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Zhu Reduction Theory for Vertex Operator Algebras on Riemann Surfaces

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics
by

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Declaration

I declare that the work presented in this thesis is my own, and has not been previously submitted for award at another degree granting institution.

Michael Welby.

Abstract

In this thesis we first develop a recursive relation for n-point functions for Vertex Operator Super Algebras (VOSAs) on a genus two Riemann surface constructed by sewing two tori. This relation is used to develop formal differential equations for n-point functions on a genus two surface, as well as for differential forms on this surface. We demonstrate the applications of this results for a well-known example of VOSA and compare them to existing results in the literature. In the second part, we develop a more general version of this identity for a Vertex Operator Algebra (VOA) on a general genus Riemann surface, using the Schottky uniformisation of a genus g Riemann surface; we then develop some geometric theory for the results that arise. We also apply this results to well-known examples of VOAs to obtain general genus identities for objects such as differential forms on a Riemann surface.

Introduction

A Vertex Operator (Super) Algebra (usually abbreviated as VO(S)A or (super-)VOA) is an algebraic structure with a strong connection to many areas of mathematics, such as conformal field theory, Lie algebras, analytic number theory and complex geometry. These objects first arose in relation to the monstrous moonshine conjectures [Bor] and have enjoyed a rich theory since.

The central idea of this thesis is that of Zhu reduction [Zhu] for a genus two (later general genus) (super-)VOA, which gives recursive relations for n-point functions (or differentials) for (super-)VOAs on a Riemann surface, relating them to $(n-1)$-point functions. These recursive relations can then be used to derive differential equations involving well-known objects from number theory. These recursive relations are well-understood at genus zero and one and recent work has greatly extended understanding of the relevant objects at genus two, along with their geometric implications.

This thesis builds on previous work by T. Gilroy, G. Mason, M. P. Tuite and A. Zuevsky, who have further developed Zhu reduction theory for (super-)VOAs and their modules at genus one, and extended the work to genus two by applying a technique of A. Yamada [Y], which involves the sewing together of tori to construct a genus two surface. The latter part of the thesis uses the Schottky approach to construct a Riemann surface of arbitrary genus.

This thesis is divided into three parts:

Part I comprises the first three sections, in which we introduce the relevant background concepts.

Section 1 introduces vertex operator super algebras, their modules and some classical examples such as the Heisenberg VOA and the Free Fermion VOSA.

Section 2 covers classical objects in number theory such as modular forms and elliptic functions, along with some generalised versions of those objects with enhanced symmetries. Differential forms on Riemann surfaces are also introduced, along with the construction of genus two Riemann surfaces from sewn tori.

Section 3 discusses work done to date on the theory of Zhu reduction. Zhu reduction for a genus one VOSA is introduced, along with results such as differential equations that can be derived from these relations.

In **Part II**, we develop a genus two generalisation of the Zhu reduction formula for genus one VOSAs derived in [MTZ]. This can also be understood as a VOSA generalisation of the results of [GT1].

In **Section 4** we develop the aforementioned generalisation, building on methods used in [GT1]. The idea combines the sewing of two tori to construct a genus two Riemann surface with genus one Zhu reduction results for VOSAs from [MTZ]. The formula is first developed for quasiprimary states of the VOA, and then extended to descendants. The method involves the relating of certain linear-algebraic objects encoding information from the sewn tori to give a higher genus analogue of the classical Weierstrass elliptic functions. We also compare this new result to the special case of that derived in [GT1]. The result also displays the preservation of counting of holomorphic N-differentials on the genus two surface as suggested by the Riemann-Roch theorem.

In **Section 5** we apply this new formula to the classical example of the Free Fermion VOSA. We compare to known results obtained via combinatorial methods in [TZ1] involving the Szegő kernel.

Part III discusses the derivation of a Zhu reduction formula for a general genus VOA, building on the theory at genus zero.

In **Section 6** we discuss the construction of a general genus Riemann surface using the Schottky uniformisation, which involves the sewing of handles to the Riemann sphere. We then discuss n-point functions and a general Zhu reduction at genus zero, extending previous ideas of [Zhu] to include more general coefficients in the final genus zero recursion relation.

Section 7 introduces the idea of a partition function (and examines its $\mathrm{SL}(2,\mathbb{C})$ symmetry) and an n-point function (or differential) at arbitrary genus using the Schottky construction. We use the genus zero Zhu reduction formula of Section 6 along with the Schottky technique to develop a genus g recursive relation for quasiprimary states. We again use infinite (block) matrices and vectors to achieve this. We obtain a generalised Zhu coefficient with an interpretation on a genus g surface, along with a spanning set for holomorphic N-differentials on the Riemann surface. These are examined in more detail in the next section.

Section 8 deals with the analysis of the aforementioned Zhu coefficients $\Psi_N(x,y)$ and holomorphic N-differentials $\Theta_a(x,\ell)$. We find that $\Psi_N(x,y)$ can be written as a Poincaré sum over the genus g Schottky group, giving a natural geometric interpretation of the Zhu reduction coefficients. To obtain a convergent Poincaré sum, we invoke ideas of Bers [Be] involving a certain $(N,1-N)$-quasidifferential on the surface. We analyse further to obtain a spanning set of holomorphic N-differentials on the genus g Riemann surface, and obtain a relationship between Ψ_N and Θ_a.

Section 9 examines applications of general genus Zhu recursion, i.e. the derivation of genus g differential equations. In particular, we derive general genus Ward identities, identities for the genus g bidifferential of the second kind and the genus g projective connection, and show the relationship between the partition function for a genus g lattice VOA and that of the genus g Heisenberg VOA. We also derive a new $\mathrm{SL}(2,\mathbb{C})$-invariant differential operator which acts with respect to the surface moduli.

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Part I
Preliminaries

1 Vertex Operator (Super) Algebras on Genus One and Two Riemann Surfaces

In this section we introduce the concept of a Vertex Operator Super Algebra (VOSA) and related structures, along with their interpretation on Riemann surfaces. We also introduce some classical examples that will feature later in applications of the results of this thesis. This review of the topic largely follows the article [MT4]; other treatments can be found in [Bor], [FHL], [FLM], [K], [LeLi] and [MN].

1.1 Local fields

Let V be a superspace, i.e. V is a vector space where each vector $v \in V$ has a *parity* $p(v) \in \mathbb{Z}_2$. For a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we can define the endomorphism space $\mathrm{End}(V)$ of all linear mappings from V to V. We define a linear map for all $a \in V$

$$a(z) : V \to \mathrm{End}(V)[[z, z^{-1}]],$$

with

$$a(z)v = \sum_{n \in \mathbb{Z}} (a(n)v) z^{-n-1} \in V[[z, z^{-1}]],$$

for $v \in V$, where $\mathrm{End}(V)[[z, z^{-1}]]$ is the ring of formal Laurent series with coefficients in $\mathrm{End}(V)$. The individual coefficients in the series $a(z)$ are known as *modes*, while z is a formal parameter. We often refer to elements of V as *states* and to V as the *state space*. Throughout the thesis, when referring to (super-)VOA elements, we will use the terms "vector" and "state" interchangeably. We will be interested in a particular subset of these series known as *local fields*.

The formal series $a(z)$ is a field if it satisfies the *truncation property*, that is, for all $v \in V$, there exists $N \in \mathbb{Z}$ such that $a(n)v = 0$ for all $n > N$. Then we define the space

$$\mathfrak{F}(V) = \{a(z) : a(z) \text{ is a field}\},$$

These operators do not generally commute, and we can define the *commutator* of two fields in distinct indeterminates x, y

$$\begin{aligned} [a(x), b(y)] &= \left[\sum_{m \in \mathbb{Z}} a(m) x^{-m-1}, \sum_{n \in \mathbb{Z}} b(n) y^{-n-1} \right] \\ &= \sum_{m,n \in \mathbb{Z}} [a(m), b(n)] x^{-m-1} y^{-n-1}, \end{aligned}$$

where the Lie bracket $[\cdot, \cdot]$ is defined with respect to the superspace grading

$$[a(m), b(n)] = a(m)b(n) - (-1)^{p(a)p(b)} b(n)a(m). \tag{1}$$

For $p(a) = \bar{0}$ or $p(b) = \bar{0}$, we obtain the usual ring-theoretic commutator. We say that two fields $a(z)$, $b(z)$ are *mutually local* if there exists a positive integer N such that

$$(x - y)^N [a(x), b(y)] = 0.$$

We denote this by $a(z) \sim_N b(z)$ or $a(z) \sim b(z)$ if N can be inferred from context.

1.2 Vertex (super) algebras

We will now review some aspects of vertex algebra theory. A vertex super algebra is a quadruple $(V, Y(\cdot,\cdot), \mathbb{1}, T)$ consisting of the following

- A vector space V where every $v \in V$ has a parity $p(v) \in \{\overline{0}, \overline{1}\}$. The sets of vectors with equal parity form vector spaces. V then enjoys a parity space decomposition $V = V_{\overline{0}} \oplus V_{\overline{1}}$, where
$$V_{\overline{a}} = \{u \in V : p(u) = \overline{a}\}.$$
- A linear map $Y(\cdot, z) : V \to \mathrm{End}(V)[[z, z^{-1}]]$ defined for all $u \in V$ by
$$Y(u, z) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1}.$$
$Y(u, z)$ is known as a *vertex operator*. The modes $u(n)$ map between parity spaces as follows
$$u(n) : V_{\overline{a}} \to V_{\overline{a}+p(u)}, \tag{2}$$
for $p(u), \overline{a} \in \mathbb{Z}_2$. That is, even modes preserve the parity subspace while odd modes send $V_{\overline{a}}$ to $V_{\overline{a}+\overline{1}}$ (the addition is performed modulo 2);
- A non-zero vacuum vector $\mathbb{1} \in V_{\overline{0}}$.

These data satisfy the following axioms for every vector $u, v \in V$

- Each vertex operator must satisfy
$$Y(u, z) \sim_N Y(v, z),$$
for an integer N sufficiently large (locality);
- We have that
$$Y(u, z)\mathbb{1} = u + O(z).$$
This property is known as *creativity*. Hence at $z = 0$ we have that $Y(u, z)\mathbb{1} = u$. We say that $Y(u, z)$ *creates* the state u;
- The vertex operator for the vacuum vector is the identity operator
$$Y(\mathbb{1}, z) = \mathrm{Id}_V.$$
- There exists an endomorphism T such that
$$[T, Y(u, z)] = \partial_z Y(u, z).$$
This is known as *translation covariance.*

Additionally, vertex operators also enjoy the commutator identity for all $u, v \in V$
$$[u(m), Y(v, z)] = \sum_{j\geq 0} \binom{m}{j} Y(u(j)v, z) z^{m-j} = \left(\sum_{j\geq 0} Y(u(j)v, z)\partial_z^{(j)}\right) z^m, \tag{3}$$
noting that $\partial_k^{(j)} z^m = \binom{m}{j} z^{m-j}$, where
$$\partial_z^{(j)} = \frac{1}{j!}\partial_z^j. \tag{4}$$

We will use this notation throughout this thesis.

We say that a state $u \in V$ generates the vertex (super) algebra V if

$$V = \mathrm{span}\{u(-k_1)u(-k_2)\dots u(-k_n)\mathbb{1}, k_i \geq 1, n \geq 0\}.$$

We can extend this notion to vertex (super) algebras with more than one generator. Vertex operators also enjoy *associativity*

$$(x+y)^M Y(Y(u,x)v,y) = (x+y)^M Y(u,x+y)Y(v,y), \tag{5}$$

for a sufficiently large integer M. If $V_{\bar{1}} = 0$, then V is a vertex algebra with the data and axioms of above with parity factors omitted.

1.3 Vertex operator (super) algebras

We will now review some aspects of Vertex Operator Super Algebras (VOSAs). A VOSA is a vertex super algebra satisfying some additional rules for all $u, v \in V$

- There exists a conformal vector $\omega \in V_{\bar{0}}$ with vertex operator

  $$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

  where the $L(n)$ operators satisfy the Virasoro Lie algebra relations

  $$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m,-n} c\mathrm{Id}_V,$$

  where $\delta_{i,j}$ is the Kronecker delta and c is a complex number known as the *central charge.*
- $T = L(-1)$, i.e. $Y(L(-1)u, z) = \partial_z Y(u,z)$ (translation);
- V decomposes into eigenspaces with half-integral eigenvalues for the operator $L(0)$

  $$V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n,$$

  with

  $$V_n = \{u \in V : L(0)v = nu\},$$

  and $\dim V_n < \infty$. The eigenvalue for $L(0)$ on a vector $u \in V$ is known as the *(conformal) weight* of u and is denoted $\mathrm{wt}(u)$. The parity spaces are associated with the weights as follows

  $$V_{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} V_n, \qquad V_{\bar{1}} = \bigoplus_{n\in\mathbb{Z}+\frac{1}{2}} V_n.$$

We have that $\mathbb{1} \in V_{\bar{0},0}$ and $\omega \in V_{\bar{0},2}$, where

$$V_{\bar{a},n} = \{v \in V : p(v) = \bar{a}, L(0)v = nv\}.$$

A vertex operator algebra (VOA), then, is a VOSA where $V_{\bar{1}} = 0$. The modes $u(n)$ for any vector $u \in V$ map between these eigenspaces as follows

$$u(n) : V_m \to V_{m+\mathrm{wt}(u)-n-1}. \tag{6}$$

More specifically, we can see how a mode $u(n)$ maps between eigenspaces and parity spaces using (2) and (6)

$$u(n) : V_{\overline{a},m} \to V_{\overline{a}+p(u),m+\mathrm{wt}(u)-n-1}. \tag{7}$$

Examining the mode $u(\mathrm{wt}(u)-1)$ we see that $u(\mathrm{wt}(u)-1)V_m \subseteq V_m$. Following this, we define the *level-preserving mode* $o(v)$ by

$$o(u) = \begin{cases} u(\mathrm{wt}(u)-1), & \mathrm{wt}(u) \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Lastly, we say a state $u \in V$ is *quasiprimary* if $L(1)u = 0$, and *primary* if $L(n)u = 0$ for $n \geq 1$.

1.4 The Free Fermion VOSA

We now consider a simple example of a $\frac{1}{2}\mathbb{Z}$-graded VOSA known as the Free Fermion, or the Neveu-Schwarz sector. The rank one case (where the algebra has one generator) is constructed from a single state ψ of weight $\frac{1}{2}$, whose modes have commutation relations

$$[\psi(m), \psi(n)] = \psi(m)\psi(n) + \psi(n)\psi(m) = \delta_{m+n+1,0}\mathrm{Id}_V.$$

Note the anticommutator given by (1) as ψ has parity $\overline{1}$. The Virasoro vector for this VOSA is given by

$$\omega = \frac{1}{2}\psi(-2)\psi(-1)\mathbb{1},$$

which gives a Virasoro algebra of central charge $\frac{1}{2}$. We can also examine the rank two Free Fermion VOSA $V \otimes V$, with generators $\psi_1 = \psi \otimes \mathbb{1}$ and $\psi_2 = \mathbb{1} \otimes \psi$. For convenience, we introduce the off-diagonal basis

$$\psi^{\pm} = \frac{1}{\sqrt{2}}\left(\psi_1 \pm \mathrm{i}\psi_2\right).$$

The modes of these states obey the bracket

$$\left[\psi^+(m), \psi^-(n)\right] = \delta_{m+n+1,0}, \quad \left[\psi^{\pm}(m), \psi^{\pm}(n)\right] = 0. \tag{9}$$

This gives the relation (for $n \geq 0$)

$$\begin{aligned} \psi^+(n)\psi^- &= \psi^+(n)\psi^-(-1)\mathbb{1} \\ &= \psi^-(-1)\psi^+(n)\mathbb{1} + \left[\psi^+(n), \psi^-(-1)\right]\mathbb{1} \\ &= \delta_{n0}\mathbb{1}. \end{aligned} \tag{10}$$

The conformal vector in the rank two case is

$$\omega = \frac{1}{2}\left(\psi^+(-2)\psi^-(-1) + \psi^-(-2)\psi^+(-1)\right)\mathbb{1}.$$

We obtain a Virasoro structure with central charge 1.

1.5 The Heisenberg VOA

This VOA is generated by a single vector h of weight one, whose modes obey the commutation relations of the Heisenberg Lie algebra

$$[h(m), h(n)] = h(m)h(n) - h(n)h(m) = m\delta_{m,-n}\mathrm{Id}_V.$$

From these relations we find that

$$h(m)h = \delta_{m1}\mathbb{1}, \tag{11}$$

for $m \geq 0$, using $h = h(-1)\mathbb{1}$, with the Virasoro vector for the VOA given by

$$\omega = \frac{1}{2}h(-1)^2\mathbb{1},$$

with central charge 1.

1.6 Modules for a VOA

As in the representation theory of groups and rings, the representation theory of (super-)VOAs admits the notion of a module. A module for a VOA V is a space satisfying axioms similar to that of a VOA. Let V be a VOA. Then a *weak module* W for V is a tuple (W, Y^W) consisting of the following [MT4]

- A vector space W;
- A linear map $Y^W : V \to \mathfrak{F}(W)$ defined by

  $$Y^W(v, z) = \sum_{n\in\mathbb{Z}} v^W(n)z^{-n-1},$$

  which satisfies for all $u, v, w \in W$

- $Y^W(\mathbb{1}, z) = \mathrm{Id}_W$ (vacuum);
- $Y^W(u, z) \sim_N Y^W(v, z)$ for some $N \gg 0$ (locality);
- $(x+y)^M Y^W(u, x+y)Y^W(v, y)w = (x+y)^M Y^W(Y(u, x)v, y)w$ (associativity);

for a sufficiently large integer M. Then we define a *V-module* as a weak $\mathbb{C}$-graded V-module (W, Y^W), $W = \bigoplus_{\alpha\in\mathbb{C}} W_\alpha$ which satisfies

- $\dim W_\alpha < \infty$;
- For all α, $W_{\alpha+n} = 0$ for $n \ll 0$;
- $L(0)w = \alpha w, w \in W_\alpha$.

We note that for $u^{W_a} \in W_\alpha$

$$u^{W_\alpha}(n) : W_m \to W_{m+\alpha-n-1}.$$

If the only submodules of W are 0 and W, we say that W is a *simple* module. An example of simple VOA modules considered in this thesis are simple Heisenberg modules M_α determined by the eigenvalue α under the Heisenberg mode $h(0)$.

2 Elliptic Functions and Riemann Surfaces

In this section we discuss modular forms and elliptic functions, functions with modular symmetries on the upper-half complex plane and a torus respectively. We also introduce differential forms on a genus g Riemann surface and related concepts such as the Riemann-Roch theorem. We follow the treatment of [Bob], [MT4], [MTZ], [Se] and [TZ2] here.

2.1 (Twisted) modular forms and elliptic functions

We will first review some aspects of modular forms and elliptic functions, along with their twisted counterparts. We first note that the modular group

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\},$$

acts on the complex upper half complex plane $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(z) > 0\}$ as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .\tau = \frac{a\tau+b}{c\tau+d}.$$

Using this, we define modular functions and forms. A *modular function* of weight n is a meromorphic function $f(\tau)$ defined on $\mathbb{H}$ such that [Se]

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n f(\tau), \tag{12}$$

for all $\tau \in \mathbb{H}$ with $ad-bc=1$. Equivalently, as the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

generate $\mathrm{SL}(2,\mathbb{Z})$, we can rephrase (12) as

$$\begin{aligned} f(\tau+1) &= f(\tau), \\ f\left(-\frac{1}{\tau}\right) &= \tau^n f(\tau). \end{aligned} \tag{13}$$

If $f(\tau)$ enjoys these transformational properties and is holomorphic on the entire upper half plane, it is called a *modular form.* We can then find a Fourier expansion for $f(\tau)$

$$f(\tau) = \sum_{n\geq 0}^{\infty} a_n q^n,$$

where $q = \exp(2\pi i\tau)$, $0 < |q| < 1$, $\tau \in \mathbb{H}$. The a_n coefficients typically have some number-theoretic significance. A classical example of a modular form is the *Eisenstein series* E_n (defined for $n \geq 2$ by)

$$E_n(\tau) = \begin{cases} -\dfrac{B_n}{n!} + \dfrac{2}{(n-1)!} \sum_{r\geq 1} \dfrac{r^{n-1}q^r}{1-q^r}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

where B_n is the nth Bernoulli number, given by the Taylor expansion [Se]

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{n\geq 1} (-1)^{n+1} B_n \frac{x^{2n}}{(2n)!}. \tag{14}$$

We note that $B_n = 0$ for n odd, $n \neq 1$. B_n is also related to certain values of the Riemann zeta function.

Then for even $n \geq 4$, E_n is a holomorphic modular form of weight n ($E_n = 0$ for odd n). We note that E_2 does not satisfy the transformational property (12), it is instead a quasimodular form which transforms with an additive term as follows

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}.$$

Then we can define the genus one *Weierstrass elliptic functions* for $k \geq 1$ by [MT2]

$$P_k(z, \tau) = (-1)^{k-1} \partial_z^{(k-1)} P_1(z, \tau) = \frac{1}{z^k} + (-1)^k \sum_{n \geq k} \binom{n-1}{k-1} E_n(\tau) z^{n-k}.$$

where $\partial_z^{(k-1)}$ is that of (4) and

$$P_1(z, \tau) = \frac{1}{z} - \sum_{n \geq 2} E_n(\tau) z^{n-1}.$$

These functions have periodicities

$$\begin{aligned} P_k(z + 2\pi i, \tau) &= P_k(z, \tau), \\ P_k(z + 2\pi i\tau, \tau) &= P_k(z, \tau) - \delta_{k1}, \\ P_k(z, \tau + 1) &= P_k(z, \tau), \end{aligned}$$

and hence have a natural geometric interpretation on a torus.

2.2 Twisted Weierstrass functions and Eisenstein series

We now introduce a more general version of the Eisenstein and Weierstrass functions which enjoy additional $\mathrm{SL}(2, \mathbb{Z})$ symmetries. Here we will follow the article [MTZ].

Let $\theta, \phi \in U(1)$, i.e. they are complex numbers of modulus one, with $\phi = \exp(2\pi i\lambda)$ for some parameter $0 \leq \lambda < 1$. Then we define *twisted Weierstrass functions* for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$

$$P_k\begin{bmatrix}\theta \\ \phi\end{bmatrix}(z, \tau) = \frac{1}{z^k} + (-1)^k \sum_{n \geq k} \binom{n-1}{k-1} E_n\begin{bmatrix}\theta \\ \phi\end{bmatrix}(\tau) z^{n-k}, \tag{15}$$

where the $E_n\left[\begin{smallmatrix}\theta \\ \phi\end{smallmatrix}\right](\tau)$ terms are *twisted Eisenstein series*

$$\begin{aligned} E_n\begin{bmatrix}\theta \\ \phi\end{bmatrix}(\tau) = -\frac{B_n(\lambda)}{n!} &+ \frac{1}{(n-1)!} \sum_{r \geq 0}{}' \frac{(r+\lambda)^{n-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} \\ &+ \frac{(-1)^n}{(n-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{n-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}, \end{aligned} \tag{16}$$

where the primed sum notation indicates omission of the $r = 0$ term when $\theta = \phi = 1$. The $B_n(\lambda)$ terms are generalised Bernoulli numbers, the coefficients of the Taylor expansion of the function

$$f(z) = \frac{q_z^\lambda}{q_z - 1},$$

with $q_z = \exp(z)$. We note the generalisation of (14). For even n and $(\theta, \phi) = (1, 1)$ we find that $E_n \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\tau) = E_n(\tau)$. For odd n we have that $E_n \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\tau) = -B_1(0)\delta_{n1} = \frac{1}{2}\delta_{n1}$, where δ_{ij} is the Kronecker delta.

These functions will feature prominently in our discussion of Zhu reduction for genus one and two VOSAs. One of the objective of this thesis is to find an analogue of (15) with genus two periodicities. As in the original elliptic case we find that

$$P_k \begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau) = (-1)^{k-1}\partial_z^{(k-1)} P_1 \begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau).$$

$P_k \left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right]$ enjoys the the $\mathrm{SL}(2,\mathbb{Z})$ symmetry

$$P_k \begin{bmatrix}\theta^a\phi^b\\ \theta^c\phi^d\end{bmatrix}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k P_k \begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau).$$

Looking at a particular example $(a = d = -1, b = c = 0)$ we see

$$P_k \begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau) = (-1)^k P_k \begin{bmatrix}\theta^{-1}\\ \phi^{-1}\end{bmatrix}(-z,\tau).$$

2.3 Differential forms on genus g Riemann surfaces

We consider a Riemann surface of genus g with $2g$ canonical homology cycles α_a, β_a for $a = 1, \ldots, g$. For convenience, we define the following indexing sets

$$\mathcal{I} = \{\pm 1, \pm 2, \ldots, \pm g\}, \quad \mathcal{I}_+ = \{1, 2, \ldots, g\}. \tag{17}$$

There exists a unique singular symmetric bidifferential form $\omega^{(g)}(x,y)$ of weight $(1,1)$, called the *normalised bidifferential of the second kind* of the form [FarK], [Y]

$$\omega^{(g)}(x,y) = \left(\frac{1}{(x-y)^2} + \text{regular terms}\right) dxdy, \tag{18}$$

for any local coordinates x, y and with normalisation

$$\oint_{\alpha_a} \omega^{(g)}(x,\cdot) = 0,$$

for $a \in \mathcal{I}_+$. From this, we construct the genus g *projective connection* [Gu]

$$s^{(g)}(x) = 6 \lim_{y\to x}\left(\omega^{(g)}(x,y) - \frac{dxdy}{(x-y)^2}\right). \tag{19}$$

We have that

$$\nu_a^{(g)}(x) = \oint_{\beta_a} \omega^{(g)}(x,\cdot), \tag{20}$$

provides a basis of g holomorphic 1-differentials with normalisation [FarK]

$$\frac{1}{2\pi \mathrm{i}} \oint_{\alpha_a} \nu_b^{(g)} = \delta_{ab}, \tag{21}$$

for $a, b \in \mathcal{I}_+$. Following this, we define the genus g *period matrix* entrywise by

$$2\pi\mathrm{i}\Omega_{ab}^{(g)} = \oint_{\beta_a} \nu_b^{(g)}. \tag{22}$$

We also define the *normalised differential of the third kind* for $P_1, P_2 \in \mathcal{S}^{(g)}$

$$\omega^{(g)}_{P_2-P_1}(x) = \int_{P_1}^{P_2} \omega^{(g)}(x, \cdot). \tag{23}$$

For the Riemann sphere $\mathcal{S}^{(0)} \cong \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $x, y, P_1, P_2 \in \widehat{\mathbb{C}}$, we have

$$\omega^{(0)}(x, y) = \frac{1}{(x-y)^2} dxdy,$$

giving

$$\omega^{(0)}_{P_2-P_1}(x) = \left(\frac{1}{x-P_2} - \frac{1}{x-P_1} \right) dx. \tag{24}$$

Lastly, we quote the Riemann-Roch theorem, which provides a counting for the dimension of the space of holomorphic N-differentials on a Riemann surface of arbitrary genus

Genus	Weight	Dimension
$g=0$	$N \leq 0$	$1-2N$
	$N > 0$	0
$g=1$	$N \in \mathbb{Z}$	1
$g \geq 2$	$N < 0$	0
	$N = 0$	1
	$N = 1$	g
	$N \geq 2$	$(g-1)(2N-1)$

Later we will show that the counting of the coefficients obtained in the genus two Zhu VOSA reduction formulas complies exactly with the counting provided by the Riemann-Roch theorem in all possible cases. We conjecture that these coefficients form a basis for holomorphic N-differentials on $\mathcal{S}^{(2)}$, but the proof of this requires more general geometric methods and is not covered in this thesis. The genus g reduction formula for VOAs provides a spanning set of holomorphic N-differentials; this set can be reduced to a basis by exploiting the $\mathrm{SL}(2,\mathbb{C})$-related properties of the genus g n-point function.

2.4 Sewing Riemann surfaces and the ϵ-formalism

We now discuss a method for sewing Riemann surfaces, developed by Yamada in [Y] and expanded on by Mason and Tuite in [MT1]. Let $\mathcal{S}^{(1)}_1$ and $\mathcal{S}^{(1)}_2$ denote genus one Riemann surfaces. The objective is to sew these tori to form a genus two Riemann surface.

It is well-known that a torus $\mathcal{S}^{(1)}_a$ can be written as $\mathbb{C}/L_a$, where $L_a = 2\pi\mathrm{i}(\mathbb{Z}\tau_a \oplus \mathbb{Z})$ is a lattice, for $\tau_a \in \mathbb{H}$. Then this lattice has some vector $\lambda_{\min}$ of non-zero minimal norm. Let p_a be a point on $\mathcal{S}^{(1)}_a$, $a = 1, 2$ and let z_a be a local coordinate in the neighbourhood of p_a. Consider the disc on $\mathcal{S}^{(1)}_a$ defined by $|z_a| \leq r_a$ with $r_a < \frac{1}{2}|\lambda_{\min}|$. We now introduce a complex sewing parameter ϵ where $|\epsilon| \leq r_1 r_2$, and remove the disc

$$\{z_a : |z_a| \leq |\epsilon|/r_{\bar{a}}\} \subset \mathcal{S}^{(1)}_a,$$

where we employ the convention that $\overline{1} = 2$, $\overline{2} = 1$. We define the annular region for each surface $\mathcal{S}_a^{(1)}$

$$\mathcal{A}_a = \{z_a : |\epsilon|/r_{\overline{a}} \leq |z_a| \leq r_a\} \subset \mathcal{S}_a^{(1)},$$

and make the identification $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ using the sewing relation

$$z_1 z_2 = \epsilon. \tag{25}$$

Using this identification we can construct the genus two surface

$$\mathcal{S}^{(2)} = \left\{\mathcal{S}_1^{(1)} \backslash \mathcal{A}_1\right\} \cup \left\{\mathcal{S}_2^{(1)} \backslash \mathcal{A}_2\right\} \cup \mathcal{A},$$

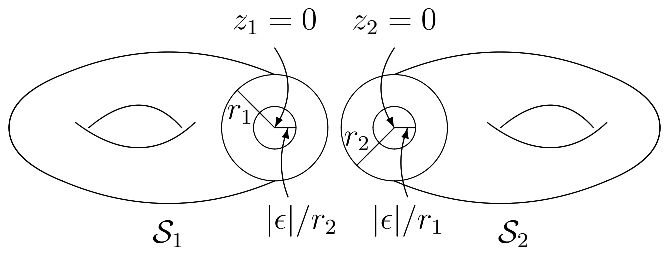

Figure 1: Sewing two tori.

given by the sewing domain

$$\mathcal{D}^{\epsilon} = \left\{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} : |\epsilon| < \frac{1}{4} D(q_1) D(q_2)\right\}.$$

Taking the limit $\epsilon \to 0$, we can see that the double torus degenerates into two separate tori.

2.5 The Szegő kernel

Lastly, we define the Szegő kernel, which will feature prominently here in the discussion of the Free Fermion VOSA. It is a holomorphic $\left(\frac{1}{2}, \frac{1}{2}\right)$-differential on a Riemann surface of the form

$$S^{(g)} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (x, y) = \left(\frac{1}{x - y} + \text{regular terms}\right) dx^{\frac{1}{2}} dy^{\frac{1}{2}},$$

where $\theta^{(g)}, \phi^{(g)} \in U(1)^g$. For $(\theta, \phi) \neq (1, 1)$, the genus one Szegő kernel is given by [TZ2]

$$S^{(1)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x, y) = \left(P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x - y, \tau)\right) dx^{\frac{1}{2}} dy^{\frac{1}{2}}.$$

We can construct higher genus Szegő kernels from lower genus data. In this thesis in particular we are interested in the genus two Szegő kernel (here we suppress the twist notation for readability) [MTZ]

$$S^{(2)}(x, y) = \begin{cases} S_a^{(1)}(x, y) + h_a(x) \left(1 - F_{\overline{a}} F_a\right)^{-1} F_{\overline{a}} \overline{h}_a(y), & x, y \in \widehat{\mathcal{S}}_a, \\ \xi (-1)^{\overline{a}} h_a(x) \left(1 - F_{\overline{a}} F_a\right)^{-1} \overline{h}_{\overline{a}}(y), & x \in \widehat{\mathcal{S}}_a, y \in \widehat{\mathcal{S}}_{\overline{a}}, \end{cases} \tag{26}$$

where $S_a^{(1)}(x,y)$ denotes $\left(P_1\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](x-y,\tau_a)\right)dx^{\frac{1}{2}}dy^{\frac{1}{2}}$, where τ_a is the modular parameter of the sewn torus $\mathcal{S}_a^{(1)}$, and the vectors $h_a, \overline{h}_a$ and the matrices F_a for $a=1,2$ are given by

$$h_a(k,x) = \epsilon^{\frac{k}{2}-\frac{1}{4}} P_k \begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x,\tau_a), \tag{27}$$

$$\overline{h}_a(k,y) = \epsilon^{\frac{k}{2}-\frac{1}{4}} P_k \begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(-y,\tau_a), \tag{28}$$

$$F_a(k,l) = \epsilon^{\frac{1}{2}(k+l-1)}(-1)^l \binom{k+l-2}{k-1} E_{k+l-1}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(\tau_a), \tag{29}$$

where $\theta_a, \phi_a \in U(1)$, $x,y \in \mathcal{S}_a$, $\tau_a \in \mathbb{H}$ are modular parameters for the left and right tori and ϵ is the sewing parameter of (25).

3 Zhu Reduction for n-Point Functions at Genus One and Two

In this section we introduce the idea of Zhu reduction for n-point functions for (super-)VOAs at genus one and two. This concept is central to this thesis and is a powerful tool for developing recursive relations from which we can extract differential equations for VOSA partition functions and differential forms on a Riemann surface. Here we follow the approach of [GT1], [MT2], [MT3], [MT4], [MTZ], [TZ1] and [Zhu].

3.1 Partition functions and n-point functions

We first define the *partition function* $Z_V^{(1)}(\tau)$ for a VOSA which is given by

$$Z_V^{(1)}(\tau) = \mathrm{STr}_V \left(q^{L(0) - \frac{c}{24}} \right), \tag{30}$$

where c is the central charge of the VOA and STr is the *supertrace*

$$\mathrm{STr}_V(A) = \mathrm{Tr}_{V_{\bar{0}}}(A) - \mathrm{Tr}_{V_{\bar{1}}}(A), \tag{31}$$

for a given operator A. For the Free Fermion VOSA of §1.4 with central charge $\frac{1}{2}$, we find that the partition function is given by [MTZ]

$$Z_V^{(1)}(\tau) = \mathrm{STr}_V \left(q^{L(0) - \frac{1}{48}} \right) = q^{-\frac{1}{48}} \prod_{n \geq 0}^{\infty} \left(1 - q^{n + \frac{1}{2}} \right) = \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)},$$

where $\eta(\tau)$ is the classical Dedekind eta function of number theory

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n).$$

This is almost a modular form, but not quite as it does not enjoy the translational symmetry of (13)

$$\eta(\tau + 1) = e^{\frac{\pi \mathrm{i}}{12}} \eta(\tau),$$
$$\eta\left(-\frac{1}{\tau} \right) = \sqrt{-\mathrm{i}\tau}\, \eta(\tau).$$

We note that $\eta\left(-\frac{1}{\tau}\right)$ has a branch cut, while η^2 is a modular form with characters. For a VOA, the supertrace becomes the usual trace

$$Z_V^{(1)}(\tau) = \mathrm{Tr}_V \left(q^{L(0) - \frac{c}{24}} \right).$$

In the case of the Heisenberg VOA of §1.5 (for central charge 1), we find that

$$Z_V^{(1)}(\tau) = \frac{1}{\eta(\tau)},$$

which can be derived from a combinatorial argument about the dimensions of the Heisenberg VOA graded spaces. Extending the idea of a partition function, we can include a VOSA automorphism g to obtain the *orbifold partition function*

$$Z_V^{(1)}(g; \tau) = \mathrm{STr}_V (g q^{L(0) - \frac{c}{24}}). \tag{32}$$

This can be naturally generalised to take an arbitrary number of vertex operators as trace arguments. The genus one n-point function for a VOSA module W, automorphism $g \in \mathrm{Aut}(V)$ and n states $v_1, \ldots, v_n \in V$ inserted at points $x_1, \ldots, x_n$ respectively on a torus is defined by

$$
\begin{aligned}
&Z_W^{(1)}(g; v_1, x_1; \ldots; v_n, x_n; \tau) \\
&= \mathrm{STr}_W \left(g Y(q_{x_1}^{L(0)} v_1, q_{x_1}) \ldots Y(q_{x_n}^{L(0)} v_n, q_{x_n}) q^{L(0)-\frac{c}{24}} \right), \qquad (33)
\end{aligned}
$$

where $q_{x_k} = \exp(x_k), q = \exp(2\pi \mathrm{i} \tau)$ and c is the central charge of the VOSA. When the trace is taken over a module W, the vertex operators are implicitly assumed to belong to the module in order to suppress $Y^W(v^W, x)$ notation, and similarly for v^W. Here we deal with traces over g-twisted modules, where g is an continuous symmetry of the VOSA generated by a Heisenberg vector (see [DZha], [Li], [MTZ]).

Setting $g = 1, v_k = \mathbb{1}$ for $k = 1, \ldots, n$ yields the partition function. In order for these n-point functions to be non-trivial, we require that the sum of the state parities is zero

$$
p(v_1) + p(v_2) + \ldots + p(v_n) = 0.
$$

For a VOA, we again have that the supertrace becomes the usual trace.

3.2 The square bracket formalism

Given a vertex operator super algebra $(V, Y(\cdot,\cdot), \mathbb{1}, \omega)$, we can find an isomorphic VOSA $(V, Y[\cdot,\cdot], \mathbb{1}, \widetilde{\omega})$ with an identical vector space and vacuum vector, however ω undergoes the transformation

$$
\omega \mapsto \widetilde{\omega} = \omega - \frac{c}{24}\mathbb{1},
$$

while the vertex operators undergo the conformal transformation $Y(v,z) \mapsto Y[v,z]$ with new modes

$$
Y[v,z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1),
$$

with $q_z = \exp(z)$. Then the vertex operator for $\widetilde{\omega}$ is given by

$$
Y[\widetilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.
$$

As in the original setting, $L[0]$ provides a $\frac{1}{2}\mathbb{Z}$-grading on V

$$
V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_{[n]},
$$

where $V_{[n]}$ is an eigenspace given by

$$
V_{[n]} = \{v \in V : L[0]v = nv\},
$$

however it is distinct from that induced by $L(0)$. Following the notation $\mathrm{wt}(v)$ for the $L(0)$-weight of v, we denote the $L[0]$-weight of v by $\mathrm{wt}[v]$ ($\widetilde{\omega}$ has $L[0]$-weight 2 as in the round bracket case). The square bracket formalism provides useful tools for working with n-point functions. Using [MTZ, Lemma 3], n-point functions can be related to 1-point functions as follows. For a VOSA module W

$$\begin{aligned}
&Z_W^{(1)}(g; v_1, x_1; \ldots; v_n, x_n; \tau) \\
&= Z_W^{(1)}(g; Y[v_1, x_1]Y[v_2, x_2]\ldots Y[v_n, x_n]\mathbb{1}, \tau) \\
&= Z_W^{(1)}(g; Y[v_1, x_{1n}]Y[v_2, x_{2n}]\ldots Y[v_{n-1}, x_{n-1n}]v_n, \tau),
\end{aligned} \tag{34}$$

where $x_{ij} = x_i - x_j$. As before, when the trace is taken over a module W, we suppress then $Y^W[v^W, z]$ notation.

3.3 Adjoint vertex operators and the Li-Zamolodchikov metric

The algebra generated by the Virasoro modes $L(-1)$, $L(0)$ and $L(1)$ acts in a natural way on vertex operators, corresponding to the Möbius maps (fractional linear transformations) relating to translation, dilation and inversion respectively, which we can write as an $\mathrm{SL}(2,\mathbb{C})$ action on V. In particular we are interested in the Möbius map of the form $z \mapsto -\frac{\lambda^2}{z}$. Then the vertex operators undergo the transformation [TZ1]

$$Y_\lambda^\dagger(v, z) = Y\left(\exp\left(\frac{z}{\lambda^2}L(1)\right)\left(-\frac{z}{\lambda}\right)^{-2L(0)} v, -\frac{\lambda^2}{z}\right).$$

This is known as the *adjoint vertex operator* with respect to the complex parameter λ. We are particularly interested in the Möbius map with $-\lambda^2 = \epsilon$, the sewing parameter, or equivalently

$$\lambda = -\xi\epsilon^{\frac{1}{2}},$$

where $\xi = \pm\sqrt{-1}$, depending on the branch chosen. These adjoint operators satisfy

$$\left(Y^\dagger\right)^\dagger(v, z) = (-1)^{2\mathrm{wt}(v)}Y(v, z).$$

In the case of an integrally graded VOA, then, we have that $(Y^\dagger)^\dagger(u, z) = Y(u, z)$, whereas for half-integral weight states of a VOSA we obtain a multiplier of -1. For a quasiprimary state v, the modes of these vertex operators $Y^\dagger(v, z) = \sum_{n\in\mathbb{Z}} v^\dagger(n)z^{-n-1}$ are of the form

$$v^\dagger(n) = \lambda^{2\mathrm{wt}(v)}(\lambda^2)^{n+1}u(2\mathrm{wt}(v) - n - 2).$$

For $\lambda = 1$ this simplifies to

$$v^\dagger(n) = (-1)^{n+1}u(2\mathrm{wt}(v) - n - 2). \tag{35}$$

The vertex operators and modes in the square bracket setting undergo analogous transformations, denoted by $Y^\dagger[v, z]$ and $v^\dagger[n]$ respectively, with the adjoint operator given by

$$v^\dagger[n] = \lambda^{2\mathrm{wt}[v]}(\lambda^2)^{n+1}u[2\mathrm{wt}[v] - n - 2]. \tag{36}$$

Following this, we define an *invariant bilinear form* $\langle\cdot,\cdot\rangle_\lambda$ on V. We say that such a form is invariant if for all $u, v, w \in V$

$$\langle Y(u, z)v, w\rangle_\lambda = (-1)^{p(u)p(v)}\langle v, Y^\dagger(u, z)w\rangle_\lambda. \tag{37}$$

Expanding both sides as formal series and examining coefficients we find that

$$\langle u(n)v, w\rangle_\lambda = (-1)^{p(u)p(v)}\langle v, u^\dagger(n)w\rangle_\lambda.$$

This form must be symmetric and only gives a non-trivial result if $\mathrm{wt}(u) = \mathrm{wt}(v)$. We assume that V is of *strong CFT-type*, that is, $V_0 = \mathbb{C}\mathbb{1}$ and $V \cong V'$, the dual space of V. Then the form is unique up to normalisation, which we choose as

$$\langle \mathbb{1}, \mathbb{1}\rangle = 1. \tag{38}$$

We refer to this unique non-degenerate invariant bilinear form as the *Li-Zamolodchikov (Li-Z) metric*. Given a basis $\{u^a\}$ for a VOSA V, we define the *dual basis* with respect to the Li-Z metric to be the basis $\{\overline{u}^b\}$ for which

$$\langle u^a, \overline{u}^b\rangle = \delta_{ab}.$$

3.4 Genus one Zhu reduction

In [Zhu], Zhu introduced a recursion formula relating n-point functions to $(n-1)$-point functions in terms of elliptic Weierstrass functions. This was later generalised in [MTZ] to incorporate $\mathbb{R}$-graded VOA modules and VOA automorphisms. Let σ denote the parity automorphism

$$\sigma v = (-1)^{p(v)}v, \tag{39}$$

for $v \in V$ and let g denote a VOSA automorphism that commutes with σ. Let W be a module which is stable under g and σ, i.e. $\sigma W = W$ and $gW = W$ (we enforce this so that the n-point functions have a modular nature) [DZha]. Then the genus one $(n+1)$-point function (33) for states $v, v_1, \ldots, v_n$, insertion points $x, y_1, \ldots, y_n$, VOSA module W and automorphism $g \in \mathrm{Aut}(V)$ obeys the following Zhu recursion formula [MTZ]

$$\begin{aligned}
&Z_W^{(1)}(g; v, x; , v_1, y_1; \ldots; v_n, y_n; \tau)\\
&= \mathrm{STr}_W\left(go(v)Y(q_{y_1}^{L(0)}v_1, q_{y_1})\ldots Y(q_{y_n}^{L(0)}v_n, q_{y_n})q^{L(0)-\frac{c}{24}}\right)\\
&\quad + \sum_{k=1}^{n}\sum_{j\geq 0} p(v, \boldsymbol{v_{k-1}})\partial_{y_k}^{(j)}P_1\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x-y_k, \tau)Z_W^{(1)}(\ldots; v[j]v_k, y_k; \ldots),
\end{aligned} \tag{40}$$

where q_{y_k}, q are as above, $\phi = \exp(2\pi \mathrm{i}\lambda)$ (with $\lambda = \mathrm{wt}_W(v) \bmod \mathbb{Z}$), g has eigenvalue θ^{-1} acting on v, STr is the supertrace of (31), while $o(v)$ is the level preserving mode (8). The $p(v, \boldsymbol{v_{k-1}})$ terms are parity factors given by [MTZ]

$$p(v, \boldsymbol{v_{k-1}}) = \begin{cases} 1, & k = 1,\\ (-1)^{p(v)\sum_{i=1}^{k-1}p(v_i)}, & k > 1.\end{cases} \tag{41}$$

We will often refer to the terms of type

$$Z_W^{(1)}(\ldots; v[j]v_k, y_k; \ldots), \tag{42}$$

as *contraction terms*. While the trace in this formula is taken over a module, there is an equivalent formula for Zhu recursion for n-point functions for genus one VOAs

$$Z_V^{(1)}(v, x; , v_1, y_1; \ldots; v_n, y_n; \tau)$$

$$
\begin{aligned}
&= \mathrm{Tr}_V \left(o(v) Y(q_{y_1}^{L(0)} v_1, q_{y_1}) \ldots Y(q_{y_n}^{L(0)} v_n, q_{y_n}) q^{L(0)-\frac{c}{24}} \right) \\
&\quad + \sum_{k=1}^{n} \sum_{j \geq 0} \partial_{y_k}^{(j)} P_1(x - y_k, \tau) Z_V^{(1)}(\ldots; u[j]v_k, y_k; \ldots).
\end{aligned} \tag{43}
$$

Taking the 1-point function $Z_V^{(1)}(\widetilde{\omega}, x)$ for a VOA V, we find

$$
\begin{aligned}
Z_V^{(1)}(\widetilde{\omega}, x) &= \mathrm{Tr} \left(o(\widetilde{\omega}) q^{L(0)-\frac{c}{24}} \right) \\
&= \mathrm{Tr} \left(\left(L(0) - \frac{c}{24} \right) q^{L(0)-\frac{c}{24}} \right) \\
&= q\partial_q Z_V^{(1)}.
\end{aligned}
$$

We extend this idea to find an application of this formula in the derivation of genus one Ward identities. We insert a Virasoro state ω at a point x on the torus and general states $v_1, v_2, \ldots, v_n$ at $y_1, y_2, \ldots, y_n$ respectively, also on the torus. For a VOA V, we find

$$
\begin{aligned}
&Z_V^{(1)}(\omega, x; v_1, y_1; \ldots; v_n, y_n) \\
&= \left(q\partial_q + \sum_{k=1}^{n} P_1(x - y_k, \tau)\partial_{y_k} + \mathrm{wt}[v_k] P_2(x - y_k, \tau) \right) Z_V^{(1)}(v_1, y_1; \ldots; v_n, y_n).
\end{aligned} \tag{44}
$$

For an n-point function where all states inserted are Virasoro states, we find

$$
\begin{aligned}
&Z_V^{(1)}(\omega, x; \omega, y_1; \ldots; \omega, y_n) \\
&= \left(q\partial_q + \sum_{k=1}^{n} P_1(x - y_k, \tau)\partial_{y_k} + 2P_2(x - y_k, \tau) \right) Z_V^{(1)}(\omega, y_1; \ldots; \omega, y_n) \\
&\quad + \frac{c}{2} \sum_{k=1}^{n} P_4(x - y_k, \tau) Z_V^{(1)}(\omega, y_1; \ldots; \widehat{\omega, y_k}; \ldots; \omega, y_n),
\end{aligned} \tag{45}
$$

where the caret denotes omission of the insertion. These identities were generalised in [GT1] to genus two and will be extended to general genus in §9.3.

3.5 Genus two partition and n-point Functions on a VOSA

Following the sewing procedure introduced in §2.4, we can define a genus two version of the partition function and n-point functions as mentioned in (30) and (33) respectively. The genus two partition function for VOSA modules W_1, W_2 is given by

$$
Z_{W_1,W_2}^{(2)} = Z_{W_1,W_2}^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_{W_1}^{(1)}(u, \tau_1) Z_{W_2}^{(1)}(\overline{u}, \tau_2), \tag{46}
$$

where u runs over a basis for V and $\overline{u}$ is the dual of u with respect to the Li-Z metric (37) with $\lambda = \epsilon$. We note that the $Z_{W_a}^{(1)}(u, \tau_a)$ terms are genus one 1-point functions. We can also naturally incorporate VOSA automorphisms g_1, g_2 to define a genus two orbifold partition function generalisation of (32)

$$
Z_{W_1,W_2}^{(2)}(g_1, g_2; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_{W_1}^{(1)}(g_1; u, \tau_1) Z_{W_2}^{(1)}(g_2; \overline{u}, \tau_2). \tag{47}
$$

We can extend this definition to include insertions on the genus two surface. The genus two n-point function (with $n = L+R+1$)for a VOSA V and automorphisms $g_1, g_2 \in \mathrm{Aut}(V)$, $L+1$ states inserted on the left torus and R states on the right is defined as

$$
\begin{aligned}
&Z_V^{(2)}(g_1, g_2; v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}) \\
&= Z_V^{(2)}(g_1, g_2; v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}; \tau_1, \tau_2, \epsilon) \\
&= \sum_{u \in V} Z_V^{(1)}(g_1 Y[v, x] \boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] u, \tau_1) Z_V^{(1)}(g_2 \boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}] \overline{u}, \tau_2). \qquad (48)
\end{aligned}
$$

Lastly, we can define a genus two n-point function for VOSA modules. The genus two n-point function (with $n = L+R+1$) for VOSA modules W_1, W_2 and VOSA automorphisms g_1, g_2, $L+1$ states inserted on the left torus and R states on the right is

$$
\begin{aligned}
&Z_{W_1, W_2}^{(2)}(g_1, g_2; v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}) \\
&= Z_{W_1, W_2}^{(2)}(g_1, g_2; v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}; \tau_1, \tau_2, \epsilon) \\
&= \sum_{u \in V} Z_{W_1}^{(1)}(g_1 Y[v, x] \boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] u, \tau_1) Z_{W_2}^{(1)}(g_2 \boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}] \overline{u}, \tau_2). \qquad (49)
\end{aligned}
$$

where the boldface vertex operators denote

$$
\begin{aligned}
\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] &= Y[a_1, x_1] \ldots Y[a_L, x_L], \\
\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}] &= Y[b_R, y_R] \ldots Y[b_1, y_1],
\end{aligned}
$$

and u runs over a basis of V, ϵ is the sewing parameter and $\overline{u}$ is the dual of u with respect to the ϵ-dependent Li-Z metric. One of the objectives of this thesis to find a genus two generalisation of (40) for this genus two n-point function.

3.6 Genus two Zhu reduction for VOAs

In [GT1], Gilroy and Tuite have developed a genus two version of Zhu's recursion formula. Define the following infinite matrices for $k, l \geq 1$

$$\Gamma(k, l) = \delta_{k, -l+2N-2}, \qquad (50)$$

$$\Delta(k, l) = \delta_{k, l+2N-2}, \qquad (51)$$

$$\Pi = \Gamma^2, \qquad (52)$$

$$\Lambda_a(k, l) = \epsilon^{(k+l)/2} (-1)^{l+1} \binom{k+l-1}{l} E_{k+l}(\tau_a), \qquad (53)$$

where $\delta_{a,b}$ denotes the Kronecker delta. Next we define the infinite vectors $\mathbb{R}(x)$, $\mathbb{P}(x, \tau_a)$

$$\mathbb{R}(x; m) = \epsilon^{\frac{m}{2}} P_{m+1}(x, \tau_a), \qquad (54)$$

$$\mathbb{P}(x, m) = \epsilon^{\frac{m}{2}} \left(P_m(x, \tau) - E_m(\tau) \right). \qquad (55)$$

Lastly, define the infinite row vector $\mathbb{Q}(x)$ for $x \in \mathcal{S}_a$ by

$$\mathbb{Q}(x) = \mathbb{R}(x) \Delta \left(I - \widetilde{\Lambda}_{\overline{a}} \widetilde{\Lambda}_a \right)^{-1},$$

where I is the infinite identity matrix, $\widetilde{\Lambda}_a = \Lambda_a \Delta$, and we use the convention $\overline{1} = 2$, $\overline{2} = 1$. Genus one Zhu recursion (43) is then employed on the left torus and then related to the right to obtain (recalling the notation (4)) to obtain

Theorem 3.1. *For a VOA V, quasiprimary state v $\mathrm{wt}[v]=N$ and general states $a_1,\ldots,a_L$ inserted on the left torus and general states $b_1,\ldots,b_R$ inserted on the right torus, the genus two n-point function satisfies the following recursive identity*

$$\begin{aligned}Z_V^{(2)}(v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r}) = {}& {}^N\varphi_1(x)o_1 + {}^N\varphi_2(x)o_2 + {}^N\varphi^{\Pi}(x)\mathbb{X}_1^{\Pi}\\ &+\sum_{l=1}^{L}\sum_{j\geq 0}\partial_{x_l}^{(j)}\,{}^N\mathcal{P}(x,x_l)Z_V^{(2)}(\ldots;v[j]a_l,x_l;\ldots)\\ &+\sum_{r=1}^{R}\sum_{j\geq 0}\partial_{y_r}^{(j)}\,{}^N\mathcal{P}(x,y_r)Z_V^{(2)}(\ldots;v[j]b_r,y_r;\ldots),\end{aligned} \tag{56}$$

where

$$o_1=\sum_{u\in V}\mathrm{Tr}_V\left(o(v)\boldsymbol{Y(q_{x_l}^{L(0)}a_l q_{x_l})}Y(q_0^{L(0)}u,q_0)q_1^{L(0)-\frac{c}{24}}\right)Z_V^{(1)}(\boldsymbol{Y[b_r,y_r]}\overline{u},q_2), \tag{57}$$

$$o_2=\sum_{u\in V}Z_V^{(1)}(\boldsymbol{Y[a_l,x_l]}u,q_1)\mathrm{Tr}_V\left(o(v)\boldsymbol{Y(q_{y_r}^{L(0)}b_r q_{y_r})}Y(q_0^{L(0)}\overline{u},q_0)q_1^{L(0)-\frac{c}{24}}\right), \tag{58}$$

and

$${}^N\varphi_a(x)=\begin{cases}1+\epsilon^{\frac{1}{2}}\left({}^N\mathbb{Q}(x)\widetilde{\Lambda}_{\overline{a}}\right)(1), & x\in\widehat{\mathcal{S}}_a,\\ (-1)^N\epsilon^{\frac{1}{2}}\left({}^N\mathbb{Q}(x)\right)(1), & x\in\widehat{\mathcal{S}}_{\overline{a}},\end{cases} \tag{59}$$

and

$${}^N\varphi^{\Pi}(x)=\left(\mathbb{R}(x)+{}^N\mathbb{Q}(x)\left(\widetilde{\Lambda}_{\overline{a}}\Lambda_a+\Lambda_{\overline{a}}\Gamma\right)\right)\Pi, \tag{60}$$

for $x\in\mathcal{S}_a$ and

$$\begin{aligned}&{}^N\mathcal{P}(x,y)\\ &=\begin{cases}P_1(x-y,\tau_a)-P_1(x,\tau_a)-{}^N\mathbb{Q}(x)\widetilde{\Lambda}_{\overline{a}}\mathbb{P}_1^T(y)\\ -\pi_N\left({}^N\mathbb{Q}(x)\Lambda_{\overline{a}}\right)(K), & x,y\in\widehat{\mathcal{S}}_a,\\ \\ (-1)^{N+1}\Big[{}^N\mathbb{Q}(x)\mathbb{P}_1^T(y)+\pi_N\epsilon^{\frac{K}{2}}P_{K+1}(x)\\ +\pi_N\left({}^N\mathbb{Q}(x)\widetilde{\Lambda}_{\overline{a}}\Lambda_a\right)(K)\Big], & x\in\widehat{\mathcal{S}}_a, y\in\widehat{\mathcal{S}}_{\overline{a}},\end{cases}\end{aligned} \tag{61}$$

where $\pi_N=1-\delta_{N1}$ and $K=2N-2$.

Remark 3.1. This formula can be naturally generalised to incorporate VOA modules.

Remark 3.2. These functions are obtained using VOA techniques, however their existence and geometric meaning can be inferred using the construction of Bers [Be] from the perspective of higher genus Green's functions. See [GT1], [GT2] for further details. In particular, $\{{}^N\varphi_a(x)dx^N, {}^N\varphi^{\Pi}(x)dx^N\}$ form a basis of holomorphic N-differentials on $\mathcal{S}^{(2)}$ and ${}^N\mathcal{P}(x,y)dx^N dy^{1-N}$ is a meromorphic bidifferential quasiform with a simple pole and $x=y$.

The aim of this thesis is to find an analogue of this formula in the VOSA setting, where we find that for v of integral weight, twisted versions (parametrised by symmetries of the VOSA) of the ${}^N\mathcal{P}$ (and its derivatives), ${}^N\mathcal{F}_a(x)$, ${}^N\mathcal{F}^{\Pi}(x;m)$ functions take the place of the coefficients in equation (56). In the half-integral weight case, one finds that the ${}^N\mathcal{F}_a(x)$ terms do not exist and are compensated for by two extra ${}^N\mathcal{F}^{\Pi}(x;m)$ terms.

Part II

Genus Two Zhu Reduction for Vertex Operator Super Algebras

4 Genus Two Zhu Reduction for VOSAs

We now extend the results of [GT1] to a twisted setting for a VOSA, where we now consider states of half-integral conformal weight. We follow the techniques of [GT1], extending the relevant linear-algebraic objects to those incorporating VOSA symmetries. This can also be viewed as an extension of the genus one Zhu reduction formula of [MTZ]. The approach involves the sewing of two tori to construct a genus two Riemann surface, and applying the idea of genus one Zhu reduction to the definition of the genus two n-point function. We develop a recursive relation between objects on the left torus and objects on the right to obtain a global result for the genus two surface. We obtain results which are a natural generalisation of Theorem 3.1 of and obtain interesting results which ensure compliance of the counting of dimension of the space of holomorphic N-differentials with the Riemann-Roch theorem.

4.1 Definitions

We develop a Zhu reduction formula for the genus two n-point function for VOSA modules (49) of § 3.5. This can be considered an extension of the genus two VOA Zhu reduction formula (56), and many of the objects we define in this section are twisted generalisations of those found in [GT1], or are appropriate modifications. We can also consider the results in this section to be generalisations of those of [MTZ], i.e. (40).

We introduce the following definitions, which are similar to those of [GT1], however these definitions incorporate additional complex parameters of modulus one (twistings), and the index range is extended to include zero, and the summation convention includes the zeroth component. We define an infinite matrix $\Lambda\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right]$ (a twisted generalisation of the matrix (53)) for $m, n \geq 0$, $\theta, \phi \in U(1)$ by

$$\Lambda\begin{bmatrix}\theta\\ \phi\end{bmatrix}(m,n) = \begin{cases} \epsilon^{\frac{m+n}{2}}(-1)^{n+1}\binom{m+n-1}{n} E_{m+n}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau), & m \geq 1, \\ 0, & m = 0. \end{cases} \tag{62}$$

for $E_n\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right]$ of (16). We will often use the shorthand Λ_a to denote $\Lambda\left[\begin{smallmatrix}\theta_a\\ \phi_a\end{smallmatrix}\right]$ for $a = 1, 2$. For convenience we define the infinite row vector $\widetilde{\mathbb{P}}\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](x)$ for $m \geq 0$ by

$$\widetilde{\mathbb{P}}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,m) = \begin{cases} (-1)^{m+1}\epsilon^{\frac{m}{2}} P_m\begin{bmatrix}\theta\\ \phi\end{bmatrix}(-x,\tau), & m \geq 1, \\ 0, & m = 0. \end{cases} \tag{63}$$

Similarly, we define the row vector $\mathbb{R}\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](x)$ for $m \geq 0$ (a twisted verion of (54)) by

$$\mathbb{R}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x;m) = \epsilon^{\frac{m}{2}} P_{m+1}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,\tau). \tag{64}$$

We define infinite twisted $\mathbb{X}_a$ vectors (analogous to those defined in [GT1]) for $a = 1, 2$, $m \geq 0$ with components

$$\mathbb{X}_1(m) = \epsilon^{-\frac{m}{2}} \sum_{u \in V} Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]} v[m]u, \tau_1) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}\overline{u}, \tau_2), \tag{65}$$

$$\mathbb{X}_2(m) = \epsilon^{-\frac{m}{2}} \sum_{u\in V} Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]}u, \tau_1) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[m]\overline{u}, \tau_2). \tag{66}$$

We note that $\sum_{m\in\mathbb{Z}} \mathbb{X}_a(m) x^{-m-1}$ is a genus two n-point function for $a = 1, 2$. We also abbreviate genus one and two contraction terms (see (42)) as follows

$$\begin{aligned}
&Z^{(2)}_{W_1,W_2}(\ldots; v[j]a_l, x_l; \ldots) \\
&= \sum_{u\in V} Z^{(1)}_{W_1}(g_1; \ldots; v[j]a_l, x_l; \ldots, \tau_1) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}\overline{u}, \tau_2),
\end{aligned}$$

where

$$\begin{aligned}
&Z^{(1)}_{W_1}(g_1; \ldots; v[j]a_l, x_l; \ldots) \\
&= Z^{(1)}_{W_1}(g_1; Y[a_1, x_1] \ldots Y[v[j]a_l, x_l] \ldots Y[a_L, x_L]u, \tau_1),
\end{aligned}$$

with a similar definition for $Z^{(2)}_{W_1,W_2}(\ldots; v[j]b_r, y_r; \ldots)$, etc. Lastly, we define O_a terms

$$\begin{aligned}
&O_1 = O_1(v, \boldsymbol{a_l, x_l}|\boldsymbol{b_r, y_r}; \tau_1, \tau_2, \epsilon) \\
&= \sum_{u\in V} \mathrm{STr}_{W_1}\left(g_1 o(v) \boldsymbol{Y(q_{x_l}^{L(0)} a_l, q_{x_l})} Y(q_0^{L(0)} u, q_0) q_1^{L(0)-\frac{c}{24}} \right) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}\overline{u}, \tau_2),
\end{aligned} \tag{67}$$

and

$$\begin{aligned}
&O_2 = O_2(v, \boldsymbol{a_l, x_l}|\boldsymbol{b_r, y_r}; \tau_1, \tau_2, \epsilon) \\
&= \sum_{u\in V} Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]}u, \tau_1) \mathrm{STr}_{W_2}\left(g_2 o(v) \boldsymbol{Y(q_{y_r}^{L(0)} b_r, q_{y_r})} Y(q_0^{L(0)} \overline{u}, q_0) q_2^{L(0)-\frac{c}{24}} \right).
\end{aligned} \tag{68}$$

where $q_0 = 1$.

4.2 A preliminary recursion formula

We define θ_a to be complex parameter in $U(1)$ such that $g_a v = \theta_a^{-1} v$ for $g_a \in \mathrm{Aut}(V)$, and $\phi_a = \exp(2\pi \mathrm{i} \lambda_a)$, where $\lambda_a = \mathrm{wt}_{W_a}[v] \bmod \mathbb{Z}$. Applying the genus one Zhu reduction formula (40) of [MTZ] to the definition of the genus two n-point function (49), we find that

$$\begin{aligned}
&Z^{(2)}_{W_1,W_2}(g_1, g_2; v, x; \boldsymbol{a_l, x_l}|\boldsymbol{b_r, y_r}) \\
&= \delta^{1,1}_{\theta_1,\phi_1} \Bigg(\sum_{u\in V} \mathrm{STr}_{W_1}\left(g_1 o(v) \boldsymbol{Y(q_{x_l}^{L(0)} a_l, q_{x_l})} Y(q_0^{L(0)} u, q_0) q_1^{L(0)-\frac{c}{24}} \right) \\
&\qquad \times Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}\overline{u}, \tau_2) \Bigg) \\
&\quad + \sum_{l=1}^{L} \sum_{j\geq 0} p(v, \boldsymbol{a_{l-1}}) \partial^{(j)}_{x_l} P_1 \begin{bmatrix} \theta_1 \\ \theta_1 \end{bmatrix} (x - x_l, \tau_1) \\
&\qquad \times \left(\sum_{u\in V} Z^{(1)}_{W_1}(g_1; \ldots; v[j]a_l, x_l; \ldots) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}\overline{u}, \tau_2) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ p_1 \sum_{m\geq 0} P_{1+m}\begin{bmatrix}\theta_1\\ \theta_1\end{bmatrix}(x,\tau_1)\\
&\quad\times\left(\sum_{u\in V} Z^{(1)}_{W_1}(g_1;\boldsymbol{Y}[\boldsymbol{a_l},\boldsymbol{x_l}]v[m]u,\tau_1)Z^{(1)}_{W_2}(g_2;\boldsymbol{Y}[\boldsymbol{b_r},\boldsymbol{y_r}]\overline{u},\tau_2)\right).
\end{aligned}
$$

This can be rewritten using the objects defined above in (64), (65) and (67) as

$$
\begin{aligned}
&Z^{(2)}_{W_1,W_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r})\\
&=\delta^{1,1}_{\theta_1,\phi_1}O_1 + p_1\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\mathbb{X}_1\\
&\quad+\sum_{l=1}^{L}\sum_{j\geq 0} p(v,\boldsymbol{a_{l-1}})\partial^{(j)}_{x_l}P_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots),
\end{aligned}
\tag{69}
$$

with $\delta^{c,d}_{a,b}=\delta_{a,c}\delta_{b,d}$, and $p(v,\boldsymbol{a_{l-1}})$ following (41). We obtain analogous results for v inserted at x on $\widehat{\mathcal{S}}_2$ in (49).

4.3 Recursion for $\mathbb{X}_a$

We assume v is quasiprimary and is of $L[0]$-weight $N\geq\frac{1}{2}$. We recall that the entries $\mathbb{X}_1(m)$ and $\mathbb{X}_2(m)$ are derived from genus two n-point functions for all $m\geq 0$. We require the left and right supertraces for $\mathbb{X}_a$, $a=1,2$ to be non-zero, for this we enforce

$$
\begin{aligned}
&p(a_1)+\ldots+p(a_L)+p(v)+p(u)=0,\\
&p(b_1)+\ldots+p(b_R)+p(\overline{u})=0.
\end{aligned}
$$

As u and $\overline{u}$ must be of the same parity, upon addition we find that

$$
p(v)+\sum_{l=1}^{L}p(a_l)+\sum_{r=1}^{R}p(b_r)=0.
$$

Multiplying across by $p(v)$ (noting that $p(v)\in\mathbb{Z}/2\mathbb{Z}$) we find

$$
p(v)+p(v)\sum_{l=1}^{L}p(a_l)+p(v)\sum_{r=1}^{R}p(b_r)=0.
$$

Hence we have

$$
(-1)^{p(v)}p_1p_2=1, \tag{70}
$$

where

$$
p_a=\begin{cases}(-1)^{p(v)[p(a_1)+\ldots+p(a_L)]}, & a=1,\\ (-1)^{p(v)[p(b_1)+\ldots+p(b_R)]}, & a=2.\end{cases}
$$

This gives the following relations

$$
\begin{aligned}
&(-1)^{p(v)p(u)}=(-1)^{p(v)}p_1=p_2,\\
&(-1)^{p(v)p(\overline{u})}=p_1.
\end{aligned}
$$

As in [GT1], we can decompose the vector $v[m]u$ over a basis $\{w\}$ for V as follows (using (36))

$$\begin{aligned} v[m]u &= \sum_{w\in V} \langle v[m]u, \overline{w}\rangle w \\ &= (-1)^{p(v)p(u)} \sum_{w\in V} \langle u, v^{\dagger}[m]\overline{w}\rangle w \\ &= p_2 \sum_{w\in V} \langle u, \xi^{2N}\epsilon^{m-K/2} v[K-m]\overline{w}\rangle w \\ &= p_2 \xi^{2N}\epsilon^{m-K/2} \sum_{w\in V} \langle u, v[K-m]\overline{w}\rangle w, \end{aligned} \tag{71}$$

where $K = 2N - 2$ and $v^{\dagger}[m]$ is the adjoint operator for $v[m]$. We refer to (71) as the genus two *adjoint relation*. Using the adjoint relation, we find that we can relate components of $\mathbb{X}_1$ to those of $\mathbb{X}_2$ and vice versa

$$\begin{aligned} \mathbb{X}_1(m) &= p_2\xi^{2N}\epsilon^{-\frac{(K-m)}{2}} \sum_{w\in V} Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]}w, \tau_1) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[K-m]\overline{w}, \tau_2) \\ &= p_2\xi^{2N}\mathbb{X}_2(K-m). \end{aligned} \tag{72}$$

Define the infinite shift matrices[1] Γ, Δ, Π indexed by $k, l \geq 0$

$$\Gamma(k,l) = \delta_{k,K-l}, \tag{73}$$
$$\Delta(k,l) = \delta_{k,l+K+1}, \tag{74}$$
$$\Pi = \Gamma^2, \tag{75}$$

i.e. Γ has a 1 where the indices k and $K - l$ agree and zeroes everywhere else, and similarly for Δ. Π is a projection matrix of the form $\left(\begin{smallmatrix} I_{K+1} & 0 \\ 0 & 0\end{smallmatrix}\right)$, where I_{K+1} is the $(K+1)$-dimensional identity matrix. These matrices obey the following identities

Lemma 4.1.

$$\Delta^T\Delta = I, \tag{76}$$
$$\Delta\Delta^T = I - \Pi, \tag{77}$$
$$\Gamma\Delta = \Delta^T\Gamma = 0, \tag{78}$$

where I is the infinite identity matrix.

We now consider $0 \leq m \leq K$. Using (72), the vectors $\mathbb{X}_1$ and $\mathbb{X}_2$ can be related as follows

$$\mathbb{X}_1(m) = p_2\xi^{2N}(\Gamma\mathbb{X}_2)(m).$$

In more general terms, we have

$$\mathbb{X}^{\Pi}_a = p_{\overline{a}}\xi^{2N}\Gamma\mathbb{X}^{\Pi}_{\overline{a}},$$

for $a = 1, 2$, where

$$\mathbb{X}^{\Pi}_a = \Pi\mathbb{X}_a, \tag{79}$$

[1] We note that Γ has the same definition as in [GT1] (albeit with an extended index range), however the definition of Δ differs from that of [GT1]. Lastly, we note that the matrix Π defined above is a $(K+1)$-dimensional projector, as opposed to the $(K-1)$-dimensional projector of [GT1].

recalling the convention that $\overline{1} = 2$ and $\overline{2} = 1$. We know that V can be decomposed into two distinct parity spaces, corresponding to parity $\overline{0}$ and $\overline{1}$ (integer and half-integer weight respectively under the Virasoro grading). This tells us that

$$p(v) = 2N \bmod 2,$$

which gives

$$\xi^{4N} = (-1)^{2N} = (-1)^{p(v)}. \tag{80}$$

We then note the consistency of this formula (using $\Pi^2 = \Pi$)

$$\mathbb{X}_a^{\Pi} = p_{\overline{a}} \xi^{2N} \Gamma \mathbb{X}_{\overline{a}}^{\Pi} = p_a p_{\overline{a}} \xi^{4N} \Gamma^2 \mathbb{X}_a^{\Pi} = \mathbb{X}_a^{\Pi},$$

by properties (70) and (80).

For $g_1 \in \mathrm{Aut}(V)$ such that $g_1 v = v$ and for $\mathrm{wt}_{W_1}[v]$ integral (i.e. when $\theta_1 = 1$ and $\phi_1 = 1$), we have that [MTZ, Prop. 6]

$$\sum_{k=1}^{n} p(v, \boldsymbol{v_{k-1}}) Z_W^{(1)}(g; v_1, y_1; \ldots; v[0]v_k, y_k; \ldots; v_n, y_n) = 0, \tag{81}$$

holds, with $p(v, \boldsymbol{v_{k-1}})$ of (41). Then the components $\mathbb{X}_1^{\Pi}(0)$, $\mathbb{X}_2^{\Pi}(K)$ can be rewritten as follows, using (81)

$$\begin{aligned}
\mathbb{X}_2^{\Pi}(K) &= p_1 \xi^{2N} \mathbb{X}_1(0) \\
&= p_1 \xi^{2N} \sum_{w \in V} Z_{W_1}^{(1)}(g_1; \boldsymbol{Y[a_l, x_l]} v[0]w, \tau_1) Z_{W_2}^{(1)}(g_2; \boldsymbol{Y[b_r, y_r]} \overline{w}, \tau_2) \\
&= -\xi^{2N} \sum_{l=1}^{L} p(v, \boldsymbol{a_{l-1}}) Z_{W_1, W_2}^{(2)}(\ldots; v[0]a_l, x_l; \ldots).
\end{aligned} \tag{82}$$

Likewise for $\mathbb{X}_2^{\Pi}(0)$, $\mathbb{X}_1^{\Pi}(K)$, when $g_1 v = v$, $\mathrm{wt}_{W_1}[v] \in \mathbb{Z}$

$$\mathbb{X}_1^{\Pi}(K) = -\xi^{2N} \sum_{r=1}^{R} p(v, \boldsymbol{b_{r-1}}) Z_{W_1, W_2}^{(2)}(\ldots; v[0]b_r, y_r; \ldots). \tag{83}$$

These expressions occur for $\mathbb{X}_a^{\Pi}(0)$, $\mathbb{X}_a^{\Pi}(K)$ when $\delta_{\theta_a, \phi_a} = 1$ and $\delta_{\theta_{\overline{a}}, \phi_{\overline{a}}}^{1,1} = 1$ respectively (this corresponds to states that are simultaneously of integral module weight and fixed under the action of the group element g_a) for $a = 1, 2$. Recall definitions (74), (75). Define the vector $\mathbb{X}_a^{\perp}$ by

$$\mathbb{X}_a^{\perp} = \Delta^T \mathbb{X}_a.$$

Using property (77), we then have that

$$\mathbb{X}_a = \mathbb{X}_a^{\Pi} + \Delta \mathbb{X}_a^{\perp}.$$

The non-trivial entries of the vector $\Delta \mathbb{X}_a^{\perp}$ occur from the index $K + 1$ onwards (recalling the indexing convention), and these entries coincide with those of $\mathbb{X}_a$ from there on, i.e.

$$\left(\Delta \mathbb{X}_a^{\perp} \right)(m) = \begin{cases} 0, & 0 \leq m \leq K, \\ \mathbb{X}_a(m - K), & m > K. \end{cases}$$

This allows us to deal with the Zhu reduction for $m > K$. We had before that (72)

$$\mathbb{X}_1(m) = \xi^{2N}\epsilon^{-\frac{K-m}{2}} \sum_{w\in V} Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]}w, \tau_1) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[K-m]\overline{w}, \tau_2).$$

To develop a full recursion relation between $\mathbb{X}_1$ and $\mathbb{X}_2$ we must examine the recursion with a negative mode in the last slot, i.e. develop a recursion formula for

$$Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[-s]\overline{w}, \tau_2),$$

with $s \geq 1$. We first note that

$$\begin{aligned} &Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}Y[v, x]Y[\overline{w}, z]\mathbb{1}, \tau_2) \\ &= p_2 Z^{(1)}_{W_2}(g_2; Y[v, x]\boldsymbol{Y[b_r, y_r]}Y[\overline{w}, z]\mathbb{1}, \tau_2), \end{aligned} \tag{84}$$

up to some multiplicative locality factors, and examine the recursion of the $(R+2)$-point function

$$Z^{(1)}_{W_2}(g_2; Y[v, x]\boldsymbol{Y[b_r, y_r]}Y[\overline{w}, z]\mathbb{1}, \tau_2).$$

We have that

$$\begin{aligned} &Z^{(1)}_{W_2}(g_2; Y[v, x]\boldsymbol{Y[b_r, y_r]}Y[\overline{w}, z]\mathbb{1}, \tau_2) \\ &= \delta^{1,1}_{\theta_2,\phi_2} \mathrm{STr}_{W_2}\left(g_2 o(v)\boldsymbol{Y(q_{y_r}^{L(0)} b_r, q_{y_r})} Y(q_z^{L(0)}\overline{w}, q_z) q_2^{L(0)-\frac{c}{24}}\right) \\ &\quad + \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}})\partial^{(j)}_{y_r} P_1\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(x-y_r, \tau_2) Z^{(1)}_{W_2}(g_2; \ldots; v[j]b_r, y_r; \ldots) \\ &\quad + p_2 \sum_{j\geq 0} P_{1+j}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(x-z, \tau_2) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}Y[v[j]\overline{w}, z]\mathbb{1}, \tau_2). \end{aligned}$$

The coefficient of z^0 gives

$$\begin{aligned} &Z^{(1)}_{W_2}(g_2; Y[v, x]\boldsymbol{Y[b_r, y_r]}\overline{w}, \tau_2) \\ &= \delta^{1,1}_{\theta_2,\phi_2} \mathrm{STr}_{W_2}\left(g_2 o(v)\boldsymbol{Y(q_{y_r}^{L(0)} b_r, q_{y_r})} Y(q_0^{L(0)}\overline{w}, q_0) q_2^{L(0)-\frac{c}{24}}\right) \\ &\quad + \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}})\partial^{(j)}_{y_r} P_1\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(x-y_r, \tau_2) Z^{(1)}_{W_2}(g_2; \ldots; v[j]b_r, y_r; \ldots) \\ &\quad + p_2 \sum_{j\geq 0} P_{1+j}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(x, \tau_2) Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[j]\overline{w}, \tau_2). \end{aligned} \tag{85}$$

We also have that (using the linearity of the n-point trace function)

$$Z^{(1)}_{W_2}(g_2; Y[v, x]\boldsymbol{Y[b_r, y_r]}\overline{w}, \tau_2) = \sum_{s\in\mathbb{Z}} Z^{(1)}_{W_2}(g_2; v[-s]\boldsymbol{Y[b_r, y_r]}\overline{w}, \tau_2)x^{s-1}.$$

Extracting, then, the coefficient of x^{s-1} in the left hand side of equation (85), we find (we are interested in the case $s \geq 1$)

$$Z^{(1)}_{W_2}(g_2; Y[v, x]\boldsymbol{Y[b_r, y_r]}\overline{w}, \tau_2) = \sum_{s\in\mathbb{Z}} Z^{(1)}_{W_2}(g_2; v[-s]\boldsymbol{Y[b_r, y_r]}\overline{w}, \tau_2)x^{s-1}.$$

Repeating the idea for the right hand side of equation (85), we obtain

$$
\begin{aligned}
&Z^{(1)}_{W_2}(g_2; v[-s]\boldsymbol{Y[b_r, y_r]}\overline{w}, \tau_2)\\
&= \delta_{s,1}\delta^{1,1}_{\theta_2,\phi_2}\mathrm{STr}_{W_2}\left(g_2 o(v)\boldsymbol{Y(q_{y_r}^{L(0)} b_r, q_{y_r})}Y(q_0^{L(0)}\overline{w}, q_0)q_2^{L(0)-\frac{c}{24}}\right)\\
&\quad + \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}})(-1)^{s+1}\binom{s+j-1}{j} P_{s+j}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(-y_r, \tau_2)\\
&\quad\times Z^{(1)}_{W_2}(g_2; \ldots; v[j]b_r, y_r; \ldots)\\
&\quad + p_2\sum_{j\geq 0}(-1)^{j+1}\binom{s+j-1}{j} E_{s+j}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(\tau_2)Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[j]\overline{w}, \tau_2),
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
&\delta_{s,1}\delta^{1,1}_{\theta_2,\phi_2}\mathrm{STr}_{W_2}\left(g_2 o(v)\boldsymbol{Y(q_{y_r}^{L(0)} b_r, q_{y_r})}Y(q_0^{L(0)}\overline{w}, q_0)q_2^{L(0)-\frac{c}{24}}\right)\\
&+ \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}})\epsilon^{-\frac{s}{2}}\partial^{(j)}_{y_r}\widetilde{\mathbb{P}}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(y_r, s)Z^{(1)}_{W_2}(g_2; \ldots; v[j]b_r, y_r; \ldots)\\
&+ p_2\sum_{j\geq 0}(-1)^{j+1}\binom{s+j-1}{j} E_{s+j}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(\tau_2)Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[j]\overline{w}, \tau_2).
\end{aligned}
$$

Up to locality factors this implies

$$
\begin{aligned}
&Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[-s]\overline{w}, \tau_2)\\
&= p_2\Bigg(\delta_{s,1}\delta^{1,1}_{\theta_2,\phi_2}\mathrm{STr}_{W_2}\left(g_2 o(v)\boldsymbol{Y(q_{y_r}^{L(0)} b_r, q_{y_r})}Y(q_0^{L(0)}\overline{w}, q_0)q_2^{L(0)-\frac{c}{24}}\right)\\
&\quad + \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}})\epsilon^{-\frac{s}{2}}\partial^{(j)}_{y_r}\widetilde{\mathbb{P}}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(y_r, s)Z^{(1)}_{W_2}(g_2; \ldots; v[j]b_r, y_r; \ldots)\\
&\quad + p_2\sum_{j\geq 0}(-1)^{j+1}\binom{s+j-1}{j} E_{s+j}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(\tau_2)Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[j]\overline{w}, \tau_2)\Bigg).
\end{aligned}
$$

Upon multiplication by $p_2\xi^{2N}\epsilon^{\frac{s}{2}}Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]}w, \tau_1)$ and summing over the basis $\{w\}$, we find that

$$
\begin{aligned}
&p_2\xi^{2N}\epsilon^{\frac{s}{2}}\sum_{w\in V} Z^{(1)}_{W_1}(g_1; \boldsymbol{Y[a_l, x_l]}w, \tau_1)Z^{(1)}_{W_2}(g_2; \boldsymbol{Y[b_r, y_r]}v[-s]\overline{w}, \tau_2)\\
&= \xi^{2N}\left(\mathbb{O}_2 + \mathbb{G}_2\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix} + p_2\left(\Lambda_2\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}\mathbb{X}_2\right)\right)(s),
\end{aligned}
$$

where $\Lambda_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}$ is as in (62) and $\mathbb{O}_a, \mathbb{G}_a = \mathbb{G}_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}$ are infinite vectors defined for $m \geq 0$ by

$$
\mathbb{O}_a(m) = \delta_{m,1}\delta^{1,1}_{\theta_a,\phi_a}\epsilon^{\frac{1}{2}}O_a, \tag{86}
$$

$$
\mathbb{G}_1(m) = \sum_{l=1}^{L}\sum_{j\geq 0} p(v, \boldsymbol{a_{l-1}})\partial^{(j)}_{x_l}\widetilde{\mathbb{P}}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x_l; m)Z^{(2)}_{W_1,W_2}(\ldots; v[j]a_l, x_l; \ldots), \tag{87}
$$

$$
\mathbb{G}_2(m) = \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}})\partial^{(j)}_{y_r}\widetilde{\mathbb{P}}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(y_r; m)Z^{(2)}_{W_1,W_2}(\ldots; v[j]b_r, y_r; \ldots), \tag{88}
$$

for O_1, O_2 of (67), (68). These objects also find analogues in [GT1]. Then setting $s = m - K$ and suppressing the twist notation for clarity, we find that

$$\begin{aligned}\mathbb{X}_1(m) &= \xi^{2N}\left(\mathbb{O}_2 + \mathbb{G}_2 + p_2\Lambda_2\mathbb{X}_2\right)(m-K)\\ &= \xi^{2N}\left(\Delta\left(\mathbb{O}_2 + \mathbb{G}_2 + p_2\Lambda_2\mathbb{X}_2\right)\right)(m),\end{aligned}$$

with a similar formula for the entries of $\mathbb{X}_2$. Then for $a = 1, 2$ we have

$$\mathbb{X}_a(m) = \xi^{2N}\left(\Delta\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}\right)\right)(m),$$

for $m \geq K+1$. Then

$$\Delta\mathbb{X}_a^{\perp} = \xi^{2N}\left(\Delta\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}\right)\right).$$

Recalling (76), multiply across by Δ^T to find

$$\begin{aligned}\mathbb{X}_a^{\perp} &= \xi^{2N}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}\right)\\ &= \xi^{2N}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}^{\Pi}\right) + \xi^{2N}p_{\overline{a}}\widetilde{\Lambda}_{\overline{a}}\mathbb{X}_{\overline{a}}^{\perp}\\ &= \xi^{2N}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}^{\Pi}\right)\\ &\quad + \xi^{2N}p_{\overline{a}}\widetilde{\Lambda}_{\overline{a}}\left(\xi^{2N}\left(\mathbb{O}_a + \mathbb{G}_a + p_a\Lambda_a\mathbb{X}_a\right)\right)\\ &= \xi^{2N}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}^{\Pi}\right)\\ &\quad + \xi^{4N}p_{\overline{a}}\widetilde{\Lambda}_{\overline{a}}\left(\mathbb{O}_a + \mathbb{G}_a + p_a\Lambda_{\overline{a}}\mathbb{X}_a^{\Pi}\right)\\ &\quad + \xi^{4N}p_a p_{\overline{a}}\widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\mathbb{X}_a^{\perp},\end{aligned}$$

where $\widetilde{\Lambda}_a = \Lambda_a\Delta$. Recalling that $(-1)^{p(v)} = \xi^{4N}$, we find

$$\begin{aligned}\mathbb{X}_a^{\perp} &= \xi^{2N}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}^{\Pi}\right)\\ &\quad + \underbrace{\xi^{4N}p_{\overline{a}}}_{=p_a}\widetilde{\Lambda}_a\left(\mathbb{O}_a + \mathbb{G}_a + p_a\Lambda_a\mathbb{X}_a^{\Pi}\right)\\ &\quad + \underbrace{(-1)^{p(v)}p_a p_{\overline{a}}}_{=1}\widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\mathbb{X}_a^{\perp},\end{aligned}$$

using relations (70) and (80). Define a formal inverse by

$$\left(I - \widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\right)^{-1} = \sum_{n\geq 0}\left(\widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\right)^n.$$

Then

$$\begin{aligned}\left(I - \widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\right)\mathbb{X}_a^{\perp} &= \xi^{2N}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}^{\Pi}\right)\\ &\quad + p_a\widetilde{\Lambda}_{\overline{a}}\left(\mathbb{O}_a + \mathbb{G}_a + p_a\Lambda_a\mathbb{X}_a^{\Pi}\right).\end{aligned}$$

Then we obtain the following recursion formula for $\mathbb{X}_a^{\perp}$

Proposition 4.1. *Let v be a quasiprimary vector with* $\mathrm{wt}[v] = N$. *Then* $\mathbb{X}_a = \mathbb{X}_a^{\Pi} + \Delta\mathbb{X}_a^{\perp}$, *where*

$$\begin{aligned}\mathbb{X}_a^{\perp} &= \xi^{2N}\left(I - \widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\right)^{-1}\left(\mathbb{O}_{\overline{a}} + \mathbb{G}_{\overline{a}} + p_{\overline{a}}\Lambda_{\overline{a}}\mathbb{X}_{\overline{a}}^{\Pi}\right)\\ &\quad + p_a\left(I - \widetilde{\Lambda}_{\overline{a}}\widetilde{\Lambda}_a\right)^{-1}\widetilde{\Lambda}_{\overline{a}}\left(\mathbb{O}_a + \mathbb{G}_a + p_a\Lambda_a\mathbb{X}_a^{\Pi}\right).\end{aligned} \tag{89}$$

4.4 Zhu reduction formula

We recall formula (69)

$$\begin{aligned}
&Z^{(2)}_{W_1,W_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r},\tau_1,\tau_2,\epsilon)\\
&=\delta^{1,1}_{\theta_1,\phi_1}O_1+p_1\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\mathbb{X}_1\\
&\quad+\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})\partial^{(j)}_{x_l}P_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots).
\end{aligned}\tag{90}$$

We also define an infinite row vector for $x\in\widehat{\mathcal{S}}_a$

$${}^{N}\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)=\mathbb{R}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x)\Delta\left(I-\widetilde{\Lambda}_{\overline{a}}\begin{bmatrix}\theta_{\overline{a}}\\ \phi_{\overline{a}}\end{bmatrix}\widetilde{\Lambda}_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}\right)^{-1},$$

where the pre-superscript N is used to reinforce the dependence on N through Δ. The $\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right]$ notation reflects dependence on the θ_a parameters of the VOSA, and the respective module weights, which give ϕ_a for $a=1,2$. Substituting the expression for $\mathbb{X}_1$ from Proposition 4.1 (noting that $x\in\widehat{\mathcal{S}}_1$), we find that

$$\begin{aligned}
&Z^{(2)}_{W_1,W_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r})\\
&=\delta^{1,1}_{\theta_1,\phi_1}O_1+p_1\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\mathbb{X}_1^{\Pi}\\
&\quad+p_1\xi^{2N}\underbrace{\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\Delta\left(I-\widetilde{\Lambda}_2\widetilde{\Lambda}_1\right)^{-1}}_{={}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)}\left(\mathbb{O}_2+\mathbb{G}_2+p_2\Lambda_2\mathbb{X}_2^{\Pi}\right)\\
&\quad+\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\Delta\left(I-\widetilde{\Lambda}_2\widetilde{\Lambda}_1\right)^{-1}\widetilde{\Lambda}_2\left(\mathbb{O}_1+\mathbb{G}_1+p_1\Lambda_1\mathbb{X}_1^{\Pi}\right)\\
&\quad+\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})P_{1+j}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots).
\end{aligned}$$

Then the Zhu reduction formula takes the form

$$\begin{aligned}
&Z^{(2)}_{W_1,W_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r})\\
&=\delta_{\theta_1,\phi_1}O_1+p_1\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x)\mathbb{X}_1^{\Pi}\\
&\quad+p_1\xi^{2N}\cdot{}^{N}\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\left(\mathbb{O}_2+\mathbb{G}_2+p_2\Lambda_2\mathbb{X}_2^{\Pi}\right)\\
&\quad+{}^{N}\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\left(\mathbb{O}_1+\mathbb{G}_1+p_1\Lambda_1\mathbb{X}_1^{\Pi}\right)\\
&\quad+\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})\partial^{(j)}_{x_l}P_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)Z^{(2)}_{W_1,W_2}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(\ldots;v[j]a_l,x_l;\ldots).
\end{aligned}\tag{91}$$

We define expressions ${}^N\mathcal{F}_a \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)$, ${}^N\mathcal{F}^\Pi \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)$ by

$$
{}^N\mathcal{F}_a \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x) = \begin{cases} 1+\epsilon^{1/2}\left({}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_{\overline{a}}\begin{bmatrix}\theta_{\overline{a}}\\ \phi_{\overline{a}}\end{bmatrix}\right)(1), & x\in\widehat{\mathcal{S}}_a, \\ \xi^{2N}\epsilon^{1/2}\left({}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\right)(1), & x\in\widehat{\mathcal{S}}_{\overline{a}}, \end{cases}
$$

and

$$
{}^N\mathcal{F}^\Pi \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x) = \left(\mathbb{R}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x) + {}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\left(\widetilde{\Lambda}_{\overline{a}}\Lambda_a + \Lambda_{\overline{a}}\Gamma\right)\right)\Pi,
$$

for $x\in\widehat{\mathcal{S}}_a$. We now examine the contribution of each of these terms to the Zhu reduction formula. The O_1 coefficient, then, is

$$
\begin{aligned}
&\delta^{1,1}_{\theta_1,\phi_1}O_1 + {}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\mathbb{O}_1 \\
&= \delta^{1,1}_{\theta_1,\phi_1}\left(1+\epsilon^{1/2}\left({}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}\right)(1)\right)O_1 \\
&= \delta^{1,1}_{\theta_1,\phi_1}\ {}^N\mathcal{F}_1\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)O_1,
\end{aligned}
$$

using the definition of $\mathbb{O}_1$ in (86). Likewise, the coefficient of the O_2 term is

$$
\begin{aligned}
&p_1\xi^{2N}\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\mathbb{O}_2 \\
&= p_1\delta^{1,1}_{\theta_2,\phi_2}\left(\xi^{2N}\epsilon^{\frac{1}{2}}\left({}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\right)(1)\right)O_2 \\
&= p_1\delta^{1,1}_{\theta_2,\phi_2}\ {}^N\mathcal{F}_2\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)O_2.
\end{aligned}
$$

The contributions from the projective terms are as follows

$$
\left(p_1\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x) + p_1\,{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\Lambda_1\right)\mathbb{X}_1^\Pi + p_1p_2\xi^{2N}\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\Lambda_2\mathbb{X}_2^\Pi.
$$

But we have that

$$
\begin{aligned}
&p_1p_2\xi^{2N}\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\Lambda_2\mathbb{X}_2^\Pi = p_1p_2\xi^{2N}\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\Lambda_2\Pi\mathbb{X}_2^\Pi \\
&= p_1p_2\xi^{2N}\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\Lambda_2\Gamma^2\mathbb{X}_2^\Pi = p_1\xi^{2N}\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\Lambda_2\Gamma\left(\xi^{2N}p_2\Gamma\mathbb{X}_2^\Pi\right) \\
&= p_1\cdot{}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\Lambda_2\Gamma\mathbb{X}_1^\Pi.
\end{aligned}
$$

Then the total contribution of $\mathbb{X}_1^\Pi$ terms is

$$
\begin{aligned}
&p_1\left(\mathbb{R}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x) + {}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\left(\widetilde{\Lambda}_2\Lambda_1 + \Lambda_2\Gamma\right)\right)\mathbb{X}_1^\Pi \\
&= p_1\cdot{}^N\mathcal{F}^\Pi\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\mathbb{X}_1^\Pi,
\end{aligned}
$$

using $\Pi^2 = \Pi$.

We now examine the contributions from the contraction terms for the modes of $Y[v, x]$ on the states inserted on the left and right torus respectively. Extracting the $\mathbb{G}_1$ terms using (87) we find

$$
\begin{aligned}
&{}^{N}\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\mathbb{G}_1\\
&=\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})\left({}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\widetilde{\Lambda}_2\partial_{x_l}^{(j)}\widetilde{\mathbb{P}}\left[\begin{smallmatrix}\theta_1\\ \phi_1\end{smallmatrix}\right](x_l)\right)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots).
\end{aligned}
$$

The other contribution for the left torus insertions is

$$
\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})\partial_{x_l}^{(j)}P_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots).
$$

Totalling the terms detailed above, we find a genus two analogue of the genus one Weierstrass functions. The coefficients of the left torus contraction terms for $j\geq 0$, $x, y \in \widehat{\mathcal{S}}_1$ are

$$
\partial_{x_l}^{(j)}\left(P_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-x_l,\tau_1)+{}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_2\widetilde{\mathbb{P}}_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x_l)\right).
$$

The contribution for the right torus insertions is (using (88))

$$
\begin{aligned}
&p_1\xi^{2N}\left({}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\mathbb{G}_2\right)\\
&=p_1\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{b_{r-1}})\left(\xi^{2N}\cdot{}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\partial_{y_r}^{(j)}\widetilde{\mathbb{P}}_1(y_r)\right)Z^{(2)}_{W_1,W_2}(\ldots;v[j]b_r,y_r;\ldots).
\end{aligned}
$$

Following this, we define genus two twisted Weierstrass functions

$$
{}^{N}\widehat{\mathcal{P}}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y)=\begin{cases}P_1\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x-y,\tau_a)+{}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\widetilde{\Lambda}_{\overline{a}}\widetilde{\mathbb{P}}_1\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(y,\tau_a), & x,y\in\widehat{\mathcal{S}}_a,\\[2ex] \xi^{2N}\cdot{}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\widetilde{\mathbb{P}}_1\begin{bmatrix}\theta_{\overline{a}}\\ \phi_{\overline{a}}\end{bmatrix}(y,\tau_{\overline{a}}), & x\in\widehat{\mathcal{S}}_a, y\in\widehat{\mathcal{S}}_{\overline{a}},\end{cases}
$$

for $a = 1, 2$. This finally leads to a genus two Zhu reduction formula for half-integrally graded VOSAs for quasiprimary states

Theorem 4.1. *For a quasiprimary state v with $\mathrm{wt}[v] = N \geq \frac{1}{2}$ inserted at $x \in \widehat{\mathcal{S}}_1$, states a_l, for $l = 1, 2, \ldots, L$ inserted at points $x_1, x_2, \ldots, x_L \in \widehat{\mathcal{S}}_1$ respectively, and states $b_1, b_2, \ldots, b_R$ inserted at points $y_1, y_2, \ldots, y_L \in \widehat{\mathcal{S}}_2$ respectively, the n-point function (for $n = L + R + 1$) satisfies the recursive identity*

$$
\begin{aligned}
&Z^{(2)}_{W_1,W_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r};\tau_1,\tau_2,\epsilon)\\
&=\delta^{1,1}_{\theta_1,\phi_1}\,{}^{N}\mathcal{F}_1\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)O_1+p_1\delta^{1,1}_{\theta_2,\phi_2}\,{}^{N}\mathcal{F}_2\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)O_2+p_1\,{}^{N}\mathcal{F}^{\Pi}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\mathbb{X}^{\Pi}_1\\
&\quad+\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})\partial_{x_l}^{(j)}\,{}^{N}\widehat{\mathcal{P}}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,x_l)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots)\\
&\quad+p_1\sum_{r=1}^{R}\sum_{j\geq 0}p(v,\boldsymbol{b_{r-1}})\partial_{y_r}^{(j)}\,{}^{N}\widehat{\mathcal{P}}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y_r)Z^{(2)}_{W_1,W_2}(\ldots;v[j]b_r,y_r;\ldots),\qquad(92)
\end{aligned}
$$

with a similar formula for $x \in \widehat{\mathcal{S}}_2$.

4.5 Zhu reduction formula II

In the case that $(\theta_a,\phi_a)=(1,1)$ for $a=1,2$, then the identities (83) and (82) hold respectively. Then the terms ${}^N\mathcal{F}^{\Pi}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x;0)$ and ${}^N\mathcal{F}^{\Pi}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x;K)$ can be absorbed into the coefficients of the contraction terms $Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots)$ and $Z^{(2)}_{W_1,W_2}(\ldots;v[j]b_r,y_r;\ldots)$ of (92). We also note that the O_a terms contribute for $\delta_{\theta_a,\phi_a}=1$. Firstly, we define the infinite column vector $\mathbb{P}\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](x)$ with entries given by

$$\mathbb{P}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x;m)=\begin{cases}\widetilde{\mathbb{P}}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x;m)+(-1)^m\delta^{1,1}_{\theta,\phi}\epsilon^{\frac{m}{2}}E_m\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau), & m\geq 1,\\ 0, & m=0,\end{cases}\tag{93}$$

for $m,j\geq 0$, $\tau\in\mathbb{H}$ and $\theta,\phi\in U(1)$. We note that

$$\partial_x^{(j)}\mathbb{P}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x)=\partial_x^{(j)}\widetilde{\mathbb{P}}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x),\tag{94}$$

for $j\geq 1$. Following this definition, we define modified genus two twisted Weierstrass functions ${}^N\mathcal{P}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x,y)={}^N\mathcal{P}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x,y,\tau_1,\tau_2,\epsilon)$ as follows

$$\begin{aligned}&{}^N\mathcal{P}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y)\\ &=\begin{cases}{}^N\widehat{\mathcal{P}}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y)-\delta^{1,1}_{\theta_a,\phi_a}\left(P_1\left[\begin{smallmatrix}\theta_a\\ \phi_a\end{smallmatrix}\right](x,\tau_a)-\pi_N\left({}^N\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\Lambda_{\overline{a}}\right)(K)\right),\\ x,y\in\widehat{\mathcal{S}}_a,\\ {}^N\widehat{\mathcal{P}}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y)-\xi^{2N}\delta^{1,1}_{\theta_a,\phi_a}\pi_N\left(\epsilon^{K/2}P_{K+1}\left[\begin{smallmatrix}\theta_a\\ \phi_a\end{smallmatrix}\right](x,\tau_a)-\left({}^N\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\widetilde{\Lambda}_{\overline{a}}\Lambda_a\right)(K)\right),\\ x\in\widehat{\mathcal{S}}_a,y\in\widehat{\mathcal{S}}_{\overline{a}},\end{cases}\end{aligned}$$

for $a=1,2$, and for $j\geq 1$, and where $\pi_N=1-\delta_{N,1}$. Note that

$$\partial_y^{(j)}\,{}^N\mathcal{P}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y)=\partial_y^{(j)}\,{}^N\widehat{\mathcal{P}}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y),$$

for $j\geq 1$, noting (94). These functions differ by additive y-independent terms for $j=0$, analogous to the genus one property

$$P_1\begin{bmatrix}1\\ 1\end{bmatrix}(x,\tau)=\frac{1}{2}+P_1(x,\tau),$$

with $P_k\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right](x,\tau)=P_k(x,\tau)$ for $k>1$.

In the case where $g_1v=v$ and $\mathrm{wt}_{W_1}[v]\in\mathbb{Z}$, i.e. $(\theta_1,\phi_1)=(1,1)$, we have that the preparatory lemma (83) applies. Extracting the relevant terms, we find

$$\begin{aligned}&{}^N\mathbb{Q}(x)\widetilde{\Lambda}_2\mathbb{G}_1+p_1\left({}^N\mathbb{Q}(x)\widetilde{\Lambda}_2\Lambda_1\right)(0)\mathbb{X}_1^{\Pi}(0)\\ &=\sum_{m\geq 0}\left({}^N\mathbb{Q}(x)\widetilde{\Lambda}_2\right)(m)\left(\mathbb{G}_1(m)+p_1\Lambda_1(m,0)\mathbb{X}_1^{\Pi}(0)\right)\end{aligned}$$

$$= \sum_{l=1}^{L}\sum_{j\geq 0} p(v,\boldsymbol{a_{l-1}})\left({}^{N}\mathbb{Q}(x)\widetilde{\Lambda}_2\partial_{x_l}^{(j)}\mathbb{P}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x_l,\tau_1)\right)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots),$$

using (62), (81), and (93). Likewise, when $(\theta_1,\phi_1)=(1,1)$, the $\mathbb{R}\left[\begin{smallmatrix}\theta_1\\ \phi_1\end{smallmatrix}\right](x)$ multiplier contributes as follows

$$\begin{aligned}
&p_1\mathbb{R}\left[\begin{smallmatrix}\theta_1\\ \phi_1\end{smallmatrix}\right](x;0)\mathbb{X}_1^{\Pi}(0)\\
&= p_1 P_1\left[\begin{smallmatrix}\theta_1\\ \phi_1\end{smallmatrix}\right](x,\tau_1)\sum_{u\in V} Z^{(1)}_{W_1}(g_1;\boldsymbol{Y[a_l,x_l]}v[0]u,\tau_1)Z^{(1)}_{W_2}(g_2;\boldsymbol{Y[b_r,y_r]}\overline{u},\tau_2)\\
&= -P_1\left[\begin{smallmatrix}\theta_1\\ \phi_1\end{smallmatrix}\right](x,\tau_1)\sum_{l=1}^{L}p(v,\boldsymbol{a_{l-1}})Z^{(1)}_{W_1}(g_1;\ldots;v[0]a_l,x_l;\ldots),
\end{aligned}$$

for $(\theta_1,\phi_1)=(1,1)$, using (81). Similarly

$$\begin{aligned}
&p_1p_2\xi^{2N}\cdot\left({}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\Lambda_2\right)(K)\mathbb{X}_2^{\Pi}(K)\\
&=(p_1p_2\xi^{2N})\left({}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\Lambda_2\right)(K)\left(-\xi^{2N}\sum_{l=1}^{L}p(v,\boldsymbol{a_{l-1}})Z^{(2)}_{W_1,W_2}(\ldots;v[0]a_l,x_l;\ldots)\right)\\
&=-\left({}^{N}\mathbb{Q}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x)\Lambda_2\right)(K)\sum_{l=1}^{L}p(v,\boldsymbol{a_{l-1}})Z^{(2)}_{W_1,W_2}(\ldots;v[0]a_l,x_l;\ldots),
\end{aligned}$$

using property (70). Then the total left torus contraction terms are as follows

$$\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{a_{l-1}})\partial_{x_l}^{(j)}\,{}^{N}\mathcal{P}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,x_l)Z^{(2)}_{W_1,W_2}(\ldots;v[j]a_l,x_l;\ldots).$$

Similarly, if we have that g_2 fixes v and $\mathrm{wt}_{W_2}[v]\in\mathbb{Z}$, i.e. $(\theta_2,\phi_2)=(1,1)$, then some components of the $\mathbb{X}_a^{\Pi}$ vectors contribute to the right torus contraction terms. The first summand on the right torus is the following

$$\begin{aligned}
&p_1\xi^{2N}\left({}^{N}\mathbb{Q}(x)\mathbb{G}_2+p_2\left({}^{N}\mathbb{Q}(x)\Lambda_2\right)(0)\mathbb{X}_2^{\Pi}(0)\right)\\
&=p_1\xi^{2N}\sum_{m\geq 0}{}^{N}\mathbb{Q}(x;m)\left(\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{b_{r-1}})\left(\partial_{y_r}^{(j)}\widetilde{\mathbb{P}}\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(y_r,m)-\delta_{j,0}\delta^{1,1}_{\theta_2,\phi_2}\Lambda_2(m,0)\right)\right)\\
&\quad\times Z^{(2)}_{W_1,W_2}(\ldots;v[j]b_r,y_r;\ldots)\\
&=p_1\xi^{2N}\sum_{l=1}^{L}\sum_{j\geq 0}p(v,\boldsymbol{b_{r-1}})\left({}^{N}\mathbb{Q}(x)\partial_{y_r}^{(j)}\mathbb{P}(y_r)\right)Z^{(2)}_{W_1,W_2}(\ldots;v[j]b_r,y_r;\ldots),
\end{aligned}$$

using (62) and (93). The $\mathbb{X}_1(K)$ contribution to the right torus sum for $(\theta_2,\phi_2)=(1,1)$ is

$$\begin{aligned}
&p_1\mathbb{R}_1(x;K)\mathbb{X}_1^{\Pi}(K)\\
&=-p_1\xi^{2N}\epsilon^{K/2}P_{K+1}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x,\tau_1)\sum_{r=1}^{R}p(v,\boldsymbol{b_{r-1}})Z^{(2)}_{W_1,W_2}(\ldots;v[0]b_r,y_r;\ldots).
\end{aligned}$$

We also have

$$p_1\left({}^{N}\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\Lambda_1\right)(K)\mathbb{X}_1^{\Pi}(K)$$

$$= -p_1\xi^{2N}\left({}^N\mathbb{Q}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\widetilde{\Lambda}_2\Lambda_1\right)(K)\sum_{r=1}^{R} p(v,\boldsymbol{b_{r-1}})Z^{(2)}_{W_1,W_2}(\ldots;v[0]b_r,y_r;\ldots),$$

under the same conditions. Then similarly to the left torus terms, the right torus contraction terms are

$$p_1\sum_{r=1}^{R}\sum_{j\geq 0} p(v,\boldsymbol{b_{r-1}})\partial^{(j)}_{y_r}\,{}^N\mathcal{P}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y_r)Z^{(2)}_{W_1,W_2}(\ldots;v[0]b_r,y_r;\ldots).$$

We aim to write the final Zhu reduction formula so that it has a structure analogous with that of (56). We do this by writing the ${}^N\mathcal{F}_a(x)$, ${}^N\mathcal{F}^{\Pi}(x)$ terms in a more independent way.

We have four main cases regarding the values of (θ_a,ϕ_a) for $a=1,2$. We have

$$\begin{aligned}&(\theta_a,\phi_a)=(1,1)\text{ for }a=1,2, &(95)\\ &(\theta_a,\phi_a)\neq(1,1)\text{ for }a=1,2,\\ &(\theta_1,\phi_1)=(1,1),(\theta_2,\phi_2)\neq(1,1),\\ &(\theta_1,\phi_1)\neq(1,1),(\theta_2,\phi_2)=(1,1).\end{aligned}$$

In the case of (95), we obtain the result (56). We now present a Zhu reduction for quasiprimary states which encapsulates these various possibilities. Define the infinite projection matrix Π^{δ} by

$$\Pi^{\delta}=\pi_{N,\frac{1}{2}}\left(\Pi-\delta^{1,1}_{\theta_1,\phi_1}E_{00}-\delta^{1,1}_{\theta_2,\phi_2}E_{KK}\right), \tag{96}$$

where $\pi_{N,\frac{1}{2}}=1-\delta_{N,\frac{1}{2}}$ and E_{ij} are infinite elementary matrices with entries given by

$$E_{ij}(m,n)=\delta^{i,j}_{m,n}.$$

Then Π^{δ} is either a $(K-1)$-, K- or $(K+1)$-dimensional projector, depending on the values of θ_a and ϕ_a. This matrix catalogues whether or not the first and last non-trivial terms of $\mathbb{X}^{\Pi}_a$ contribute to the sum via the contraction terms outlined above, or through the $\mathbb{X}^{\Pi}_a$ terms with $v[0],v[K]$ acting in the last slot. Recalling that the zeroth and K-th terms of $\mathbb{X}^{\Pi}_a$ may or may not appear in this Zhu reduction term, we see that the relevant projection matrix takes the form of Π^{δ}. The projective term contribution is then

$$p_1\cdot\,{}^N\mathcal{F}^{\Pi}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x)\mathbb{X}^{\delta}_1,$$

where $\mathbb{X}^{\delta}_a=\Pi^{\delta}\mathbb{X}_a$. Then we can write the Zhu reduction formula in the most general form for the four possible cases as follows

Theorem 4.2 (Quasiprimary Genus Two Zhu reduction for VOSAs)**.** *For a quasiprimary state v with $\mathrm{wt}[v]=N\geq\frac{1}{2}$ inserted at $x\in\widehat{\mathcal{S}}_1$, states a_l, for $l=1,2,\ldots,L$ inserted at points $x_1,x_2,\ldots,x_L\in\widehat{\mathcal{S}}_1$ respectively, and states $b_1,b_2,\ldots,b_R$ inserted at points $y_1,y_2,\ldots,y_L\in\widehat{\mathcal{S}}_2$ respectively, the n-point function (for $n=L+R+1$) satisfies the recursive identity*

$$Z^{(2)}_{W_1,W_2}(g_1,g_2;v,x;\boldsymbol{a_l},\boldsymbol{x_l}|\boldsymbol{b_r},\boldsymbol{y_r};\tau_1,\tau_2,\epsilon)$$

$$
\begin{aligned}
&= \delta^{1,1}_{\theta_1,\phi_1}\ {}^{N}\mathcal{F}_1 \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x) O_1 + p_1 \delta^{1,1}_{\theta_2,\phi_2}\ {}^{N}\mathcal{F}_2 \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x) O_2 + p_1\ {}^{N}\mathcal{F}^{\Pi} \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x) \mathbb{X}^{\delta}_1 \\
&\quad + \sum_{l=1}^{L}\sum_{j\geq 0} p(v, \boldsymbol{a_{l-1}}) \partial^{(j)}_{x_l}\ {}^{N}\mathcal{P} \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x, x_l) Z^{(2)}_{W_1,W_2}(\ldots; v[j]a_l, x_l; \ldots) \\
&\quad + p_1 \sum_{r=1}^{R}\sum_{j\geq 0} p(v, \boldsymbol{b_{r-1}}) \partial^{(j)}_{y_r}\ {}^{N}\mathcal{P} \begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x, y_r) Z^{(2)}_{W_1,W_2}(\ldots; v[j]b_r, y_r; \ldots). \qquad (97)
\end{aligned}
$$

Remark 4.1. We do not prove existence and convergence of ${}^{N}\mathcal{F}_a(x)$, ${}^{N}\mathcal{F}^{\Pi}(x)$ and ${}^{N}\mathcal{P}(x, y)$ here, but intend to do from a general geometric construction in future work. We aim to do this using the approach of Bers [Be], such as is done in [GT2], but extended to half-integer weights and periodicities, which we conjecture involves a twisted version of the Green's function seen in [McIT]. Convergence in the case $N = \frac{1}{2}$ is discussed in the next section. We intend to complete this work in a future paper containing the above results and some geometric analysis and interpretations.

5 Weight-$\frac{1}{2}$ Zhu reduction: The Rank Two Free Fermion VOSA

We will now examine how this formula compares to known results for VOSAs. We first recall the genus one and two Szegő kernel as in [TZ1]. For $(\theta,\phi)\neq(1,1)$ the genus one Szegő kernel is

$$S^{(1)}(x,y)=S^{(1)}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,y)=P_1\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x-y,\tau)dx^{\frac{1}{2}}dy^{\frac{1}{2}}.$$

The genus two Szegő kernel is given by (26). We will often abbreviate $h_a=h_a\left[\begin{smallmatrix}\theta_a\\ \phi_a\end{smallmatrix}\right]$, $\overline{h}_a=\overline{h}_a\left[\begin{smallmatrix}\theta_a\\ \phi_a\end{smallmatrix}\right]$, $F_a=F_a\left[\begin{smallmatrix}\theta_a\\ \phi_a\end{smallmatrix}\right]$ etc. in this section for clarity.

We will now discuss the 2-point function for the rank-two Free Fermion VOSA V. Here we consider the untwisted case (i.e. $\theta_1=\theta_2=1$). Both ψ^+ and ψ^- each have conformal weight $\frac{1}{2}$ (giving $\phi_1=\phi_2=-1$). We first note that for $j\geq 0$

$$\begin{aligned}\psi^+[j]\psi^- &= \psi^+[j]\psi^-[-1]\mathbb{1}\\ &= \psi^-[-1]\psi^+[j]\mathbb{1}+\left[\psi^+[j],\psi^-[-1]\right]\mathbb{1}\\ &= \delta_{j0}\mathbb{1}.\end{aligned}$$

Using the Zhu formula 4.2 for the 2-point function with both states inserted on the left, we obtain

$$\begin{aligned}&Z_V^{(2)}(\psi^+,x;\psi^-,y|\tau_1,\tau_2,\epsilon)\\ &=\sum_{j\geq 0}\partial_y^{(j)}\,{}^{\frac{1}{2}}\mathcal{P}\left[\begin{smallmatrix}1^{(2)}\\ -1^{(2)}\end{smallmatrix}\right](x,y)Z_V^{(2)}(\psi^+[j]\psi^-,y)\\ &=\ {}^{\frac{1}{2}}\mathcal{P}\left[\begin{smallmatrix}1^{(2)}\\ -1^{(2)}\end{smallmatrix}\right](x,y)Z_V^{(2)}(\mathbb{1},y)\\ &=\ {}^{\frac{1}{2}}\mathcal{P}\left[\begin{smallmatrix}1^{(2)}\\ -1^{(2)}\end{smallmatrix}\right](x,y)Z_V^{(2)},\end{aligned}$$

where $\left[\begin{smallmatrix}1^{(2)}\\ -1^{(2)}\end{smallmatrix}\right]$ denotes $\theta_a=1$, $\phi_a=-1$ for $a=1,2$. The vector ψ^+ is of conformal weight $N=\frac{1}{2}$, which gives $2N-2=-1$, $\Delta=I$ and $o(\psi^+)=0$ (from (8)), which gives trivial O_a terms. We have no ${}^N\mathcal{F}(x)$ terms as $N=\frac{1}{2}$ in (96). This gives that $\widetilde{\Lambda}_a=\Lambda_a$ for $a=1,2$. Then we know that ${}^{\frac{1}{2}}\mathcal{P}\left[\begin{smallmatrix}1^{(2)}\\ -1^{(2)}\end{smallmatrix}\right](x,y)$ for $x,y\in\widehat{\mathcal{S}}_1$ is given by

$$\begin{aligned}&P_1\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right](x-y,\tau_1)+{}^{\frac{1}{2}}\mathbb{Q}\left[\begin{smallmatrix}1^{(2)}\\ -1^{(2)}\end{smallmatrix}\right](x)\mathbb{P}_1\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right](y,\tau_2)\\ &=P_1\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right](x-y,\tau_1)+\sum_{m,n\geq 0}\mathbb{R}\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right](x;m)\left(I-\Lambda_2\Lambda_1\right)^{-1}(m,n)\mathbb{P}_1\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right](y;n).\end{aligned}$$

We first note that (29) and (62) give

$$\Lambda\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(m-1,n)=F_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(m,n),\tag{98}$$

and (27) and (64) give

$$\mathbb{R}\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x;m)=\epsilon^{-\frac{1}{4}}h_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x;m)dx^{-\frac{1}{2}}.\tag{99}$$

Lastly (28) and (63) tell us that

$$\mathbb{P}_1 \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (y; m) = \epsilon^{\frac{1}{4}} \overline{h}_a^T \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (y; m) dy^{-\frac{1}{2}}. \tag{100}$$

Shifting index in m and noting that $\mathbb{P}_1(y, 0) = 0$ and $\Lambda_a(m, 0) = 0$ for $a = 1, 2$, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\mathcal{P} \left[\begin{smallmatrix} 1^{(2)} \\ -1^{(2)} \end{smallmatrix} \right] (x, y) \\
&= P_1 \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] (x - y, \tau_1) + \sum_{m,n \geq 1} \mathbb{R} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] (x; m - 1) \left(I - \Lambda_2 \Lambda_1\right)^{-1} (m - 1, n) \mathbb{P}_1 \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] (y; n) \\
&= P_1 \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] (x - y, \tau_1) + \sum_{m,n \geq 1} \epsilon^{-\frac{1}{4}} h_1(x; m) \left(I - F_2 F_1\right)^{-1} (m, n) \epsilon^{\frac{1}{4}} \bar{h}_1^T (y; n) \\
&= P_1 \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] (x - y, \tau_1) + h_1(x) \left(I - F_2 F_1\right)^{-1} \bar{h}_1^T (y) \\
&= S^{(2)} \left[\begin{smallmatrix} 1^{(2)} \\ -1^{(2)} \end{smallmatrix} \right] (x, y) dx^{-\frac{1}{2}} dy^{-\frac{1}{2}},
\end{aligned}$$

using (98), (99) and (100). Then the equation for the formal $(\frac{1}{2}, \frac{1}{2})$-differential is given by

$$Z_V^{(2)}(\psi^+, x; \psi^-, y) dx^{\frac{1}{2}} dy^{\frac{1}{2}} = S^{(2)} \left[\begin{smallmatrix} 1^{(2)} \\ -1^{(2)} \end{smallmatrix} \right] (x, y) Z_V^{(2)}(\tau_1, \tau_2, \epsilon), \tag{101}$$

which agrees with Theorem 2 of [TZ1] for the case of the 2-point function. This gives

Lemma 5.1. *For $x, y \in \mathcal{S}^{(2)}$, we have*

$$\tfrac{1}{2}\mathcal{P} \begin{bmatrix} 1^{(2)} \\ -1^{(2)} \end{bmatrix} (x, y) dx^{\frac{1}{2}} dy^{\frac{1}{2}} = S^{(2)} \begin{bmatrix} 1^{(2)} \\ -1^{(2)} \end{bmatrix} (x, y).$$

We now examine the Heisenberg vector for the rank two Free Fermion VOSA

$$h = \psi^+[-1]\psi^-. \tag{102}$$

We now use ideas of [GT1] to develop an equation for the Heisenberg 1-point function

$$\begin{aligned}
Z_V^{(2)}(\psi^+, x; \psi^-, y) &= Z_V^{(2)}(Y[\psi^+, x] Y[\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon) \\
&= Z_V^{(2)}(Y[Y[\psi^+, x - y]\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon) \\
&= \sum_{m \in \mathbb{Z}} Z_V^{(2)}(Y[Y[\psi^+, x - y]\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon) \\
&= \sum_{m \in \mathbb{Z}} Z_V^{(2)}(Y[\psi^+[m]\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon)(x - y)^{-m-1} \\
&= \sum_{m \leq 0} Z_V^{(2)}(Y[\psi^+[m]\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon)(x - y)^{-m-1} \\
&= Z_V^{(2)}(Y[\psi^+[0]\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon)(x - y)^{-1} \\
&\quad + Z_V^{(2)}(Y[\psi^+[-1]\psi^-, y] \mathbb{1}, \tau_1, \tau_2, \epsilon) + \mathcal{O}(x - y) \\
&= \frac{Z_V^{(2)}(\tau_1, \tau_2, \epsilon)}{x - y} + Z_V^{(2)}(h, x, \tau_1, \tau_2, \epsilon) + \ldots,
\end{aligned}$$

using (9), (10). Then

$$Z_V^{(2)}(\psi^+, x; \psi^-, y) - \frac{Z_V^{(2)}(\tau_1, \tau_2, \epsilon)}{x - y} = Z_V^{(2)}(h, y, \tau_1, \tau_2, \epsilon) + \mathcal{O}(x - y).$$

Multiplying across by $dx^{\frac{1}{2}}dy^{\frac{1}{2}}$, letting $y \to x$ and using (101) and (102), we obtain

$$Z_V^{(2)}(h, x, \tau_1, \tau_2, \epsilon)dx = \kappa^{(2)}(x) Z_V^{(2)}(\tau_1, \tau_2, \epsilon),$$

where $\kappa^{(2)}(x)$ is a holomorphic 1-differential on the genus two surface

$$\kappa^{(2)}(x) = \lim_{y \to x} \left(S^{(2)} \left[\begin{smallmatrix} 1^{(2)} \\ -1^{(2)} \end{smallmatrix} \right] (x, y) - \frac{dx^{\frac{1}{2}}dy^{\frac{1}{2}}}{x - y} \right).$$

By examining the Heisenberg modules of the Free Fermion VOSA, we can obtain an expression for $\kappa^{(2)}(x)$. We write the 2-point function as a sum over these modules

$$\begin{aligned} Z_V^{(2)}(h, x) &= \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^2} Z_{M_{\boldsymbol{\alpha}}}^{(2)}(h, x) \\ &= \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^2} \nu_{\boldsymbol{\alpha}}(x) Z_{M_{\boldsymbol{\alpha}}}^{(2)}, \end{aligned}$$

from [GT1]. Expanding the module partition function, we can write

$$Z_V^{(2)}(h, x) = \left(\sum_{\boldsymbol{\alpha} \in \mathbb{Z}^2} \nu_{\boldsymbol{\alpha}}(x) e^{\pi \mathrm{i} \boldsymbol{\alpha}.\Omega.\boldsymbol{\alpha}} \right) Z_M^{(2)},$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, where Ω is the genus two period matrix of (22). Here we introduce the genus two *Jacobi theta function* for a rank two integral lattice

$$J^{(2)}(\Omega, \boldsymbol{\zeta}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^2} e^{\pi \mathrm{i} (\boldsymbol{\alpha}.\Omega.\boldsymbol{\alpha} + \boldsymbol{\alpha}.\boldsymbol{\zeta})},$$

where $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathbb{C}^2$. Then for the free fermion we obtain the following relation

$$Z_V^{(2)}(h, x) = \left(\partial_{\zeta_1} \left(J^{(2)}(\Omega, \boldsymbol{\zeta}) \right)_{\boldsymbol{\zeta}=(0,0)} \nu_1(x) + \partial_{\zeta_2} \left(J^{(2)}(\Omega, \boldsymbol{\zeta}) \right)_{\boldsymbol{\zeta}=(0,0)} \nu_2(x) \right) Z_M^{(2)},$$

giving

$$\kappa^{(2)}(x) = \partial_{\zeta_1} \left(J^{(2)}(\Omega, \boldsymbol{\zeta}) \right)_{\boldsymbol{\zeta}=(0,0)} \nu_1(x) + \partial_{\zeta_2} \left(J^{(2)}(\Omega, \boldsymbol{\zeta}) \right)_{\boldsymbol{\zeta}=(0,0)} \nu_2(x).$$

Part III

Genus g Zhu Reduction for Vertex Operator Algebras

6 Riemann Surfaces from a Sewn Sphere

In this section we will discuss a method for constructing a genus g surface from genus zero data, involving the usual Schottky uniformisation. This method involves sewing g handles to the Riemann sphere, yielding a genus g surface. We also extend some existing results of Zhu [Zhu] to obtain a more general Zhu reduction formula at genus zero.

6.1 The Schottky uniformisation of a Riemann surface

Consider the Riemann sphere $\mathcal{S}^{(0)} \cong \widehat{\mathbb{C}}$ and recall $\mathcal{I} = \{\pm 1, \pm 2, \ldots, \pm g\}$ of (17). Let $\{\mathcal{C}_a\}$ for $a \in \mathcal{I}$ be a set of $2g$ disjoint Jordan curves on $\mathcal{S}^{(0)}$. Points z', z on the curves $\mathcal{C}_{-a}$ and $\mathcal{C}_a$ respectively are identified using the Schottky relation [Bob]

$$\frac{z' - W_{-a}}{z' - W_a} \cdot \frac{z - W_a}{z - W_{-a}} = q_a, \tag{103}$$

for complex parameter $q_a = q_{-a}$ with $0 < |q_a| < 1$ and points W_{-a}, W_a on the sphere which are fixed under the action of the Möbius transformation γ_a, where γ_a is given by $z' = \gamma_a z$. In particular, W_a (resp. W_{-a}) is an attracting (resp. repelling) fixed point for $a \in \mathcal{I}_+ = \{1, 2, \ldots, g\}$ of (17). We also have that γ_a is conjugate to $\left(\begin{smallmatrix} q_a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$. The genus g Schottky group Γ is then given by the free group on the generators γ_a for $a = 1, \ldots, g$. The generators also satisfy $\gamma_{-a} = \gamma_a^{-1}$ and the curves $\mathcal{C}_a$ obey the relation $\gamma_a \mathcal{C}_a = -\mathcal{C}_{-a}$. Let w_a be such that $\gamma_a . w_a = \infty$. Noting this and using (103), we see that

$$w_a = \frac{W_a - q_a W_{-a}}{1 - q_a}.$$

We can rewrite relation (103) in a simpler format

$$(z' - w_{-a})(z - w_a) = \rho_a, \tag{104}$$

where ρ_a is given by

$$\rho_a = -\frac{q_a (W_a - W_{-a})^2}{(1 - q_a)^2}.$$

We note that $\rho_a = \rho_{-a}$. It is convenient to choose the Jordan curves $\mathcal{C}_a$ to be circles of radius $|\rho_a|^{\frac{1}{2}}$ with centre w_a. This choice guarantees that points outside $\mathcal{C}_a$ will be mapped to the interior of the disc bounded by $\mathcal{C}_{-a}$ under γ_a. This can be seen by rearranging (104)

$$\gamma_a z = w_{-a} + \frac{\rho_a}{z - w_a}. \tag{105}$$

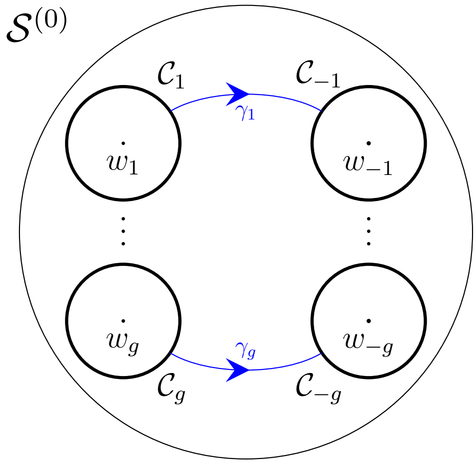

Figure 2: Constructing a genus g surface using the Schottky uniformisation

Then for a point y in the exterior of the disk, we note that

$$|\gamma_a y - w_{-a}| = \frac{|\rho_a|}{|y - w_a|} < |\rho_a|^{\frac{1}{2}},$$

i.e. $\gamma_a y$ maps points in the exterior of $\mathcal{C}_a$ to the interior of $\mathcal{C}_{-a}$. The converse also applies to interior points of $\mathcal{C}_a$. We then obtain a genus g Riemann surface with homology cycles α_a corresponding to the contours $\mathcal{C}_{-a}$ and β_a corresponding to paths from $\mathcal{C}_a$ to $\mathcal{C}_{-a}$ for $a = 1, \ldots, g$.

Let $\Lambda(\Gamma)$ denote the *limit set* of Γ, the set of accumulation points of the action of Γ on $\widehat{\mathbb{C}}$. We refer to the individual points of this set as *limit points*. Then the set of *ordinary points* of Γ is given by $\Omega(\Gamma) = \widehat{\mathbb{C}} - \Lambda(\Gamma)$. Then $\mathcal{S}^{(g)}$ is given by the quotient $\Omega(\Gamma)/\Gamma$.

Remark 6.1. We note that we can also approach this construction using ideas of Yamada [Y].

6.2 Genus zero Zhu theory

For $v_1, v_2, \ldots, v_n \in V$, define the genus zero n-point (correlation) function by

$$Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}) = \langle \mathbb{1}, \boldsymbol{Y}(\boldsymbol{v}, \boldsymbol{y})\mathbb{1}\rangle,$$

with

$$Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}) = Z^{(0)}(v_1, y_1; \ldots; v_n, y_n),$$

and

$$\boldsymbol{Y}(\boldsymbol{v}, \boldsymbol{y}) = Y(v_1, y_1) \ldots Y(v_n, y_n),$$

where $\langle \cdot, \cdot \rangle$ is the Li-Zamolodchikov (Li-Z) metric of §3.3 with $\lambda = 1$. We note the independence of this n-point function from the VOA; the dependence is solely on the given vertex operators. For convenience, we will also use the following notation

$$Z^{(0)}(\ldots; v_k, y_k; \ldots) = Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}).$$

Let

$$\boldsymbol{dy}^{\mathbf{wt}(\boldsymbol{v})} = \prod_{k=1}^{n} dy_k^{\mathrm{wt}(v_k)}.$$

We will also work with the differential form

$$\mathcal{F}^{(0)}(\boldsymbol{v}, \boldsymbol{y}) = Z^{(0)}(\boldsymbol{v}, \boldsymbol{y})\boldsymbol{dy}^{\mathbf{wt}(\boldsymbol{v})}, \tag{106}$$

with similar notational conventions. We note that for a quasiprimary state u of weight N, the following simultaneously hold

$$\begin{aligned} u(\ell)\mathbb{1} &= 0, \\ u^{\dagger}(\ell)\mathbb{1} &= 0, \end{aligned}$$

when $0 \leq \ell \leq 2N-2$, where $u^{\dagger}(\ell)$ is the standard adjoint operator (with $\lambda = 1$) for $u(\ell)$, which is given by

$$u^{\dagger}(\ell) = (-1)^{N} u(2N-2-\ell).$$

Then we find that

$$\langle \mathbb{1}, u(\ell)\boldsymbol{Y}(\boldsymbol{v}, \boldsymbol{y})\mathbb{1}\rangle = 0,$$

and

$$\langle \mathbb{1}, \boldsymbol{Y}(\boldsymbol{v}, \boldsymbol{y})u(\ell)\mathbb{1}\rangle = 0,$$

for $0 \leq \ell \leq 2N-2$. Then commuting the mode $u(\ell)$ through the product of vertex operators, we see by using the commutator identity (3) that for $0 \leq \ell \leq 2N-2$, the genus zero n-point function obeys the following identity

$$\sum_{k=1}^{n} \sum_{j\geq 0} \binom{\ell}{j} y_k^{\ell-j} Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots) = 0. \tag{107}$$

This serves as a genus zero VOA analogue of (81). We can see from (3) that the $2N-1$ identities can be rewritten as a differential condition

$$\sum_{k=1}^{n} \left(\sum_{j\geq 0} Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots)\partial_k^{(j)} \right) y_k^{\ell} = 0,$$

for $0 \leq \ell \leq 2N-2$, with $\partial_k^{(j)} = \partial_{y_k}^{(j)}$. Let $\Pi_{2N-2}(y)$ denote the space of polynomials in y of degree $2N-2$. Then we have the following lemma

Lemma 6.1. *For a quasiprimary state u of weight N, the genus zero n-point function obeys the differential condition*

$$\sum_{k=1}^{n} \left(\sum_{j\geq 0} Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots)\partial_k^{(j)} \right) p_{2N-2}(y_k) = 0, \tag{108}$$

for any polynomial $p_{2N-2}(y) \in \Pi_{2N-2}(y)$.

We will now develop a genus zero Zhu reduction formula. We first examine the $(n+1)$-point function

$$Z^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y}). \tag{109}$$

For quasiprimary u of weight N, we know that

$$\begin{aligned}
&\langle \mathbb{1}, u(r+N-1)\boldsymbol{Y(v,y)}\mathbb{1}\rangle \\
&= \langle u^{\dagger}(r+N-1)\mathbb{1}, \boldsymbol{Y(v,y)}\mathbb{1}\rangle \\
&= (-1)^N\langle u(N-r-1)\mathbb{1}, \boldsymbol{Y(v,y)}\mathbb{1}\rangle = 0,
\end{aligned}$$

for $r \leq N-1$. Then (109) can be written as

$$\begin{aligned}
&\sum_{r\geq N} x^{-r-N}\langle \mathbb{1}, u(r+N-1)\boldsymbol{Y(v,y)}\mathbb{1}\rangle \\
&= \sum_{s\geq 0} x^{-s-2N}\langle \mathbb{1}, u(2N-1+s)\boldsymbol{Y(v,y)}\mathbb{1}\rangle \\
&= \sum_{k=1}^{n}\sum_{s\geq 0} x^{-s-2N}\langle \mathbb{1}, \ldots \sum_{j\geq 0}\partial_k^{(j)}(y_k^{2N-1+s})Y(u(j)v_k,y_k)\ldots \mathbb{1}\rangle \\
&= \sum_{k=1}^{n}\sum_{j\geq 0}\sum_{s\geq 0}\partial_k^{(j)}(y_k^{2N-1+s})x^{-s-2N}\langle \mathbb{1}, \ldots Y(u(j)v_k,y_k)\ldots \mathbb{1}\rangle \\
&= \sum_{k=1}^{n}\sum_{j\geq 0}\sum_{s\geq 0} x^{-s-2N}\partial_k^{(j)}(y_k^{2N-1+s})Z^{(0)}(\ldots;u(j)v_k,y_k;\ldots).
\end{aligned}$$

Examining the coefficient

$$\sum_{s\geq 0} x^{-s-2N}\partial_k^{(j)}(y_k^{2N-1+s}),$$

we can rewrite it as follows

$$\begin{aligned}
&\sum_{s\geq 0} x^{-s-2N}\partial_k^{(j)}(y_k^{2N-1+s}) = \partial_k^{(j)}\left(\sum_{s\geq 0}\left(\frac{y_k}{x}\right)^s\cdot\left(\frac{y_k}{x}\right)^{2N-1}\cdot x^{-1}\right) \\
&= \partial_k^{(j)}\left(\frac{1}{1-\frac{y_k}{x}}\cdot\left(\frac{y_k}{x}\right)^{2N-1}\cdot x^{-1}\right) = \partial_k^{(j)}\left(\left(\frac{y_k}{x}\right)^{2N-1}\cdot\frac{1}{x-y_k}\right) = \partial_k^{(j)}\zeta_N(x,y_k),
\end{aligned}$$

with $\zeta_N(x,y)$ given by

$$\begin{aligned}
\zeta_N(x,y) &= \left(\frac{y}{x}\right)^{2N-1}\cdot\frac{1}{x-y} \\
&= \frac{1}{x-y} - \sum_{\ell=0}^{2N-2}\frac{y^\ell}{x^{\ell+1}}.
\end{aligned}$$

Define the operator $\partial^{(i,j)}$ as follows

$$\partial^{(i,j)}f(x,y) = \partial_x^{(i)}\partial_y^{(j)}f(x,y),$$

for a function $f(x,y)$, with $\partial_x^{(i)} = \frac{1}{i!}\partial_x^i$ as before. Putting this all together, we obtain the genus zero recursion formula

$$Z^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y}) = \sum_{k=1}^{n}\sum_{j\geq 0}\partial^{(0,j)}\zeta_N(x,y_k)Z^{(0)}(\ldots;u(j)v_k,y_k;\ldots). \tag{110}$$

As mentioned above and outlined in equation (108), we can add any polynomial $p_{2N-2}(y) \in \Pi_{2N-2}(y)$ with arbitrary coefficients to $\zeta_N(x,y)$ and the above formula (110) will still hold. In particular, we can choose to replace $\zeta_N(x,y)$ by

$$\psi_N^{(0)}(x,y) = \frac{1}{x-y} + \sum_{\ell=0}^{2N-2} f_\ell(x) y^\ell, \tag{111}$$

where $f_\ell(x)$ is any Laurent series in x. Then we can write genus zero Zhu reduction in more generality

Theorem 6.1 (Quasiprimary Genus Zero Zhu reduction)**.** *For u quasiprimary of weight N, the genus zero $(n+1)$-point function obeys the following Zhu reduction formula*

$$Z^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y}) = \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(0,j)}\psi_N^{(0)}(x,y_k) Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots). \tag{112}$$

Let $\Psi_N^{(0)}(x,y) = \psi_N^{(0)}(x,y)dx^N dy^{1-N}$. Multiplying by appropriate differentials, we obtain a Zhu reduction relation for the differential $\mathcal{F}^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y})$ of (106)

Theorem 6.2 (Quasiprimary Genus Zero Zhu reduction for n-point differentials)**.** *For u quasiprimary of weight N, the genus zero $(n+1)$-point function obeys the following Zhu reduction formula*

$$\mathcal{F}^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y}) = \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(0,j)}\Psi_N^{(0)}(x,y_k)dy_k^j\, \mathcal{F}^{(0)}(\ldots; u(j)v_k, y_k; \ldots). \tag{113}$$

We can extend the relation (112) to quasiprimary descendants $\frac{1}{i!}L(-1)^i u$, $i \geq 0$

$$\begin{aligned}
&\tfrac{1}{i!}Z^{(0)}(L(-1)^i u, x;\boldsymbol{v},\boldsymbol{y}) = \partial_x^{(i)} Z^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y})\\
&= \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(i,j)}\psi_N^{(0)}(x,y_k) Z^{(0)}(\ldots; u(j)v_k, y_k \ldots),
\end{aligned}$$

which gives us the corollary

Corollary 6.1 (General Genus Zero Zhu reduction)**.** *For u quasiprimary of weight N, the genus zero $(n+1)$-point function obeys the following Zhu reduction formula*

$$\begin{aligned}
&\tfrac{1}{i!}Z_V^{(0)}(L(-1)^i u; x; v_1, y_1; \ldots; v_n, y_n)\\
&= \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(i,j)}\psi_N^{(0)}(x,y_k) Z_V^{(0)}(\ldots; u(j)v_k; y_k; \ldots).
\end{aligned}$$

Remark 6.2. The flexibility afforded to us by condition (108) allows us to choose a wide variety of coefficients in formula (112). For example, we can choose $f_\ell(x) = 0$ in (111) for all $0 \leq \ell \leq 2N-2$ to obtain

$$Z^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y}) = \sum_{k=1}^{n}\sum_{j\geq 0} \frac{1}{(x-y_k)^{1+j}} Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots).$$

This can be useful in genus zero calculations.

Recall the definition of $\psi_N^{(0)}(x,y)$ given in (111); then note the expansion of $\partial^{(0,j)}\psi_N^{(0)}(x,y)$

$$\partial^{(0,j)}\psi_N^{(0)}(x,y) = \frac{1}{(x-y)^{1+j}} + \sum_{\ell=0}^{2N-2}\binom{\ell}{j} f_\ell(x)y^{\ell-j}.$$

Extending this to $\partial^{(0,j)}\psi_N^{(0)}(x+y,y)$ we find

$$\partial^{(0,j)}\psi_N^{(0)}(x+y,y) = \frac{1}{x^{1+j}} + \sum_{\ell=0}^{2N-2}\binom{\ell}{j} f_\ell(x+y)y^{\ell-j} = \frac{1}{x^{1+j}} + \sum_{t\geq 0}\mathcal{E}_t^j(y)x^t,$$

where

$$\mathcal{E}_t^j(y) = \sum_{\ell=0}^{2N-2}\binom{\ell}{j}\partial^{(t)}f_\ell(y)y^{\ell-j}. \tag{114}$$

We note that $\mathcal{E}_t^j(y) = 0$ for $j > 2N-2$. Using these data, we can examine the Zhu reduction of an n-point function with a mode acting on the state in the first slot

$$Z^{(0)}(u,x+y_1;\boldsymbol{v},\boldsymbol{y}) = \sum_{k=1}^{n}\sum_{j\geq 0}\partial^{(0,j)}\psi_N^{(0)}(x+y_1,y_k)Z^{(0)}(\ldots;u(j)v_k,y_k;\ldots).$$

The $k=1$ term must be treated differently due to its singular nature at $x=0$. We separate the previous expression as follows

$$\begin{aligned}Z^{(0)}(u,x+y_1;\boldsymbol{v},\boldsymbol{y}) &= \sum_{k=2}^{n}\sum_{j\geq 0}\partial^{(0,j)}\psi_N^{(0)}(y_1+x,y_k)Z^{(0)}(\ldots;u(j)v_k,y_k;\ldots)\\ &\quad+\sum_{j\geq 0}\partial^{(0,j)}\psi_N^{(0)}(y_1+x,y_1)Z^{(0)}(u(j)v_1,y_1;\ldots).\end{aligned}$$

Here we make use of the associativity property (5) of vertex operators. Then we can rewrite the $(n+1)$-point function as follows

$$\begin{aligned}Z^{(0)}(u,x+y_1;\boldsymbol{v},\boldsymbol{y}) &= \langle \mathbb{1}, Y(u,x+y_1)Y(v_1,y_1)\ldots Y(v_n,y_n)\mathbb{1}\rangle\\ &= \langle \mathbb{1}, Y(Y(u,x)v_1,y_1)\ldots Y(v_n,y_n)\mathbb{1}\rangle\\ &= \sum_{t\in\mathbb{Z}}\langle \mathbb{1}, Y(u(-t-1)v_1,y_1)\ldots Y(v_n,y_n)\mathbb{1}\rangle x^t.\end{aligned}$$

To find the entry corresponding to the $u(-t-1)$ mode (for $t\geq 0$), we extract the coefficient of x^t (and multiply by sufficiently high powers of $(x-y_k)$, $k=1,\ldots,n$ to obtain

Theorem 6.3 (Quasiprimary Genus Zero Zhu reduction II)**.** *For a quasiprimary state u of weight N, the genus zero n-point function obeys the following Zhu reduction formula for $t\geq 0$*

$$\begin{aligned}&Z^{(0)}(u(-t-1)v_1,y_1;v_2,y_2;\ldots;v_n,y_n)\\ &=\sum_{k=2}^{n}\sum_{j\geq 0}\partial^{(t,j)}\psi_N^{(0)}(y_1,y_k)Z^{(0)}(\ldots;u(j)v_k,y_k;\ldots)\\ &\quad+\sum_{j\geq 0}\mathcal{E}_t^j(y_1)Z^{(0)}(u(j)v_1,y_1;\ldots).\end{aligned}$$

Remark 6.3. Using locality, we can allow the mode to act on any v_k, $k = 1, \ldots, n$ and adjust the right hand side accordingly.

Corollary 6.2 (General Genus Zero Zhu reduction II)**.** *For a quasiprimary descendant state $\frac{1}{i!}L(-1)^i u$, with u of conformal weight N, the genus zero n-point function obeys the following Zhu reduction formula*

$$\begin{aligned}
&Z^{(0)}(\tfrac{1}{i!}L(-1)^i u(-t-1)v_1, y_1; v_2, y_2; \ldots; v_n, y_n) \\
&= \sum_{k=2}^{n} \sum_{j\geq 0} \binom{t+i}{i} \partial^{(t+i,j)} \psi_N^{(0)}(y_1, y_k) Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots) \\
&\quad + \sum_{j\geq 0} \binom{t+i}{i} \mathcal{E}_{t+i}^{j}(y_1) Z^{(0)}(u(j)v_1, y_1; \ldots).
\end{aligned}$$

Remark 6.4. In particular, for the trivial choice $\psi_N^{(0)}(x, y) = \frac{1}{x-y}$, we have the following Zhu reduction relation

$$\begin{aligned}
&Z^{(0)}(\tfrac{1}{i!}(L(-1)^i u(-t-1))v_1, y_1; v_2, y_2; \ldots; v_n, y_n) \\
&= \sum_{k=2}^{n} \sum_{j\geq 0} \binom{t+i}{i} \binom{t+i+j}{j} \frac{(-1)^{t+i}}{(y_1 - y_k)^{t+i+j+1}} Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots).
\end{aligned}$$

7 Genus g Zhu reduction for Formal n-Point Differentials

We now develop a general genus Zhu reduction formula by using the Schottky construction of a genus g Riemann surface. Using results from genus zero Zhu theory and an appropriate definition of a genus g n-point function for the Schottky formalism, we extend the approach of [GT1] using infinite block matrices and vectors for to obtain a recursive relation for objects on the handles of a general genus Riemann surface. This is then developed to obtain a global Zhu recursion formula for n-point differentials on this surface.

7.1 Genus g formal n-point functions

We will now examine some properties of the genus g partition and formal n-point functions[2]. We revisit the Li-Zamolodchikov (Li-Z) metric of (37) in the simpler case for a VOA. We recall that $\langle u, v\rangle = 0$ for $\mathrm{wt}(u) \neq \mathrm{wt}(v)$. Let $\boldsymbol{b}_+$ denote a basis for $V^{\otimes g}$, i.e. g copies of V. For a basis vector $b_a \in V$ with $a \in \mathcal{I}_+$, let $\overline{b}_a$ denote its dual with respect to the Li-Z metric for $\lambda = 1$. For convenience, for a basis vector b_a of weight n_a, we define the scaled dual vector b_{-a}

$$b_{-a} = \rho_a^{n_a}\overline{b}_a. \tag{115}$$

Similarly, let $\overline{\boldsymbol{b}}_+$ denote the dual basis $\{\overline{b}_a\}$ with respect to the ρ_a-dependent Li-Z metric $\langle \cdot, \cdot\rangle_a$ (i.e. for $\lambda = \rho_a$), given by

$$\langle u, v\rangle_{\rho_a} = \rho_a^{-N}\langle u, v\rangle_1, \tag{116}$$

for u, v of weight N, The adjoint operator for this metric $u^\dagger(m)$ for quasiprimary u of weight N is given by

$$u^\dagger(m) = (-1)^N \rho_a^{m-N+1} u(2N-2-m). \tag{117}$$

We now extend ideas from Part II and develop a genus g adjoint relation. Recall that the Li-Z metric $\langle \cdot, \cdot\rangle$ is invertible and that $\langle u, v\rangle = 0$ for $\mathrm{wt}(u) \neq \mathrm{wt}(v)$ for homogeneous u, v. Let $\{b\}$ be a homogeneous basis for a VOA V with Li-Z dual basis $\{\overline{b}\}$. We note the following

Lemma 7.1. *For quasiprimary u of weight N we have*

$$\sum_{b\in V_n} (u(m)b) \otimes \overline{b} = \sum_{b\in V_{n+N-m-1}} b \otimes (u^\dagger(m)\overline{b}). \tag{118}$$

Proof. Since $\mathrm{wt}(u(m)b) = n + N - m + 1$ for $b \in V_n$ we find

$$u(m)b = \sum_{c\in V_{n+N-m-1}} \langle \overline{c}, u(m)b\rangle c = \sum_{c\in V_{n+N-m-1}} \langle u^\dagger(m)\overline{c}, b\rangle c.$$

Similarly for $c \in V_{n+N-m-1}$ we find that $\mathrm{wt}(u^\dagger(m)\overline{c}) = n$ (using (35)). This gives

$$u^\dagger(m)\overline{c} = \sum_{b\in V_n} \langle u^\dagger(m)\overline{c}, b\rangle \overline{b}.$$

Hence the result follows on relabelling. □

[2]From this section onwards, for clarity, with the exception of n-point functions and n-point formal differentials, general genus objects carry no genus label. Genus zero objects will continue to be highlighted with a zero superscript label, e.g. $\psi_N^{(0)}$.

Remark 7.1. Suppose that U is a subVOA of V and $W \subset V$ is a U-module. For $u \in U$ and homogeneous W-basis $\{w\}$ we may then extend (118) to obtain

$$\sum_{w\in W_n} (u(m)w) \otimes \overline{w} = \sum_{w\in W_{n+N-m-1}} w \otimes (u^\dagger(m)w).$$

Now we will consider Zhu reduction for genus g formal n-point functions. The genus g partition function is defined by

$$\begin{aligned} Z_V^{(g)} = Z_V^{(g)}(\rho_a, w_{\pm a}) &= \sum_{n_a\geq 0} \sum_{b_a\in V_{n_a}} \rho_a^{n_a} Z^{(0)}(\bar{b}_1, w_{-1}; b_1, w_1; \ldots; \bar{b}_{-g}, w_{-g}; b_g, w_g) \\ &= \sum_{\boldsymbol{b}_+} Z^{(0)}(\boldsymbol{b}, \boldsymbol{w}), \end{aligned} \tag{119}$$

where $\sum_{\boldsymbol{b}_+}$ denotes summation over any basis of $V^{\otimes g}$ and

$$Z^{(0)}(\boldsymbol{b}, \boldsymbol{w}) = Z^{(0)}(b_{-1}, w_{-1}; b_1, w_1; \ldots; b_{-g}, w_{-g}; b_g, w_g).$$

Remark 7.2. We note that for VOAs V_1, V_2, the genus g partition function for the tensor product $V_1 \otimes V_2$ is the product $Z_{V_1}^{(g)} Z_{V_2}^{(g)}$.

Extending this definition to a formal n-point function we have

$$Z_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = \sum_{\boldsymbol{b}_+} Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}), \tag{120}$$

where

$$Z^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}) = Z^{(0)}(v_1, y_1; \ldots; v_n, y_n; b_{-1}, w_{-1}; \ldots; b_g, w_g).$$

Furthermore, let U be a subVOA of V, where V has a U-module decomposition $V = \bigoplus_\alpha W_\alpha$ for $\alpha \in A$, where A is some indexing set. Then

$$Z_V^{(g)} = \sum_{\boldsymbol{\alpha}} Z_{W_{\boldsymbol{\alpha}}}^{(g)},$$

where the sum ranges over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in A^{\otimes g}$ and $Z_{W_{\boldsymbol{\alpha}}}^{(g)} = \sum_{\boldsymbol{b}_+ \in W_{\boldsymbol{\alpha}}} Z^{(0)}(\boldsymbol{b}, \boldsymbol{w})$ for g-fold tensor product $W_{\boldsymbol{\alpha}} = \bigotimes_{a=1}^g W_{\alpha_a}$. We can naturally extend this to an n-point function for genus g VOA modules

$$Z_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = \sum_{\boldsymbol{\alpha}} Z_{W_{\boldsymbol{\alpha}}}^{(g)}(\boldsymbol{v}, \boldsymbol{y}). \tag{121}$$

7.2 SL$(2, \mathbb{C})$-invariance for the genus g partition function

We now derive some identities for the genus g partition function which come about due to the SL$(2, \mathbb{C})$-invariance of the Riemann sphere. Define the following operators

$$\begin{aligned} l_{-1} &= -\sum_{a\in\mathcal{I}} \partial_{w_a}, \\ l_0 &= -\sum_{a\in\mathcal{I}} \left(w_a \partial_{w_a} + \rho_a \partial_{\rho_a}\right), \\ l_1 &= -\sum_{a\in\mathcal{I}} \left(w_a^2 \partial_{w_a} + 2w_a\rho_a\partial_{\rho_a} + \rho_a \partial_{w_{-a}}\right). \end{aligned}$$

Remark 7.3. We find l_{-1}, l_0 and l_1 provide a representation of $\mathrm{SL}(2,\mathbb{C})$ (corresponding to translation, dilation and inversion respectively) with the usual bracket

$$[l_m, l_n] = (m-n)l_{m+n}.$$

We have that

Proposition 7.1 ($\mathrm{SL}(2,\mathbb{C})$-invariance of the genus g partition function)**.** *For $i = -1, 0, 1$, we find*

$$l_i Z_V^{(g)} = 0. \tag{122}$$

Proof. For $\ell = 0$ and $u = \omega$ (the Virasoro vector) in (107) we have

$$\sum_{a\in\mathcal{I}} Z^{(0)}(\ldots;\omega(0)b_a, w_a;\ldots) = 0,$$

which upon noting that $\omega(0) = L(-1)$ we obtain the translation property of the n-point function

$$\sum_{a\in\mathcal{I}} \partial_{w_a} Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) = 0,$$

which implies (122) for $i=-1$. We now set $\ell = 1$, $u = \omega$ in (107) to find

$$\sum_{a\in\mathcal{I}} w_a Z^{(0)}(\ldots;\omega(0)b_a, w_a;\ldots) + \sum_{a\in\mathcal{I}} Z^{(0)}(\ldots;\omega(1)b_a, w_a;\ldots) = 0.$$

Rewriting in terms of Virasoro modes we find

$$\begin{aligned}&\sum_{a\in\mathcal{I}} w_a Z^{(0)}(\ldots;L(-1)b_a, w_a;\ldots) + \sum_{a\in\mathcal{I}} Z^{(0)}(\ldots;L(0)b_a, w_a;\ldots)\\ &= \sum_{a\in\mathcal{I}} w_a\partial_{w_a} Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + \sum_{a\in\mathcal{I}} n_a Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) = 0,\end{aligned}$$

where $n_a = \mathrm{wt}(b_a)$. Multiplying by appropriate factors again and summing over the bases and levels we find

$$\begin{aligned}&\sum_{a\in\mathcal{I}} w_a\partial_{w_a} Z_V^{(g)} + \sum_{a\in\mathcal{I}}\left(\sum_{b_a\in V_{n_a}} n_a\rho_a^{n_a} Z^{(0)}(\boldsymbol{b},\boldsymbol{w})\right)\\ &= \sum_{a\in\mathcal{I}} \left(w_a\partial_{w_a} + \rho_a\partial_{\rho_a}\right) Z_V^{(g)} = 0,\end{aligned}$$

as required, noting that $n_a\rho_a^{n_a} = \rho_a\partial_{\rho_a}(\rho_a^{n_a})$. Lastly, the case $\ell = 2$, $u = \omega$ in (107) gives

$$\begin{aligned}\sum_{a\in\mathcal{I}} \Big(&w_a^2 Z^{(0)}(\ldots;L(-1)b_a, w_a;\ldots) + 2w_a Z^{(0)}(\ldots;L(0)b_a, w_a;\ldots)\\ &+ Z^{(0)}(\ldots;L(1)b_a, w_a;\ldots)\Big) = 0,\end{aligned}$$

or

$$\sum_{a\in\mathcal{I}} \left(w_a^2\partial_{w_a} Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + 2w_a n_a Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + Z^{(0)}(\ldots;L(1)b_a, w_a;\ldots)\right) = 0.$$

We now use the genus g adjoint relation (Lemma 7.1 and the relation $L^\dagger(1) = \rho_a L(-1)$ (for the ρ_a-dependent Li-Z metric) to rewrite the $L(1)$ term as

$$\begin{aligned}
&\sum_{a\in\mathcal{I}}\Big(w_a^2\partial_{w_a}Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + 2w_a n_a Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + \rho_a Z^{(0)}(\ldots;L(-1)b_{-a},w_{-a};\ldots)\Big)\\
&\quad = \sum_{a\in\mathcal{I}}\Big(w_a^2\partial_{w_a}Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + 2w_a n_a Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) + \rho_a \partial_{-w_a}Z^{(0)}(\boldsymbol{b},\boldsymbol{w})\Big) = 0,
\end{aligned}$$

as required. As before we find

$$\sum_{a\in\mathcal{I}}\left(w_a^2\partial_{w_a} + 2w_a\rho_a\partial_{\rho_a} + \rho_a\partial_{-w_a}\right)Z_V^{(g)} = 0.$$

□

7.3 A preliminary Zhu reduction formula

Examining the recursion of a genus zero $(2g+n+1)$-point function using equation (112) for quasiprimary u of weight N we find

$$\begin{aligned}
&Z_V^{(g)}(u,x;\boldsymbol{v},\boldsymbol{y})\\
&= \sum_{k=1}^{n}\sum_{j\geq 0}\partial^{(0,j)}\psi_N^{(0)}(x,y_k)\sum_{\boldsymbol{b}_+}Z^{(0)}(\ldots;u(j)v_k,y_k;\ldots)\\
&\quad + \sum_{a\in\mathcal{I}}\sum_{j\geq 0}\partial^{(0,j)}\psi_N^{(0)}(x,w_a)\sum_{\boldsymbol{b}_+}Z^{(0)}(\ldots;u(j)b_a,w_a;\ldots).
\end{aligned}\tag{123}$$

Define the doubly indexed column vector[3] $X = (X_a(m))$ with entries

$$X_a(m) = \rho_a^{-\frac{m}{2}}\sum_{\boldsymbol{b}_+}Z^{(0)}(\ldots;u(m)b_a,w_a;\ldots),\tag{124}$$

for $m\geq 0$, $a\in\mathcal{I}$. Similarly, $p(x) = (p_a(x,m))$ is an infinite row vector with entries given by

$$p_a(x,m) = \rho_a^{\frac{m}{2}}\partial^{(0,m)}\psi_N^{(0)}(x,w_a).\tag{125}$$

Then the Zhu reduction formula (123) can be rewritten as

$$Z_V^{(g)}(u,x;\boldsymbol{v},\boldsymbol{y}) = p(x)X + \sum_{k=1}^{n}\sum_{j\geq 0}\partial^{(0,j)}\psi_N^{(0)}(x,y_k)Z_V^{(g)}(\ldots;u(j)v_k,y_k;\ldots).\tag{126}$$

We now want to develop a recursive formula for X.

7.4 Recursion for X

Using linearity, we can apply Lemma 7.1 to the summands of $X_a(m)$ to find a recursive relation for the vector. Using equations (117), (124) and Lemma 7.1 we have

[3] We note that the infinite vectors used here again begin their indexing at *zero*.

Lemma 7.2.

$$X_a(m) = (-1)^N \rho_a^{m-N+1} \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots; u(2N-2-m)b_{-a}, w_{-a}; \ldots), \tag{127}$$

where $u^\dagger(m)$ is the adjoint operator to $u(m)$ with respect to the ρ_a-dependent Li-Z metric.

We separately analyse this expression for $0 \le m \le 2N-2$ and $m > 2N-2$. For $0 \le m \le 2N-2$, (127) implies

$$X_a(m) = (-1)^N X_{-a}(2N-2-m). \tag{128}$$

For $m > 2N-2$, set $t = m - 2N + 1 \ge 0$. Then (127) becomes

$$X_a(m) = (-1)^N \rho_a^{\frac{t+1}{2}} \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots; u(-t-1)b_{-a}, w_{-a}; \ldots).$$

Theorem 6.3 and Remark 6.3 imply

$$\begin{aligned}
X_a(m) &= (-1)^N \rho_a^{\frac{t+1}{2}} \sum_{\boldsymbol{b}_+} \sum_{k=1}^{n} \sum_{j\ge 0} \partial^{(t,j)} \psi_N^{(0)}(w_{-a}, y_k) Z^{(0)}(\ldots; u(j)v_k, y_k; \ldots) \\
&+ (-1)^N \rho_a^{\frac{t+1}{2}} \sum_{\boldsymbol{b}_+} \sum_{\substack{b\in\mathcal{I},\\ b\neq -a}} \sum_{j\ge 0} \partial^{(t,j)} \psi_N^{(0)}(w_{-a}, w_b) Z^{(0)}(\ldots; u(j)b_b, w_b; \ldots) \\
&+ (-1)^N \rho_a^{\frac{t+1}{2}} \sum_{\boldsymbol{b}_+} \sum_{j\ge 0} \mathcal{E}_t^j(w_{-a}) Z^{(0)}(\ldots; u(j)b_{-a}, w_{-a}; \ldots),
\end{aligned}$$

with $\mathcal{E}_t^j$ of (114). We can streamline these expressions as follows

$$X_a(m) = (G + RX)_a(t) = (G + RX)_a(m - 2N + 1), \tag{129}$$

for $m > 2N-2$ and for infinite doubly indexed column vector $G = (G_a(m))$ and matrix $R = (R_{ab}(m,n))$ which are defined for $a, b \in \mathcal{I}$ and $m, n \ge 0$ as follows

$$G_a(m) = \sum_{k=1}^{n} \sum_{j\ge 0} \partial_k^{(j)} q_a(y_k; m) Z_V^{(g)}(\ldots; u(j)v_k, y_k; \ldots),$$

with column vector $q(y) = (q_a(y; m))$ given by (similarly to $p(x)$)

$$q_a(y; m) = (-1)^N \rho_a^{\frac{m+1}{2}} \partial^{(m,0)} \psi_N^{(0)}(w_{-a}, y). \tag{130}$$

The matrix R has entries for $k, l \ge 0$

$$R_{ab}(m,n) = \begin{cases} (-1)^N \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n}{2}} \partial^{(m,n)} \psi_N^{(0)}(w_{-a}, w_b), & a \neq -b \\ (-1)^N \rho_a^{\frac{m+n+1}{2}} \mathcal{E}_m^n(w_{-a}), & a = -b. \end{cases} \tag{131}$$

Define the matrix Δ by

$$\Delta_{ab}(m,n) = \delta_{m,n+2N-1}\delta_{ab}. \tag{132}$$

Note that Δ satisfies the identities

$$\Delta^T \Delta = I, \tag{133}$$

$$\Delta\Delta^T = I - \Pi, \tag{134}$$

where Π is a $(2N-1)$-dimensional projection matrix

$$\begin{pmatrix} I_{2N-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Equation (129) tells us that

$$X_a(m) = (\Delta(G + RX))_a(m),$$

for $m \geq 2N - 1$. Now define

$$X^\perp = \Delta^T X,$$

and

$$X^\Pi = \Pi X.$$

Then using (133) we have

$$X^\perp = G + RX.$$

Then we can use the relation

$$X = (\Pi + (I - \Pi))X = X^\Pi + \Delta X^\perp,$$

to obtain

$$\begin{aligned} X^\perp &= G + R\left(X^\Pi + \Delta X^\perp\right) \\ &= G + RX^\Pi + \widetilde{R}X^\perp, \end{aligned}$$

where $\widetilde{R} = R\Delta$. Formally solving for $X^\perp$ gives

$$X^\perp = \left(I - \widetilde{R}\right)^{-1} RX^\Pi + \left(I - \widetilde{R}\right)^{-1} G,$$

where the formal inverse $(I - \widetilde{R})^{-1}$ is given by

$$\left(I - \widetilde{R}\right)^{-1} = \sum_{n\geq 0} \widetilde{R}^n.$$

Altogether we have

Lemma 7.3. *Let u be a quasiprimary vector with* $\mathrm{wt}(u) = N$. *Then $X = X^\Pi + \Delta X^\perp$ where*

$$X^\perp = \left(I - \widetilde{R}\right)^{-1} RX^\Pi + \left(I - \widetilde{R}\right)^{-1} G.$$

7.5 Genus g Zhu reduction

Recalling the preliminary Zhu reduction formula (126), we can substitute for X using Lemma 7.3 to obtain

$$Z_V^{(g)}(u, x; \boldsymbol{v}, \boldsymbol{y}) = \phi(x)o(u; \boldsymbol{v}, \boldsymbol{y}) + \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(0,j)}\psi_N(x, y_k) Z_V^{(g)}(\ldots; u(j)v_k, y_k; \ldots), \tag{135}$$

where $\phi(x) = (\phi_a(x;m))$ is a finite doubly indexed row vector given by [4]

$$\phi_a(x;m) = \rho_a^{-\frac{m}{2}} \left(p(x) + \widetilde{p}(x) \left(I - \widetilde{R} \right)^{-1} R \right)_a (m),$$

for $a \in \mathcal{I}$, $0 \leq m \leq 2N-2$ with $o(u;\boldsymbol{v},\boldsymbol{y})$ similarly defined for $0 \leq m \leq 2N-2$ by

$$o_a(u;\boldsymbol{v},\boldsymbol{y};m) = \rho_a^{\frac{m}{2}} X_a(m). \tag{136}$$

Lastly, $\widetilde{p}(x) = p(x)\Delta$ and $\psi_N(x,y)$ is given by

$$\psi_N(x,y) = \psi_N^{(0)}(x,y) + \widetilde{p}(x)(I - \widetilde{R})^{-1} q(y).$$

Where no superscript genus label is attached, $\psi_N(x,y)$ is understood to refer to the general genus function. Now define the following vectors of formal differential forms

$$P(x) = p(x)dx^N; \quad Q(y) = q(y)dy^{1-N},$$

with $\widetilde{P}(x) = P(x)\Delta$ etc. Then $\Psi_N(x,y) = \psi_N(x,y)dx^N dy^{1-N}$ can be written as

$$\Psi_N(x,y) = \Psi_N^{(0)}(x,y) + \widetilde{P}(x)(I - \widetilde{R})^{-1} Q(y), \tag{137}$$

for $a \in \mathcal{I}$, $0 \leq \ell \leq 2N-2$ with

$$\Psi_N^{(0)}(x,y) = \psi_N^{(0)}(x,y)dx^N dy^{1-N}.$$

Define $\Phi(x) = \phi(x)dx^N$ and $O(u;\boldsymbol{v},\boldsymbol{y}) = o(u;\boldsymbol{v},\boldsymbol{y})\boldsymbol{dy}^{\boldsymbol{wt(v)}}$. Then multiplying (135) by the appropriate differential factors, we obtain the main result

Theorem 7.1 (Quasiprimary Genus g Zhu Reduction). *The genus g n-point formal differential for a quasiprimary vector u of weight* $\mathrm{wt}(u) = N$ *inserted at $x \in \mathcal{S}^{(g)}$ and general vectors $v_1, v_2 \ldots, v_n$ inserted at $y_1, y_2 \ldots, y_n \in \mathcal{S}^{(g)}$ respectively, satisfies the recursive identity*

$$\begin{aligned}\mathcal{F}_V^{(g)}(u,x;\boldsymbol{v},\boldsymbol{y}) &= \Phi(x)O(u;\boldsymbol{v},\boldsymbol{y}) \\ &\quad + \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(0,j)}\Psi_N(x,y_k)dy_k^j\, \mathcal{F}_V^{(g)}(\ldots;u(j)v_k,y_k;\ldots).\end{aligned} \tag{138}$$

Remark 7.4. The dy_k^j factor is balanced by the differential factor of $dy_k^{\mathrm{wt}(v_k)-j+1-N}$ in $\Psi_N(x,y_k)$ given by the weight of $u(j)v_k$, noting how $u(j)$ maps between the $L(0)$-eigenspaces V_n in (7) for $\overline{a} = p(u) = \overline{0}$.

We also have the following corollary for descendants of quasiprimary vectors

Corollary 7.1 (General Genus g Zhu Reduction). *The genus g formal n-point differential for a for a level $i \geq 0$ descendant $\frac{1}{i!}L(-1)^i u$ of a quasiprimary vector u of weight* $\mathrm{wt}(u) = N$ *inserted at $x \in \mathcal{S}^{(g)}$ and general vectors $v_1, v_2 \ldots, v_n$ inserted at $y_1, y_2 \ldots, y_n \in \mathcal{S}^{(g)}$ respectively, satisfies the recursive identity*

$$\begin{aligned}&\mathcal{F}_V^{(g)}(\tfrac{1}{i!}L(-1)^i u,x;\boldsymbol{v},\boldsymbol{y}) \\ &= \partial_x^{(i)}\Phi(x)\, dx^i O(u;\boldsymbol{v},\boldsymbol{y}) \\ &\quad + \sum_{k=1}^{n}\sum_{j\geq 0} \partial^{(i,j)}\Psi_N(x,y_k)dx^i dy_k^j\, \mathcal{F}_V^{(g)}(\ldots;u(j)v_k,y_k;\ldots).\end{aligned}$$

[4]Note the finite indexing here due to the Π projection matrix.

We can extend Theorem 7.1 to U-modules for a subVOA U of V. Let V be a VOA with subVOA U, where V has a U-module decomposition $V = \bigoplus_{\alpha\in A} W_\alpha$, where A is some indexing set. Let $\boldsymbol{W_\alpha}$ denote the g-fold tensor product $\boldsymbol{W_\alpha} = \bigotimes_{a=1}^{g} W_{\alpha_a}$, for some choice of modules $W_{\alpha_1}, W_{\alpha_2}, \ldots, W_{\alpha_g}$. Then we obtain the following reduction formula for n-point functions of (121)

Theorem 7.2. *The* $(n+1)$*-point function* $\mathcal{F}^{(g)}_{\boldsymbol{W_\alpha}}(u, x; \boldsymbol{v}, \boldsymbol{y})$ *for a quasiprimary vector* $u \in U$ *of weight* N *inserted at* $x \in \mathcal{S}^{(g)}$ *and* $v_1, \ldots, v_n \in V$ *inserted at* $y_1, \ldots, y_n \in \mathcal{S}^{(g)}$ *respectively, satisfies the recursive identity*

$$\begin{aligned}\mathcal{F}^{(g)}_{\boldsymbol{W_\alpha}}(u, x; \boldsymbol{v}, \boldsymbol{y}) &= \Phi(x) O_{\boldsymbol{W_\alpha}}(u; \boldsymbol{v}, \boldsymbol{y}) \\ &\quad + \sum_{k=1}^{n} \sum_{j\geq 0} \partial^{(0,j)} \Psi_N(x, y_k) dy_k^j\, \mathcal{F}^{(g)}_{\boldsymbol{W_\alpha}}(\ldots; u(j)v_k, y_k; \ldots). \qquad (139)\end{aligned}$$

8 The Geometric Meaning of $\Psi_N(x,y)$ and $\Phi(x)$

We now examine the geometry of $\Psi_N(x,y)$ and $\Phi(x)$. In particular, we find that the genus g $(N,1-N)$-differential Ψ_N can be written as a Poincaré sum over the genus g Schottky group. We introduce a genus zero $(N,1-N)$-differential of Bers [Be] for which this Poincaré sum converges. Lastly, we develop a relationship between Ψ_N, its periods over a β-cycle and a spanning set of holomorphic N-differentials for the genus g Riemann surface.

8.1 Moment analysis of $\Psi_N^{(0)}(x,y)$

We first note some Laurent expansions of $\psi_N^{(0)}(x,y)$. Let us assume that the $f_\ell(x)$ coefficient functions of (111) are holomorphic in a neighbourhood of w_a with radius of convergence r_a for all $a\in\mathcal{I}$. Then

$$\psi_N^{(0)}(x,y)=\frac{1}{x-y}+\sum_{\ell=0}^{2N-2}f_\ell(x)y^\ell,$$

can be expanded in a neighbourhood of (w_a,w_b) for $a\neq b$. Recall the definitions of the vectors $p(x)$ and $q(y)$ and the matrix R

$$\begin{aligned}
p_a(x;m)&=\rho_a^{\frac{m}{2}}\partial^{(0,m)}\psi_N^{(0)}(x,w_a),\\
q_a(y;m)&=(-1)^N\rho_a^{\frac{m+1}{2}}\partial^{(m,0)}\psi_N^{(0)}(w_{-a},y),\\
R_{ab}(m,n)&=\begin{cases}(-1)^N\rho_a^{\frac{m+1}{2}}\rho_b^{\frac{n}{2}}\partial^{(m,n)}\psi_N^{(0)}(w_{-a},w_b), & a\neq -b,\\ (-1)^N\rho_a^{\frac{m+n+1}{2}}\mathcal{E}_m^n(w_{-a}), & a=-b.\end{cases}
\end{aligned}$$

We can take Laurent expansions of these objects in an appropriate domain

$$\begin{aligned}
\psi_N^{(0)}(x,w_a+y_a)&=\sum_{m\geq 0}\rho_a^{-\frac{m}{2}}p_a(x;m)y_a^m \text{ for } |y_a|<|x-w_a|,\\
\psi_N^{(0)}(w_{-a}+x_{-a},y)&=(-1)^N\sum_{m\geq 0}\rho_a^{-\frac{m+1}{2}}q_a(y;m)x_{-a}^m \text{ for } |x_{-a}|<|y-w_{-a}|,
\end{aligned}$$

for $a\in\mathcal{I}$. Similarly, for $a\neq -b$ we have

$$\begin{aligned}
p_b(w_{-a}+x_{-a};n)&=(-1)^N\sum_{m\geq 0}\rho_a^{-\frac{m+1}{2}}R_{ab}(m,n)x_{-a}^m,\ \text{ for } |x_{-a}|<|w_{-a}-w_b|,\\
q_a(w_b+y_b;n)&=\sum_{m\geq 0}\rho_b^{-\frac{n}{2}}R_{ab}(m,n)y_b^m,\ \text{ for } |y_b|<|w_{-a}-w_b|.
\end{aligned}$$

8.2 $\Psi_N(x,y)$ as a formal Poincaré sum

We want to show that $\Psi_N(x,y)$ can be obtained via a Poincaré sum over the genus g Schottky group. We recall the Schottky sewing relations

$$\begin{aligned}
(z'-w_{-a})(z-w_a)&=\rho_a,\\
z'=w_{-a}+\frac{\rho_a}{z-w_a}&=\gamma_a z,
\end{aligned}$$

with associated differential

$$d(\gamma_a z)=\frac{d}{dz}\left(w_{-a}+\frac{\rho_a}{z-w_a}\right)dz=\frac{-\rho_a}{(z-w_a)^2}dz.$$

Recall that γ_a maps the contour $\mathcal{C}_a$ to $-\mathcal{C}_{-a}$. Let D_a denote the disc bounded by the contour $\mathcal{C}_a$, so that $\mathcal{C}_a = \partial D_a$. We consider the region $\mathcal{D} \subset \mathcal{S}^{(g)}$, where $\mathcal{D}$ is given by

$$\mathcal{D} = \widehat{\mathbb{C}} - \bigcup_{a\in\mathcal{I}} D_a.$$

Recall that γ_a maps points in the exterior of $\mathcal{C}_a$ to the interior of $\mathcal{C}_{-a}$ and vice versa. Then for any $x \in \mathcal{D}$ (i.e. exterior to all discs D_a), we have that $|\gamma_a x - w_{-a}| < |w_b - w_{-a}|$ for $a \neq -b$.

We first recall that $\psi_N^{(0)}$ has the expansion

$$\psi_N^{(0)} = \frac{1}{x-y} + \sum_{\ell=0}^{2N-2} f_\ell(x) y^\ell.$$

Recall that the genus g object $\Psi_N(x,y)$ is given by the formal sum (137). Using definitions (125), (131) and (132) we find

$$\begin{aligned}
&\left(\widetilde{P}_a(x)\widetilde{R}_{ab}\right)(n) \\
&= \sum_{m\geq 0} \left(\rho_a^{\frac{m+2N-1}{2}} \partial^{(0,m+2N-1)} \psi_N^{(0)}(x, w_a)\right) \\
&\quad \times \left((-1)^N \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n+2N-1}{2}} \partial^{(m,n+2N-1)} \psi_N(w_{-a}, w_b)\right) dx^N.
\end{aligned}$$

We again note that the higher order derivative terms annihilate the $f_\ell(w_a)y^\ell$ terms (as $0 \leq \ell \leq 2N-2$) and leave a single pole

$$\begin{aligned}
\widetilde{P}_a(x)\widetilde{R}_{ab}(n) &= \sum_{m\geq 0} \frac{(-1)^N \rho_a^{m+N}}{(x-w_a)^{m+2N}} \partial^{(m,n+2N-1)} \psi_N^{(0)}(w_{-a}, w_b) dx^N \\
&= \sum_{m\geq 0} \left(\frac{\rho_a}{x-w_a}\right)^m \partial^{(m,n+2N-1)} \psi_N^{(0)}(w_{-a}, w_b) \left(\frac{-\rho_a dx}{(x-w_a)^2}\right)^N.
\end{aligned}$$

This is convergent in the domain $|\gamma_a x - w_{-a}| < |w_{-a} - w_b|$. Reading this sum as a Taylor series we find

$$\begin{aligned}
\left(\widetilde{P}_a(x)\widetilde{R}_{ab}\right)(n) &= \partial^{(0,n+2N-1)} \psi_N^{(0)} \left(w_{-a} + \frac{\rho_a}{x-w_a}, w_b\right) d(\gamma_a x)^N \\
&= \partial^{(0,n+2N-1)} \psi_N^{(0)} (\gamma_a x, w_b)\, d(\gamma_a x)^N \\
&= p_b(\gamma_a x; n+2N-1) d(\gamma_a x)^N \\
&= P_b(\gamma_a x; n+2N-1) \\
&= \widetilde{P}_b(\gamma_a x; n),
\end{aligned}$$

so that

$$\widetilde{P}_a(x)\widetilde{R}_{ab} = \widetilde{P}_b(\gamma_a x),$$

for $a \neq -b$. Let $x \in \mathcal{D}$ so that $x \notin D_a$, hence $\gamma_a x \in D_{-a}$. Then we must have $\gamma_a x \notin D_b$ and hence the inequality $|\gamma_a x - w_{-a}| < |w_{-a} - w_b|$ is satisfied. We can now extend this to higher powers of $\widetilde{R}$. Using the same ideas, we find that for $a \neq -b$, $b \neq -c$

$$\widetilde{P}_a(x)\widetilde{R}_{ab}\widetilde{R}_{bc} = \widetilde{P}_b(\gamma_a x)\widetilde{R}_{bc} = \widetilde{P}_c(\gamma_b\gamma_a x),$$

since $\gamma_a x \in D_{-a}$ implying $\gamma_a x \notin D_b$ so that $\gamma_b\gamma_a x \in D_{-b}$ and hence $\gamma_b\gamma_a x \notin D_c$. We extend this using induction to find that

$$\widetilde{P}_{a_k}(x)\widetilde{R}_{a_k a_{k-1}}\widetilde{R}_{a_{k-1}a_{k-2}}\ldots\widetilde{R}_{a_2 a_1} = \widetilde{P}_{a_1}(\gamma_{a_2}\ldots\gamma_{a_k}x). \tag{140}$$

We now examine the term $\widetilde{P}_a(x)Q_a(y)$ (using (125), (130), (132))

$$\widetilde{P}_a(x)Q_a(y) = \sum_{m\geq 0}\widetilde{P}_a(x;m)Q_a(y;m) = \sum_{m\geq 0}P_a(x;m+2N-1)Q_a(y;m)$$
$$= \sum_{m\geq 0}\left(\rho_a^{\frac{m+2N-1}{2}}\partial^{(0,m+2N-1)}\psi_N^{(0)}(x,w_a)\right)\left((-1)^N\rho_a^{\frac{m+1}{2}}\partial^{(m,0)}\psi_N^{(0)}(w_{-a},y)\right)dx^N dy^{1-N}.$$

We note that the $\widetilde{P}_a(x;l)$ terms are again independent of the f_ℓ functions, however the $\partial^{(l,0)}$ derivative does not necessarily annihilate the $f_l(x)$ terms and we still need to ensure the convergence of these functions. We assume that for all $0 \leq \ell \leq 2N-2$, $f_\ell(x)$ is holomorphic in the region $|x - w_a| < r_a$, $a \in \mathcal{I}$ for some r_a. This gives

$$\begin{aligned}\widetilde{P}_a(x)Q_a(y) &= \sum_{m\geq 0}\frac{(-1)^N\rho_a^{m+N}}{(x-w_a)^{m+2N}}\partial^{(m,0)}\psi_N^{(0)}(w_{-a},y)dx^N dy^{1-N}\\ &= \left(\frac{-\rho_a}{(x-w_a)^2}\right)^N\sum_{m\geq 0}\left(\frac{\rho_a}{x-w_a}\right)^m\partial^{(m,0)}\psi_N^{(0)}(w_{-a},y)dx^N dy^{1-N}.\end{aligned}$$

We notice that this is a Taylor expansion in the domain

$$\{x,y : |\gamma_a x - w_{-a}| < |y - w_{-a}| \text{ and } |\gamma_a x - w_{-a}| < r_{-a}\}. \tag{141}$$

The first inequality is guaranteed for $x, y \in \mathcal{D}$, and the second gives convergence of $\psi_N^{(0)}(w_{-a}, y)$. We can then rewrite the above

$$\begin{aligned}\widetilde{P}_a(x)Q_a(y) &= \psi_N^{(0)}\left(w_{-a} + \frac{\rho_a}{x-w_a}, y\right)\left(\frac{-\rho_a}{(x-w_a)^2}dx\right)^N dy^{1-N}\\ &= \psi_N^{(0)}\left(\gamma_a x, y\right)d(\gamma_a x)^N dy^{1-N}\\ &= \Psi_N^{(0)}(\gamma_a x, y),\end{aligned}$$

giving the following

$$\widetilde{P}_a(x)Q_a(y) = \Psi_N^{(0)}\left(\gamma_a x, y\right),$$

in the domain (141). Using this result and (140), we find that for $x, y \in \mathcal{D}$, $a_i \in \mathcal{I}$, $i = 1, \ldots k$ and $a_i \neq -a_{i+1}$

$$\widetilde{P}_{a_k}(x)\widetilde{R}_{a_k a_{k1}}\widetilde{R}_{a_{k1}a_{k2}}\ldots\widetilde{R}_{a_2 a_1}Q_{a_1}(y) = \Psi_N(\gamma_{a_1}\gamma_{a_2}\ldots\gamma_{a_k}x, y), \tag{142}$$

for $|\gamma_a x - w_{-a}| < r_{-a_1}$ with $\gamma_{a_1}\gamma_{a_2}\ldots\gamma_{a_k}$ a reduced Schottky word of length k. Then assuming that the $\Psi_N(\gamma x, y)$ terms converge, we sum over all $a_k \in \mathcal{I}$ (noting that $\widetilde{R}_{-a,a} = \widetilde{R}_{a,-a} = 0$) to find

$$\widetilde{P}(x)R^n Q(y) = \sum_{\gamma\in\Gamma_n}\Psi_N^{(0)}(\gamma x, y),$$

where Γ_n denotes the set of reduced Schottky words of length n. Then the formal sum (137) reads

$$\begin{aligned}\Psi_N^{(g)}(x,y) &= \widetilde{P}(x)(I-\widetilde{R})^{-1}Q(y)\\ &= \sum_{n\geq 0} P(x)\widetilde{R}^n Q(y)\\ &= \sum_{n\geq 0}\sum_{\gamma\in\Gamma_n} \Psi_N^{(0)}(\gamma x,y)\\ &= \sum_{\gamma\in\Gamma} \Psi_N^{(0)}(\gamma x,y),\end{aligned}$$

i.e. a Poincaré sum over all reduced words of the Schottky group. This gives the result

Theorem 8.1. *$\Psi(x,y)$ of (137) can be written as the formal Poincaré sum*

$$\Psi_N(x,y) = \sum_{\gamma\in\Gamma} \Psi_N^{(0)}(\gamma x,y), \tag{143}$$

where Γ is the genus g Schottky group.

Remark 8.1. Later in §8.4 we show that this Poincaré sum is convergent for an appropriate choice of $\Psi_N^{(0)}(x,y)$.

8.3 Relating $\Phi(x)$ and $\Psi_N(x,y)$

We show that the genus g coefficient $\Psi_N(x,y)$ is a formal N-differential in x and a quasi-$(1-N)$-differential in y. We have that

Proposition 8.1. *Let $y = w_a + z_a$. Then $\Psi_N(x,y)$ is quasiperiodic in y over a β_a cycle determined by the Schottky group generator γ_a*

$$\Psi_N(x,y) - \Psi_N(x,\gamma_a y) = \sum_{\ell=0}^{2N-2} \Theta_a(x,\ell) z_a^{\ell} dz_a^{1-N}, \tag{144}$$

where the coefficients $\Theta_a(x,\ell)$ are formal N-differentials in x given by

$$\Theta_a(x,\ell) = \sum_{\gamma\in\Gamma} T_a(\gamma x,\ell) d(\gamma x)^N, \tag{145}$$

with

$$\begin{aligned}T_a(x,\ell) = {}& \frac{1}{(x-w_a)^{\ell+1}} + \sum_{k=\ell}^{2N-2} \binom{k}{\ell} f_k(x) w_a^{k-\ell}\\ &+ (-1)^N \sum_{k=0}^{\ell} \rho_a^{N-1-\ell} \binom{2N-2-k}{k-\ell} f_{2N-2-k}(x) w_{-a}^{\ell-k}.\end{aligned} \tag{146}$$

for $a\in\mathcal{I}_+$, $0\leq \ell\leq 2N-2$ and $f_\ell(x)$ of (111).

Proof. We can write the difference as

$$\Psi_N(x,y) - \Psi_N(x,\gamma_a y)$$

$$= \sum_{\gamma\in\Gamma} \left(\psi_N^{(0)}(\gamma x, y) d(\gamma x)^N dy^{1-N}) - \psi_N^{(0)}(\gamma x, \gamma_a y) d(\gamma x)^N d(\gamma_a y)^{1-N} \right). \tag{147}$$

This can be rewritten as

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \left(\psi_N^{(0)}(\gamma x, y) d(\gamma x)^N dy^{1-N} - \psi_N^{(0)}((\gamma_a\gamma) x, \gamma_a y) d((\gamma_a\gamma) x)^N d(\gamma_a y)^{1-N} \right) \\
&= \sum_{\gamma\in\Gamma} \left(\frac{d(\gamma x)^N dy^{1-N}}{\gamma x - y} - \frac{d((\gamma_a\gamma)x)^N d(\gamma_a y)^{1-N}}{\gamma_a\gamma x - \gamma_a y} \right) \\
&\quad + \sum_{\gamma\in\Gamma} \sum_{\ell=0}^{2N-2} \left(f_\ell(\gamma x) y^\ell d(\gamma x)^N dy^{1-N} - f_\ell(\gamma_a\gamma x)(\gamma_a y)^\ell d(\gamma_a\gamma x)^N d(\gamma_a y)^N \right).
\end{aligned}$$

We evaluate these two terms separately. Recall that for a Möbius transformation $\gamma x = \frac{ax+b}{cx+d}$ with $ad - bc = 1$ we have

$$d(\gamma x) = \frac{1}{(cx+d)^2} dx,$$

and

$$\gamma x - \gamma y = \frac{x-y}{(cx+d)(cy+d)}.$$

We examine the genus zero summands, i.e.

$$\begin{aligned}
&\frac{dx^N dy^{1-N}}{x-y} - \frac{d(\gamma_a x)^N d(\gamma_a y)^{1-N}}{\gamma_a x - \gamma_a y} \\
&= \frac{dx^N dy^{1-N}}{x-y} - \frac{\left(-\frac{\rho_a}{(x-w_a)^2}\right)^N \left(-\frac{\rho_a}{(y-w_a)^2}\right)^{1-N} dx^N dy^{1-N}}{w_{-a} + \frac{\rho_a}{x-w_a} - w_{-a} - \frac{\rho_a}{y-w_a}} \\
&= \frac{1}{x-y} dx^N dy^{1-N} - \frac{1}{x-y} \left(\frac{y-w_a}{x-w_a} \right)^{2N-1} dx^N dy^{1-N} \\
&= \frac{1}{x-y} \left(1 - \frac{y-w_a}{x-w_a} \right)^{2N-1} dx^N dy^{1-N},
\end{aligned} \tag{148}$$

using $\gamma_a x = w_{-a} + \frac{\rho_a}{x-w_a}$, $d(\gamma_a x) = -\frac{\rho_a}{(x-w_a)^2} dx$, and similarly for y. We can expand (148) as follows

$$\begin{aligned}
&\frac{dx^N dy^{1-N}}{x-y} - \frac{d(\gamma_a x)^N d(\gamma_a y)^{1-N}}{\gamma_a x - \gamma_a y} \\
&= \frac{1}{x-y} \left(1 - \frac{y-w_a}{x-w_a} \right) \sum_{\ell=0}^{2N-2} \left(\frac{y-w_a}{x-w_a} \right)^\ell dx^N dy^{1-N} \\
&= \frac{1}{x-y} \cdot \frac{x-y}{x-w_a} \sum_{\ell=0}^{2N-2} \left(\frac{y-w_a}{x-w_a} \right)^\ell dx^N dy^{1-N} \\
&= \sum_{\ell=0}^{2N-2} \frac{(y-w_a)^\ell}{(x-w_a)^{\ell+1}} dx^N dy^{1-N}.
\end{aligned}$$

Letting $y = z_a + w_a$, we find

$$\frac{dx^N dz_a^{1-N}}{x - z_a} - \frac{d(\gamma_a x)^N d(\gamma_a y)^{1-N}}{\gamma_a x - \gamma_a z_a} = \sum_{\ell=0}^{2N-2} \frac{z_a^\ell}{(x-w_a)^{\ell+1}} dx^N dz_a^{1-N}.$$

The remaining genus zero components are (letting $y = w_a + z_a$)

$$\begin{aligned}
&\sum_{\ell=0}^{2N-2} \Big(f_\ell(x) y^\ell dx^N dy^{1-N} - f_\ell(\gamma_a x)(\gamma_a y)^\ell d(\gamma_a x)^N d(\gamma_a y)^{1-N} \Big) \\
&= \sum_{\ell=0}^{2N-2} \Bigg(f_\ell(x)(w_a + z_a)^\ell \\
&\qquad\qquad -f_\ell(\gamma_a x) \left(\frac{-\rho_a}{(x-w_a)^2} \right)^N \left(w_{-a} + \frac{\rho_a}{z_a} \right)^\ell \left(-\frac{\rho_a}{z_a^2} \right)^{1-N} \Bigg) dx^N dz_a^{1-N} \\
&= \sum_{\ell=0}^{2N-2} \Bigg(f_\ell(x)(w_a + z_a)^\ell + \\
&\qquad\qquad (-1)^N \rho_a^{1-N} f_\ell(\gamma_a x) \left(\frac{-\rho_a}{(x-w_a)^2} \right)^N (w_{-a} z_a + \rho_a)^\ell z_a^{2N-2-\ell} \Bigg) dx^N dz_a^{1-N}.
\end{aligned}$$

Putting everything together, we find

$$\begin{aligned}
&\Psi_N(x, y) - \Psi_N(x, \gamma_a y) \\
&= \sum_{\gamma\in\Gamma} \sum_{\ell=0}^{2N-2} f_\ell(\gamma x) \Big((w_a + z_a)^\ell + (-1)^N \rho_a^{1-N} (w_{-a} z_a + \rho_a)^\ell z_a^{2N-2-\ell} \Big) d(\gamma x)^N dz_a^{1-N} \\
&\quad + \sum_{\gamma\in\Gamma} \sum_{\ell=0}^{2N-2} \frac{z_a^\ell}{(\gamma x - w_a)^{\ell+1}} d(\gamma x)^N dz_a^{1-N} \\
&= \sum_{\ell=0}^{2N-2} \Theta_a(x, \ell) z_a^\ell dz_a^{1-N},
\end{aligned}$$

for a formal N-differential $\Theta_a(x, \ell)$, exploiting the property of the shifted Poincaré sum which allows us to write $f_\ell(\gamma\gamma_a x)$ as $f_\ell(\gamma x)$. Then we have

$$\Psi_N(x, \gamma_a y) - \Psi_N(x, y) = \sum_{\gamma\in\Gamma} F_N(\gamma x, z_a) d(\gamma x)^N dz_a^{1-N},$$

with

$$\begin{aligned}
F_N(x, z_a) &= \sum_{\ell=0}^{2N-2} \frac{z_a^\ell}{(x-w_a)^{\ell+1}} \\
&\qquad + \sum_{k=0}^{2N-2} f_k(x) \Big((w_a + z_a)^k + (-1)^N \rho_a^{1-N} (w_{-a} z_a + \rho_a)^k z_a^{2N-2-k} \Big).
\end{aligned}$$

We want to rewrite this expression as a polynomial in z_a. Expanding the binomials we find

$$\begin{aligned}
F_N(x, z_a) &= \sum_{\ell=0}^{2N-2} \frac{z_a^\ell}{(x-w_a)^{\ell+1}} \\
&\qquad + \sum_{k=0}^{2N-2} f_k(x) \sum_{\ell=0}^{k} \binom{k}{\ell} \Big(w_a^{k-\ell} z_a^\ell + (-1)^N \rho_a^{1-N+\ell} w_{-a}^{k-\ell} z_a^{2N-2-\ell} \Big) \\
&= \sum_{\ell=0}^{2N-2} \frac{z_a^\ell}{(x-w_a)^{\ell+1}} + \sum_{k=0}^{2N-2} f_k(x) \sum_{\ell=0}^{k} \binom{k}{\ell} w_a^{k-\ell} z_a^\ell
\end{aligned}$$

$$+(-1)^N\sum_{k=0}^{2N-2} f_k(x)\sum_{\ell=0}^{k}\binom{k}{\ell}\rho_a^{1-N+\ell}w_{-a}^{k-\ell}z_a^{2N-2-\ell}.$$

Exploiting the symmetry of the binomial coefficient on the second summand and reindexing the third under $k\to 2N-2-k$, $\ell\to 2N-2-\ell$, we find

$$\begin{aligned}
F_N &= \sum_{\ell=0}^{2N-2}\frac{z_a^\ell}{(x-w_a)^{\ell+1}}+\sum_{k=0}^{2N-2}\sum_{\ell=0}^{k}\binom{k}{\ell}f_k(x)w_a^{k-\ell}z_a^\ell\\
&\quad+(-1)^N\sum_{\ell=0}^{2N-2}\sum_{k=0}^{\ell}\binom{2N-2-k}{2N-2-\ell}f_{2N-2-k}(x)\rho_a^{N-1-\ell}w_{-a}^{\ell-k}z_a^\ell\\
&=\sum_{\ell=0}^{2N-2}\left(\frac{1}{(x-w_a)^{\ell+1}}+\sum_{k=\ell}^{2N-2}f_k(x)\binom{k}{\ell}w_a^{k-\ell}\right.\\
&\quad\left.+(-1)^N\sum_{k=0}^{\ell}\binom{2N-2-k}{\ell-k}f_{2N-2-k}(x)\rho_a^{N-1-\ell}w_{-a}^{\ell-k}\right)z_a^\ell\\
&=\sum_{\ell=0}^{2N-2}T_a(x,\ell)z_a^\ell,
\end{aligned}$$

with $T_a(x,\ell)$ as given in (146). □

We can further analyse Θ of Proposition 8.1. Consider the term

$$\Phi(x)O(u;\boldsymbol{v},\boldsymbol{y})=\sum_{a\in\mathcal{I}}\sum_{\ell=0}^{2N-2}\Phi_a(x,\ell)O_a(u;\boldsymbol{v},\boldsymbol{y};\ell), \tag{149}$$

in the genus g Zhu reduction formula (135), which can be rewritten as

$$\Phi(x)O(u;\boldsymbol{v},\boldsymbol{y})=\sum_{a\in\mathcal{I}_+}\left(\sum_{\ell=0}^{2N-2}\Phi_a(x,\ell)O_a(\ell)+\sum_{\ell=0}^{2N-2}\Phi_{-a}(x,\ell)O_{-a}(\ell)\right).$$

Recalling relation (128) we find the following relation for O_a

$$O_a(\ell)=(-1)^N\rho_a^{\ell+1-N}O_{-a}(2N-2-\ell),$$

then (149) becomes

$$\Phi(x)O(u;\boldsymbol{v},\boldsymbol{y})=\sum_{a\in\mathcal{I}_+}\sum_{\ell=0}^{2N-2}\Big(\Phi_a(x,\ell)O_a(\ell)+(-1)^N\rho_a^{\ell+1-N}\Phi_{-a}(x,\ell)O_a(2N-2-\ell)\Big),$$

which upon reindexing for the second summand gives

$$\Phi(x)O(u;\boldsymbol{v},\boldsymbol{y})=\sum_{a\in\mathcal{I}_+}\sum_{\ell=0}^{2N-2}\Big(\Phi_a(x,\ell)+(-1)^N\rho_a^{N-1-\ell}\Phi_{-a}(x,2N-2-\ell)\Big)O_a(\ell).$$

We now have

$$\Phi(x)O(u,\boldsymbol{v},\boldsymbol{y})=\sum_{a\in\mathcal{I}_+}\Theta_a(x)O_a(u;\boldsymbol{v},\boldsymbol{y}). \tag{150}$$

We also notice that $\Psi_N(x,y)$ enjoys the Laurent expansion

$$\Psi_N(x, w_a + z_a) = \sum_{m\in\mathbb{Z}} \rho_a^{-\frac{m}{2}} \left(P(x) + \widetilde{P}(x)(I-\widetilde{R})^{-1}R\right)_a (m) z_a^m dz_a^{1-N}. \tag{151}$$

Then in the neighbourhood of w_a, $\psi_N(x,y)$ enjoys the following Laurent expansion

$$\Psi_N(x,y) = \ldots + \sum_{\ell=0}^{2N-2} \Phi_a(x,\ell) z_a^\ell dz_a^{1-N} + \ldots, \tag{152}$$

for $y = w_a + z_a$. Now consider the change in $\psi_N(x,y)$ under the Möbius map

$$y \to \gamma_a y = w_{-a} + \frac{\rho_a}{y - w_a},$$

so that

$$w_a + z_a \mapsto w_{-a} + z_{-a},$$

with $z_a z_{-a} = \rho_a$ as usual. Using (152) we find (letting $y = w_a + z_a$)

$$\begin{aligned}
&\Psi_N(x,y) - \Psi_N(x,\gamma_a y)\\
&= \Psi_N(x, w_a + z_a) - \Psi_N(x, w_{-a} + z_{-a})\\
&= \sum_{\ell=0}^{2N-2} \left(\Phi_a(x,\ell) z_a^\ell dz_a^{1-N} - \Phi_{-a}(x,\ell) z_{-a}^\ell dz_{-a}^{1-N}\right)\\
&= \sum_{\ell=0}^{2N-2} \left(\Phi_a(x,\ell) z_a^\ell - \Phi_{-a}(x,\ell)\left(\frac{\rho_a}{z_a}\right)^\ell \left(\frac{-\rho_a}{z_a^2}\right)^{1-N}\right) dz_a^{1-N}\\
&= \sum_{\ell=0}^{2N-2} \left(\Phi_a(x,\ell) z_a^\ell + (-1)^N \rho_a^{\ell+1-N} \Phi_{-a}(x,\ell) z_a^{2N-2-\ell}\right) dz_a^{1-N}\\
&= \sum_{\ell=0}^{2N-2} \left(\Phi_a(x,\ell) + (-1)^N \rho_a^{N-1-\ell} \Phi_{-a}(x, 2N-2-\ell)\right) z_a^\ell dz_a^{1-N}\\
&= \sum_{\ell=0}^{2N-2} \Theta_a(x,\ell) z_a^\ell dz_a^{1-N}.
\end{aligned} \tag{153}$$

This gives

Proposition 8.2. *The formal differentials $\Phi(x)$ and $\Theta_a(x,\ell)$ enjoy the following relationship*

$$\Theta_a(x,\ell) = \Phi_a(x,\ell) + (-1)^N \rho_a^{N-1-\ell} \Phi_{-a}(x, 2N-2-\ell).$$

Remark 8.2. We note that equation (150) gives a refinement of the leading terms of the main Zhu reduction formula in Theorem 7.1.

Remark 8.3. We note a correspondence with the structure of $\Psi_N(x,y)$ observed in [Be], [T2]. In [T2] we see that

$$\Psi_N(x,y) - \Psi_N(x,\gamma y) = \sum_{r=1}^{d_N} \Phi_r^\vee(x) \widehat{\Xi}_r[\gamma](y),$$

where d_N is the dimension of the space of holomorphic N-differentials on the Riemann surface, $\Phi_r^\vee(x)$ denotes an holomorphic N-differential in x in the dual basis of holomorphic N-differentials with respect to the Petersson inner product, and $\widehat{\Xi}_r$ denotes a cocycle for the Schottky group element γ. The $\Phi_r^\vee(x)$ objects comprise a basis for the space of Petersson dual holomorphic N-differentials and the cocycles $\widehat{\Xi}_r[\gamma](y)$ similarly provides a basis. Using the Bers map [Be], from any cocycle basis element, we can retrieve an holomorphic N-differential. Here we have an overdetermined set $\{\Theta_a(x,\ell)\}$, so we can say that the $\Theta_a(x,\ell)$ objects provide a spanning set for the space of holomorphic N-differentials on $\mathcal{S}^{(g)}$. In [T2], we see that the cocycle

$$\Xi_{a\ell}[\gamma_b](z) = \delta_{ab} z_a^\ell dz^{1-N},$$

matches exactly the term that each $\Theta_a(x,\ell)$ multiplies.

We also obtain a corollary to Theorem 7.1

Corollary 8.1. *The genus g 1-point formal differential for a quasiprimary vector u of weight $\mathrm{wt}(u) = N$ inserted at $x \in \mathcal{S}^{(g)}$ is given by*

$$\mathcal{F}_V^{(g)}(u,x) = \sum_{a\in\mathcal{I}_+} \sum_{\ell=0}^{2N-2} \Theta_a(x,\ell) O_a(u;\ell),$$

where the $\Theta_a(x,\ell)$ are formal N-differentials in x for $x \in \mathcal{S}^{(g)}$.

Remark 8.4. We find that the holomorphic N-differentials $\Theta_a(x,\ell)$ and $\Phi_a(x,\ell)$ along with the $(N, 1-N)$-differentials $\Psi_N(x,y)$ are convergent objects for the Bers choice of Ψ_N outlined in § 8.4.

8.4 The Bers choice

We now make a choice for $\Psi_N^{(0)}$ for which Ψ_N and Θ are convergent objects. We will often refer to this choice of $\Psi_N^{(0)}$ as the *Bers choice.* For $N = 1$ we choose

$$\Psi_1^{(0)}(x,y) = \left(\frac{1}{x-y} - \frac{1}{x}\right) dx, \tag{154}$$

which is the genus zero 1-differential of the third kind (24) with $P_1 = 0$, $P_2 = y$ with simple poles at $x = y$ and $x = 0$; similarly, $\Psi_1^{(g)}(x,y)$ is the genus g 1-differential of the third kind $\omega_{y-0}^{(g)}$ of (23). The Poincaré sum (143) is convergent for $x \neq y$, $x \neq 0$. For $N \geq 2$ we make the choice [Be]

$$\Psi_N^{(0)}(x,y) = \frac{1}{x-y} \prod_{j=0}^{2N-2} \frac{y-A_j}{x-A_j} dx^N dy^{1-N}, \tag{155}$$

where the $A_j \in \mathbb{C}$ are $2N-1$ distinct limit points for Γ. $\Psi_N^{(0)}$ is $\mathrm{SL}(2,\mathbb{C})$-invariant for $N \geq 1$, where $\mathrm{SL}(2,\mathbb{C})$ acts on x, y and A_j. The Poincaré sum (143) is convergent on $\mathcal{S}^{(g)}$ for this choice and yields a holomorphic (for $x \neq y$) N-differential in x and a $(1-N)$-quasidifferential in y with cocycles over the cycles β_a of $\mathcal{S}^{(g)}$.

Likewise the coefficients $\Theta_a(x,\ell)$ are genus g holomorphic N-differentials which span the set of holomorphic N-differentials on $\mathcal{S}^{(g)}$. The Bers choice gives a convergent Poincaré sum in (145).

Remark 8.5. We note that (154) is not a special case of (155) for $N = 1$, $A_0 = 0$ as 0 is an ordinary point of Γ as opposed to a limit point.

Theorem 8.2. *For the Bers choice* $\Psi_N^{(0)}$ *of* (154), (155), *the coefficients* $\Theta(x)$, $\Psi_N(x, y)$ *in the genus* g *Zhu reduction formula (Theorem 7.1) are convergent on* $\mathcal{S}^{(g)}$ *for* $x \neq y$, *using* (150) *and Remark 8.4. Furthermore,* $\{\Theta_a(x, \ell)\}$ *for* $a = 1, \ldots, g$, $\ell = 0, \ldots, 2N - 2$ *spans the vector space of holomorphic* N*-differentials on the* $\mathcal{S}^{(g)}$.

9 Some Applications of Genus g Zhu Reduction

We now examine some applications of Theorem 8.2. As mentioned before, Zhu reduction is a powerful technique for developing differential equations for partition functions, n-point differentials and differential forms on a Riemann surface. Using this new formula, we can develop these equations for arbitrary genus $g \geq 2$. We generalise known results at weight one and for the Heisenberg VOA and its modules. We analyse the relations given by the Virasoro vector for the Heisenberg VOA, and develop general genus Ward identities which extend the results of (44), (45) and the genus two results of [GT1]. We also obtain a general genus identity for the Siegel theta function using modules for general rank lattice Heisenberg VOAs.

From this section forward, we use the Bers choice (154), (155) for which $\Psi_N(x, y)$ and $\Theta_a(x, \ell)$ are convergent.

9.1 The weight one 1-point differential and the Heisenberg VOA

We first examine Zhu reduction for the 1-point formal differential $\mathcal{F}_V^{(g)}(u, x)$ given by a weight one state u. By Corollary 8.1, we have that

$$\mathcal{F}_V^{(g)}(u, x) = \sum_{a \in \mathcal{I}_+} \Theta_a(x) O_a(u).$$

Here we omit the zero index for clarity. Then

$$\Theta_a(x) = \Phi_a(x) - \Phi_{-a}(x) = \Psi_1(x, y) - \Psi_1(x, \gamma_a y),$$

using Lemma 8.2. We recall that for $N = 1$, the 1-differentials are given by the change in $\Psi_1(x, y)$ over a β-cycle

$$\Theta_a(x) = \nu_a(x),$$

where $\nu_a(x)$ is the normalised genus g 1-differential. Recall that for $N = 1$, $\Psi_1(x, y)$ is given by the Poincaré sum of the genus zero differential of the third kind (24) over the Schottky group

$$\begin{aligned}\Psi_1(x, y) &= \sum_{\gamma \in \Gamma} \left(\frac{1}{\gamma x - y} - \frac{1}{\gamma x} \right) d(\gamma x) \\ &= \theta(x, y) - \theta(x, 0),\end{aligned}$$

in the notation of [Bu], where[5]

$$\theta(x, y) = \sum_{\gamma \in \Gamma} \frac{1}{\gamma x - y} d(\gamma x).$$

By op. cit. p.59, we have that

$$\Psi_1(x, y) - \Psi_1(x, \gamma y) = \theta(x, -\tfrac{d}{c}),$$

[5]We note the slight streamlining of notation and the inclusion of differential factors as compared to [Bu].

where γ is a Möbius transformation of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For $\gamma = \gamma_a$ (a Schottky generator), $-\frac{d}{c} = w_a$ which gives

$$\Psi_1(x,y) - \Psi_1(x,\gamma_a y) = \theta(x, w_a). \tag{156}$$

Hence we make the identification

$$\nu_a(x) = \theta(x, w_a),$$

with normalisation

$$\frac{1}{2\pi\mathrm{i}} \oint_{\alpha_a} \nu_a = \delta_{ab},$$

where α_a is identified with the contour $\mathcal{C}_{-a}$ (e.g. see [Bob] p.57). Hence the Zhu reduction formula yields

Proposition 9.1.

$$\mathcal{F}_V^{(g)}(u,x) = \sum_{a\in\mathcal{I}_+} \nu_a(x) O_a(u).$$

Remark 9.1. This result gives us a method of computing $O_a(u)$

$$O_a(u) = \frac{1}{2\pi\mathrm{i}} \oint_{\alpha_a} \mathcal{F}_V^{(g)}(u,x),$$

using the normalisation property (21) of ν_a.

We now examine genus g Zhu reduction for the rank one Heisenberg VOA M. For odd n, the n-point differentials are zero, however for even n we obtain geometric results. We note that for a Heisenberg state, $O_a = 0$ for all $a = 1\ldots, g$. The proof then follows from the Zhu reduction formula of Theorem 7.1. Then for $n = 2$ we find

$$\begin{aligned}\mathcal{F}_M^{(g)}(h,x;h,y) &= \partial^{(0,1)}\Psi_1(x,y)dy\, Z_M^{(g)} \\ &= \omega(x,y) Z_M^{(g)},\end{aligned}$$

using relation (11), where $\omega(x,y)$ is the genus g bidifferential of the second kind (18) and $Z_M^{(g)}$ is the genus g partition function. We can generalise this to any even n

Proposition 9.2.

$$\mathcal{F}_M^{(g)}(h,x_1;h,x_2;\ldots;h,x_n) = (\mathrm{Sym}_n\omega)\, Z_M^{(g)},$$

where $\mathrm{Sym}_n\omega$ *is the symmetric product*

$$\mathrm{Sym}_n\omega = \sum_\sigma \prod_{(r,s)} \omega(x_r,x_s),$$

where σ *is a fixed-point-free involution* $\sigma = \ldots(rs)\ldots$ *of the labels* $\{1,2,\ldots,n\}$.

Proof.

$$\mathcal{F}_M^{(g)}(h,x_1;h,x_2;\ldots;h,x_n)$$

$$= \sum_{k=2}^{n} \sum_{j \geq 0} \partial^{(0,j)} \Psi_1(x_1, x_k) dx_k^j \mathcal{F}_M^{(g)}(h, x_2; \ldots; h(j)h, x_k; \ldots; h, x_n)$$
$$= \sum_{k=2}^{n} \sum_{j \geq 0} \omega(x_1, x_k) \mathcal{F}_M^{(g)}(h, x_2; \ldots; h(j)h, x_k; \ldots; h, x_n)$$
$$= \sum_{k=2}^{n} \omega(x_1, x_k) \mathcal{F}_M^{(g)}(h, x_2; \ldots; \widehat{h, x_k}; \ldots; h, x_n),$$

where the caret denotes omission of the vector. The proof then continues in an inductive fashion to give

$$\mathcal{F}_M^{(g)}(h, x_1; h, x_2; \ldots; h, x_n) = (\mathrm{Sym}_n \omega)\, Z_M^{(g)}.$$

□

Remark 9.2. We note that $\mathcal{F}_M^{(g)}(h, x_1; h, x_2; \ldots; h, x_n)$ is a generating function for all n-point differentials of the Heisenberg VOA [MT2] (Prop. 3.8). This is due to the fact that the VOA is generated by the state h.

We now use the generating function to develop a differential equation for the partition function $Z_M^{(g)}$ using the 1-point function for the Virasoro vector. For the Heisenberg VOA, this vector is given by $\omega = \frac{1}{2}h(-1)h$. We first note using the limit method used in Section 5

$$\begin{aligned}\mathcal{F}_M^{(g)}(h, x; h, y) &= \sum_{\boldsymbol{b}_+} \langle \mathbb{1}, Y(h, x) Y(h, y) \boldsymbol{Y(b, w)} \mathbb{1} \rangle dxdy \\ &= \sum_{\boldsymbol{b}_+} \langle \mathbb{1}, Y(Y(h, x-y) Y(h, y)) \boldsymbol{Y(b, w)} \mathbb{1} \rangle dxdy \\ &= \sum_{\boldsymbol{b}_+} \sum_{m \in \mathbb{Z}} \langle \mathbb{1}, Y(h(m)h, y) \boldsymbol{Y(b, w)} \mathbb{1} \rangle (x-y)^{-m-1} dxdy,\end{aligned}$$

using VOA associativity. For $m > 1$ we have $h(m)h = 0$. We then find that

$$Z_M^{(g)}(h, x; h, y) - \frac{Z_M^{(g)}}{(x-y)^2} = Z_M^{(g)}(h(-1)h, y) + \mathcal{O}(x-y),$$

using $h(0)h = 0$ and $h(1)h = \mathbb{1}$. Multiplying by appropriate differential factors and taking the limit $y \to x$ we obtain

$$\lim_{y \to x} \frac{1}{2} \left(Z_M^{(g)}(h, x; h, y) - \frac{Z_M^{(g)}}{(x-y)^2} \right) dxdy = Z_M^{(g)}(\tfrac{1}{2}h(-1)h, x) dx^2,$$

giving

$$\lim_{y \to x} \frac{1}{2} \left(\omega(x, y) - \frac{dxdy}{(x-y)^2} \right) Z_M^{(g)} = Z_M^{(g)}(\tfrac{1}{2}h(-1)h, x) dx^2,$$

which yields the equation for the Virasoro vector ω

$$\mathcal{F}_M^{(g)}(\omega, x) = \frac{1}{12} s(x) Z_M^{(g)}. \tag{157}$$

where $s(x)$ is the genus g projective connection (19).

9.2 The Virasoro 1-point differential

We also obtain results for the genus g weight-two differential $\mathcal{F}_V^{(g)}(\omega, x)$

$$\mathcal{F}_V^{(g)}(\omega, x) = \sum_{a\in\mathcal{I}_+}\sum_{\ell=0}^{2}\Theta_a(x,\ell)O_a(\omega;\ell),$$

where $\Theta = (\Theta_a(x,\ell))$ is a vector of $3g$ holomorphic 2-differentials. We note

$$\begin{aligned}O_a(\omega;0) &= \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots;\omega(0)b_a, w_a;\ldots) = \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots;L(-1)b_a, w_a;\ldots)\\ &= \sum_{\boldsymbol{b}_+}\partial_{w_a}Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) = \partial_{w_a}Z_V^{(g)}.\end{aligned}$$

For $O_a(\omega;1)$ we find

$$O_a(\omega;1) = \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots;L(0)b_a, w_a;\ldots) = \sum_{\boldsymbol{b}_+} n_a Z^{(0)}(\boldsymbol{b},\boldsymbol{w}),$$

where n_a is the conformal weight of b_a. Using the definition of the genus g partition function (119) we see that

$$\begin{aligned}O_a(\omega;1) &= \sum_{n_a\geq 0}\sum_{b_a\in V_{n_a}} n_a\rho_a^{n_a}Z^{(0)}(\overline{b}_1, w_{-1}; b_1, w_1;\ldots;\overline{b}_g, w_{-g}; b_g, w_g)\\ &= \rho_a\partial_{\rho_a}Z_V^{(g)},\end{aligned}$$

using (115) and (116) where we recall $\overline{b}_a$ is the unscaled ρ_a-dependent dual vector to b_a. Lastly we find

$$\begin{aligned}O_a(\omega;2) &= \sum_{\boldsymbol{b}_+} Z^{(0)}(\ldots;L(1)b_a, w_a;\ldots)\\ &= \sum_{\boldsymbol{b}_+}\rho_a Z^{(0)}(\ldots;L(-1)b_{-a}, w_{-a};\ldots)\\ &= \sum_{\boldsymbol{b}_+}\rho_a\partial_{w_{-a}}Z^{(0)}(\boldsymbol{b},\boldsymbol{w}) = \rho_a\partial_{w_{-a}}Z_V^{(g)}.\end{aligned}$$

using the genus g adjoint relation. Let $\partial_{a,\ell}$ for $a\in\mathcal{I}_+$, $\ell = 0,1,2$ denote the following operators

$$\partial_{a,0} = \frac{\partial}{\partial w_a};\quad \partial_{a,1} = \rho_a\frac{\partial}{\partial\rho_a};\quad \partial_{a,2} = \rho_a\frac{\partial}{\partial w_{-a}}.$$

We define the differential operator

$$\nabla(x) = \sum_{a\in\mathcal{I}_+}\sum_{\ell=0}^{2}\Theta_a(x,\ell)\partial_{a,\ell}. \tag{158}$$

Remark 9.3. This is a natural generalisation of the $q\partial_q$ operator of genus one.

Remark 9.4. Using the SL$(2,\mathbb{C})$-invariance of the partition function (122) we can write $\nabla(x)$ as a sum of $3g-3$ differential operators with coefficients given by $3g-3$ independent holomorphic 2-differentials.

Then we have the following result

Proposition 9.3. *The Virasoro* 1*-point differential satisfies the differential equation*

$$\mathcal{F}_V^{(g)}(\omega, x) = \nabla(x) Z_V^{(g)}. \tag{159}$$

In the particular case of the genus g rank one Heisenberg VOA M, using (157) and (159) we obtain

Theorem 9.1. *The genus g partition function for the rank one Heisenberg VOA satisfies the following differential equation*

$$\left(\nabla(x) - \frac{1}{12}s(x)\right) Z_M^{(g)} = 0. \tag{160}$$

Remark 9.5. This is a genus g analogue of the classical modular-invariant differential equation

$$\left(q\partial_q - \frac{1}{2}E_2(q)\right) Z_M^{(1)} = 0,$$

with $q = e^{2\pi \mathrm{i}\tau}$ for the genus one partition function. We also find this to be a natural generalisation of Proposition 7.1 of [GT1].

9.3 Genus g Ward identities

Using 8.2, we can derive genus g Ward identities which are generalisations of those found in [GT1] (see Propositions 6.2 and 6.3) and equations (44), (45) by setting $u = \omega$, the Virasoro vector. We find

Theorem 9.2 (Genus g Ward Identity)**.** *For primary states v_k of weight* $\mathrm{wt}(v_k)$ *for $k = 1, \ldots, n$, the $(n+1)$-point formal differential satisfies the following Ward identity*

$$\mathcal{F}_V^{(g)}(\omega, x; \boldsymbol{v}, \boldsymbol{y}) = \left(\nabla(x) + \sum_{k=1}^{n} dy_k(\Psi_2(x, y_k)\partial_{y_k} + \mathrm{wt}(v_k) dy_k\, \partial_{y_k}\Psi_2(x, y_k))\right) \mathcal{F}_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}).$$

We can also examine, for example, Ward identities for non-primary states such as the Virasoro vector ω. We insert states at $x, y_1, y_2, \ldots, y_n \in \mathcal{S}$ to find

Proposition 9.4. *The genus g Virasoro n-point differential obeys the following Zhu reduction relation*

$$\begin{aligned}\mathcal{F}_V^{(g)}(\omega, x; \boldsymbol{\omega}, \boldsymbol{y}) = &\left(\nabla(x) + \sum_{k=1}^{n}(\Psi_2(x, y_k)\partial_{y_k} + 2dy_k\, \partial_{y_k}\Psi_2(x, y_k))\right) \mathcal{F}_V^{(g)}(\boldsymbol{\omega}, \boldsymbol{y}) \\ &+ \frac{c}{2}\sum_{k=1}^{n} dy_k^3 \partial^{(0,3)}\Psi_2(x, y_k)\mathcal{F}_V^{(g)}(\ldots; \widehat{\omega, y_k}; \ldots),\end{aligned}$$

where the caret denotes omission of the state insertion.

Remark 9.6. The factor of $\frac{c}{2}$ arises due to the commutation relations of the Virasoro modes $L(n)$.

9.4 Genus g differential equations

We now examine some differential equations for the Heisenberg n-point correlation function in order to derive relations for genus g objects. We obtain the following differential equations

Theorem 9.3. *The genus g objects $\Psi_2(x,y)$, $\omega(x,y)$, $s(x)$, $\nu_a(x)$ and Ω_{ab} satisfy the following differential equations for all genera $g \geq 2$ and for all $a, b \in \mathcal{I}_+$*

$$(\nabla(x) + dy\Psi_2(x,y)\partial_y + 2dy\partial_y\Psi_2(x,y))\left(\tfrac{1}{6}s(y)\right) + dy^3\partial^{(0,3)}\Psi_2(x,y) = \omega(x,y)^2, \tag{161}$$

$$\nabla(x)\omega(y,z) + d_y(\Psi_2(x,y)\omega(y,z)) + d_z(\Psi_2(x,z)\omega(y,z)) = \omega(x,y)\omega(x,z), \tag{162}$$

$$\nabla(x)\nu_b(y) + d_y(\Psi_2(x,y)\nu_b(y)) = \omega(x,y)\nu_b(x), \tag{163}$$

$$2\pi \mathrm{i}\nabla(x)\Omega_{ab} = \nu_a(x)\nu_b(x), \tag{164}$$

where $d_x f(x) = \partial_x f\, dx$.

Remark 9.7. These differential equations are generalisations of those found in [GT1] to arbitrary genus $g \geq 2$. We note that we can obtain these equations by integration techniques and by exploiting the quasiperiodicity of the $(N, 1-N)$-differential Ψ_N, as compared to the approach in op. cit. where Heisenberg modules are used.

Remark 9.8. Equation (164) is equivalent to Rauch's formula [R], [McIT].

Proof. We examine Zhu reduction for the Virasoro 2-point formal differential. We again consider the rank one Heisenberg VOA M. Using similar ideas to before, we note that

$$\begin{aligned}
&\mathcal{F}_M^{(g)}(\omega, x; \omega, y)\\
&= \frac{1}{2}\lim_{x_i\to x}\left(\mathcal{F}_M^{(g)}(h, x_1; h, x_2; \omega, y) - \frac{\mathcal{F}_M^{(g)}(\omega, y)dx_1dx_2}{(x_1-x_2)^2}\right)\\
&= \tfrac{1}{2}\lim_{x_i\to x}\left(\tfrac{1}{2}\lim_{y_i\to y}\left(\mathcal{F}_M^{(g)}(h, x_1; h, x_2; h, y_1; h, y_2) - \tfrac{\mathcal{F}_M^{(g)}(h,x_1;h,x_2)dy_1dy_2}{(y_1-y_2)^2}\right) - \tfrac{\mathcal{F}_M^{(g)}(\omega,y)dx_1dx_2}{(x_1-x_2)^2}\right)\\
&= \frac{1}{2}\lim_{x_i\to x}\left(\frac{1}{2}\lim_{y_i\to y}\left(\omega(x_1,x_2)\omega(y_1,y_2) + \omega(x_1,y_1)\omega(x_2,y_2)\right.\right.\\
&\qquad\left.\left. +\omega(x_1,y_2)\omega(x_2,y_1) - \frac{\omega(x_1,x_2)dy_1dy_2}{(y_1-y_2)^2}\right) - \frac{1}{12}s(y)\frac{dx_1dx_2}{(x_1-x_2)^2}\right)Z_M^{(g)}\\
&= \frac{1}{2}\lim_{x_i\to x}\left(\frac{1}{12}s(y)\omega(x_1,x_2) - \frac{1}{12}s(y)\frac{dx_1dx_2}{(x_1-x_2)^2} + \omega(x_1,y)\omega(x_2,y)\right)Z_M^{(g)}\\
&= \left(\frac{1}{144}s(x)s(y) + \frac{1}{2}\omega(x,y)^2\right)Z_M^{(g)}.
\end{aligned} \tag{165}$$

On the other hand, using Proposition 9.4 (with $c=1$) we have

$$\begin{aligned}
\mathcal{F}_M^{(g)}(\omega, x; \omega, y) &= (\nabla(x) + dy\Psi_2(x,y)\partial_y + 2dy\partial_y\Psi_2(x,y))\, Z_M^{(g)}(\omega, y)\\
&\quad + \tfrac{1}{2}dy^3\partial^{(0,3)}\Psi_2(x,y)Z_M^{(g)}\\
&= (\nabla(x) + \Psi_2(x,y)\partial_y + dy\partial_y\Psi_2(x,y))\left(\tfrac{1}{12}s(y)Z_M^{(g)}\right)\\
&\quad + \tfrac{1}{2}dy^3\partial^{(0,3)}\Psi_2(x,y)Z_M^{(g)}.
\end{aligned}$$

But (160) implies

$$
\begin{aligned}
&\mathcal{F}_M^{(g)}(\omega, x; \omega, y) \\
&= \left(\tfrac{1}{144}s(x)s(y) + \nabla(x)\left(\tfrac{1}{12}s(y)\right)\right. \\
&\quad \left.+\tfrac{1}{12}\Psi_2(x,y)\partial_y s(y) + \tfrac{1}{6}dy\partial_y\Psi_2(x,y)s(y) + dy^3\partial^{(0,3)}\Psi_2(x,y)\right) Z_M^{(g)}.
\end{aligned} \tag{166}
$$

Comparing (165) and (166), we obtain (161). To prove (162), we first examine Zhu reduction for $\mathcal{F}_M^{(g)}(\omega, x; h, y; h, z)$

$$
\begin{aligned}
&\mathcal{F}_M^{(g)}(\omega, x; h, y; h, z) \\
&= \left(\nabla(x) + dy\Psi_2(x,y)\partial_y + dy\partial_y\Psi_2(x,y) + \Psi_2(x,z)\partial_z + dz\partial_z\Psi_2(x,z)\right) Z_M^{(g)}(h, y; h, z) \\
&= \nabla(x)\left(\omega(y,z)Z_M^{(g)}\right) \\
&\quad + \left(dy\Psi_2(x,y)\partial_y + dy\partial_y\Psi_2(x,y) + \Psi_2(x,z)\partial_z + dz\partial_z\Psi_2(x,z)\right)\omega(y,z)Z_M^{(g)} \\
&= \nabla(x)(\omega(y,z))Z_M^{(g)} + \omega(y,z)\nabla(x)Z_M^{(g)} \\
&\quad + \left(dy\Psi_2(x,y)\partial_y + dy\partial_y\Psi_2(x,y) + \Psi_2(x,z)\partial_z + dz\partial_z\Psi_2(x,z)\right)\omega(y,z)Z_M^{(g)} \\
&= \left(\nabla(x)\omega(y,z) + d_y(\Psi_2(x,y)\omega(y,z)) + d_z(\Psi_2(x,z)\omega(y,z) + \frac{1}{12}s(x)\omega(y,z)\right) Z_M^{(g)},
\end{aligned} \tag{167}
$$

using Theorem 9.2 and Proposition 160. Alternatively, we examine the expansion of the 4-point differential $\mathcal{F}_M^{(g)}(h, x_1; h, x_2; h, y; h, z)$, taking the limit as before

$$
\mathcal{F}_M^{(g)}(\omega, x; h, y; h, z) = \lim_{x_1, x_2 \to x}\left(\mathcal{F}_M^{(g)}(h, x_1; h, x_2; h, y; h, z) - \frac{\mathcal{F}_M^{(g)}}{(x_1 - x_2)^2}\right),
$$

giving

$$
\mathcal{F}_M^{(g)}(\omega, x; h, y; h, z) = \left(\frac{1}{12}s(x)\omega(y,z) + \omega(x,y)\omega(x,z)\right) Z_M^{(g)}. \tag{168}
$$

Comparing (167) and (168) yields (162). Next we will integrate (162) to derive a differential equation for ν_a, $a \in \mathcal{I}_+$. We first note that for the moduli derivatives derived from the Virasoro 1-point function analysis we have

$$
\partial_{a,\ell}(\gamma_b z) = \partial_{a,\ell}\left(w_{-b} + \frac{\rho_a}{y - w_b}\right) = \frac{\rho_a}{(y - w_a)^{2-\ell}}\delta_{ab}. \tag{169}
$$

Using this, we find

Lemma 9.1.

$$
\nabla(x)(\gamma_a y) = \rho_a \sum_{\ell=0}^{2} \Theta_a(x, \ell)(y - w_a)^{\ell - 2}. \tag{170}
$$

Let $\nu_b(y) = n_b(y)dy$. As ν_b is coordinate-independent, we find that

$$
\nu_b(y) = \nu_b(\gamma_a y) = n_b(\gamma_a y)d(\gamma_a y).
$$

This tells us that

$$
n_b(\gamma_a y) = \left(\frac{d(\gamma_a y)}{dy}\right)^{-1} n_b(y) = -\frac{(y - w_a)^2}{\rho_a} n_b(y). \tag{171}
$$

We then find using (170) and (171) that

$$\nabla(x)(\gamma_a y)n_b(\gamma_a y) = -\sum_{\ell=0}^{2}\Theta_a(x,\ell)(y-w_a)^\ell\nu_b(y)dy^{-1}.$$

Hence Lemma 8.1 implies that

$$\nabla(x)(\gamma_a y)n_b(\gamma_a y) = \left(\Psi_2(x,\gamma_a y) - \Psi_2(x,y)\right)\nu_b(y), \tag{172}$$

using equation (153).

Now let $\omega(y,z) = f(y,z)dz$. Similarly we find

$$\nabla(x)(\gamma_a z)f(y,\gamma_a z) = \left(\Psi_2(x,\gamma_a z) - \Psi_2(x,z)\right)\omega(y,z). \tag{173}$$

Lastly, we use the Leibniz rule for integration and use (173) to find that

$$\begin{aligned}\oint_{\beta_a}\nabla(x)\omega(y,z) &= \nabla(x)\left(\oint_{\beta_a}\omega(y,z)\right) - \nabla(x)(\gamma_a z)f(y,\gamma_a z)\\ &= \nabla(x)\nu_a(y) - \nabla(x)(\gamma_a z)f(y,\gamma_a z),\end{aligned} \tag{174}$$

where $\oint_{\beta_a}$ denotes integration over the cycle β_a from a base point z to $\gamma_a z$. We now integrate equation (162) in z over a β_a cycle to find

$$\begin{aligned}\omega(x,y)\nu_a(y) &= \nabla(x)\nu_a(y) - \nabla(x)(\gamma_a z)f(y,\gamma_a z) + \oint_{\beta_a} d_y(\Psi_2(x,y)\omega(y,z))\\ &\quad + \oint_{\beta_a} d_z(\Psi_2(x,z)\omega(y,z))\\ &= \nabla(x)\nu_a(y) - \nabla(x)(\gamma_a z)f(y,\gamma_a z) + d_y(\Psi_2(x,y)\nu_a(y))\\ &\quad + (\Psi_2(x,\gamma_a z) - \Psi_2(x,z))\omega(y,z).\end{aligned}$$

as $\omega(x,y)$ is a bidifferential. Using equation (173), we obtain (163). Using (163), we can obtain a differential equation for the period matrix. We now integrate (163) over a β_a cycle to find

$$\nabla(x)\left(\oint_{\beta_a}\nu_b\right) - \nabla(x)(\gamma_a y)n_b(\gamma_a y) + \oint_{\beta_a} d_y(\Psi_2(x,y)\nu_b(y)) = \nu_a(x)\nu_b(x).$$

On evaluation of the integrals and using (20), (22), and (172) we find that the entries of the genus g period matrix satisfy (164). □

9.5 Lattice VOA partition functions and the genus g Siegel theta function

We now derive the partition function for a lattice VOA in terms of the partition function for the Heisenberg VOA and the genus g Siegel lattice theta function. Let Θ_L be the theta function for an even lattice L of rank d

$$\Theta_L(\Omega) = \sum_{\boldsymbol{\alpha}\in L^g} e^{\pi\mathrm{i}\,\boldsymbol{\alpha}.\Omega.\boldsymbol{\alpha}}, \tag{175}$$

where $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_g) \in L^g$, Ω is the period matrix and

$$\boldsymbol{\alpha}.\Omega.\boldsymbol{\alpha} = \sum_{a,b\in\mathcal{I}_+}\alpha_a\Omega_{ab}\alpha_b.$$

Consider g modules M_{α_a} for the rank d Heisenberg VOA M^d determined by g lattice vectors $\alpha_a \in L$. Let $Z^{(g)}_{M_{\boldsymbol{\alpha}}}$ denote the genus g partition function for a set of g modules

$$Z^{(g)}_{M_{\boldsymbol{\alpha}}} = \sum_{\boldsymbol{b}_+ \in M_{\boldsymbol{\alpha}}} Z^{(0)}(\boldsymbol{b}, \boldsymbol{w}),$$

where $M_{\boldsymbol{\alpha}} = \bigotimes_{a=1}^{g} M_{\alpha_a}$ where the M_{α_a} are modules for the above lattice vectors. Then the following holds

Theorem 9.4. *Let L be an even lattice. Then the genus g partition function for g modules $M_{\boldsymbol{\alpha}} = \bigotimes_{a=1}^{g} M_{\alpha_a}$ for lattice vectors $\alpha_1, \ldots, \alpha_g \in L$ is*

$$Z^{(g)}_{M_{\boldsymbol{\alpha}}} = e^{\pi \mathrm{i}\, \boldsymbol{\alpha}.\Omega.\boldsymbol{\alpha}} \left(Z^{(g)}_{M} \right)^d,$$

where $Z^{(g)}_{M}$ is the rank one Heisenberg partition function.

Proof. The rank d Heisenberg VOA is generated by Heisenberg states $h_1, h_2, \ldots, h_d$ with commutation relations

$$[h_i(m), h_j(n)] = m\delta_{ij}\delta_{m,-n}.$$

For the rank d Heisenberg Virasoro vector we have

$$\omega = \frac{1}{2}\sum_{i=1}^{d} h_i(-1)^2 \mathbb{1}.$$

Taking the appropriate limit (similar to the derivation of (160)) of the 2-point function $Z_{M_{\boldsymbol{\alpha}}}(h_1, x_1; h_2, x_2)$ we find

$$\mathcal{F}^{(g)}_{M_{\boldsymbol{\alpha}}}(\omega, x) = \left(\frac{1}{2}\nu^2_{\boldsymbol{\alpha}}(x) + \frac{d}{12}s(x) \right) Z^{(g)}_{M_{\boldsymbol{\alpha}}}. \tag{176}$$

where

$$\nu_{\boldsymbol{\alpha}} = \sum_{a \in \mathcal{I}_+} \alpha_a \nu_a(x).$$

Next

$$\begin{aligned}\nabla(x)(e^{\pi \mathrm{i}\, \alpha.\Omega.\alpha}(Z^{(g)}_{M})^d) &= \nabla(x)(\pi \mathrm{i}\, \alpha.\Omega.\alpha) e^{\pi \mathrm{i}\, \alpha.\Omega.\alpha}(Z^{(g)}_{M})^d + e^{\pi \mathrm{i}\, \alpha.\Omega.\alpha}\frac{d}{12}s(x)(Z^{(g)}_{M})^d \\ &= \left(\nabla(x)(\pi \mathrm{i}\, \alpha.\Omega.\alpha) + \frac{d}{12}s(x) \right) e^{\pi \mathrm{i}\, \alpha.\Omega.\alpha}(Z^{(g)}_{M})^d, \end{aligned} \tag{177}$$

using equation (160). Let $M_{\boldsymbol{\alpha}}$ denote the collection of modules $M_{\alpha_1}, \ldots, M_{\alpha_g}$. Theorems 7.2 and 9.2 tell us that

$$\mathcal{F}^{(g)}_{M_{\boldsymbol{\alpha}}}(\omega, x) = \nabla(x) Z^{(g)}_{M_{\boldsymbol{\alpha}}}. \tag{178}$$

Lastly, Rauch's formula (164) implies

$$2\pi \mathrm{i}\, \nabla(x) \left(\sum_{a,b \in \mathcal{I}_+} \alpha_a \Omega_{ab} \alpha_b \right) = \sum_{a,b \in \mathcal{I}_+} \alpha_a \alpha_b \nu_a(x) \nu_b(x),$$

which gives

$$\nabla(x)(\pi \mathrm{i}\, \alpha.\Omega.\alpha) = \frac{1}{2}\nu^2_{\boldsymbol{\alpha}}(x).$$

On comparison of (176), (177) and (178), the result obtains. □

From this we therefore obtain

Theorem 9.5. *The genus g partition function for a lattice VOA V_L is*

$$Z_{V_L}^{(g)} = \Theta_L(\Omega)\left(Z_M^{(g)}\right)^d,$$

for the Siegel lattice theta function of (175).

$$\Theta_L(\Omega) = \sum_{\boldsymbol{\alpha}\in L^g} e^{\pi \mathrm{i}\, \boldsymbol{\alpha}.\Omega.\boldsymbol{\alpha}}.$$

Proof. The result is immediate from Theorem 9.4 upon summation over all $\boldsymbol{\alpha} \in L^g$ and using (175). $\square$

Remark 9.9. We note that the above result is a genus g generalisation of Theorem 11 in genus two of [MT3]. Furthermore, we have that [T2], [TZ3].

$$Z_M^{(g)} = \det\left(1 - \widetilde{R}\right)^{-\frac{1}{2}}.$$

We again note the independence of the $f_\ell(x)$ terms due to the presence of $\widetilde{R}$. This will be developed further in [T2].

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