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On the Axiomatic Ranking of Opportunity Sets in a Logical Framework

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Abstract.

The paper presents an axiomatic analysis of opportunity ranking within a logical framework. It establishes that any opportunity or freedom ranking is completely determined by a subjacent logical chain that is interpreted as a freedom standard. However, a preferential interpretation of the chain is also possible and it leads to the results similar to those already obtained by Nehring and Puppe (1999).

**Keywords:** opportunity ranking, logical framework

**JEL Classification:** D71
1. Introduction

In an attempt to explore the intrinsic value of freedom, researchers have proposed various systems of axioms expressing their different intuitions about freedom, see Bossert et al. (1994), Gravel (1994) Jones and Sugden (1982) Klemisch-Ahlert (1993) Pattanaik and Xu (1990, 1998), Puppe (1996) Sen (1988, 1991, 1993), Sugden (1998) Suppes (1987), among others). The earlier papers have utilized axioms, which were completely independent of an underlying preference relation among basic alternatives, see Klemisch-Ahlert (1993), Pattanaik and Zu (1990) or Suppes (1987) among others). However, following Sen’s (1991) insistence on the importance of preference as an essential ingredient in the evaluation of freedom as opportunity, many authors recently have incorporated individual preferences in their axiomatic frameworks. For example, Pattanaik and Xu (1998) argue that the preferences that are important in assessing the intrinsic value of freedom are not the actual preferences that the agent currently has, but “the preference orderings that a reasonable person in the agent’s situation can possibly have.” Similarly, Sugden (1998) emphasizes the importance of potential or counterfactual preferences in evaluating opportunity sets.

On the other hand, Nehring and Puppe (1999) and Puppe (1998), utilizing a fundamental representation theorem due to Kreps (1979), view the different preference orders not as given (a priori) but as induced by the agent’s assessment of opportunity sets. In particular, Nehring and Puppe (1999) provide a general framework for analyzing “preference for opportunity” using two rather mild conditions – monotonicity and contraction consistency. Given those two conditions, they establish that an opportunity ranking can be generated by a set of weak orders corresponding to different assessments of the alternatives. Hence, they argue that Kreps’ representation theorem may be given a more general interpretation than Kreps himself has intended. For instance, it is possible to interpret the induced orderings as any potential counterfactual preferences and not necessarily as probable future preferences as Kreps has suggested in his 1979 seminal paper.

The aim of this paper is to provide an axiomatic analysis of opportunity ranking within a general logical framework. We propose a minimal set of axioms which structurally are similar to those of Nehring and Puppe (1999). The central result of the
paper establishes that any opportunity, or freedom, ranking is completely determined by a subjacent logical chain which can be interpreted as a freedom standard. On the other hand, a logical chain can also incorporate different individual preferences which may lead to a representation result similar to Theorem 3.1 of Nehring and Puppe (1999).

The plan of this paper is as follows. Section 2 introduces some basic logical concepts and the system of axioms that implicitly defines an opportunity (freedom) ranking. Section 3, based on the work on preferential logic (see Freund (1998)), establishes a one to one correspondence between freedom rankings and logical chains that are viewed in this paper as freedom standards. Section 4 concludes with some brief remarks.

2. Basic concepts and definitions

We denote by $L$ a propositional language consisting of a finite number of atomic sentences $p_1, p_2, \ldots, p_n$ and the usual connectives of negation ($\neg$), disjunction ($\lor$) and conjunction ($\land$). We also include a constant falsehood ($\bot$). Each atomic sentence may denote a possible alternative or possible choice situation for an agent. For instance, the disjunction $p_i \lor p_j$ represents a true choice between alternatives $p_i$ and $p_j$ (or between choice situations) for an agent. The negation $\neg p_i$ may be interpreted then as a rejection of possible alternative $p_i$. A literal is a formula of the form $p_i$ or $\neg p_i$. Any formula of $L$ could be represented as a disjunction of conjunctions of literals. We will denote by $\vdash$ the classical consequence operator in $L$ and read $\alpha \vdash \beta$ as ‘$\alpha$ logically implies $\beta$’. A formula $\alpha$ is said to be consistent if it does not imply $\bot$.

The freedom (opportunity) ranking of an agent $\geq$ is defined on the set of all consistent formulas, $\alpha \geq \beta$ could be interpreted as ‘$\alpha$ offers at least as much freedom (opportunity) as $\beta$’. The asymmetric and symmetric part of $\geq$ will be denoted by $>$ and $\sim$, respectively.

Next, we impose a minimal set of requirements on $\geq$ expressed in a propositional language $L$. Perhaps it is possible to translate the results of this paper into a set-
theoretic language. However, following Freund (1998), we believe that a language of propositional logic provides a more general and flexible framework. First, we make a standard assumption that $\geq$ is reflexive and transitive. This is reflected in our axioms Ref and Tran. Moreover, we require that $\geq$ should satisfy the fairly uncontroversial conditions of monotonicity and dominance. These requirements are expressed as axioms Mon and Dom.

Ref (Reflexivity): $\alpha \geq \alpha$.

Tran (Transitivity): if $\alpha \geq \beta$ and $\beta \geq \gamma$, then $\alpha \geq \gamma$.

Mon (Monotonicity): if $\alpha \models \beta$ and $\alpha \geq \gamma$, then $\beta \geq \gamma$.

Dom (Dominance): if $\alpha \geq \beta$, then $\alpha \geq \alpha \lor \beta$.

The monotonicity condition simply says that if $\alpha$ offers at least as much freedom as $\gamma$, then, all the consequences of $\alpha$ also offer at least as much freedom as $\gamma$. In our framework, Mon implies the following condition (see Lemma 1): if $\alpha$ implies $\beta$, then $\beta$ offers at least as much freedom as $\alpha$. Of course, given Tran this condition also implies Mon. The dominance condition formulated above is a weaker version of the similar condition in Kreps (1979), see also Puppe (1996, 1998). It says that if $\alpha$ offers at least as much freedom as $\beta$ then adding $\beta$ to $\alpha$ would not increase freedom. As Lemma 1 illustrates we can strengthen the consequent of Dom.

Lemma 1.  
(i) if $\alpha \geq \gamma$ or $\beta \geq \gamma$, then $\alpha \lor \beta \geq \gamma$.
(ii) if $\alpha \models \beta$, then $\beta \geq \alpha$.
(iii) if $\alpha \models \beta$ and $\gamma \geq \beta$, then $\gamma \geq \alpha$.
(iv) $\alpha \geq \beta$ if and only if (iff) $\alpha \sim \alpha \lor \beta$.

Proof: (i) Without loss of generality suppose $\alpha \geq \gamma$. But we also have $\alpha \models \alpha \lor \beta$. By Mon then, $\alpha \lor \beta \geq \gamma$.

(ii) and (iii) are fairly easy and left to the reader.

(iv) Since $\alpha \models \alpha \lor \beta$, by (ii) we have $\alpha \lor \beta \geq \alpha$. Hence, if $\alpha \geq \beta$, then $\alpha \sim \alpha \lor \beta$. Conversely, suppose $\alpha \geq \alpha \lor \beta$. Since $\beta \models \alpha \lor \beta$, we can conclude by (iii), that $\alpha \geq \beta$. Q.E.D.
Since \( \sim \) is an equivalence relation on \( L \), we can define a map that associates with every consistent formula \( \alpha \) its equivalence class \([\alpha]\). Hence \([\alpha] \geq [\beta] \) iff \( \alpha \geq \beta \). This map \( \rho \) is a function from \( L \) onto a finite totally ordered set which can be taken equal to the interval \([0, k] = \{0, 1, 2, \ldots, k\}\). We can refer to \( \rho \) as the *freedom ranking function* associated with the freedom ranking \( \geq \), and the integer \( \rho(\alpha) \) as the *rank* of \( \alpha \). By the definition, \( \rho(\alpha) \geq \rho(\beta) \) iff \( \alpha \geq \beta \), which can be interpreted as ‘\( \alpha \) offers at least as much freedom (opportunity) as \( \beta \) iff the rank of \( \alpha \) is at least as big as the rank of \( \beta \). Obviously, we have \( \rho(\alpha) = \rho(\beta) \) whenever \( \alpha \models \beta \) and \( \beta \models \alpha \).

**Lemma 2.** For every consistent formulas \( \alpha, \beta \), we have

\[
\rho(\alpha \lor \beta) = \max (\rho(\alpha), \rho(\beta)).
\]

**Proof.** Assume without loss of generality that \( \max (\rho(\alpha), \rho(\beta)) = \rho(\alpha) \). But this means that \( \rho(\alpha) \geq \rho(\beta) \), hence \( \alpha \geq \beta \). By Lemma 1(iv), we can derive \( \alpha \sim \alpha \lor \beta \), that is, \( [\alpha] = [\alpha \lor \beta] \). Therefore, these formulas have the same rank. Q.E.D.

Suppose we have a map \( \rho \) from \( L \) to \([0, k]\) that satisfies \( \rho(\alpha) = \rho(\beta) \) whenever \( \alpha \) and \( \beta \) are logically equivalent. Moreover also assume that \( \rho \) satisfies \( \rho(\alpha \lor \beta) = \max (\rho(\alpha), \rho(\beta)) \). Define the binary relation \( \geq \) on \( L \) as follows: \( \alpha \geq \beta \) iff \( \rho(\alpha) \geq \rho(\beta) \). Then we can easily verify that our four conditions are satisfied. The verification of Ref and Tran is left to the reader. To verify Mon, suppose \( \alpha \models \beta \) and \( \alpha \geq \gamma \), that is, \( \rho(\alpha) \geq \rho(\gamma) \). By Lemma 1(ii), we also have \( \beta \geq \alpha \), that is, \( \rho(\beta) \geq \rho(\alpha) \). By transitivity then, \( \rho(\beta) \geq \rho(\gamma) \), as required. To verify Dom, suppose \( \alpha \geq \beta \), that is \( \rho(\alpha) \geq \rho(\beta) \). By Lemma 2, we have \( \rho(\alpha \lor \beta) = \max (\rho(\alpha), \rho(\beta)) \). Since \( \rho(\alpha) \geq \rho(\beta) \), \( \rho(\alpha \lor \beta) = \rho(\alpha) \). By Ref, we also have \( \rho(\alpha) \geq \rho(\alpha \lor \beta) \) as required. Clearly its associated ranking function is equal to \( \rho \). Therefore we have established the following corollary of Lemma 2.

**Corollary.** There is a one to one mapping between the family of freedom (opportunity) relations \( \geq \) and the family of ranking functions \( \rho \) that satisfy the following conditions:
(i) \( \rho(\alpha) = \rho(\beta) \) if \( \alpha \) and \( \beta \) are logically equivalent;
(ii) \( \rho(\alpha \lor \beta) = \max(\rho(\alpha), \rho(\beta)) \).

3. Standards and Freedom

Standards will be expressed in a propositional language \( L \) as a chain of non-equivalent consistent formulas of \( L \), that is, \( \Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \) where, for each \( i > 1 \), \( \sigma_i \models \sigma_{i-1} \). Such sequence of elements will be called a standard of length \( k \).

Let \( \Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \). For any consistent formulas \( \alpha \) and \( \beta \), define the relation \( \geq_{\Sigma} \) as follows: \( \alpha \geq_{\Sigma} \beta \) iff there exists an index \( i \) such that if \( \sigma_i \) implies \( \beta \), then \( \sigma_i \) implies \( \alpha \) (symbolically: \( \exists \sigma_i \in \Sigma \) such that if \( \sigma_i \models \beta \), then \( \sigma_i \models \alpha \). We want to establish that \( \geq_{\Sigma} \) is a freedom (opportunity) ranking and we shall refer to it as the opportunity ranking induced by the standard \( \Sigma \).

**Theorem 1.** For any standard \( \Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \), the relation \( \geq_{\Sigma} \) on \( L \) defined as above is an opportunity ranking.

**Proof.** The verification of Ref and Tran is easy and left to the reader. To prove Mon, suppose \( \alpha \models \beta \) and \( \alpha \geq_{\Sigma} \gamma \) but it is not the case that \( \beta \geq_{\gamma} \gamma \). Then, for any arbitrary index \( i \) we have, that \( \sigma_i \) implies \( \gamma \) but \( \sigma_i \) does not imply \( \beta \). Since \( \alpha \geq_{\Sigma} \gamma \), there is an index \( i \) such that, if \( \sigma_i \) implies \( \gamma \), then \( \sigma_i \) implies \( \alpha \). Hence, we have that \( \sigma_i \) implies \( \alpha \). But \( \alpha \models \beta \) and, therefore, we can also deduce that \( \sigma_i \) implies \( \beta \), a contradiction.

To prove Dom, suppose \( \alpha \geq_{\Sigma} \beta \) but it is not the case that \( \alpha \geq_{\Sigma} \alpha \lor \beta \). Then, for any arbitrary index \( i \) we have that \( \sigma_i \) implies \( \alpha \lor \beta \) but \( \sigma_i \) does not imply \( \alpha \). If \( \sigma_i \) implies \( \alpha \lor \beta \), then either \( \sigma_i \) implies \( \alpha \) or \( \sigma_i \) implies \( \beta \). If \( \sigma_i \) implies \( \alpha \) we have reached a contradiction. Therefore, assume that \( \sigma_i \) implies \( \beta \). Since \( \alpha \geq_{\Sigma} \beta \), there exists an index \( i \) such that if \( \sigma_i \) implies \( \beta \), then \( \sigma_i \) implies \( \alpha \). Hence, we can deduce that \( \sigma_i \) implies \( \alpha \), a contradiction. Q.E.D.
Hence, the set $\Sigma$ could be interpreted as a standard according to which we can rank consistent formulas of $L$. It can be verified that the ranking function $\rho_\Sigma$ associated with the opportunity relation $\geq_\Sigma$ is defined as follows:

$$\rho_\Sigma(\alpha) = \max \{i : \sigma_i \text{ implies } \alpha\} \text{ if the set is not empty, and }$$

$$\rho_\Sigma(\alpha) = 0 \text{ if } \sigma_1 \text{ implies } \neg \alpha.$$ 

The following example illustrates the role of a standard in ranking consistent formulas of $L$. Suppose an agent contemplates a choice between being a monk and serving God or being a philosopher engaging in critical thinking. We denote these two alternatives by $m$ and $p$. Hence, the language $L$ is built on two propositional sentences $m$ and $p$ and consists of 14 non-equivalent consistent formulas. We shall rank these formulas according to the following standard $\Sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 = m \lor p$ and $\sigma_2 = m \land p$. This standard has an obvious interpretation: the agent wants to combine religion with philosophy but failing to achieve it will settle for specialization. We determine the induced freedom (opportunity) ranking of this agent by evaluating the rank of each formula.

First, we start with formulas of rank 0. By definition, such a formula $\alpha$ must imply $\neg m \land \neg p$. Therefore, the only consistent formula $\alpha$ of rank 0 is $\neg m \land \neg p$.

In order to determine the formulas of rank greater than 0, we have to find formulas $\alpha$ that are not of rank 0 and satisfy $\sigma_2 \models \neg \alpha$, that is, $\alpha \not\models \neg m \lor \neg p$. There are exactly seven such formulas: $\neg m$, $\neg p$, $\neg m \land p$, $m \land \neg p$, $\neg m \lor \neg p$, $(\neg m \land p) \lor (\neg m \land \neg p)$, $(m \land \neg p) \lor (\neg m \land \neg p)$. Notice, however, that $\neg m$ implies $\neg m \lor \neg p$ and $m \land \neg p$ implies $\neg p$. Similarly, $\neg p$ implies $\neg m \lor \neg p$ and $\neg m \land p$ implies $\neg m$. Also $(\neg m \land p) \lor (\neg m \land \neg p)$ implies $\neg m \lor \neg p$ as well as $\neg m$, while $(m \land \neg p) \lor (\neg m \land \neg p)$ implies $\neg m \lor \neg p$ and $\neg p$. Hence, we have the following hierarchy of formulas with rank greater than 0

$$\neg m \land p, m \land \neg p$$

$$(\neg m \land p) \lor (\neg m \land \neg p), (m \land \neg p) \lor (\neg m \land \neg p)$$

$$\neg m, \neg p$$

$$\neg m \lor \neg p.$$
We can add the remaining six formulas to this hierarchy in the following order:

\[ m \land p \]
\[ m, p \]
\[ m \lor p, \lnot m \lor p, m \lor \lnot p. \]

The freedom ranking among the formulas is given as follows: \( \alpha \geq \beta \) iff the rank of \( \alpha \) is greater than or equal to the rank of \( \beta \). Therefore, we can see that, given this standard, the agent will have at least as much freedom from his significant choice between being monk or philosopher as he will have from his specialization, that is, \( m \lor p \geq \lnot m \land p \) and \( m \lor p \geq m \land \lnot p \). Notice, however, that \( m \geq \lnot m \lor \lnot p \) and \( p \geq \lnot m \lor \lnot p \). Also \( m \geq \lnot m \land p \) and \( p \geq m \land \lnot p \).

Next, we will show that any freedom ranking of an agent can be completely determined by his or her standard, that is, all opportunity rankings are indeed induced by a standard.

**Theorem 2.** For any opportunity ranking \( \geq \) defined on a propositional language \( L \), there is a standard \( \Sigma \) such that \( \geq = \geq \Sigma \).

**Proof.** First, we want to point out that any formula \( \alpha \) can be presented in its normal disjunctive form as \( \alpha = \lor \sigma \), where the disjunction is taken over all complete formulas \( \sigma \) such that \( \sigma \models \alpha \). Let \( \geq \) be an opportunity ranking on \( L \), \( \rho \) its associated ranking function and \( k \) the height of \( \rho \). For \( 1 \leq i \leq k \), let \( \sigma_i \) be the disjunction of all complete formulas of rank \( \geq i \), that is \( \sigma_i = \lor \{ \sigma : \sigma \text{ a complete formula and } \rho(\sigma) \geq i \} \). The sequence \( \Sigma = (\sigma_i, 1 \leq i \leq k) \) is then a standard of \( L \). We want to show that this standard induces \( \geq \Sigma \), that is, that \( \geq = \geq \Sigma \).

Suppose, first, that it is not the case that \( \alpha \geq \beta \), and set \( i = \rho(\beta), j = \rho(\alpha) \). Then we have \( i > j \). Given the equivalence of \( \alpha \) to the disjunction of all complete formulas \( \sigma \), with \( \sigma \models \alpha \), we have by Corollary (ii) that \( \rho(\alpha) = j \geq \rho(\sigma) \) for any complete formula \( \sigma \) such that \( \sigma \models \alpha \). If a complete formula \( \sigma \) has a rank strictly greater that \( j \), one has
therefore $\sigma \vdash \neg \alpha$. It follows then that $\sigma_i$ does not imply $\alpha$. We can also present $\beta$ as the disjunction of all complete formulas $\sigma$ such that $\sigma \vdash \beta$. Notice that $i = \rho(\beta) = \max \{\rho(\sigma) : \sigma$ a complete formula and $\sigma \vdash \beta\}$. But this implies that there is a complete formula $\sigma$, with $\sigma \vdash \beta$ and $\rho(\sigma) = i$, that is, $\sigma_i \vdash \beta$. Hence, we have established that there is an index $i$ such that $\sigma_i$ implies $\beta$ but fails to imply $\alpha$, that is, it is not the case that $\alpha \geq \Sigma \beta$.

Conversely, suppose that it is not the case that $\alpha \geq \Sigma \beta$. Then, there exists an index $i$ such that $\sigma_i$ implies $\beta$ and $\sigma_i$ does not imply $\alpha$. Since $\sigma_i$ does not imply $\alpha$, we have $\sigma_i \vdash \neg \alpha$ and, therefore, $\rho(\alpha) < i$. But $\sigma_i$ implies $\beta$, and this means that $\rho(\beta) \geq i$. Hence, it follows that $\rho(\beta) > \rho(\alpha)$, that is, it is not the case that $\alpha \geq \beta$. Q.E.D.

4. Concluding remarks

First, we want to draw an attention to the fact that our propositional language $L$ is built on a finite number of atomic sentences. This limitation is essential for the proof of Theorem 2 and cannot be removed.

Second, an opportunity or freedom ranking is defined only on consistent formulas of $L$. Hence, this analysis can not handle inconsistent choice situations violating the contraction consistency property.

Finally, in their recent paper van Hees and Wissenburg (1998) have criticized three preferential approaches to freedom as opportunity: in terms of fixed preferences, in term of possible future preferences, and in terms of the preferences of reasonable persons. They conclude, on a philosophical ground, that any concept of freedom as opportunity ‘must ultimately find recourse in some moral standard’. The present paper seems to lend some formal ammunition to their claim (though we do not insist that standards must be moral: morality does not enter into considerations in various comparisons of opportunity sets). Notice, though that Nehring and Puppe (1999) have suggested another possible interpretation for individual orderings as a concept deduced or “revealed” by the agent’s evaluations of opportunities. However, this
interpretation is also not free of its own problems, see Puppe (1998) for a comprehensive survey.
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