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Morishima's Nonlinear Model of the Cycle: Simplifications and Generalizations

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Abstract*

Michio Morishima's nonlinear model of the trade cycle (Morishima, 1958) is simplified and generalized to show, by means of the Andronov-Hopf bifurcation theorem, the existence of a periodic orbit. In addition, an attempt is made to place Morishima's contribution in the broader tradition of the work of the Japanese School of Nonlinear Trade Cycle Theorists and to place this, in turn, within the context and development of the tradition of mathematical economics in early post-war Japan.

JEL Classification Codes: B31, C61, E32

Key Words: Business Cycles, Nonlinear Dynamics, Endogenous Cycles, Bifurcation Theory

* In honour and to the memory of Michio Morishima who died last year, fittingly but sadly, exactly in the centennial year of the birth of John Hicks. To the best of my knowledge, Morishima's only contribution to mathematical trade cycle theory in English was in Morishima, 1958. However, his contributions in Japanese to this subject were extensive during the early part of his academic life. He gave me a copy of his Japanese doctoral dissertation, which was on the subject of mathematical trade cycle theories, and inscribed in it (in Japanese): 'Watashi wa mohaya suri keizai gakusei deva arimasen' ('I am no longer a mathematical economist' - my translation.)

1. INTRODUCTION¹

Michio Morishima's *Contribution to the Nonlinear Theory of the Trade Cycle* [28] was his only formal foray into the field of business cycle theory. However, this does not mean he did not consider cyclical issues in his massive and impressive contributions to equilibrium growth theories (cf., in particular, Parts III and IV of [29]), linear models of Marxian dynamics (for eg., [27]) and, most importantly, in many of his earlier writings in Japanese. In this one *Contribution* he was squarely in a remarkable tradition of Japanese mathematical economists (in particular, Yasui, [38]², and Ichimura, [16], [17]) who formalized the pioneer nonlinear trade cycle theories of Kaldor ([19]), Hicks ([14]) and Goodwin ([8]) in the form of second-order, nonlinear, *differential* equations. The mathematical formalizations with which they encapsulated Kaldor's nonlinear model³ and the way they classified and formalized the trade cycle theories of Kaldor, Hicks and Goodwin defined future research in the field.

Moreover, Morishima (op.cit) and Ichimura (op.cit) were the pioneers in investigating, formally, the question of the *existence* of a limit cycle in a nonlinear trade cycle model. Goodwin came close to posing the question of the existence and stability of a (unique) limit cycle for the reduced form equation of his nonlinear model of the trade cycle⁴, but did not take the final steps that Morishima and Ichimura took, a few years later. More than a full decade had to elapse, after the pioneering efforts of Morishima and Ichimura, before Hugh Rose [31], introduced the full paraphernalia of the *Poincaré–Bendixson Theory for 2-dimensional autonomous system* to pose the problem of the existence of a limit cycle in an aggregative model of fluctuations⁵.

In this paper I try to simplify and generalize Morishima's model, both from an economic and a mathematical viewpoint. The main economic infelicity I aim to remove is the controversial assumption of additively separable induced and autonomous components in investment behaviour. This assumption, most decisively and consistently maintained in Hicks (op.cit), was severely criticised from empirical, statistical and theoretical viewpoints by almost all reviewers and commentators and, eventually accepted as dubious even by Hicks. Quite apart from the difficulties of statistically separating and identifying the induced component of investment from the autonomous part, there was an important mathematical infelicity in the way autonomous investment was finally included in the reduced form equation of Morishima's model⁶. The way I remove this infelicity makes it easy to apply the Andronov-Hopf bifurcation theorem to prove the existence of a periodic orbit in the modified Morishima model. Morishima had relied on less general theorems and fairly *ad hoc* economic assumptions to demonstrate the existence of a cycle in his model.

The paper, therefore, is organised as follows. In the next section a brief excursus on what I have come to call the 'Japanese Tradition in Nonlinear Trade Cycle Theory' is presented just to provide a backdrop for Morishima's unique contribution to the field. Next, in §3, the basic Morishima model is described and in §4 the extensions and generalizations are formulated and the main result is stated and proved. The concluding §5 discusses some general historical, methodological and epistemological modelling issues.

2. THE JAPANESE TRADITION IN NONLINEAR TRADE CYCLE THEORY

At the AEA meetings of 1956, Martin Bronfenbrenner attempted to provide an ‘interim’⁷ report on *The State of Japanese Economics* ([3]). In a perceptive observation he noted:

"[Younger men, their thinking and writing for the most part abstract and mathematical, predominantly Anglo-American in their training⁸, who are ‘modern’ or ‘pure’ economists] equally at home in micro- and macroeconomics, [have] tended to concentrate along lines of Walrasian general equilibrium, Keynesian income and employment theory, and Harrodian dynamics, to the neglect of the standard pabulum of partial equilibrium.

This is the group which, in ten years, has caught up with England and America from a position of nearly hopeless inferiority. It is also the group which has come furthest toward positive contributions to the international body of economic thought. Its works are appearing somewhat belatedly, and in bits and pieces, in many journals familiar to Western readers. .. I am assured that their larger works, available thus far only in Japanese, rank much higher in depth, breadth, and originality."

[3], pp. 392-3.

Nikaido’s classic paper on general equilibrium theory ([30]), completed independently and prior to the appearance of ‘Arrow-Debreu’ ([2]), was published almost simultaneously with Bronfenbrenner’s own piece. Kose ([21]), Uzawa, Furuya and others were already doing serious work on gradient dynamics, investigating stability problems in a maximization context, Inada ([18]) on social choice theory, works that were to mature into fundamental contributions and create traditions for the generation that came in the 60s and later, before the end of the 50s. Masazo Sono [33] and Takuma Yasui [37] had opened mathematical economic pathways, with their works in demand analysis and consumer theory, for the ‘Young Turks’ (eg., Morishima [26], Yokoyama [39] and Uzawa [36]) who were to blossom in the 50s.

If much of the above was mathematical economics ‘along lines of Walrasian general equilibrium’, and the about to emerge growth theoretic contributions of Uzawa, Inada and Morishima were to dramatically widen the horizons of ‘Harrodian dynamics’ in multisectoral variations of neoclassical growth theory, then the nonlinear trade cycle theorists’ work was on ‘Keynesian income and employment theory’. I am not sure a parallel can be found in any other centre of economic research, nationally or trans-nationally, where a coherent, almost cohesive, group of mathematically minded economists contributed to all three of the central areas of what came to be the frontiers of theoretical research in economics: general equilibrium theory, growth theory and business cycle theory.

The work on nonlinear trade cycle theory should, therefore, be viewed as one leg of the mathematical ‘tripod’ on which Japanese economic theoretical research was based, proceeding simultaneously on three fronts with fundamental contributions from those who later were to be acknowledged as pioneers of various strands of mathematical economics.

Thus, Morishima’s Nonlinear Model was continuing a ‘Japanese tradition of research’ in the field of nonlinear trade cycle theory that had been initiated by Yasui

(op.cit)⁹. Yasui observing that the general economic hypotheses underpinning the Hicks model in [14], particularly the reliance on a non-linear acceleration principle and the (artificial) separation between autonomous and induced investments, *even though it was formulated in discrete time*, were substantially equivalent to those in the Goodwin model ([8]), began a tradition of referring to the Hicks-Goodwin Model being represented by the (*forced*) Rayleigh equation. In contrast, there was the non-linear Kaldor model, relying on an investment function depending on the level of income and the stock of capital. Yasui was the pioneer who reduced it to the (*unforced*) van der Pol-type equation in income, Y (cf. [38], equation 2.17, p.232)¹⁰.

The essential point here is that Yasui (op.cit), having identified the *economic underpinnings* of the models in Goodwin (op.cit) ([8]) and in Hicks (op.cit) went on, also, to identify their totally different *mathematical formalisms in terms of nonlinear differential equations*. Goodwin theorized and formalized in terms of continuous time and (non-linear) differential equations; Hicks, at least in [14], in terms of *linear* difference equations and *discrete* time. Indeed, Hicks was explicit about the reasons for eschewing continuous time and non-linearities. Economically, the Hicksian discussion in [14] had proceeded in terms of ‘period analysis’, a method he had probably absorbed from the Swedes; hence, it was natural, he claimed ([14], p.169), to mathematise in terms of discrete time, even though it may not have been mathematically necessary to do so. On the other hand, there was the economic question of *lags*; here he felt that the medium of continuous time would be able to handle only the very simplest kind and anything remotely realistic, in continuous time, would lead to *Integral Equation* formalisms which were ‘easiest to deal with’ as ‘limiting cases of difference equations’ (ibid). Anyone familiar with the elementary decision lag in Goodwin’s model, and the approximations he had to resort to, so that the reduced form of the model could avoid being a non-linear difference-differential equation, should have no difficulty in appreciating Hicks’s reasons for working with difference equations (cf. also, below, §5). As for *linearity*, on this, too, Hicks was quite explicit, but less categorical¹¹ (cf. also [15], pp. 212-3).

The ‘third’ of the three Japanese contributions – after Yasui and Morishima – was by Ichimura ([16]). He summarized the ‘Japanese tradition’ in his much quoted chapter in the Kurihara book in the following way¹²:

"The present paper is intended as an attempt to explore further the possibility of post-Keynesian nonlinear theories of economic fluctuations, and also to re-examine the well-known classical theories of trade cycles in the light of recent theoretical developments along the lines of nonlinear macrodynamics. As will be shown, *most classical theories can be formulated in terms of nonlinear differential equations ...*"

[16], p.195; italics added.

Once the die was cast, however, the consequences were inevitable: harnessing of standard theorems of non-linear differential equations to establish the existence of (at least one) limit cycle. In particular, the economics of the functional forms were subject to those hypotheses that were required for the validity of the relevant theorems to which Yasui, Ichimura and Morishima appealed – especially the famous theorem of Levinson and Smith ([23]), the Bendixson negative criterion and, eventually, the Poincaré-Bendixson theorem.

Not long after the Hicks book, [14], and the Goodwin *classic*, ([8]), were published, Roy Allen ([1]) codified them for textbook presentation, but did so in distinct

chapters, maintaining their essential differences from both of the above points of view; similarly, Gandolfo's textbook presentations retained fidelity to the originals in these two respects ([6]), as did most textbooks of the 60s and 70s. Thus, it is not entirely clear that it is quite legitimate to interpret and categorize the *Hicks* model in the non-linear class and then to identify and 'equate' it to the *Goodwin* model of 1951.

Whether it is legitimate or not, it is that identification that indelibly etched itself in the collective tradition of non-linear macrodynamics a place for the Hicks model on the same footing as the models of Kaldor and Goodwin as being encapsulated by one or another of a famous nonlinear differential equation.

3. THE MORISHIMA MODEL

The bare bones of Morishima's model consist of a dynamic consumption function, a nonlinear investment function against the backdrop of a closed real economy¹³. The derivation of the reduced from nonlinear dynamical equation is, therefore, a simple question of deft algebraic manipulations.

The dynamic consumption relation between Consumption at time t , C_t and real output at time $t - \tau$, $Y_{t-\tau}$, is:

$$C_t = \alpha Y_{t-\tau} + \beta(t) \quad (1)$$

where α is a variant of the usual MPC and $\beta(t)$ is 'the historically given upward drift of the consumption function' (ibid, p.168). Next he approximates C_t by taking only the first two terms of a Taylor series expansion to get:

$$C = \alpha Y - \alpha\tau\dot{Y} + \beta(t) \quad (2)$$

Morishima then denotes the desired capital stock by Θ , the part of the capital stock required for innovational investment to be L , and the actual capital stock by K , the Morishima version of the Hicks-Goodwin investment functions become:

$$I = \nu(\Theta + L - K); \nu \text{ (accelerator coefficient)} > 0 \quad (3)$$

$$\dot{\Theta} = \varphi(\dot{Y}); \varphi' \gg 0 \quad (4)$$

Denoting by $l(t)$ 'autonomous or innovational investment' (ibid, p. 168)¹⁴ and using the flow identity for expenditure in a closed economy and (2), (3) and (4), with some simple and straightforward algebraic manipulations the following reduced form dynamics in real output can easily be derived:

$$\alpha\tau\ddot{Y} + \{\alpha\tau\nu + (1 - \alpha)\}\dot{Y} + \nu\frac{\varphi(\dot{Y})}{\dot{Y}} + \nu(1 - \alpha)Y - (\beta(t) + l(t)) = \beta'(t) - l'(t) \quad (5)$$

At this point, faced with the daunting task of analysing a *forced*, second-order, nonlinear differential equation, Morishima resorted to an infelicitous economic assumption¹⁵ by simply postulating that $\beta(t)$ and $l(t)$ are constants, given by β^* and l^* , respectively! Such an assumption enabled Morishima to simplify his reduced form equation to a homogeneous, second-order, nonlinear differential equation and,

thence, simply appeal to standard (but highly restrictive) theorems to demonstrate the existence of a limit cycle.

Firstly, there was no point in assuming, in (2), a separate time dependent parameter acting as a kind of Duesenberry-type ratchet effect on consumption if, eventually, it was to be assumed a constant. Why not simply assume, *ab initio*, a relevant higher level of intercept to encapsulate the effect, especially since there is no loss of generality in taking such a route towards the derivation of the final, reduced form, equation?

Secondly, even more crucially, from an economic point of view, there is no logic – empirically or theoretically – of making an explicit assumption of additive separability of autonomous from innovational investment and, then, eliminating the time dependency of the former simply to avoid the unpalatable analytical consequence of having to deal with a forced, second-order, nonlinear differential equation. Such a strategy makes the economics a poor handmaiden of an analytical straitjacket, but Morishima was, of course, simply following a tradition that had been initiated by Hicks and Goodwin.

However, when he came to reflect, in the ‘third impression’ of his justly famous book, [14], on this strict, identifiable separation assumption, Hicks himself expressed grave doubts:

"Of all the concepts which are used in [14], that which has caused the most trouble is *Autonomous Investment*; and here I must admit to having brought the trouble upon myself, for I do not think that I was entirely consistent in the use which I made of the term. I am afraid that I do occasionally talk as if one could tell whether a particular piece of investment was autonomous just by looking at it; this is quite wrong."

[14], pp.vi-viii [Preface to the Third Impression of [14]]

The assumption of strict and identifiable separability between autonomous and induced investment had been severely criticized by a galaxy of business cycle theorists who reviewed the book in leading Journals, literally within weeks and months of the book having been published. I shall not go through a catalogue but it may be useful to record the views of just two of them: Duesenberry in the *QJE* ([5]) and Lundberg in the *Ekonomisk Tidskrift* ([24]). Duesenberry pointed out, in rather measured tones, that (*ibid*, p.473; italics added):

"Hicks's argument and many similar ones are based on a division of investment into three classes: autonomous investment, induced investment, and replacement investment. Like many other concepts in business cycle theory the above classification is somewhat poorly related to the underlying micro-theory of investment.

In fact, we *cannot make a clear distinction between these three types of investment* except in certain rather special cases."

Lundberg, in contrast, was more pungent in expressing his doubts about such a separation assumption:

"[There] is the question of the distinction between induced and non-induced ('autonomous') investment. Hicks gives an extremely unsatisfactory description of the latter, and all that we can discover is that it is not determined by the increase in production from year to year,

and that it is a necessary condition for continuous expansion that autonomous investment should increase in step with national income. ... But as far as I can see *there is no firm basis for dividing total investment into these two categories*. ..[T]his division [into induced and autonomous investment] can be expected to vary during the course of expansion. I consider, therefore, that this division of investment activity into categories, which is used by Harrod and Hicks, is a useless method for empirical investigation, and therefore probably an unfruitful hypothesis for a business cycle model."

[24], p.103; italics added.

In defense of Morishima, however, it can be pointed out that he was only trying to generalize and formulate a ‘master equation’ for the already developed nonlinear theories of the trade cycle. He was not trying to develop his own theory of the phenomenon of aggregate fluctuations – at least not in the contribution in focus here; nor was he trying to extend or generalize the Kaldor, Hicks or Goodwin contributions.

Eventually, therefore, the reduced form dynamical system, in terms of deviations from equilibrium, which Morishima analyses has the following form:

$$\frac{\alpha\tau}{\nu}\ddot{z} + \left\{ \alpha\tau + (1-\alpha)\frac{1}{\nu} - \frac{\varphi(\dot{z})}{\dot{z}} \right\} \dot{z} + (1-\alpha)z = 0 \quad (6)$$

where, $z = Y - Y^*$ and equilibrium income, $Y^* = \frac{\beta^* + I^*}{(1-\alpha)}$

4. SIMPLIFICATIONS, GENERALIZATIONS AND A THEOREM

Given the above strictures against the arbitrary assumption of additively separating induced from autonomous investment, the simplification I shall suggest below, via a straightforward parametrization, facilitates a more general analysis of the existence of a cycle in the modified Morishima model. Moreover, the *ad hoc* assumption of an equally arbitrary assumption of a constancy of innovational autonomous investment can with one fell swoop be removed, as well. Thus, (3) and (4) become, respectively:

$$I = \nu(Q - K) \quad (7)$$

and:

$$\dot{Q} = \varphi(\dot{Y}; \eta) \quad (8)$$

where:

η :variable parameter encapsulating innovational (autonomous) *and* induced investment;

φ satisfies all the assumptions in the original Morishima model;

This simplification, together with the rescaling of the consumption function (2) to eliminate β , leads to the following more general, but also simpler, reduced form dynamics in real output:

$$\ddot{Y} + \left(\frac{1-\alpha-\nu\varepsilon}{\varepsilon} \right) \dot{Y} + \frac{\nu}{\varepsilon} \varphi(\dot{Y}; \eta) + \frac{\nu}{\varepsilon} (1-\alpha)Y = 0 \quad (9)$$

Where $\varepsilon = \alpha\tau$ and defining $m = \frac{\nu}{\varepsilon}$, we get:

$$\ddot{Y} = \left[m - \left(\frac{1-\alpha}{\varepsilon} \right) \right] \dot{Y} - m\varphi(\dot{Y}; \eta) - m(1-\alpha)Y \quad (10)$$

A state space representation of (10), for $x_2 = \dot{Y}$, gives:

$$\dot{x}_1 = x_2 (\equiv \dot{Y}) \quad (11)$$

and:

$$\dot{x}_2 = m(\alpha-1)x_1 + \left[\left\{ m - \left(\frac{1-\alpha}{\varepsilon} \right) \right\} - m\varphi(x_2; \eta) \right] x_2 \quad (12)$$

Taking a Taylor series expansion of $\varphi(x_2; \eta)$ and, without loss of generality, scaling units so that the constant term in the expansion is used to eliminate the terms within the curly brackets in (12), we get the linearized system:

$$\dot{x}_1 = x_2 \quad (13)$$

$$\dot{x}_2 = m(\alpha-1)x_1 + m\varphi'(\bar{x}_2; \eta)x_2 \quad (14)$$

The characteristic equation of the linearized system is given by:

$$\lambda^2 - B\lambda + C = 0 \quad (15)$$

where:

$$B \equiv m\varphi'(\bar{x}_2; \eta) \quad (16)$$

$$C \equiv -m(\alpha-1) \quad (17)$$

I can, now, state and prove the main theorem:

THEOREM 1. *In any neighbourhood U of the equilibrium Y^* of (10) in \mathbb{R}^2 , $\forall \eta_0 > 0, \exists \bar{\eta}$ with $|\bar{\eta}| < \eta_0$ such that the system (11) and (12) [or (10)] has a nontrivial periodic orbit in U .*

Proof. The proof consists in verifying the conditions of the Andronov-Hopf theorem (cf. [11], p.344, ff). First of all the eigenvalues of the linear part of (11)-(12), denoting the real and imaginary parts by γ and ρ , are: ■

$$\lambda_{1,2} = \gamma(\eta) \pm \rho(\eta) \quad (18)$$

It is easy to compute and verify that:

$$\gamma(0) = 0 \text{ and } \rho(0) \neq 0 \quad (19)$$

Moreover, it is clear that the real part of the eigenvalues crosses the imaginary axis at nonzero speed; i.e.,

$$\frac{d\gamma}{d\eta}(\eta=0) \neq 0 \quad (20)$$

Since the nonlinear and linear parts of (11)-(12), as functions of the parameter η , at the origin, are identically zero for all sufficiently small $|\eta|$, the conditions of the Andronov-Hopf theorem are verified and the existence of a nontrivial periodic orbit is proved.

Remark 1. There are several advantages in the formal approach to demonstrate the existence of a cycle using the above simple bifurcation approach over the traditional methods used by Morishima (and Ichimura). Their methods were confined to planar dynamics; the bifurcation approach utilised above, even though of the simplest variety, can easily be used to analyse higher dimensional systems. Slightly more general bifurcational analysis can also be harnessed to demonstrate the existence of cycles in systems depending on several parameters.

Remark 2. The requirement of a vanishing vector field at ‘the origin’ is not crucial; whatever the equilibrium values, an appropriate change of variables can be effected around any relevant equilibrium configuration (as, indeed, Morishima did, in his own analysis - and as has been standard practice, at least since [8]. The same comments apply to my own rescaling when taking a Taylor series expansion of $\varphi(x_2; \eta)$.

5. CONJECTURES AND REFLECTIONS

To the best of my knowledge, it was in 1928 that the idea of interpreting economic cycles as being generated by a *non-linear differential equation* capable of *relaxation oscillations* was first hypothesized by Hamburger [12], p.112:

"The present writer would like to point out that the applicability of the principle of relaxation-oscillations to economic cycles was first emphasized by him in 1928 [at the May 7, 1928, Meeting of the Batavian Society of Logic Empirical Philosophy] in a discussion following a paper read by Messrs. Van der Pol and J. van der Mark on ‘The Heartbeat Considered as a Relaxation-Oscillation, and an Electrical Model of the Heart.’"

Independently, in 1933, Philippe Le Corbeiller had suggested something similar in the very first volume of the **Econometrica**:

"Le problème des crises, et plus généralement des oscillations des prix, est assurément l’un des plus difficiles de l’Économie Politique; il ne sera sans doute pas de trop, pour approcher de sa solution, de la mise en commun de *toutes les ressources de la théorie des oscillations* et de la théorie économique. C’est pourquoi j’ai pensé pouvoir vous présenter un compte-rendu succinct d’un avance récente, que je crois importante, de la théorie des oscillations: celle apportée au problème des *systèmes autoentretenus par la découverte des oscillations de relaxation*, due à un savant hollandais, le Dr Balth. van der Pol."

[22]pp.328-9; italics added.

Goodwin referred to this ‘admonition’ in the opening page of his pioneering piece which was one of the starting points for the impressive and important work of the Japanese Nonlinear Trade Cycle theorists [8], p.1; italics added:

"[E]conomists will be led, as natural scientists have been led, to seek in nonlinearities an explanation of the maintenance of oscillation. Advice to this effect, given by Professor Le Corbeiller is one of the earliest issues of this journal, has gone largely unheeded"

An unanswerable question, for the context of the themes broached in this paper is why the Japanese Nonlinear Trade Cycle Theorists confined their attention to modelling the nonlinear *economic* dynamics exclusively in terms of nonlinear *differential* equations. Morishima, following the tradition initiated by Yasui, continued and codified the practice of modelling macroeconomic theories of fluctuations in terms of *nonlinear second order differential equations*. Ichimura, for example, was well aware of the frontiers of research in nonlinear difference-differential equations (cf. [16], p.201, footnote 9). Why, in particular, did Morishima adhere and identify himself with this tradition in trade cycle theory but not in growth theory or other aspects of mathematical economics? I do not have a clear answer to these ponderings.

As mentioned above, both explicitly and implicitly, there were at least two core economic – theoretical and empirical – infelicities in this tradition. Firstly, the empirically indefensible assumption of additively separating induced from autonomous investment components; secondly, having assumed, in the specification of the behavioural investment function, a time-varying autonomous investment component, to discard it to avoid having to do the hard work of analysing an intractable forced nonlinear differential equation, does not inspire confidence in such a tradition. Serious students, sympathetic to the nonlinear, endogenous, approach to the modelling of macroeconomic fluctuations, may be forgiven for thinking that *ad hoc* assumptions were harnessed to make a reduced form system amenable to interpretations in terms of standard theorems of nonlinear dynamics; that the reduced form system was not a representation of natural economic dynamics. Perhaps this kind of research strategy also contributed to the demise of the nonlinear tradition of modelling fluctuations after the initial euphoria of the early 50s and the eventual emergence of the stochastic, exogenous, approach to modelling business cycles. The burgeoning literature on endogenous cycles, in the aftermath of the ‘chaotic excitements’ of the 80s, seemed to have suffered the same fate as the first, perhaps stillborn, nonlinear approach of the 50s.

Secondly, mathematically, too, it does not seem wholly defensible or fair to identify an explicit discrete-time, piece-wise linear economic modelling strategy, adopted with measured deliberation by Hicks, with a continuous-time, nonlinear, approach favoured and practised by Goodwin and Kaldor. Hicks’s own reservations against the use of continuous-time modelling of aggregate fluctuations may well be worth remembering:

"The verbal discussion in the text has proceeded in terms of ‘period analysis’ - with time divided into discontinuous periods - and the mathematical theory which follows will do the same. ... *The other reason is more fundamental*. It is actually only the simpler sorts of lags which it is convenient to study by means of differential equations; some of the problems I want to study would not reduce to differential equations at all, if time were treated as continuous. They would reduce to *Integral Equations*; [I] suspect that the easiest way of dealing with these integral equations would be to treat them as limiting cases of difference equations; .."

[14], pp.170-1; first set of italics, added.

As for assuming linear behavioural relations, the Hicksian justification was characteristically pragmatic¹⁶:

"[I]t may be questioned whether we derive any advantage from extensions into non-linearity, when we come to the more complex cases,....
. There may be some special cases where [the assumption of linearity] is not true; but I think that these cases can usually be covered in more convenient ways than by assuming non-linearity.

Two relevant observations on these issue might also be worth mentioning in the context of the Hicksian stances. In his stimulating and highly original review article of [14], Goodwin expressed, with characteristic clarity and directness, serious worries about nonlinear discrete time modelling of aggregate economic fluctuations:

"Another questionable feature of [Hicks's **Contribution to the Theory of the Trade Cycle**] is that in a book on non-linear cycle we are presented with nothing but *linear* theory, even in the mathematical appendix. It is, however, much easier to cavil than to suggest what might have been done, for non-linear difference equations represent virgin territory mathematically *Combining the difficulties of difference equations with those of non-linear theory, we get an animal of a ferocious character and it is wise not to place too much confidence in our conclusions as to behavior.*"

[7], p.319; the initial italics in the original.

How exactly was a non-linear theory of the cycle presented linearly? That is the second point I wish to make - that a mathematically faithful formalization of the economics and methodology of Hicks would imply the following relation:

$$Y_n = \min \left\{ Y_n^{cng}, A_n + \sum_{i=1}^m c_i Y_{n-i} + \max \left\{ d_n, \sum_{i=1}^{m-1} \nu_i (Y_{n-i} - Y_{n-i-1}) \right\} \right\}, (\forall m \geq 2) \quad (21)$$

where:

Y_n : real output in period n ;

$c_i \in [0, 1)$: coefficients of the marginal propensity to consume, distributed appropriately over m lags;

$\nu_i \geq 0$: accelerator coefficients distributed over m lags;

$Y_n^{cng} = Y^{cng}(1 + \zeta)^n$: output ceiling growing at the exogenously given trend rate of $\zeta \geq 0$;

$A_n = A_0(1 + \zeta)^n$: autonomous investment, also growing at an exogenously given rate $\zeta \geq 0$;

Naturally, the *nonlinear theory* of the cycle is encapsulated in the piecewise linear nature of the global system given above. To find a nonlinear, continuous-time, dynamical system that is formally equivalent to (21) in the geometry of its behaviour is, I think, formally impossible. I might go further and conjecture that it is an effectively undecidable question - especially since the second part of Hilbert's 16th problem remains unsolved, more than a century after it was formulated.

As Haavelmo eloquently argued, at the very dawn of the mathematical approach to business cycle modelling¹⁷, these are not issues that can be settled by appealing to satisfactory ‘mimicking of the stylized facts of business cycles’. These are fundamental questions of epistemology - theories of how best to obtain knowledge of observable phenomena of theoretical and policy relevance.

Michio Morishima’s equilibrium growth models were, in general, formulated in terms of linear, discrete-time, systems. But he was also a master of non-linear, discrete-time, modelling with fundamental mathematical contributions to the non-linear eigenvalue problem for non-negative square matrices. The vast canvas on which he described, with erudition and idiosyncratic originality, his interpretations of the classical and neo-classical masters of economic theory – Ricardo, Walras, Marx, von Neumann, Hicks – were almost invariably in terms of highly innovative linear, discrete-time models.

However, in modelling aggregate fluctuations, he followed the tradition that had been broached by his senior contemporary, Takuma Yasui. It is noteworthy, all the same, that he never returned either to the theme or the method in his subsequent, vast, contributions to economic theory. I am not sure I am able to infer or discern a particular modelling strategy, motivated by a methodological or epistemological credo.

It is remarkable, therefore, that the relatively inaccessible but highly original contributions of Yasui, Morishima and Ichimura seem to have determined a whole tradition of mathematical modelling of a defining phenomenon of macroeconomic dynamics.

Notes

¹MANY STYLISTIC, CONCEPTUAL AND TECHNICAL INFELICITIES WERE WEEDED OUT AS A RESULT OF THE CRITICAL COMMENTS ON AN EARLIER DRAFT BY PROFESSOR NICO GARRIDO, MR STEPHEN KINSELLA, DR SRINIVAS RAGHAVENDRA AND PROFESSOR STEFANO ZAMBELLI. PROFESSOR ZAMBELLI'S COMMENTS, SUGGESTIONS AND INCISIVE CRITICISMS WERE CRUCIAL IN CLARIFYING SEVERAL DUBIOUS POINTS IN THE EARLIER DRAFT. MR KINSELLA'S DETAILED COMMENTS AND CONFIRMING MATHEMATICA SIMULATIONS OF MY 'SIMPLIFIED AND EXTENDED MORISHIMA MODEL' FOR PLAUSIBLE VALUES OF THE PARAMETERS WERE MOST REASSURING. THE MATHEMATICA CODE AND SIMULATIONS ARE AVAILABLE ON REQUEST FROM THE AUTHOR. THE REMAINING INFELICITIES ARE, ALAS, ENTIRELY MY OWN RESPONSIBILITY.

²I am using, for purposes of reference in this paper, the 'somewhat condensed version' of [38] that was mimeographed and circulated in 1961. I am most grateful to Professor Masanao Aoki of the department of economics at UCLA for making this available to me. It was Yasui's paper of 1953 that seems to have brought to the attention of 'western' scholars the significant work on nonlinear trade cycle theory that had been going on in Japan in the early post-war years.

3

'Yasui showed that the model could be translated into mathematical terms with the aid of van der Pol's theory of relaxation oscillations.' [20], p.9

When, in 1973, I was assigned Kaldor as my PhD supervisor at Cambridge and expressed interest in working on nonlinear trade cycle theories, he asked me to read the papers by Yasui and Morishima. Many years later, in Siena in 1983, when we discussed his 1940 model, he still maintained the importance of Yasui's early formalization of his model. Interestingly, my own first exposure to the van der Pol equation for relaxation oscillations came during undergraduate lectures on electric circuit theory, at Kyoto University in 1968, by one of the great pioneers of nonlinear dynamics: Chihiro Hayashi. This is the one thing I am proud to share with Michio Morishima - to have graduated from Kyoto University!

⁴The closest Goodwin came to the issue of proving existence of a limit cycle was when he stated, [8], pp.13-4 (*italics in original*):

"It is intuitively clear that [the system] will settle down to a [limit cycle], although the proof requires the rigorous methods developed by Poincaré. ... Perfectly general conditions for the stability of motion are complicated and difficult to formulate, but what we can say is that any curve of the general shape of $X(\dot{x})$ [or $\psi(\dot{y})$] will give rise to a *single, stable* limit cycle."

⁵Rose relied almost exclusively on the classic text of Coddington and Levinson ([4], especially, Chapter 16 of this outstanding textbook) for the mathematical results that he used. Neither Morishima, nor Ichimura, seem to have been aware of these results. In this context it might be useful to point out that the initial application of 'fix point theorems' was for the purpose of proving the existence of solutions for ordinary and partial differential equations (cf.[34], in particular, pp. 119-120). The penchant for posing 'existence' problems in economic theory by the pioneering Japanese mathematical economists – primarily, Nikaido, Uzawa, Morishima, Ichimura and Negishi – can, with hindsight, be given a 'Whig justification', if a full scale history of thought exercise was to be attempted.

⁶An infelicity he may have inherited from a tradition initiated by Goodwin ([8]), p.12 of assuming, for 'simplicity', that induced investment is a constant. Without such an assumption they would have had to deal with a *forced* Rayleigh equation, the global analytics of which remains as mysterious as ever, despite sterling efforts by a series of eminent mathematicians from Cartwright and Littlewood all the way to Hayashi and Smale.

⁷'Interim' in the sense that it was a report based on studies and seminars during a summer sojourn at Keio University in 1955, ten years after Japan's defeat and surrender, in August, 1945.

⁸Ichimura, for example, had obtained his doctorate at MIT in 1953.

⁹As must be clear by now, my concern here is not the richer and better known mathematical economics and orthodox Keynesian traditions fostered with great success by many of the eminent Japanese mathematical economists mentioned above. I am concentrating strictly and narrowly on the particular mathematical business cycle theory tradition that led to the codification of modelling aggregate fluctuations in terms of *nonlinear differential equations* and the posing of *existence* questions mathematically with regard to cycles. With the notable and remarkable exception of Richard Goodwin, the Japanese trio of Yasui, Ichimura and Morishima were almost solely responsible for codifying the tradition of modelling aggregate fluctuations as second-order nonlinear differential equations. Goodwin himself did not return to formal nonlinear differential equation modelling of aggregate fluctuations in a sustained way, after [8], till the famous ‘Dobb Festschrift’ paper [10], early versions of which had been presented in Cambridge from about 1963. He did, however, elaborate, clarify, modify and finesse the model of [8] in several significant and influential essays in the ‘50s (a notable example is [9]).

¹⁰The standard, parametric, forced Rayleigh equation is:

$$\ddot{y} + \dot{y}^3 - 2\lambda\dot{y} + y = g(t) \quad (22)$$

In the economic context in discussion in this paper, the forcing term, $g(t)$, encapsulated the content of an identifiably separate autonomous part of investment behaviour.

On the other hand, the standard, parametric, forced van der Pol equation is given by:

$$\ddot{y} - \alpha(1 - y^2)\dot{y} + \omega^2 y = g(t) \quad (23)$$

The choice between a Rayleigh and a van der Pol equation to formalise nonlinear business cycle theories has, as an economic backdrop, a precise stance on policy, as well. The former emphasises derivative and the latter proportional policy controls. Formally, however, the two equations can be transformed into each other by a simple redefinition of variables.

¹¹My own first teacher of macrodynamics, Professor Björn Thalberg, in his contribution to the Goodwin Festschrift ([35], pp. 103-4) observed, quite perceptively, that:

"It is worth noting that in a linear model it is generally difficult to point out any immediate or particular cause of a turning point, the explanation of each turn being in principle the whole model. This is in contrast with analyses by means of nonlinear cycle models where it often seems natural to link, for example, the upper turning-point to the existence of a kind of ‘ceiling’. ...[However] the borderline between linear and nonlinear models, from the point of view of the economic content, may not be very sharp." (italics added)

I have a feeling this is a view with which Hicks may have agreed.

¹²Ichimura was handsome in his acknowledgement of Yasui’s role in his own famous synthetic contribution [16], p.195, footnote 8):

"A mathematical formulation of the Kaldor theory was given by Professor T. Yasui, to which contribution the discussion of the present paper owes not the least:..."

¹³For the sake of ‘simplicity’, but also because the point I wish to emphasise is the ‘illegitimate’ dichotomy between identifiable, separable, autonomous and induced investment components, I shall concentrate on Morishima’s ‘Hicks-Goodwin model’. The interested reader can easily verify that there is no loss of generality in this simplification. As pointed out in footnote 11, above, there is a formal mathematical similarity between the Rayleigh and van der Pol equations in that the one can be transformed into the other by a simple redefinition of variables. Morishima’s ‘fundamental equation of nonlinear trade cycle theory’ (ibid, eqⁿ (9), p. 169) is simply a linear combination of these two types of equations. Therefore a mathematical formalization to demonstrate the existence of a cycle in one of them can easily be adapted for a linear combination of two of them. Thus,

for both the above mentioned economic reason and this mathematical one, I shall concentrate on Morishima's Hicks-Goodwin formalism in this paper.

¹⁴Where $l(t)$ is given by the first derivative of L .

¹⁵In this he was following Goodwin's equal infelicity.

¹⁶Hicks, together with Frisch and many others, are the precursors to the newclassical conviction on a similar methodological credo:

"Most of the econometric work implementing equilibrium models has involved fitting statistical models that are linear in the variables (but often highly nonlinear in the parameters). This feature is subject to criticism on the basis of the indisputable principle that there generally exist nonlinear models that provide better approximations than linear models. ... It is an open question whether for explaining the central features of the business cycle there will be a big reward to fitting nonlinear models"

[25], p.314; italics added.

Of course, Lucas and Sargent were advocating the Frisch-Slutsky methodology of linear stochastic difference equation modelling - i.e., the exogenous theory of business cycles; Hicks, on the other hand, was an enlightened, though not uncompromising, proponent of the endogenous vision for modelling aggregate fluctuations. The Japanese School of nonlinear trade cycle theorists were, of course, squarely in the endogenous tradition.

17

"The degree of conformity between .. theoretical solutions and the corresponding observed time series is used as a test of the validity of the model. In particular, since most economic time series show cyclical movements, one is led to consider only mathematical models the solutions of which are cycles corresponding approximately to those appearing in the data. ...

This condition for a 'good' theory is of course not a sufficient one, since there are in general many different a priori setups of theory which are capable of reproducing approximately the observed cycles. But, what is more important, it may not even be a necessary condition, and its application may result in a dangerous and misleading discrimination between theories."

[13], p.312; italics in original.

Haavelmo's wise scepticism on justifying theories of the business cycle by appealing to 'data mimicry' must sound almost archaic to modern theorists who routinely express self-approval of their model's adequacy on these very grounds.

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