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<tr>
<td><strong>Author(s)</strong></td>
<td>Ha Van, Hieu</td>
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<tr>
<td><strong>Publication Date</strong></td>
<td>2019-03-29</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>NUI Galway</td>
</tr>
<tr>
<td><strong>Item record</strong></td>
<td><a href="http://hdl.handle.net/10379/15081">http://hdl.handle.net/10379/15081</a></td>
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ENTRY PATTERN MATRICES

PhD Thesis

by

Hieu Ha Van

Supervisor: Dr. Rachel Quinlan

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway

February 2019
Abstract

An entry pattern matrix (EPM for short) is a rectangular matrix in which each entry is an indeterminate. The same indeterminate may appear in multiple positions, but different indeterminates are independent. For a field $\mathbb{F}$, an $\mathbb{F}$-completion of an EPM is the matrix that results from assigning a specific value from $\mathbb{F}$ to each indeterminate that appears as an entry of the matrix. In this thesis, we are concerned with the set of $\mathbb{F}$-completions of an entry pattern matrix which can be considered as a vector space whose dimension is equal to the number of distinct indeterminates appearing in the entry pattern matrix.

Chapter 1 presents some background to the content of Chapters 3, 4, and 5. We discuss linear subspaces of square matrices in which every non-zero element is either nonsingular, and in which every element is nilpotent. In particular, we consider bounds on the dimensions of such spaces.

In Chapter 2 we will introduce the concept of an entry pattern matrix and discuss some general properties.

In Chapter 3 we will consider the maximum rank of a completion of a given entry pattern matrix over a field $\mathbb{F}$. We will show that this number can depend on the field under consideration, and focus on cases where it does. We will define the generic rank and the maximal completion rank of an entry pattern matrix and introduce the concept "EPM-rank-tight" field and prove that every finite field of characteristic less than 17, except $\mathbb{F}_2$, is EPM-rank-tight.

In Chapter 4 we will introduce the concept of an $\mathbb{F}$-almost-nonsingular EPM as an EPM whose completions are all nonsingular provided that their entries are not all equal. We present constructions for entry pattern matrices that are almost-nonsingular over the real, the rational fields and over finite fields, and obtain lower bounds for their numbers of indeterminates.

In Chapter 5 we will give bounds for the number of indeterminates in $n \times n$ nilpotent entry pattern matrices over fields of positive characteristic. We also give the classification of such entry pattern matrices attaining the bounds.
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Declaration

I declare that this thesis represents my own work and I have not obtained a degree in NUI Galway, or elsewhere, on the basis of the work described in the thesis. The collaborative contributions have been indicated clearly and acknowledged. Due references have been provided on all supporting literatures and resources.

Ha Van Hieu
February, 2019
Acknowledgment

Firstly, I would like to thank my supervisor, Dr. Rachel Quinlan, for having introduced me to this fascinating and beautiful subject. I thank her from the bottom of my heart for her enthusiasm, her wonderful explanations and guidance.

Besides my supervisor, I would like to thank my PhD viva committee members: Prof. Dane Flannery, Dr. Roberto Canogar McKenzie and Prof. Graham Ellis for their time, insightful comments and interesting questions.

I also gratefully thank the staff of School of Maths, Applied Maths and Statistics for their support during the last four years. Also I would like to offer my gratitude to the College of Science, National University of Ireland, Galway and Vietnam National University, HCMC for their kind financial supports.

Finally, thanks to my family for their support and encouragement, especially my wife Nguyen Thanh Thuy.
Chapter 1

Introduction

This thesis is concerned with algebraic properties of entry pattern matrices. In an entry pattern matrix (EPM), every entry is an indeterminate and while a single indeterminate may appear in multiple positions, different indeterminates are independent of each other. Thus an entry pattern matrix encodes a relation of a particular type amongst the entries of a matrix, namely that entries in certain specified positions are equal to each other. Given an entry pattern matrix $A$ and a field $F$, we may consider the set of all matrices obtained by assigning a value from $F$ to each indeterminate that appears in $A$. Such matrices are referred to as $F$-completions of $A$, and they form an $F$-vector space whose dimension is the number of distinct indeterminates appearing in $A$. This thesis considers properties of spaces of matrices that arise in this manner, and their dependence on the choice of field.

The concept of an entry pattern matrix was proposed by Huang and Zhan in 2015 [19]. These authors observed that many important classes of matrices have defining properties that are characterized by entry patterns, for example symmetric matrices, Toeplitz matrices, Hankel matrices and circulant matrices. They also observed that such matrix classes often exhibit properties of special interest. For example, real symmetric matrices have real spectrum and are normal, and circulant matrices form commutative algebras.

In their article [19], Huang and Zhan investigate entry pattern matrices whose real completions all have real spectrum, and whose real completions are normal. They also propose wider investigation of the existence and classification of entry pattern matrices whose completions (over any field, or over particular fields) all share specified properties. This thesis reports on some new contributions in this direction, in respect of properties involving rank distributions and nilpotency.

Three particular questions are investigated. The first concerns the maximum rank of a completion of a given entry pattern matrix $A$, over a field $F$. An upper bound for this maximum is given by the rank of $A$ when considered as a matrix written over the transcendental extension of $F$ generated by its distinct indeterminates. It is shown that if $F$ is a small finite field, then it is possible for this bound not to be attained by any $F$-completion of $A$. This phenomenon is investigated in Chapter 3,
which has been published in Linear Algebra and its Applications as an article co-authored with Rachel Quinlan [33].

The second question is the identification of the maximum number of indeterminates that can occur in a square entry pattern matrix whose \( F \)-completions are all nonsingular, provided that at least two indeterminates are assigned different values from \( F \). This investigation is highly field dependent and closely related to the classical problem of determining dimension bounds for linear spaces of square matrices whose nonzero elements are all nonsingular. This is the main theme of Chapter 4. Some background on this topic is presented in this introductory chapter. Another joint article based on the content of Chapter 4 has recently been submitted for publication.

Chapter 5 is concerned with entry pattern matrices all of whose completions are nilpotent, which can exist over fields of positive characteristic. Tight upper bounds are identified for the numbers of indeterminates in EPMs with this property. The results presented in this chapter connect to the extensive literature on linear spaces of nilpotent matrices.

This thesis may be interpreted within the context of matrix completion problems, which are broadly concerned with the identification of properties that a matrix may or may not possess, based on partial knowledge of its entries. In various contexts, "partial knowledge" might mean that some entries are known and some are not, that some relations are known to exist between entries whose values are not known, or that some overall structural information is known, for example about the sign pattern or the positions of zero and nonzero entries.

The techniques employed in the thesis are mostly classical, involving the theory of determinants as well as polynomials and field extensions. Although none of the proofs ultimately rely on computational approaches, the computer algebra system Maple has been an extremely valuable tool in the development and verification of the constructions that are presented in this work.

### 1.1 Nonsingular vector spaces of matrices

Let \( F \) denote an arbitrary field, \( M_{m \times n}(F) \) the space of \( m \times n \) matrices over \( F \), \( I_n \) the identity matrix of size \( n \times n \) and \( \mathbb{F}^n \) the space of column vectors of length \( n \). We also write \( M_n(F) \) for \( M_{n \times n}(F) \). In this section, we consider linear subspaces of \( M_n(F) \) in which every non-zero element is nonsingular and show how they are related to presemields. We also present the maximum possible dimension of such a space over certain fields, which will be needed for reading Chapter 4.

**Definition 1.1.1.** Let \( V \) be a vector subspace of \( M_n(F) \). We say that \( V \) is a **nonsingular vector space** over \( F \) (or an \( F \)-nonsingular vector space) if every non-zero matrix in \( V \) is nonsingular.

We note that if \( F \) is an algebraically closed field (for example, \( \mathbb{C} \)) and \( A, B \) are nonsingular \( n \times n \) matrices then \( \det(A + \lambda B) \) is a polynomial
of degree $n$, which has at least one root in $\mathbb{F}$. Therefore, if $\mathbb{F}$ is an algebraically closed field then the dimension of any nonsingular vector subspace of $M_n(\mathbb{F})$ is just 1. Hence, any nonsingular vector subspace of $n \times n$ matrices over an algebraically closed field is spanned by one nonsingular matrix. We will give some examples of nonsingular vector subspaces of dimension greater than 1. The second example below will show us how nonsingular vector subspaces relate to nonsingular entry pattern matrices defined in later chapters.

**Example 1.1.1.** Let $\mathcal{V}$ be the 2-dimensional $\mathbb{R}$-vector space spanned by $I_2$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then

$$\mathcal{V} = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$ 

Since $\det \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = x^2 + y^2$, we easily see that the only singular matrix in $\mathcal{V}$ is the zero matrix, which proves that $\mathcal{V}$ is an $\mathbb{R}$-nonsingular vector subspace of $M_2(\mathbb{R})$ of dimension 2.

**Example 1.1.2.** Let

$$A(x, y, z) = \begin{bmatrix} x & y & x & z & z & x \\ y & x & y & x & z & z \\ x & z & x & z & y & y \\ z & y & x & x & y & y \\ z & z & y & y & z & z \\ x & z & y & y & z & z \end{bmatrix},$$

and

$$\mathcal{V} = \{A(x, y, 0) : x, y \in \mathbb{R}\},$$
$$\mathcal{V}' = \{A(x, y, z) : x, y, z \in \mathbb{R}\}.$$ 

Since $\det A(x, y, 0) = -\frac{1}{5} ((x - y)^2 + y^2)^3$, the only singular matrix in $\mathcal{V}$ is the zero matrix. In other words, $\mathcal{V}$ is an $\mathbb{R}$-nonsingular vector subspace of $M_6(\mathbb{R})$.

We note that $\mathcal{V}$ is a subspace of $\mathcal{V}'$. Moreover, since $\det A(x, y, z) = -\frac{1}{5} ((x - y)^2 + (y - z)^2 + (z - x)^2)^3$, every matrix in $\mathcal{V}'$ has rank either 0 (if it is the zero matrix) or 1 (if it is a scalar multiple of the $6 \times 6$ all-one matrix $J_6$) or 6 (otherwise). Hence, any two-dimensional subspace of $\mathcal{V}'$ which does not contain $J_6$ is $\mathbb{R}$-nonsingular.

Next, we will investigate upper bounds for the dimension of nonsingular vector subspaces of $M_n(\mathbb{F})$.

**Lemma 1.1.1.** Let $\mathbb{F}$ be an arbitrary field and let $\mathcal{V}$ be a nonsingular vector subspace of $M_n(\mathbb{F})$. Then $\dim \mathcal{V} \leq n$.

**Proof.** Let $\mathcal{V}^c$ be the space of $n \times n$ matrices whose last row is zero. Then $\dim \mathcal{V}^c = n(n - 1)$ and every matrix of $\mathcal{V}^c$ has rank at most $n - 1$. Hence, $\mathcal{V}^c \cap \mathcal{V} = \{O_{n \times n}\}$ where $O_{n \times n}$ is the $n \times n$ zero matrix. Since $\dim \mathcal{V}^c + \dim \mathcal{V} \leq n^2$, $\dim \mathcal{V} \leq n$. 

$\square$
Generally, the inequality \( \dim V \leq n \) above is not sharp for every \( n \) and for every field \( F \). We will prove that the existence of an \( n \)-dimensional nonsingular subspace of \( M_n(F) \) is equivalent to the existence of an \( n \)-dimensional presemield \( S \) over \( F \).

First of all, let \( V \subseteq M_n(F) \) be an \( n \)-dimensional nonsingular vector space. Then there is an \( F \)-linear isomorphism \( \sigma : F^n \to V \). Via this isomorphism, we may define an operation \( \circ \) on \( F^n \) by \( u \circ v := \sigma(u)v \) for every \( u, v \in F^n \). Since every non-zero matrix of \( V \) is nonsingular and \( \sigma \) is an isomorphism, the operation \( \circ \) satisfies the following properties

\[ (S1) \quad u \circ (v + w) = u \circ v + u \circ w \quad \text{and} \quad (u + v) \circ w = u \circ w + v \circ w, \quad \text{for all} \quad u, v, w \in F^n, \]

\[ (S2) \quad u \circ v = 0 \quad \text{implies that either} \quad u = 0 \quad \text{or} \quad v = 0, \quad \text{and} \]

\[ (S3) \quad \text{for all} \quad u, v \in F^n \setminus \{0\}, \quad \text{there exist unique} \quad u', v' \in F^n \quad \text{such that} \quad u \circ u' = v \quad \text{and} \quad v' \circ u = v. \]

These properties lead us to the definition of presemield as follows

**Definition 1.1.2.** A presemield \( (S, +, \circ) \) is a set with two operations \( + \) (addition) and \( \circ \) (multiplication) so that \( (S, +) \) is an abelian group with identity 0 and \( (S, +, \circ) \) satisfies \( (S1) \), \( (S2) \) and \( (S3) \).

Therefore, if there exists an \( n \)-dimensional nonsingular vector subspace of \( M_n(F) \) then there exists a presemield \( S = F^n \) which is an \( n \)-dimensional algebra over \( F \).

Conversely, if there exists a presemield \( (S, +, \circ) \) which is an \( n \)-dimensional algebra over \( F \), then \( S \) has an \( F \)-basis of \( n \) vectors, say \( \{v_1, v_2, \ldots, v_n\} \). For each element \( u \in S \), we write \( L_u \) for the left regular representation on \( S \) defined by \( L_u(v) = u \circ v \). Then it is easy to see that \( L_u \) is a linear isomorphism on \( S \) provided that \( u \neq 0 \). Let \( M_u \) denote the representation matrix of \( L_u \) with respect to the basis \( \{v_1, \ldots, v_n\} \). Then \( M_u := \{M_u : u \in S\} \) is an \( F \)-nonsingular linear subspace of \( M_n(F) \) which has a basis \( \{M_{v_1}, M_{v_2}, \ldots, M_{v_n}\} \).

**Theorem 1.1.2.** There exists an \( n \)-dimensional nonsingular vector subspace \( V \) of \( M_n(F) \) if and only if there exists a presemield \( S \) which is an \( n \)-dimensional algebra over \( F \).

In particular, if a field \( F \) admits an extension field of degree \( n \), then there is an \( n \)-dimensional nonsingular vector subspace of \( M_n(F) \).

We note that a presemield is called a semifield if the operation \( \circ \) has a unit, i.e.,

\[ (S4) \quad \text{there exists} \quad 1 \in S \quad \text{such that} \quad 1 \circ u = u \circ 1 = u \quad \text{for all} \quad u \in S. \]

A semifield is a skewfield (also known as a division ring) if the operator \( \circ \) is associative, i.e.,

\[ (S5) \quad u \circ (v \circ w) = (u \circ v) \circ w \quad \text{for all} \quad u, v, w \in S. \]
And a skewfield is a field if the operation \( \circ \) is commutative, i.e.,

\[
(S6) \quad u \circ v = v \circ u \quad \text{for all } u, v \in S.
\]

**Example 1.1.3.** The only presemifields over \( \mathbb{R} \) have dimension 1 (for example, the real field \( \mathbb{R} \)), 2 (for example, the complex field \( \mathbb{C} \)), 4 (for example, the quaternion division algebra \( \mathbb{H} \)) or 8 (for example, the octonion semifield \( \mathbb{O} \)) [7]. Hence, there exists an \( n \)-dimensional nonsingular vector subspace of \( M_n(\mathbb{R}) \) if and only if \( n \in \{1, 2, 4, 8\} \). And such vector spaces are spanned by the matrices representing the left multiplication by basis elements in these semifields. For example, we know that the quaternion division algebra \( \mathbb{H} \) has the \( \mathbb{R} \)-basis \( \{1, i, j, k\} \). Let \( L_\alpha : \mathbb{H} \to \mathbb{H} \) denote the left multiplication defined by \( L_\alpha(x) = \alpha x \) for every \( x \in \mathbb{H} \) and let \( M_\alpha \) denote the matrix of \( L_\alpha \) with respect to the basis \( \{1, i, j, k\} \). Then a representation of \( \mathbb{H} \) in \( M_4(\mathbb{R}) \) is given by

\[
\mathbb{H} = \begin{bmatrix}
  a & b & c & d \\
  b & a & -d & c \\
  c & d & a & -b \\
  d & -c & b & a
\end{bmatrix} = aM_1 + bM_i + cM_j + dM_k : a, b, c, d \in \mathbb{R}.
\]

We are interested in finding the maximum possible dimension of an \( \mathbb{R} \)-nonsingular vector subspace of \( M_n(\mathbb{R}) \) for \( n \notin \{1, 2, 4, 8\} \). In [1], J.F. Adams gave the maximum possible dimension of an \( \mathbb{R} \)-nonsingular vector subspace of \( M_n(\mathbb{R}) \) for any integer \( n \). He also gave the maximum possible dimension of such an \( \mathbb{R} \)-nonsingular vector subspace in which every matrix is symmetric. Constructions of those subspaces were given in [2]. We will present briefly their constructions here and note that those spaces are spanned by matrices whose entries are either 0, or \( \pm 1 \). We will revisit this theme in Chapter 4 where we will give constructions of such a nonsingular vector subspace of \( M_n(\mathbb{R}) \) of the same dimension but having a spanning set as \((0, 1)\)-matrices, provided \( n \) has an odd divisor greater than 3.

Let \( \rho(n) \) be the Radon-Hurwitz number defined for a positive integer \( n \) by

\[
\rho(n) = 2^b + 8c,
\]

where \( b \) and \( c \) are the unique integers for which \( 0 \leq b \leq 3 \) and \( 2^{b+4c} \) is the highest power of 2 that divides \( n \). And let \( \rho_\mathbb{R}(n) \) be the maximum possible dimension of a nonsingular vector subspace of \( M_n(\mathbb{F}) \). Then

**Theorem 1.1.3.** [1] Let \( n \) be a positive integer. Then

\[
\rho_\mathbb{R}(n) = \rho(n).
\]

As we discussed in Example 1.1.3, the following representations of \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \) in \( M_1(\mathbb{R}) \), \( M_2(\mathbb{R}) \), \( M_4(\mathbb{R}) \) and \( M_8(\mathbb{R}) \) respectively are \( \mathbb{R} \)-nonsingular vector spaces of dimension \( \rho(1) \), \( \rho(2) \), \( \rho(4) \) and \( \rho(8) \) re-
respectively.

$$R = \{ [x_1] : x_1 \in \mathbb{R} \}, \ C = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\} ,$$

$$O = \begin{bmatrix} x_1 - x_2 - x_3 - x_4 \\ x_2 x_1 - x_4 x_3 \\ x_3 x_4 x_1 - x_2 \\ x_4 - x_3 x_2 x_1 \end{bmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} ,$$

$$H = \begin{bmatrix} x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 \\ x_2 x_1 - x_4 x_3 - x_6 x_5 - x_8 - x_7 \\ x_3 x_4 x_1 - x_2 - x_7 - x_8 x_5 x_6 \\ x_4 - x_3 x_2 x_1 - x_8 x_7 - x_6 x_5 x_6 \\ x_5 x_6 x_7 x_8 x_1 - x_2 - x_3 - x_4 \\ x_6 x_5 x_8 - x_7 x_2 x_1 x_4 - x_3 \\ x_7 - x_8 - x_5 x_6 x_3 - x_4 x_1 x_2 \\ x_8 x_7 - x_6 - x_5 x_4 x_3 - x_2 x_1 \end{bmatrix} : x_1, \ldots, x_8 \in \mathbb{R} .$$

For $n = 2^k = 16 \cdot 2^{k-4}$, where $k \geq 4$, an $\mathbb{R}$-nonsingular vector subspace of $M_n(\mathbb{R})$ of dimension $\rho(n) = 8 + \rho(2^{k-4})$ is constructed by induction as follows. Let $V$ be a $\rho(2^{k-4})$-dimensional nonsingular vector subspace of $M_{2^{k-4}}(\mathbb{R})$. Define

$$V_1 := \left\{ \begin{bmatrix} xI_n & P \\ P^T & -xI_n \end{bmatrix} : P \in V, x \in \mathbb{R} \right\} ,$$

where $A^T$ is the transpose of the matrix $A$. And define

$$V_2 = \{ Q \otimes I_8 + I_{2n} \otimes C : Q \in V_1, C \in \} ,$$

where $\otimes$ is the subspace of $O$ consisting of matrices whose entries on the main diagonals are zero and $\otimes$ is the Kronecker product of two matrices. Recall that if $A = (a_{ij})$ is an $m \times n$ matrix and $B$ is a $p \times q$ matrix, then $A \otimes B$ is the $mp \times nq$ matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} .$$

Then $V_2$ is $\mathbb{R}$-nonsingular subspace of $M_n(\mathbb{R})$ of dimension $\rho(n)$.

Finally, if $n = (2m + 1)2^r$ and $V$ is an $\mathbb{R}$-nonsingular vector subspace of $M_{2^r}(\mathbb{R})$ of dimension $\rho(2^r)$, then

$$V' = \{ \text{diag}(A, \ldots, A) : A \in V \} \subseteq M_n(\mathbb{R})$$

is an $\mathbb{R}$-nonsingular vector space whose dimension is equal to $\dim V$ or $\rho(2^r)$ or $\rho(n)$.

Furthermore, it is shown in [2] that the maximum possible dimension of a nonsingular subspace of $M_n(\mathbb{R})$ consisting of symmetric matrices, which is denoted by $\rho_s^\circ(n)$, is $\rho_s^\circ(n) = \rho \left( \frac{n}{2} \right) + 1$ if $n$ is even and $\rho_s^\circ(n) = 1$. 

elsewhere. If \( n \) is even and \( V \) is an \( \mathbb{R} \)-nonsingular vector subspace of \( M_n(\mathbb{R}) \) then
\[
V_1 := \left\{ \begin{bmatrix} xI_n & P \\ Pt & -xI_n \end{bmatrix} : P \in V, x \in \mathbb{R} \right\}
\]
is an \( \mathbb{R} \)-nonsingular vector subspace of \( M_n(\mathbb{R}) \) which has dimension \( \rho(\frac{n}{2})+1 = \rho_S(n) \).

On the other hand, if \( F \) admits a field extension \( E \) of degree \( n \), then the left multiplication by any element \( \alpha \) of \( E \) defines an element of \( \text{End}_\mathbb{F}(E) \), which is invertible provided that \( \alpha \neq 0 \). The set of all such endomorphisms is a nonsingular space of dimension \( n \), and hence \( \rho_E(n) = n \). Thus \( \rho_E(n) = n \) for all \( n \) if \( F \) admits field extensions of all degrees. The following theorem summarizes our knowledge of the function \( \rho_F \).

**Theorem 1.1.4.** Let \( n \) be a positive integer and \( F \) be a field. Then

1. If \( F \) is algebraically closed, then \( \rho_F(n) = 1 \) for all \( n \).
2. If \( F \) admits a field extension of degree \( n \), then \( \rho_F(n) = n \) for all \( n \). Classes of fields having this property for every \( n \) include finite fields and finite extensions of \( \mathbb{Q} \).
3. \( \rho_\mathbb{R}(n) = \rho(n) \).
4. \( \rho_\mathbb{S}(n) = \rho(\frac{n}{2}) \) if \( n \) is even and \( \rho_\mathbb{S}(n) = 1 \) elsewhere.

### 1.2 Nilpotent vector spaces of matrices

A square matrix \( A \in M_n(F) \) is said to be nilpotent if there exists an integer \( k \) so that \( A^k \) is equal to the zero matrix. The least such integer \( k \) is called the nilpotent index of \( A \). In 1958, Gerstenhaber studied the linear spaces and algebras of nilpotent matrices. He gave an upper bound for the dimension of a vector space of \( n \times n \) matrices in which every matrix was nilpotent and also proved that any space which attains the bound is similar to the space of strictly upper triangular matrices, provided that the field \( F \) has at least \( n \) elements [16].

We first give some definitions and terminology.

**Definition 1.2.1.** A vector space \( V \subseteq M_n(F) \) is said to be nilpotent (or \( F \)-nilpotent) if every matrix in \( V \) is nilpotent. Furthermore, \( V \) is said to be nilpotent of index \( r \) if \( r \) is the greatest nilpotent index of elements of \( V \).

**Example 1.2.1.** Let \( F \) be a field and let \( SU_n(F) \subseteq M_n(F) \) be the space of strictly upper triangular matrices of size \( n \times n \). Then \( SU_n(F) \) is nilpotent of index \( n - 1 \).

**Example 1.2.2.** Let

\[
A(x,y,z,r,s,t) := \begin{bmatrix}
  x & x & y \\
  x & z & z \\
  s & r & t \\
  s & r & t 
\end{bmatrix},
\]

where
and let
\[ V = \{ A(x, y, z, r, s, t) : x, y, z, r, s, t \in F_2 \}. \]

Then \( V \) is a 6-dimensional linear vector subspace of \( M_4(F_2) \). Moreover,
\[
A(x, y, z, r, s, t)^3 \mod 2 = x(y + z)(r + s) \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 
\end{bmatrix}.
\]

Therefore, every matrix in \( V \) is nilpotent and \( A(1, 1, 0, 1, 0, 0) \) is nilpotent of index 4. Therefore \( V \) is \( F_2 \)-nilpotent of index 4.

**Theorem 1.2.1.** [16] If \( F \) has at least \( n \) elements then the maximal dimension of a nilpotent subspace of \( M_n(F) \) is \( \frac{n(n-1)}{2} \). Furthermore, any \( \frac{n(n-1)}{2} \)-dimensional nilpotent subspace of \( M_n(F) \) is similar to the strictly upper triangular space \( SU_n(F) \).

In 1985, Serezhkin proved that the theorem still holds for every field \( F \) [29]. Alternative proofs of this result have been given in [22, 21, 8]. In 1993, Brualdi and Chavez gave an upper bound for the dimension of a nilpotent vector subspace of \( M_n(F) \) in which the rank of every matrix is bounded above. They also considered the case where the nilpotent index of every matrix is bounded above as well. Their results required that the field \( F \) was sufficiently large. In 2010, Macdonald, Macdougall and Sweet gave the maximum dimension of a nilpotent vector subspace of \( M_n(F) \) in which the nilpotent index and the rank of every matrix are bounded above, provided that the field \( F \) has more than \( n \) elements.

**Theorem 1.2.2.** [21] Suppose \( \mathcal{V} \subseteq M_n(F) \) is an \( F \)-nilpotent vector space of nilpotent index \( k \) in which the rank of every matrix is at most \( r \). If \( \text{card}(F) > n \), setting \( q = \left\lceil \frac{r}{k-1} \right\rceil \), then
\[
\dim \mathcal{V} \leq nr - \frac{r^2}{2} - \frac{r}{2} + \frac{q^2}{2}(k - 1) + \frac{q}{2}(-2r + k - 1).
\]

In the case of nilpotent index 2, i.e. \( k = 2 \), we have
\[
\dim \mathcal{V} \leq r(n - r).
\]

In [32], this result was proven to still hold if we reduce the condition of the cardinality of \( F \) to be greater than \( \frac{n}{2} \). The authors in [32] also classified all nilpotent spaces which attain the maximum dimension as presented below.

**Theorem 1.2.3.** Let \( \mathcal{V} \subseteq M_n(F) \) be an \( F \)-nilpotent vector space of index 2 where \( |F| > \frac{n}{2} \). If every matrix in \( \mathcal{V} \) has rank at most \( r \), then
\[
\dim \mathcal{V} \leq r(n - r).
\]
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Furthermore, if \( \dim \mathbb{V} = r(n-r) \), then \( \mathbb{V} \) is similar to one of the following spaces.

\[
\mathbb{V}_1 = \left\{ \begin{bmatrix} O & A & B \\ O & O & O \\ O & O & O \end{bmatrix} : A \in M_{r \times (n-2r)}(\mathbb{F}), B \in M_{r \times r}(\mathbb{F}) \right\},
\]

\[
\mathbb{V}_2 = \left\{ \begin{bmatrix} O & O & B \\ O & O & A \\ O & O & O \end{bmatrix} : A \in M_{(n-2r) \times r}(\mathbb{F}), B \in M_{r \times r}(\mathbb{F}) \right\},
\]

\[
\mathbb{V}_3 = \left\{ \begin{bmatrix} O & A \\ O & O \end{bmatrix} : A \in M_2(\mathbb{F}), n \text{ is even} \right\},
\]

where \( O \) is the zero block of appropriate size.

In Chapter 5, we will compare these results with the maximum possible number of indeterminates of an \( n \times n \) \( \mathbb{F} \)-nilpotent entry pattern matrix where \( \mathbb{F} \) is a finite field.
Chapter 2

Entry Pattern Matrices

In this chapter, we introduce the class of entry pattern matrices and discuss some of their features. An entry pattern matrix (EPM) is a rectangular matrix in which each entry is an element of a specified set of independent indeterminates. A completion of an entry pattern matrix $A$ over a field $\mathbb{F}$ is the matrix that results from assigning a value from $\mathbb{F}$ to each indeterminate that appears as an entry of $A$. The set of all $\mathbb{F}$-completions of $A$ is a vector space over $\mathbb{F}$, whose dimension is equal to the number of distinct indeterminates appearing in $A$.

The concept of an entry pattern matrix was introduced by Huang and Zhan in 2015 [19]. These authors note that several important classes of matrices, including symmetric, circulant, Toeplitz and Hankel matrices, have defining properties that can be expressed in terms of entry patterns. They investigate the particular case of entry pattern matrices whose real completions all have real spectrum, and they propose a general study of matrix-theoretic properties of entry patterns and their completions. This introduces a new focus to the subject of matrices with indeterminate entries. Other classes of such objects that have attracted recent attention include partial matrices [23] and affine column independent (ACI) matrices [6, 9].

2.1 Definitions and examples

Given a finite set $S = \{x_1, x_2, \ldots, x_k\}$, we denote by $M_{m \times n}(x_1, \ldots, x_k)$, or $M_{m \times n}(S)$ the set of $m \times n$ matrices whose entries are from $S$.

**Definition 2.1.1.** [19] An entry pattern matrix (EPM) is a matrix in which every entry is an indeterminate. Distinct indeterminates in an EPM are independent, and the same indeterminate may appear in multiple positions.

We may write an entry pattern matrix $A$ as $A(x_1, \ldots, x_k)$ if $x_1, \ldots, x_k$ are the indeterminates appearing in $A$. An EPM is said to have $k$ indeterminates if there are exactly $k$ indeterminates appearing in $A$. For a square entry pattern $A$ of size $n \times n$, we denote by $\det_{\mathbb{F}}(A)$ the determinant of $A$ where the coefficients are considered as elements of the field $\mathbb{F}$. 


For abbreviation, we let \( \det(A) \) stand for the determinant of \( A \) where the coefficients are considered in characteristic zero. For any field \( \mathbb{F} \), \( \det_\mathbb{F}(A) \) is either the zero polynomial or a homogeneous polynomial of degree \( n \) with coefficients in the prime field of \( \mathbb{F} \).

**Example 2.1.1.** An \( m \times n \) EPM in which each indeterminate appears exactly in one position has \( mn \) indeterminates. We refer to such an EPM as a free \( m \times n \) EPM.

An \( n \times n \) EPM is called a symmetric EPM if the \((i,j)\)-entry and \((j,i)\)-entry are the same indeterminate for every \( i, j \). An \( n \times n \) symmetric EPM which has exactly \( \frac{n(n+1)}{2} \) indeterminates is said to be a free symmetric EPM.

Similarly, an \( n \times n \) circulant EPM is an \( n \times n \) EPM in which the indeterminate appearing in the \((i,j)\)-position also appears in \((i+1 \mod n, j+1 \mod n)\)-position for every \( i, j \). A free \( n \times n \) circulant EPM is an \( n \times n \) circulant EPM with \( n \) indeterminates.

**Example 2.1.2.** Let

\[
A = \begin{bmatrix} x & y & z \\ y & z & y \end{bmatrix}, \quad B = \begin{bmatrix} x & x + y \\ 0 & y \end{bmatrix}.
\]

Then \( A \) is an entry pattern matrix with three indeterminates \( x, y, z \), or \( A \in M_{2 \times 3}(x,y,z) \), while \( B \) is not an EPM since \( B \) has an entry 0 which is not an indeterminate, and \( x + y \) which is not independent of the two indeterminates \( x, y \).

**Definition 2.1.2.** Let \( A(x_1, x_2, \ldots, x_k) \in M_{m \times n}(S) \) be an entry pattern matrix and let \( \mathbb{F} \) be a field. Then an \( \mathbb{F}\)-completion of \( A \) is an \( m \times n \) matrix over \( \mathbb{F} \) which is obtained by specifying the values of the indeterminates in \( A \) as elements in the field \( \mathbb{F} \). The \( \mathbb{F} \)-completion which results from setting \( x_1 = a_1, \ldots, x_k = a_k \) is written as \( A(a_1, \ldots, a_k) \).

**Example 2.1.3.** Let

\[
A(x, y) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}.
\]

Then \( I_2 \) is an \( \mathbb{F} \)-completion of \( A \) since \( I_2 = A(1, 0) \).

Let \( A(x_1, \ldots, x_k) \) be an \( m \times n \) EPM and let \( \mathbb{F} \) be a field. The completions of \( A \) obtained by assigning the same value from \( \mathbb{F} \) to all the indeterminates are called uniform \( \mathbb{F} \)-completions of \( A \). It is obvious that the rank of a uniform completion of \( A \) is either 0 or 1. The set of all \( \mathbb{F} \)-completions of an entry pattern matrix \( A \) is a linear subspace of \( M_{m \times n}(\mathbb{F}) \) which has dimension \( k \). We will call this space the \( \mathbb{F} \)-pattern class of \( A \) and denote it by \( C_\mathbb{F}(A) \). The completion matrix obtained by assigning the value 1 to \( x_i \) and 0 to \( x_j \) for \( j \neq i \) is said to be the coefficient matrix of \( A \) with respect to the indeterminate \( x_i \). It is easy to see that

- every coefficient matrix is a \((0,1)\)-matrix,

- no two of the coefficient matrices have a non-zero entry at the same position,
and the sum of all coefficient matrices of any entry pattern matrix is the all-one matrix (of the same size with the EPM).

Therefore, the coefficient matrices are linearly independent over \( \mathbb{F} \) and so they comprise an \( \mathbb{F} \)-basis of \( C_{\mathbb{F}}(A) \). Thus,

\[
\dim_\mathbb{F} C_{\mathbb{F}}(A(x_1, \ldots, x_k)) = k.
\]

Moreover, by assigning the value 1 to all indeterminates in \( A \), we easily see that the \( m \times n \) all-one matrix \( J_{m \times n} \) belongs to \( C_{\mathbb{F}}(A) \) for all fields \( \mathbb{F} \).

The set of non-negative integers occurring as ranks of \( \mathbb{F} \)-completions of an entry pattern matrix \( A \) is denoted by \( r_{\mathbb{F}}(A) \). We have

\[
J_{m \times n} \in C_{\mathbb{F}}(A) \Rightarrow \{0, 1\} \subseteq r_{\mathbb{F}}(A).
\]

**Lemma 2.1.1.** Let \( A \) be an EPM and let \( \mathbb{F} \) be a field. Then \( r_{\mathbb{F}}(A) = \{0, 1\} \) if and only if the rows of \( A \) are all equal or the columns of \( A \) are.

**Proof.** If the columns of \( A \) are not all equal then there is a row, say \( R_i \), which contains at least two distinct indeterminates among its entries. Furthermore, if the rows of \( A \) are not all equal then there is another row, say \( R_j \), of \( A \) which is not equal to \( R_i \). Hence, there is a \( 2 \times 2 \) submatrix \( B \) of \( A \) so that \((B)_{11} \neq (B)_{12}\) and the second row of \( B \) is not equal to the first row of \( B \). Without loss of generality, assume \( B \) has the form

\[
B = \begin{bmatrix}
x & y \\
* & 
\end{bmatrix},
\]

where either \((B)_{21} \neq x\) or \((B)_{22} \neq y\). If \((B)_{21} \neq x\), then by assigning the value 0 to \( x \), and the value 1 to \( y \) and to \((B)_{21}\) (if \((B)_{21} \neq y\)), we obtain a nonsingular completion of \( B \). Similarly, there is a nonsingular completion of \( B \) if \((B)_{22} \neq y\). Therefore, if the rows of \( A \) are not all equal and so are the columns of \( A \) then \( A \) has a completion of rank at least 2. \( \square \)

An immediate consequence of Lemma 2.1.1 is that if \( A \) is \( m \times n \) and \( r_{\mathbb{F}}(A) = \{0, 1\} \) then the maximum number of indeterminates in \( A \) is \( \max\{m, n\} \).

**Example 2.1.4.** Let

\[
A = \begin{bmatrix}
x & z & z & y \\
z & x & y & z \\
z & y & x & y \\
y & z & y & x 
\end{bmatrix}.
\]

Then the \( \mathbb{F}_2 \)-pattern class of \( A \) is

\[
\{O, I_4, J_4, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \}.
\]

By simple calculations, we may see easily that

\[
r_{\mathbb{F}_2}(A) = \{0, 1, 4\}.
\]
Example 2.1.5. Let $A$ be a free symmetric entry pattern matrix of size $n \times n$. Then the $F$-pattern class of $A$ is the space of all symmetric matrices of size $n \times n$ with entries in $F$.

Remark. Let $A(x_1, \ldots, x_k)$ be an $m \times n$ entry pattern matrix and let $F$ be a field. Then $A$ can be considered as a matrix with entries in $F = F(x_1, \ldots, x_k)$, the field of rational functions in the variables $x_1, \ldots, x_k$ over $F$. Hence, it is possible to consider $A$ as a usual matrix over $F(x_1, \ldots, x_k)$. We say that $A$ is nonsingular over $F(x_1, \ldots, x_k)$, or $F$-nonsingular, if its determinant is not the zero polynomial over $F$. In Chapter 3, the rank of $A$ when considered as a matrix over $F(x_1, \ldots, x_k)$ will be called the generic $F$-rank of $A$. It is easy to see that the rank of any $F$-completion of $A$ cannot exceed the generic $F$-rank of $A$. In Chapter 3 we will investigate for which fields the generic rank can be attained by a completion. The case when $A$ is an entry pattern matrix with exactly one indeterminate is identical since every $F$-completion of $A$ is a scalar multiple of the all-one matrix $J$. In the next chapters, we will see that entry pattern matrices with exactly two indeterminates are much more interesting. For example, in Chapter 3, we will prove that the maximum rank of any $F$-completion of any entry pattern matrix with two indeterminates is equal to its generic $F$-rank for any field $F$. And in Chapter 4, we will prove the existence of a series of square entry pattern matrices of all order $\geq 4$ with 2 indeterminates such that any matrix in the $F$-pattern class is singular only if it is a scalar multiple of $J$ for any field $F$. Throughout this thesis, for a given field $F$ and two entry pattern matrices $A$ of size $m \times n$ and $B$ of size $n \times q$, the product $A \cdot B$ (or $AB$) is defined as the usual product of two matrices $A$ and $B$ over the field of rational functions over $F$ generated by the indeterminates appearing in $A$ and $B$.

Definition 2.1.3. Let $A$ and $B$ be two entry pattern matrices of the same size. Then we say that $A$ and $B$ are pattern equivalent (written $A \equiv B$) if their $F$-pattern classes are equal for every field $F$.

It is easy to see that the relation "pattern equivalent" is an equivalence relation. Furthermore, recall that the number of indeterminates in two EPMs $A$ and $B$ are the dimension of the $F$-pattern classes of $A$ and $B$, respectively. Therefore, if $A \equiv B$ as entry pattern matrices, then $A$ and $B$ have the same number of indeterminates. Suppose that $A(x_1, \ldots, x_k)$ and $B(y_1, \ldots, y_k)$ are two entry pattern matrices of the same size such that $A$ is obtained from $B$ by firstly relabelling the indeterminates appearing in $A$ from $\{x_1, \ldots, x_k\}$ to $\{y_1, \ldots, y_k\}$ and secondly by permuting the indeterminates (if necessary). That is, there exists a bijective function $f : \{x_1, \ldots, x_k\} \to \{y_1, \ldots, y_k\}$ so that the $(i,j)$-entry of $A(f(x_1), \ldots, f(x_k))$ is the same as the $(i,j)$-entry of $B$. Then the coefficient matrix of $A$ with respect to $x_i$ is equal to the coefficient matrix of $B$ with respect to $f(x_i)$ for every $i$. It follows that $A \equiv B$.

Moreover, suppose that $A(x_1, \ldots, x_k) \equiv B(x_1, \ldots, x_k)$. For $i = 1, \ldots, k$, write $A_i$ and $B_i$ for the coefficient matrices of $A$ and $B$ with
2.2. The number of pattern equivalence classes of $m \times n$ EPMs

respect to $x_i$, respectively. Then for every field $F$,
\[
\langle A_1, \ldots, A_k \rangle_F = \langle B_1, \ldots, B_k \rangle_F,
\]
where $\langle A_1, \ldots, A_k \rangle_F$ and $\langle B_1, \ldots, B_k \rangle_F$ are the $F$-vector spaces generated over $F$ by $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$, respectively.

It follows that $A_i$ is a linear combination of $B_1, \ldots, B_k$ for every $i$. Moreover, recall that

- every coefficient matrix is a $(0,1)$-matrix,
- and no two of the coefficient matrices of an EPM have non-zero entry at the same position.

Therefore, $A_i = \sum \lambda_{ij} B_j$ where $\lambda_{ij} \in \{0,1\}$ are not all zero for each $1 \leq i \leq k$. Furthermore, since no two of the matrices $\{A_1, \ldots, A_k\}$ have non-zero entry at the same position, either $\lambda_{ij} = 0$ or $\lambda_{ij} = 0$ for any two different integers $i \neq j$ and for any integer $j$. Therefore, if there exist integers $i$ and $j \neq j'$ so that $\lambda_{ij} = \lambda_{ij'} = 1$ then both $B_j$ and $B_{j'}$ are not contained in $\langle A_1, \ldots, A_k \rangle$. It follows that $A_i \in \{B_1, \ldots, B_k\}$ for every $i$, or $\{A_1, \ldots, A_k\} = \{B_1, \ldots, B_k\}$. This proves the following lemma.

**Lemma 2.1.2.** Let $A$ and $B$ be two entry pattern matrices of the same size. Then $A \equiv B$ if and only if they have the same number of indeterminates and $A$ can be obtained from $B$ by labelling indeterminates.

This lemma shows that the definition of pattern equivalence of entry pattern matrices can be expressed in terms of entry patterns. And therefore, two entry pattern matrices are pattern equivalent if their $F$-pattern classes are equal for some field $F$.

**Example 2.1.6.** The following entry pattern matrices are pattern equivalent.
\[
A = \begin{bmatrix} x & y \\ x & x \end{bmatrix}, B = \begin{bmatrix} z & t \\ z & z \end{bmatrix}, C = \begin{bmatrix} y & x \\ y & y \end{bmatrix}.
\]

2.2 The number of pattern equivalence classes of $m \times n$ EPMs

In this section, we will enumerate the pattern equivalence classes of $m \times n$ entry pattern matrices with $k$ indeterminates $x_1, \ldots, x_k$. Since this number depends only on the integers $m, n, k$, we will denote it by $\Delta_{m,n}(k)$.

Let $a_1, a_2, \ldots, a_k$ be positive integers such that
\[
\begin{align*}
\{ a_1 + \cdots + a_k &= mn, \\
\{ a_i &\geq 1, \text{ for all } i. 
\end{align*}
\]

We firstly note that the number of elements of $M_{m \times n}(x_1, \ldots, x_k)$ where the indeterminate $x_i$ appears in exactly $a_i$ positions for every $i$ is equal to the multinomial coefficient $\binom{mn}{a_1, a_2, \ldots, a_k}$, or $\frac{(mn)!}{a_1!a_2!\cdots a_k!}$. Therefore, the
number of \( m \times n \) entry pattern matrices with \( k \) indeterminates, which is denoted by \( \delta_{m,n}(k) \), is

\[
\delta_{m,n}(k) = \sum_{a_1 + \cdots + a_k = mn} \frac{(mn)!}{a_1!a_2! \cdots a_k!},
\]

where \( a_i \geq 1 \forall i \).

By Lemma 2.1.2, we have \( \Delta_{m,n}(k) = \frac{1}{k!} \delta_{m,n}(k) \).

Moreover, for each integer \( i : 0 \leq i \leq k \), let \( S_i \) be the set of vectors \( (b_1, \ldots, b_k) \in \mathbb{N}^k \) which have exactly \( i \) zero entries and \( b_1 + b_2 + \cdots + b_k = mn \). Obviously, \( S_k = \emptyset \), \( S_0 = \{(b_1, \ldots, b_k) \in \mathbb{N}^k : b_1 + \cdots + b_k = mn, b_i \geq 1 \forall i\} \) and the sets \( S_1, S_2, \ldots, S_k \) are pairwise disjoint. Therefore,

\[
S_0 = S \setminus \bigcup_{1 \leq i \leq k} S_i,
\]

where \( S = \{(b_1, \ldots, b_k) \in \mathbb{N}^k : b_1 + \cdots + b_k = mn\} \).

Since

\[
\delta_{m,n}(k) = \sum_{(a_1, \ldots, a_k) \in S_0} \frac{(mn)!}{a_1!a_2! \cdots a_k!},
\]

it follows that

\[
\delta_{m,n}(k) = \sum_{(a_1, \ldots, a_k) \in S} \frac{(mn)!}{a_1!a_2! \cdots a_k!} - \sum_{i=1}^{k-1} \sum_{(a_1, \ldots, a_k) \in S_i} \frac{(mn)!}{a_1!a_2! \cdots a_k!}.
\]

The multinomial formula shows that

\[
\sum_{(a_1, \ldots, a_k) \in S} \frac{(mn)!}{a_1!a_2! \cdots a_k!} = k^{mn}.
\]

On the other hand, for each \( i : 1 \leq i \leq k - 1 \), \( S_i \) is the disjoint union of \( \binom{k}{i} \) subsets of \( S_i \) in which each subset corresponds to a way of choosing \( i \) entries of \( (a_1, \ldots, a_k) \) to be zero. By changing the indices if necessary, we see that \( \delta_{m,n}(k-i) = \sum_{a_1a_2 \cdots a_k} \frac{(mn)!}{a_1!a_2! \cdots a_k!} \) where the sum ranges over all elements of any subset above.

This shows that

\[
\delta_{m,n}(k) = k^{mn} - \sum_{i=0}^{k-1} \binom{k}{k-i} \delta_{m,n}(i) = k^{mn} - \sum_{i=1}^{k-1} \binom{k}{i} \delta_{m,n}(i).
\]

**Theorem 2.2.1.** Let \( m, n, k \) be positive integers. Then the number of pattern equivalence classes of \( m \times n \) entry pattern matrices with \( k \) indeterminates is

\[
\Delta_{m,n}(k) = \frac{1}{k!} \delta_{m,n}(k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^{mn}. \tag{2.1}
\]

**Proof.** We fix \( m, n \) now and prove (2.1) by induction on \( k \). It is obvious that (2.1) holds for \( k = 1 \). Assume that it holds for any positive integer up to \( k \). Or

\[
\delta_{m,n}(i) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} (i-j)^{mn} \quad \forall i \leq k.
\]
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Recall that $\delta_{m,n}(k)$ satisfies the following recurrence formula

$$\delta_{m,n}(k + 1) = (k + 1)^{mn} - \sum_{i=1}^{k} \binom{k + 1}{i} \delta_{m,n}(i).$$

Therefore,

$$\delta_{m,n}(k + 1) = (k + 1)^{mn} - \sum_{i=1}^{k} \binom{k + 1}{i} \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} (i - j)^{mn}.$$

Let $l$ be an integer. Then for each integer $i$, there exists unique $j$ so that $i - j = k + 1 - l$. Therefore,

$$\delta_{m,n}(k + 1) = (k + 1)^{mn} - \sum_{i=1}^{k+1} \sum_{i=k+1-l}^{k} \binom{k + 1}{i} (-1)^{i+l-1} \binom{i}{i} (k+1-l)^{mn}.$$

Moreover,

$$\sum_{i=k+1-l}^{k} \binom{k + 1}{i} (-1)^{i+l-1} \binom{i}{i} = \frac{(k + 1)!}{(k - l + 1)! \sum_{j=0}^{i} (-1)^{j} \frac{1}{j!(l-j)!}} = (-1)^{l} \frac{(k + 1)!}{(k - l + 1)! \cdot l!} = (-1)^{l} \binom{k + 1}{l}.$$

Therefore,

$$\delta_{m,n}(k + 1) = (k + 1)^{mn} - \sum_{l=1}^{k+1} (-1)^{l} \binom{k + 1}{l} (k+1-l)^{mn}.$$ 

Thus

$$\delta_{m,n}(k + 1) = \sum_{i=0}^{k+1} (-1)^{i} \binom{k + 1}{i} (k + 1 - i)^{mn}.$$

This completes the proof.

**Corollary 2.2.2.** Let $S$ be a set of $k$ independent indeterminates. Then the number of pattern equivalence classes of EPMs of dimension $m \times n$ is

$$|M_{m \times n}(S)| = \left| \sum_{i=1}^{k} \Delta_{m,n}(i) \right| = \sum_{i=1}^{k} \frac{1}{i!} \sum_{l=0}^{i} (-1)^{l} \binom{l}{i} (i - l)^{mn}.$$

**Proof.** We know that $\binom{k}{l}$ is the number of ways to choose $l$ indeterminates from the set of $k$ indeterminates. Since there are exactly $\Delta_{m,n}(l)$ entry pattern matrices with $l$ indeterminates,

$$|M_{m \times n}(S)| = \left| \sum_{i=1}^{k} \frac{1}{i!} \sum_{l=0}^{i} (-1)^{l} \binom{l}{i} (l - i)^{mn} \right|.$$
Example 2.2.1. \(\delta_{2,2}(2) = 2^4 - 2 = 14\). Hence, there are exactly \(\Delta_{2,2}(2) = 7\) pattern equivalence classes of \(2 \times 2\) EPMs which have two indeterminates \(x, y\) as listed below.

\[
\begin{bmatrix} y & x \\ x & y \end{bmatrix} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \quad \begin{bmatrix} y & x \\ x & x \end{bmatrix} = \begin{bmatrix} y & y \\ y & y \end{bmatrix}, \quad \begin{bmatrix} x & y \\ x & x \end{bmatrix} = \begin{bmatrix} x & y \\ y & x \end{bmatrix},
\]

\[
\begin{bmatrix} x & x \\ y & y \end{bmatrix}, \quad \begin{bmatrix} y & x \\ x & x \end{bmatrix} = \begin{bmatrix} y & y \\ y & x \end{bmatrix}, \quad \begin{bmatrix} y & x \\ x & y \end{bmatrix} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}.
\]

2.3 Normal entry pattern matrices

This section presents the interesting class of normal entry pattern matrices, which is investigated by Huang and Zhan in \([19]\). Recall that a real matrix \(A\) is said to be normal if \(AA^T = A^T A\) where \(A^T\) is the transpose of \(A\).

**Definition 2.3.1.** A square entry pattern matrix \(A\) is said to be *normal* if every matrix in \(C_2(A)\) is normal.

Since every real symmetric matrix is normal, we easily see that every symmetric entry pattern matrix is normal. However, there exist non-symmetric entry pattern matrices which are still normal.

**Example 2.3.1.** Let

\[
A = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}.
\]

Then \(AA^T = A^T A\). Hence \(A\) is normal.

It is trivial to see that any entry pattern matrix with 1 indeterminate is normal. Moreover, if \(A(x_1, x_2)\) is an entry pattern matrix then

\[
A(x_1, x_2) = (x_1 - x_2)A_1 + x_2J,
\]

where \(A_1\) is the coefficient matrix of \(A\) with respect to \(x_1\), i.e., \(A_1 = A(1, 0)\). Therefore,

\[
\begin{cases}
AA^T = (x_1 - x_2)^2A_1A_1^T + x_2(x_1 - x_2)(A_1J + JA_1^T) + x_2^2J^2, \\
A^TA = (x_1 - x_2)^2A_1^TA_1 + x_2(x_1 - x_2)(A_1^TJ + JA_1) + x_2^2J^2.
\end{cases}
\]

Hence, if \(A\) is normal then \(A_1\) is normal. Moreover, if \(A_1\) is normal then \(A_1A_1^T = A_1^TA_1\). So, for every \(i\),

\[
\sum_k (A_1)_{ik}^2 = (A_1A_1^T)_{ii} = (A_1^TA_1)_{ii} = \sum_k (A_1)_{ki}^2.
\]

Hence, the number of entries equal to one in the \(i\)-th row of \(A_1\) must be equal to the number of entries equal to one in the \(i\)-th column of \(A_1\) for all \(i\). On the other hand,

\[
\begin{align*}
(A_1J + JA_1^T)_{ij} &= \sum_k [(A_1)_{ik} + (A_1^T)_{kj}], \\
(A_1^TJ + JA_1)_{ij} &= \sum_k [(A_1^T)_{ik} + (A_1)_{kj}].
\end{align*}
\]

Thus if \(A_1\) is normal then \(A_1J + JA_1^T = A_1^TJ + JA_1\) and therefore, \(A\) is normal. This proves the following lemma.
Lemma 2.3.1. Let \( A \) be an entry pattern matrix with at most two indeterminates. Then \( A \) is normal if and only if a coefficient matrix of \( A \) is normal.

We remark that if \( A \) is normal then every coefficient matrix is normal but the converse is not true in general. For example, let’s consider the following entry pattern matrix

\[
A = \begin{bmatrix}
a & y & z & z \\
b & y & z & \\
z & c & y & \\
y & z & d & \\
\end{bmatrix}
\]

Then each coefficient matrix of \( A \) is normal. But the completion of \( A \) obtained by assigning the value 1 to \( a, y \) and the value zero to the other indeterminates is not normal, which proves that the entry pattern matrix \( A \) is not normal.

In 2015, Huang and Zhan have shown that there are many nonsymmetric normal entry pattern matrices. They also determined the maximum possible number of indeterminates in a normal EPM of a given order and characterize the entry pattern matrices which attain that maximum number of indeterminates as follows.

Theorem 2.3.2. [19] Let \( n \geq 3 \) be an integer, and let \( A \) be a nonsymmetric \( \mathbb{R} \)-normal entry pattern of order \( n \) with \( k \) distinct entries. Then \( k \leq \frac{n(n-3)}{2} + 3 \), where equality holds if and only if \( A \) is similar to a pattern of the form

\[
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1,n-3} & y_1 & y_1 & y_1 \\
x_{12} & x_{22} & \cdots & x_{2,n-3} & y_2 & y_2 & y_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
x_{1,n-3} & x_{2,n-3} & \cdots & x_{n-3,n-3} & y_{n-3} & y_{n-3} & y_{n-3} \\
y_1 & y_2 & \cdots & y_{n-3} & z & u & v \\
y_1 & y_2 & \cdots & y_{n-3} & v & z & u \\
y_1 & y_2 & \cdots & y_{n-3} & u & v & z \\
\end{pmatrix}
\]

where \( u, v, z, y_i, x_{ij}, 1 \leq i \leq j \leq n - 3 \), are distinct indeterminates.

We note that if \( A \) is a square EPM with the property that \( CC^T = C^T C \) for all \( C \in C_F(A) \) for some field \( F \) then \( AA^T = A^T A \). Indeed, assume that \( A = A(x_1, \ldots, x_k) \). Then each entry of \( AA^T - A^T A \) is either the zero polynomial or a homogeneous quadratic polynomial. Since \( CC^T = C^T C \) for every \( C \in C_F(A) \), the quadratic polynomial appearing in each entry of \( AA^T - A^T A \) (if any) must vanish on \( F \) which is impossible since \( F \) has at least two elements. Therefore, \( AA^T = A^T A \). Hence, if there is any field \( F \) for which every element of \( C_F(A) \) commutes with its transpose then this is true for every field, in particular \( A \) is normal according to Definition 2.3.1. In the next chapters, we consider properties of entry pattern matrices which do depend on the choice of \( F \).
Chapter 3

On the Maximum Rank of Completions of Entry Pattern Matrices

This chapter is concerned with rank bounds for square entry pattern matrices. The basic theme is the determination of the maximum rank of completions of a given EPM over a specified field $\mathbb{F}$. An upper bound for this number is given by the \textit{generic $\mathbb{F}$-rank}, which is the rank of the EPM when considered as a matrix written over the function field generated over $\mathbb{F}$ by the indeterminate entries. Provided that the field $\mathbb{F}$ has high enough order, we will see that the generic $\mathbb{F}$-rank can be attained by some $\mathbb{F}$-completion and so the generic and maximum ranks coincide. However, examples exist over small finite fields of entry pattern matrices whose generic rank exceeds their maximum rank. We will show that this can occur only if the matrix order exceeds the field order. And we will prove that over any finite field of characteristic $2, 3, 5, 7, 11, 13$, except $\mathbb{F}_2$, there exists a generically nonsingular square EPM of order $|\mathbb{F}| + 1$ whose $\mathbb{F}$-completions are all singular.

3.1 Definitions and examples

\textbf{Definition 3.1.1.} Let $A(x_1, \ldots, x_k)$ be an EPM with $k$ indeterminates $x_1, \ldots, x_k$. Then the \textit{maximum $\mathbb{F}$-rank} of $A$, denoted by $\text{mF-rank}(A)$, is the maximum rank of all $\mathbb{F}$-completions of $A$,

$$\text{mF-rank}(A(x_1, \ldots, x_k)) = \max_{a_1, \ldots, a_k \in \mathbb{F}} \text{rank}(A(a_1, \ldots, a_k)).$$

\textbf{Definition 3.1.2.} Let $A$ be an EPM. Then the \textit{generic $\mathbb{F}$-rank} of $A$, denoted by $\text{gF-rank}(A)$, is the rank of $A$ when considered as a matrix in $M_{m \times n}(\mathbb{F})$, where $\mathbb{F}$ is the function field generated over $\mathbb{F}$ by the indeterminate entries of $A$. We say that a square EPM $A \in M_n(S)$ is $\mathbb{F}$-nonsingular if $\text{gF-rank}(A) = n$.

We remark that the generic $\mathbb{F}$-rank of an EPM may depend on the characteristic of the field $\mathbb{F}$. For example the following matrix has generic
rank 4 over any field of characteristic different from 2, and generic rank 3 over any field of characteristic 2. Since the row sums are all equal to $2x + 2y$, it is clear that this expression is a factor of the determinant and hence that the determinant vanishes in characteristic 2.

$$\begin{bmatrix} x & x & y & y \\ x & y & y & x \\ x & y & x & y \\ y & x & x & y \end{bmatrix}.$$  

We will say that an entry pattern matrix $A$ is $F$-rank-discrepant (sometimes abbreviated to discrepant) if its generic $F$-rank exceeds its maximum $F$-rank. The following is an example of a $4 \times 4$ $F_3$-rank-discrepant EPM.

**Example 3.1.1.** Let $F = F_3$ and let

$$A_3(x, y, z) = \begin{bmatrix} x & y & y & y \\ y & x & z & z \\ z & z & x & x \\ y & y & z & y \end{bmatrix}.$$  

Then $\det_{F_3}(A_3) = (x - y)(x - z)(y - z)(x + y + z)$ vanishes on $F_3$. Hence, $m_{F_3}(A_3) = 3$, $g_{F_3}(A_3) = 4$, and the matrix $A_3$ demonstrates that the field $F_3$ is EPM-rank-tight.

### 3.2 Conditions for coincidence of the generic and maximum $F$-ranks

Let $A(x_1, \ldots, x_k)$ be a square EPM of order $n$. In this section we show that the generic and maximum ranks of $A$ coincide over any field $F$ for which $|F| \geq n$, but may differ if $|F| < n$, provided that the number of indeterminates is at least three. We also show that for every prime power $q$, there exists a $(q + 2) \times (q + 2)$ EPM whose generic $F_q$-rank exceeds its maximum $F_q$-rank. Subsequent sections are concerned with the question of when $q + 2$ can be improved to $q + 1$ in this statement.

We note that we lose nothing by restricting our attention to generically nonsingular square entry pattern matrices. If a (rectangular) EPM has generic $F$-rank $r$ and maximum $F$-rank $r'$, then it possesses an $r \times r$ submatrix that has generic $F$-rank $r$ and maximum $F$-rank $r'$. Indeed, if an EPM $A$ has generic $F$-rank $r$ and maximum $F$-rank $r'$ where $r' \leq r$ then there is an $r' \times r' \times r'$ submatrix $M$ of $A$ so that $m_{F}(M) = r'$. Since the maximum rank of any EPM can not exceed its generic rank over any field, $g_{F}(M) = r'$. This implies that $M$ is $F$-nonsingular. Since $g_{F}(A) = r$, we may extend the submatrix $M$ to an $r \times r$ submatrix $N$ of $A$ so that $g_{F}(N) = r$. On the other hand, $r' = m_{F}(M) \leq m_{F}(N) \leq m_{F}(A) = r'$. Thus $g_{F}(N) = r$ and $m_{F}(N) = r'$, as required.
Throughout this section and the next we make frequent use of both parts of the following statement, which are Theorems 6.13 and 6.15 of [20].

**Theorem 3.2.1.** Let \( f \in \mathbb{F}_q[x_1, \ldots, x_k] \) with \( \deg(f) = d \geq 0 \).

1. Then the equation \( f(x_1, \ldots, x_k) = 0 \) has at most \( dq^{k-1} \) solutions in \( \mathbb{F}_q^k \).

2. If \( f \) is homogeneous and \( d \geq 1 \), then the equation \( f(x_1, \ldots, x_k) = 0 \) has at most \( d(q^{k-1} - 1) + 1 \) solutions in \( \mathbb{F}_q^k \).

**Theorem 3.2.2.** Let \( A(x_1, x_2, \ldots, x_k) \) be an \( n \times n \mathbb{F} \)-nonsingular square entry pattern matrix. If the field \( \mathbb{F} \) has at least \( n \) elements then the maximum \( \mathbb{F} \)-rank of \( A \) is \( n \).

**Proof.** We write \( f \) for the determinant of \( A \), which is a homogeneous polynomial of degree \( n \) in \( \mathbb{F}[x_1, \ldots, x_k] \). If \( \mathbb{F} \) is infinite, it is clear that \( f \) cannot vanish on \( \mathbb{F}^k \). If \( \mathbb{F} \) is finite of order \( q \), then by Theorem 3.2.1 (2.), \( f \) vanishes on at most \( n(q^{k-1} - 1) + 1 \) elements of \( \mathbb{F}_q^k \). If \( n \leq q \) then this number is less than \( q^k \) and there exists an \( \mathbb{F} \)-completion of \( A \) with non-zero determinant and rank \( n \).

It follows from Theorem 3.2.2 that the maximum and generic rank of an entry pattern matrix always coincide over an infinite field. Our next observation is that, regardless of field considerations, the generic and maximum ranks always coincide for EPMs with fewer than three indeterminates.

**Theorem 3.2.3.** Let \( \mathbb{F} \) be any field and \( A \) be an EPM with fewer than 3 indeterminates. Then the maximum \( \mathbb{F} \)-rank of \( A \) is equal to its generic \( \mathbb{F} \)-rank.

**Proof.** If the number of indeterminates is 1, all entries of \( A \) are equal and the maximum \( \mathbb{F} \)-rank and generic \( \mathbb{F} \)-rank of \( A \) are both equal to 1.

If \( A \) has two distinct indeterminates \( x \) and \( y \), denote the generic \( \mathbb{F} \)-rank of \( A \) by \( g \), and assume that \( A \) is a \( g \times g \) EPM over \( \mathbb{F} \) with determinant \( f(x, y) \), a non-zero polynomial in \( \mathbb{F}[x, y] \). After the row operation \( R_1 \to R_1 - R_2 \), the entries of the first row are in \( \{x-y, y-x, 0\} \).

By Laplace's expansion, \( f(x, y) = (x-y) \sum \pm \det(A_i) \) where the \( A_i \) are \( (g-1) \times (g-1) \) entry pattern submatrices of \( A \). After repeating this expansion step \( g-1 \) times, we conclude that

\[
f(x, y) = (x-y)^{g-1}(\alpha x + \beta y)
\]

for some non-zero \((\alpha, \beta) \in \mathbb{F}^2\). Note that either \( f(1, 0) \) or \( f(0, 1) \) is not zero, so the maximum and generic \( \mathbb{F} \)-ranks of \( A \) coincide. \( \square \)

The above theorems show that the generic \( \mathbb{F} \)-rank and maximum \( \mathbb{F} \)-rank of an \( n \times n \) entry pattern matrix are equal if \( n \leq |\mathbb{F}| \) or if the number of indeterminates is 1 or 2. For a prime power \( q = p^k \), this raises the question of identifying the least possible order of an \( \mathbb{F}_q \)-rank-discrepant
EPM, if any exist. By Theorem 3.2.2, $q+1$ is a lower bound for the answer to this question and any example that attains this bound must have maximum $F_q$-rank $q$ and generic $F_q$-rank $q+1$. Indeed, assume that $A$ is a $(q+1) \times (q+1)$ $F_q$-rank-discrepant EPM. Denote $g = g_{F_q}(A)$. If $g \leq q$ then there is a $g \times g$ $F_q$-nonsingular submatrix $M$ of $A$ and therefore, $m_{F_q}(M) = g$ by Theorem 3.2.2. Hence,

$$g = m_{F_q}(M) \leq m_{F_q}(A) \leq g_{F_q}(A) = q.$$ 

This implies that $m_{F_q}(A) = g_{F_q}(A)$, a contradiction to the fact that $A$ is $F_q$-rank-discrepant. Therefore, $g_{F_q}(A) = q+1$, or $A$ is $F_q$-nonsingular. Thus $A$ has a $(q) \times (q)$ submatrix $N$ which is also $F_q$-nonsingular. By Theorem 3.2.2 again, $m_{F_q}(N) = q$. So $m_{F_q}(A) \geq q$. The equality occurs because $A$ is $F_q$-rank-discrepant.

Our next theorem shows that $F_q$-rank-discrepant entry pattern matrices of order $q+2$ exist for all prime powers $q$.

**Theorem 3.2.4.** For every finite field $F_q$, there exists an $F_q$-rank-discrepant entry pattern matrix $A_{q+2}(x, y, z)$ of order $q+2$.

**Proof.** In $F_q[x, y, z]$, let

$$C_{q+1}(x, y, z) = xy(x^{q-1} - y^{q-1}) + yz(y^{q-1} - z^{q-1}) + zx(z^{q-1} - x^{q-1}).$$

It is easily observed that $C_{q+1}(x, y, z)$ vanishes on $F_q^3$, i.e. $C_{q+1}(a, b, c) = 0$ for all $a, b, c \in F_q$. We note that (over $F_q$)

$$C_{q+1}(x, y, z) = (y - z)(x^q - z^q) - (x - z)(y^q - z^q) = (x - z)(y - z)[(x - z)^{q-1} - (y - z)^{q-1}].$$

(3.1)

For $q > 2$, we also observe that $(x - z)^{q-1} - (y - z)^{q-1}$ is the determinant of the $(q - 1) \times (q - 1)$ matrix $A_1$ which has all entries on the main diagonal equal to $x - z$, all entries on the superdiagonal equal to $y - z$, $y - z$ in the $(q - 1, 1)$-position and zeros elsewhere.

$$(x - z)^{q-1} - (y - z)^{q-1} = \det \begin{bmatrix} x - z & y - z & 0 & \ldots & \ldots & 0 \\ \\
0 & x - z & y - z & \ldots & \ldots & 0 \\ \\
0 & 0 & x - z & \ddots & \ddots & \vdots \\ \\
\vdots & \ddots & \ddots & \ddots & y - z & 0 \\ \\
0 & 0 & 0 & \ldots & x - z & y - z \\ \\
y - z & 0 & 0 & \ldots & 0 & x - z \end{bmatrix}. \quad (q-1) \times (q-1)$$

If $q = 2$, we take $A_1 = (x + y)_{1 \times 1}$. We augment $A_1$ to a $(q + 2) \times (q + 2)$ matrix $A_2$ by adjoining three initial rows and three initial columns as follows:

- Every entry of Row 1 is $z$, except that the last entry is $y$ in the case $q = 2$;
- The $(2, 2)$ and $(3, 3)$ entries are respectively $x - z$ and $y - z$;
3.3 Extensions of EPM-rank-tight fields

- All other entries in the first three rows and first three columns are equal to zero.

Clearly \( A_2 \) has determinant \( z C_{q+1}(x, y, z) \), which vanishes on \( \mathbb{F}_3^q \). To obtain an entry pattern matrix \( A(x, y, z) \) having the same determinant as \( A_2 \), we add Row 1 to all subsequent rows of \( A_2 \). The result of this process has the same general pattern for all \( q \); the example for \( q = 9 \) is displayed below.

\[
\begin{bmatrix}
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
zzzzzzzzz \\
\end{bmatrix}
\]

For every prime power \( q \), this construction yields a \((q + 2) \times (q + 2)\) entry pattern matrix whose generic \( \mathbb{F}_q \)-rank is \( q + 2 \) and whose maximum \( \mathbb{F}_q \)-rank is \( q + 1 \).

The question of whether the \( q + 2 \) of Theorem 3.2.4 can be improved to \( q + 1 \) motivates the following definition.

**Definition 3.2.1.** The finite field \( \mathbb{F}_q \) is called *EPM-rank-tight* if there exists a \((q + 1) \times (q + 1)\) entry pattern matrix whose generic \( \mathbb{F}_q \)-rank is \( q + 1 \) and whose maximum \( \mathbb{F}_q \)-rank is \( q \).

The remainder of this chapter is concerned with the property of EPM-rank-tightness for finite fields. In view of Theorem 3.2.3, we restrict our attention to entry pattern matrices with three indeterminates \( x, y \) and \( z \). We remark however that the determination of the maximum possible number of indeterminates in a rank-discrepant EPM of specified order is a problem of potential interest.

### 3.3 Extensions of EPM-rank-tight fields

In this section we prove that finite extensions of EPM-rank-tight fields retain the property of being EPM-rank-tight. Recall from Example 3.1.1 that the field \( \mathbb{F}_3 \) is EPM-rank-tight, so our theorem will establish that all finite fields of characteristic 3 are EPM-rank-tight. We will show in Section 3.4 that the fields \( \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13} \), and hence all of their finite extensions, are EPM-rank-tight.

For any positive integer \( d \), we write \( C_{d+1} \) for the polynomial

\[
xy(x^{d-1} - y^{d-1}) + yz(y^{d-1} - z^{d-1}) + zx(z^{d-1} - x^{d-1}),
\]

which may be interpreted as being written over any field according to context. The polynomial \( C_{q+1}(x, y, z) \) vanishes on \( \mathbb{F}_3^q \).
Lemma 3.3.1. Let \( p \) be a prime and suppose that the field \( \mathbb{F}_p \) is EPM-rank-tight. Then there exists an \( \mathbb{F}_p \)-nonsingular EPM \( A(x, y, z) \in M_{p+1}(x, y, z) \) whose determinant is a nonzero scalar multiple of \( C_{p+1}(x, y, z) \).

Proof. Let \( T \) be the set of \( (p+1) \times (p+1) \) EPMs of generic \( \mathbb{F}_p \)-rank \( p+1 \) and maximum \( \mathbb{F}_p \)-rank \( p \). Since \( \mathbb{F}_p \) is EPM-rank-tight, \( T \) is a nonempty set. Let \( k \) be the least number of indeterminates in an element of \( T \), and choose an element \( A \) of \( T \) having \( k \) indeterminates. By Lemma 3.2.3, \( k \geq 3 \). We will show that \( k = 3 \). If \( k > 3 \), let \( x, y, z \) and \( t \) be independent indeterminates appearing as entries of \( A \). The result of identifying any pair of indeterminates in \( A \) must be a generically singular matrix, otherwise we would have a contradiction to the minimality of \( k \).

It follows that \( \det(A) \) is a multiple of \( p(x, y, z, t) = (x-y)(x-z)(x-t)(y-z)(y-t)(z-t) \). If so, \( \det(A) \) is a homogeneous polynomial whose degree in the variable \( x \) is at least 3 and at most \( p-2 \). Since the total degree is \( p+1 \), Theorem 3.2.1 (1.) shows that it is possible to assign values from \( \mathbb{F}_p \) to the remaining indeterminates in \( \det(A) \) so that the coefficient of the leading power of \( x \) does not vanish. Since the result of such an assignment is a polynomial in \( \mathbb{F}_p[x] \) of degree between 3 and \( p-2 \) which does not vanish on \( \mathbb{F}_p \), it follows that \( \det(A) \) cannot vanish on \( \mathbb{F}_p^k \) which means that the maximum \( \mathbb{F}_p \)-rank of \( A \) is \( p+1 \). From this contradiction we conclude that \( k = 3 \), and we write \( x, y, z \) for the indeterminates in \( A \).

Next, we will show that

\[
\det(A) = \alpha(xy(x^{p-1} - y^{p-1}) + yz(y^{p-1} - z^{p-1}) + zx(z^{p-1} - x^{p-1}))
\]

for some \( 0 \neq \alpha \in \mathbb{F}_p \).

As above, the result of identifying any pair of indeterminates in \( A \) must be a generically singular matrix, whence \( (x-y)(y-z)(x-z) \) divides \( \det(A) \). It follows that each of \( x, y \) and \( z \) has degree at least 2 and at most \( p \) in \( \det(A) \). Because \( \det A \) vanishes on \( \mathbb{F}_p^3 \), it follows from Theorem 3.2.1 (2.) that the degree in all three cases is \( p \).

Now write \( \det(A) = x^py_1(y, z) + \cdots + x^pg_p(y, z) + g_{p+1}(y, z) \) where \( g_i(y, z) = \alpha y + \beta z \neq 0 \). Let

\[
Q := \{(a, b) : a \in \mathbb{F}_p^2 : \alpha a + \beta b \neq 0 \}.
\]

For all \( (y_0, z_0) \in Q \), \( \det A(x, y_0, z_0) \) vanishes on \( \mathbb{F}_p \). Hence

\[
\det A(x, y_0, z_0) = (\alpha y_0 + \beta z_0)(x^p - x),
\]

which means that \( g_i(y_0, z_0) = 0 \) for all \( (y_0, z_0) \in Q \) and for \( 2 \leq i \leq p-1 \). Since \( g_i \) is homogeneous of degree at most \( p-1 \), the equation \( g_i(y, z) \) has at most \( (p-1)^2 \) nontrivial solutions in \( \mathbb{F}_p^2 \) by Theorem 3.2.1. However \( Q \) is the complement of a line in \( \mathbb{F}_p^2 \), so \( |Q| = p^2 - p \) and \( (0, 0) \notin Q \). Since \( p^2 - p > (p-1)^2 \) it follows that \( g_i(y, z) = 0 \) for \( 2 \leq i \leq p-1 \).

Hence,

\[
\det A(x, y, z) = (\alpha y + \beta z)x^p + g_p(y, z)x + g_{p+1}(y, z).
\]
Because \( \det A(0, y, z) = 0 \) for all \( y, z \in \mathbb{F}_p \), \( g_{p+1}(y, z) = \delta y z(y^{p-1} - z^{p-1}) \) for some \( \delta \in \mathbb{F}_p \).

Now for all \((y_0, z_0) \in \mathbb{F}_p^2\), the expression \((\alpha y_0 + \beta z_0)x^p + g_p(y_0, z_0)x\) in \( \mathbb{F}_p[x]\) is either equal to zero or to \((\alpha y_0 + \beta z_0)(x^p - x)\). It follows that \(g_p(y_0, z_0) = -\alpha y_0 - \beta z_0\) for all \( y_0, z_0 \), and \(g_p(y, z) = -\alpha y^p - \beta z^p\). Now

\[
\det A(x, y, z) = (\alpha y + \beta z)x^p - (\alpha y^p + \beta z^p)x + \delta y z(y^{p-1} - z^{p-1})
= \alpha(x^p y - xy^p) - \beta(x^p z - xz^p) + \delta(y^p z - yz^p).
\]

Since 3 is the least number of indeterminates of a matrix in \( T \), setting \( x = y, x = z \) or \( y = z \) in \( \det(A) \) must result in the zero polynomial. It easily follows that \( \alpha = \beta = \delta \). Therefore,

\[
\det A(x, y, z) = \alpha C_{p+1}(x, y, z),
\]

for some nonzero \( \alpha \in \mathbb{F}_p \).

We remark that the proof of Lemma 3.3.1 also shows that if \( A(x, y, z) \) is an \( \mathbb{F}_p \)-rank-discrepant EPM of size \( n \times n \) then \( \det A \) is a scalar multiple of \( C_{p+1} \).

Let \( \mathbb{F}_q \) be a finite extension of \( \mathbb{F}_p \). We may rewrite \( C_{q+1} \) using the formulation of (3.1), and note the following useful factorization, which applies in characteristic \( p \).

\[
C_{q+1}(x, y, z) = C_{p+1}(x, y, z)H_{\frac{q-1}{p-1}}((x - z)^{p-1}, (y - z)^{p-1}), \tag{3.2}
\]

where \( H_d(X, Y) := \sum_{i=0}^d X^i Y^{d-i} \).

**Lemma 3.3.2.** There exists a \((q-p) \times (q-p)\) matrix with entries drawn from \( \{\pm(x - z), \pm(y - z), 0\} \) whose determinant is \( H_{\frac{q-1}{p-1}}((x - z)^{p-1}, (y - z)^{p-1}) \).

**Proof.** As noted earlier in the proof of Theorem 3.2.4, \((x - z)^{p-1} + (y - z)^{p-1} = H_1((x - z)^{p-1}, (y - z)^{p-1})\) is the determinant of the following matrix \( B_1 \), provided that \( p \neq 2 \).

\[
B_1 = \begin{bmatrix}
x - z & y - z & 0 & \cdots & 0 \\
0 & x - z & y - z & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & y - z \\
z - y & 0 & 0 & \cdots & x - z \\
\end{bmatrix}_{(p-1) \times (p-1)}
\]

If \( p = 2 \), we take \( B_1 = (x + y)_{1 \times 1} \).

Over any field, the polynomials \( H_d \) satisfy the following recursion, for \( d \geq 2 \):

\[
H_d(X, Y) = (X + Y)H_{d-1}(X, Y) - XYH_{d-2}(X, Y). \tag{3.3}
\]

We use this recursion and the matrix \( B_1 \) above to construct a sequence \( B_1, B_2, \ldots, B_{\frac{q-1}{p-1}} = B \), where \( B_i \) is a matrix of order \( i(p-1) \) whose
determinant is \( H_i((x-z)^{p-1}, (y-z)^{p-1}) \) and whose entries come from the set \( \{\pm(x-z), \pm(y-z), 0\} \). First we define \( B_2 \) to be the matrix of order \( 2(p-1) \) that has \( B_1 \) as the upper left and lower right \((p-1)\times(p-1)\) blocks, has \( z-y \) in the \((p-1, p)\) position and \( x-z \) in the \((p, 1)\) position and is otherwise full of zeros. The determinant of \( B_2 \) is

\[
\det(B_1) \det((x-z)^{p-1}(y-z)^{p-1}) = H_2((x-z)^{p-1}, (y-z)^{p-1}), \text{ by (3.3)}.
\]

Now for \( i \geq 3 \), define \( B_i \) to be the matrix of order \( i(p-1) \) that has \( B_1 \) as its upper left \((p-1)\times(p-1)\) block, \( B_{i-1} \) in the lower right block of order \((i-1)(p-1)\), the entry \( z-y \) in the \((p-1, p)\) position, \( x-z \) in the \((p, 1)\) position, and zeros elsewhere. Using (3.3) and the fact that the lower right block of order \((i-2)(p-1)\) in \( B_i \) is \( B_{i-2} \), it is straightforward to confirm by induction that \( \det(B_i) = H_i((x-z)^{p-1}, (y-z)^{p-1}) \). In particular \( B = B_{\frac{q-1}{p-1}} \) has determinant \( H_{\frac{q-1}{p-1}}((x-z)^{p-1}, (y-z)^{p-1}) \). The matrix \( B \) has \( \frac{q-1}{p-1} \) copies of \( B_1 \) along its diagonal, has the entry \( z-y \) in positions \((i,j)\) where \((p-1)\mid i \) and \( j = i+1 \), has the entry \( x-z \) in positions \((i,j)\) where \( j \equiv 1 \mod (p-1) \) and \( i = j+p-1 \), and is otherwise full of zeros.

**Example 3.3.1.** For \( p = 3 \), \( q = 9 \), the \( 6 \times 6 \) matrix \( B \) is given by

\[
B = \begin{bmatrix}
  x-z & y-z & 0 & 0 & 0 & 0 \\
  z-y & x-z & z-y & 0 & 0 & 0 \\
  x-z & 0 & x-z & y-z & 0 & 0 \\
  0 & 0 & z-y & x-z & z-y & 0 \\
  0 & 0 & x-z & 0 & x-z & y-z \\
  0 & 0 & 0 & 0 & z-y & x-z
\end{bmatrix}.
\]

We now are in a position to prove the main theorem of this section. The construction in the theorem does not apply in the case \( p = 2 \), where the matrix \( B \) has an exceptional structure, and we show in Section 3.4 that \( \mathbb{F}_2 \) is not EPM-rank-tight. We state and prove the theorem for ground fields \( \mathbb{F}_p \) of prime order, but we remark that the theorem and proof hold if \( \mathbb{F}_p \) is replaced by any EPM-rank-tight finite field (including in characteristic 2). Indeed the proof shows that if \( E \) is a field extension of \( \mathbb{F} \) of degree \( k \), and there exists an \( \mathbb{F} \)-rank-discrepant entry pattern matrix of order \( n \), then there exists an \( \mathbb{E} \)-rank-discrepant matrix of order \( n + |\mathbb{E}| - |\mathbb{F}| \).

**Theorem 3.3.3** (Extension theorem). Let \( p \) be an odd prime. If the field \( \mathbb{F}_p \) is EPM-rank-tight then so is any finite extension of \( \mathbb{F}_p \).

**Proof.** Assume that \( \mathbb{F}_p \) is EPM-rank-tight and let \( \mathbb{F}_q \) be a finite extension of \( \mathbb{F}_p \). By Lemma 3.3.1, there exists a \((p+1)\times(p+1)\) entry pattern matrix \( A(x,y,z) \) for which \( \det(A) = \alpha C_{p+1}(x,y,z) \) for some nonzero \( \alpha \in \mathbb{F}_p \). Let \( B \) be the matrix constructed in Lemma 3.3.2. Then for any \((p+1)\times(q-p)\) matrix \( A' \) we have

\[
\det \begin{bmatrix} A & A' \\ 0 & B \end{bmatrix} = \det(A) \times \det(B) = \alpha C_{p+1}(x,y,z) \times H_{\frac{q-1}{p-1}}((x-z)^{p-1}, (y-z)^{p-1}) = \alpha C_{q+1}(x,y,z).
\]
We now demonstrate that, for a suitable choice of $A'$, the above matrix can be converted to entry pattern form by a sequence of elementary row operations that does not affect its determinant. Let $\mathbb{F}_p$ denote the function field $\mathbb{F}_p(x, y, z)$. Take $A'$ to be the matrix whose $j$-th column is equal to the first column of $A$ if $j \equiv 1 \mod (p - 1)$ and equal to the second column of $A$ otherwise. Since $A$ is a nonsingular $\mathbb{F}_p$-matrix, some $\mathbb{F}_p$-linear combination of the rows of $A$ is the first standard basis vector $e_1$, and another $\mathbb{F}_p$-linear combination of the rows of $A$ has 0 as its first entry and 1 in all other positions. The same $\mathbb{F}_p$-linear combinations of the rows of the matrix $[A \ A']$ respectively give the row vectors $u_{1 \times (q+1)}$ and $v_{1 \times (q+1)}$, where the $j$-th entry of $u$ is 1 if $j = 1$ or $j - (p + 1) \equiv 1 \mod (p - 1)$ and zero otherwise, and where $u + v$ is the all-ones vector of length $q + 1$.

Now apply the following elementary row operations to Rows $p + 2$ through $q + 1$ of the matrix $\begin{bmatrix} A & A' \\ 0 & B \end{bmatrix}$.

1. Add $(y \times u) + (z \times v)$ to Row $i$ if $i \equiv p + 1 \mod (p - 1)$. This replaces every appearance of $z - y$ in these rows with $z$, every $x - z$ with $x$, and every zero entry with $z$.

2. For all other $i$ in the range $p + 2$ to $q + 1$, add $z \times (u + v)$ to Row $i$. This replaces the $x - z$ entries with $x$, the $y - z$ entries with $y$, and the zero entries with $z$.

The result of this process is an entry pattern matrix $A_q$ of order $q + 1$, whose determinant over $\mathbb{F}_q$ is $\alpha C_{q+1}(x, y, z)$. Thus $\mathbb{F}_q$ is EPM-rank-tight.

Example 3.3.2. Applying the procedure of Theorem 3.3.3 to the field $\mathbb{F}_9$, using the matrices of Examples 3.1.1 and 3.3.1 we construct the following $\mathbb{F}_9$-rank-discrepant EPM of order 10.

$$\begin{bmatrix} x & y & y & y & x & x & y & y & x & x \\ y & x & z & z & y & x & y & x & y & x \\ z & z & x & x & z & z & z & z & z & z \\ y & y & z & y & y & y & y & y & y & y \\ z & z & z & x & y & z & z & z & z & z \\ y & z & z & z & x & z & z & y & z & z \\ z & z & z & z & x & x & y & z & z & z \\ y & z & z & z & y & z & z & z & z & z \\ z & z & z & z & z & x & z & x & y & z \\ y & z & z & y & y & z & z & z & z & x \end{bmatrix}$$

3.4 Minimal constructions

In this section we will show that all finite fields of characteristics 3, 5, 7, 11 or 13 are EPM-rank-tight. So also are all fields of characteristic 2 with the exception of $\mathbb{F}_2$ itself.
Lemma 3.4.1. The field $\mathbb{F}_2$ is not EPM-rank-tight.

Proof. Assume that there exist $3 \times 3$ EPMs that are generically nonsingular in characteristic 2 but have maximum $\mathbb{F}_2$-rank less than 3. Let $k$ be the least possible number of indeterminates in such a matrix and let $A$ be an example having $k$ indeterminates. By Lemma 3.2.3, $k \geq 3$, so let $x, y$ and $z$ be independent indeterminates appearing as entries of $A$. The result of identifying any pair of indeterminates in $A$ must be a generically singular matrix, otherwise we would have a contradiction to the minimality of $k$. It follows that $\det(A)$ is a multiple of $p(x, y, z) = (x + y)(x + z)(y + z) = x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2$. Since $\det(A)$ is homogeneous of degree 3 and has at most six terms, it must be equal to $p(x, y, z)$. Thus each entry of $A$ is either $x, y$ or $z$. Since $x^2y, xz^2$ and $xy^2$ appear in $\det A$, we may assume, after permuting the rows and columns of $A$ if necessary, that the three entries on the main diagonal of $A$ are $x, x, y$ (in order) and that the entries in the $(1, 3)$ and $(3, 1)$ positions are either both equal to $y$ or both equal to $z$. It is easily checked that no completion of either of these partial patterns has the desired determinant.

We now present a general construction to show that every finite field of characteristic 2 and order $q \geq 4$ is EPM-rank-tight. For $q = 2^k$ and $k \geq 2$, we construct an EPM $A(x, y, z)$ of order $2^k + 1$, whose determinant is $C_{q+1}(x, y, z)$. By (3.2), we have

$$C_{q+1}(x, y, z) = x^qy + xy^q + x^qz + xz^q + y^qz + yz^q = XY(X+Y)H_{q-2}(X, Y),$$

where $X = x - z$, $Y = y - z$ and $H_{q-2}(X, Y) = \sum_{i=0}^{q-2} X^iY^{q-2-i}$.

We begin by noting that $H_{q-2}(X, Y)$ is the determinant of the tridiagonal $(q - 2) \times (q - 2)$ matrix with non-zero entries defined as follows:

- Each entry on the main diagonal is $X + Y$;
- Each entry on the first superdiagonal is $X$;
- Each entry on the first subdiagonal is $Y$.

Now let $B_1$ be the $(q - 1) \times (q - 1)$ matrix obtained from the above tridiagonal matrix by appending an initial row in which every entry is $X + Y$, and an initial column in which every entry except the first is zero. Our construction essentially involves applying elementary row operations to $B_1$ to convert it to a form which more closely resembles an entry pattern matrix, then appending two additional rows and columns to account for the factors $X$ and $Y$, and applying further row and column operations to obtain an EPM. We describe the steps below and use the case $q = 8$ as a representative example.

**Step 1:** Adjustments to $B_1$. Let $B'_1$ be the matrix that coincides with $B_1$ except in the first row where all of its entries are 1. Then $\det(B_1) = (X + Y) \det(B'_1)$. We apply the following elementary row operations to $B'_1$:
3.4. Minimal constructions

- For \( l = 2, \ldots, q - 2 \), add Row \((l + 1)\) and \( z \times (\text{Row } 1)\) to Row \( l \).
- Add \( x \times (\text{Row } 1)\) to the final row.

Finally replace each entry in Row 1 with \( x + y \) to obtain a matrix with determinant \((x + y)H_{p-2}(x + z, y + z)\) whose entries outside of Row 1 are elements of the set \( \{x, y, z\} \). In the case \( q = 8 \) we have

\[
B'_1 = \begin{pmatrix}
0 & x + y & y + z & 0 & 0 & 0 & 0 & 0 \\
0 & x + z & x + y & y + z & 0 & 0 & 0 & 0 \\
0 & 0 & x + z & x + y & y + z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x + z & x + y & y + z & 0 \\
0 & 0 & 0 & 0 & 0 & x + z & x + y & y + z \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
x + y & x + y & x + y & x + y & x + y & x + y \\
z & y & y & z & z & z \\
z & y & x & y & z & z \\
z & z & x & y & x & y \\
z & z & z & x & y & x \\
x & x & x & x & x & y \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
z & y & x & y & z & z & z \\
z & x & y & x & y & z & z \\
z & z & x & y & x & y & z \\
z & z & z & x & y & x & z \\
z & z & z & z & x & y & x \\
x & x & x & x & x & y & z \\
\end{pmatrix}
\]

Step 2: Incorporation of the factors \( x + z \) and \( y + z \). We introduce two new initial rows and columns to the above matrix as follows.

1. First append an initial column consisting of zeros, to obtain a \((q - 1) \times q\) matrix.

2. Next append an initial row whose first entry is \( y + z \) and whose remaining entries are all equal to \( y \). The result of this is a \( q \times q \) matrix whose determinant is \((y + z)(x + y)H_{q-2}(x + z, y + z)\).

3. Now append an initial row of zeros, resulting in a \((q + 1) \times q\) matrix.

4. Finally append an initial column in which the first entry is \( x + z \), the third entry is 0, and all remaining entries are given by \( z \). The result of this is a \((q + 1) \times (q + 1)\) matrix with the desired determinant \( C_{q+1}(x, y, z)\).

In the case \( q = 8 \) this construction yields the following matrix.

\[
\begin{pmatrix}
x + z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z & y + z & y & y & y & y & y & y \\
0 & 0 & x + y & x + y & x + y & x + y & x + y & x + y \\
z & 0 & z & y & x & y & z & z \\
z & 0 & z & z & x & y & z & z \\
z & 0 & z & z & z & x & y & x \\
z & 0 & z & z & z & z & x & y \\
z & 0 & x & x & x & x & x & z \\
\end{pmatrix}
\]
Step 3: Conversion to EPM form. We apply the following row and column operations to the matrix obtained at Step 2.

1. Add Row 2 to Row 3. This converts all the $x + y$ entries in Row 3 to $x$ and changes the $(3, 1)$ entry to $z$ and the $(3, 2)$ entry to $y + z$.

2. Add Row 4 to Row 1. After this step Row 1 has $x$ as its first entry and otherwise coincides with Row 4.

3. Finally add Column 1 to Column 2, to clear the zero and $y + z$ entries that appear in Column 2.

Since none of these operations affect the determinant, the result is an entry pattern matrix $A_q(x, y, z)$ whose determinant (in characteristic 2) is $C_{q+1}(x, y, z)$. In the case $q = 8$ we have

$$A_8(x, y, z) = \begin{bmatrix}
  x & x & z & y & x & y & z & z \\
  z & y & y & y & y & y & y & y \\
  z & y & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & x & y & x & y & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\
  z & z & x & x & x & x & x & x \\
  z & z & z & y & x & y & z & z \\
  z & z & z & z & x & y & x & y \\
  z & z & z & z & z & x & y & x \\end{bmatrix}.$$

We have proved the following theorem.

Theorem 3.4.2. If $q = 2^k$, then the field $\mathbb{F}_q$ is EPM-rank-tight if and only if $k \geq 2$.

We now move to the case of odd characteristic and consider the fields $\mathbb{F}_p$ with $p \in \{3, 5, 7, 11, 13\}$. We will show that these fields, and hence all of their finite extensions, are EPM-rank-tight, through explicit constructions of entry pattern matrices of order $p + 1$, with generic $\mathbb{F}_p$-rank exceeding their maximum $\mathbb{F}_p$-rank. Our constructions for these primes are not identical as each one depends on the relevant characteristic, but they all involve the same key idea. We do not know whether $\mathbb{F}_p$ is EPM-rank-tight for $p \geq 17$, and we show that the methodology that we have developed for these small primes cannot resolve this question.

For $p \in \{3, 5, 7, 11, 13\}$, we exhibit a $(p + 1) \times (p + 1)$ entry pattern matrix having determinant a scalar multiple of $C_{p+1}(x, y, z)$ and having entries in the set $\{x, y, z\}$. Writing $X = x - z$ and $Y = y - z$ as before, we note by (3.1) that

$$C_{p+1}(x, y, z) = X^p Y - X Y^p = X Y (X^{p-1} - Y^{p-1}) = X Y \prod_{\alpha \in \mathbb{F}_p^*} (X + \alpha Y).$$

The idea of our constructions is to organize all but two of these $p + 1$ linear factors into pairs, so that the product of the two factors within each pair is a quadratic polynomial that arises as the determinant of a
2 × 2 matrix with entries drawn from \( \{-X, \pm Y, \pm (X - Y)\} \). The factors omitted from this arrangement are a \( p(X, Y) = X + \alpha Y \) for a suitably chosen \( \alpha \neq 0 \) in \( \mathbb{F}_p \), and one of \( X \) and \( Y \). The first step is to write a \((p - 1) \times (p - 1)\) matrix having these 2 × 2 submatrices as blocks, and then to augment this to a \((p + 1) \times (p + 1)\) matrix in which the first row has entries alternating between 0 and 1, starting with 1, the second row has entries alternating between 0 and 1, starting with 0, and the first two columns have zeros outside of the first two rows. The entries of rows 3 to \( p + 1 \) are then resolved to entry pattern form by the addition of suitable multiples of the first two rows to subsequent rows. The last step is to introduce the two omitted factors to the determinant, and the key to this is to assign entries from \( \{x, y, z\} \) to the positions of row 2 so that the sum of entries 2 through \( p + 1 \) in each column is the omitted factor \( X + \alpha Y = x + \alpha y - (\alpha + 1)z \). The feasibility of this step depends on judicious selection of the entries in the 2 × 2 blocks.

The construction for \( p = 5 \) is presented below, followed by key details of solutions for \( p = 3, 7, 11 \) and 13 that are constructed using the same methods. In all cases the order in which 2 × 2 matrices realizing particular quadratic determinants are presented corresponds to the order in which they may be arranged along the main diagonal in the construction of the solution that we exhibit below.

\[ p = 5 : \quad C_6(X, Y) = XY(X + Y)(X + 2Y)(X + 3Y)(X + 4Y). \]

Note that

- \( X(X + 3Y) = X^2 + 3XY = (X - Y)^2 - Y^2 \)
  \[ = \begin{pmatrix} X - Y & Y \\ Y & X - Y \end{pmatrix} = \begin{pmatrix} x - y & y - z \\ y - z & x - y \end{pmatrix}, \]

- \( (X + Y)(X + 4Y) = X^2 - Y^2 \)
  \[ = \begin{pmatrix} X & -Y \\ -Y & X \end{pmatrix} = \begin{pmatrix} x - z & z - y \\ z - y & x - z \end{pmatrix}, \]

- \( p(X, Y) = X + 2Y = x + 2y + 2z. \)

The second matrix below is obtained from the first by adding multiples of rows 1 and 2 to subsequent rows. Entries denoted by \( * \) are yet to be decided and do not affect the determinant.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & x - y & y - z & * & * \\
0 & 0 & y - z & x - y & * & * \\
0 & 0 & 0 & x - z & z - y & * \\
0 & 0 & 0 & z - y & x - z & *
\end{bmatrix} \quad \rightarrow \quad
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
y & z & x & * & * & * \\
z & y & x & * & * & * \\
z & y & z & x & * & * \\
y & z & y & z & x & *
\end{bmatrix}
\]

Next, on the second matrix above, we will do the following operations (in orders):

1. convert every entry of Row 2 to 1 by adding Row 1 to Row 2,

2. multiply Row 1 by the omitted factor \( Y \), and add \( z \times (\text{Row 2}) \),
3. assign indeterminate entries to the undecided positions so that in every column, the sum of the last four entries includes four of the five terms in the omitted factor $x + 2y + 2z$.

Results of the step (3.) above may differ slightly, but one possibility is the following matrix

$$
A' = \begin{bmatrix}
y z y z y z \\
1 1 1 1 1 \\
y z x y y z \\
z y y x z y \\
z y z y x z \\
y z y z z x 
\end{bmatrix},
$$

whose determinant is

$$
XY(X + Y)(X + 3Y)(X + 4Y).
$$

Finally, rewrite the entries of Row 2 as below

$$
A = \begin{bmatrix}
y z y z y z \\
x x z y z \\
y z x y y z \\
z y y x z y \\
z y z y x z \\
y z y z z x 
\end{bmatrix}
$$

to obtain an entry pattern matrix $A$ in which the sum of the last five entries in each column is $x + 2y + 2z$. By adding all subsequent rows of $A$ to Row 2, we confirm that

$$
\det(A) = (x + 2y + 2x) \det(A') = C_6(X, Y).
$$

$p = 3 :$ $C_4(X, Y) = XY(X + Y)(X + 2Y)$.

Factorization: $p(X, Y) = X + Y = x + y + z$,

$$
Y(X + 2Y) = XY - Y^2 = \begin{vmatrix}
x - z & z - y & y - z
\end{vmatrix}.
$$

$$
A_3(x, y, z) = \begin{bmatrix}
x z x z \\
x x y x \\
z y x z \\
y z y z 
\end{bmatrix}.
$$

$$
\det A_3(x, y, z) = C_4(x, y, z) \text{ in } \mathbb{F}_3[x, y, z].
$$


Factorization: $p(X, Y) = X + 3Y = x + 3y + 3z$,

$$
(X + Y)(X + 6Y) = X^2 - Y^2 = \begin{vmatrix}
x - z & z - y & y - z
\end{vmatrix}.
$$
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\[ X(X + 5Y) = X^2 - 2XY = (X - Y)^2 - Y^2 = \begin{vmatrix} x - y & y - z \\ y - z & x - y \end{vmatrix} \]

\[ (X + 2Y)(X + 4Y) = X^2 - XY + Y^2 = \begin{vmatrix} x - z & y - z \\ z - y & x - y \end{vmatrix} \]

\[ A_7(x, y, z) = \begin{bmatrix} y & z & y & z & y & z & y & z \\ x & x & y & y & y & y & x & x \\ y & y & z & y & y & x & y & y \\ y & z & x & y & z & y & x & y \\ y & y & y & x & z & y & y & y \\ y & y & y & x & y & z & y & y \\ x & y & x & z & x & z & y & y \\ x & z & z & z & x & z & y & z \\ z & y & z & y & z & x & y & z \\ z & z & z & y & z & x & x & y \end{bmatrix} \]

\[ \det A_7(x, y, z) = C_8(x, y, z) \text{ in } \mathbb{F}_7[x, y, z]. \]

\[ p = 11: \ C_{12}(X, Y, Z) = XY \prod_{\alpha=1}^{10} (X + \alpha Y). \]

Factorization: \( p(X, Y) = X + 5Y, \ 3p(X, Y) = 3x + 4y + 4z, \)
\[ Y(Y + 9Y) = XY - 2Y^2 = Y(X - Y) - Y^2 = \begin{vmatrix} z - y & y - z \\ y - z & x - y \end{vmatrix} \]

\[ (X + Y)(X + 10Y) = X^2 - Y^2 = \begin{vmatrix} x - z & z - y \\ z - y & x - z \end{vmatrix} \]

\[ (X + 2Y)(X + 6Y) = X^2 - 3XY + Y^2 = (X - Y)^2 - XY = \begin{vmatrix} x - y & z - y \\ z - x & x - y \end{vmatrix} \]

\[ (X + 3Y)(X + 7Y) = (X - Y)X - Y^2 = \begin{vmatrix} x - y & z - y \\ z - x & x - y \end{vmatrix} \]

\[ (X + 4Y)(X + 8Y) = X^2 + XY - Y^2 = \begin{vmatrix} x - z & z - y \\ z - y & x - z \end{vmatrix} \]

\[ A_{11}(x, y, z) = \begin{bmatrix} x & z & x & z & x & z & x & z & x & z \\ x & x & y & x & y & x & x & x & x & x \\ y & y & z & y & y & x & x & y & y & x \\ y & z & x & y & y & z & y & x & y & y \\ y & z & x & y & y & z & y & x & y & y \\ y & z & x & z & x & z & y & y & y & y \\ z & x & z & x & y & z & z & z & z & z \\ z & x & z & x & z & x & y & z & z & z \\ z & x & z & x & z & x & y & z & z & z \\ y & z & z & x & y & z & y & z & x & x \end{bmatrix} \]

\[ \det A_{11}(x, y, z) = 8C_{12}(x, y, z) \text{ in } \mathbb{F}_{11}[x, y, z]. \]

\[ p = 13: \ C_{14}(X, Y, Z) = XY \prod_{\alpha=1}^{13} (X + \alpha Y). \]

Factorization: \( p(X, Y) = X + 6Y, \ 3p(X, Y) = 3x + 5y + 5z, \)
\[ (X + Y)(X + 12Y) = X^2 - Y^2 = \begin{vmatrix} x - z & z - y \\ z - y & x - z \end{vmatrix} \]
\[(X + 3Y)(X + 9Y) = X^2 + Y^2 - XY = \begin{vmatrix} y - z & z - x \\ x - y & y - z \end{vmatrix},\]
\[Y(X + 11Y) = XY - 2Y^2 = \begin{vmatrix} z - y & y - z \\ z - y & y - x \end{vmatrix},\]
\[(X + 4Y)(X + 7Y) = X^2 - 2XY + 2Y^2 = \begin{vmatrix} y - x & y - z \\ y - z & x - y \end{vmatrix},\]
\[(X + 5Y)(X + 8Y) = X^2 + Y^2 = \begin{vmatrix} x - z & z - y \\ z - y & z - x \end{vmatrix}.\]

\[A_{13}(x, y, z) = \begin{vmatrix} x & z & x & z & x & x & x & x & z \\ x & y & y & x & x & x & x & x & x \\ z & y & z & z & x & x & x & x & y \\ y & z & z & x & y & y & x & x & z \\ z & z & x & z & y & y & y & y & y \\ y & z & y & y & z & z & y & y & y \\ y & y & y & z & z & z & y & y & y \\ y & y & y & z & z & z & z & z & y \\ y & z & y & y & y & y & y & y & y \end{vmatrix}.\]

\[\det A_{13}(x, y, z) = 7C_{14}(x, y, z) \text{ in } \mathbb{F}_{13}[x, y, z].\]

From these constructions and Theorem 3.3.3 we reach the following conclusion.

**Theorem 3.4.3.** Every finite field of characteristic \(p\) is EPM-rank-tight if \(p \in \{3, 5, 7, 11, 13\}.

We note that \(p = 13\) is the limit for which the method employed above can determine whether \(\mathbb{F}_p\) is EPM-rank-tight. The method relies on the splitting of the polynomial \(C_{p+1}(X, Y)\) into a pair of linear factors (at least one of which is either \(X, Y\) or \(X - Y\)) and a product of \(\frac{p-1}{2}\) quadratic factors each of which arises as the determinant of a \(2 \times 2\) matrix whose entries are drawn from \(\{0, \pm X, \pm Y, \pm (X - Y)\}\). Since \(C_{p+1}(X, Y)\) is the product of \(p + 1\) distinct linear factors in \(\mathbb{F}_p[X, Y]\), these \(\frac{p-1}{2}\) quadratic determinants must be reducible as polynomials in \(\mathbb{F}_p\), and pairwise relatively prime over \(\mathbb{F}_p\). Moreover they can collectively include at most two of \(X, Y\) and \(X - Y\) as factors.

The feasible range of primes to which the method may apply can be investigated by enumerating (in characteristic zero and up to scalar multiplication) the quadratic polynomials that arise as determinants of \(2 \times 2\) matrices with entries from the specified list and do not have repeated factors. A straightforward count reveals 19 possibilities, 12 of which are generically reducible with either \(X, Y\) or \(X - Y\) as a repeated factor. At most two of these 12 can occur amongst our choice of \(\frac{p-1}{2}\) pairwise relatively prime quadratic factors, since one of \(X, Y\) or \(X - Y\) must be
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reserved for the first row. Thus the number of $2 \times 2$ determinants that can appear in our construction is bounded above by $2 + 7 = 9$, and $p - 1$ cannot exceed 18, which gives 19 as the maximum value of $p$ to which our method could possibly apply. We now consider whether it can apply to $p = 17$ or $p = 19$.

The seven quadratic $2 \times 2$ determinants that do not (necessarily) have $X, Y$ or $X - Y$ as factors are enumerated below; these are reducible in some characteristics and not in others.

1. $X^2 + Y^2$,
2. $X^2 + (X - Y)^2$,
3. $X^2 - 2Y^2 - 2XY$,
4. $X^2 + Y^2 - XY$,
5. $X^2 - Y^2 + XY$,
6. $X^2 - Y^2 - XY$,
7. $X^2 + Y^2 - 3XY$.

A successful construction for $p = 17$ would need to involve at least six of the above factors, and would thus require at least six of them to be reducible over $\mathbb{F}_{17}$. However, only the first two are, since none of $3, 5, 12$ or $14$ is a square in $\mathbb{F}_{17}$. A solution using this method for $p = 19$ would need to involve all the above factors. This is not feasible since for example the first two are irreducible in $\mathbb{F}_{19}$, since $-1$ is not a square modulo 19.

We conclude with the following open problems.

- Is every finite field of order 3 or more EPM-rank-tight?
- What is the maximum possible number of indeterminates in a rank-discrepant EPM of specified order? In several of our constructions there are numerous entries that do not affect the determinant and could be replaced by further independent indeterminates.
Chapter 4

Almost-nonsingular Entry Pattern Matrices

Given an $n \times n$ entry pattern matrix $A$ and a field $F$, one may consider the set of non-negative integers occurring as ranks of $F$-completions of $A$, which we denote by $r_F(A)$. In Chapter 3 we considered the maximum rank of completions of an entry pattern matrix $A$ with indeterminates $x_1, \ldots, x_k$ over a field $F$. This is naturally bounded above by the generic $F$-rank of $A$, which is the rank of $A$ when considered as a matrix over the function field $F(x_1, \ldots, x_k)$. However, over a finite field, the maximum rank of an $F$-completion may be strictly less than the generic $F$-rank, and some constructions of entry patterns exhibiting this phenomenon are presented in [33].

We note that every entry pattern matrix has the zero matrix as an $F$-completion over every field $F$. Moreover, completions of rank 1 can always be obtained by assigning the same non-zero value from $F$ to every indeterminate. However, for an $m \times n$ entry pattern matrix $A$, the minimum element of $r_F(A) \setminus \{0, 1\}$ may be any integer from 2 to $\min(m, n)$. In this chapter we consider the case of square entry patterns where every $F$-completion in which two indeterminates are assigned different values is nonsingular. We refer to such a pattern as an $F$-almost-nonsingular EPM.

For a given field $F$ and positive integer $n$, one may ask for the maximum possible number $\tau_F(n)$ of indeterminates in an $n \times n$ entry pattern matrix that is almost-nonsingular over $F$. We present some results on this question in the case where $F$ is the field of real numbers, the field of rational numbers, or a finite field. The question is obviously related to the problem of determining the maximum possible dimension $\rho_F(n)$ of a linear subspace of $M_n(F)$ in which every non-zero element is nonsingular, which is 1 if $F$ is algebraically closed [Theorem 1.1.4] and at most $n$ for any field [Lemma 1.1.1]. It is easily seen that $\tau_F(n) \leq \rho_F(n) + 1$ for all $n$ and for every field $F$. This inequality is strict except in a some special cases (including when $F = \mathbb{R}$ and $n$ has an odd divisor strictly greater than 3).

Given a finite set $S = \{x_1, x_2, \ldots, x_k\}$, a field $F$ and an EPM $A \in M_{m \times n}(S)$, we recall that the $F$-completions of $A$ obtained by assigning
all the indeterminates by the same value in \( \mathbb{F} \), which have rank 0 or 1, are called uniform \( \mathbb{F} \)-completions of \( A \).

**Definition 4.0.1.** A square EPM \( A(x_1, \ldots, x_k) \) is said to be almost-nonsingular over a field \( \mathbb{F} \) (or \( \mathbb{F} \)-almost-nonsingular) if every non-uniform \( \mathbb{F} \)-completion of \( A \) is nonsingular, or equivalently if

\[
\text{for all } (a_1, \ldots, a_k) \in \mathbb{F}^k, \det A(a_1, \ldots, a_k) = 0 \iff a_1 = \cdots = a_k.
\]

**Example 4.0.1.** Let \( \mathbb{F} = \mathbb{F}_3 \) be the field of 3 elements. Write

\[
A(x, y, z) = \begin{bmatrix} y & z & x & z \\ z & y & x & x \\ x & x & y & z \\ z & x & y & y \\ x & z & y & y \end{bmatrix}.
\]

Then \( \det A = x^5 + x^4z + 2x^3y^2 + x^3yz + x^3z^2 + x^2y^3 + x^2z^3 + 2xy^3z + y^5 + y^4z + y^3z^2 + yz^4 + z^5 \). Since \( \alpha^3 = \alpha \) for all \( \alpha \in \mathbb{F}_3 \), we have

\[
\det A(a, b, c) = a + b + c + ab(a - b) + bc(b - c) + ac(c - a),
\]

for \((a, b, c) \in \mathbb{F}_3^3\). If any of \(a, b, c\) is zero, it is clear that the above expression evaluates to zero only if \(a = b = c = 0\). On the other hand if all three are non-zero then \(a^2 = b^2 = c^2 = 1\) and the above expression reduces to \(a + b + c\) which is zero in \(\mathbb{F}_3\) only if \(a = b = c\). We conclude that \(A\) is almost-nonsingular over \(\mathbb{F}_3\).

We remark that the matrix \(A\) in the above example is not almost-nonsingular over \(\mathbb{F}_5\), and has a singular completion over \(\mathbb{F}_5\) when \(x, y, z\) are respectively assigned the values 0, 1, 2.

### 4.1 Universally almost-nonsingular EPMs

In this section we demonstrate the existence of almost-nonsingular \(n \times n\) entry pattern matrices with two indeterminates, for all fields and for all \(n \geq 4\). Section 4.3 will consider how and when the number of indeterminates may be increased. Lemma 4.1.1 below is an elementary observation which is central to many of our constructions throughout this chapter.

Let \(A\) be an \(n \times n\) entry pattern matrix with \(k\) indeterminates \(x_1, \ldots, x_k\), that is almost-nonsingular over a field \(\mathbb{F}\). Suppose that \(X\) is an \(\mathbb{F}\)-completion of \(A\) in which not all entries are equal. Then \(X + \alpha J\) is a nonsingular matrix in \(M_n(\mathbb{F})\) for all \(\alpha \in \mathbb{F}\), where \(J\) is the \(n \times n\) matrix whose entries are all equal to 1.

The following lemma, which characterizes square matrices \(B\) for which \(B + \alpha J\) is always nonsingular, plays an important role throughout this chapter.

**Lemma 4.1.1.** Let \(B\) be a nonsingular matrix in \(M_n(\mathbb{F})\) for some field \(\mathbb{F}\). Then \(B + \alpha J\) is nonsingular for all \(\alpha \in \mathbb{F}\) if and only if the sum of the entries of \(B^{-1}\) is 0.
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Proof. Let $j$ be the vector whose entries are all equal to 1 and $v$ be the unique vector in $F^n$ for which $Bv = j$, and let $\beta$ denote the sum of the entries of $v$. Note that $Jv = \beta j$.

- If $\beta \neq 0$, then $(B - \beta^{-1}J)v = 0$ and $B - \beta^{-1}J$ is singular.

- If $\beta = 0$, let $\alpha \in F$. Since $B$ is nonsingular and the column space of $J$ is $\langle j \rangle$, any vector in the right nullspace of $B + \alpha J$ belongs to $\langle v \rangle$. However, $Jv = \beta v = 0$ and so $(B + \alpha J)v = Bv \neq 0$. Thus $B + \alpha J$ is nonsingular for all $\alpha \in F$.

Thus $B + \alpha J$ is nonsingular for all $\alpha \in F$ if and only if $j^T v = 0$. Since $v = B^{-1}j$, this expression is $j^T B^{-1}j$, which is the sum of the entries of $B^{-1}$.

The conclusion of Lemma 4.1.1 is stated in a manner that refers only to the matrix $B$, but is equivalent to the statement that $j^T v = 0$ where $Bv = j$. This formulation will be convenient in our applications of the lemma.

We note two immediate consequences of Lemma 4.1.1.

Corollary 4.1.2. Let $F$ be a field and $A(x_1, \ldots, x_k)$ be an EPM. Then $A$ is $F$-almost-nonsingular if and only if the matrix $A(a_1, \ldots, a_{k-1}, 0)$ is nonsingular, and $j^T : A(a_1, \ldots, a_{k-1}, 0)^{-1} \cdot j = 0$ for all non-zero choices of $(a_1, \ldots, a_{k-1})$ in $F^{n-1}$.

Theorem 4.1.3. Let $n$ be a positive integer and write $A(x, y) = xI_n + y(J_n - I_n)$. For a field $F$, $A(x, y)$ is $F$-almost-nonsingular if and only if $\text{char } F$ divides $n$.

We will say that a square entry pattern matrix is universally almost-nonsingular if it is $F$-almost-nonsingular for every field $F$. We note that the maximum dimension of a nonsingular linear vector space over an algebraically closed field is 1 [Theorem 1.1.4]. It follows that a universally almost-nonsingular entry pattern matrix can have at most 2 indeterminates. If $T(x, y)$ is a square EPM with two indeterminates, we may write $T(x, y) = xA + yB$, where $A = T(1, 0)$ and $B = T(0, 1)$ are $(0, 1)$-matrices with $A + B = J$. Then $T(x, y)$ is almost-nonsingular over the field $F$ if and only if $A + \lambda J$ is nonsingular over $F$ for every $\lambda \in F$, which occurs precisely if $A$ is nonsingular over $F$ and the sum of the entries of its inverse is zero in $F$. We remark that if there is a field $E$ of characteristic zero over which $T(x, y)$ is almost-nonsingular then the sum of entries of $A^{-1}$ is zero in $E$ [Lemma 4.1.1]. Hence, that sum of entries of $A^{-1}$ is zero in any field. Therefore, it follows from Corollary 4.1.2 that $T(x, y)$ is universally almost-nonsingular if and only if there is a field of characteristic zero over which $T(x, y)$ is almost-nonsingular.

We now present a construction for $n \geq 4$ of an $n \times n$ EPM that is universally almost-nonsingular. We also provide examples of entry pattern matrices in two indeterminates that are almost-nonsingular only in
particular positive characteristics. It follows from Lemma 4.1.1 that if an
entry pattern matrix with two indeterminates is almost-nonsingular over
a single field of a particular characteristic, then it is almost-nonsingular
over all fields of that characteristic.

We begin with the following observation, which was noted in another
context in Theorem 3.2.3.

**Lemma 4.1.4.** Let \( T(x, y) \) be an \( n \times n \) entry pattern matrix. Then
\[
\det_Q(T(x, y)) = (x - y)^{n-1}(\alpha x + \beta y),
\]
for some \( \alpha, \beta \in \mathbb{Z} \).

**Proof.** After subtracting Row \( n \) of \( T(x, y) \) from each of the previous rows,
we obtain a matrix in which every entry in Rows 1, 2, \ldots, \( n-1 \) is either
0 or \( \pm(x - y) \). The determinant of this matrix is either zero or is a sum
of terms of the form \( \pm(x - y)^{n-1} \cdot z \), where \( z \) is either \( x \) or \( y \).

Lemma 4.1.4 enables us to describe the possible rank distributions of
EPMs with two indeterminates. If \( \det(T(x, y)) = (x - y)^{n-1}(\alpha x + \beta y) \),
then for the field \( F \) we have the following observations.

- If \( \det(A) \) is the zero polynomial in \( F[x, y] \), then \( T \) has no nonsing-
  ular \( F \)-completion.
- If \( \alpha = -\beta \neq 0 \) in \( F \), then \( T(x, y) \) is \( F \)-almost-nonsingular.
- If \( \alpha \neq -\beta \), then suppose without loss of generality that \( \alpha \neq 0 \).
  Then \( T(1, 0) \) is nonsingular and \( T(x, y) = (x - y)T(1, 0) + yJ \), and
  since \( J \) has rank 1, every \( F \)-completion of \( T \) in which \( x \) and \( y \) are
  assigned different values has rank \( n \) or \( n-1 \). Since \( \det(T(-\frac{\beta}{\alpha}, 1)) = 0 \),
  there exist completions of \( T \) that have rank \( n-1 \). Thus \( r_F(T) = \{0, 1, n-1, n\} \)
in this case.

Thus if \( T(x, y) \) has an \( F \)-completion of rank \( n \), then \( r_F(T) \subseteq \{0, 1, n-1, n\} \).

We proceed now to the construction of universally almost-nonsingular
\( n \times n \) matrices for \( n \geq 4 \). Suppose that \( T(x, y) = xA + yB \) is an \( F \)-
almost-nonsingular matrix of size \( n \times n \). Then \( A \) and \( B \) are nonsingular
\( (0, 1) \)-matrices with \( A + B = J \), and we may assume that the number of
entries equal to 1 in \( A \) is at most equal to the number in \( B \). Moreover,
after permuting rows if necessary, we may assume that the entries on the
main diagonal of \( A \) are all equal to 1. By Lemma 4.1.1, the sum in \( F \) of
the entries in \( A^{-1} \) is zero.

- If \( n = 2 \), then the only candidate for \( A \) is \( I_2 \), whose inverse has the
  required property only if \( \text{char} F = 2 \).

- If \( n = 3 \), then \( A \) has either three or four entries equal to 1. In
  the first case, \( A = I_3 \) and the entries of \( A^{-1} \) sum to zero only if
  \( \text{char} F = 3 \).
In the second case, we may assume (after transposing if necessary) that $A$ is a lower triangular matrix with exactly one entry below the main diagonal equal to 1. Then the sum of the entries of $A^{-1}$ is 2, and $T(x, y)$ is $F$-almost-nonsingular only if $\text{char} F = 2$.

From these observations it is clear that a universally nonsingular EPM must have size at least $4 \times 4$. The matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is evidently nonsingular (over every field) and satisfies $Av = j$ where $v = (1, -1, 1, -1)^T$. Since the entries of $v$ sum to zero in $\mathbb{Q}$ it follows from Corollary 4.1.2 that the entry pattern matrix $T_4(x, y) = xA + y(J - A)$ is universally nonsingular. We remark that $T_4(x, y)$ is also symmetric, and may be used as the starting point for a construction of symmetric universally nonsingular EPMs of all higher orders, as outlined in the following lemma.

**Lemma 4.1.5.** Let $n$ be a positive integer with $n \geq 4$. Then there exists an $n \times n$ universally almost-nonsingular entry pattern matrix which is symmetric.

**Proof.** The matrix $T_4(x, y)$ is symmetric and universally nonsingular, with determinant $(x - y)^4$. For $n \geq 4$, assume that $T_n(x, y)$ is a universally almost-nonsingular $n \times n$ EPM. We write

$$T_{n+1}(x, y) = \begin{cases} [T_n(x, y) \ c_n] & \text{if } n \text{ is even}, \\ [T_n(x, y) \ c_n] & \text{if } n \text{ is odd}, \end{cases}$$

where $c_n, r_n$ are respectively the $n$-th column and $n$-th row of $T_n$. After subtracting Row $n$ of $T_{n+1}$ from Row $(n + 1)$ and Column $n$ from Column $(n + 1)$, we obtain a matrix in which the blocks on the main diagonal of $T_{n+1}$ are $T_n$ and $\pm(x - y)$ and the off-diagonal entries are all zero. Therefore, $\det T_{n+1} = \pm(x - y) \det T_n$. Since $T_n(x, y)$ is universally almost-nonsingular, so is $T_{n+1}$. By induction on $n$, $T_n$ is universally almost-nonsingular for every $n \geq 4$. \qed

### 4.2 From a nonsingular pseudo-EPM to an almost-nonsingular EPM

The examples in Section 4.1 of entry pattern matrices that are almost-nonsingular over particular fields all have exactly two independent indeterminates. In this section we present our main technical device for the
construction of almost-nonsingular EPMs having greater numbers of indeterminates. The family \( \{ T_m, m \geq 4 \} \) of symmetric universally almost-nonsingular EPMs constructed in Section 4.1 is one of the ingredients; the other is a nonsingular pseudo-EPM, defined as follows:

**Definition 4.2.1.** A pseudo-entry pattern matrix (pseudo-EPM) \( P \) on a finite set of indeterminates \( X \) is a matrix in which every entry is either 0 or \( x \) or \(-x\) for some \( x \in X \). We say that \( P \) has \( k \) indeterminates if \( k \) is the number of elements \( x \) of \( X \) for which either \( x \) or \(-x\) is an entry of \( P \).

An \( n \times n \) pseudo-EPM is said to be \( \mathbb{F} \)-nonsingular for a field \( \mathbb{F} \) if each of its nonzero \( \mathbb{F} \)-completions is a nonsingular matrix in \( M_n(\mathbb{F}) \).

For example, the matrix \( \begin{bmatrix} -x & y \\ y & x \end{bmatrix} \) is a \( \mathbb{R} \)-nonsingular pseudo-EPM with two indeterminates.

Nonsingular pseudo-EPMs are not as elusive as almost-nonsingular entry patterns over specified fields, and may for example be constructed from regular representations of suitably chosen field or semifield extensions. In this section we show how to transform an \( n \times n \) \( \mathbb{F} \)-nonsingular pseudo-EPM with \( k \) indeterminates into an \( mn \times mn \) \( \mathbb{F} \)-almost-nonsingular entry pattern matrix with \( k + 1 \) indeterminates, for any \( m \geq 4 \) and for any field \( \mathbb{F} \). Our construction will be used in Section 4.3 to demonstrate the existence of almost-nonsingular entry pattern matrices with arbitrarily high numbers of indeterminates over certain fields, including all real fields and all finite fields.

**Definition 4.2.2.** Let \( P \) be an \( n \times n \) pseudo-EPM with indeterminates \( x_1, \ldots, x_k \). Then, for \( m \geq 4 \), \( \Phi_m(P)(x_1, \ldots, x_n, x) \) is the EPM of size \( mn \times mn \), with indeterminates \( x_1, \ldots, x_k, x \), in which the \( m \times m \) block in the \((i, j)\)-position is given by

\[
\begin{align*}
T_m(x_t, x) & \text{ if } P_{ij} = x_t, \\
T_m(x, x_t) & \text{ if } P_{ij} = -x_t, \\
xJ_m & \text{ if } P_{ij} = 0,
\end{align*}
\]

where \( T_m \) is defined as in Section 4.1.

Thus \( \Phi_m(P) \) is constructed from \( P \) by first replacing every appearance of \( x_t \) with \( x_t - x \), every appearance of \( -x_t \) with \( x - x_t \) and every appearance of 0 with \( x - z \), and then replacing every difference of the form \( y - z \) with the \( m \times m \) block \( T_m(y, z) \). We now state our main theorem, which is proved by using standard properties of Kronecker products. For matrices \( S \) and \( T \) of sizes \( n \times n \) and \( m \times m \) respectively, \( \det(S \otimes T) = (\det S)^m (\det T)^n \). We also note that \( (S_1 \otimes T_1)(S_2 \otimes T_2) = (S_1 S_2) \otimes (T_1 T_2) \), for \( n \times n \) matrices \( S_1 \) and \( S_2 \) and \( m \times m \) matrices \( T_1 \) and \( T_2 \).

**Theorem 4.2.1.** Let \( P(x_1, \ldots, x_k) \) be a nonsingular \( n \times n \) pseudo-EPM with \( k \) indeterminates on a field \( \mathbb{F} \) and let \( m \geq 4 \) be a positive integer. Then \( \Phi_m(P)(x_1, \ldots, x_k, x) \) is an \( \mathbb{F} \)-almost-nonsingular EPM.
4.2. Nonsingular pseudo-EPMs

Let $a_1, \ldots, a_k, a$ be elements of $\mathbb{F}$ and write $Q = \Phi_n(P)(a_1, \ldots, a_k, a)$. We consider the $mn \times mn$ matrix $Q$ to be partitioned into $m \times m$ blocks. We write $A = T_m(1, 0)$ and $B = T_m(0, 1)$. Then the block in the $(i, j)$-position of $Q$ is given by

$$
\begin{cases}
a_i A + a B = (a_i - a) A + a J & \text{if } P_{ij} = x_i, \\
a A + a_i B = (a - a_i) A + a_i J & \text{if } P_{ij} = -x_i, \\
a A + a B = a J & \text{if } P_{ij}(i, j) = 0.
\end{cases}
$$

Hence

$$Q = P(a_1 - a, \ldots, a_k - a) \otimes A + M \otimes J,$$

for some $n \times n$ matrix $M$. We write $P_1$ for $P(a_1 - a, \ldots, a_k - a)$. If $P_1$ is nonsingular, then

$$Q = (P_1 \otimes A) \left( I_{mn} + P_1^{-1} M \otimes A^{-1} J \right).$$

From the universal nonsingularity of $T_n$ it follows from Lemma 4.1.1 that $JA^{-1} J = O_m$, hence that $(A^{-1} J)^2 = O_m$. Then

$$(P_1^{-1} M \otimes A^{-1} J)^2 = (P_1^{-1} M)^2 \otimes O_{m \times m} = O_{m \times mn},$$

and in particular $N = P_1^{-1} M \otimes A^{-1} J$ is nilpotent. It follows that $I + N$ is similar to an upper unitriangular matrix and that $\det(I + N) = 1$. Finally,

$$\det Q = (\det P_1)^m (\det(A(I + N)))^n = (\det P_1)^m \det(A)^n.$$

We conclude for $a_1, \ldots, a_k, a \in \mathbb{F}$ that $\Phi_n(P)(a_1, \ldots, a_k, a)$ is nonsingular if $P(a_1 - a, \ldots, a_k - a)$ is nonsingular. Thus every non-uniform $\mathbb{F}$-completion of $\Phi_n(P)(x_1, \ldots, x_k, x)$ is nonsingular if every non-zero $\mathbb{F}$-completion of $P(x_1, \ldots, x_k)$ is nonsingular, as required. \hfill \Box

The following corollary is an immediate consequence of Theorem 4.2.1, since an $\mathbb{F}$-nonsingular pseudo-EPM may be obtained from any $\mathbb{F}$-almost-nonsingular EPM by assigning the value zero to one indeterminate and leaving the others unchanged.

**Corollary 4.2.2.** Let $m \geq 4$ be a positive integer. If there exists an $n \times n$ $\mathbb{F}$-almost-nonsingular entry pattern matrix, then there exists a $(mn) \times (mn)$ $\mathbb{F}$-almost-nonsingular entry pattern matrix with the same number of indeterminates.

An alternative formulation states that for any field $\mathbb{F}$, and integers $n$ and $m$ with $m \geq 4$,

$$\tau_\mathbb{F}(n) \leq \tau_\mathbb{F}(mn),$$

where $\tau_\mathbb{F}(n)$ is the maximum possible number of indeterminates in an $n \times n$ $\mathbb{F}$-almost-nonsingular EPM.

We complete this section by proving a general form of the inequality 4.1 and therefore showing that it is still true if we drop the assumption $m \geq 4$. 


Lemma 4.2.3. Let $n$ be a positive integer and $\mathbb{F}$ be a field. Assume that $n = n_1 + n_2 + \cdots + n_k$ where $n_i$ is a positive integer for all $i$. Then $\tau_{\mathbb{F}}(n) \geq \min\{\tau_{\mathbb{F}}(n_i) : i = 1, \ldots, k\}$.

Proof. Write $m = \min\{\tau_{\mathbb{F}}(n_i) : i = 1, \ldots, k\}$. Then for each $i$, there is an $n_i \times n_i$ $\mathbb{F}$-almost-nonsingular EPM $A_i$ which has $m$ indeterminates. By using the same collection of indeterminates for all $A_i$, we may write $A_i = A_i(x_1, \ldots, x_m)$ for every $i$. For each $i$, let $B_i$ denote the pseudo-EPM obtained from $A_i$ by setting $x_m = 0$. Then $B_i$ is $\mathbb{F}$-nonsingular with $m - 1$ indeterminates for all $i$. Let $A = \text{diag}(B_1, \ldots, B_m)$ be the block $mn \times mn$ diagonal matrix which has $B_1, \ldots, B_m$ on the main diagonal. Then

- $A$ is an $\mathbb{F}$-nonsingular pseudo-EPM since $\det A = \prod_i \det B_i$,
- $j^T A^{-1} j = \sum_i j^T B_i^{-1} j = 0$.

Hence, the entry pattern matrix obtained from $A$ by replacing every appearance of zero with $x_m$ is $\mathbb{F}$-almost-nonsingular [see Corollary 4.1.2]. This proves that there is an $n \times n$ $\mathbb{F}$-almost-nonsingular EPM with $m$ indeterminates. Therefore,

$$\tau_{\mathbb{F}}(n) \geq m \text{ or } \tau_{\mathbb{F}}(n) \geq \min\{\tau_{\mathbb{F}}(n_i) : i = 1, \ldots, k\}.$$

By writing $mn = n + n + \cdots + n$, we easily see the inequality 4.1 holds for every positive integer $m$.

$$\tau_{\mathbb{F}}(n) \leq \tau_{\mathbb{F}}(mn) \text{ for all } m \geq 1.$$

### 4.3 Constructions over particular fields

In this section, we consider lower bounds for the numbers $\tau_{\mathbb{F}}(n)$, where $\mathbb{F}$ is either the field of real numbers or the field of rational numbers or a finite field. We will show that $\tau_\mathbb{R}$, and $\tau_\mathbb{Q}$ are unbounded functions on the natural numbers. We recall that $\rho_{\mathbb{F}}(n)$ denotes the maximum dimension of a linear subspace of $M_n(\mathbb{F})$ in which every non-zero element is nonsingular (we will refer to such a subspace as a nonsingular space).

We consider the relationship between the functions $\tau$ and $\rho$, over the fields of real and rational numbers and over finite fields.

We note some properties of the function $\rho_{\mathbb{F}}$. For any field $\mathbb{F}$, $\rho_{\mathbb{F}}(n)$ is bounded above by $n$ [Lemma 1.1.1]. If $\mathbb{F}$ admits a field extension $\mathbb{E}$ of degree $n$, then $\rho_{\mathbb{E}}(n) = n$. Thus $\rho_{\mathbb{F}}(n) = n$ for all $n$ if $\mathbb{F}$ admits field extensions of all degrees, which occurs for example if $\mathbb{F}$ is a finite field or if $\mathbb{F}$ is a finite extension of $\mathbb{Q}$ [Item 2., Theorem 1.1.4].

At the other extreme are algebraically closed fields, for which $\rho$ has the constant value 1 [Item 1., Theorem 1.1.4].

The situation over the real numbers is considerably more complicated as well as more interesting, and is briefly outlined in Section 4.3.1 below.
Writing $\tau_F(n)$ for the maximum possible number of indeterminates in an $n \times n \mathbb{F}$-nonsingular pseudo-EPM, we note the following inequalities which apply for all fields $\mathbb{F}$.

- $\tau_F(n) \leq \rho_F(n)$.
  We remark that this inequality may be strict, for example $1 = \tau_{\mathbb{Q}(\sqrt{-1})}(2) < \rho_{\mathbb{Q}(\sqrt{-1})}(2) = 2$.

- $\tau_F(n) \leq \tau_F(n) + 1$.
  If $P(x_1, \ldots, x_k)$ is an $\mathbb{F}$-almost-nonsingular EPM for some field $\mathbb{F}$, then $P(0, x_2, \ldots, x_k)$ is an $\mathbb{F}$-nonsingular pseudo-EPM.

- $\tau_F(n) \leq \rho_F(n) + 1$.
  We will identify some cases where equality is attained here.

- $\tau_F(mn) \geq \tau_F(n) + 1$, for every integer $m \geq 4$.
  This is immediate from Theorem 4.2.1.

### 4.3.1 Constructions over $\mathbb{R}$

The values of the function $\rho_{\mathbb{R}}$, which are given by the Radon-Hurwitz numbers, were determined in a celebrated paper of Adams in 1962 [1]. Recall that for an integer $n$, the Radon-Hurwitz number $\rho(n)$, which is equal to $\rho_{\mathbb{R}}(n)$, is given by

$$\rho(n) = 2^b + 8c,$$

where $b$ and $c$ are the unique integers for which $0 \leq b \leq 3$ and $2^{b+4c}$ is the highest power of 2 that divides $n$. The existence of a nonsingular space of dimension $\rho_{\mathbb{R}}(n)$ in $M_n(\mathbb{R})$ had been established by Radon in 1922 [27]. An alternative construction of such spaces is given by Adams, Lax and Phillips in [2], showing that in certain cases the bound can be attained by a space of symmetric matrices [see Theorem 1.1.3]. It is their construction that we employ in this section, along with Theorem 4.2.1, to provide examples of $\mathbb{R}$-almost-nonsingular entry pattern matrices with unbounded numbers of indeterminates.

The following theorem summarizes our knowledge of the function $\tau_{\mathbb{R}}$.

**Theorem 4.3.1.** Let $n$ be a positive integer. Then

1. $\tau_{\mathbb{R}}(1) = \tau_{\mathbb{R}}(2) = \tau_{\mathbb{R}}(3) = 1$, $\tau_{\mathbb{R}}(4) = 2$ and $\tau_{\mathbb{R}}(6) = 3$.
2. If $n$ has an odd divisor greater than 3, then $\tau_{\mathbb{R}}(n) = \rho(n) + 1$.
3. If $n = 3 \cdot 2^k$, then $\rho(\frac{n}{3}) + 1 \leq \tau_{\mathbb{R}}(n) \leq \rho(n) + 1$.
4. If $n = 2^k$, then $\rho(\frac{n}{4}) + 1 \leq \tau_{\mathbb{R}}(n) \leq \rho(n) + 1$.

That $\tau_{\mathbb{R}}(1) = \tau_{\mathbb{R}}(2) = \tau_{\mathbb{R}}(3) = 1$ follows from the demonstration in Section 4.1 that an EPM that has two or more indeterminates and is almost-nonsingular over a field of characteristic zero must have size at least $4 \times 4$. The existence of the universally nonsingular EPM $T_4$
establishes that $\tau_{R}(4) \geq 2$. To see that $\tau_{R}(4) \leq 2$, let $P(x, y, z)$ be a $4 \times 4$ EPM with three indeterminates. Then we may assume that $x$ occurs at most five times as an entry of $P(x, y, z)$. Then $P(1, 0, 0)$ is a $(0, 1)$-matrix that has at most five entries equal to 1. It is easily confirmed that such a matrix, if nonsingular, cannot satisfy the condition of Lemma 4.1.1 in characteristic zero. Thus $P(x, y, z)$ cannot be $R$-almost-nonsingular, and we conclude that $\tau_{R}(4) = 2$. That $\tau_{R}(6) \leq 3$ is clear since $\rho(6) = 2$. The equality is demonstrated by the $6 \times 6$ $R$-almost-nonsingular EPM with three indeterminates in Example 1.1.2. This proves Item 1. in Theorem 4.3.1.

The remaining items depend on Theorem 4.2.1, which provides for all $m \geq 4$ and for any field $F$ a means of passing from an $n \times n$ $F$-nonsingular pseudo-EPM with $k$ indeterminates to an $mn \times mn$ $F$-almost-nonsingular EPM with $k+1$ indeterminates. We complete the proof of Theorem 4.3.1 by establishing that the dimension bound $\rho(2^r)$ for a $2^r \times 2^r$ $R$-nonsingular space is attained by the space of completions of a pseudo-EPM, for all $r \geq 0$. Since the value of $\rho(n)$ depends only on the maximal 2-power divisor of $n$, Item 2. of Theorem 4.3.1 will follow immediately by applying Theorem 4.2.1 with the greatest odd divisor of $n$ as the value of $m$. Items 3. and 4. will follow similarly, taking $m = 6$ and $m = 4$ respectively in the application of Theorem 4.2.1.

It remains to prove the following lemma.

**Lemma 4.3.2.** Let $r$ be a nonnegative integer. Then there exists a $R$-nonsingular $2^r \times 2^r$ pseudo-entry pattern matrix with $\rho(2^r)$ indeterminates.

In order to prove Lemma 4.3.2, we first note that $\rho(n) = n$ only if $n \in \{1, 2, 4, 8\}$. Examples of $R$-nonsingular spaces of dimension $\rho(n)$ in these cases arise from the four distinct semifields that are defined over the real numbers; namely the fields $R$ and $C$ of real and complex numbers in dimensions 1 and 2, the quaternion division algebra $H$ in dimension 4 and the (non-associative) octonion semifield $O$ in dimension 8. The additive groups of these structures may respectively be identified with the spaces of real completions of the pseudo-entry pattern matrices below. In each case the pseudo-entry pattern matrix represents the $R$-linear endomorphism of $R, C, H$ or $O$ defined as left multiplication by a generic element $\sum x_i e_i$ with respect to a suitably chosen basis $\{e_i\}$. The semifield structures on $R, C, H$ and $O$ guarantee that the pseudo-entry pattern matrices $A_0, A_1, A_2$ and $A_3$ defined below are $R$-nonsingular. In the first three cases, the spaces of real completions of $A_0, A_1$ and $A_2$ are closed under multiplication and isomorphic as associative rings to $R, C$ and $H$.

\[
A_0 = \begin{bmatrix} x_1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix},
\]
Thus Lemma 4.3.2 holds for $r = 0, 1, 2, 3$. To prove the lemma we note that

\[ \rho(2^{r+1}) = \rho(2^r) + 8, \]

for every non-negative integer $r$. To complete the proof of Lemma 4.3.2, it is now sufficient to demonstrate a means of passing from an $n \times n$ \( \mathbb{R} \)-nonsingular pseudo-EPM with $k$ indeterminates to a $16n \times 16n$ \( \mathbb{R} \)-nonsingular pseudo-EPM with $k + 8$ indeterminates. To achieve this we present a two-step construction due to Adams, Lax and Phillips [2], adapted slightly to the context of pseudo-entry pattern matrices.

**Lemma 4.3.3.** Let $P$ be an $n \times n$ \( \mathbb{R} \)-nonsingular pseudo-EPM with $k$ indeterminates. Write $P'$ for the symmetric $2n \times 2n$ pseudo-EPM

\[
\begin{bmatrix}
xI_n & P \\
P^T & -xI_n
\end{bmatrix},
\]

where $x$ represents an indeterminate not appearing in $P$. Then $P'$ is an \( \mathbb{R} \)-nonsingular symmetric pseudo-EPM with $k + 1$ indeterminates.

**Proof.** Let $P_0 = \begin{bmatrix} \lambda I_n & A \\ A^T & -\lambda I_n \end{bmatrix}$ be an \( \mathbb{R} \)-completion of $P'$. Suppose that $P_0$ is singular and let $u$ and $v$ be vectors in \( \mathbb{R}^n \) for which \( \begin{bmatrix} u \\ v \end{bmatrix} \) is a non-zero vector in the right nullspace of $P_0$. Then

\[ \lambda u + Av = 0, \text{ and } A^T u - \lambda v = 0. \]

It follows that $\lambda u^T u = -u^T Av = -v^T A^T u = -\lambda v^T v$. Since $u^T u$ and $v^T v$ are not both zero, it follows that $\lambda = 0$. Then $A = 0$; otherwise $A$ is nonsingular and so is $P_0 = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$. We conclude that the only singular \( \mathbb{R} \)-completion of $P'$ is the zero matrix. \( \square \)

We now write $C$ for the matrix obtained from $A_3$ by setting $x_1 = 0$. Then $C$ is an $8 \times 8$ \( \mathbb{R} \)-nonsingular skew-symmetric pseudo-EPM with seven indeterminates.

**Lemma 4.3.4.** Let $Q$ be an $n \times n$ symmetric \( \mathbb{R} \)-nonsingular pseudo-EPM with $k$ indeterminates, none of which appear in $C$. Then the Kronecker sum of $Q$ and $C$, given by

\[ Q \oplus C = Q \otimes I_8 + I_n \otimes C, \]

is an \( \mathbb{R} \)-nonsingular $8n \times 8n$ pseudo-EPM with $k + 7$ indeterminates.
Proof. That $Q \oplus C$ is a pseudo-EPM is clear from the definition of the Kronecker product and the fact that the entries on the main diagonal of $C$ are all zero. Let $A$ be a non-zero $\mathbb{R}$-completion of $Q \oplus C$. Then $A = Q_0 \oplus C_0$, where $Q_0$ and $C_0$ are $\mathbb{R}$-completions of $Q$ and $C$, not both zero. Since $Q$ and $C$ are $\mathbb{R}$-nonsingular, at least one of $Q_0$ and $C_0$ is nonsingular. The spectrum of $Q_0 \oplus C_0$ consists of all numbers of the form $\lambda + \mu$ where $\lambda$ and $\mu$ are eigenvalues of $Q_0$ and $C_0$ respectively [Theorem 4.4.5, [17]]. Since every eigenvalue of $Q_0$ is real, every non-zero eigenvalue of $C$ is pure imaginary and at least one of $Q_0$ and $C_0$ has no zero eigenvalue, so it follows that $A$ is nonsingular, as required. \hfill \square

By composing the constructions of Lemmas 4.3.3 and 4.3.4, we may extend any $n \times n$ $\mathbb{R}$-nonsingular pseudo-EPM with $k$ indeterminates to a $16n \times 16n$ $\mathbb{R}$-nonsingular pseudo-EPM with $k + 8$ indeterminates. By applying this extension repeatedly to each of $A_0, A_1, A_2$ and $A_3$, we obtain for every $r \geq 0$ a $2^r \times 2^r \mathbb{R}$-nonsingular pseudo-EPM with $\rho(2^r)$ indeterminates. This completes the proof of Lemma 4.3.2 and hence also the proof of Theorem 4.3.1.

We conclude this section with a remark on the symmetric case. We write $\tau^S_\mathbb{R}(n)$ for the maximum possible number of indeterminates in a symmetric $n \times n$ entry pattern matrix that is $\mathbb{R}$-almost-nonsingular. Then $\tau^S_\mathbb{R}(n) \leq \rho^S_\mathbb{R}(n) = \rho\left(\frac{n}{2}\right)$ if $n$ is even [Theorem 1.1.4]. Since the operation of Theorem 4.2.1 preserves symmetry, we obtain the following "symmetric version" of Theorem 4.3.1.

**Theorem 4.3.5.** Let $n$ be a positive integer. Then

1. $\tau^S_\mathbb{R}(1) = \tau^S_\mathbb{R}(2) = \tau^S_\mathbb{R}(3) = 1$, $\tau^S_\mathbb{R}(4) = 2$ and $\tau^S_\mathbb{R}(6) = 3$.

2. If $n$ is odd and $n > 3$, then $\tau^S_\mathbb{R}(n) = \tau_\mathbb{R}(n) = 2$.

3. If $n$ is even and $n$ has an odd divisor greater than 3, then $\tau^S_\mathbb{R}(n) = \rho\left(\frac{n}{2}\right) + 2$.

4. If $n = 3 \cdot 2^k$, then $\rho\left(\frac{n}{12}\right) + 2 \leq \tau^S_\mathbb{R}(n) \leq \rho\left(\frac{n}{2}\right) + 2$.

5. If $n = 2^k$, then $\rho\left(\frac{n}{8}\right) + 2 \leq \tau^S_\mathbb{R}(n) \leq \rho\left(\frac{n}{2}\right) + 2$.

Proof. The first item is a consequence of the proof of Item 1. in Theorem 4.3.1 since $T_4$ and the $6 \times 6$ EPM mentioned in that proof are symmetric. In order to prove the second item, we first note that $\tau^S_\mathbb{R}(n) \leq \tau_\mathbb{R}(n)$ for every $n$. Therefore, if $n > 3$ is odd then $\tau^S_\mathbb{R}(n) \leq 2$. The equality is demonstrated by the existence of the symmetric universally almost-nonsingular EPM $T_4$.

If $n = m2^k$ where $m \geq 4$ and $k \geq 1$ then there exists a $\mathbb{R}$-nonsingular $2^{k-1} \times 2^{k-1}$ pseudo-entry pattern matrix with $\rho(2^{k-1})$ indeterminates [Lemma 4.3.3]. By Lemma 4.3.3, there is a $\mathbb{R}$-nonsingular $2^k \times 2^k$ pseudo-entry pattern matrix with $\rho(2^{k-1}) + 1$ indeterminates. Since the operation of Theorem 4.2.1 preserves symmetry, the lower bounds for $\tau^S_\mathbb{R}(n)$ in Item 3. and Item 4. follow by taking $m = 6$ and $m = 4$ respectively. The upper bounds in those items are immediate from the fact that $\tau^S_\mathbb{R}(n) \leq \rho^S_\mathbb{R}(n)$ for every integer $n$. \hfill \square
4.3.2 Constructions over \( \mathbb{Q} \)

As in Section 4.3.1, our primary method for constructing examples of \( \mathbb{Q} \)-almost-nonsingular entry pattern matrices, and hence obtaining lower bounds for the values of the function \( \tau_{\mathbb{Q}} \), is to apply Theorem 4.2.1 to \( \mathbb{Q} \)-nonsingular pseudo-EPMs. Our strategy for constructing a \( \mathbb{Q} \)-nonsingular pseudo-EPM is to use the left regular representation of a field extension of \( \mathbb{Q} \) obtained by adjoining a root of an irreducible polynomial with many zero coefficients and with all nonzero coefficients equal to 1 or -1.

Let \( p(x) = \sum_{i=0}^{n} a_i x^i \) be a \( \mathbb{Q} \)-irreducible polynomial of degree \( n \), and let \( \alpha \) be a root of \( p(x) \). Let \( \mathbb{F} \) denote the field \( \mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/\langle p(x) \rangle \). Then \( \mathcal{B} = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\} \) is a \( \mathbb{Q} \)-basis of \( \mathbb{F} \). For every element \( a \) of \( \mathbb{F} \) we write \( \Phi_a \) for the \( \mathbb{Q} \)-linear transformation of \( \mathbb{F} \) defined for \( b \in \mathbb{F} \) by \( \Phi_a(b) = ab \), and we write \( M_a \) for the matrix of \( \Phi_a \) with respect to \( \mathcal{B} \).

Since \( \mathbb{F} \) is a field, \( \Phi_a \) is invertible for every non-zero \( a \). Moreover the mapping that takes \( a \in \mathbb{F} \) to \( \Phi_a \) is an \( \mathbb{F} \)-linear transformation, and hence \( \{M_a : a \in \mathbb{F}\} \) is an \( \mathbb{F} \)-nonsingular space of dimension \( n \).

The matrix \( M_a \) has the entry 1 in every position of the first subdiagonal, has \(-a_i\) in the \((i,n)\)-position and has zeros elsewhere. More generally \( M_{\alpha^k} = (M_a)^k \) has the entry 1 in every position of the \( k \)-th subdiagonal and otherwise has zero entries in the first \( n-k \) columns. If \( a_i \in \{0,1,-1\} \) for each \( i \), and if there are \( r \) values of \( k \) for which the matrices \( M_{\alpha^k} \) have entries in \( \{0,1,-1\} \), and have the property that the sets of positions in which their nonzero entries occur are pairwise disjoint, then we may use these data to construct a \( \mathbb{Q} \)-nonsingular pseudo-EPM with \( r \) indeterminates.

We employ this technique to construct

- an \( n \times n \) \( \mathbb{Q} \)-nonsingular pseudo-EPM with \( n \) indeterminates, when \( n \) is a power of 2, from the \( \mathbb{Q} \)-irreducible polynomial \( x^n + 1 \), and
- for any positive integer \( n \), an \( n \times n \) \( \mathbb{Q} \)-nonsingular pseudo-EPM with \( \lceil \frac{n}{2} \rceil \) indeterminates, from the \( \mathbb{Q} \)-irreducible polynomial \( x^n - x - 1 \).

**Lemma 4.3.6.** Let \( n \) be a power of 2. Then there exists a \( \mathbb{Q} \)-nonsingular \( n \times n \) pseudo-entry pattern matrix with \( n \) indeterminates.

**Proof.** Let \( p_n(x) = x^n + 1 \). Then

\[
p_n(x + 1) = (x + 1)^n + 1 = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + 2.
\]

Since \( n \) is a power of 2, \( \binom{n}{i} \) is even for all \( i \) with \( 1 \leq i \leq n - 1 \). However 4 does not divide the constant term in \( p_n(x + 1) \). Thus \( p_n(x + 1) \) is irreducible over \( \mathbb{Q} \) by the Eisenstein criterion, hence so also is \( p_n(x) \). If \( \alpha \) is a root of \( p_n(x) \), then \( (M_{\alpha^k})_{ij} = \begin{cases} 
1 & \text{if } j = i - k, \\
-1 & \text{if } j = i - k + n, \\
0 & \text{otherwise}.
\end{cases} \)
It is easily observed that no two of the matrices \( \{ M_1, M_\alpha, \ldots, M_\alpha^{n-1} \} \) have a non-zero entry in the same position. This implies that
\[
x_0 M_1 + x_1 M_\alpha + \cdots + x_{n-1} M_\alpha^{n-1}
\]
is an \( n \times n \) \( \mathbb{Q} \)-nonsingular pseudo-entry pattern matrix with \( n \) indeterminates, completing the proof.

We remark that the matrix of Lemma 4.3.6 is the generic skew-circulant matrix of size \( n \times n \). This pseudo-entry pattern matrix is \( \mathbb{Q} \)-nonsingular precisely when \( n \) is a power of 2.

When \( n \) has an odd divisor, we use the polynomial \( q_n(x) = x^n - x - 1 \) to construct an \( n \times n \) \( \mathbb{Q} \)-nonsingular pseudo-EPM with \( \left\lfloor \frac{n}{2} \right\rfloor \) indeterminates. The \( \mathbb{Q} \)-irreducibility of \( q_n(x) \) for all \( n \geq 2 \) is demonstrated in [28].

Lemma 4.3.7. For every integer \( n \geq 2 \), there is a \( \mathbb{Q} \)-nonsingular \( n \times n \) pseudo-entry pattern matrix with \( \left\lfloor \frac{n}{2} \right\rfloor \) indeterminates.

Proof. Let \( \alpha \) be a root of \( q_n(x) \) and let \( \mathbb{F} = \mathbb{Q}(\alpha) \). Then \( \{ 1, \alpha, \ldots, \alpha^{n-1} \} \) is a \( \mathbb{Q} \)-basis of \( \mathbb{F} \), with respect to which we may write the matrix of left multiplication by \( \alpha^k \) as follows, for \( 1 \leq k \leq n-1 \).
\[
(M_\alpha^k)_{ij} = \begin{cases} 
1 & \text{if } i - j = k, \text{ or } j - i = n - k, \text{ or } j - i = n - k - 1 \text{ and } i \neq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

For odd values of \( i \) in the range \( 1, \ldots, n-1 \), the sets of positions in which \( M_\alpha^i \) have non-zero entries are pairwise disjoint, and disjoint also from the main diagonal. It follows that
\[
x_0 I_n + \sum_{1 \leq i \leq n-1, \ i \ odd} x_i M_\alpha^i
\]
is a \( \mathbb{Q} \)-nonsingular pseudo-entry pattern matrix with \( \left\lfloor \frac{n}{2} \right\rfloor \) indeterminates, completing the proof.

By applying Theorem 4.2.1 to the pseudo-entry pattern matrices constructed in Lemma 4.3.6 and Lemma 4.3.7, we obtain partial lower bounds for \( \tau_\mathbb{Q} \) as follows

Theorem 4.3.8. Let \( n \) be a positive integer.

- If \( n = m \cdot 2^k \) where \( m \geq 4 \), then \( \tau_\mathbb{Q}(n) \geq 2^k + 1 \). In particular, \( \tau_\mathbb{Q}(3 \cdot 2^k) = \tau_\mathbb{Q}(6 \cdot 2^{k-1}) \geq 2^{k-1} + 1 \) and \( \tau_\mathbb{Q}(2^k) = \tau_\mathbb{Q}(4 \cdot 2^{k-2}) \geq 2^{k-2} + 1 \).

- If \( n \) has a divisor \( d \geq 4 \), then \( \tau_\mathbb{Q}(n) \geq \left\lfloor \frac{n}{2d} \right\rfloor + 1 \).

We conclude this section by showing the function \( \tau_\mathbb{Q} \) is bounded below by a function that increases linearly with \( n \).

Theorem 4.3.9. Let \( n \) be a non-negative integer. Then \( \tau_\mathbb{Q}(n) \geq \left\lfloor \frac{n-2}{18} \right\rfloor \).
Proof. The statement clearly holds if $n < 20$. For $n \geq 20$ (in fact, for $n \geq 12$) we may observe that there exist non-negative integers $s, t$ such that $n = 4s + 5t$. To see this, it suffices to note that this property holds for $n \in \{12, 13, 14, 15\}$, and that $n + 4$ has the property whenever $n$ does.

If $n = 4s + 5t$ then by Lemma 4.2.3 we have

$$\tau_Q(n) \geq \min \{\tau_Q(4s), \tau_Q(5t)\} \geq \min \left\{\left\lfloor \frac{s}{2} \right\rfloor, \left\lfloor \frac{t}{2} \right\rfloor \right\} + 1.$$ 

To optimize a lower bound for $\tau_Q(n)$ obtained via this method, we need to choose $s, t$ so that $\min \{s, t\}$ is as great as possible.

We may adapt an expression of the form $n = 4s + 5t$ by writing

$$n = 4(s + 5k) + 5(t - 4k),$$

for any integer $k$. By choosing $k$ appropriately we may arrange that $|s - t| \leq 4$.

With this choice of $s, t$, we have

$$n \leq \min \{9s + 20, 9t + 16\} \leq 9 \min \{s, t\} + 20.$$ 

Thus $\min \{\frac{s}{2}, \frac{t}{2}\} \geq \frac{n - 20}{18}$. Finally

$$\tau_Q(n) \geq \min \left\{\left\lfloor \frac{s}{2} \right\rfloor, \left\lfloor \frac{t}{2} \right\rfloor \right\} + 1 \geq \left\lfloor \frac{n - 2}{18} \right\rfloor.$$ 

\[\square\]

While the lower bounds described in Theorem 4.3.8 exceed those of Theorem 4.3.9 when $n$ has small divisors, Theorem 4.3.9 has the advantage that it applies uniformly to all natural numbers. Thus it demonstrates a qualitative difference in the behaviour of the functions $\tau_Q$ and $\tau_R$, namely that $\tau_Q$ is bounded below by an increasing linear function on $\mathbb{N}$.

### 4.3.3 Constructions over finite fields

The techniques of Section 4.3.2 can in principle be used to construct examples of almost-nonsingular entry pattern matrices over the finite field $\mathbb{F}_q$, from an $\mathbb{F}_q$-irreducible polynomial whose coefficients belong to the set $\{0, 1, -1\}$ and which has few non-zero coefficients. However, no general constructions for irreducible polynomials of prescribed degree with these properties are known. Chapter 3 of [26] presents some constructions for irreducible polynomials over finite fields, as well as some long-standing conjectures on the existence of irreducible polynomials with specified properties. For example it is conjectured that for every positive integer $n$ and every finite field $\mathbb{F}_q$ of odd characteristic, there exists an $\mathbb{F}_q$-irreducible polynomial of degree $n$ with at most four non-zero coefficients.

For a prime $p$, we write $f_p(x) = x^p - x - 1$. It is clear that $f_p(x)$ has no root in $\mathbb{F}_p$. Let $\alpha$ be a root of $f_p(x)$ in some extension $\mathbb{F}_{p^n}$. Let $\sigma$ be
the Frobenius automorphism of the extension, i.e. \( \sigma(x) = x^p \) for \( x \in \mathbb{F}_p^n \). Then \( \sigma^i(\alpha) = \alpha + i \) and \( \langle \sigma \rangle \) acts transitively on the set of roots of \( f_p(x) \) which is \( \{\alpha, \alpha + 1, \ldots, \alpha + p - 1\} \). It follows that \( f_p(x) \) is irreducible in \( \mathbb{F}_p[x] \), that \( f_p(x) \) splits completely over any extension of \( \mathbb{F}_p \) where it factorizes, and that this occurs precisely when \( p \) divides the degree of the extension over \( \mathbb{F}_p \).

Using the construction of Section 4.3.2, we have the following restricted analogue of Theorem 4.3.8, for extensions of \( \mathbb{F}_p \) over which \( f_p(x) \) is irreducible.

**Theorem 4.3.10.** Let \( p \) be a prime and let \( n \) be a positive integer which is not a multiple of \( p \). Then

\[
\tau_{\mathbb{F}_p^n}(mp) \geq \left\lfloor \frac{P}{2} \right\rfloor + 1, \text{ for } m \geq 4.
\]

Moreover, the technique used in Section 4.3.2 to construct nonsingular pseudo-entry pattern matrices (and hence, almost-nonsingular entry pattern matrices) from the irreducibility over \( \mathbb{Q} \) of \( x^n - x - 1 \) may be applied to any irreducible trinomial of the form \( x^n \pm x^k \pm 1 \) over any field, as shown in the following lemma.

**Lemma 4.3.11.** Let \( n, k \) be positive integers with \( k < n \). Suppose that there exists a trinomial \( f_{n,k}(x) \) of the form \( x^n \pm x^k \pm 1 \) that is irreducible in \( \mathbb{F}[x] \) for a field \( \mathbb{F} \). Then there exists an \( n \times n \) \( \mathbb{F} \)-nonsingular pseudo-EPm whose number of indeterminates is

\[
\bullet \quad 1 + k \left\lfloor \frac{n-1}{2k} \right\rfloor \quad \text{if } \left\lfloor \frac{n-1}{k} \right\rfloor \text{ is even},
\]

\[
\bullet \quad n - k \left\lfloor \frac{n-k-1}{2k} \right\rfloor \quad \text{if } \left\lfloor \frac{n-1}{k} \right\rfloor \text{ is odd}
\]

**Proof.** Let \( \alpha \) be a root of \( f_{n,k}(x) \) and \( L = \mathbb{F}(< \alpha) \). Then \( B = \{1, \alpha, \ldots, \alpha^{n-1}\} \) is an \( \mathbb{F} \)-basis of \( L \). We write \( M_a \) for the matrix representing left multiplication by \( a \) with respect to \( B \). In particular, \( M_1 \) is the identity matrix. For \( r \) from 1 to \( n - k - 1 \), \( (M_{\alpha^r})_{ij} \) is non-zero if and only if one of the following holds.

\[
\bullet \quad i = j + r
\]

\[
\bullet \quad j = n - r + i
\]

\[
\bullet \quad j = n - r + i - k \text{ and } i > k.
\]

Thus \( M_{\alpha^r} \) has one non-zero entry in each of its first \( n - r \) columns and two in each subsequent column. For \( 1 \leq r < t \leq n - k - 1 \), \( M_{\alpha^r} \) and \( M_{\alpha^t} \) have a non-zero entry in the same position if and only if \( t - r = k \). We write

\[
S = \{2ak + b : a, b \in \mathbb{Z}, 1 \leq b \leq k\} \cap \{1, 2, \ldots, n - k - 1\}.
\]

Then no two elements of \( S \) differ by \( k \), and

\[
\{I_n\} \cup \{M_{\alpha^i} : i \in S\}
\]
is a set of $|S|$ nonsingular $(0, 1)$-matrices that determines an $\mathbb{F}$-nonsingular pseudo-EPM.

If $\left\lfloor \frac{n-1}{k} \right\rfloor$ is even then $|S| = k \left\lfloor \frac{n-1}{2k} \right\rfloor$ and we obtain an $\mathbb{F}$-nonsingular pseudo-EPM whose number of indeterminates is

$$1 + k \left\lfloor \frac{n - 1}{2k} \right\rfloor.$$

If $\left\lfloor \frac{n-1}{k} \right\rfloor$ is odd then the number of indeterminates is

$$1 + k \left\lfloor \frac{n - 1}{2k} \right\rfloor + (n - 1) - 2k \left\lfloor \frac{n - 1}{2k} \right\rfloor - k = n - k \left\lfloor \frac{n + k - 1}{2k} \right\rfloor.$$

This completes the proof of the lemma.

We note the following special cases of Lemma 4.3.11.

- If $x^n \pm x^k \pm 1$ is $\mathbb{F}$-irreducible and $2k$ divides $n - 1$, then there exists an $\mathbb{F}$-nonsingular $n \times n$ pseudo-EPM with $\frac{n+1}{2}$ indeterminates.
- If $x^n \pm x^k \pm 1$ is $\mathbb{F}$-irreducible and $2k$ divides $n$, then there exists an $\mathbb{F}$-nonsingular $n \times n$ pseudo-EPM with $\frac{n}{2}$ indeterminates.
- If $x^n \pm x^k \pm 1$ is $\mathbb{F}$-irreducible and $n$ is divisible by $k$ but not by $2k$, then there exists an $\mathbb{F}$-nonsingular $n \times n$ pseudo-EPM with $\frac{n-k}{2} + 1$ indeterminates.
- If $x^n \pm x \pm 1$ is $\mathbb{F}$-irreducible, then there exists an $\mathbb{F}$-nonsingular $n \times n$ pseudo-EPM with $\left\lfloor \frac{n}{2} \right\rfloor$ indeterminates (as in Lemma 4.3.7).

Little general information is known about the existence of irreducible trinomials of the form $f_{n,k}$ for a given integer $n$. Some criteria for irreducibility have been given in [3, 34, 12, 31], some constructions of irreducible trinomials from known examples of lesser degree are given in [3, 34]. Lemma 4.3.12 below appears in [34] and is quoted in [26] as Theorem 3.2.5. For an irreducible polynomial $f(x) \in \mathbb{F}_q[x]$, the order of $f(x)$, denoted by $\text{ord}(f)$, is the multiplicative order of a root of $f(x)$ in an extension. This can also be described as the least integer $m$ for which $f(x)$ divides $x^m - 1$.

**Lemma 4.3.12.** Let $f(x) \in \mathbb{F}_q[x]$ be an irreducible polynomial over $\mathbb{F}_q$ of degree $n$ and let $t$ be a positive integer. Then $f(x^t)$ is irreducible over $\mathbb{F}_q$ if and only if

- $\gcd \left( t, \frac{q^n - 1}{\text{ord}(f)} \right) = 1$;
- each prime factor of $t$ divides $\text{ord}(f)$; and
- if $4|t$, then $4|q^n - 1$. 

Now let $\alpha$ be a root of $f_p(x) = x^p - x - 1$. Then $\alpha^{p^n} = \alpha + n$ for every integer $n$. Hence,

$$\frac{\alpha^{p^n}}{\alpha^{p^i-1}} = \alpha^{p^{n-1} + p^{n-2} + \cdots + 1} = \prod_{i=0}^{p-1} (\alpha + i) = \alpha^p - \alpha = 1.$$  

Write $e_p = \frac{p^{p-1}}{p-1}$. Then $\text{ord}(f_p)$ must divide $e_p$. Obviously, $\text{ord}(f_p) = e_p$ if $e_p$ is prime. Moreover, since $e_p = p^{p-1} + \cdots + p + 1$, $e_p$ is an odd integer for every $p$. Therefore, each prime factor of $\text{ord}(f_p)$ is odd. Hence, by applying Lemma 4.3.12 to $f_p$, we have the following.

**Lemma 4.3.13.** Let $p$ be a prime and $f_p(x) = x^p - x - 1$ and let $q = p^k$ where $p$ is not a divisor of $k$. Then $f_p(x^q)$ is irreducible over $\mathbb{F}_q$ if and only if

- $\gcd\left(t, \frac{p^{p-1}}{\text{ord}(f_p)}\right) = 1$;
- each prime factor of $t$ divides $\text{ord}(f_p)$.

In particular, if $e_p$ is prime then $f_p(x^{e_p})$ is irreducible over $\mathbb{F}_p$ (and over any extension of $\mathbb{F}_p$ whose degree is not a multiple of $p$) for all $j$. It is easily checked for example that $e_2, e_3, e_{19}$ are prime numbers. For $p = 5$ we note that $e_5 = 781 = 11 \cdot 71$. If $\alpha$ is a root of $f_5(x)$ then neither $\alpha^{11}$ nor $\alpha^{71}$ is 1. It follows that $f_5(x^s)$ is irreducible over any field of the form $\mathbb{F}_{5^i}$, where $s$ has the form $11^i71^j$ for integers $i$ and $j$ nor both zero, and for any $k$ with $5 \nmid k$.

It is conjectured in [30] that $\text{ord}(f_p) = e_p$ for every prime number $p$, and this is confirmed for all primes $p \leq 41$. If the conjecture holds then $\frac{p^{p-1}}{\text{ord}(f_p)} = p - 1$ which is coprime to $e_p$ for every prime number $p$, and so Lemma 4.3.13 applies with any divisor of $\text{ord}(f_p)$ as the value of $t$. We summarize the connection to almost-nonsingular entry pattern matrices in the following statement.

**Theorem 4.3.14.** Let $p$ be a prime number, $f_p(x) = x^p - x - 1$ and $t$ be an integer satisfying the conditions of Lemma 4.3.13, with $t > 1$. Let $q = p^k$ where $p$ is not a divisor of $k$. Then for every integer $m \geq 4$,

$$\begin{cases} \tau_{q^m}(mpt) \geq t + 1, & \text{if } p = 2, \\ \tau_{q^m}(mpt) \geq \left\lfloor \frac{(p-1)}{2} \right\rfloor + 2, & \text{if } p \text{ is odd}. \end{cases}$$

*Proof.* The polynomial $f_p(x)$ is irreducible in $\mathbb{F}_q(x)$ since $p$ does not divide $k$. So also is $f_p(x^t)$ by Lemma 4.3.13. If $p = 2$ then $f_p(x^t) = x^{2t} - x^t - 1$. It is easy to check that $\left\lfloor \frac{2t+1}{t} \right\rfloor = 1$ is odd. Therefore, by Lemma 4.3.11, there exists a $2t \times 2t$ $\mathbb{F}_q$-nonsingular pseudo-EPM whose number of indeterminates is

$$2t - t \left\lfloor \frac{2t + t - 1}{2t} \right\rfloor = 2t - t = t.$$
If $p$ is odd then $f_p(x^t) = x^{pt} - x^t - 1$. In this case, $\left[ \frac{pt-1}{t} \right] = p - 1$ is even. Hence, by Lemma 4.3.11, there exists a $pt \times pt$ $\mathbb{F}_q$-nonsingular pseudo-EPM whose number of indeterminates is

$$1 + t \left[ \frac{pt-1}{2t} \right] = 1 + \frac{t(p - 1)}{2}.$$ 

The conclusion follows immediately from applying Theorem 4.2.1 to the $\mathbb{F}_q$-nonsingular pseudo-EPMs above.

Since any candidate for $t$ in Lemma 4.3.13 may be replaced by any positive integer power of itself, it follows from Theorem 4.3.14 that $\tau_{\mathbb{F}_q}$ is an unbounded function on $\mathbb{N}$, subject to the following conditions on $q = p^k$:

- $p$ does not divide $k$;
- Not all of the prime divisors of $\text{ord}(f_p)$ are divisors of $\frac{p^p - 1}{\text{ord}(f_p)}$.

We conclude this section by noting that for every positive integer $n$ and for every prime $p$, there exist infinitely many finite fields $\mathbb{F}$ of characteristic $p$ with the property that an $n \times n$ almost-nonsingular entry pattern matrix over $\mathbb{F}$ can possess no more than two distinct indeterminates. This follows from the observation that every polynomial of degree at most $n$ in $\mathbb{F}_p[x]$ splits over the unique extension of $\mathbb{F}_p$ of degree $n!$.

**Theorem 4.3.15.** Let $n$ be a positive integer with $n \geq 4$, and let $p$ be a prime number. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ with the property that $n!$ divides the degree $[\mathbb{F} : \mathbb{F}_p]$. Then $\tau_{\mathbb{F}}(k) = 2$ for all $4 \leq k \leq n$.

**Proof.** Suppose that $P(x, y, z)$ is a $k \times k$ entry pattern matrix with 3 indeterminates, where $k \leq n$. Write $A = P(1, 0, 0)$ and $B = P(0, 1, 0)$. Suppose that $A$ and $B$ are nonsingular in $M_k(\mathbb{F}_p)$. Then det($A + xB$) is a polynomial of degree $k$ in $\mathbb{F}_p[x]$, which has a root in $\mathbb{F}$. It follows that the $\mathbb{F}$-span of $A$ and $B$ contains a non-zero matrix that is singular in $M_k(\mathbb{F})$, and hence that there is no $\mathbb{F}$-almost-nonsingular $k \times k$ EPM with 3 indeterminates. Therefore, there is no $\mathbb{F}$-almost-nonsingular $k \times k$ EPM with at least 3 indeterminates. This shows that $\tau_{\mathbb{F}}(k) \leq 2$. That $\tau_{\mathbb{F}}(k) \geq 2$ for $k \geq 4$ is immediate from the existence of $k \times k$ universally almost-nonsingular EPM with 2 indeterminates constructed in Lemma 4.1.5.

## 4.4 Concluding remarks

The main contribution of this chapter is to provide lower bounds for the maximum number $\tau_p(n)$ of indeterminates that may appear in an $n \times n$ entry pattern matrix whose $\mathbb{F}$-completions are nonsingular whenever at least two indeterminates are assigned different values. Several questions remain open about the relationships between the functions $\tau_{\mathbb{F}}$ and $\rho_{\mathbb{F}}$ for a field $\mathbb{F}$. In Section 4.3 of this chapter, we have proven that $\tau_{\mathbb{F}}(n) \leq \tau_{\mathbb{F}_q}$ for
\( \rho \mathcal{F} (n) + 1 \) for any field \( \mathbb{F} \) and that for any natural number \( n \), with equality in the following cases:

- \( \mathbb{F} \) is algebraically closed.
- \( \mathbb{F} = \mathbb{R} \) and \( n = 6 \), or \( n \) has an odd divisor greater than 3.

It would be of interest to know whether the equality \( \tau \mathcal{F} (n) = \rho \mathcal{F} (n) + 1 \) can be satisfied in other cases, and more generally to determine the values of \( \tau \mathcal{F} \) on 2-powers and to identify whether \( \tau \mathcal{F} (n) \) depends only on the 2-power part of \( n \), as \( \rho \mathcal{F} (n) \) does.

Other matters for further investigation include the tightness of our bounds \( \tau \mathcal{Q} \) as stated in Theorems 4.3.1, 4.3.8 and 4.3.9. It is likely that the lower bounds for \( \tau \mathcal{Q} (n) \) given in 4.3.8 and 4.3.9 in particular can be improved upon.

Upper bounds for \( \tau \mathcal{F} (n) \) can be obtained by considering the minimum possible number of non-zero entries in an \( n \times n \) \((0, 1)\)-matrix \( A \) for which \( \langle A, J \rangle \) is a nonsingular space. By Lemma 4.1.1, a characterizing property of such matrices is that the sum of the entries of their inverse is zero. Let \( \sigma \mathcal{F} (n) \) be the minimum possible number of non-zero entries in an \( n \times n \) \( \mathbb{F} \)-nonsingular \((0, 1)\)-matrix with this property. Then \( \tau \mathcal{F} (n) \leq \frac{n^2}{\sigma \mathcal{F} (n)} \). If \( \text{char}(\mathbb{F}) \) is positive and divides \( n \) then it is easy to see that \( \sigma \mathcal{F} (n) = n \).

Since any \( n \times n \) nonsingular \((0, 1)\)-matrix which has exactly \( n \) non-zero entries is a permutation matrix and the difference of any two permutation matrices is singular, we easily see that \( \tau \mathcal{F} (n) = n \) if and only if \( n = 2 \) and \( \text{char} \mathbb{F} = 2 \). The problem of determining the values of \( \sigma \mathcal{F} \) for a given field \( \mathbb{F} \) is of possible interest. We note that if \( n = 2m \) is even then \( \sigma \mathcal{F} (n) \leq 2n \) for every field \( \mathbb{F} \) since the \( n \times n \) matrix with \( I_m \) in the upper left and lower right \( m \times m \) blocks, \( O_m \) in the upper right, and 1s in the first two columns of the lower left \( m \times m \) block has the required property.
Chapter 5

Nilpotent Entry Pattern Matrices

In this chapter, we will discuss properties of square entry pattern matrices whose completions over a field $\mathbb{F}$ are all nilpotent. The $\mathbb{F}$-pattern class of a nilpotent entry pattern matrix with $k$ indeterminates is a nilpotent $\mathbb{F}$-vector space of dimension $k$. In this chapter, we will give upper bounds for the maximal number with indeterminates in such a nilpotent entry pattern matrix and compare the results with the dimension bounds for nilpotent vector spaces presented in Chapter 1.

Definition 5.0.1. A square entry pattern matrix $A$ of size $n \times n$ is nilpotent over a field $\mathbb{F}$ (or $\mathbb{F}$-nilpotent) if every matrix in the pattern class $C_{\mathbb{F}}(A)$ is nilpotent. Furthermore, the maximum of the nilpotent indices of the matrices in $C_{\mathbb{F}}(A)$ is called the nilpotent index of $A$.

Since $J$ is an element of $C_{\mathbb{F}}(A)$ for every field $\mathbb{F}$ and for every $n \times n$ entry pattern matrix $A$, there is no $\mathbb{F}$-nilpotent entry pattern matrix if $\text{char}\mathbb{F} = 0$. Indeed, it is easy to check that $J^n = n^{n-1}J$. Therefore, $J$ is nilpotent only if $n = 0$ in $\mathbb{F}$. This happens only if $\text{char}\mathbb{F}$ is positive and divides $n$. In that case, we have $J^2 = nJ = 0$, so $J$ is nilpotent of index two.

Lemma 5.0.1. Let $A$ be an $n \times n$ entry pattern matrix which is nilpotent over a field $\mathbb{F}$. Then the characteristic of $\mathbb{F}$ is positive and divides $n$.

Example 5.0.1. Let $\mathbb{F} = \mathbb{F}_2$ and let

$$A(x, y, z, r, s, t) = \begin{bmatrix} x & x & y & y \\ x & x & z & z \\ r & s & t & t \\ r & s & t & t \end{bmatrix}.$$ 

As presented in Example 1.2.2, the pattern class $C_{\mathbb{F}_2}(A)$ is nilpotent of index 4. Hence, $A$ is $\mathbb{F}_2$-nilpotent of index 4. Note that $A$ is also nilpotent over any field of characteristic 2.
Chapter 5. Nilpotent entry pattern matrices

5.1 Completions of nilpotent entry pattern matrices

Let $A$ be an $n \times n$ entry pattern matrix with $k$ indeterminates which is nilpotent over a field $\mathbb{F}$ of positive characteristic $p$ and let $M$ be a completion of $A$. In this section, we will discuss properties of $M$ and from there, give an upper bound for the number of indeterminates appearing in $A$. First of all, the $\mathbb{F}$-span of $M$ and $J$ is a subspace of the $\mathbb{F}$-pattern class of $A$ and hence, it is nilpotent. Therefore, $M, J$ and $M + J$ are nilpotent. Lemma 1 in [22] shows that for any two matrices $N, P$ of the same order, if $N, P$ and $N + P$ are nilpotent then $\text{trace}(NP) = 0$. Therefore,

$$\text{trace}(MJ) = 0.$$ 

This proves the following lemma.

Lemma 5.1.1. Let $A$ be an $\mathbb{F}$-nilpotent entry pattern matrix and $M$ be an $\mathbb{F}$-completion of $A$. Then

$$\text{trace}(MJ) = 0,$$

or equivalently,

$$JMJ = O.$$ 

This means that the sum of the entries of $M$ is 0 in $\mathbb{F}$. In particular, if $\mathbb{F}$ is a field of positive characteristic $p$ then the number of non-zero entries of each coefficient matrix of $A$ is a multiple of $p$. The following observation is now immediate.

Lemma 5.1.2. Let $A$ be a nilpotent entry pattern matrix over a field of positive characteristic $p$. Then the number of non-zero entries in each coefficient matrix of $A$ is at least $p$. Therefore, if $A$ is of size $n \times n$ and has $k$ indeterminates, then

$$k \leq \frac{n^2}{p}.$$ 

Example 5.1.1. Let

$$A = \begin{bmatrix} x & y & y & x \\ x & x & x & x \\ x & x & x & x \\ x & y & x & x \end{bmatrix}.$$ 

Then $A$ is $\mathbb{F}_5$-nilpotent of index 4. The coefficient matrix $A_2$ of $A$ with respect to $y$ has exactly 5 non-zero entries.

We remark that the upper bound in Lemma 5.1.2 is not sharp since there is no $n \times n$ $\mathbb{F}_p$-nilpotent entry pattern matrix which has $\frac{n^2}{p}$ indeterminates. Indeed, suppose that such an entry pattern matrix exists, then one of its coefficient matrices $A_i$ has at least one non-zero entry on the main diagonal and has exactly $p$ non-zero entries by Lemma 5.1.1. Moreover, since $A_i$ is nilpotent, $\text{trace}(A_i) = 0$. Therefore, $A_i$ has exactly
non-zero entries on the main diagonal and 0 elsewhere. This clearly forces that $A_t$ is not nilpotent, which is a contradiction.

Next, we will show that the sum of the entries of any power of any completion of a nilpotent entry pattern matrix is 0.

**Theorem 5.1.3.** Let $\mathbb{F}$ be a field and let $A$ be an $\mathbb{F}$-completion of an $n \times n$ entry pattern matrix which is nilpotent over $\mathbb{F}$. Then $\text{trace}(A^kJ) = 0$ or equivalently, $JA^kJ = 0$ for every integer $k$.

In order to prove this theorem, we need to recall a well-known result on the coefficients of the characteristic polynomial of a sum of matrices. Let $S$ be the free semi-group generated by $k$ letters $x_1, x_2, \ldots, x_k$. That is, $S = \{x_{i_1}x_{i_2}\cdots x_{i_s} : 1 \leq i_1, i_2, \ldots, i_s \leq k\}$ and the product of any two words $x = x_{i_1}\cdots x_{i_s}$ and $y = x_{j_1}\cdots x_{j_t}$ in $S$ is defined by $xy := x_{i_1}\cdots x_{i_s}x_{j_1}\cdots x_{j_t}$. Two words $m_1, m_2$ are said to be (cyclically) equivalent if one is obtained from the other by a cyclic permutation, i.e., there are two words $u, v \in S$ so that $m_1 = uv$ and $m_2 = vu$. This cyclical equivalence is reflexive, symmetric and transitive. Hence, it is an equivalence relation. The length of a word is the number of letters appearing in the word. The length of the word $m$ is denoted by $l(m)$. A product of copies of a word $m$ is called a power of $m$. A word is said to be indecomposable if it is not a power of a word of smaller length. For a word $m = x_{i_1}\cdots x_{i_s} \in S$, we will denote by $v(m)$ the vector in $\mathbb{N}^k$ whose $i$-th entry is equal to the number of times the letter $x_i$ appears in $m$. For example, let $m = x_1x_2x_1$. Then $l(m) = 3$, $v(m) = (2, 1, 0, \ldots, 0) \in \mathbb{N}^k$ and $m$ is equivalent to $x_1^2x_2$ which is indecomposable. For any $n \times n$ matrix $M$, let

$$
\det(\lambda I_n - M) = \lambda^n - c_1(M)\lambda^{n-1} + \cdots + (-1)^nc_n(M).
$$

**Lemma 5.1.4.** [4] Let $\mathbb{F}$ be a field and let $x_1, \ldots, x_k \in M_n(\mathbb{F})$. Let $S$ denote the free semi-group generated by $k$ letters $x_1, \ldots, x_k$. Then for $t_1, \ldots, t_k \in \mathbb{F}$, we have

$$
c_m(t_1x_1 + \cdots + t_kx_k) = \sum (-1)^{m-(j_1+\cdots+j_r)}t_1^{v_1(p_1)}\cdots t_k^{v_k(p_r)}c_{j_1}(p_1)\cdots c_{j_r}(p_r)
$$

where $t^v = t_1^{v_1}t_2^{v_2}\cdots t_k^{v_k}$ for $v = (v_1, \ldots, v_k)$ and the sum ranges over all representatives of indecomposable words $p_i \in S$ and $j_1, \ldots, j_r \in \{0, \ldots, m\}$ such that $j_1l(p_1) + \cdots + j_rl(p_r) = m$.

**Proof of Theorem 5.1.3.** We will prove $JA^kJ = 0$ by induction on $k$. First of all, it is easy to see that all the representatives of indecomposable words of length at most two in the free semi-group $S$ generated by $A$ and $J$ are $\{A, J, AJ\}$. Hence, by applying Lemma 5.1.4, we obtain

$$
0 = c_2(A + J) = c_2(A) + c_2(J) + c_1(A)c_1(J) - c_1(AJ).
$$

It follows that $c_1(AJ) = 0$ or equivalently, $JAJ = 0$.

Now assume that $JA^iJ = 0$ for all $i \leq k - 1$, and so $A^iJ$ is nilpotent of index two for all $i \leq k - 1$. Let $S'$ be the set of all representatives of indecomposable words of length at most $k + 1$ in $S$. Since $J^2 = O$ and...
Chapter 5. Nilpotent entry pattern matrices

JA^t J = O for all $i \leq k - 1$, it suffices to consider elements of $S'$ of the form $A^t J^\delta$ where $t$ is a positive integer not exceeding $k + 1$ and $\delta$ is either 0 or 1. Besides, $A$ and $A^t J$ are nilpotent for every $i \leq k - 1$. Therefore, applying Lemma 5.1.4 to $c_{k+1}(A + J)$, we get

$$0 = c_{k+1}(A + J) = c_1(A^k J) = \text{trace}(A^k J)$$

Or equivalently, $JA^k J = O$, as required.

**Corollary 5.1.5.** Let $F$ be a field. If $A$ is a completion of an $F$-nilpotent entry pattern matrix, then the algebra generated over $F$ by $\{A, J\}$ is nilpotent.

**Proof.** By Theorem 5.1.3, $JA^k J = O$ for every non-negative integer $k$. So any element of the algebra can be written in the form

$$B = \sum x^\delta_{ij} A^t J^\delta A^j$$

where $x^\delta_{ij} \in F$, and $\delta$ is either 0 or 1. Let $n$ be the order of $A$. Then $B^{2n}$ is the sum of monomials $A^t J^\delta A^j$ of degree $2n$ or greater. It follows that either $i \geq n$ or $j \geq n$. Therefore, $B$ is nilpotent, as required.

We conclude this section with a remark on upper bounds for the number of indeterminates in a nilpotent entry pattern matrix. By Lemma 5.1.2, if $k$ is the number of indeterminates in an $n \times n$ nilpotent entry pattern matrix $A$ over a field $F$ of characteristic $p$ then $k \leq \frac{n^2}{p}$. This bound is less than or equal to the Gerstenhaber bound $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ in Theorem 1.2.1 unless $p = 2$. We will show that there is an $n \times n$ $F$-nilpotent entry pattern matrix with $\frac{n^2 + n - np}{p}$ indeterminates. In particular, if $p = 2$ then this number coincides with the Gerstenhaber bound in Theorem 1.2.1.

Let $F$ be a field of positive characteristic $p$ and let $n$ be an integer divisible by $p$. Set $m = \frac{n}{p}$. Let $A$ be the $n \times n$ entry pattern matrix which is partitioned into $m^2 p \times p$ blocks $A_{11}, A_{12}, \ldots, A_{1m}, \ldots, A_{mm}$ so that

- the $(i, j)$-block $A_{ij}$ is the $p \times p$ entry pattern matrix with one indeterminate if $i = j$,
- the $(i, j)$-block $A_{ij}$ is the $p \times p$ entry pattern matrix which has exactly $p$ different indeterminates and the indeterminates appearing in the same row are all equal if $i < j$,
- the $(i, j)$-block $A_{ij}$ is the $p \times p$ entry pattern matrix which has exactly $p$ different indeterminates and the indeterminates appearing in the same column are all equal if $i > j$, and
- all the indeterminates appearing in each matrix $A_{ij}$ are independent to each other and independent to the indeterminates in the other block matrices.

Then the entry pattern matrix $A$ has $\frac{n}{p} + \frac{n^2 - np}{p}$ indeterminates. Moreover, by direct computing, we easily prove the followings.
1. The entries of \( A_{ij}A_{ji} \) are all equal. In particular, \( A_{ij}A_{ji} = O \) if \( i \leq j \).

2. The entries of each matrix of \( \{A_{ii}A_{ij}, A_{ji}A_{ii}\} \) are all equal. In particular, \( A_{ii}A_{ij} = A_{ji}A_{ii} = O \) if \( i \geq j \).

3. If \( i < j < k \) then the entries in each row of \( A_{ij}A_{jk} \) are all equal.

4. If \( i > j > k \) then the entries in each column of \( A_{ij}A_{jk} \) are all equal.

5. Otherwise, the entries of \( A_{ij}A_{jk} \) are all equal. In particular, if \( j > i \) and \( j > k \) then \( A_{ij}A_{jk} = O \).

Therefore, the \((1,1)\)-block of \( A^2 \) is zero while the other blocks have the same property as the corresponding blocks of \( A \). That is, the entries of the \((i,j)\)-block of \( A^2 \) are all equal if \( i = j > 1 \) and the entries of each row and column of the \((i,j)\)-block and the \((j,i)\)-block respectively of \( A \) are all equal if \( i < j \). By applying the same argument to \( A^4 = (A^2)^2 \), we easily see that the \((1,1)\)-block, \((1,2)\)-block, \((2,1)\)-block and \((2,2)\)-block of \( A^4 \) are zero while the other blocks have the same property as the corresponding blocks of \( A \) and \( A^2 \). By repeating this step up to \( m = \frac{n}{p} \) times, we get that \( A^{2m} = O \). Therefore, \( A \) is a nilpotent entry pattern matrix. This proves the following theorem.

**Theorem 5.1.6.** Let \( n \) be an integer which is divisible by a prime number \( p \) and let \( F \) be a field of characteristic \( p \). Then there is an \( F \)-nilpotent entry pattern matrix of size \( n \times n \) which has \( \frac{n^2-np+n}{p} \) indeterminates.

In particular, if \( p = 2 \) then \( \frac{n^2-np+n}{p} = \frac{n^2-n}{2} = \binom{n}{2} \). On the other hand, it follows from Theorem 1.2.1 that the maximum dimension of nilpotent vector subspaces of \( M_n(F) \) is \( \binom{n}{2} \) and such a \( \binom{n}{2} \)-dimensional nilpotent vector space is similar to the space of strictly upper triangular matrices. The following corollary is an immediate consequence.

**Corollary 5.1.7.** Let \( F \) be a field of characteristic 2 and let \( n \) be an even positive integer. Then the maximum number of indeterminates of an \( n \times n \) \( F \)-nilpotent entry pattern matrix is \( \binom{n}{2} \). Furthermore, there is an entry pattern matrix attaining that bound and the \( F \)-pattern class of such an \( F \)-nilpotent entry pattern matrix with \( \binom{n}{2} \) indeterminates is similar to the strictly upper triangular space \( SU_n(F) \).

**Example 5.1.2.** Let \( F \) be a field of characteristic \( p = 2 \) and let \( A \) be the \( 6 \times 6 \) entry pattern matrix defined by

\[
A = \begin{bmatrix}
  x & x & a & a & b & b \\
  x & x & c & c & d & d \\
  r & s & y & y & e & e \\
  r & s & y & y & f & f \\
  t & k & l & m & z & z \\
  t & k & l & m & z & z 
\end{bmatrix}
\]
Then \( A \) has \( 15 = \binom{6}{2} \) indeterminates. Moreover,

\[
A^5 \mod 2 = x(a + c)(e + f)(r + s)(l + m) \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 
\end{bmatrix}.
\]

Therefore, \( A \) is an \( \mathbb{F} \)-nilpotent EPM of index 6.

### 5.2 Nilpotent entry pattern matrices of index 2

In this section, we will study entry pattern matrices which are nilpotent of index 2. We will determine the maximum number of indeterminates in such a square entry pattern matrix of a given order and identify the entry pattern matrices which attain these bounds.

**Lemma 5.2.1.** Let \( A \) be a non-zero \( n \times n \) \((0,1)\)-matrix and let \( p \) be a prime number such that \( p \mid n \). If every non-zero element of the \( \mathbb{F}_p \)-span of \( A \) and \( J_n \) is \( \mathbb{F}_p \)-nilpotent of index two, then the number of non-zero entries of \( A \) is at least \( p^2 \).

**Proof.** Since every non-zero element of the \( \mathbb{F}_p \)-span of \( A \) and \( J_n \) is \( \mathbb{F}_p \)-nilpotent of index two,

\[
0 = (A + xJ_n)^2 = A^2 + x(AJ_n + J_nA) \quad \forall x \in \mathbb{F}_p.
\]

It follows that \( AJ_n + J_nA = 0 \), i.e., \( R_i + j^T C_j = 0 \) for every \( i, j \) where \( R_i \) and \( C_j \) denote the \( i \)-th row and the \( j \)-th column of \( A \), respectively. That is, the number of entries equal to 1 in each row of \( A \) is congruent to a constant \( r \) and the number of entries equal to 1 in each column of \( A \) is congruent to a constant \( s \) with \( r + s = 0 \mod p \).

Suppose there is a row, say \( R_i \), of \( A \) in which the number of entries equal to 1 is between 1 and \( p - 1 \). Then \( r \neq 0 \), and \( s \neq 0 \), so every row and every column of \( A \) are non-zero. It follows that there are \( j \) and \( k \) so that \( A_{ij} = 1 = A_{jk} \). Thus,

\[
(A^2)_{ik} = \sum_l (A)_{il}(A)_{lk} = 1 + \sum_{l \neq j} (A)_{il}(A)_{lk}
\]

is a positive number which is at most \( p - 1 \). This is not equal to zero in any field of characteristic \( p \), which is a contradiction to the hypothesis that \( A \) is nilpotent of index two. Therefore, every non-zero row of \( A \) has at least \( p \) non-zero entries.

Moreover, it is easy to see that \( \langle A, J \rangle_{\mathbb{F}_p} \) is nilpotent of index two if and only if \( \langle A^T, J \rangle_{\mathbb{F}_p} \) is nilpotent of index two.

Hence, each non-zero row of \( A^T \) has at least \( p \) non-zero entries. This proves that \( A \) has at least \( p^2 \) non-zero entries, completing the proof. \( \square \)
Lemma 5.2.1 shows that any coefficient matrix of an $n \times n$ entry pattern matrix which is nilpotent of index 2 over a field of characteristic $p$ has at least $p^2$ non-zero entries. Since any two of the coefficient matrices of an entry pattern matrix have no non-zero entries at the same position, we easily see that the number of indeterminates is at most $\frac{n^2}{p}$ as shown in the following theorem. We also show that any such nilpotent entry pattern matrix which attains that maximum bound is permutation similar to the Kronecker product of the free $\frac{n}{p} \times \frac{n}{p}$ entry pattern matrix and $J_p$.

**Theorem 5.2.2.** Let $\mathbb{F}$ be a field of positive characteristic $p$, and let $n$ be a positive integer divisible by $p$. Assume that $A$ is an $n \times n$ entry pattern matrix with $k$ indeterminates which is nilpotent of index 2 over $\mathbb{F}$. Then $k \leq \frac{n^2}{p^2}$. Moreover, if $k = \frac{n^2}{p^2}$ then $A$ is permutation similar to the following entry pattern matrix

$$B \otimes J_p = \begin{bmatrix}
    x_{11}J_p & x_{12}J_p & \ldots & x_{1,\frac{n}{p}}J_p \\
    x_{21}J_p & x_{22}J_p & \ldots & x_{2,\frac{n}{p}}J_p \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{\frac{n}{p}1}J_p & x_{\frac{n}{p}2}J_p & \ldots & x_{\frac{n}{p},\frac{n}{p}}J_p
\end{bmatrix},$$

where $B = (x_{ij})_{\frac{n}{p}, \frac{n}{p}}$ is the free $\frac{n}{p} \times \frac{n}{p}$ entry pattern matrix and $\otimes$ is the Kronecker product.

**Proof.** Assume that $A(x_1, \ldots, x_k)$ is an $n \times n$ $\mathbb{F}$-nilpotent entry pattern matrix. For $i$ from 1 to $k$, let $A_i$ be the coefficient matrix of $A$ with respect to $x_i$. Then the $\mathbb{F}$-span of $A_1$ and $J_n$ is nilpotent of index two for every $i$. Therefore, the number of non-zero entries of $A_i$ is at least $p^2$ for every $i$ [Lemma 5.2.1]. Since no two of the matrices $\{A_1, \ldots, A_k\}$ have a non-zero entry in the same position, $kp^2 \leq n^2$. It follows that

$$k \leq \frac{n^2}{p^2}.$$ 

Furthermore, if $k = \frac{n^2}{p^2}$, then each matrix $A_i$ has exactly $p$ non-zero rows, and in each of these rows there are exactly $p$ non-zero entries. Rewrite

$$A = x_1A_1 + \cdots + x_pA_p + \cdots + x_{\frac{n^2}{p^2}}A_{\frac{n^2}{p^2}}.$$ 

Without loss of generality (by permuting the indices of the indeterminates if necessary), we may assume that the $(1,1)$-entry of $A_1$ is non-zero, i.e., $(A_1)_{11} = 1$. Since $A_1$ is nilpotent of index 2,

$$0 = (A_1^2)_{11} = \sum_j (A_1)_{1j} (A_1)_{j1} = 1 + \sum_{j \neq 1} (A_1)_{1j} (A_1)_{j1}. \quad (5.1)$$

Since each matrix $A_i$ has exactly $p$ non-zero rows and each non-zero row has exactly $p$ non-zero entries, (5.1) implies that $(A_1)_{1j} = (A_1)_{j1}$ for every $j$. Hence, up to similarity, we may assume that the first $p$ entries of the first row of $A_1$ are all equal to 1 while the other entries of that row
are zero. Similarly, for $i$ from 2 to $p$, we have $(A_1)_{ij} = (A_1)_{ji}$ for every $j$ since

$$0 = (A_1^2)_{ii} = \sum_j (A_1)_{ij}(A_1)_{ji} = 1 + \sum_{j \neq 1} (A_1)_{2j}(A_1)_{j2}.$$

Therefore, up to permutation similarity, we may assume that $A_1 = \begin{bmatrix} J_p & O \\ O & O \end{bmatrix}$.

By applying the same technique to the coefficient matrices which have non-zero entry on their main diagonals, we easily see that, up to permutation similarity, they contain the block $J_p$ on their main diagonals and 0 elsewhere. There are exactly $\frac{n}{p}$ such coefficient matrices. Denote these coefficient matrices by $A_1, A_2, \ldots, A_{2p}$, where the $(i, i)$-block of $A_i$ is equal to $J_p$. Let $A_s$ be the coefficient matrix of $A$ whose $(1, p+1)$-entry is non-zero: $(A_s)_{1,p+1} = 1$. Since $A_s + A_2$ is nilpotent of index 2,

$$0 = ((A_s + A_2)^2)_{1,p+1} = 1 + \sum_{p+2 \leq j \leq 2p} (A_s)_{1j}(A_2)_{j,p+1}.$$

Therefore, the entries of the first row of $A_s$ are equal to 1 in the positions $p+1$ through $2p$ and 0 elsewhere.

Moreover, since $A_s + A_1$ is nilpotent of index 2,

$$0 = ((A_s + A_1)^2)_{1,p+1} = 1 + \sum_{2 \leq j \leq p} (A_1)_{1j}(A_s)_{j,p+1}.$$

Therefore, the entries of the $(p+1)$-th column of $A_s$ are equal to 1 in the first $p$ positions and 0 elsewhere.

Similarly, by computing the entries in the positions $(1, p+2), (1, p+3), \ldots$ of $(A_s + A_1)^2$ and $(A_s + A_2)^2$, we have

$$A_s = \begin{bmatrix} O_p & J_p & O \\ O & O & O \end{bmatrix}.$$

Hence, by applying the same technique, we easily see that if a coefficient matrix $A_i$ has non-zero entry in the position $(i, j)$ where $i = p\alpha + 1, j = pb + 1$, then $A_i$ is the $n \times n$ block matrix which contains $J_p$ in the position $(a + 1, b + 1)$. So the entry pattern matrix $A$ is permutation similar to the Kronecker product $B \otimes J_p$, where $B$ is the $\frac{n}{p} \times \frac{n}{p}$ entry pattern matrix with $\frac{n^2}{p^2}$ indeterminates.

It is clear that the maximum completion rank of $B \otimes J_p$ above is equal to the maximum completion rank of $B$ which is equal to $r = \frac{n}{p}$. Therefore, the $\mathbb{F}$-pattern class of $B \otimes J_p$ is a nilpotent vector subspace of index 2 of $M_n(\mathbb{F})$ which has dimension $\frac{n^2}{p^2}$, and has maximal rank $r = \frac{n}{p}$. As demonstrated in Theorem 1.2.3, the maximum dimension of an $\mathbb{F}$-nilpotent vector subspace of nilpotent index 2 of $M_n(\mathbb{F})$ which has rank at most $r$ is $r(n - r)$ if $|\mathbb{F}| > \frac{n}{2}$. Generally, if $p$ divides $n > p$ and $r = \frac{n}{p}$ then $\frac{n^2}{p^2} \leq r(n - r)$. The equality occurs if and only if $p = 2$. This proves the following corollary.
Corollary 5.2.3. Let $\mathbb{F}$ be a field of characteristic $p = 2$ and let $n$ be an even number. Then the maximum number of indeterminates of an $n \times n \mathbb{F}$-nilpotent EPM of index 2 is $\frac{n^2}{4}$. Moreover, any $\mathbb{F}$-nilpotent EPM of index 2 which has $\frac{n^2}{4}$ indeterminates is permutation similar to the following EPM

$$B \otimes J_2 = \begin{bmatrix} x_{11}J_2 & x_{12}J_2 & \ldots & x_{1,\frac{n}{2}}J_2 \\ x_{21}J_2 & x_{22}J_2 & \ldots & x_{2,\frac{n}{2}}J_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{\frac{n}{2},1}J_2 & x_{\frac{n}{2},2}J_2 & \ldots & x_{\frac{n}{2},\frac{n}{2}}J_2 \end{bmatrix},$$

where $B = (b_{ij})_{\frac{n}{2} \times \frac{n}{2}}$ is the free $\frac{n}{2} \times \frac{n}{2}$ EPM and $\otimes$ is the Kronecker product. Moreover, the $\mathbb{F}$-pattern class of the EPM $B \otimes J_2$ above is similar to one of the spaces $V_1, V_2, V_3$ defined in Theorem 1.2.3.
Chapter 6

Conclusion and Discussion

The aim of this thesis is to study a class of linear spaces of matrices over a field with interesting properties. More precisely, the following contributions are noted.

- We have studied entry pattern matrices and their spaces of completions over various fields. In particular, we have distinguished the concepts of generic rank and maximum completion rank, and introduced the concept of EPM-rank-tight.

- We have proven that finite extensions of EPM-rank-tight fields retain the property of being EPM-rank-tight. Hence, we have shown that all finite fields of characteristics 3, 5, 7, 11 or 13 are EPM-rank-tight. So also are all fields of characteristic 2 with the exception of $\mathbb{F}_2$ itself.

- We have introduced the concept of a pseudo-entry pattern matrix and demonstrated an extension from a nonsingular pseudo-EPM to an almost-nonsingular EPM. We have presented constructions of almost-nonsingular entry pattern matrices over the real field, the rational field and some finite fields. Lower bounds for the maximum number of indeterminates in an almost-nonsingular EPM are determined on those fields.

- We have investigated some interesting properties of nilpotent entry pattern matrices and given the upper bound for the number of indeterminates of nilpotent entry pattern matrices, in general and of nilpotent entry pattern matrices of index 2. Especially, in the latter case, we have proven that the bound is sharp by proving the existence and uniqueness (up to similarity) of such entry pattern matrices in which the upper bound is attained.

We also note some directions for further work.

- In Chapter 3, the method used to prove the EPM-rank-tightness of the fields of characteristic less than 17 can not be applied to investigate the EPM-rank-tightness of fields of greater characteristic.
It would be of interest to adapt the method to prove the conjecture: Every finite field is EPM-rank-tight unless it is the field of two elements.

- In Chapter 4, we have obtained the exact value of $\tau_{\mathbb{F}}(n) = \rho(n) + 1$ only if $n$ has an odd divisor greater than 3. We also proved that if $n = 3 \cdot 2^k$ then $\rho\left(\frac{n}{3}\right) + 1 \leq \tau_{\mathbb{F}}(n) \leq \rho(n) + 1$. In particular, $2 \leq \tau_{\mathbb{F}}(6) \leq 3$. We have an example to demonstrate that $\tau_{\mathbb{F}}(6) = 3$. Hence, it is reasonable to ask whether $\tau_{\mathbb{F}}(n) = \rho(n) + 1$ for every integer $n$ which has an odd divisor. Generally, the problem of determining the value of $\tau_{\mathbb{F}}(n)$ for every field $\mathbb{F}$ and for every integer $n$ is still open.

- In Chapter 5, we have obtained the upper bound for the number $k$ of indeterminates in an $n \times n$ nilpotent entry pattern matrix over a field $\mathbb{F}$ of characteristic $p$ which divides $n$ [Lemma 5.1.2]. That bound is less than the bound in Theorem 1.2.1 if $p \neq 2$. That bound is not sharp. However, we have obtained the sharp upper bound for the number of indeterminates of an $\mathbb{F}$-nilpotent $n \times n$ entry pattern matrix of index two and classified the entry pattern matrices attaining the bound. There is also potential for investigating the maximal number of indeterminates of such nilpotent entry pattern matrices without restriction on the index.
References


