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VERTEX ALGEBRAS ACCORDING TO ISAAC NEWTON

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ABSTRACT. We give an introduction to vertex algebras using elementary forward difference methods originally due to Isaac Newton.

1. INTRODUCTION

In this paper we present an introduction to the theory of vertex algebras [B], [FLM], [K], [LL], [FHL], [LZ], [MN], [MT]. A cursory examination of the literature of vertex algebras reveals a variety of identities involving binomial coefficients

$$\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i!},$$

for all $n \in \mathbb{Z}$. We describe how all of these arise from Newton's binomial theorem either directly or else through elementary Newton finite difference identities [N]. In particular, our approach provides both a motivation and a new understanding of the fundamental axioms of locality and lower truncation for vertex operators. We also obtain a simplified and stronger proof of the Borcherds-Frenkel-Lepowsky-Meurmann identity.

2. NEWTON FORWARD DIFFERENCES AND FORMAL SERIES

2.1. **Forward Differences.** We consider an elementary but very relevant illustration of formal series techniques used in vertex algebra theory. Our example comes from Newton's theory of finite differences.[1]

Let U be a vector space over a field of characteristic zero. Let $U^{\mathbb{Z}}$ denote the set of doubly infinite sequences $\alpha = \{\alpha_n\}_{n\in\mathbb{Z}}$ with components $\alpha_n \in U$. Define the *(first) forward difference operator* $\Delta : U^{\mathbb{Z}} \to U^{\mathbb{Z}}$

$$(\Delta\alpha)_n = \alpha_{n+1} - \alpha_n, \quad \alpha \in U^{\mathbb{Z}}. \tag{1}$$

The N^{th} *forward difference operator* is defined for all integers $N \geq 2$ by

$$\Delta^N = \Delta \circ \Delta^{N-1}.$$

Example 1. *For a real function $f(x)$ define $\alpha_n = f(n) \in \mathbb{R}$. Then $(\Delta\alpha)_n$ is the classical Newton forward difference used in the polynomial interpolation of $f(x)$.*

The action of Δ^N on $U^{\mathbb{Z}}$ is given by:

Lemma 2. *The N^{th} forward difference of $\alpha \in U^{\mathbb{Z}}$ has components*

$$(\Delta^N\alpha)_n = \sum_{i\geq 0}(-1)^i\binom{N}{i}\alpha_{n+N-i}. \tag{2}$$

Key words and phrases. Vertex Algebras, Conformal Field Theory, Finite Differences.

[1]Borcherds [B] defined vertex algebras whilst at Trinity College Cambridge exactly 300 years after Newton [N] invented finite differences at the same institution!

Proof. Write $\Delta = F - I$ where F is the forward shift operator

$$(F\alpha)_n = \alpha_{n+1}, \tag{3}$$

and I is the identity operator. The result follows from Newton's binomial identity

$$(F-I)^N\alpha = \sum_{i\geq 0}\binom{N}{i}(-1)^i F^{N-i}\alpha.$$

$\square$

We now consider $\ker\Delta^N$, the space of sequences with zero N^{th} forward difference.

Proposition 3 (Newton's Forward Difference Formula). *Let $\alpha \in U^{\mathbb{Z}}$ with components α_n. If $\alpha \in \ker\Delta^N$ for some $N \geq 1$ then for all $n \in \mathbb{Z}$*

$$\alpha_n = \sum_{i\geq 0}\binom{n}{i}\left(\Delta^i\alpha\right)_0. \tag{4}$$

Conversely, if $\alpha \in U^{\mathbb{Z}}$ has components $\alpha_n = \sum\limits_{i=0}^{N-1}\binom{n}{i}R_i$ for $R_i \in U$ and some $N \geq 1$ then $\alpha \in \ker\Delta^N$.

Proof. Assume $\alpha \in \ker\Delta^N$. We have $\alpha_n = (F^n\alpha)_0$ for all $n \in \mathbb{Z}$ with F the forward shift operator of (3). Then (4) follows from a binomial expansion of $F^n\alpha = (I+\Delta)^n\alpha$

$$F^n\alpha = \sum_{i\geq 0}\binom{n}{i}\Delta^i\alpha, \tag{5}$$

for all $n \in \mathbb{Z}$. For $n \geq 0$, (5) is obvious whereas for $n = -k < 0$ we can verify that

$$\alpha = F^k\sum_{i\geq 0}\binom{-k}{i}\Delta^i\alpha,$$

for all $\alpha \in \ker\Delta^N$. Hence (5) holds and therefore (4) results.

Conversely, if $\alpha_n = \sum_{i=0}^{N-1}\binom{n}{i}R_i$ then noting that for $\beta_n = n^k$ and $k > 0$

$$(\Delta\beta)_n = kn^{k-1} + O\left(n^{k-2}\right),$$

we find $\alpha \in \ker\Delta^N$ since $\binom{n}{i} = \frac{1}{i!}n^i + O\left(n^{i-1}\right)$. $\square$

We also note the following result:

Corollary 4. *$\alpha \in \ker\Delta^N$ iff $\alpha_n = p_{N-1}(n)$ where $p_{N-1}(n)$ is a degree $N-1$ polynomial in n with coefficients in U.*

Example 5. *Let $p_{N-1}(x)$ be a polynomial of degree $N-1$ with coefficients in $\mathbb{R}$. Then Proposition 3 is Newton's forward difference formula expressing $p_{N-1}(n)$ for all $n \in \mathbb{Z}$ in terms of $p_{N-1}(i)$ for $i = 0, \ldots, N-1$. Replacing n by x on the right hand side of* (4) *gives the Newton interpolating polynomial for a real function $f(x)$ in terms of $\alpha_i = f(i)$ for $i = 0, 1, \ldots, N-1$.*

2.2. **Formal Generating Series.** Define a *formal generating series* $\alpha(z)$ for $\alpha \in U^{\mathbb{Z}}$ by

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} \in U[[z, z^{-1}]], \tag{6}$$

where $U[[z, z^{-1}]]$ denotes the space of formal Laurent series in an indeterminate parameter z with coefficients in U. We associate the component α_n with z^{-n-1} for reasons that become clearer below e.g. (16), Lemma 8 and Theorem 24.

Define the *formal derivative* ∂ of $\alpha(z)$ by

$$\partial\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n(-n-1)z^{-n-2} = \sum_{n\in\mathbb{Z}} \left(-n\alpha_{n-1}\right) z^{-n-1}. \tag{7}$$

We also define $\partial^{(i)} := \dfrac{1}{i!}\,\partial^i$. The *formal residue* of the Laurent series (6) is defined by

$$\operatorname{Res}_z \alpha(z) = \alpha_0. \tag{8}$$

Lemma 6. *The formal series $\alpha(z)$ satisfies versions of the fundamental theorem of calculus, the Leibniz rule and integration by parts:*

$$\operatorname{Res}_z \partial\alpha(z) = 0, \tag{9}$$

$$\partial\left(z^k\alpha(z)\right) = kz^{k-1}\alpha(z) + z^k\partial\alpha(z), \tag{10}$$

$$\operatorname{Res}_z z^k\partial\alpha(z) = -k\operatorname{Res}_z z^{k-1}\alpha(z), \tag{11}$$

for all $k \in \mathbb{Z}$.

Proof. (7) immediately implies (9) and (10). $\operatorname{Res}_z \partial\left(z^k\alpha(z)\right) = 0$ implies (11). □

The formal nature of a series $\alpha(z)$ is well-illustrated by the constant sequence $\alpha_n = \alpha_0 \in U$, for all n, for which

$$\alpha(z) = \alpha_0\,\delta(z),$$

for formal *delta series* defined by

$$\delta(z) = \sum_{m\in\mathbb{Z}} z^m. \tag{12}$$

The delta series is analogous to the Dirac delta function in the sense that

$$z^k\delta(z) = \delta(z), \tag{13}$$

for all $k \in \mathbb{Z}$. In particular, we note that

$$(z-1)\delta(z) = 0. \tag{14}$$

We also define a family of formal delta series indexed by integers $i \geq 0$ as follows:

$$\delta^{(i)}(z) = (-1)^i\partial^{(i)}\delta(z) = \sum_{m\in\mathbb{Z}} (-1)^i \binom{m}{i} z^{m-i}, \tag{15}$$

with $\delta^{(0)}(z) = \delta(z)$. On relabelling, we note that (15) can be rewritten as

$$\delta^{(i)}(z) = \sum_{n\in\mathbb{Z}} \binom{n}{i} z^{-n-1}, \tag{16}$$

i.e. $\delta^{(i)}(z)$ is the formal series for the integer sequence $\left\{\binom{n}{i}\right\}_{n\in\mathbb{Z}}$. We further find that (14) generalises to:

Lemma 7. $(z-1)\delta^{(i)}(z) = \delta^{(i-1)}(z)$ *for all* $i \geq 1$.

Since $(F\alpha)(z) = z\alpha(z)$ for the forward shift operator of (3), it follows that the formal series for $\Delta^i\alpha$ is

$$\left(\Delta^i\alpha\right)(z) = (z-1)^i\alpha(z). \tag{17}$$

Thus $\alpha \in \ker\Delta^N$ iff $(z-1)^N\alpha(z) = 0$. Noting that

$$(\Delta^i\alpha)_0 = \mathrm{Res}_z(z-1)^i\alpha(z), \tag{18}$$

we may reformulate Newton's forward difference formula Proposition 3 in terms of formal series using (15) and (18) to find:

Lemma 8. *Let $\alpha \in U^{\mathbb{Z}}$. Then $\alpha \in \ker\Delta^N$ iff*

$$\alpha(z) = \sum_{i=0}^{N-1} R_i\delta^{(i)}(z), \tag{19}$$

for $R_i = \mathrm{Res}_z(z-1)^i\alpha(z) \in U$.

In numerous classical applications of generating series with $U = \mathbb{C}$, the formal parameter z can be taken to be a complex number in some domain on which the generating series converges. However, the formal delta series $\delta^{(i)}(z)$ diverges everywhere on the complex plane. Nevertheless, if we decompose $\delta^{(i)}(z) = \delta^{(i)}(z)_+ + \delta^{(i)}(z)_-$ with

$$\delta^{(i)}(z)_+ = \sum_{n\geq 0}\binom{n}{i}z^{-n-1}, \qquad \delta^{(i)}(z)_- = \sum_{n\leq -1}\binom{n}{i}z^{-n-1}, \tag{20}$$

where the $\pm$ subscripts refer to the sign of the sequence index n. Then the series $\delta^{(i)}(z)_+$ and $\delta^{(i)}(z)_-$ converge on disjoint complex domains as follows:

$$\delta^{(i)}(z)_+ = \frac{1}{(z-1)^{i+1}}, \quad |z|>1, \qquad \delta^{(i)}(z)_- = \frac{-1}{(z-1)^{i+1}}, \quad |z|<1.$$

We utilise these expansions for formal z by adopting the following convention:

Definition 9 (Expansion Convention)**.** *For $m \in \mathbb{Z}$ and formal variables x, y we define*

$$(x+y)^m = \sum_{k\geq 0}\binom{m}{k}x^{m-k}y^k, \tag{21}$$

i.e. we expand in the second variable. For $m \geq 0$, $(x+y)^m = (y+x)^m$, with a finite sum, whereas for $m < 0$, $(x+y)^m$ and $(y+x)^m$ are distinct infinite series.

Following this convention we write

$$\delta^{(i)}(z)_+ = \frac{1}{(z-1)^{i+1}}, \qquad \delta^{(i)}(z)_- = -\frac{1}{(-1+z)^{i+1}}, \tag{22}$$

so that

$$\delta^{(i)}(z) = \frac{1}{(z-1)^{i+1}} - \frac{1}{(-1+z)^{i+1}}.$$

We may similarly decompose any formal series as $\alpha(z) = \alpha(z)_+ + \alpha(z)_-$ where

$$\alpha(z)_+ = \sum_{n\geq 0}\alpha_n z^{-n-1}, \qquad \alpha(z)_- = \sum_{n\leq -1}\alpha_n z^{-n-1}. \tag{23}$$

The $\pm$ subscripts refer to the sign of the sequence index n.[2] Thus Lemma 8 and (22) imply $\alpha \in \ker(\Delta^N)$ iff

$$\alpha(z)_+ = \sum_{i=0}^{N-1} \frac{R_i}{(z-1)^{i+1}}, \qquad \alpha(z)_- = -\sum_{i=0}^{N-1} \frac{R_i}{(-1+z)^{i+1}}.$$

Altogether the following theorem summarises our discussion thus far.

Theorem 10. *Let $\alpha \in U^{\mathbb{Z}}$ with formal series $\alpha(z)$ and let*

$$R_i = \left(\Delta^i \alpha\right)_0 = \operatorname{Res}_z (z-1)^i \alpha(z) \in U,$$

for integers $i \geq 0$. The following are equivalent:

(i) $\alpha \in \ker \Delta^N$,

(ii) $(z-1)^N \alpha(z) = 0$,

(iii) $\alpha_n = \sum_{i=0}^{N-1} \binom{n}{i} R_i$ *for all* $n \in \mathbb{Z}$,

(iv) $\alpha_n = p_{N-1}(n)$, *a degree* $N-1$ *polynomial in* n *with coefficients in* U,

(v) $\alpha(z) = \sum_{i=0}^{N-1} R_i \delta^{(i)}(z)$,

(vi) $\alpha(z)_+ = \sum_{i=0}^{N-1} \frac{R_i}{(z-1)^{i+1}}$ *and* $\alpha(z)_- = -\sum_{i=0}^{N-1} \frac{R_i}{(-1+z)^{i+1}}$.

2.3. **Calculus of Formal Series.** We gather a compendium of results concerning formal series that we make use of later.

Lemma 11 (Taylor's Theorem)**.** *For formal series $\alpha(z)$ then $\alpha(x+y)$ has formal Taylor expansion in y*

$$\alpha(x+y) = e^{y\partial}\alpha(x), \tag{24}$$

where $e^{y\partial} := \sum_{i\geq 0} y^i\, \partial^{(i)}$.

Proof. Using the formal expansion convention Definition 9 we find

$$\begin{aligned}\alpha(x+y) &= \sum_{n\in\mathbb{Z}} \alpha_n (x+y)^{-n-1} = \sum_{n\in\mathbb{Z}} \alpha_n \sum_{i\geq 0} \binom{-n-1}{i} x^{-n-1-i} y^i \\ &= \sum_{n\in\mathbb{Z}} \alpha_n \sum_{i\geq 0} y^i \partial^{(i)} \left(x^{-n-1}\right) = \sum_{i\geq 0} y^i\, \partial^{(i)} \alpha(x).\end{aligned}$$

□

Lemma 12. *For $\alpha(z)_\pm$ of* (23) *we have*

$$\partial\left(\alpha(z)_\pm\right) = \left(\partial\alpha(z)\right)_\pm . \tag{25}$$

Lemma 13 (Residue Theorem)**.** *Let $\alpha(z)$ be a formal series. For integer $k \geq 0$ we have*

$$\operatorname{Res}_x \frac{\alpha(x)}{(x-z)^{k+1}} = \partial^{(k)}\alpha(z)_-, \qquad \operatorname{Res}_x \frac{\alpha(x)}{(-z+x)^{k+1}} = -\partial^{(k)}\alpha(z)_+. \tag{26}$$

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[2]We use the convention usually adopted in physics which is opposite to that chosen in [K].

Proof. Consider
$$\mathrm{Res}_x\, \frac{\alpha(x)}{x-z} = \mathrm{Res}_x \sum_{n\in\mathbb{Z}}\sum_{r\geq 0} \alpha_n x^{-n-r-2} z^r = \sum_{r\geq 0} \alpha_{-r-1}\, z^r = \alpha(z)_-.$$
Similarly we find $\mathrm{Res}_x\, \dfrac{\alpha(x)}{-z+x} = -\alpha(z)_+$. Thus (26) holds for $k=0$. The general result follows on applying $\partial_z^{(k)}$ and using (25). □

3. Locality

3.1. **Locality of Formal Series.** Let us now assume that U is an associative algebra over a field of characteristic zero i.e. U is a vector space equipped with an associative bilinear product $AB \in U$ for all $A, B \in U$. In the next section we will consider U to be the algebra of endomorphisms of a vector space V.

We define the formal product and commutator for formal generating series $\alpha(z), \beta(z) \in U[[z, z^{-1}]]$ by

$$\alpha(x)\beta(y) = \sum_{m,n\in\mathbb{Z}} \alpha_m\beta_n x^{-m-1}y^{-n-1} \in U[[x, x^{-1}, y, y^{-1}]], \tag{27}$$

$$[\alpha(x), \beta(y)] = \sum_{m,n\in\mathbb{Z}} [\alpha_m, \beta_n] x^{-m-1}y^{-n-1} \in U[[x, x^{-1}, y, y^{-1}]], \tag{28}$$

for independent indeterminates x and y and commutator $[\alpha_m, \beta_n] = \alpha_m\beta_n - \beta_n\alpha_m$. In the language of Chapter 2, the bivariate series (27) and (28) are generating series for *doubly indexed* sequences $\{\alpha_m\beta_n\}_{m,n\in\mathbb{Z}}$ and $\{[\alpha_m, \beta_n]\}_{m,n\in\mathbb{Z}}$, respectively.

We now define the fundamental notion of *locality* – one of the most important properties enjoyed by vertex operators [Li],[G],[LZ]. For formal series $\alpha(x), \beta(y)$ and integers $n \geq 0$ we define the formal bivariate series [3]

$$C^n(\alpha(x), \beta(y)) = (x-y)^n[\alpha(x), \beta(y)]. \tag{29}$$

Definition 14 (Locality). $\alpha(z), \beta(z) \in U[[z, z^{-1}]]$ *are called* mutually local *if for some integer* $n \geq 0$

$$C^n(\alpha(x), \beta(y)) = 0. \tag{30}$$

The *order of locality* of $\alpha(z)$ and $\beta(z)$ is the *least* integer $n = N \geq 0$ for which (30) holds, in which case we say that $\alpha(z)$ and $\beta(z)$ are *mutually local of order* N and write $\alpha(z) \overset{N}{\sim} \beta(z)$ (or simply $\alpha(z) \sim \beta(z)$ if N is not specified). We also say that $\alpha(z)$ is *local* if $\alpha(z) \sim \alpha(z)$.

Lemma 15. *If* $\alpha(z) \overset{N}{\sim} \beta(z)$ *then* $\partial\alpha(z) \overset{N+1}{\sim} \beta(z)$.

Proof. $0 = \partial_x C^{N+1}(\alpha(x), \beta(y)) = NC^N(\alpha(x), \beta(y)) + C^{N+1}(\partial\alpha(x), \beta(y))$. □

We define the n^{th} *residue product* $*_n$ for $n \geq 0$ of formal series $\alpha(z), \beta(z)$ to be the formal series [4]

$$(\alpha *_n \beta)(z) = \mathrm{Res}_x\, C^n(\alpha(x), \beta(z)) = \sum_{k=0}^{n} \binom{n}{k} (-z)^k\, [\alpha_{n-k}, \beta(z)]. \tag{31}$$

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[3]This is a well-defined formal series in x and y since $n \geq 0$.

[4]The n^{th} residue product is often also notated by $\alpha(z)_{(n)}\beta(z)$ e.g. [K, MN].

For $\alpha(z) \stackrel{N}{\sim} \beta(z)$ it follows that

$$(\alpha *_n \beta)(z) = 0 \text{ for all } n \geq N. \tag{32}$$

The locality condition (30) is closely related to Theorem 10 of the last section for an appropriate choice of vector space and sequence. Let $W = U[[y, y^{-1}]]$ be the vector space of formal series in y with coefficients in U. If $\alpha(z) \stackrel{N}{\sim} \beta(z)$ then

$$0 = y^{1-N} C^N(\alpha(x), \beta(y)) = (z-1)^N \sum_{n\in\mathbb{Z}} \gamma_n z^{-n-1}, \tag{33}$$

for $z = \dfrac{x}{y} = xy^{-1}$ and

$$\gamma_n = y^{-n} \left[\alpha_n, \beta(y)\right]. \tag{34}$$

γ_n determines a sequence $\gamma \in W^{\mathbb{Z}}$ with formal series

$$\gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n-1} = y \left[\alpha(yz), \beta(y)\right],$$

where $\alpha(yz) = \sum_{n\in\mathbb{Z}} \alpha_n (yz)^{-n-1}$. But (33) implies

$$(z-1)^N \gamma(z) = 0,$$

which is Property (ii) of Theorem 10. Therefore γ satisfies the equivalent properties following from Newton's forward difference formula. In particular, Theorem 10 (i) implies $\gamma \in \ker \Delta^N$ which together with Lemma 2 implies that for all $n \in \mathbb{Z}$

$$\sum_{i=0}^{N} \binom{N}{i} (-y)^i \left[\alpha_{n-i}, \beta(y)\right] = 0. \tag{35}$$

Theorem 10 (iii) determines γ in terms of the N residues

$$R_i(y) = \mathrm{Res}_z (z-1)^i \gamma(z) = y^{-i} \left(\alpha *_i \beta\right)(y),$$

for i^{th} residue product (31) with $0 \leq i \leq N-1$. Thus Theorem 10 (iii) implies locality is equivalent to

$$[\alpha_m, \beta(y)] = \sum_{i=0}^{N-1} \binom{m}{i} y^{m-i} (\alpha *_i \beta)(y). \tag{36}$$

In terms of components, (36) reads

$$[\alpha_m, \beta_n] = \sum_{i=0}^{N-1} \binom{m}{i} (\alpha *_i \beta)_{m+n-i}. \tag{37}$$

Theorem 10 (iv)–(vi) describe further corresponding properties equivalent to locality. Recalling (23), Theorem 10 (vi) implies

$$[\alpha(x)_+, \beta(y)] = \sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(x-y)^{i+1}}, \qquad [\alpha(x)_-, \beta(y)] = -\sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(-y+x)^{i+1}},$$

employing the formal expansion convention (21). Altogether we therefore find Theorem 10 implies the following list of properties equivalent to locality [K]:

Theorem 16. *Let $\alpha(z), \beta(z) \in U[[z,z^{-1}]]$ and let $(\alpha *_i \beta)(z)$ be the i^{th} residue product for $i \geq 0$. The following are equivalent:*

(i) $\alpha(z) \overset{N}{\sim} \beta(z)$,

(ii) $\sum_{i=0}^{N} \binom{N}{i} (-y)^i \, [\alpha_{n-i}, \beta(y)] = 0$ *for all* $n \in \mathbb{Z}$,

(iii) $[\alpha_m, \beta(y)] = \sum_{i=0}^{N-1} \binom{m}{i} y^{m-i} \, (\alpha *_i \beta)(y)$,

(iv) $[\alpha_m, \beta_n] = \sum_{i=0}^{N-1} \binom{m}{i} (\alpha *_i \beta)_{m+n-i}$,

(v) $y^{-m}[\alpha_m, \beta(y)] = p_{N-1}(m)$, *where* $p_{N-1}(m)$ *is a degree* $N-1$ *polynomial in* m *with coefficients in* $U[[y,y^{-1}]]$,

(vi) $[\alpha(x), \beta(y)] = \sum_{i=0}^{N-1} y^{-i-1} \delta^{(i)}\left(\frac{x}{y}\right) (\alpha *_i \beta)(y)$,

(vii) $[\alpha(x)_+, \beta(y)] = \sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(x-y)^{i+1}}$ *and* $[\alpha(x)_-, \beta(y)] = -\sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(-y+x)^{i+1}}$.

We also define the *normally ordered product* of $\alpha(x)$ and $\beta(y)$ by

$$:\alpha(x)\beta(y): = \alpha(x)_-\beta(y) + \beta(y)\alpha(x)_+. \tag{38}$$

Thus we find

$$\begin{aligned} \alpha(x)\beta(y) &= [\alpha(x)_+, \beta(y)] \ + :\alpha(x)\beta(y):, \\ \beta(y)\alpha(x) &= -[\alpha(x)_-, \beta(y)] \ + :\alpha(x)\beta(y):, \end{aligned}$$

which imply

Corollary 17. *(OPE) Let $\alpha(z), \beta(z) \in U[[z,z^{-1}]]$. Then $\alpha(z) \overset{N}{\sim} \beta(z)$ if and only if*

$$\begin{aligned} \alpha(x)\beta(y) &= \sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(x-y)^{i+1}} + :\alpha(x)\beta(y):, \\ \beta(y)\alpha(x) &= \sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(-y+x)^{i+1}} + :\alpha(x)\beta(y):. \end{aligned}$$

Remark 18. *The above expressions for $\alpha(x)\beta(y)$ and $\beta(y)\alpha(x)$ are related to the* Operator Product Expansion (OPE) *in chiral conformal field theory (e.g.* [BPZ], [FMS]*)*

$$\alpha(x)\beta(y) \overset{\text{OPE}}{\sim} \sum_{i=0}^{N-1} \frac{(\alpha *_i \beta)(y)}{(x-y)^{i+1}}, \tag{39}$$

to indicate the "pole structure" in the "complex domain" $|x| > |y|$. The remaining "non-singular parts" are not displayed since the pole terms determine the commutation relations of the components in Theorem 16 (iii)–(vii). We also note that N, the order of locality, determines the highest pole order.

3.2. **Examples of Locality.** Suppose that U is a Lie algebra where for all $u, v, w \in U$ the commutator satisfies the Jacobi identity

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0. \tag{40}$$

3.2.1. *The Heisenberg Algebra.* Consider the vector space

$$\widehat{H} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}h_n \oplus \mathbb{C}K,$$

with basis h_n and central element K obeying the Lie algebra commutation relations:

$$[h_m, h_n] = m\delta_{m,-n}K, \quad [h_m, K] = 0. \tag{41}$$

The formal series $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}$ obeys

$$y^{-m}\left[h_m, h(y)\right] = y^{-m}\sum_{n\in\mathbb{Z}}\left[h_m, h_n\right] y^{-n-1} = K\binom{m}{1}y^{-1},$$

a degree 1 polynomial in m. Thus Property (v) of Theorem 16 holds (with $\alpha = \beta = h$) which implies $h(z)$ is local of order $N = 2$. Equivalently, we have Property (iii) of Theorem 16 with

$$(h *_0 h)(y) = 0, \quad (h *_1 h)(y) = K.$$

Considering the Lie algebra components h_n as being elements of the universal enveloping algebra of $\widehat{H}$ we obtain the OPE

$$h(x)h(y) \overset{\mathrm{OPE}}{\sim} \frac{K}{(x-y)^2}. \tag{42}$$

This example is known in vertex algebra theory as the Heisenberg or free boson algebra and in conformal field theory as the bosonic string e.g. [P],[FMS].

3.2.2. *Affine Kac-Moody Algebras.* Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Lie bracket $[\ ,\ ]$ equipped with an invariant symmetric, bilinear form $\langle\ ,\ \rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$ i.e.

$$\langle [a,b], c\rangle = \langle a, [b,c]\rangle, \tag{43}$$

for all $a, b, c \in \mathfrak{g}$. The *affine Lie algebra* or *Kac-Moody algebra* associated to $(\mathfrak{g}, \langle\ ,\ \rangle)$ is the vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}K = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}\otimes t^n \oplus \mathbb{C}K,$$

with Lie algebra commutators

$$\begin{aligned} &[a\otimes t^m, b\otimes t^n] = [a,b]\otimes t^{m+n} + m\langle a,b\rangle\delta_{m,-n}K, \\ &[a\otimes t^m, K] = 0, \end{aligned} \tag{44}$$

for all $a, b \in \mathfrak{g}$. Define the formal series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ for $a \in \mathfrak{g}$ where $a_n := a\otimes t^n$. Then we find that for all $a, b \in \mathfrak{g}$

$$y^{-m}\left[a_m, b(y)\right] = \left[a,b\right](y) + K\langle a,b\rangle\binom{m}{1}y^{-1}.$$

This is Property (iii) of Theorem 16 for $N = 2$ with

$$(a *_0 b)(y) = [a,b]\,(y), \quad (a *_1 b)(y) = K\langle a,b\rangle.$$

Hence $a(z)$ and $b(z)$ are mutually local of order 2 if $\langle a,b\rangle \neq 0$, of order 1 if $\langle a,b\rangle = 0$ and $[a,b] \neq 0$ and of order 0 if $\langle a,b\rangle = 0$ and $[a,b] = 0$. With a suitable universal enveloping algebra interpretation we obtain the OPE

$$a(x)b(y) \overset{\mathrm{OPE}}{\sim} \frac{K\langle a,b\rangle}{(x-y)^2} + \frac{[a,b]\,(y)}{x-y}, \tag{45}$$

This example is known as an affine Kac-Moody algebra theory or as a current algebra in conformal field theory. The Heisenberg algebra (41) corresponds to a 1-dimensional subalgebra generated by a for which $\langle a, a\rangle \neq 0$.

3.2.3. *The Virasoro Algebra.* Consider the vector space

$$\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}K,$$

with basis L_n and central element K obeying the Virasoro algebra with commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{2}K\binom{m+1}{3}\delta_{m+n,0}, \quad [L_m, K] = 0. \tag{46}$$

(The factor of $\frac{1}{2}$ is conventional.) Define the formal series [5]

$$\omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{47}$$

Note that $\omega(z)$ is the formal series for a sequence with components

$$\omega_n = L_{n-1}.$$

Recalling the formal derivative (7) it follows that

$$\begin{aligned} y^{-m}\left[\omega_m, \omega(y)\right] &= \sum_{n\in\mathbb{Z}} \left[(m-n)\omega_{m+n-1} + \frac{1}{2}K\binom{m}{3}\delta_{m+n,2}\right] y^{-m-n-1} \\ &= \partial\omega(y) + 2\omega(y)\binom{m}{1}y^{-1} + \frac{1}{2}K\binom{m}{3}y^{-3}, \end{aligned}$$

a polynomial in m of degree 3. Hence $\omega(z)$ is local of order 4 from Theorem 16 (v) and with a suitable universal enveloping algebra we obtain the OPE

$$\omega(x)\omega(y) \overset{\mathrm{OPE}}{\sim} \frac{\frac{1}{2}K}{(x-y)^4} + \frac{2\omega(y)}{(x-y)^2} + \frac{\partial\omega(y)}{x-y}. \tag{48}$$

4. Creative Fields

4.1. **Fields.** Let V be a vector space over $\mathbb{C}$. We shall often refer to an element of V as a *state*. Let $\mathrm{End}(V)$ denote the algebra of endomorphisms of V i.e. linear maps from V to V. $\mathrm{End}(V)$ is an associative algebra with unit given by the identity map I_V over the field $\mathbb{C}$ with bilinear product given by the composition of linear maps $AB = A\circ B$ for all $A, B \in \mathrm{End}(V)$.

Consider a formal series $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$ with components $\alpha_n \in \mathrm{End}(V)$. $\alpha(z)$ is called a *field* if for any $v \in V$

$$\alpha_n v = 0 \text{ for } n \gg 0, \tag{49}$$

i.e. for n sufficiently large. The property (49) is called *lower truncation* and plays a vital role in vertex algebras.

[5] $\omega(z)$ is usually notated by $T(z)$ in conformal field theory and is called the energy momentum tensor.

For fields $\alpha(x),\beta(y)$ we may extend the definition of the bivariate formal series (29) to all $n\in\mathbb{Z}$ by defining

$$C^n(\alpha(x),\beta(y)) = (x-y)^n\alpha(x)\beta(y) - (-y+x)^n\beta(y)\alpha(x). \tag{50}$$

Clearly, this agrees with (29) for $n\geq 0$. In general, applying the formal expansion convention of (21) we find

$$C^n(\alpha(x),\beta(y)) = \sum_{l,m\in\mathbb{Z}} C^n_{lm}(\alpha,\beta)x^{-l-1}y^{-m-1},$$

where for all $l,m,n\in\mathbb{Z}$

$$C^n_{lm}(\alpha,\beta) = \sum_{i\geq 0}(-1)^i\binom{n}{i}\left(\alpha_{l+n-i}\beta_{m+i} - (-1)^n\beta_{m+n-i}\alpha_{l+i}\right). \tag{51}$$

Remark 19. *By lower truncation, $C^n_{lm}(\alpha,\beta)v$ reduces to a finite sum of terms for each $v\in V$ and hence $C^n(\alpha(x),\beta(y))$ is a well-defined formal bivariate series.*

Lemma 20. *For all $k\geq 0$ and all $n\in\mathbb{Z}$ we have*

$$(x-y)^kC^n(\alpha(x),\beta(y)) = C^{n+k}(\alpha(x),\beta(y)). \tag{52}$$

Proof. The expansion convention (21) implies $(x-y)^k(x-y)^n = (x-y)^{n+k}$ and $(x-y)^k(-y+x)^n = (-y+x)^k(-y+x)^n = (-y+x)^{n+k}$ for $k\geq 0$. □

By Remark 19, for fields $\alpha(x),\beta(y)$ we may similarly extend the definition of the n^{th} residue product $*_n$ to all $n\in\mathbb{Z}$ with

(53)
$$(\alpha *_n \beta)(z) = \mathrm{Res}_x\, C^n(\alpha(x),\beta(z)) = \sum_{i\geq 0}\binom{n}{i}\left((-z)^i\alpha_{n-i}\beta(z) - (-z)^{n-i}\beta(z)\alpha_i\right),$$

with components

$$(\alpha *_n \beta)_m = C^n_{0m}(\alpha,\beta) = \sum_{i\geq 0}(-1)^i\binom{n}{i}\left(\alpha_{n-i}\beta_{m+i} - (-1)^n\beta_{m+n-i}\alpha_i\right). \tag{54}$$

Lemma 21. *If $\alpha(z),\beta(z)$ are fields then $(\alpha *_n \beta)(z)$ is a field for all $n\in\mathbb{Z}$.*

Proof. (54) implies $(\alpha *_n \beta)(z)$ is a field provided $C^n_{0m}(\alpha,\beta)v = 0$ for any $v\in V$ for $m\gg 0$. But $\beta_{m+i}v = 0$ and $\beta_{m+n-i}\alpha_i v = 0$ for $m\gg 0$ for some v dependent finite range of i following Remark 19. □

For $n<0$, $(\alpha *_n \beta)(z)$ is related to the normally ordered product (38) as follows:

Lemma 22. *For fields $\alpha(x),\beta(z)$ and $k\geq 0$ we have*

$$(\alpha *_{-k-1} \beta)(z) = \,:\partial^{(k)}\alpha(z)\beta(z):. \tag{55}$$

Proof. The result follows directly from Lemma 13. □

Lemma 23. *∂ is a derivation of the n^{th} residue product of two fields $\alpha(z),\beta(z)$ i.e.*

$$\partial(\alpha *_n \beta)(z) = (\partial\alpha *_n \beta)(z) + (\alpha *_n \partial\beta)(z). \tag{56}$$

Proof. From (50) we directly find

$$(\partial_x + \partial_z)\, C^n(\alpha(x),\beta(z)) = C^n(\partial\alpha(x),\beta(z)) + C^n(\alpha(x),\partial\beta(z)).$$

Taking Res_x, the result follows since $\mathrm{Res}_x\,\partial_x C^n(\alpha(x),\beta(z)) = 0$ from (9). □

The next theorem is fundamental to the theory of vertex algebras. It is often stated either as a foundational axiom [B, FLM, Li] or else is proved subject to some further assumed properties [Li, K, MN]. However, here we only assume that $\alpha(x), \beta(y)$ are local fields.

Theorem 24 (Borcherds-Frenkel-Lepowsky-Meurmann identity)**.** *Let $\alpha(x), \beta(y)$ be mutually local fields. Then for all $l, m, n \in \mathbb{Z}$ we have*

$$\sum_{i\geq 0}\binom{l}{i}(\alpha *_{n+i}\beta)_{l+m-i} = \sum_{i\geq 0}(-1)^i\binom{n}{i}\left(\alpha_{l+n-i}\beta_{m+i} - (-1)^n\beta_{m+n-i}\alpha_{l+i}\right). \tag{57}$$

Proof. Let $\alpha(z) \overset{N}{\sim} \beta(z)$ for $N \geq 0$. Thus (57) is the trivial identity $0 = 0$ for $n \geq N$ by locality and (32) so that we need only consider $n < N$. In a similar fashion to the proof of Theorem 16, the identity (57) is a consequence of Newton forward differences applied to an appropriate choice of sequence. Note that the right hand side of (57) is $C^n_{lm}(\alpha,\beta)$ of (51). For each $n < N$ we define a sequence γ^n with components in $W := \mathrm{End}(V)[[y, y^{-1}]]$ labelled by $l \in \mathbb{Z}$ as follows

$$(\gamma^n)_l = y^{-l}\sum_{m\in\mathbb{Z}} C^n_{lm}(\alpha,\beta)y^{-m-1},$$

with formal series

$$\gamma^n(z) = y\, C^n(\alpha(yz), \beta(y)).$$

Since $\alpha(x) \overset{N}{\sim} \beta(y)$ and using (52) we find for each $n < N$ that

$$(z-1)^{N-n}\gamma^n(z) = y^{1+n-N}C^N(\alpha(yz), \beta(y)) = 0.$$

Thus applying Newton's forward difference formula Theorem 10 (iii) we find

$$(\gamma^n)_l = \sum_{i\geq 0}\binom{l}{i}R^n_i, \tag{58}$$

with R^n_i for $i \geq 0$ given by

$$\begin{aligned}
R^n_i &= y\,\mathrm{Res}_z(z-1)^i C^n(\alpha(yz), \beta(y))\\
&= y^{1-i}\,\mathrm{Res}_z\, C^{n+i}(\alpha(yz), \beta(y))\\
&= y^{-i}\,\mathrm{Res}_x\, C^{n+i}(\alpha(x), \beta(y)) = y^{-i}(\alpha *_{n+i}\beta)(y),
\end{aligned}$$

using (52) and that $\mathrm{Res}_z\,\rho(yz) = y^{-1}\,\mathrm{Res}_x\,\rho(x)$ for any formal series $\rho(x)$. We have therefore shown that $(\gamma^n)_l\, y^l$ is given by

$$\sum_{m\in\mathbb{Z}} C^n_{lm}(\alpha,\beta)y^{-m-1} = \sum_{i\geq 0}\binom{l}{i}y^{l-i}(\alpha *_{n+i}\beta)(y). \tag{59}$$

The result follows on computing the coefficients of y^{-m-1} in (59). □

(57) specializes to the commutator formula Theorem 16 (iv) for $n = 0$ and to the residue product formula (53) when $l = 0$. There are a number of equivalent ways of writing (57).

Proposition 25. *Both of the following identities are equivalent to the Borcherds-Frenkel-Lepowsky-Meurmann identity:*

(60)
$$\sum_{i\geq 0} y^{-i-1}\delta^{(i)}\left(\frac{x}{y}\right)(\alpha *_{n+i}\beta)(y) = (x-y)^n\alpha(x)\beta(y) - (-y+x)^n\beta(y)\alpha(x),$$

(61)
$$y^{-1}\delta\left(\frac{x-z}{y}\right)\sum_{m\in\mathbb{Z}}(\alpha *_m \beta)(y)z^{-m-1} = z^{-1}\delta\left(\frac{x-y}{z}\right)\alpha(x)\beta(y) - z^{-1}\delta\left(\frac{-y+x}{z}\right)\beta(y)\alpha(x).$$

Proof. (59) is equivalent to

$$C^n(\alpha(x),\beta(y)) = \sum_{l\in\mathbb{Z}} x^{-l-1}\sum_{i\geq 0}\binom{l}{i}y^{l-i}(\alpha *_{n+i}\beta)(y).$$

Recalling (16) this can be written as (60). This in turn is equivalent to

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} z^{-n-1}\sum_{i\geq 0} y^{-i-1}\delta^{(i)}\left(\frac{x}{y}\right)(\alpha *_{n+i}\beta)(y)\\
&= z^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{x-y}{z}\right)^n\alpha(x)\beta(y) - z^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{-y+x}{z}\right)^n\beta(y)\alpha(x)\\
&= z^{-1}\delta\left(\frac{x-y}{z}\right)\alpha(x)\beta(y) - z^{-1}\delta\left(\frac{-y+x}{z}\right)\beta(y)\alpha(x),
\end{aligned}$$

recalling (12). Finally, Taylor's Theorem of Lemma 11 and (15) imply

$$\delta\left(\frac{x-z}{y}\right) = \sum_{i\geq 0}\delta^{(i)}\left(\frac{x}{y}\right)\left(\frac{z}{y}\right)^i,$$

so that, after relabelling, we obtain

$$\sum_{n\in\mathbb{Z}} z^{-n-1}\sum_{i\geq 0} y^{-i-1}\delta^{(i)}\left(\frac{x}{y}\right)(\alpha *_{n+i}\beta)(y) = y^{-1}\delta\left(\frac{x-z}{y}\right)\sum_{m\in\mathbb{Z}}(\alpha *_m\beta)(y)z^{-m-1}.$$

Thus the result holds. □

Remark 26. *For $n \geq 0$ the Borcherds-Frenkel-Lepowsky-Meurmann identity* (60) *follows from locality using Theorem 16 (vi) and Lemma 7.*

The next result is very useful for the construction of local fields.

Lemma 27 (Dong's Lemma)**.** *Let $\alpha(z), \beta(z), \gamma(z)$ be mutually local fields. Then $(\alpha *_n \beta)(z)$ and $\gamma(z)$ are mutually local fields for all $n \in \mathbb{Z}$.*

Proof. For some orders of locality $K, L, M \geq 0$ we have

$$\alpha(z) \overset{K}{\sim} \beta(z), \quad \alpha(z) \overset{L}{\sim} \gamma(z), \quad \beta(z) \overset{M}{\sim} \gamma(z).$$

In particular, $C^n(\alpha(x),\beta(z)) = 0$ and $(\alpha *_n \beta)(z) = 0$ for $n \geq K$. Hence we need only consider $n \leq K-1$. Let $N = K+L+M-n-1$ and define

$$D(x,y,z) = (y-z)^N\left[\gamma(y), C^n(\alpha(x),\beta(z))\right].$$

Note that $N \geq 0$ since $L, M, K-n-1 \geq 0$. Using (52) we find

$$D(x,y,z) = (y-z)^M (y-x+x-z)^{N-M} \left[\gamma(y), C^n(\alpha(x),\beta(z))\right]$$
$$= (y-z)^M \sum_{r=0}^{K-n-1} \binom{N-M}{r} (y-x)^{N-M-r} \left[\gamma(y), C^{n+r}(\alpha(x),\beta(z))\right],$$

where $r \leq K-n-1$ in the sum since $C^{n+r}(\alpha(x),\beta(z)) = 0$ for $n+r \geq K$. Therefore $N-M-r \geq L$ for each r in the sum so that

$$(y-z)^M (y-x)^{N-M-r} \left[\gamma(y), C^{n+r}(\alpha(x),\beta(z))\right] = 0,$$

since $\alpha(z) \overset{L}{\sim} \gamma(z)$ and $\beta(z) \overset{M}{\sim} \gamma(z)$. Thus $D(x,y,z) = 0$ which implies

$$C^N(\gamma(y), (\alpha *_n \beta)(z)) = \mathrm{Res}_x\, D(x,y,z) = 0,$$

i.e. $\gamma(z) \sim (\alpha *_n \beta)(z)$ with order of locality at most N. □

4.2. **Creative Fields.** Let $\mathbf{1} \in V$ denote a distinguished state called the *vacuum vector*.[6] A *creative field* for $a \in V$ is a field which we notate by

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},$$

with components or *modes* $a_n \in \mathrm{End}(V)$ such that

$$a_{-1}\,\mathbf{1} = a, \tag{62}$$
$$a_n\,\mathbf{1} = 0, \text{ for all } n \geq 0. \tag{63}$$

(63) is equivalent to $a(z)_+\,\mathbf{1} = 0$ (cf. (23)). We sometimes write (62) and (63) together as [7]

$$a(z)\,\mathbf{1} = a + O(z) \in V[[z]],$$

where $V[[z]]$ is the space of formal power series in z with coefficients in V.

The following lemma describes several important examples of creative fields.

Lemma 28. *Let $a(z), b(z)$ be creative fields for states $a, b \in V$, respectively.*

(i) $za(z)$ *creates the zero vector* 0*,*
(ii) $I(z) = \mathrm{Id}_V$*, the identity* V *endomorphism, creates the vacuum* $\mathbf{1}$*,*
(iii) $\lambda a(z) + \mu b(z)$ *creates* $\lambda a + \mu b$ *for* $\lambda, \mu \in \mathbb{C}$*,*
(iv) $(a *_n b)(z)$ *creates* $a_n b$ *for* $n \in \mathbb{Z}$*,*
(v) $:\partial^{(k)} a(z) b(z):$ *creates* $a_{-k-1} b$ *for* $k \geq 0$*,*
(vi) $\partial^{(k)} a(z)$ *creates* $a_{-k-1}\,\mathbf{1}$*.*

Proof. (i)–(iii) are trivially true. (53) implies that

$$(a *_n b)(z)\,\mathbf{1} = \sum_{i\geq 0} \binom{n}{i} \left((-z)^i\, a_{n-i} b(z) - (-z)^{n-i} b(z) a_i\right)\mathbf{1}$$
$$= \sum_{i\geq 0} \binom{n}{i} (-z)^i\, a_{n-i}\,(b + O(z)) = a_n b + O(z),$$

using creativity of $a(z)$ and $b(z)$. Hence (iv) holds. (iv) implies (v) on using Lemma (22). (vi) follows from (ii) and (v) on choosing $b = \mathbf{1}$ and $b(z) = I(z)$. □

[6] The vacuum vector is usually denoted by $|0\rangle$ in CFT.
[7] This is usually written in CFT as $\lim_{z\to 0} a(z)|0\rangle = a$.

Remark 29. *A creative field $a(z)$ for $a \in V$ is clearly not unique since, by Lemma 28 (i), $a(z) + zb(z)$ also creates a for any creative field $b(z)$.*

The lower truncation property (49) is refined for local creative fields as follows:

Corollary 30 (Lower Truncation)**.** *Let $a(z), b(z)$ be local creative fields for $a, b \in V$ respectively. Then $a(z) \overset{N}{\sim} b(z)$ implies*

$$a_n b = 0 \text{ for all } n \geq N. \tag{64}$$

Proof. $(a *_n b)(z) = 0$ for $n \geq N$ by (32) so that Lemma 28 (v) implies the result. □

5. Vertex Algebras

5.1. **Uniqueness and Translation Covariance.** Consider a vector space V with vacuum vector $\mathbf{1} \in V$ and a set of mutually local creative fields $\mathcal{F} := \{a(z) : a \in V\}$. By Remark 29, $a(z) \in \mathcal{F}$ is not the unique creative field for $a \in V$.

Proposition 31. *Suppose that $\phi(z) \in \mathcal{F}$ is a creative field for the zero state 0. Then*

$$\phi(z)\,\mathbf{1} = 0 \Leftrightarrow \phi(z) = 0. \tag{65}$$

Proof. Assume that $\phi(z)\,\mathbf{1} = 0$. Let $a \in V$ with a creative field $a(z) \in \mathcal{F}$ where $a(z) \overset{N}{\sim} \phi(z)$ for some $N \geq 0$. Then

$$0 = x^{-N} C^N(\phi(x), a(y))\,\mathbf{1} = x^{-N}(x-y)^N \phi(x)a(y)\,\mathbf{1} = \phi(x)a + O(y),$$

i.e. $\phi(x)a = 0$. This is true for any $a \in V$ so that $\phi(x) = 0$. The converse is trivial. □

This result immediately implies:

Corollary 32. *Let $a(z), \widetilde{a}(z) \in \mathcal{F}$ be creative fields for $a \in V$. Then*

$$a(z) = \widetilde{a}(z) \Leftrightarrow a(z)\,\mathbf{1} = \widetilde{a}(z)\,\mathbf{1}\,.$$

We now describe a uniqueness criterion for $\mathcal{F}$. Let $T \in \mathrm{End}(V)$ such that

$$T\,\mathbf{1} = 0, \tag{66}$$

$$[T, a(z)] = \partial a(z) \text{ for all } a(z) \in \mathcal{F}. \tag{67}$$

In terms of modes, (67) is equivalent to

$$[T, a_n] = -n a_{n-1}. \tag{68}$$

T is called a *translation operator* and $\mathcal{F}$ is said to be *translation covariant* if (66) and (67) are satisfied for a translation operator T.

Theorem 33 (Uniqueness)**.** *Let $\mathcal{F}$ be a set of mutually local creative fields for V. The elements of $\mathcal{F}$ are unique if and only if $\mathcal{F}$ is translation covariant.*

Proof. Assume that the elements of $\mathcal{F}$ are unique. Define $T \in \mathrm{End}(V)$ by

$$Ta = a_{-2}\,\mathbf{1}, \tag{69}$$

for each $a \in V$ with unique creative field $a(z)$. By Lemma 28 (ii) we know that $I(z) = \mathrm{Id}_V$ is a creative field for $\mathbf{1}$ and is therefore unique by assumption. Thus (69) implies (66). By Dong's Lemma 27 and Lemma 28 (iv) we also know that $(a *_n b)(z) \in \mathcal{F}$ is a creative field for $a_n b$ for each $a, b \in V$. Hence, by the assumed uniqueness property

$$(a_n b)\,(z) = (a *_n b)(z). \tag{70}$$

In particular, using (54) we find that for all $a, b \in V$

$$T(a_n b) = (a_n b)_{-2}\,\mathbf{1} = \sum_{i\geq 0}(-1)^i\binom{n}{i}\left(a_{n-i}b_{i-2} - (-1)^n b_{n-i-2}\,a_i\right)\mathbf{1}$$
$$= a_n b_{-2}\,\mathbf{1} - n a_{n-1}b_{-1}\,\mathbf{1} = a_n Tb - n a_{n-1}b.$$

Hence $\mathcal{F}$ is translation covariant using (68).

Conversely, assume that $\mathcal{F}$ is translation covariant with some translation operator T. Thus for $a(z) \in \mathcal{F}$, (66) and (67) imply that $Ta_{-k}\,\mathbf{1} = k a_{-k-1}\,\mathbf{1}$ for all $k \in \mathbb{Z}$. Hence $T^n a = T^n a_{-1}\,\mathbf{1} = n! a_{-n-1}\,\mathbf{1}$ for all $n \geq 0$ so that

$$a(z)\,\mathbf{1} = e^{zT}a. \tag{71}$$

But if $\widetilde{a}(z)$ is another translation covariant creative field for a then $\widetilde{a}(z)\,\mathbf{1} = e^{zT}a = a(z)\,\mathbf{1}$. Hence by Corollary 32 we conclude that $a(z) = \widetilde{a}(z)$. Therefore the elements of $\mathcal{F}$ are unique. □

5.2. **Vertex Algebras.** We have now gathered all the requisite concepts to define a vertex algebra. Let $Y(a,z)$ denote the unique translation covariant creative field for $a \in V$ of Theorem 33. $Y(a,z)$ is called the *vertex operator* for a. Y can also be construed as a mapping

$$\begin{aligned} Y : V &\to \mathrm{End}(V)[[z,z^{-1}]], \\ a &\mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \end{aligned} \tag{72}$$

called the *state-field correspondence*.

Definition 34. *A* Vertex Algebra *consists of the data* $(V, Y, T, \mathbf{1})$ *where* V *is a vector space, a distinguished vacuum vector* $\mathbf{1} \in V$*, a translation operator* $T \in \mathrm{End}(V)$ *and a state-field correspondence* Y *with the following properties:*

locality: $Y(a,z) \sim Y(b,z)$ *for all* $a, b \in V$,

creativity: $Y(a,z)\,\mathbf{1} = a + O(z)$,

translation covariance: $[T, Y(a,z)] = \partial Y(a,z), \quad T\,\mathbf{1} = 0.$

Lemma 35. *The state-field correspondence is an injective linear map.*

Proof. Linearity follows from Lemma 28 (iii). Suppose $Y(a,z) = Y(b,z)$ for $a, b \in V$. Then $a_{-1} = b_{-1}$ so that $a = a_{-1}\,\mathbf{1} = b_{-1}\,\mathbf{1} = b$. Hence Y is injective. □

We describe a number of important properties of vertex operators:

Proposition 36. *Let* $a(z) = Y(a,z)$ *and* $b(z) = Y(b,z)$ *be the vertex operators for* $a, b \in V$.

(i) $Y(\mathbf{1}, z) = \mathrm{Id}_V$*,*
(ii) $Y(a,z)\,\mathbf{1} = e^{zT}a$*,*
(iii) $Y(a_n b, z) = (a *_n b)(z)$ *for all* $n \in \mathbb{Z}$*,*
(iv) $Y(Ta, z) = \partial Y(a,z)$.

Proof. $\mathrm{Id}_V \in \mathcal{F}$ creates $\mathbf{1}$, by Lemma 28 (ii), and is translation covariant giving (i). Property (ii) was shown in (71) in the proof of the Uniqueness Theorem 33. $(a *_n b)(z) \in \mathcal{F}$ is a local creative field for $a_n b$ by Lemma 21 and Lemma 28 (iii). Translation covariance of $a(z)$ and $b(z)$ implies

$$[T, C^n(a(x), b(z))] = C^n(\partial a(x), b(z)) + C^n(a(x), \partial b(z)),$$

so that

$$\begin{aligned}[T,(a*_n b)(z)] &= \mathrm{Res}_x\,[T, C^n(a(x),b(z))] \\ &= (\partial a *_n b)(z) + (a *_n \partial b)(z) = \partial(a *_n b)(z),\end{aligned} \tag{73}$$

by Lemma 23. Thus $(a *_n b)(z)$ is translation covariant and so (iii) holds.

Lemma 15 and Lemma 28 (vi) imply $\partial Y(a,z)$ is a local creative field for $a_{-2}\mathbf{1} = Ta$. Translation covariance for $Y(a,z)$ implies

$$[T,\partial Y(a,z)] = \partial\,[T,Y(a,z)] = \partial(\partial Y(a,z)),$$

so that $\partial Y(a,z)$ is also translation covariant. Therefore (iv) follows from the Uniqueness Theorem 33. □

Corollary 37. *T is a derivation of the vertex algebra where for all $a,b\in V$:*

$$T(a_n b) = (Ta)_n b + a_n Tb.$$

Proof. $(Ta)_n = -na_{n-1}$ from Proposition 36 (iii). (68) implies $[T,a_n]b = (Ta)_n b$. □

Proposition 36 (iii) implies that, for a vertex algebra, we may replace all n^{th} residue products $(a *_n b)(z)$ by the unique vertex operator $Y(a_n b, z)$ in the previous sections. Thus the locality Theorem 16 implies the *Commutator Formulas*

$$[a_m, Y(b,z)] = \sum_{i\geq 0}\binom{m}{i} Y(a_i b, z) z^{m-i}, \tag{74}$$

$$[a_m, b_n] = \sum_{i\geq 0}\binom{m}{i}(a_i b)_{m+n-i}, \tag{75}$$

for all $m,n\in\mathbb{Z}$. Similarly (54) implies the A*ssociator Formula*

$$(a_n b)_m = \sum_{i\geq 0}(-1)^i\binom{n}{i}\left(a_{n-i}b_{m+i} - (-1)^n b_{m+n-i}a_i\right). \tag{76}$$

These can be combined into the Borcherds-Frenkel-Lepowsky-Meurmann identity

$$\sum_{i\geq 0}\binom{l}{i}(a_{n+i}b)_{l+m-i} = \sum_{i\geq 0}(-1)^i\binom{n}{i}\left(a_{l+n-i}b_{m+i} - (-1)^n b_{m+n-i}a_{l+i}\right), \tag{77}$$

for all $l,m,n\in\mathbb{Z}$ (cf. (57)). This in turn is equivalent to (cf. (61))

$$\begin{aligned}&z^{-1}\delta\left(\frac{x-y}{z}\right)Y(a,x)Y(b,y) - z^{-1}\delta\left(\frac{-y+x}{z}\right)Y(b,y)Y(a,x) \\ &= y^{-1}\delta\left(\frac{x-z}{y}\right)Y(Y(a,z)b,y).\end{aligned} \tag{78}$$

(75) and (76) are axioms in the original formulation of vertex algebras by Borcherds in [B]. These were shown to be equivalent to the identity (78), called the Jacobi identity by Frenkel, Lepowsky and Meurmann [FLM].

5.3. **Translation and Skewsymmetry.**

Lemma 38 (Translation Symmetry). *T is a generator of translation symmetry:*
$$e^{yT}Y(a,x)e^{-yT}=Y(a,x+y).$$

Proof. The Baker-Campbell-Hausdorff formula for linear operators A, B states that
$$e^{A}Be^{-A}=e^{\mathrm{ad}_A}B,$$
where $\mathrm{ad}_A(\cdot)=[A,\cdot]$ is the adjoint operator. Thus we find
$$e^{yT}Y(a,x)e^{-yT}=e^{y\,\mathrm{ad}_T}Y(a,x)=e^{y\partial}Y(a,x),$$
by translation covariance (67). The result follows from Taylor's theorem (24). □

Lemma 39 (Skew-Symmetry). *Let $a,b\in V$, a vertex algebra. Then*
$$Y(a,z)b=e^{zT}Y(b,-z)a, \tag{79}$$
or in terms of components:
$$a_nb{=}(-1)^{n+1}\sum_{k\geq 0}(-1)^kT^kb_{n+k}a. \tag{80}$$

Proof. Let $Y(a,z)\overset{N}{\sim}Y(b,z)$ so that
$$(z-y)^NY(a,z)Y(b,y)\,\mathbf{1}=(z-y)^NY(b,y)Y(a,z)\,\mathbf{1}\,.$$
By Proposition 36 (ii) and translation symmetry we have
$$\begin{aligned}(z-y)^NY(a,z)e^{yT}b&=(z-y)^NY(b,y)e^{zT}a\\ &=(z-y)^Ne^{zT}Y(b,y-z)a.\end{aligned}\tag{81}$$
Lower truncation (64) implies $x^NY(b,x)a$ contains no negative powers of x. Thus $(z-y)^NY(b,y-z)a$ also contains no negative powers of y. Taking $y=0$ in (81) we obtain (79) on multiplying by z^{-N}. (80) follows immediately. □

5.4. **Examples of Vertex Algebras.** We have the following very useful generating theorem [FKRW], [MP].

Theorem 40 (Generating Theorem). *Let V be a vector space with $\mathbf{1}\in V$ and $T\in\mathrm{End}(V)$. Let $\{a^i(z)\}_{i\in\mathcal{I}}$ for some indexing set $\mathcal{I}$ be a set of mutually local, creative, translation-covariant fields which* generates *V i.e.*
$$V=\mathit{span}\{a^{i_1}_{n_1}\ldots a^{i_k}_{n_k}\,\mathbf{1}\ |n_1,\ldots,n_k\in\mathbb{Z},i_1,\ldots i_k\in\mathcal{I}\}.$$
Then there is a unique vertex algebra $(V,Y,\mathbf{1},T)$ with vertex operators defined on the spanning set by
$$Y\left(a^{i_1}_{n_1}\ldots a^{i_k}_{n_k}\,\mathbf{1},z\right)=a^{i_1}*_{n_1}\left(a^{i_1}*_{n_2}\left(\ldots\left(a^{i_k}*_{n_k}I\right)\right)\right)(z), \tag{82}$$
a composition of k residue products and where $I(z)=Y(\mathbf{1},z)=\mathrm{Id}_V$.

Proof. $\mathcal{F}=\{a^{i_1}*_{n_1}(a^{i_1}*_{n_2}(\ldots(a^{i_k}*_{n_k}I)))(z)\}$ is a set of mutually local creative fields for V by repeated use of Lemma 21, Dong's Lemma 27 and Lemma 28 (iv). Furthermore, $a^{i_1}*_{n_1}(a^{i_1}*_{n_2}(\ldots(a^{i_k}*_{n_k}I)))(z)$ is translation covariant by (73). Hence, by the Uniqueness Theorem 33, $\mathcal{F}$ forms a set of unique vertex operators on the spanning set and therefore by linearity on V. □

5.4.1. *The Heisenberg Vertex Algebra.* The Heisenberg vertex algebra is constructed from the Verma module[8] M_0 of the Heisenberg Lie algebra (41) given by

$$M_0 = \text{span}\{h_{-n_1} \dots h_{-n_k} v_0 | n_1, \dots, n_k \geq 1\},$$

where $h_n v_0 = 0$ for all $n \geq 0$ and $K v_0 = v_0$. Then with $V = M_0$ and $\mathbf{1} = v_0$ we find $h(z)$ is a creative field for $h = h_{-1}\,\mathbf{1}$ which is translation covariant for

$$T = \sum_{n \geq 0} h_{-n-1} h_n.$$

Thus Theorem 40 and Lemma 28 imply that $h(z)$ generates a vertex algebra with

$$Y(h_{-n_1} \dots h_{-n_k}\,\mathbf{1}, z) = \mathbf{:}\,\partial^{(n_1)} h(z)\mathbf{:}\,\partial^{(n_2)} h(z) \dots \mathbf{:}\,\partial^{(n_{k-1})} h(z) \partial^{(n_k)} h(z)\,\mathbf{::} \dots \mathbf{:},$$

for $n_1, \dots, n_k \geq 1$.

5.4.2. *The Virasoro Vertex Algebra.* The Virasoro vertex algebra is constructed from a Verma module $M_{C,0}$ of the Virasoro Lie algebra (46) defined by

$$M_{C,0} = \text{span}\{L_{-n_1} \dots L_{-n_k} v_0 | n_1, \dots, n_k \geq 1\},$$

where $L_n v_0 = 0$ for all $n \geq 0$ and $K v_0 = C v_0$. Then $\omega(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ is translation covariant for $T = L_{-1}$ but is *not a creative field* with vacuum v_0 since

$$\omega(z) v_0 = z^{-1} L_{-1} v_0 + L_{-2} v_0 + O(z).$$

But since $L_1 L_{-1} v_0 = 0$ it follows that

$$M_{C,1} = \text{span}\{L_{-n_1} \dots L_{-n_k} L_{-1} v_0 | n_1, \dots, n_k \geq 1\},$$

is submodule of $M_{C,0}$. Abusing notation by identifying states, operators and fields associated with $M_{C,0}$ with the corresponding states, operators and fields induced on the quotient $\mathrm{Vir} = M_{C,0}/M_{C,1}$ we find that $Y(\omega, z) = \omega(z)$ generates a vertex algebra with $T = L_{-1}$, $\mathbf{1} = v_0$ and $V = \mathrm{Vir}$ with vertex operators

$$Y(L_{-n_1} \dots L_{-n_k}\,\mathbf{1}, z) = \mathbf{:}\,\partial^{(n_1)} L(z)\mathbf{:}\,\partial^{(n_2)} L(z) \dots \mathbf{:}\,\partial^{(n_{k-1})} L(z) \partial^{(n_k)} L(z)\,\mathbf{::} \dots \mathbf{:},$$

for $n_1, \dots, n_k \geq 2$.

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[8] e.g. See [K], [MT] for further details

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