Genus Two Virasoro Correlation Functions for Vertex Operator Algebras

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December 1, 2016

Abstract

We consider all genus two correlation functions for the Virasoro vacuum descendants of a vertex operator algebra. These are described in terms of explicit generating functions that can be combinatorially expressed in terms of graph theory.

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1 Introduction

A Vertex Operator Algebra (VOA) (e.g. [FLM], [K], [LL], [MT1]) is an algebraic structure closely related to Conformal Field Theory (CFT) in physics e.g. [DMS]. An essential ingredient of a VOA or CFT is the existence of a Virasoro vector whose vertex operator modes generate a Virasoro subalgebra of central charge c.

A connection between VOAs and modular forms is manifested through n-point correlation trace functions. Zhu recursion expresses genus one n-point correlation functions in terms of $(n-1)$-point correlation functions [Z]. Correlation functions of Virasoro descendant vectors are of particular importance. The Zhu recursion of genus one Virasoro correlation functions gives rise to genus one Ward Identities. Given some conditions on the VOA, the Ward Identities then give rise to genus one modular differential equations.

A complete description of the Virasoro correlation functions at genus zero and genus one has been described [HT]. In particular, it has been shown that the generating functions for all such Virasoro correlation functions may be expressed symmetrically by taking sums of weights of an appropriate set of graphs.

Correlation functions for VOAs on a genus two Riemann surface have been defined, and in some cases calculated based on explicit sewing procedures (citations). In recent work, Zhu reduction for genus two n-point correlation functions has been described [GT1]. The recursion of the genus two Virasoro correlation functions gives rise to the genus two Ward Identities. The existence of Virasoro

The purpose of this paper is to describe all genus two correlation functions for Virasoro descendants of the vacuum vector in terms of explicit generating functions and to describe the effects analytic and modular transformations on these generating functions.

We begin in Section 2 with a brief review of ideas from the theories of Riemann Surfaces and Vertex Operator Algebras which allow the definition of the genus two correlation functions we consider. These genus two correlation functions satisfy a Zhu recursion relation [GT1].

In Section 3 we consider the genus two correlation functions for Virasoro vacuum descendants. We describe a set of symmetric generating functions for all such genus two Virasoro correlation functions. These generating functions satisfy an explicit genus two Ward Identity, involving genus two *generalised Weierstrass functions* [GT1]. The resulting expressions for the generating functions are not manifestly symmetric. By means of the differential equation for the normalised $(1,1)$-bidifferential (Proposition **??**) and by defining a sequence of differential operators, we demonstrate in Theorem 3.8 how to express the generating functions as sums of weights of appropriate graphs. In this way, we achieve symmetric expansions of the genus two generating functions, similar to the symmetric expansions of [HT] for genus one Virasoro correlation functions.

In Section (***suggest section for analytic and modular transformations***), we consider analytic and modular transformations of the Virasoro generating functions. Proposition 4.1 demonstrates the effect of any analytic transformation on the sequence of differential operators defined in Section 3. The effect of the analytic transformation on the Virasoro generating function floows from this result. We then consider the effect of a general $\mathrm{Sp}(4,\mathbb{Z})$ transformation on the Virasoro generating functions. Theorem 4.2 demonstrates that the generalised Weierstrass functions apearing in the genus two Ward Identity are $\mathrm{Sp}(4,\mathbb{Z})$ invariant. Applying this result to the genus two Ward Identity, we obtain Theorem 4.6, which describes the the modular transformations of the Virasoro generatin functions.

The principal aim of this paper is to obtain from the recursive Ward identity (3.6) a combinatoric expression for $G_n(z_1,\ldots,z_n)$ which is symmetric and coordinate free.

**** Everything based on Ward id and diff eqn for omega

2 Vertex operator algebras on genus two Riemann surfaces

2.1 Genus two Riemann surfaces

We briefly review some concepts in genus two Riemann surface theory e.g. [FK, F, Mu]. Let $\mathcal{S}^{(2)}$ be a compact genus two Riemann surface with canonical homology basis α^i, β^i for $i=1,2$. There exists a unique holomorphic symmetric bidifferential $(1,1)$-form $\omega(x,y)$, the *normalised bidifferential of the second kind*, where for $x\neq y\in\mathcal{S}^{(2)}$

$$\omega(x,y)=\frac{dxdy}{(x-y)^2}+\frac{1}{6}s(x)+O\left((x-y)\right), \tag{2.1}$$

$$\oint_{\alpha^i}\omega(x,\cdot)=0,\quad i=1,2.$$

$s(x)$ is called the *projective connection*. We further have

$$\nu_i(x)=\oint_{\beta^i}\omega(x,\cdot),\qquad \Omega_{ij}=\frac{1}{2\pi i}\oint_{\beta^i}\nu_j,\quad i,j=1,2.$$

for *holomorphic differentials* $\nu_i(x)$ normalised by $\oint_{\beta^i}\nu_j=2\pi i\delta_{ij}$ and *period matrix* $\Omega\in\mathbb{H}_2$, the genus two Siegel upper half plane i.e. $\Omega=\Omega^T$ and $\Im(\Omega)>0$

We now consider the genus two Riemann surface $\mathcal{S}^{(2)}$ constructed by sewing two genus one tori $\mathcal{S}_a=\mathbb{C}/\Lambda_a$, for lattice $\Lambda_a=2\pi i(\mathbb{Z}\tau_a\oplus\mathbb{Z})$ with modular parameter $\tau_a\in\mathbb{H}_1$ for $a=1,2$ [MT2]. Let $z_a\in\mathcal{S}_a$, $\epsilon\in\mathbb{C}$ and define punctured tori

$$\widehat{\mathcal{S}}_1=\mathcal{S}_1\backslash\left\{z_1,|z_1|\leq|\epsilon|/r_2\right\},\qquad \widehat{\mathcal{S}}_2=\mathcal{S}_2\backslash\left\{z_2,|z_2|\leq|\epsilon|/r_1\right\},$$

where $|\epsilon|\leq r_1r_2$. We identify the annular regions $\{z_1,|\epsilon|/r_2\leq|z_1|\leq r_1\}$ and $\{z_2,|\epsilon|/r_1\leq|z_2|\leq r_2\}$ via the sewing relation $z_1z_2=\epsilon$. Then $\mathcal{S}^{(2)}$ is parameterized by the sewing domain

$$\mathcal{D}_{\mathrm{sew}}=\left\{(\tau_1,\tau_2,\epsilon)\in\mathbb{H}_1\times\mathbb{H}_1\times\mathbb{C}:|\epsilon|<\frac{1}{4}D(q_1)D(q_2)\right\}, \tag{2.2}$$

where $q_a=e^{2\pi i\tau_a}$ and $D(q_a)=\min_{\lambda_a\in\Lambda_a,\lambda_a\neq0}|\lambda_a|$. We may then obtain explicit expressions for $\omega(x,y)$, $\nu_i(x)$ and Ω_{ij} on $\mathcal{S}^{(2)}$ for $x,y\in\widehat{\mathcal{S}}_1\cup\widehat{\mathcal{S}}_2$ described in [MT2].

2.2 Vertex operator algebras on a torus

We review aspects of vertex operator algebras (e.g. [FLM, K, LL, MT1]). A Vertex Operator Algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$ where $\dim V_{(n)} < \infty$ for each $n \in \mathbb{Z}$, a linear map $Y : V \to \mathrm{End}\,(V)[[z, z^{-1}]]$ for a formal parameter z and pair of distinguished vectors: the vacuum $\mathbf{1} \in V_{(0)}$ and the conformal vector $\omega \in V_{(2)}$. For each $v \in V$, the image under the map Y is the *vertex operator*

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1},$$

with *modes* $v(n) \in \mathrm{End}\,(V)$, where $Y(v, z)\mathbf{1} = v + O(z)$. Vertex operators satisfy *locality* i.e. for all $u, v \in V$ there exists an integer $k \geq 0$ such that

$$(z_1 - z_2)^k \,[Y(u, z_1), Y(v, z_2)] = 0.$$

The vertex operator of the conformal vector ω is $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ where the modes $L(n)$ satisfy the Virasoro algebra with *central charge* c

$$[L(m), L(n)] = (m - n)L(m + n) + c\,\frac{m^3 - m}{12}\delta_{m,-n}\,\mathrm{Id}_V.$$

Furthermore, $L(0)v = kv$ for *conformal weight* $\mathrm{wt}\,(v) = k$ for all $v \in V_{(k)}$ and $Y(L(-1)u, z) = \partial_z Y(u, z)$.

In order to describe VOAs on a torus, Zhu [Z] introduced an isomorphic VOA $(V, Y[\ ,\ ], \mathbf{1}, \widetilde{\omega})$ with "square bracket" vertex operators

$$Y[v, z] = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1} = Y\left(e^{L(0)}v, e^z - 1\right),$$

and conformal vector $\widetilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$ with modes $L[n]$.

Define the *genus one partition function* by the formal trace $Z^{(1)}_V(\tau) = \mathrm{Tr}_V\left(q^{L(0)-c/24}\right)$, with $q = e^{2\pi i\tau}$, the *genus one correlation one point function* by the formal trace

$$Z^{(1)}_V(v; \tau) = \mathrm{Tr}_V\left(o(v)q^{L(0)-c/24}\right), \quad v \in V,$$

where $o(v) = v(k - 1)$ for $v \in V_{(k)}$ and the *genus one n-point correlation function* for $v_1, \ldots, v_n \in V$ inserted at $z_1, \ldots, z_n \in \mathbb{C}/(2\pi i(\mathbb{Z}\tau \oplus \mathbb{Z}))$ by

$$Z^{(1)}_V(v_1, z_1; \ldots; v_n, z_n; \tau) = Z^{(1)}_V(Y[v_1, z_1] \ldots Y[v_n, z_n]\mathbf{1}; \tau).$$

Zhu describes a general recursion formula for expressing any genus one n-point correlation function as a linear combination of $(n - 1)$-point functions with coefficients given by explicit universal elliptic functions [Z].

2.3 VOAs on a genus two Riemann surface

We define the genus two partition function and n-point correlation function for a VOA based on the sewing scheme for $\mathcal{S}^{(2)}$ constructed from two tori $\mathcal{S}_1$ and $\mathcal{S}_2$ [MT3, GT1]. The *genus two partition function* for V of strong CFT-type is defined by

$$Z_V^{(2)}(\tau_1,\tau_2,\epsilon)=\sum_{u\in V} Z_V^{(1)}(u;\tau_1)Z_V^{(1)}(\overline{u};\tau_2), \tag{2.3}$$

where the formal sum is taken over any V-basis and $\overline{u}$ is the dual of u with respect to an invariant invertible bilinear form $\langle\ ,\ \rangle$ associated with the Mobius map $z\to\epsilon/z$ (see [GT1] for more details).

The *genus two n-point correlation function* for $a_1,\ldots,a_L\in V$ and $b_1,\ldots,b_R\in V$ formally inserted at $x_1,\ldots,x_L\in\widehat{\mathcal{S}}_1$ and $y_1,\ldots,y_R\in\widehat{\mathcal{S}}_2$, respectively, is defined by

$$\begin{aligned}&Z_V^{(2)}(a_1,x_1;\ldots;a_L,x_L|b_1,y_1;\ldots;b_R,y_R;\tau_1,\tau_2,\epsilon)\\&=\sum_{u\in V}Z_V^{(1)}(Y[a_1,x_1]\ldots Y[a_L,x_L]u;\tau_1)Z_V^{(1)}(Y[b_R,y_R]\ldots Y[b_1,y_1]\overline{u};\tau_2).\end{aligned} \tag{2.4}$$

Convergent expressions have been found for such correlation functions for particular VOAs such as the Heisenberg VOA and lattice VOAs [MT3]. A formal Zhu recursion formula for a genus two n-point function in terms of $(n-1)$-point functions is described in [GT1] where the coefficients are formal series, called *generalised Weierstrass functions*, which depend on the conformal weight of the recursion vector but are otherwise universal. For conformal weight 1 or 2, these series are holomorphic on appropriate domains [GT1]. In this paper, we concentrate on genus two n-point functions for n copies of the Virasoro vector $\widetilde{\omega}$ for any VOA V.

3 Genus Two Virasoro Correlation Functions

3.1 The Virasoro generating function

Consider the genus two Virasoro n-point correlation function for $z_1,\ldots,z_n\in\widehat{\mathcal{S}}_1$

$$Z_V^{(2)}(\widetilde{\omega},z_1;\ldots;\widetilde{\omega},z_n)=\sum_{u\in V}Z_V^{(1)}(Y[\widetilde{\omega},z_1]\ldots Y[\widetilde{\omega},z_n]u;\tau_1)\,Z_V^{(1)}(\overline{u};\tau_2), \tag{3.1}$$

(where we suppress the dependence on τ_1,τ_2,ϵ). The formal differential

$$G_n(z_1,\ldots,z_n)=Z_V^{(2)}(\widetilde{\omega},z_1;\ldots;\widetilde{\omega},z_n)\,d\mathbf{z}^2, \tag{3.2}$$

where $d\mathbf{z}^2=dz_i^2\ldots dz_n^2$, is independent of whether we formally insert $\widetilde{\omega}$ at $z_i\in\widehat{\mathcal{S}}_1$ or at $\epsilon/z_i\in\widehat{\mathcal{S}}_2$ (Proposition 6 of [MT3]). Similarly to [HT] we find

Proposition 3.1. *$G_m(z_1,\ldots,z_m)$ is symmetric and is a generating function for all genus two n–point correlation functions for Virasoro vacuum descendants.*

Proof. $G_m(z_1,\ldots,z_m)$ is a symmetric in $z_1,\ldots,z_m$ by locality. Consider the genus two n-point function for n Virasoro vacuum descendants $v_i = L[-k_1^i]\ldots L[-k_{m_i}^i]\mathbf{1}$ inserted at $z_i \in \widehat{\mathcal{S}}_1$ for $i=1,\ldots,n$ and $k_j^i \geq 2$ given by

$$Z_V^{(2)}(v_1,z_1;\ldots;v_n,z_n) = \sum_{u\in V} Z_V^{(1)}(Y[v_1,z_1]\ldots Y[v_n,z_n]u;\tau_1)\, Z_V^{(1)}(\overline{u};\tau_2).$$

$Z_V^{(1)}(Y[v_1,z_1]\ldots Y[v_n,z_n]u;\tau_1)$ is the coefficient of $\prod_{i=1}^n\prod_{j=1}^{m_i}(x_j^i)^{k_j^i-2}$ in

$$Z_V^{(1)}\left(Y[Y[\widetilde{\omega},x_1^1]\ldots Y[\widetilde{\omega},x_{m_1}^1]\mathbf{1},z_1]\ldots Y[Y[\widetilde{\omega},x_1^n]\ldots Y[\widetilde{\omega},x_{m_n}^n]\mathbf{1},z_n]u;\tau_1\right).$$

Using associativity and lower truncation (e.g. [K, LL, MT1]) we find for $N\gg 0$ that

$$\prod_{i=1}^n\prod_{j=1}^{m_i}(x_j^i+z_i)^N Y[Y[\widetilde{\omega},x_1^1]\ldots Y[\widetilde{\omega},x_{m_1}^1]\mathbf{1},z_1]\ldots Y[Y[\widetilde{\omega},x_1^n]\ldots Y[\widetilde{\omega},x_{m_n}^n]\mathbf{1},z_n]u$$
$$=\prod_{i=1}^n\prod_{j=1}^{m_i}(x_j^i+z_i)^N Y[\widetilde{\omega},z_1+x_1^1]\ldots Y[\widetilde{\omega},z_1+x_{m_1}^1]\ldots Y[\widetilde{\omega},z_n+x_1^n]\ldots Y[\widetilde{\omega},z_n+x_{m_n}^n]u.$$

Thus the genus two n-point function for $v_1,\ldots,v_n$ is the coefficient of $\prod_{i=1}^n\prod_{j=1}^{m_i}(x_j^i)^{k_j^i-2}$ of the formal expansion of $Z_V^{(2)}\left(\widetilde{\omega},z_1+x_1^1;\ldots;\widetilde{\omega},z_n+x_{m_n}^n\right)$ for $M=\sum_{i=1}^n m_i$. □

3.2 The Ward identity

Define a genus two modular derivative operator

$$\nabla_x = \sum_{1\leq a\leq b\leq 2}\nu_a(x)\nu_b(x)\partial_{ab}, \tag{3.3}$$

where $\partial_{ab} := \frac{\partial}{\partial\Omega_{ab}}$, for period matrix Ω_{ab}, and normalised holomorphic 1–differentials ν_a where $a,b=1,2$. There exists an injective but non-surjective holomorphic map F^Ω

$$F^\Omega : \mathcal{D}_{\text{sew}} \to \mathbb{H}_2,$$
$$(\tau_1,\tau_2,\epsilon)\mapsto \Omega(\tau_1,\tau_2,\epsilon),$$

from the sewing domain $\mathcal{D}_{\text{sew}}$ into the Siegel upper half plane [GT1]. Thus we may interpret the action of the operator ∇_x on $\mathcal{D}_{\text{sew}}$ as $\left(F^\Omega\right)^{-1}\circ\nabla_x\circ F^\Omega$.

In Section 5 of [GT1] we describe a genus two Ward identity for the genus two Virasoro n-point correlation function. This is expressed in terms of generalised Weierstrass functions ${}^2\mathcal{P}_k(x,y)$ for $k\geq 1$ defined as follows. Let $\boldsymbol{\nu}(x)=[\nu_1(x),\nu_2(x)]$ denote a row vector of holomorphic 1–differentials and define

$$\Psi(x,y) = {}^2\mathcal{P}_1(x,y)dx^2(dy)^{-1} = -\frac{\omega(x,y)\begin{vmatrix}\boldsymbol{\nu}(x)\\ \boldsymbol{\nu}(y)\end{vmatrix}+\begin{vmatrix}\boldsymbol{\nu}(y)\\ \nabla_x\boldsymbol{\nu}(y)\end{vmatrix}}{\begin{vmatrix}\boldsymbol{\nu}(y)\\ \partial_y\boldsymbol{\nu}(y)\end{vmatrix}dy}. \tag{3.4}$$

Proposition 3.2 ([GT1]). *$\Psi(x,y)$ is a holomorphic $(2,-1)$–bidifferential for $x \neq y$ where, for any local coordinates x, y*

$$\Psi(x,y) = \left(\frac{1}{x-y} + \textit{regular terms}\right) dx^2 dy^{-1}.$$

We also define

$$ {}^2\mathcal{P}_k(x,y) = \frac{1}{(k-1)!}\partial_y^{k-1}\left({}^2\mathcal{P}_1(x,y)\right) = \frac{1}{(x-y)^k} + \text{regular terms}, \qquad (3.5)$$

for $k \geq 1$, which is holomorphic for $x \neq y$.

Proposition 3.3 ([GT1]). *$G_n(z_1, \ldots, z_n)$ of (3.2) obeys the formal Ward identity for for $z_1, \ldots, z_n \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$*

$$\begin{aligned} &G_n(z_1, \ldots, z_n) \\ &= \left(\nabla_{z_1} + dz_1^2 \sum_{k=2}^{n} \left({}^2\mathcal{P}_1(z_1, z_k)\partial_{z_k} + 2 \cdot {}^2\mathcal{P}_2(z_1, z_k)\right)\right) G_{n-1}(z_2, \ldots, z_n) \\ &\quad + \frac{c}{2}\sum_{k=2}^{n} {}^2\mathcal{P}_4(z_1, z_k) G_{n-2}(z_2, \ldots, \widehat{z}_k, \ldots, z_n)\, dz_1^2 dz_k^2, \end{aligned} \qquad (3.6)$$

where $\widehat{z}_k$ denotes the omission of the given term.

3.3 Some analytic differential equations

For the genus two bidifferential $\omega(x,y)$, normalised holomorphic 1–differentials ν_a for $a = 1, 2$ and the projective connection $s(x)$ we find from Section 6 of [GT1] that

Proposition 3.4. *The bidifferential $\omega(x,y)$, the normalised holomorphic 1–differentials ν_a for a=1,2 and the projective connection $s(x)$ satisfy the following analytic differential equations on the sewing domain $\mathcal{D}_{\mathrm{sew}}$ and for $x, y_1, y_2 \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$*

$$\left(\nabla_x + dx^2 \sum_{r=1}^{2} \left({}^2\mathcal{P}_1(x, y_r)\partial_{y_r} + {}^2\mathcal{P}_2(x, y_r)\right)\right)\omega(y_1, y_2) = \omega(x, y_1)\omega(x, y_2), \qquad (3.7)$$

$$\left(\nabla_x + dx^2 \left({}^2\mathcal{P}_1(x,y)\partial_y + {}^2\mathcal{P}_2(x,y)\right)\right)\nu_a(y) = \omega(x,y)\nu_a(x), \qquad (3.8)$$

$$\left(\nabla_x + dx^2 \left({}^2\mathcal{P}_1(x,y)\partial_y + 2\, {}^2\mathcal{P}_2(x,y)\right)\right)\frac{s(y)}{6} + {}^2\mathcal{P}_4(x,y) dx^2 dy^2 = \omega(x,y)^2. \qquad (3.9)$$

(3.8) and (3.9) can be obtained from (3.7) by appropriate integration and limits, respectively [GT1]. We may also rewrite (3.7) and (3.8) using (3.4) in a coordinate independent way as follows:

Corollary 3.5. *$\omega(x,y)$ and ν_a for a=1,2 satisfy the following coordinate independent analytic differential equations on the sewing domain $\mathcal{D}^{\rm sew}$*

$$\nabla_x\, \omega(y_1,y_2) + \sum_{r=1}^{2} \partial_{y_r} \left(\Psi(x,y_r)\omega(y_1,y_2)\right) dy_r = \omega(x,y_1)\omega(x,y_2), \tag{3.10}$$

$$\nabla_x\, \nu_a(y) + \partial_y \left(\Psi(x,y)\nu_a(y)\right) dy = \omega(x,y)\nu_a(x). \tag{3.11}$$

We finally note that the genus two partition function $Z_M^{(2)}(\tau_1,\tau_2,\epsilon)$ for the Heisenberg VOA M obeys [GT1]

Proposition 3.6. *$Z_M^{(2)}(\tau_1,\tau_2,\epsilon)$ is holomorphic on the sewing domain $\mathcal{D}^{\rm sew}$ and, for $x \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$, satisfies*

$$\nabla_x Z_M^{(2)} = \frac{1}{12} s(x) Z_M^{(2)}. \tag{3.12}$$

(3.7)-(3.12) are the genus two analogues of differential equations for elliptic and modular functions described in [HT]. Thus (3.12) corresponds to

$$q\frac{\partial}{\partial q}\left(\frac{1}{\eta(q)}\right) = \frac{1}{2}E_2(q)\left(\frac{1}{\eta(q)}\right),$$

for the weight 2 quasi-modular Eisenstein series $E_2(q) = -\frac{1}{12} + 2\sum_{m,n\geq 1} nq^{mn}$.

3.4 The main theorem

We show below in Theorem 3.8 how to express G_n in a manifestly symmetric fashion as a sum of weights of appropriate graphs. The graph configurations are precisely those exploited in [HT] to describe genus one Virasoro n–point functions. Furthermore, many of the arguments below mirror the genus one case. However, the weights are differently defined in the genus two case and the technicalities are more involved. In order to achieve this we define, for central charge c

$$\Theta_V(\tau_1,\tau_2,\epsilon) := Z_M^{(2)}(\tau_1,\tau_2,\epsilon)^{-c} Z_V^{(2)}(\tau_1,\tau_2,\epsilon), \tag{3.13}$$

where $Z_M^{(2)}$ is the genus two partition function for the Heisenberg VOA. For example, for a lattice VOA V_L for an even lattice L of rank c we find $\Theta_V = \Theta_L(\Omega)$, the genus two Siegel lattice theta function [MT3]. We also define a linear differential operator $\mathcal{O}_n(z_1,\ldots,z_n)$ which acts on differentiable functions of Ω by

$$\mathcal{O}_n(z_1,\ldots,z_n)\Theta_V := Z_M^{(2)}(\tau_1,\tau_2,\epsilon)^{-c} G_n(z_1,\ldots,z_n). \tag{3.14}$$

For $n=1$ we find $G_1(z_1) = \nabla_{z_1} Z_V^{(2)}(\tau_1,\tau_2,\epsilon)$ so that, using (3.12), we have

$$\mathcal{O}_1(z_1) = \nabla_{z_1} + \frac{c}{12}s(z_1). \tag{3.15}$$

It is useful to define the linear operator

$$\mathcal{D}_{z_1,z_2} = \nabla_{z_1} + dz_1^2 \left({}^2\mathcal{P}_1(z_1,z_2)\partial_{z_2} + 2\ {}^2\mathcal{P}_2(z_1,z_2) \right). \tag{3.16}$$

Then for $n = 2$, the Ward identity (3.6) implies

$$\begin{aligned}
\mathcal{O}_2(z_1,z_2)\Theta_V =& \left(Z_M^{(2)} \right)^{-c} \mathcal{D}_{z_1,z_2} \nabla_{z_2} Z_V^{(2)}(\tau_1,\tau_2,\epsilon) + \frac{c}{2}\, {}^2\mathcal{P}_4(z_1,z_2)\, \Theta_V\, dz_1^2 dz_2^2 \\
=& \mathcal{D}_{z_1,z_2} \left(\nabla_{z_2}\Theta_V + \frac{c}{12} s(z_2)\Theta_V \right) + \frac{c}{12} s(z_1) \nabla_{z_2}\Theta_V \\
&+ \frac{c^2}{144} s(z_1)s(z_2)\Theta_V + \frac{c}{2}\, {}^2\mathcal{P}_4(z_1,z_2)\, dz_1^2 dz_2^2\, \Theta_V.
\end{aligned}$$

From (3.8) we note that

$$\mathcal{D}_{z_1,z_2}\nu_a(z_2)\nu_b(z_2) = \omega(z_1,z_2) \left(\nu_a(z_1)\nu_b(z_2) + \nu_a(z_2)\nu_b(z_1) \right). \tag{3.17}$$

(3.17) together with (3.9) imply that

$$\begin{aligned}
\mathcal{O}_2(z_1,z_2) =& \sum_{1\le a\le b\le 2} \sum_{1\le c\le d\le 2} \nu_a(z_1)\nu_b(z_1)\nu_c(z_2)\nu_d(z_2)\partial_{ab}\partial_{cd} \\
&+ \frac{c}{12} s(z_1)\nabla_{z_2} + \frac{c}{12} s(z_2)\nabla_{z_1} + \frac{c^2}{144} s(z_1)s(z_2) \\
&+ 2\omega(z_1,z_2) \sum_{1\le a\le b\le 2} \nu_a(z_1)\nu_b(z_2)\partial_{ab} + \frac{c}{2}\omega(z_1,z_2)^2.
\end{aligned} \tag{3.18}$$

This expression is clearly symmetric in z_1, z_2 in accordance with Proposition 3.1. Furthermore, each term in (3.18) is now written in coordinate independent way.

Similarly to Section 3 of [HT] we now develop a graphical/combinatorial approach for computing $\mathcal{O}_n$ for all n. We define an *Order n Virasoro Graph* to be a directed graph $\mathcal{G}^n$ with n vertices labelled by $z_1, \ldots, z_n$. Each z_i-vertex has degree $\deg(z_i) = 0, 1$ or 2. The degree 1 vertices can have either unit indegree or outdegree whereas the degree 2 vertices have both unit indegree and outdegree. Thus, the connected subgraphs of $\mathcal{G}^n$ consist of r-cycles, with $r \geq 1$ degree 2 vertices, and chains with two degree 1 end-vertices with all vertices of degree 2. We regard a single degree 0 vertex as being a degenerate chain.

Remark 3.7. *The set of non-isomorphic order n Virasoro graphs is in one to one correspondence with the set of partial permutations of the label set $\{1, \ldots, n\}$. This is described in further detail in [HT].*

We define a genus two weight $W(\mathcal{G}^n)$ on $\mathcal{G}^n$ as follows. For each directed edge $\mathcal{E}_{ij}$ we define an edge weight

$$W(\mathcal{E}_{ij}) = W(z_i \circ\!\!\longrightarrow\!\!\circ z_j) = \begin{cases} \frac{1}{6}s(z_i) & \text{for } i = j, \\ \omega(z_i,z_j) & \text{for } i \neq j. \end{cases} \tag{3.19}$$

Let $\mathcal{C}_{k\ell}$ denote a chain in $\mathcal{G}^n$ with end-vertices z_k and z_ℓ

and assign a chain weight (including the degenerate chain)

$W(\mathcal{C}_{k\ell}) = W\Big($  $\Big) = A(z_k, z_\ell), \qquad (3.20)$

where $A(z_k, z_\ell) = \sum_{1\le a\le b\le 2} \nu_a(z_k)\nu_b(z_\ell)\alpha_{ab}$ for free parameters $\alpha_{ab} = \alpha_{ba}$. Let K be the number of cycles and define a weight for $\mathcal{G}^n$ by

$$W(\mathcal{G}^n) = \left(\frac{c}{2}\right)^K \prod_{\mathcal{E}_{ij}} W(\mathcal{E}_{ij}) \prod_{\mathcal{C}_{k\ell}} W(\mathcal{C}_{k\ell}), \tag{3.21}$$

where the first product ranges over all the edges and the second product ranges over all the chains of $\mathcal{G}^n$. Thus the weight depends on c, $\omega(z_i, z_j)$, $s(z_i)$, $\nu_a(z_i)$ and α_{ab}. We also note that W is multiplicative on the disconnected components of $\mathcal{G}^n$.

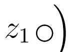
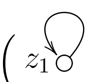
Lastly, define a linear map $\mathcal{L}_\alpha$ from $\mathbb{C}[\alpha_{ab}]$, the vector space of complex coefficient polynomials in α_{ab}, to the complex vector space spanned by ∂_{ab} derivatives with

$$\mathcal{L}_\alpha \left(\alpha_{a_1 b_1} \ldots \alpha_{a_M b_M}\right) = \partial_{a_1 b_1} \ldots \partial_{a_M b_M}. \tag{3.22}$$

Let p^n_{KM} be the number of inequivalent order n Virasoro graphs containing K cycles and M chains. In [HT] the following generating function is established

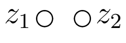
$$p^n(\alpha, \beta) = \sum_{K\ge 0, M\ge 0} p^n_{KM} \alpha^M \beta^K = (-1)^n n! \sum_{i=0}^{n} \frac{(-\alpha)^i}{i!} \binom{-\beta - i}{n - i}, \tag{3.23}$$

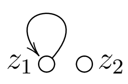
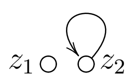
for chain and cycle counting parameters α and β respectively. Thus for $n = 1$ we find $p^1(\alpha, \beta) = \alpha + \beta$ corresponding to two inequivalent graphs with weights

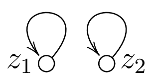
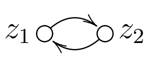
$W\Big($  $\Big) = A(z_1, z_1), \qquad W\Big($  $\Big) = \frac{c}{2}\frac{s(z_1)}{6},$

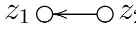
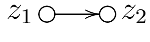
whose weight sum under the action of $\mathcal{L}_\alpha$ is $\mathcal{O}_1(z_1)$ using (3.15).

For $n = 2$ we have $p^2(\alpha, \beta) = \alpha^2 + 2\alpha\beta + \beta^2 + \beta + 2\alpha$ for 7 graphs with weights:

$W\Big($  $\Big) = A(z_1, z_1)A(z_2, z_2),$

$W\Big($  $\Big) = \frac{c}{2}\frac{s(z_1)}{6}A(z_2, z_2), \qquad W\Big($  $\Big) = \frac{c}{2}\frac{s(z_2)}{6}A(z_1, z_1),$

$W\Big($  $\Big) = \left(\frac{c}{2}\right)^2 \frac{s(z_1)}{6}\frac{s(z_2)}{6}, \qquad W\Big($  $\Big) = \frac{c}{2}\omega(z_1, z_2)^2,$

$W\Big($  $\Big) = W\Big($  $\Big) = A(z_1, z_2),$

whose weight sum under the action of $\mathcal{L}_\alpha$ using (3.18) is

$$\sum_{\mathcal{G}^2} \mathcal{L}_\alpha \left(W(\mathcal{G}^2) \right) = \mathcal{O}_2(z_1, z_2).$$

These examples illustrate the general result:

Theorem 3.8. *The order n genus two Virasoro generating function is determined by*

$$G_n(z_1, \ldots, z_n) = Z_M^{(2)}(\tau_1, \tau_2, \epsilon)^c \, \mathcal{O}_n(z_1, \ldots, z_n) \Theta_V(\tau_1, \tau_2, \epsilon),$$

for linear differential operator $\mathcal{O}_n(z_1, \ldots, z_n)$ given by

$$\mathcal{O}_n(z_1, \ldots, z_n) = \sum_{\mathcal{G}^n} \mathcal{L}_\alpha(W(\mathcal{G}^n)), \tag{3.24}$$

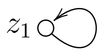
where the sum is taken over all inequivalent order n Virasoro graphs $\mathcal{G}^n$.

Proof. We prove the result by induction in n. We have already shown the result holds for $n = 1$ and $n = 2$ and employ the Ward identity (3.6) to inductively prove (3.24) for $n \geq 2$.

Every inequivalent order n Virasoro graph $\mathcal{G}^n$ can be characterized, according to the nature of the z_1 vertex, in terms of following five types:

(i) $\deg(z_1) = 0$: $z_1 \circ \; \cdots$

(ii) $\deg(z_1) = 1$: $z_1 \circ\!\longrightarrow\!\circ\, z_a \; \cdots$ or $z_1 \circ\!\longleftarrow\!\circ\, z_a \; \cdots$

(iii) $\deg(z_1) = 2$ where the z_1–vertex forms a 1-cycle:  $\cdots$

(iv) $\deg(z_1) = 2$ where the z_1–vertex is an element of a 2-cycle:  $\cdots$

(v) $\deg(z_1) = 2$ where either the z_1–vertex is a non end–vertex of a chain or an element of an r-cycle with $r \geq 3$: $\cdots \; z_a \circ\!\longrightarrow\!\overset{z_1}{\circ}\!\longrightarrow\!\circ\, z_b \; \cdots$

The Ward identity (3.6) and (3.12) imply we may recursively describe $\mathcal{O}_n$ as follows:

$$\begin{aligned} \mathcal{O}_n(z_1, \ldots, z_n) =& \frac{c}{12} s(z_1) \mathcal{O}_{n-1}(z_2, \ldots, z_n) \\ &+ \left(\nabla_{z_1} + dz_1^2 \sum_{k=2}^{n} \left({}^2\mathcal{P}_1(z_1, z_k) \partial_{z_k} + 2 \cdot {}^2\mathcal{P}_2(z_1, z_k) \right) \right) \mathcal{O}_{n-1}(z_2, \ldots, z_n) \\ &+ \frac{c}{2} \sum_{k=2}^{n} {}^2\mathcal{P}_4(z_1, z_k) \, dz_1^2 dz_k^2 \, \mathcal{O}_{n-2}(z_2, \ldots, \widehat{z}_k, \ldots, z_n), \end{aligned} \tag{3.25}$$

We now show how the parts of (3.25) relate to Virasoro graph weights by using induction in n. Thus given $\mathcal{O}_{n-1}$ and $\mathcal{O}_{n-2}$ satisfy (3.24), we see that the $\frac{c}{12} s(z_1) \mathcal{O}_{n-1}$ term of (3.25) arises from the sum over all $\mathcal{G}^n$ graphs of type (iii).

Let $\mathcal{G}^{n-1}$ denote an order $n-1$ Virasoro graph labelled by $z_2,\ldots,z_n$ of weight $W(\mathcal{G}^{n-1})$. This gives a contribution to (3.25) of

$$\nabla_{z_1}\mathcal{L}_\alpha\left(W(\mathcal{G}^{n-1})\right)=\mathcal{L}_\alpha\left(W(\mathcal{G}^{n-1})A(z_1,z_1)\right)+\mathcal{L}_\alpha\left(\nabla_{z_1}W(\mathcal{G}^{n-1})\right), \tag{3.26}$$

using the Leibniz rule for ∇_x. In particular, all terms of the form $W(\mathcal{G}^{n-1})A(z_1,z_1)$ arise as weights of $\mathcal{G}^n$ graphs of type (i).

Let us examine the contributions that arise from $\nabla_{z_1}W(\mathcal{G}^{n-1})$ in (3.26) and the remaining terms in (3.25) and show that these can be expressed in terms of a sum of the weights of graphs of type (ii), (iv) and (v). Let z_k be a given vertex in $\mathcal{G}^{n-1}$ for $k=2,\ldots,n$. Then, much as before, we can characterize $\mathcal{G}^{n-1}$ according to (a) z_k is a degree 0 vertex (b) z_k is a disconnected vertex of degree 2 (c) z_k is a degree 1 vertex or (d) z_k is a degree 2 vertex in a chain or an r-cycle for $r\geq 2$.

Case (a). $\mathcal{G}^{n-1}$ consists of a z_k vertex of degree 0 and an order $n-2$ Virasoro graph $\mathcal{G}^{n-2}$ (with vertices $z_2,\ldots,\widehat{z}_k,\ldots,z_n$) of weight

$$W(\mathcal{G}^{n-1})=A(z_k,z_k)W(\mathcal{G}^{n-2}).$$

Using (3.17) this contributes to (3.25) the term

$$\mathcal{D}_{z_1,z_k}A(z_k,z_k)W(\mathcal{G}^{n-2})=2A(z_1,z_k)\omega(z_1,z_k)W(\mathcal{G}^{n-2}),$$

the sum of the weights of two $\mathcal{G}^n$ graphs of type (ii) where z_1 and z_k form a disconnected chain of length 2.

Case (b). $\mathcal{G}^{n-1}$ consists of a disconnected degree 2 vertex z_k and an order $n-2$ Virasoro graph $\mathcal{G}^{n-2}$ of weight $W(\mathcal{G}^{n-1})=\frac{c}{12}s(z_k)W(\mathcal{G}^{n-2})$ which contributes $\frac{c}{12}\mathcal{D}_{z_1,z_k}(s(z_k))W(\mathcal{G}^{n-2})$ to (3.25). Summing with the $\frac{c}{2}\,{}^2\mathcal{P}_4(z_1,z_k)W(\mathcal{G}^{n-2})$ contribution to (3.25) gives

$$\frac{c}{2}\omega(z_1,z_k)^2W(\mathcal{G}^{n-2}),$$

using (3.9), the weight of a $\mathcal{G}^n$ graph of type (iv) where z_1 and z_k form a 2–cycle.

Case (c). z_k is an end–vertex of a chain $\mathcal{C}_{k\ell}$ so that $W(\mathcal{G}^{n-1})=A(z_k,z_\ell)\omega(z_k,z_m)\ldots$, where z_k is joined to z_m and the ellipsis denotes the factors independent of z_k. Using (3.7) and (3.8) this contributes terms to (3.25) of the form

$$\begin{aligned}&\Bigg(\bigg(\nabla_{z_1}+dz_1^2\Big({}^2\mathcal{P}_1(z_1,z_k)\partial_{z_k}+2\cdot{}^2\mathcal{P}_2(z_1,z_k)\\&+{}^2\mathcal{P}_1(z_1,z_m)\partial_{z_m}+{}^2\mathcal{P}_2(z_1,z_m)\Big)\bigg)A(z_k,z_\ell)\omega(z_k,z_m)\Bigg)\ldots\\&=A(z_1,z_\ell)\omega(z_1,z_k)\omega(z_k,z_m)\ldots+A(z_k,z_\ell)\omega(z_k,z_1)\omega(z_1,z_m)\ldots.\end{aligned} \tag{3.27}$$

Note that we have omitted in (3.27) contributions to (3.25) of the form:

$$A(z_k,z_\ell)\omega(z_k,z_m)\left(\nabla_{z_1}+{}^2\mathcal{P}_1(z_1,z_m)\partial_{z_m}+{}^2\mathcal{P}_2(z_1,z_m)\right)(\ldots)$$

which contribute to case (d) for z_m. The first term in (3.27) is the weight of a $\mathcal{G}^n$ graph of type (ii):

and the second term is the weight of a graph of type (v):

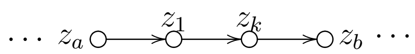
Case (d). If $\deg(z_k) = 2$ then $W(\mathcal{G}^{n-1}) = \omega(z_a, z_k)\omega(z_k, z_b)\ldots$, where z_k is joined to z_a and z_b and the ellipsis denotes the factors independent of z_k. This contributes terms to (3.25) of the form

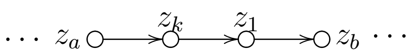
$$
\begin{aligned}
&\Bigg(\Big(\nabla_{z_1} + dz_1^2\Big({}^2\mathcal{P}_1(z_1, z_k)\partial_{z_k} + {}^2\mathcal{P}_1(z_1, z_a)\partial_{z_a} + {}^2\mathcal{P}_1(z_1, z_b)\partial_{z_b} \\
&+ {}^2\mathcal{P}_2(z_1, z_a) + 2\cdot {}^2\mathcal{P}_2(z_1, z_k) + {}^2\mathcal{P}_2(z_1, z_b)\Big)\Big)\omega(z_a, z_k)\omega(z_k, z_b)\Bigg)\ldots \\
&= \omega(z_a, z_1)\omega(z_1, z_k)\omega(z_k, z_b)\ldots + \omega(z_a, z_k)\omega(z_k, z_1)\omega(z_1, z_b)\ldots \qquad (3.28)
\end{aligned}
$$

using (3.7). Note that we have omitted in (3.7) contributions to (3.25) of the form:

$$
\begin{aligned}
\omega(z_a, z_k)\omega(z_k, z_a)\Big(\nabla_x &+ {}^2\mathcal{P}_1(z_1, z_a)\partial_{z_a} + {}^2\mathcal{P}_1(z_1, z_b)\partial_{z_b} \\
&+ {}^2\mathcal{P}_2(z_1, z_a) + {}^2\mathcal{P}_2(z_1, z_b)\Big)(\ldots)
\end{aligned}
$$

which contribute to case (c) and case (d) for z_a or z_b. The two terms in (3.28) are weights of a $\mathcal{G}^n$ graphs of type (v):



,


Thus, altogether, we find that the weights of all $\mathcal{G}^n$ graphs of type (i)–(v) contribute and hence (3.24) holds. □

Remark 3.9. *The differential operator $\mathcal{O}_n(z_1, \ldots, z_n)$ is symmetric in $z_1, \ldots, z_n$ and is expressed in a coordinate free way in terms of $\omega(x, y)$, $\nu_1(x)$, $\nu_2(x)$, $s(x)$ and $\partial^k_{\Omega_{ab}}$ i.e. we may promote $z_1, \ldots, z_n$ to the status of general coordinates on a genus 2 Riemann surface. The **only** dependence on the original VOA is the central charge parameter c.*

Remark 3.10. *Theorem 3.8 can be readily generalized for any pair of ordinary V–modules W_1, W_2 with genus two n-point function*

$$
\begin{aligned}
Z^{(2)}_{W_1,W_2}(\widetilde{\omega}, z_1; \ldots; \widetilde{\omega}, z_n)\, d\mathbf{z}^2 &= d\mathbf{z}^2 \sum_{u\in V} Z^{(1)}_{W_1}(Y[\widetilde{\omega}, z_1]\ldots Y[\widetilde{\omega}, z_n]u; \tau_1) Z^{(1)}_{W_2}(\overline{u}; \tau_2) \\
&= \mathcal{O}_n(z_1, \ldots, z_n)\Theta_{W_1,W_2}(\tau_1, \tau_2, \epsilon),
\end{aligned}
$$

where $\Theta_{W_1,W_2}(\tau_1, \tau_2, \epsilon) = Z^{(2)}_M(\tau_1, \tau_2, \epsilon)^{-c} Z^{(2)}_{W_1,W_2}(\tau_1, \tau_2, \epsilon)$.

4 Analytic and modular transformations

Propositions 3.3–3.6 are based on a genus two Riemann surface constructed by the particular sewing scheme for two tori of Section 2.1. Despite this, Remark 3.9 implies that we may express $\mathcal{O}_n(z_1,\ldots,z_n)$ in any coordinate system on an arbitrary genus two Riemann surface. In particular, we can consider the behaviour of $\mathcal{O}_n(z_1,\ldots,z_n)$ under a general analytic transformation:

Proposition 4.1. *Let $z \to \phi(z)$ be an analytic map then we have*

$$\mathcal{O}_n(z_1,\ldots,z_n) = \mathcal{O}_n(z_1,\ldots,\phi(z_i),\ldots,z_n) + \frac{c}{12}\{\phi(z_i),z_i\}dz_i^2\,\mathcal{O}_{n-1}(z_1,\ldots,\widehat{z}_i,\ldots,z_n),$$

for $i \in \{1,\ldots,n\}$ where $\{\phi(z),z\} = \frac{\phi'''(z)}{\phi'(z)} - \frac{3}{2}\left(\frac{\phi''(z)}{\phi'(z)}\right)^2$ is the Schwarzian derivative.

Proof. Choose $i=1$ wlog by Proposition 3.1. $\omega(z_1,z_j),\nu_1(z_1)$ and $\nu_2(z_1)$ are invariant under analytic transformations whereas for the projective connection [Mu, F]

$$s(z_1) = s(\phi(z_1)) + \{\phi(z_1),z_1\}dz_1^2.$$

Such a term only arises in the Virasoro graphs $\mathcal{G}^n$ of type (iii), where the z_1–vertex forms a 1-cycle of weight $W(\mathcal{G}^n) = \frac{c}{12}s(z_1)W(\mathcal{G}^{n-1})$. Thus the result follows. □

In order to describe the genus two modular properties of $\mathcal{O}_n(z_1,\ldots,z_n)$ we analyse the modular properties of the $(2,-1)$-form $\Psi(x,y)$ of (3.4). The genus two modular group $\mathrm{Sp}(4,\mathbb{Z})$ consists of integral block matrices $\gamma := \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ where A,B,C,D obey:

$$\begin{aligned} &A^TD - C^TB = I, \quad AB^T = BA^T, \quad CD^T = DC^T, \\ &A^TC = C^TA, \quad B^TD = D^TB, \end{aligned} \tag{4.1}$$

for identity matrix I. It is convenient to define for $\Omega \in \mathbb{H}_2$

$$M = C\Omega + D, \quad N = (C\Omega+D)^{-1}.$$

The holomorphic differentials $\boldsymbol{\nu}(x)$, the period matrix Ω and the bidifferential $\omega(x,y)$ transform under $\gamma \in \mathrm{Sp}(4,\mathbb{Z})$ as follows [F, Mu, GT1]

$$\boldsymbol{\nu}^\gamma(x) = \boldsymbol{\nu}N, \tag{4.2}$$

$$\Omega^\gamma = (A\Omega+B)N, \tag{4.3}$$

$$\nabla_x^\gamma = \nabla_x, \tag{4.4}$$

$$\omega^\gamma(x,y) = \omega(x,y) - \frac{1}{2}\sum_{1\le i\le j\le 2}(\nu_i(x)\nu_j(y)+\nu_j(x)\nu_i(y))\,\partial_{ij}\log\det M, \tag{4.5}$$

$$s^\gamma(x) = s(x) - 6\,\nabla_x \log\det M. \tag{4.6}$$

Applying these modular transformations we will show:

Theorem 4.2. $\Psi(x, y)$ *is* $\mathrm{Sp}(4, \mathbb{Z})$ *modular invariant.*

In order to prove Theorem 4.2 we prove two lemmas. The first lemma concerns the second term on the right hand side of (4.5):

Lemma 4.3.

$$\frac{1}{2} \sum_{1 \le i \le j \le 2} \left(\nu_i(x)\nu_j(y) + \nu_j(x)\nu_i(y)\right) \partial_{ij} \log\det M = \boldsymbol{\nu}(x) N C \boldsymbol{\nu}^T(y). \tag{4.7}$$

Proof. Using the $\mathrm{Sp}(4, \mathbb{Z})$ relations (4.1) we find $N = A^T - C^T \Omega^\gamma$ so that

$$(NC)^T = NC. \tag{4.8}$$

The result follows by direct calculation using (4.8) where we find

$$\begin{aligned}
\partial_{11} \log\det M &= (M_{22}C_{11} - M_{12}C_{21})/\det M = (NC)_{11}, \\
\partial_{22} \log\det M &= (NC)_{22}, \\
\partial_{12} \log\det M &= 2(NC)_{12}.
\end{aligned}$$

□

Lemma 4.4. *For* $\boldsymbol{\nu}^\gamma(x) = \boldsymbol{\nu} N$ *of* (4.2) *we have*

$$\begin{vmatrix} \boldsymbol{\nu}^\gamma(x) \\ \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{vmatrix} \det N, \tag{4.9}$$

$$\begin{vmatrix} \boldsymbol{\nu}^\gamma(y) \\ \partial_y \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(y) \\ \partial_y \boldsymbol{\nu}(y) \end{vmatrix} \det N, \tag{4.10}$$

$$\begin{vmatrix} \boldsymbol{\nu}^\gamma(y) \\ \nabla_x^\gamma \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{vmatrix} \det N + \boldsymbol{\nu}(x) N C \boldsymbol{\nu}^T(y) \begin{vmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{vmatrix} \det N. \tag{4.11}$$

Proof. $\begin{vmatrix} \boldsymbol{\nu}^\gamma(x) \\ \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(x)N \\ \boldsymbol{\nu}(y)N \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{vmatrix} \det N$ and similarly for (4.10). To prove (4.11), we first note that

$$\nabla_x M = C \boldsymbol{\nu}(x)^T \boldsymbol{\nu}(x).$$

Furthermore, $(\nabla_x N)M = -N \nabla_x M$ so that

$$\nabla_x N = -N(\nabla_x M)N = -NC\boldsymbol{\nu}(x)^T \boldsymbol{\nu}(x) N. \tag{4.12}$$

Hence, using (4.4) and (4.8), we find that

$$\begin{aligned}
\begin{vmatrix} \boldsymbol{\nu}^\gamma(y) \\ \nabla_x^\gamma \boldsymbol{\nu}^\gamma(y) \end{vmatrix} &= \begin{vmatrix} \boldsymbol{\nu}(y) N \\ \nabla_x \left(\boldsymbol{\nu}(y)\right) N + \boldsymbol{\nu}(y) \nabla_x N \end{vmatrix} \\
&= \begin{vmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{vmatrix} \det N + \begin{vmatrix} \boldsymbol{\nu}(y) N \\ -\boldsymbol{\nu}(y) N C \boldsymbol{\nu}(x)^T \boldsymbol{\nu}(x) N \end{vmatrix} \\
&= \begin{vmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{vmatrix} \det N + \boldsymbol{\nu}(x) N C \boldsymbol{\nu}^T(y) \begin{vmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{vmatrix} \det N.
\end{aligned}$$

□

Proof of Theorem 4.2. Combining Lemmas 4.3 and 4.4 we immediately obtain

$$\omega^\gamma(x,y)\begin{vmatrix}\boldsymbol{\nu}^\gamma(x)\\ \boldsymbol{\nu}^\gamma(y)\end{vmatrix}+\begin{vmatrix}\boldsymbol{\nu}^\gamma(y)\\ \nabla_x^\gamma\,\boldsymbol{\nu}^\gamma(y)\end{vmatrix}=\left[\omega(x,y)\begin{vmatrix}\boldsymbol{\nu}(x)\\ \boldsymbol{\nu}(y)\end{vmatrix}+\begin{vmatrix}\boldsymbol{\nu}(y)\\ \nabla_x\,\boldsymbol{\nu}(y)\end{vmatrix}\right]\det N.$$

Thus using (4.10) we find $\Psi(x,y)$ is $\mathrm{Sp}(4,\mathbb{Z})$ modular invariant. □

Corollary 4.5. *The differential equations* (3.10) *and* (3.11) *are* $\mathrm{Sp}(4,\mathbb{Z})$ *invariant.*

Proof. Under the action of $\gamma\in\mathrm{Sp}(4,\mathbb{Z})$, the change in the left hand side of (3.10) is

$$\begin{aligned}&-\nabla_x\left(\boldsymbol{\nu}(y_1)NC\boldsymbol{\nu}^T(y_2)\right)-\sum_{r=1}^{2}\partial_{y_r}\left(\Psi(x,y_r)\boldsymbol{\nu}(y_1)NC\boldsymbol{\nu}^T(y_2)\right)dy_r\\ &=-\sum_{r=1}^{2}\omega(x,y_r)\boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y_r)-\boldsymbol{\nu}(y_1)(\nabla_xN)C\boldsymbol{\nu}^T(y_2),\end{aligned}$$

using (3.11). But (4.8) and (4.12) imply

$$-\boldsymbol{\nu}(y_1)(\nabla_xN)C\boldsymbol{\nu}^T(y_2)=\left(\boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y_1)\right)\left(\boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y_2)\right).$$

Thus the total change in the left hand side of (3.10) is

$$\omega^\gamma(x,y_1)\omega^\gamma(x,y_2)-\omega(x,y_1)\omega(x,y_2),$$

as required. A similar method shows that (3.11) is modular invariant. □

Let us now consider the modular properties of the operator $\mathcal{O}_n(z_1,\ldots,z_n)$. Following Remark 3.9 we know that $\mathcal{O}_n(z_1,\ldots,z_n)$ depends on $\omega(x,y)$, $\nu_1(x)$, $\nu_2(x)$, $s(x)$ and ∂_{ab}. These terms transform under $\gamma\in\mathrm{Sp}(4,\mathbb{Z})$ as in (4.2)-(4.6) so that

$$\mathcal{O}_n:=\mathcal{O}_n(z_1,\ldots,z_n)\to\mathcal{O}_n^\gamma(z_1,\ldots,z_n)=:\mathcal{O}_n^\gamma,$$

where here we suppress the dependence on $z_1,\ldots,z_n$ for convenience. We then find

Theorem 4.6. *For differentiable* $F=F(\Omega)$ *we have for each* $\gamma\in\mathrm{Sp}(4,\mathbb{Z})$ *that*

$$\mathcal{O}_n^\gamma\left(\det(M)^{c/2}F\right)=\det(M)^{c/2}\mathcal{O}_n\left(F\right).\tag{4.13}$$

Proof. We prove the result by induction in n. The result is trivially true for $n=0$. For $n=1$ we use (3.15) to find

$$\begin{aligned}\mathcal{O}_1^\gamma\left(\det(M)^{c/2}F\right)&=\left(\nabla_{z_1}^\gamma+\frac{c}{12}s^\gamma(z_1)\right)\left(\det(M)^{c/2}F\right)\\ &=\det(M)^{c/2}\mathcal{O}_1(F),\end{aligned}$$

using (4.4) and (4.6). (3.25) implies by induction that for $n \geq 2$

$$\begin{aligned}
&\mathcal{O}_n^\gamma\left(\det(M)^{c/2}F\right) \\
&= \left(\nabla_{z_1}^\gamma + \frac{c}{12}s^\gamma(z_1) + dz_1^2\sum_{k=2}^{n}\left({}^2\mathcal{P}_1(z_1,z_k)\partial_{z_k} + 2\cdot {}^2\mathcal{P}_2(z_1,z_k)\right)\right)\left(\det(M)^{c/2}\mathcal{O}_{n-1}F\right) \\
&\quad + \frac{c}{2}\det(M)^{c/2}\sum_{k=2}^{n} {}^2\mathcal{P}_4(z_1,z_k)\,dz_1^2dz_k^2\,\mathcal{O}_{n-2}(F) \\
&= \det(M)^{c/2}\mathcal{O}_n\left(F\right),
\end{aligned}$$

using (4.4) and (4.6) again. □

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