\in\Gamma} \mathrm{H}_i(M_\gamma).$$

Proof. An immediate consequence of Theorem 4.3. □

4.3 Conjecture on the multiplication of the Hochschild homology module

The definition of the multiplication of the Hochschild homology module has been recalled in Chapter 1. This multiplication is induced from the shuffle product. We can express the shuffle product via the reduced bar resolution as follows:

$$\overset{\mathrm{sh}}{\times} : \bar{B}_p \otimes \bar{B}_q \to \bar{B}_{p+q}$$
$$(a_1\otimes\cdots\otimes a_p)\otimes(a_{p+1}\otimes\cdots\otimes a_{p+q}) \mapsto \sum_{\sigma\in S_{p,q}} \epsilon(\sigma) a_{\sigma^{-1}(1)}\otimes\cdots\otimes a_{\sigma^{-1}(p+q)}$$

The corresponding map for our case, the resolution $\mathbf{F}$, is denoted by $*$:

$$* : F_* \otimes F_* \to F_*,$$

which we need to identify. And the relation between these maps is shown in the following diagram:

$$\begin{tikzcd} F_*\otimes F_* \arrow[r, "\iota\otimes\iota"] \arrow[d, "*"] & \bar{B}_*\otimes\bar{B}_* \arrow[d, "\overset{\mathrm{sh}}{\times}"] \\ F_* \arrow[r, "\iota"] & \bar{B}_* \end{tikzcd}$$

The chain map ι between the resolution $(\mathbf{F}, d)$ and the reduced bar complex $(\bar{\mathcal{B}}, b)$ can be inductively defined on the A^e-basis elements by the composite

$$\iota_i = s_i \circ \iota_{i-1} \circ d_i$$

and extended linearly for other elements, where s is a contracting homotopy of $\bar{\mathcal{B}}$, see Section 1.3.1 for the formula of s.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d_2"] \arrow[d, "\iota_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "\iota_1"] & F_0 \arrow[r] \arrow[d, "\iota_0"] & 0 \\
\cdots \arrow[r, shift right] & \bar{B}_2 \arrow[l, shift right] \arrow[r, shift right, "b_2"'] & \bar{B}_1 \arrow[l, shift right, "s_2"'] \arrow[r, shift right, "b_1"'] & \bar{B}_0 \arrow[l, shift right, "s_1"'] \arrow[r] & 0
\end{tikzcd}$$

Conjecture 4.5. *We have that*

$$\iota(e_I t^{(p)}) * \iota(e_J t^{(q)}) = \begin{cases} \operatorname{sgn}(I, J)\dfrac{(p+q)!}{p!q!}\iota(e_{I\cup J} t^{(p+q)}) & \textit{if } I \cap J = \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

If this is true, we suggest to define a multiplication on $A \otimes \mathbf{V}$ by

$$\mathbf{x}^\alpha \otimes e_I t^{(p)} * \mathbf{x}^\beta \otimes e_J t^{(q)} = \begin{cases} \operatorname{sgn}(I, J)\dfrac{(p+q)!}{p!q!}\mathbf{x}^{\alpha+\beta} \otimes e_{I\cup J} t^{(p+q)} & \text{if } I \cap J = \emptyset, \\ 0 & \text{if } I \cap J \neq \emptyset. \end{cases}$$

We present some explicit computations on a small example $A = k[x, y]/\langle xy\rangle$. The A^e-resolution of A is interpreted as follows:

$$\mathbf{F}: \qquad \cdots \longrightarrow A^e t \oplus A^e e_1 e_2 \xrightarrow{d_2} A^e e_1 \oplus A^e e_2 \xrightarrow{d_1} A^e \longrightarrow 0$$

with the differential d given by

- $d_1(e_1) = x \otimes 1 - 1 \otimes x$, $d_1(e_2) = y \otimes 1 - 1 \otimes y$,
- $d_2(t) = 1 \otimes y \cdot e_1 + x \otimes 1 \cdot e_2$ and $d_{i+j}(uv) = d(u)v + (-1)^{|u|} u d(v)$ for $u \in F_i$, $v \in F_j$.

We compute the chain map ι at some first degrees as follows. Also, we will check the conjecture of multiplication at these degrees. To save the space, we often use the notation $a_0[a_1|\cdots|a_n]a_{n+1}$ in place of $a_0 \otimes a_1 \otimes \cdots a_n \otimes a_{n+1}$ for the elements in $\bar{\mathcal{B}}$.

Computations of ι at degree 0. For any $a \otimes b \in F_0 = A^e$, we define

$$\iota_0(a \otimes b) = a[\ ]b.$$

Computations of ι at degree 1. We have that:

$$d(e_1) = x \otimes 1 - 1 \otimes x;$$

$$\iota(x \otimes 1 - 1 \otimes x) = x[\ ]1 - 1[\ ]x; \text{ and}$$

$$s(x[\ ]1 - 1[\ ]x) = 1[x]1.$$

So we define $\iota_1(e_1) = 1[x]1$ and similarly $\iota_1(e_2) = 1[y]1$.

Computations of ι at degree 2. We have that:

$$d(e_1e_2) = (x \otimes 1 - 1 \otimes x)e_2 - (y \otimes 1 - 1 \otimes y)e_1;$$

$$\begin{aligned}\iota_1\left(d(e_1e_2)\right) &= (x \otimes 1 - 1 \otimes x)1[y]1 - (y \otimes 1 - 1 \otimes y)1[x]1\\ &= x[y]1 - 1[y]x - y[x]1 + 1[x]y;\end{aligned}$$

$$\iota_2(e_1e_2) := s\left(\iota_1(d(e_1e_2))\right) = 1[x|y]1 - 1[y|x]1.$$

By a similar computation, we have that $\iota_2(t) = 1[x|y]1$.
Now we can check that: $\iota_1(e_1) * \iota_1(e_2) = 1[x]1 * 1[y]1 = 1[x|y]1 - 1[y|x]1 = \iota_2(e_1e_2)$.

Computations of ι at degree 3. We can check that $\iota(e_1t) = 1[x|y|x]1$ and $\iota(e_2t) = 1[y|x|y]1$ and the conjecture holds for this degree.

Discussions. The conjecture on the multiplication of the Hochschild homology is true as long as we have checked in some lower degrees. In general, we can prove that the conjecture holds for the three first degrees for any algebra $k[x_1, \ldots, x_n]/\langle x_1 \cdots x_n\rangle$ with $n \geq 2$. For the future work, we need to check that the conjecture is true at all degrees or we have to modify the formula of the multiplication if there are some unwanted examples on the conjecture.

Conclusion

The aim of this thesis is to study the ring structure of the Hochschild cohomology of two families of complete intersections: the square-free monomial compete intersections and the numerical semigroup algebras of embedding dimension two. The following contributions are noted.

1. We have presented a concrete method to describe the ring structure of the Hochschild cohomology of the family of the square-free monomial compete intersections in terms of generators and relations. In particular, we worked out an explicit formula for the multiplication of the Hochschild cohomology module. Then we provided a full and detailed description of the generators and relations of the Hochschild cohomology ring. Besides that, we suggested a decomposition of the Hochschild cohomology ring and computed its Hilbert series with respect to this decomposition.

2. We have also obtained the corresponding results for the family of the numerical semigroup algebras of embedding dimension two including the theorems on descriptions of the rings structure of the Hochschild cohomology rings in terms of generators and relations; and the computations of the Hilbert series with respect to the decomposition we provided.

3. We have worked out on the module structure of the Hochschild homology version of the family of the square-free monomial complete intersections. We gave a conjecture on the multiplication and constructed some illustrative examples to check that the conjecture makes sense.

4. We provided the details of the author's Macaulay2 code in order to compute and check back the Hilbert series in examples.

We now mention some open questions raised by this thesis, and suggest some possible directions for further research.

1. As the Hochschild cohomology of algebras has the structure of an associative algebra and a Lie algebra, naturally we would like to consider the structure of the Gerstenhaber bracket of the Hochschild cohomology of these algebras.

2. We would like to have a full description of the ring structure of the Hochschild homology of the family of the square-free monomial complete intersections in terms of generators and relations. For this, we need to show that the conjecture in Chapter 4 is true or we have to modify the multiplication to get a right formula. The other family of algebras should be proceeded analogously, i.e., the homology version for the family of numerical semigroup algebras of embedding dimension two.

3. We may ask about the Hochschild (co)homology of the family of numerical semigroup algebras of embedding dimension three or more. We do not have an answer for these cases so far because our method only works for complete intersections while in most cases the numerical semigroup algebras of embedding dimension three or more are not the case of complete intersections.

4. The resolution of Guccione et al. which we used to construct the Hochschild cohomology only works for complete intersections so far. To deal with the cases of non complete intersections, we need to find out some alternative resolution which is fruitful in computing. The first case we may think of are the almost complete intersections, that is, the algebras of the form $R/[f_1, f_2, \ldots, f_r, f_{r+1}]$ where the sequence $f_1, f_2, \ldots, f_r, f_{r+1}$ is not regular, but the sequence f_1, f_2, $\ldots$, f_r is regular.

5. We can consider to use our method to investigate the Hochschild cohomology of some other families of complete intersections, for example the family of the parity binomial edge ideals. We can see that the concrete approach we have gone through depends significantly on the features of the Hochschild complex. This means that it might not

be feasible to generalize our method for other algebras analogously. However, there may be some potential interest in the structure of the Hochschild cohomology of other complete intersections.

6. The motivation of these problems theoretically comes from the beauty of the structure of the Hochschild cohomology and the internal needs of the Hochschild theory. The author wishes to find some real-world models or some constructions in applied maths to see how these structures work.

Appendix: Macaulay2 code

Appendix A

We present in this section the Macaulay2 code to compute the Hilbert series of the Hochschild cohomology of the algebra $A = k[x_1, x_2, \ldots, x_n]/\langle x_1 x_2 \cdots x_n\rangle$ for the case $n = 2$ where k is a field with characteristic 2; and for the case $n = 3$ where k is a field with characteristic 101.

Example 1

$n = 2$: By Theorem 2.16 we have that

$$\mathrm{HH}^*(A) \cong k[x_1, x_2, y_1, y_2, z]/I,$$

where the ideal I is generated by x_1x_2, x_1y_2, y_1x_2, y_1y_2, y_1^2, y_2^2, x_1z, x_2z, $(y_1 + y_2)z$.

We write down here the multidegrees of the variables based on the decomposition in Section 2.7, Chapter 2:

Variable	Multidegree
x_1	$(0, 1, 0)$
x_2	$(0, 0, 1)$
y_1	$(1, 0, 0)$
y_2	$(1, 0, 0)$
z	$(2, -1, -1)$

Macaulay2 code is shown in the next pages where the first part is the code to compute the Hilbert series and the second part is the code to check that our formula of Hilbert series in Theorem 2.19 is computed correctly.

Note: Due to the limited space, some of the outputs (o-) will be omitted.

In practice, the full outputs will show up when using the input commands (i-).

In the below Macaulay2 session, we are going to do the following:

i1: define a polynomial ring over a field of characteristic 2

i2: define the ideal I by giving generators of I

i3: compute the Gröbner basis of the ideal I based on the ordering given in i1

i4: compute the Hilbert series s of the algebra $R[x_1, x_2, y_1, y_2, z]/I$

i5: reduce the Hilbert series s

```
----------------------------------------------- Macaulay2 code -----------------------------------------------
i1 : R=ZZ/2[x1,x2,y1,y2,z,Degrees=>{{0,1,0},{0,0,1},{1,0,0},{1,0,0},{2,-1,-1}}]

o1 = R

o1 : PolynomialRing

i2 : I=ideal(x1*x2,x1*y2,y1*x2,y1*y2,y1^2,y2^2,x1*z,x2*z,(y1+y2)*z);

o2 : Ideal of R

i3 : gens gb I

o3 = | x2z x1z y2^2 x1y2 x1x2 y1z+y2z y1y2 x2y1 y1^2 |

             1       9
o3 : Matrix R  <--- R


i4 : s = hilbertSeries I

o4 : Expression of class Divide
i5 : reduceHilbert s
                2 -1    2 -1           3 -1 -1                  2    3
     1 + 2T  - T T   - T T   - T T  - T T  T   - T T  - T T  + 2T  + T
```

```
           0    0 2     0 1     1 2    0 1  2     0 1    0 2     0    0
o5 = -----------------------------------------------------------------
                                                2 -1 -1
                        (1 - T )(1 - T )(1 - T T  T  )
                              2       1       0 1  2


----------------------------------------------------- End -----------------------------------------------------
```

```
-------------------------------------------------- Macaulay2 code ------------------------------------------------
i1 : R=QQ[x,y,z];
i2 : g=(1+2x-(x^2)*(1/z)-(x^2)*(1/y)-y*z-(x^3)*(1/y)*(1/z)-x*y-x*z+2*x^2+x^3)/((1-z)*(1-y)*(1-(x^2)*(1/y)*(1/z)))

        3        2         2         2    2 2    3    2     2
     - x y*z - 2x y*z + x*y z + x*y*z  + y z  + x  + x y + x z - 2x*y*z - y*z
o2 = ------------------------------------------------------------------------
                 2       2 2    2     2     2       2    2
                x y*z - y z  - x y - x z + y z + y*z  + x  - y*z

o2 : frac(R)

i3 : h=(((x+1)^3)*y*z-(x+y)*(x+z)*(x+y*z))/((y*z-x^2)*(1-y)*(1-z))

        3        2         2         2    2 2    3    2     2
     - x y*z - 2x y*z + x*y z + x*y*z  + y z  + x  + x y + x z - 2x*y*z - y*z
o3 = ------------------------------------------------------------------------
                 2       2 2    2     2     2       2    2
                x y*z - y z  - x y - x z + y z + y*z  + x  - y*z

o3 : frac(R)

i4 : g==h

o4 = true
```

```
------------------------------------------------------------ End ------------------------------------------------------------
```

Comments:

g: the Hilbert computed by Macaulay2

h: the Hilbert series obtained from formula in Theorem 2.19

g==h: check that whether they are equal or not

Example 2

$n = 3$: We have

$$\mathrm{HH}^*(A) \cong k[x_1, x_2, x_3, y_1, y_2, y_3, z]/I,$$

where the ideal I is generated by

$$\begin{gathered} x_1x_2x_3,\ x_1x_2y_3,\ x_1y_2x_3,\ y_1x_2x_3, \\ y_1y_2x_3,\ y_1x_2y_3,\ x_1y_2y_3,\ y_1y_2y_3, \\ y_1^2,\ y_2^2,\ y_3^2, \\ x_1x_2z,\ x_1x_3z,\ x_2x_3z, \\ (y_1+y_2)x_3z,\ (y_1+y_3)x_2z,\ (y_2+y_3)x_1z, \\ (y_1y_2+y_2y_3+y_1y_3)z. \end{gathered}$$

The multidegrees of the variables are in the following table:

Variable	Multidegree
x_1	$(0, 1, 0, 0)$
x_2	$(0, 0, 1, 0)$
x_3	$(0, 0, 0, 1)$
y_1	$(1, 0, 0, 0)$
y_2	$(1, 0, 0, 0)$
y_3	$(1, 0, 0, 0)$
z	$(2, -1, -1, -1)$

Macaulay2 code is shown in the next pages.

```
------------------------------------------------- Macaulay2 code -------------------------------------------------
i1 : R = ZZ/101[x1,x2,x3,y1,y2,y3,z,Degrees=>{{0,1,0,0},{0,0,1,0},{0,0,0,1},{1,0,0,0},{1,0,0,0},{1,0,0,0},{2,-1,-1,-1}}]

o1 = R

o1 : PolynomialRing

i2 : I = ideal(x1*x2*x3,x1*x2*y3,x1*y2*x3,y1*x2*x3,y1*y2*x3,y1*x2*y3,
x1*y2*y3,y1*y2*y3,y1^2,y2^2,y3^2,x1*x2*z,x1*x3*z,x2*x3*z,(y1+y2)*x3*z,(y1+y3)*x2*z,(y2+y3)*x1*z,(y1*y2+y2*y3+y1*y3)*z);

o2 : Ideal of R

i3 : gens gb I

o3 = | x2x3z x1x3z x1x2z x1x2x3 x1y2z+x1y3z x3y1z+x3y2z x2y1z+x2y3z y3^2 x1x2y3 y2^2 x1x3y2 y1^2 x2x3y1
 y1y2z+y1y3z+y2y3z x1y2y3 x2y1y3 x3y1y2 y1y2y3 |

             1       18
o3 : Matrix R  <--- R

i4 : s = hilbertSeries I
o4 : Expression of class Divide

i5 : reduceHilbert s
o5 : Expression of class Divide
```

```
               2 -1    2 -1    2 -1             3 -1 -1    3 -1 -1    3 -1 -1     2                             4 -1 -1 -1    2      2      2       3    4
     1 + 3T  - T T   - T T   - T T   - T T T  - T T  T   - T T  T   - T T  T   + 6T  - T T T  - T T T  - T T T  - T T  T  T   - T T  - T T  - T T  + 3T  + T
         0    0 3     0 2     0 1     1 2 3    0 2  3     0 1  3     0 1  2      0    0 1 2    0 1 3    0 2 3    0 1  2  3     0 1    0 2    0 3     0    0
o5 = -----------------------------------------------------------------------------------------------------------------------------------------------------------
                                                                             2 -1 -1 -1
                                                        (1 - T )(1 - T )(1 - T )(1 - T T  T  T  )
                                                              3      2      1      0 1  2  3
```

restart

i1 : R=QQ[x,y,z,t]

o1 = R

o1 : PolynomialRing

i2 : g=(1+3*x-(x^2)*(1/t)-(x^2)*(1/z)-(x^2)*(1/y)-y*z*t-(x^3)*(1/z)*(1/t)-(x^3)*(1/y)*(1/t)-(x^3)*(1/y)*(1/z)+6*x^2

-x*y*z-x*y*t-x*z*t-(x^4)*(1/y)*(1/z)*(1/t)-(x^2)*y-(x^2)*z-(x^2)*t+3*(x^3)+x^4)/((1-y)*(1-z)*(1-t)*(1-(x^2)*(1/y)

(1/z)(1/t)))

o2: Expression of class Divide

o2 : frac(R)

i3 : h=((x+1)^4*y*z*t-(x+y)*(x+z)*(x+t)*(x+y*z*t))/((y*z*t-x^2)*(1-y)*(1-z)*(1-t))

```
o3: Expression of class Divide
o3 : frac(R)

i4 : g==h

o4 = true

------------------------------------------------------ End ------------------------------------------------------
```

Appendix B

Example 1

We present the Macaulay2 code to calculate the Hilbert series of the Hochschild cohomology of the algebra $k[s^2, s^3]$ where k is the prime field of characteristic 101.
Then we have $a = 2$, $b = 3$, $m_1 = 3$ and $m_2 = 4$. By Theorem 3.14, we get that

$$\mathrm{HH}^*(A) \cong k[x_1, x_2, y_1, y_2, t]/I$$

where the ideal I is generated by

$$x_1^3 - x_2^2,$$

$$x_1^2 t, x_2 t,$$

$$y_2 t, y_1^2, y_2^2, y_1 y_2,$$

$$x_1 y_2 - x_2 y_1, x_2 y_2 - x_1^2 y_1.$$

The multidegrees of the variables are shown in the following:

Variable	Multidegree
x_1	$(0, 2)$
x_2	$(0, 3)$
y_1	$(1, 0)$
y_2	$(1, 1)$
t	$(2, -6)$

Macaulay2 code is shown in the next pages.

```
------------------------------------------------ Macaulay2 code ------------------------------------------------
i1 : R = ZZ/101[t,y2,y1,x2,x1,Degrees=>{{2,-6},{1,1},{1,0},{0,3},{0,2}},MonomialOrder => Lex]

o1 = R

o1 : PolynomialRing

i2 : I = ideal(x1^3-x2^2,(x1^2)*t,x2*t,y2*t,y1^2,y2^2,y1*y2,x1*y2-x2*y1,x2*y2-(x1^2)*y1)

                2     3      2                2    2                                     2
o2 = ideal (- x2  + x1 , t*x1 , t*x2, t*y2, y1 , y2 , y2*y1, y2*x1 - y1*x2, y2*x2 - y1*x1 )

o2 : Ideal of R

i3 : gens gb I

o3 = | x2^2-x1^3 y1^2 y2x1-y1x2 y2x2-y1x1^2 y2y1 y2^2 tx1^2 tx2 ty2 |

             1       9
o3 : Matrix R  <--- R

i4 : s = hilbertSeries I

o4 : Expression of class Divide
```

```
i5 : reduceHilbert s

          3         2 -3           2 -2    3 -5    3 -2
     1 + T  + T  - T T   + T T  - T T   - T T   - T T
          1    0    0 1     0 1    0 1     0 1     0 1
o5 = --------------------------------------------------
                           2       2 -6
                     (1 - T )(1 - T T  )
                           1       0 1

o5 : Expression of class Divide
------------------------------------------------------ End ------------------------------------------------------
```

```
------------------------------------------------ Macaulay2 code ------------------------------------------------
i1 : R=QQ[x,y]

o1 = R

o1 : PolynomialRing

i2 : h1 = (1-y^6)/((1-y^2)*(1-y^3))

        2
     - y  + y - 1
o2 = ------------
         y - 1

o2 : frac(R)

i3 : d = ((x^2)*(1/(y^6)))/(1-(x^2)*(1/(y^6)))

         2
        x
o3 = -------
      6    2
     y  - x

o3 : frac(R)
```

```
i4 : h = (1-y^6)/((1-y^2)*(1-y^3))+d*h1*(1-y^3-y^4+y^6)+d*y^7+x*y+x*h1+x*d*h1-x*y*d*(h1-1)

        8      6    7    3 3    6    3 2    2 3    3
     - y  - x*y  + y  + x y  - y  - x y  + x y  + x y
o4 = ------------------------------------------------
                     7    6    2     2
                    y  - y  - x y + x

o4 : frac(R)

i5 :
     g = (1+y^3+x-(x^2)*(1/(y^3))+x*y-(x^2)*(1/(y^2))-(x^3)*(1/(y^5))-(x^3)*(1/(y^2)))/((1-y^2)*(1-(x^2)*(1/(y^6))))

        8      6    7    3 3    6    3 2    2 3    3
     - y  - x*y  + y  + x y  - y  - x y  + x y  + x y
o5 = ------------------------------------------------
                     7    6    2     2
                    y  - y  - x y + x

o5 : frac(R)

i6 : g==h

o6 = true
```

-- End --

Example 2

We present the Macaulay2 code to calculate the Hilbert series of the Hochschild cohomology of the algebra $k[s^2, s^3]$ where k is the prime field of characteristic $\mathrm{char}(k) = 2$.
We have $a = 2$, $b = 3$, $m_1 = 3$ and $m_2 = 4$. In this case, we have that $\mathrm{char}(k) \mid a$. Then by Theorem 3.18, we get that

$$\mathrm{HH}^*(A) \cong k[x_1, x_2, y_1, y_2, t]/I$$

where the ideal I is generated by $x_1^3 - x_2^2, y^2 - t, x_1^2 t$. The multidegrees of the variables are shown in the following:

Variable	Multidegree
x_1	$(0, 2)$
x_2	$(0, 3)$
y	$(1, 2)$
t	$(2, -1)$

Macaulay2 code is shown in the next pages.

```
------------------------------------------------ Macaulay2 code ------------------------------------------------
i1 : R=ZZ/2[y,x1,x2,t, Degrees=>{{1,2},{0,2},{0,3},{2,-1}}]

o1 = R

o1 : PolynomialRing

i2 : I = ideal(x1^3-x2^2,y^2-t,(x1^2)*t)

              3     2   2        2
o2 = ideal (x1  + x2 , y  + t, x1 t)

o2 : Ideal of R

i3 : gens gb I

o3 = | x1^2t x1^3+x2^2 y2+t x2^2t |

             1       4
o3 : Matrix R  <--- R

i4 : s =  hilbertSeries I

          2 3    2 4    6    4 7    2 9    2 10    4 13
     1 - T T  - T T  - T  + T T  + T T  + T T   - T T
```

```
          0 1    0 1    1    0 1    0 1    0 1    0 1
o4 = -------------------------------------------------
                  3         2       2       2 -1
            (1 - T )(1 - T T )(1 - T )(1 - T T  )
                  1       0 1       1       0 1

o4 : Expression of class Divide

i5 : reduceHilbert s

            2    3    2 3      5    3 5    2 6    3 8
     1 + T T  + T  - T T  + T T  - T T  - T T  - T T
          0 1    1    0 1    0 1    0 1    0 1    0 1
o5 = ------------------------------------------------
                          2       2 -1
                    (1 - T )(1 - T T  )
                          1       0 1
------------------------------------------------------ End ------------------------------------------------------
```

```
------------------------------------------------ Macaulay2 code -----------------------------------------------
i1 : R=QQ[x,y];

i2 : h1 = ((1+x*(y^2))*(1-y^6))/((1-y^2)*(1-y^3))

          4      3      2    2
     - x*y  + x*y  - x*y  - y  + y - 1
o2 = ---------------------------------
                  y - 1

o2 : frac(R)

i3 : h2 = (((x^2)*(1/y))*(1-y^4))/(1-(x^2)*(1/y))

      2 4    2
     x y  - x
o3 = ---------
        2
       x  - y

o3 : frac(R)

i4 : h = h1*(1+h2)

        3 8    3 7    3 6    2 6    2 5    2 4     5      4      3    3    2
```

```
     - x y  + x y  - x y  - x y  + x y  - x y  + x*y  - x*y  + x*y  + y  - y  + y
o4 = ----------------------------------------------------------------------------
                                    2     2    2
                                   x y - x  - y  + y
g = (1+x*(y^2)+y^3-(x^2)*(y^3)+x*(y^5)-(x^3)*(y^5)-(x^2)*(y^6)-(x^3)*(y^8))/((1-y^2)*(1-(x^2)*(1/y)))

        3 8    3 7    3 6    2 6    2 5    2 4      5      4      3    3    2
     - x y  + x y  - x y  - x y  + x y  - x y  + x*y  - x*y  + x*y  + y  - y  + y
o6 = ----------------------------------------------------------------------------
                                    2     2    2
                                   x y - x  - y  + y

o6 : frac(R)

i7 : g==h

o7 = true
------------------------------------------------------ End ------------------------------------------------------
```

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