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$N{=}2$ AND $N{=}4$ SUBALGEBRAS OF SUPER VERTEX OPERATOR ALGEBRAS

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ABSTRACT. We develop criteria to decide if an $N{=}2$ or $N{=}4$ superconformal algebra is a subalgebra of a super vertex operator algebra in general, and of a super lattice theory in particular. We give some specific examples.

1. INTRODUCTION

The advent of Mathieu Moonshine [EOT] in recent years has brought renewed interest in $N{=}4$ superconformal algebras. Mathieu Moonshine concerns an unexpected connection between the largest Mathieu group M_{24} and the complex elliptic genus of a $K3$ surface. The connection between the $K3$ elliptic genus and $N=4$ superperconformal algebras has been known for some time in string theory [EOTY, KYY]. The new observation of Mathieu Moonshine [EOT] is that when the $K3$ elliptic genus is decomposed into characters of the $N{=}4$ central charge $c{=}6$ superconformal algebra [ET1, ET2], the coefficients are simply related to M_{24} irreducible module dimensions. This conjecture is now supported by a wealth of studies e.g. [EH1, C, GHV, G, EH2]. Recently, Song has also shown [S] that the $c{=}6$, $N{=}4$ superconformal algebra is the algebra of global sections of the chiral de Rham complex on a Kummer surface (following earlier results for hyperkähler manifolds [BZHS]), further clarifying the connection between $K3$ surfaces and $N{=}4$ superconformal algebras. However, despite these developments, an explicit construction of a vertex operator super algebra involving a $c{=}6$, $N{=}4$ superconformal subalgebra that 'explains' Mathieu Moonshine is still unknown.

All of this means that one can expect $N{=}4$ superconformal algebras to play a ubiquitous rôle in the further investigation of these subjects, much as the Virasoro algebra does in general CFT. Now the precise definition of the $N{=}4$ superconformal algebra is awkward, to say the least. For $c{=}6$, it is usually described as a subtheory of the algebra of 12 free fermions (or 4 free fermions and 4 free bosons) e.g. [ET1, KW]. But in a CFT that contains no free fermions, such as a super lattice theory for an odd lattice with minimum norm 2, one needs some other device to decide if there is an $N{=}4$ subalgebra present. Such an odd lattice construction is exploited in further work concerning $N{=}4$ superconformal algebras, odd Niemeier lattices and elliptic genera including the $K3$ elliptic genus [MTY].

The purpose of the present paper is to describe two general recognition theorems which allow one to readily identify the existence of an $N{=}4$ superconformal algebra as a subalgebra of a suitable Super Vertex Operator Algebra (SVOA) with just a few well-chosen axioms. Some effort is required to obtain efficient characterizations, i.e., without too many assumptions. It would be of interest if a genuine reduction in

the number of axioms needed can be achieved in our recognition theorems. Our main results are as follows (unexplained notation is clarified below).

Theorem 1. *Let U be a SVOA of CFT–type. Let $V \subseteq U$ be the subalgebra generated by 4 primary vectors of weight $\frac{3}{2}$ in U, so that*

$$V = \mathbb{C}\mathbf{1} \oplus V_{\frac{1}{2}} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus \dots$$

is a conformally graded subspace of U. Assume that the following hold:

(I) *The subspace of $V_{\frac{3}{2}}$ spanned by the four generators decomposes as $A \oplus B$, a pair of 2-dimensional vector representations for $\mathfrak{sl}_2$, where*

(II) $A(1)B \cong \mathfrak{sl}_2$,

(III) $A(0)A = B(0)B = 0$,

(IV) $T\mathfrak{sl}_2 \cap A(0)B \neq 0$.

Then V is an $N=4$ superconformal algebra with central charge $c=6k$, where $k \in \mathbb{C}$ is the level of the $\widehat{\mathfrak{sl}}_2$ Kac-Moody subalgebra generated by $A(1)B$.

Theorem 2. *Let L be a positive-definite, odd, integral lattice of minimum norm 2 with V_L the corresponding SVOA. Let $V \subseteq V_L$ be the subalgebra generated by 4 vectors of weight $\frac{3}{2}$, so that*

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus \dots,$$

and assume that the following hold:

(I) *The subspace of $V_{\frac{3}{2}}$ spanned by the four generators decomposes as $A \oplus B$, a pair of 2-dimensional vector representations for $\mathfrak{sl}_2$, where*

(II) $A(1)B \cong \mathfrak{sl}_2$,

(III) $A(1)A = B(1)B = 0$,

(IV) $\mathfrak{sl}_2$ *contains a root of L.*

Then V is an $N = 4$ superconformal algebra with central charge $c=6$.

We note that there are many articles in the literature, e.g., [K], [dS], that discuss constructions of $N=4$ vertex algebras from given generating sets. However, Theorems 1 and 2 in this article rather describe the existence of an $N=4$ subalgebra of a given SVOA, so that, for example the central charge of the $N=4$ subalgebra may differ from that of U or V_L respectively. Furthermore, we do not discuss the simplicity of the $N=4$ subalgebra. Our results appear to shed no light on this interesting question. For results in this direction, see [GK].

The paper is organized as follows. After reviewing some background about $N=4$ algebras in Section 2, we give the proof of Theorems 1 and 2 in Sections 3 and 4 respectively. We also briefly describe the more elementary analogous case of the $N=2$ superconformal subalgebra. This is contained in Section 5. In Section 6 we illustrate how the main Theorems may be applied to some known examples. In particular, we give (Proposition 26) a painless new construction of the $N=4$ algebra in a certain rank 6 lattice theory V_L containing no free fermions. This provides an effective way to decide the existence of the $N=4$ algebra in a general super lattice theory by reducing the problem to one about lattice embeddings. We include some appendices in Section 7 containing technical background in SVOA theory that we assume and use throughout the paper.

2. $N{=}4$ SUPERCONFORMAL ALGEBRAS

The abstract generators and relations for the $N{=}4$ superconformal algebra V of central charge c are as follows. It is generated by 4 states $G^\pm, \overline{G}^\pm$ of conformal weight $\frac{3}{2}$. The nontrivial relations can be expressed as follows (e.g. [ET1, K])

(a) $\quad J^0(0)J^\pm{=}\pm 2J^\pm,$ (b) $\quad J^0(1)J^0{=}\dfrac{c}{3}\mathbf{1},$ (c) $\quad J^+(0)J^-{=}J^0,$

(d) $\quad J^+(1)J^-{=}\dfrac{c}{6}\mathbf{1},$ (e) $\quad J^0(0)G^\pm{=}\pm G^\pm$ (f) $\quad J^0(0)\overline{G}^\pm = \pm\overline{G}^\pm,$

(g) $\quad J^\pm(0)G^\mp{=}G^\pm,$ (h) $\quad J^\pm(0)\overline{G}^\mp{=}-\overline{G}^\pm,$ (i) $\quad G^\pm(1)\overline{G}^\pm{=}2J^\pm,$

(j) $\quad G^\pm(1)\overline{G}^\mp{=}\pm J^0,$ (k) $\quad G^\pm(2)\overline{G}^\mp{=}\dfrac{c}{3}\mathbf{1},$ (l) $\quad G^\pm(0)\overline{G}^\pm{=}TJ^\pm,$

(m) $\quad G^\pm(0)\overline{G}^\mp{=}\omega\pm\dfrac{1}{2}TJ^0.$

Here, ω is a Virasoro element of central charge c, T is the translation operator and $J^\pm, J^0$ are weight one vectors. Relations (a)–(m) hold in addition to the usual Virasoro relations between $J^\pm, J^0$ and ω. The initial segment of the Fock space of V is

$$V{=}\mathbb{C}\mathbf{1} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus V_2 \oplus \dots$$

where

$$V_1{=}\langle J^\pm, J^0\rangle{\cong}\mathfrak{sl}_2$$
$$V_{\frac{3}{2}}{=}\langle G^\pm\rangle\oplus\langle\overline{G}^\pm\rangle, \text{ a pair of vector representations for } \mathfrak{sl}_2$$
$$V_2{=}T\,\mathfrak{sl}_2\oplus\mathbb{C}\omega.$$

For arbitrary subsets $X, Y{\subseteq}U$ of a SVOA U and any integer n, we have already used, and will continue to use, the convenient notation $X(n)Y$ for the linear subspace of U spanned by all products $u(n)v$ ($u{\in}X, v{\in}Y$). For the $N{=}4$ algebra V we set

$$A:{=}\langle G^\pm\rangle, \ B:{=}\langle\overline{G}^\pm\rangle.$$

Then it follows from the relations (a)–(m) that

$$A(n)A{=}B(n)B{=}0 \ (n \geq 0), \quad A(0)B{=}V_2, \quad A(1)B{=}\mathfrak{sl}_2, \quad A(2)B{=}\mathbb{C}c\mathbf{1}.$$

3. PROOF OF THEOREM 1

Let ω^U be the Virasoro element of the given SVOA U, with vertex operator

$$Y(\omega^U, z) = \sum_{n\in\mathbb{Z}} L^U(n)z^{-n-2}.$$

3.1. **Assumptions (I) and (II).** In the interests of keeping track of just which of the axioms (I)–(IV) in the statement of Theorem 1 are needed when, we begin by recording some consequences of axioms (I) and (II) alone. We set

(1) $$\mathfrak{sl}_2 := \langle h, x^{\pm} \rangle \subseteq V_1$$

(2) $$A := \langle \tau^+, \tau^- \rangle, \quad B := \langle \overline{\tau}^+, \overline{\tau}^- \rangle.$$

Here,

(3) $$h(0)x^{\pm} = \pm 2x^{\pm}, \ x^+(0)x^- = h$$

are standard generators and relations for $\mathfrak{sl}_2$, and $\tau^{\pm}, \overline{\tau}^{\pm}$ are weight vectors for $h(0)$ with weights ± 1, with

(4) $$x^{\pm}(0)\tau^{\mp} = \tau^{\pm}, \quad x^{\pm}(0)\overline{\tau}^{\mp} = \overline{\tau}^{\pm},$$

(5) $$x^{\pm}(0)\tau^{\pm} = x^{\pm}(0)\overline{\tau}^{\pm} = 0.$$

All of this is just a choice of notation based on the hypotheses of the Theorem 1. It amounts to the existence of an isomorphism of $\mathfrak{sl}_2$-modules

$$\varphi : A \rightarrow B, \ \ \tau^{\pm} \mapsto \overline{\tau}^{\pm}.$$

Note that some latitude in scaling the generators of A and B remains - a fact that we make use of later. In any case, there is a canonical $\mathfrak{sl}_2$-invariant decomposition into trivial and adjoint modules

$$A \otimes B = \Lambda \oplus \Sigma,$$

where

(6) $$\Lambda = \mathbb{C}(\tau^+ \otimes \overline{\tau}^- - \tau^- \otimes \overline{\tau}^+),$$

(7) $$\Sigma = \langle \tau^+ \otimes \overline{\tau}^+, \tau^- \otimes \overline{\tau}^-, \tau^+ \otimes \overline{\tau}^- + \tau^- \otimes \overline{\tau}^+ \rangle.$$

Of course, there is a similar decomposition of $A \otimes A = \Lambda^2(A) \oplus S^2(A)$.

Note that $A(1)B$ and $A \oplus B$ consist of primary states. This is clear for $A \oplus B$, while for $A(1)B$ we note that $L^U(1) : V_1 \rightarrow \mathbb{C}\mathbf{1}$ is a morphism of $\mathfrak{sl}_2$-modules. Restriction to $A(1)B = \mathfrak{sl}_2$ must therefore be trivial, and the assertion follows. Thus we have

(8) $$[L^U(m), u(n)] = -nu(m+n),$$

(9) $$[L^U(m), v(n)] = \left(\frac{1}{2}(m+1) - n \right) v(m+n),$$

for all for $u \in A(1)B$ and $v \in A \oplus B$. In particular, (9) implies

Lemma 3. $A(n+1)A = L^U(1)A(n)A$ for all $n \neq 1$ and $A(n+2)A = L^U(2)A(n)A$ for all n. Similar relations hold for $B(n+1)B$, $B(n+2)B$, $A(n+1)B$ and $A(n+2)B$.   □

For $u, v \in A(1)B = \mathfrak{sl}_2$ we have from commutativity that

(10) $$[u(m), v(n)] = (u(0)v)(m+n) + m\langle\!\langle u, v \rangle\!\rangle \delta_{m+n,0},$$

with $\langle\!\langle u, v \rangle\!\rangle$ given by $u(1)v = \langle\!\langle u, v \rangle\!\rangle \mathbf{1}$. Define $k \in \mathbb{C}$ by

(11) $$\langle\!\langle h, h \rangle\!\rangle = 2k.$$

Lemma 4. V is a module for the Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ of level k.

Proof. Since $h(1)x^{\pm} \in V_0 = \mathbb{C}\mathbf{1}$ we find

$$0 = h(0)(h(1)x^{\pm}) = h(1)h(0)x^{\pm} = \pm 2h(1)x^{\pm},$$

so that $\langle\!\langle h, x^{\pm}\rangle\!\rangle = 0$. Using $x^{+}(0)x^{-}{=}h$ this implies

(12) $$2k\mathbf{1}{=}h(1)(x^{+}(0)x^{-}){=}2x^{+}(1)x^{-},$$

by commutativity. Hence, for $k \neq 0$, $\langle\!\langle\ ,\ \rangle\!\rangle$ is a nondegenerate, invariant, bilinear form on $\mathfrak{sl}_2$, so that for all $u, v \in \mathfrak{sl}_2$

(13) $$\langle\!\langle u, v\rangle\!\rangle{=}k(u, v),$$

where here $(\ ,\ )$ is the standard invariant bilinear form normalized to $(h, h) = 2$. (13) also holds for $k = 0$. Hence, since $u(0)v = -v(0)u$ defines a commutator bracket on $\mathfrak{sl}_2$, (10) implies the desired result. $\qquad\square$

Lemma 5. *Properties (a)–(h) above hold for $c = 6k$.*

Proof. We set

(14) $$J^0{:=}h,\ \ J^{\pm} := x^{\pm},\ \ G^{\pm} :{=} \tau^{\pm},\ \text{ and } \overline{G}^{\pm} := \mp\overline{\tau}^{\pm}.$$

By (3)–(5), (11)–(12), we have

$$J^0(0)J^{\pm} = \pm 2J^{\pm}, \quad J^0(1)J^0 = 2k\mathbf{1},$$
$$J^+(0)J^- = J^0, \quad J^+(1)J^- = k\mathbf{1},$$
$$J^0(0)G^{\pm} = \pm G^{\pm}, \quad J^0(0)\overline{G}^{\pm} = \pm\overline{G}^{\pm},$$
$$J^{\pm}(0)G^{\mp} = G^{\pm}, \quad J^{\pm}(0)\overline{G}^{\mp} = -\overline{G}^{\pm}.$$

These are the needed relations. $\qquad\square$

Remark 6. In addition, we have $J^{\pm}(0)G^{\pm} = J^{\pm}(0)\overline{G}^{\pm} = 0$.

Next, we will obtain properties (i) and (j):

Lemma 7. *We can choose the normalization for $\tau^{\pm}, \overline{\tau}^{\pm}$ so that*

(15) $$h{=}\tau^{\pm}(1)\overline{\tau}^{\mp}, \quad \tau^{\pm}(1)\overline{\tau}^{\pm}{=}\mp 2x^{\pm}.$$

In particular, we have $G^{\pm}(1)\overline{G}^{\pm} = 2J^{\pm}$, and $G^{\pm}(1)\overline{G}^{\mp} = \pm J^0$.

Proof. Thanks to the $\mathfrak{sl}_2$-morphism $u{\otimes}v{\mapsto}u(1)v$ from $A{\otimes}B$ on to $A(1)B$, we get generators of $A(1)B$ just by replacing $u{\otimes}v$ by $u(1)v$ in the generators for Σ in (7). Thus

$$A(1)B{=}\mathfrak{sl}_2 ={\langle}\tau^{+}(1)\overline{\tau}^{+}, \tau^{-}(1)\overline{\tau}^{-}, \tau^{+}(1)\overline{\tau}^{-}{+}\tau^{-}(1)\overline{\tau}^{+}{\rangle}.$$

Similarly, the generator for Λ in (6) maps to 0, leading to $\tau^{+}(1)\overline{\tau}^{-}{-}\tau^{-}(1)\overline{\tau}^{+}{=}0$.

$\tau^{+}(1)\overline{\tau}^{-}{+}\tau^{-}(1)\overline{\tau}^{+} = 2\tau^{\pm}(1)\overline{\tau}^{\mp}$ is a nonzero semisimple element of $\mathfrak{sl}_2$. As such, it is a nonzero multiple of h. Since we are free to simultaneously scale the τ's by a nonzero constant, we may choose the scale so that $h{=}\tau^{+}(1)\overline{\tau}^{-} = \tau^{-}(1)\overline{\tau}^{+}$. Since $x^{\pm}(0)h = -h(0)x^{\pm} = \mp 2x^{\pm}$ we find, using super commutativity, that

$$-2x^{+} = x^{+}(0)\left(\tau^{+}(1)\overline{\tau}^{-}\right)$$
$$= \left(x^{+}(0)\tau^{+}\right)(1)\overline{\tau}^{-} + \tau^{+}(1)\left(x^{+}(0)\overline{\tau}^{-}\right)$$
$$= 0 + \tau^{+}(1)\overline{\tau}^{+}.$$

Similarly, $\tau^{-}(1)\overline{\tau}^{-} = 2x^{-}$. $\qquad\square$

3.2. **Assumption (III).** Lemma 3 implies

$$(16) \qquad\qquad A(1)A{=}A(2)A{=}B(1)B{=}B(2)B{=}0.$$

Thus super commutativity implies

Lemma 8. $[v(m), w(n)] = 0$ *for all* $v, w \in A$ *for all* $v, w \in B$.

Lemma 9. $u(1)v{=}0$ *for all* $u \in \mathfrak{sl}_2$ *and* $v \in A \oplus B$.

Proof. The vector space $\mathfrak{sl}_2(1)A$ is spanned by the 6 vectors of the form $u(1)\tau^{\pm}$ for $u = x^{\pm}$ or h. Recall that $\tau^+(1)\overline{\tau}^+ = -2x^+$ from (15). Hence, by super associativity

$$x^+(1)\tau^+ = -\frac{1}{2}\sum_{i \geq 0}(-1)^i\binom{1}{i}\left(\tau^+(1{-}i)\overline{\tau}^+(1{+}i){-}\overline{\tau}^+(2{-}i)\tau^+(i)\right)\tau^+$$

$$= -\frac{1}{2}\tau^+(1)\overline{\tau}^+(1)\tau^+ + 0 = -\tau^+(1)x^+,$$

using (16), $\overline{\tau}^+(2)\tau^+ \in \mathbb{C}\mathbf{1}$ and that $\overline{\tau}^+(1)\tau^+ = -\tau^+(1)\overline{\tau}^+ = 2x^+$ by super skew symmetry. But, by super skew symmetry again we directly have $x^+(1)\tau^+ = +\tau^+(1)x^+$. Thus $x^+(1)\tau^+ = 0$. Furthermore, $(x^-(0))^k x^+(1)\tau^+ = 0$ for $k = 1, 2, 3$ results in the identities

$$h(1)\tau^+ - x^+(1)\tau^- = x^-(1)\tau^+ + h(1)\tau^- = x^-(1)\tau^- = 0.$$

We can then repeat a similar argument based on super associativity and skewsymmetry using $h = \tau^-(1)\overline{\tau}^+$ to show that $h(1)\tau^{\pm} = 0$. Thus $\mathfrak{sl}_2(1)A = 0$. By an identical argument we also find $\mathfrak{sl}_2(1)B = 0$. $\qquad\square$

We next establish property (k):

Lemma 10. *We have*

$$\tau^{\pm}(2)\overline{\tau}^{\pm} = 0, \quad \tau^{\pm}(2)\overline{\tau}^{\mp} = \pm 2k\mathbf{1}.$$

Consequently, for $c = 6k$ *we have*

$$G^{\pm}(2)\overline{G}^{\pm} = 0, \text{ and } G^{\pm}(2)\overline{G}^{\mp} = \frac{c}{3}\mathbf{1}.$$

Proof. The image of $\Sigma \to A(2)B{=}\mathbb{C}\mathbf{1}$ is $\mathfrak{sl}_2$-invariant, hence is 0. Then

$$\tau^{\pm}(2)\overline{\tau}^{\pm}{=}\tau^+(2)\overline{\tau}^-{+}\tau^-(2)\overline{\tau}^+ = 0.$$

The invariant $\tau^+(2)\overline{\tau}^- - \tau^-(2)\overline{\tau}^+$ is computed as follows: $h(1)\tau^{\pm}{=}h(1)\overline{\tau}^{\pm}{=}0$ (Lemma 9) together with commutativity imply

$$(17) \qquad [h(1), \tau^{\pm}(n)]{=}\pm\tau^{\pm}(n+1), \quad [h(1), \overline{\tau}^{\pm}(n)]{=}\pm\overline{\tau}^{\pm}(n+1).$$

Using (15) we thus find

$$2k\mathbf{1} = h(1)h = h(1)\left(\tau^{\pm}(1)\overline{\tau}^{\mp}\right) = \pm\tau^{\pm}(2)\overline{\tau}^{\mp}.$$

$\qquad\square$

Lemma 11. *Let*

$$\sigma := \frac{1}{2}(\tau^+(0)\overline{\tau}^- - \tau^-(0)\overline{\tau}^+).$$

Then $\sigma(0)u = Tu$, $\sigma(1)u{=}u$ *and* $\sigma(2)u{=}0$ *for all* $u \in \mathfrak{sl}_2$. *In particular,* σ *is a* nonzero $\mathfrak{sl}_2$-*invariant.*

Proof. σ is an $\mathfrak{sl}_2$-invariant in $A(0)B \subseteq V_2$ because it generates the image of Λ under the $\mathfrak{sl}_2$-morphism defined by $A \otimes B \to A(0)B$.

$\sigma(2)\,\mathfrak{sl}_2 \in \mathbb{C}\mathbf{1}$ implies $\sigma(2)\,\mathfrak{sl}_2 = 0$ since $\mathbb{C}\mathbf{1}$ is a trivial $\mathfrak{sl}_2$ representation. By skewsymmetry, $\sigma(1)u = u(1)\sigma$ for all $u \in \mathfrak{sl}_2$. Hence, using (17) and Lemma 9, we find

$$\sigma(1)h = \frac{1}{2}h(1)\left(\tau^+(0)\overline{\tau}^- - \tau^-(0)\overline{\tau}^+\right)$$
$$= \frac{1}{2}\left(\tau^+(1)\overline{\tau}^- + \tau^-(1)\overline{\tau}^+\right) = h,$$

from (15). This proves, in particular, that $\sigma \neq 0$. Since σ is $\mathfrak{sl}_2$-invariant and h generates $A(1)B$ as an $\mathfrak{sl}_2$–module, it follows that $\sigma(1)u = u$ for all $u \in \mathfrak{sl}_2$. Furthermore, from skew-symmetry we find that

$$\sigma(0)u = -u(0)\sigma + T(u(1)\sigma)$$
$$= 0 + T(\sigma(1)u) = Tu,$$

using the $\mathfrak{sl}_2$-invariance of σ. $\qquad\square$

Lemma 12. $\sigma(1)v = \frac{3}{2}v$ *and* $\sigma(2)v = 0$ *for all* $v \in A \oplus B$.

Proof. Using superassociativity we find

$$\sigma(1) = \frac{1}{2}[\tau^+(0), \overline{\tau}^-(1)] - \frac{1}{2}[\tau^-(0), \overline{\tau}^+(1)].$$

Hence

$$\sigma(1)\tau^+ = \frac{1}{2}\tau^+(0)\overline{\tau}^-(1)\tau^+ - \frac{1}{2}\tau^-(0)\overline{\tau}^+(1)\tau^+$$
$$= -\frac{1}{2}\tau^+(0)h - \frac{1}{2}\tau^-(0)(2x^+)$$
$$= \frac{1}{2}\tau^+ + \tau^+ = \frac{3}{2}\tau^+,$$

using $\tau^\pm(0)\tau^+ = 0$ (by assumption (III), (15) and super skew-symmetry). Using superassociativity, we similarly find

$$\sigma(2) = \frac{1}{2}[\tau^+(0), \overline{\tau}^-(2)] - \frac{1}{2}[\tau^-(0), \overline{\tau}^+(2)].$$

Hence

$$\sigma(2)\tau^+ = \frac{1}{2}\tau^+(0)\overline{\tau}^-(2)\tau^+ - \frac{1}{2}\tau^-(0)\overline{\tau}^+(2)\tau^+ = 0.$$

Similar results follow for τ^- by $\mathfrak{sl}_2$-symmetry and for $\overline{\tau}^\pm$. $\qquad\square$

3.3. **Assumption (IV).** We next obtain properties (l) and (m).

Lemma 13. *We have*

$$\tau^\pm(0)\overline{\tau}^\pm = \mp\, Tx^\pm, \quad \tau^+(0)\overline{\tau}^- + \tau^-(0)\overline{\tau}^+ = Th.$$

Consequently,

$$G^\pm(0)\overline{G}^\pm = TJ^\pm, \text{ and } G^\pm(0)\overline{G}^\mp = \sigma \pm \frac{1}{2}TJ^0.$$

Proof. By assumption (IV), there exists $u \in \mathfrak{sl}_2$ such that $Tu \in A(0)B$. Since $A(0)B$ is an $\mathfrak{sl}_2$-module and $T\,\mathfrak{sl}_2 \cong \mathfrak{sl}_2$ it follows that $T\,\mathfrak{sl}_2 \subseteq A(0)B$. Thus there is a morphism of $\mathfrak{sl}_2$ adjoint-modules $\Sigma \to T\,\mathfrak{sl}_2 \subseteq A(0)B$. In particular

$$\tau^-(0)\overline{\tau}^- = \kappa T x^-,$$

for some $\kappa \neq 0$. Using (9) we have

$$L^U(1)\tau^-(0)\overline{\tau}^- = \tau^-(1)\overline{\tau}^- = 2x^-,$$

from (15). This implies

$$2x^- = L^U(1)\kappa L^U(-1)x^- = 2\kappa x^-.$$

Hence $\kappa = 1$ and $\tau^-(0)\overline{\tau}^- = Tx^-$. The other relations follow by $\mathfrak{sl}_2$-symmetry. □

Lemma 14. *The modes of $\tau^\pm, \overline{\tau}^\pm$ satisfy the commutator relations*

(18) $\quad [\tau^\pm(m), \overline{\tau}^\pm(n)] = \pm(n-m)x^\pm(m+n-1),$

(19) $\quad [\tau^\pm(m), \overline{\tau}^\mp(n)] = \pm\sigma(m+n) + \frac{1}{2}(m-n)h(m+n-1) \pm m(m-1)k\delta_{m+n+1,0}.$

Proof. Properties (k)–(m), which we have already established, show that

$$\tau^\pm(0)\overline{\tau}^\pm = \mp Tx^\pm, \quad \tau^\pm(1)\overline{\tau}^\pm = \mp 2x^\pm, \quad \tau^\pm(2)\overline{\tau}^\pm = 0,$$

$$\tau^\pm(0)\overline{\tau}^\mp = \frac{1}{2}Th \pm \sigma, \quad \tau^\pm(1)\overline{\tau}^\mp = h, \quad \tau^\pm(2)\overline{\tau}^\mp = \pm 2k\mathbf{1},$$

which imply the result from the super commutator formula. □

Lemma 15. $\sigma(0)v = Tv$ *for all $v \in A \oplus B$.*

Proof. Using (19) we have

$$\sigma(0) = [\tau^+(0), \overline{\tau}^-(0)].$$

Thus we find using super skew-symmetry that

$$\begin{aligned}
\sigma(0)\tau^+ &= \tau^+(0)\overline{\tau}^-(0)\tau^+ + 0 \\
&= \tau^+(0)\left(\tau^+(0)\overline{\tau}^- - T(\tau^+(1)\overline{\tau}^-)\right) \\
&= 0 - \tau^+(0)Th \\
&= (Th)(0)\tau^+ - T((Th)(1)\tau^+) + 0 \\
&= 0 + T\tau^+,
\end{aligned}$$

using $(Th)(0) = 0$ and $(Th)(1) = -h(0)$. A similar argument applies to the remaining elements of $A \oplus B$. □

Lemma 16. *For all $u \in A(1)B$ and $v \in A \oplus B$ we have*

(20) $\qquad\qquad [\sigma(m), u(n)] = -nu(m+n-1),$

(21) $\qquad\qquad [\sigma(m), v(n)] = \left(\frac{1}{2}m - n\right)v(m+n-1).$

Proof. Using Lemma 11 we have $\sigma(2)u = 0$, $\sigma(1)u = u$ and $\sigma(0)u = Tu$ for $u \in \mathfrak{sl}_2$ the result (20) follows from the commutator formula. Likewise, (21) follows from $\sigma(2)v = 0$, $\sigma(1)v = \frac{3}{2}v$ and $\sigma(0)v = Tv$ for all $v \in A \oplus B$ using Lemmas 12 and 15. □

Lemma 17. σ *is a Virasoro vector for central charge $c = 6k$.*

Proof. We have to check the relations

$$\sigma(0)\sigma=T\sigma, \quad \sigma(1)\sigma=2\sigma, \quad \sigma(2)\sigma=0, \quad \sigma(3)\sigma=3k\mathbf{1}.$$

Using (21) we find

$$\begin{aligned}
\sigma(1)\sigma &= \frac{1}{2}\left(\sigma(1)\tau^+(0)\overline{\tau}^- - \sigma(1)\tau^-(0)\overline{\tau}^+\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\tau^+(0)\overline{\tau}^- + \tau^+(0)\frac{3}{2}\overline{\tau}^- - \frac{1}{2}\tau^-(0)\overline{\tau}^+ - \tau^-(0)\frac{3}{2}\overline{\tau}^+\right) = 2\sigma, \\
\sigma(2)\sigma &= \frac{1}{2}\left(\sigma(2)\tau^+(0)\overline{\tau}^- - \sigma(2)\tau^-(0)\overline{\tau}^+\right) \\
&= \frac{1}{2}\left(\tau^+(1)\overline{\tau}^- + 0 - \tau^-(1)\overline{\tau}^+ - 0\right) = 0, \\
\sigma(3)\sigma &= \frac{1}{2}\left(\sigma(3)\tau^+(0)\overline{\tau}^- - \sigma(3)\tau^-(0)\overline{\tau}^+\right) \\
&= \frac{1}{2}\left(\frac{3}{2}\tau^+(2)\overline{\tau}^- + 0 - \frac{3}{2}\tau^-(2)\overline{\tau}^+ - 0\right) = 3k\mathbf{1}.
\end{aligned}$$

Lastly, by skew symmetry

$$\begin{aligned}
\sigma(0)\sigma &= -\sigma(0)\sigma + T(\sigma(1)\sigma) \\
&= -\sigma(0)\sigma + 2T\sigma,
\end{aligned}$$

so that $\sigma(0)\sigma = T\sigma$. $\qquad\square$

Thus Theorem 1 holds since all the defining relations for the $N=4$, $c=6k$ superconformal algebra are satisfied.

Remark 18. If U is C_2–cofinite and of strong CFT type then k is a positive integer by [DM] since $\mathfrak{sl}_2 \subseteq U_1$. We also note that the $N=4$ Virasoro vector σ (of Lemma 11) and ω^U (the Virasoro element of U) can be independent vectors of different central charges. Thus Theorem 1 is not a generating theorem for $N=4$ algebras (such as in [K] or [dS]) but rather describes the existence of an $N=4$ subalgebra of a given SVOA.

Finally we note that the automorphism group of the $N=4$ superconformal algebra contains an involution g defined by

$$\begin{aligned}
g : A \oplus B &\to B \oplus A \\
\left(\tau^\pm, \overline{\tau}^\pm\right) &\mapsto \left(\overline{\tau}^\pm, -\tau^\pm\right).
\end{aligned}$$

This follows by directly verifying that the defining relations (a)–(m) are preserved by g by use of super skew–symmetry i.e. $u(1)v = v(1)u$ and $u(0)v - v(0)u = -T(u(1)v) \in T\mathfrak{sl}_2$ for all $u \in A$ and $v \in B$. Furthermore, Assumption (IV) of Theorem 1 therefore has the following reformulation

Lemma 19. $T\mathfrak{sl}_2 \cap A(0)B \neq 0$ *if and only if* $A(0)B = B(0)A$. $\qquad\square$

4. PROOF OF THEOREM 2

In this Subsection we assume the hypotheses and notation of Theorem 2, in particular V is contained in a super lattice VOA V_L (see Subsection 7.3 for the definition and relevant properties). In particular, $(V_L)_1$ is a reductive Lie algebra and each of its components is a simple Lie algebra of type ADE and of level 1.

Now by hypothesis (IV) of Theorem 2 our Lie algebra $\mathfrak{sl}_2$ contains a root of L. It follows that $\mathfrak{sl}_2$ is contained in one of the components of V_1 and therefore it also has level $k=1$. Adopting the notation of the previous Section, it follows that $(h,h)=2$ (cf. Lemma 4). Thus h is a root of L and we have $\mathfrak{sl}_2=\langle h, e^{\pm h}\rangle$.

We will deduce Theorem 2 from Theorem 1. To this end, notice that states of weight $\frac{3}{2}$ in V_L are primary because L has no vectors of norm 1. Thus it suffices to take $U:=V_L$ in Theorem 1 and show that hypotheses (I)–(IV) of Theorem 1 hold. Then Theorem 1 shows that V is the $N=4$ superconformal algebra of central charge $6k=6$. Parts (I) and (II) hold by assumption, so we only have to establish (III) and (IV).

We need some additional notation. Let $(\ ,\ ):L\times L\to\mathbb{Z}$ be the bilinear form on L. Note that this is *not* the notation used in the proof of Lemma 4, where $(\ ,\ )$ denoted the invariant bilinear form on $\mathfrak{sl}_2$.

The vectors in L of norm n are denoted by

$$L_n:=\{\alpha\in L|(\alpha,\alpha)=n\},$$

in particular L_2 is the root system of L. Fix a multiplicative bicharacter $\varepsilon:L\times L\to\{\pm1\}$ that defines the central extension $\widehat{L}$ occurring in the short exact sequence

$$1\to\{\pm1\}\to\widehat{L}\to L\to 0.$$

See Subsection 7.2 for more details on the ε-formalism, in particular for the justification that

$$\text{(22)}\qquad\qquad \varepsilon(\alpha,\alpha)=\varepsilon(\alpha,-\alpha)=\left\{\begin{array}{ll}-1 & \alpha\in L_2\\ 1 & \alpha\in L_3\cup L_4\end{array}\right.$$

Recall (2) that $\tau^+,\overline{\tau}^+$ are highest weight vectors in A and B respectively. We have already mentioned that $U_{\frac{3}{2}}$ is spanned by states e^β $(\beta\in L_3)$. Thus there are nonempty subsets $X,Y\subseteq L_3$ and scalars c_α,d_λ such that

$$\text{(23)}\qquad\qquad \tau^+:=\sum_{\alpha\in X}c_\alpha e^\alpha,\ \ \overline{\tau}^+:=\sum_{\lambda\in Y}d_\lambda e^\lambda.$$

Because $h(0)\tau^+=\tau^+$ then we have $(h,\alpha)=1$ $(\alpha\in X)$, and similarly $(h,\lambda)=1$ $(\lambda\in Y)$.

As a result, we have the following useful facts that hold for $\alpha,\beta\in X$. $|(\alpha,\beta)|\leq3$ by the Schwarz inequality, moreover

$$(\alpha,\beta)=\left\{\begin{array}{ll}\pm3 & \text{iff } \alpha=\pm\beta\\ 2 & \text{iff } \alpha-\beta=\gamma \text{ (root } \gamma\perp h)\\ -2 & \text{iff } \alpha+\beta=h\\ -1 & \text{iff } \alpha+\beta=h+\gamma \text{ (root } \gamma\perp h)\end{array}\right.$$

Identical formulas hold in case $\alpha,\beta\in Y$. We use these formulas in later calculations.

The elements $x^\pm\in\mathfrak{sl}_2$ may be identified (recall that $\varepsilon(h,h)=-1$ by (22)) as $x^\pm=\mp e^{\pm h}$. Because $x^-(0)\tau^+=\tau^-$ we have

$$\tau^-=\sum_{\alpha\in X}c_\alpha e^{-h}(0)e^\alpha=\sum_{\alpha\in X}c_\alpha\varepsilon(h,\alpha)e^{\alpha-h},$$

and similarly

$$\overline{\tau}^- = \sum_{\lambda \in Y} d_\lambda \varepsilon(h, \lambda) e^{\lambda - h}.$$

We now consider the consequences of assumption (III) of Theorem 2.

Lemma 20. $A(1)A = 0$ *implies*

$$(a) \quad \sum_{\alpha \in X} c_\alpha c_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma){=}0 \quad (each\ root\ \gamma \perp h),$$

$$(b) \quad \sum_{\alpha \in X} c_\alpha c_{h-\alpha} \varepsilon(h, \alpha)\alpha{=}0,$$

with a corresponding statement concerning $B(1)B = 0$.

Proof. $A(1)A$ contains the element $\tau^+(1)\tau^-$, which is equal to

$$\sum_{\alpha,\beta \in X} c_\alpha c_\beta \varepsilon(h, \beta) e^\alpha(1) e^{\beta - h} = \sum_{(\alpha,\beta-h)=-2,-3} c_\alpha c_\beta \varepsilon(h, \beta) e^\alpha(1) e^{\beta - h}$$

$$= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta)\alpha + \sum_{(\alpha,\beta)=-1} c_\alpha c_\beta \varepsilon(h, \beta)\varepsilon(\alpha, \beta-h) e^{\alpha+\beta-h}$$

$$= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta)\alpha + \sum_{\gamma \perp h} \left\{ \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(h, \beta)\varepsilon(\alpha, \gamma-\alpha) \right\} e^\gamma$$

$$= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta)\alpha - \sum_{\gamma \perp h} \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(h, \gamma+\alpha)\varepsilon(\alpha, \gamma) e^\gamma$$

$$= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta)\alpha - \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(h, \alpha)\varepsilon(\alpha, \gamma) e^\gamma$$

$$= - \sum_{\alpha} c_\alpha c_{h-\alpha} \varepsilon(h, \alpha)\alpha + \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha} c_\alpha c_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) e^\gamma,$$

using $\varepsilon(h, \beta){=}\varepsilon(h, h-\alpha){=}{-}\varepsilon(h, \alpha)$. Hence $A(1)A{=}0$ if and only if

$$\sum_{\alpha \in X} c_\alpha c_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma){=}0,$$

for each root $\gamma \perp h$, and

$$\sum_{\alpha} c_\alpha c_{h-\alpha} \varepsilon(h, \alpha)\alpha = 0.$$

A similar analysis applies for $B(1)B = 0$. $\qquad\square$

Lemma 21. *We have* $A(0)A{=}B(0)B{=}0$.

Proof. We prove that $A(0)A{=}0$. The proof that $B(0)B{=}0$ is similar. Assume, then, that $A(0)A{\neq}0$. Because $A(1)A{=}0$, by assumption (III), then $A(0)A$ has dimension ≤ 3 by super skew-symmetry, indeed because we are assuming that $A(0)A{\neq}0$ then the image of $A{\otimes}A{\to}A(0)A$ is the adjoint module for $\mathfrak{sl}_2$. Now it follows that $0{\neq}\tau^+(0)\tau^+{\in}A(0)A$.

But we also have

$$\tau^+(0)\tau^+= \sum_{\alpha,\beta\in X} c_\alpha c_\beta e^\alpha(0)e^\beta= \sum_{(\alpha,\beta)=-1} c_\alpha c_\beta \varepsilon(\alpha,\beta)e^{\alpha+\beta}+ \sum_{(\alpha,\beta)=-2} c_\alpha c_\beta \varepsilon(\alpha,\beta)\alpha(-1)e^{\alpha+\beta}$$

$$=\sum_\gamma \left\{\sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(\alpha,h+\gamma)\right\} e^{h+\gamma}+ \left\{\sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(\alpha,h)\alpha\right\} (-1)e^h=0,$$

where we used Lemma 20. This contradiction completes the proof of the Lemma. $\square$

This establishes assumption (III) of Theorem 1.

We now consider consequences of assumptions (II) and (IV) of Theorem 2.

Lemma 22. $A(1)B \cong \mathfrak{sl}_2$ with $\mathfrak{sl}_2 = \langle h, e^{\pm h}\rangle$ if and only if

$$(a) \quad h = -\sum_{\alpha\in X} c_\alpha d_{h-\alpha}\varepsilon(h,\alpha)\alpha,$$

$$(b) \quad \sum_{\gamma\perp h} \varepsilon(h,\gamma)\left\{\sum_{\alpha\in X} c_\alpha d_{h+\gamma-\alpha}\varepsilon(\alpha,h+\gamma)\right\} e^\gamma = 0,$$

where the γ sum is taken over each root $\gamma \perp h$.

Proof. By calculations similar to those of Lemma 20 we have

$$\tau^+(1)\overline{\tau}^- = \sum_{\alpha\in X,\lambda\in Y} c_\alpha d_\lambda \varepsilon(h,\lambda)e^\alpha(1)e^{\lambda-h} = \sum_{(\alpha,\lambda-h)=-2,-3} c_\alpha d_\lambda \varepsilon(h,\lambda)e^\alpha(1)e^{\lambda-h}$$

$$= -\sum_{\alpha\in X} c_\alpha d_{h-\alpha}\varepsilon(h,\alpha)\alpha + \sum_{\gamma\perp h}\varepsilon(h,\gamma)\sum_{\alpha\in X} c_\alpha d_{h+\gamma-\alpha}\varepsilon(\alpha,h+\gamma)e^\gamma.$$

But from (15) we have that $h = \tau^+(1)\overline{\tau}^-$ iff (a) and (b) hold. Note that taking the inner product of (a) with h implies

$$(24) \qquad\qquad \sum_{\alpha\in X} c_\alpha d_{h-\alpha}\varepsilon(h,\alpha) = -2.$$

This implies

$$\tau^-(1)\overline{\tau}^+ = \sum_{\alpha\in X,\lambda\in Y} c_\alpha d_\lambda \varepsilon(h,\alpha)e^{\alpha-h}(1)e^\lambda = \sum_{(\alpha-h,\lambda)=-2,-3} c_\alpha d_\lambda \varepsilon(h,\alpha)e^{\alpha-h}(1)e^\lambda$$

$$= \sum_{\alpha\in X} c_\alpha d_{h-\alpha}\varepsilon(h,\alpha)(\alpha-h) + \sum_{\gamma\perp h}\varepsilon(h,\gamma)\sum_{\alpha\in X} c_\alpha d_{h+\gamma-\alpha}\varepsilon(\alpha,h+\gamma)e^\gamma$$

$$= -h + 2h + 0 = h,$$

iff (a) and (b) hold. The remaining relations in (15) follow from $\mathfrak{sl}_2$ symmetry. $\square$

Lemma 23. $A(1)B \cong \mathfrak{sl}_2$ implies $T\mathfrak{sl}_2 \cap A(0)B \neq 0$.

Proof. $A(0)B$ contains the element

$$\tau^+(0)\overline{\tau}^+= \sum_{(\alpha,\lambda)=-1,-2} c_\alpha d_\lambda e^\alpha(0)e^\lambda$$

$$= -\frac{1}{2}\left\{\sum_{\alpha\in X} c_\alpha d_{h-\alpha}\varepsilon(h,\alpha)\right\} h(-1)e^h = h(-1)e^h = Te^{-h},$$

by (24) of Lemma 22. Thus $T\mathfrak{sl}_2 \cap A(0)B \neq 0$. $\square$

This completes the proof of hypothesis(IV) of Theorem 1, and with it the proof of Theorem 2.

5. $N{=}2$ SUPERCONFORMAL ALGEBRAS

The $N{=}2$ SVOA of central charge c is generated by a pair of states $\tau^{\pm}$ of conformal weight $\frac{3}{2}$ satisfying the non-zero relations e.g. [K]

(i) $\quad \tau^{\pm}(2)\tau^{\mp}{=}\dfrac{c}{3}\mathbf{1}, \qquad$ (ii) $\quad \tau^{\pm}(1)\tau^{\mp}{=}\pm h, \qquad$ (iii) $\quad \tau^{\pm}(0)\tau^{\mp}{=}\omega\pm\dfrac{1}{2}T,$

(iv) $\quad h(0)\tau^{\pm}{=}\pm\tau^{\pm}, \qquad$ (v) $\quad h(1)h{=}\dfrac{c}{3}\mathbf{1},$

together with the standard Virasoro relations between $\tau^{\pm}, h$ and the Virasoro vector ω of central charge c. Similarly to Theorem 1 we have

Theorem 24. *Let U be a SVOA of CFT–type Let $V{\subseteq}U$ be the subalgebra generated by 2 primary vectors $\tau^{\pm}$ of weight $\frac{3}{2}$ in U, so that*
$$V{=}\mathbb{C}\mathbf{1}\oplus V_{\frac{1}{2}}\oplus V_1\oplus V_{\frac{3}{2}}\oplus\dots$$
is a conformally graded subspace of U. Assume that:
  (I) $h(0)\tau^{\pm}{=}\pm\tau^{\pm}$ *where* $h{:=}\tau^{+}(1)\tau^{-}$,
  (II) $\tau^{\pm}(0)\tau^{\pm}{=}0.$
Then V is an $N{=}2$ SVOA with central charge $c{=}6k$ where $h(1)h = 2k\mathbf{1}$.

Proof. We sketch the proof, which is similar in many respects to that for Theorem 1. We firstly note that $u(1)v{=}-v(1)u$ for all $u, v \in \langle\tau^{\pm}\rangle$ by super skew-symmetry. Thus Assumptions (I) and (II) above imply the properties (ii), (iv) and (v) for $c{=}6k$. As for Lemma 9, we find $h(1)\tau^{\pm} = 0$ which in turn implies (as in Lemma 10) that $\tau^{\pm}(2)\tau^{\mp}{=}2k\mathbf{1}$. Thus property (i) holds.

Using super skew-symmetry we have
$$\tau^{+}(0)\tau^{-}{=}\tau^{-}(0)\tau^{+}{-}T(\tau^{-}(1)\tau^{+}){=}\tau^{-}(0)\tau^{+}{+}Th.$$
Thus, in this case, we define

(25) $$\sigma{:=}\frac{1}{2}\left(\tau^{+}(0)\tau^{-}{+}\tau^{-}(0)\tau^{+}\right),$$

so that $\tau^{\pm}(0)\tau^{\mp}{=}\sigma\pm\frac{1}{2}Th$. Hence

(26) $$[\tau^{+}(m), \tau^{-}(n)] = \sigma(m + n){+}\frac{1}{2}(m - n)h(m{+}n{-}1){+}m(m{-}1)k\delta_{m+n+1,0}.$$

It remains to show that σ is a Virasoro vector of central charge $c{=}6k$. As in Lemma 11, we find $\sigma(0)h{=}Th$, $\sigma(1)h{=}h$ and $\sigma(2)h{=}0$. (26) implies
$$\sigma(0){=}[\tau^{+}(0), \tau^{-}(0)], \quad \sigma(1){=}[\tau^{+}(0), \tau^{-}(1)]{+}\frac{1}{2}h(0), \quad \sigma(2){=}[\tau^{+}(0), \tau^{-}(2)] + h(1),$$

from which it follows that $\sigma(0)\tau^{+} = T\tau^{+}$, $\sigma(1)\tau^{+} = \frac{3}{2}\tau^{+}$ and $\sigma(2)\tau^{+} = 0$ (cf. Lemmas 12 and 15). Similar results follow for $\sigma(n)\tau^{-}$ for $n = 0, 1, 2$. Hence (cf. Lemma 16)
$$[\sigma(m), \tau^{\pm}(n)]{=}\left(\frac{1}{2}m{-}n\right)\tau^{\pm}(m{+}n{-}1),$$

which implies that σ is a Virasoro vector of central charge $6k$ (cf. Lemma 17). $\qquad\square$

6. EXAMPLES

We provide some constructions of $N=4$ and $N=2$ subalgebras of a lattice SVOA V_L for an odd lattice L. These examples illustrate Theorems 1, 2 and 24. Throughout, we let L_n denote the set of lattice vectors in L of norm n.

6.1. **Example 1.**

Consider the lattice SVOA V_L for $L=\mathbb{Z}^6$ – the well-known rank 12 free fermion construction. Let $L=\mathbb{Z}^6$ be generated by $\gamma_1,\cdots,\gamma_6 \in L_1$ with $(\gamma_i,\gamma_j)=\delta_{ij}$. Then V_L is generated by 12 weight $\frac{1}{2}$ fermion vectors $e^{\pm\gamma_i}$.

We firstly note from Section 7.2 that

$$\varepsilon(\gamma_i,\gamma_j) = \begin{cases} -\varepsilon(\gamma_j,\gamma_i) & \text{for } i \neq j, \\ -1 & \text{for } i = j. \end{cases}$$

In addition, for convenience, we choose $\varepsilon(\gamma_1,\gamma_2) = 1$ so that $\varepsilon(\gamma_2,\gamma_1) = -1$.

Define $\mathfrak{sl}_2$ generators $h:=\gamma_1+\gamma_2$, $x^\pm:=\mp e^{\pm h} \in (V_{\mathbb{Z}^6})_1$ where $h(1)h=2\mathbf{1}$ i.e. $k=1$ in (13). Then $\langle e^{\gamma_1}, e^{-\gamma_2}\rangle$ and $\langle e^{\gamma_2}, e^{-\gamma_1}\rangle$ form a pair of $\mathfrak{sl}_2$–representations, where using Subsection 7.3, we find

$$h(0)e^{\pm\gamma_1} = \pm e^{\pm\gamma_1}, \quad h(0)e^{\pm\gamma_2} = \pm e^{\pm\gamma_2},$$

(27)
$$x^\pm(0)e^{\mp\gamma_1} = \mp e^{\pm\gamma_2}, \quad x^\pm(0)e^{\mp\gamma_2} = \pm e^{\pm\gamma_1}.$$

Define $a,\bar{a},b,\bar{b} \in (V_{\mathbb{Z}^6})_1$ by

$$a = \frac{1}{\sqrt{2}}(\gamma_3+i\gamma_4), \quad \bar{a} = \frac{1}{\sqrt{2}}(\gamma_3-i\gamma_4),$$

$$b = \frac{1}{\sqrt{2}}(\gamma_5+i\gamma_6), \quad \bar{b} = \frac{1}{\sqrt{2}}(\gamma_5-i\gamma_6),$$

which satisfy non-zero relations $a(1)\bar{a}=b(1)\bar{b}=\mathbf{1}$. Lastly, define $\tau^\pm,\bar{\tau}^\pm$ by

$$\tau^+ = a(-1)e^{\gamma_1} + b(-1)e^{\gamma_2}, \quad \tau^- = a(-1)e^{-\gamma_2} - b(-1)e^{-\gamma_1},$$

$$\bar{\tau}^+ = \bar{a}(-1)e^{\gamma_2} - \bar{b}(-1)e^{\gamma_1}, \quad \bar{\tau}^- = -\bar{a}(-1)e^{-\gamma_1} - \bar{b}(-1)e^{-\gamma_2}.$$

We now show that the sub-SVOA generated by $\tau^\pm$, $\bar{\tau}^\pm$ is the $N=4$ superconformal algebra for central charge $c=6$ by use of Theorem 1. $\tau^\pm$, $\bar{\tau}^\pm$ are clearly primary vectors of weight $\frac{3}{2}$. It is straightforward to confirm (4) and (5) by using (27). Thus Axiom (I) of Theorem 1 holds.

In order to confirm Axioms (II)–(IV) of Theorem 1 we note, using superassociativity, that for all $u,v\in\{a,b\}$ and $\lambda,\nu\in\{\pm\gamma_1,\pm\gamma_2\}$

(28) $$\left(u(-1)e^\lambda\right)(1)v(-1)e^\nu = \langle\!\langle u,v\rangle\!\rangle e^\lambda(-1)e^\nu,$$

(29) $$\left(u(-1)e^\lambda\right)(0)v(-1)e^\nu = u(-1)v(-1)e^\lambda(0)e^\nu + \langle\!\langle u,v\rangle\!\rangle e^\lambda(-2)e^\nu,$$

where $u(1)v=\langle\!\langle u,v\rangle\!\rangle\mathbf{1}$. (28) and Subsection 7.3 imply that

$$\tau^+(1)\bar{\tau}^- = -\varepsilon(\gamma_1,-\gamma_1)\gamma_1(-1)\mathbf{1} - \varepsilon(\gamma_2,-\gamma_2)\gamma_2(-1)\mathbf{1} = h,$$

$$\tau^-(1)\bar{\tau}^+ = \varepsilon(-\gamma_2,\gamma_2)(-\gamma_2)(-1)\mathbf{1} + \varepsilon(-\gamma_1,\gamma_1)(-\gamma_1)(-1)\mathbf{1} = h.$$

Therefore $\tau^+(1)\bar{\tau}^+=-2x^+$ and $\tau^-(1)\bar{\tau}^-=2x^-$ using $x^\pm(0)h=\mp 2x^\pm$ and hence $A(1)B \cong \mathfrak{sl}_2$ i.e. Axiom (II) holds. Axiom (III) follows from

$$\tau^+(0)\tau^-=-\varepsilon(-\gamma_1,-\gamma_1)a(-1)b(-1)\mathbf{1}+\varepsilon(\gamma_2,-\gamma_2)b(-1)a(-1)\mathbf{1}=0.$$

using (29). By skew-symmetry $\tau^-(0)\tau^+=\tau^+(0)\tau^-=0$ and so $A(0)A=0$ using $\mathfrak{sl}_2$ symmetry. A similar argument applies showing that $B(0)B=0$. Lastly

$$\tau^+(0)\bar{\tau}^+=-(\gamma_1+\gamma_2)(-1)e^{\gamma_1+\gamma_2}=-Tx^+,$$

so that Axiom (IV) holds. Hence the SVOA generated by $\tau^\pm$, $\bar{\tau}^\pm$ is an $N=4$ superconformal algebra for central charge $c=6k=6$ by Theorem 1.

To finish, we show that $\sigma=\omega$, the standard V_L Virasoro vector of central charge 6. From (29) we find

$$\sigma = \frac{1}{2}\left(\tau^+(0)\bar{\tau}^--\tau^-(0)\bar{\tau}^+\right)$$

$$= -\frac{1}{2}\Bigg(\varepsilon(\gamma_1,-\gamma_1)a(-1)\bar{a}+e^{\gamma_1}(-2)e^{-\gamma_1} + \varepsilon(\gamma_2,-\gamma_2)b(-1)\bar{b}+e^{\gamma_2}(-2)e^{-\gamma_2}$$

$$+\varepsilon(-\gamma_2,\gamma_2)a(-1)\bar{a}+e^{-\gamma_2}(-2)e^{\gamma_2}+\varepsilon(-\gamma_1,\gamma_1)b(-1)\bar{b}+e^{-\gamma_1}(-2)e^{\gamma_1}\Bigg)$$

$$= a(-1)\bar{a}+b(-1)\bar{b}+\frac{1}{2}\left(\gamma_1(-1)\gamma_1+\gamma_2(-1)\gamma_2\right) = \frac{1}{2}\sum_{i=1}^{6}\gamma_i(-1)\gamma_i = \omega.$$

Thus we conclude the well known result e.g. [ET1]

Proposition 25. $V_{\mathbb{Z}^6}$ *contains an* $N{=}4$ *superconformal subalgebra with the standard lattice Virasoro vector for central charge* $c{=}6$.

6.2. **Example 2.** In this example, we construct an $N=4$ algebra from lattice theories that do not contain free fermions. We also illustrate how to reduce the problem of constructing $N=4$ subalgebras of lattices theories to questions about lattices.

Let $\alpha_1,\ldots,\alpha_6$ be an orthogonal basis for $\mathbb{R}^6$ consisting of vectors of norm 3, and let L be the lattice spanned by the α_i together with

$$h:=\frac{1}{3}(\alpha_1+\ldots+\alpha_6)\in L_2.$$

Then L is an odd, positive-definite, integral lattice with *theta function*

$$\theta_L(\tau)=1+2q+24q^{\frac{3}{2}}+\ldots$$

In particular, there are no vectors of norm 1, and $\pm h$ are the only roots.

Proposition 26. V_L *contains an* $N{=}4$ *superconformal subalgebra with standard lattice Virasoro vector of central charge* $c{=}6$.

Proof. Let $X\subseteq L_3$ consist of the 6 vectors α_i. In the formalism of Section 4, especially (23), we take Y to consist of the vectors $\{h-\alpha_i\}$, so that the four generating states of A of weight $\frac{3}{2}$ will be chosen to take the form

$$\tau^+:=\sum_{\alpha\in X}c_\alpha e^\alpha, \quad \tau^-:=\sum_{\alpha\in X}c_\alpha\varepsilon(h,\alpha)e^{\alpha-h},$$

$$\bar{\tau}^+:=\sum_{\alpha\in X}d_{h-\alpha}e^{h-\alpha}, \quad \bar{\tau}^-:=-\sum_{\alpha\in X}d_{h-\alpha}\varepsilon(h,\alpha)e^{-\alpha}.$$

We show that the hypotheses of Theorem 2 hold for certain choices of scalars $c_\alpha, d_{h-\alpha}, (\alpha\in X)$. Conditions (a) and (b) of Lemma 20 and condition (b) of Lemma 22

automatically hold since $X \cap Y = 0$ and $\pm h$ are the only roots in L. We may check by direct calculation that these facts *imply* the assumptions of these two Lemmas (cf. the proofs of the Lemmas), i.e., $A(1)A = B(1)B = 0$. Now choose the scalars $c_\alpha, d_{h-\alpha}$ so that

$$c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) = -\frac{1}{3}$$

for each $\alpha_i \in X$, implying condition (a) of Lemma 22. Hence $A(1)B \cong \mathfrak{sl}_2$.

Because $\pm h$ are the only roots of L, there is a unique simple component of the Lie algebra $(V_L)_1$, and it is isomorphic to sl_2. It follows that this component is our Lie algebra $A(1)B = \mathfrak{sl}_2$, and in particular $h \in \mathfrak{sl}_2$ and $A(1)B = \langle h, e^{\pm h} \rangle$. It is then straightforward to check that A and B are indeed vector representations of $\mathfrak{sl}_2$, so that all hypotheses of Theorem 2 are satisfied. This completes the proof that $\tau^\pm$, $\overline{\tau}^\pm$ generate an $N{=}4$ subalgebra of V_L with $c{=}6$.

The Virasoro vector in this example is (cf. Lemmas 11, 17)

$$\begin{aligned}
\sigma &= \frac{1}{2}(\tau^+(0)\overline{\tau}^- - \tau^-(0)\overline{\tau}^+) \\
&= -\frac{1}{2}\sum_{\alpha,\beta \in X}\left\{ c_\alpha d_{h-\beta}\varepsilon(h,\beta)e^\alpha(0)e^{-\beta} + c_\alpha d_{h-\beta}\varepsilon(h,\alpha)e^{\alpha-h}(0)e^{h-\beta}\right\} \\
&= -\frac{1}{2}\sum_{\alpha \in X}\left\{ c_\alpha d_{h-\alpha}\varepsilon(h,\alpha)e^\alpha(0)e^{-\alpha} + c_\alpha d_{h-\alpha}\varepsilon(h,\alpha)e^{\alpha-h}(0)e^{h-\alpha}\right\} \\
&= \frac{1}{12}\sum_{\alpha \in X}\left(\alpha(-2) + \alpha(-1)^2 + (\alpha - h)(-2) + (\alpha - h)(-1)^2 \right)\mathbf{1} \\
&= \frac{1}{12}\sum_{\alpha \in X}\left(2\alpha(-1)^2 + h(-1)^2 - 2\alpha(-1)h(-1) + 2\alpha(-2) - h(-2) \right)\mathbf{1} \\
&= \frac{1}{6}\sum_\alpha \alpha(-1)\alpha.
\end{aligned}$$

This is indeed the standard Virasoro element of V_L, and the proof is complete. □

We next consider three known examples of $N{=}2$ superconformal subalgebras of odd lattice SVOAs, illustrating Theorem 24.

6.3. Example 3. Consider the lattice SVOA $V_{\mathbb{Z}^3}$, the rank 6 free fermion construction. Let $L = \mathbb{Z}^3$ be generated by $\gamma_1, \gamma_2, \gamma_3 \in L_1$ with $(\gamma_i, \gamma_j) = \delta_{ij}$. Define $h, a^\pm \in (V_{\mathbb{Z}^3})_1$ by

$$h = \gamma_1, \qquad a^\pm = \frac{1}{\sqrt{2}}(\gamma_2 \pm i\gamma_3),$$

with $a^+(1)a^- = \mathbf{1}$. Lastly, define $\tau^\pm = a^\pm(-1)e^{\pm\gamma_1}$.

Using (29) and (28) we find that Axioms (I) and (II) of Theorem 24 hold. Since $h(1)h = \mathbf{1}$, the central charge is $c{=}3$. Using (25) one finds that

$$\sigma = a^+(-1)a^- + \frac{1}{2}\gamma_1(-1)\gamma_1 = \frac{1}{2}\sum_{i=1}^3 \gamma_i(-1)\gamma_i,$$

the standard lattice Virasoro vector for $V_{\mathbb{Z}^3}$. This establishes the well-known free fermion construction

Proposition 27. $V_{\mathbb{Z}^3}$ *contains an* $N{=}2$ *superconformal subalgebra with the standard lattice Virasoro vector for central charge* $c{=}3$.

6.4. **Example 4.** In this example we show that every odd lattice SVOA for which $L_3 \neq \emptyset$ contains an $N{=}2$ SVOA with central charge $c{=}1$.

Proposition 28 ([K], Example 5.9c). *Let* $\gamma{\in}L_3$ *and define* $\tau^{\pm}{:=}\frac{1}{\sqrt{3}}e^{\pm\gamma}$ *and* $h{:=}\frac{1}{3}\gamma$. *Then* $\tau^{\pm}$ *generate an* $N{=}2$ *subalgebra of* V_L *with Virasoro vector* $\omega{=}\frac{1}{6}\gamma(-1)\gamma$ *and* $c{=}1$.

Proof. We find $h{=}\tau^{\pm}(1)\tau^{\mp}$ with $h(0)\tau^{\pm}{=}\pm\,\tau^{\pm}$ and $\tau^{\pm}(0)\tau^{\pm}{=}0$. Thus Axioms (I) and (II) of Theorem 24 hold, so $\tau^{\pm}$ generate an $N{=}2$ superconformal algebra. Since $h(1)h{=}\frac{1}{3}\mathbf{1}$, the central charge is $c{=}1$ with Virasoro vector $\frac{1}{6}\gamma(-1)\gamma$ from (25). $\qquad\square$

6.5. **Example 5.**

Proposition 29 ([A], Proposition 6.1). *Let* $\alpha,\beta{\in}L_3$ *with* $(\alpha,\beta){=}1$, *and let* λ,μ *be nonzero scalars. Define*

$$\tau^{+}{=}\frac{1}{2}(\lambda e^{\alpha}{+}\mu e^{\beta}), \quad \tau^{-}{=}\frac{1}{2}(\lambda^{-1}e^{-\alpha}{+}\mu^{-1}e^{-\beta}), \quad h{=}\frac{1}{4}(\alpha{+}\beta),$$

$$\omega{=}\frac{1}{8}\left\{\alpha(-1)\alpha{+}\beta(-1)\beta{+}\frac{2\lambda}{\mu}\varepsilon(\alpha,\beta)e^{\alpha-\beta}{+}\frac{2\mu}{\lambda}\varepsilon(\beta,\alpha)e^{\beta-\alpha}\right\}.$$

Then $\tau^{\pm}$ *generate an* $N{=}2$ *subalgebra of* V_L *with* $c{=}\frac{3}{2}$.

Proof. Axioms (I) and (II) of Theorem 24 are easily seen to hold. $h(1)h = \frac{1}{2}\mathbf{1}$ implies the central charge $c{=}\frac{3}{2}$ and ω is as given on applying (25). $\qquad\square$

7. APPENDICES

7.1. **Axioms for super VOAs.** The underlying Fock space is a $\frac{1}{2}\mathbb{Z}$–graded $\mathbb{C}$-linear super vector space

$$V= \oplus_{k\in\frac{1}{2}\mathbb{Z}} V_k,$$

with *parity* operator $p(u) = 2k \bmod 2$ for $u \in V_k$. Each state $u{\in}V$ has a vertex operator $Y(u,z){=}\sum_{n\in\mathbb{Z}} u(n)z^{-n-1}$; $u(n){\in}\operatorname{End}(V)$ is the nth *mode* of u.

There is a distinguished *vacuum state* $\mathbf{1}{\in}V_0$ with vertex operator $Y(\mathbf{1},z){=}\operatorname{Id}_V$; and a distinguished *Virasoro state* $\omega{\in}V_2$ with vertex operator

$$Y(\omega,z){=}\sum_{n\in\mathbb{Z}} L(n)z^{-n-2},$$

whose modes satisfy the Virasoro relations with *central charge* c:

$$[L(m),L(n)]{=}(m{-}n)L(m{+}n) + \frac{1}{2}\binom{m+1}{3}\delta_{m+n,0}c\operatorname{Id}_V.$$

We distinguish the endomorphism $T{\in}\operatorname{End}(V)$ defined by $T(u){:=}u(-2)\mathbf{1}{=}L(-1)u$. The $\frac{1}{2}\mathbb{Z}$ grading is determined by $L(0)$ with $L(0)u = ku$ for $u \in V_k$.

Modes satisfy the following axioms, the third being the super Jacobi identity:

(a) $u(n)v = 0$ for all $n \geq n_0$,

(b) $u(-1)\mathbf{1} = u$; $u(n)\mathbf{1} = 0$ for $n \geq 0$,

(c) $\forall\, r, s, t \in \mathbb{Z}$,

$$\sum_{i \geq 0} \binom{r}{i} (u(t+i)v)(r+s-i)w =$$

$$\sum_{i \geq 0} (-1)^i \binom{t}{i} \left\{ u(r+t-i)v(s+i) - (-1)^{t+p(u)p(v)} v(s+t-i)u(r+i) \right\} w,$$

for all $u, v, w \in V$. The special cases $t=0, r=0$ give respectively super commutativity

$$u(r)v(s) - (-1)^{p(u)p(v)} v(s)u(r) = \sum_{i \geq 0} \binom{r}{i} (u(i)v)(r+s-i),$$

and super associativity

$$(u(t)v)(s) = \sum_{i \geq 0} (-1)^i \binom{t}{i} \left\{ u(t-i)v(s+i) - (-1)^{t+p(u)p(v)} v(s+t-i)u(i) \right\}.$$

Taking $r=-1, s=0, w=\mathbf{1}$ leads to super skew-symmetry

$$v(t)u = -(-1)^{t+p(u)p(v)} \sum_{i \geq 0} (-1)^i T^i u(t+i)v.$$

7.2. **The ε-formalism.** Fix a finitely generated free abelian group L. We are interested in groups $\widehat{L}$ which are *central extensions of L by $\mathbb{Z}_2$*. So there is a short exact sequence of groups

(30) $1 \to \{\pm 1\} \to \widehat{L} \xrightarrow{-} L \to 0,$

and $\widehat{L}$ can be identified with $L \times \{\pm 1\}$ as a set, with multiplication

$$(\alpha, e)(\beta, f) = (\alpha+\beta, \varepsilon(\alpha, \beta)ef) \qquad (\alpha, \beta \in L, e, f \in \{\pm 1\}),$$

where

$$\varepsilon : L \times L \to \{\pm 1\}.$$

We may, and shall, take ε to be *bimultiplicative*, i.e.,

$$\varepsilon(\alpha+\beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma), \quad \varepsilon(\alpha, \beta+\gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma).$$

This ensures that $\varepsilon \in Z^2(L, \{\pm 1\})$ is a *2-cocycle* and that multiplication in L is *associative*. In particular we note that $\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1$ and $\varepsilon(\alpha, \beta) = \varepsilon(\alpha, -\beta) = \varepsilon(-\alpha, \beta)$.

If L is a positive-definite integral lattice with bilinear form (,), then we may further choose ε ([K] P. 155) so that it satisfies

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)+(\alpha, \alpha)(\beta, \beta)}, \quad \varepsilon(\alpha, \alpha) = (-1)^{((\alpha, \alpha)+(\alpha, \alpha)^2)/2}.$$

Remark 30. Depending on context, various choices for ε are used in the literature, although they give equivalent theories. The one used in [FLM], for example, is different to the one we generally use here.

7.3. **Super lattice theories.** Let L be a positive-definite integral lattice equipped with a bilinear form (,), with ε as in Subsection 7.2. The *twisted group algebra* $\mathbb{C}^\varepsilon[L]$ has basis $e^\alpha(\alpha\in L)$ and multiplication

$$e^\alpha e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta} \quad (\alpha,\beta \in L).$$

It is $\frac{1}{2}\mathbb{Z}$-graded by $wt(e^\alpha):=\frac{1}{2}(\alpha,\alpha)$.

There are Lie algebras (the first is abelian)

$$\mathfrak{h}:=\mathbb{C} \otimes_\mathbb{Z} L, \ \ \widehat{\mathfrak{h}}:=\mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c, \ \ \mathfrak{h}^+ = \mathfrak{h} \otimes t\mathbb{C}[t], \ \ \widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}],$$

with brackets $[x \otimes t^m, y \otimes t^n]=(x,y)m\delta_{m+n,0}c$, $[c,\widehat{\mathfrak{h}}]=0$, and an induced $\widehat{\mathfrak{h}}$-module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C} \cong S(\widehat{\mathfrak{h}}^-) \text{ (linearly)},$$

$\mathfrak{h} \otimes \mathbb{C}[t]$ acting trivially on $\mathbb{C}$ and c acting as 1. Fock space for the lattice theory is

$$V_L = M(1)\otimes\mathbb{C}^\varepsilon[L] \cong S(\widehat{\mathfrak{h}}^-)\otimes\mathbb{C}[L] \text{ (linearly)}$$

with the usual tensor product grading. The Virasoro vector is $\frac{1}{2}\sum_i h_i(-1)h_i$, the sum ranging over *any* orthonormal basis $\{h_i\}$ of $\mathfrak{h}$.

For $\alpha\in\mathfrak{h}$ write $\alpha(n):=\alpha \otimes t^n$, $\alpha(z):=\sum_{n\in\mathbb{Z}} \alpha(n)z^{-n-1}$, $z^\alpha:e^\beta\mapsto z^{(\alpha,\beta)}e^\beta$, and set

$$Y(e^\alpha, z):= \exp\left(\sum_{m=1}^\infty \alpha(-m)\frac{z^m}{m}\right) \exp\left(-\sum_{m=1}^\infty \alpha(m)\frac{z^{-m}}{m}\right) e^\alpha z^\alpha,$$

and for $v = \alpha_1(-n_1)...\alpha_k(-n_k)\otimes e^\alpha \in V_L$ $(n_i\geq 1)$ set

$$Y(v,z):=: \left(\frac{1}{(n_1 - 1)!}\left(\frac{d}{dz}\right)^{n_1-1}\alpha_1(z)\right) ... \left(\frac{1}{(n_k - 1)!}\left(\frac{d}{dz}\right)^{n_k-1}\alpha_k(z)\right) Y(e^\alpha, z):,$$

with the usual normal ordering conventions.

For $\gamma,\rho\in L$ we have

$$e^\gamma(n)e^\rho = \begin{cases} 0 & \text{if } (\gamma,\rho) \geq -n \\ \varepsilon(\gamma,\rho)e^{\gamma+\rho} & \text{if } (\gamma,\rho) = -n-1 \\ \varepsilon(\gamma,\rho)\gamma(-1)e^{\gamma+\rho} & \text{if } (\gamma,\rho) = -n-2 \\ \frac{1}{2}\varepsilon(\gamma,\rho)(\gamma(-2)e^{\gamma+\rho} + \gamma(-1)^2e^{\gamma+\rho}) & \text{if } (\gamma,\rho) = -n-3 \end{cases}$$

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