Arrow's theorem and max-star transitivity*

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Abstract

In the literature on social choice with fuzzy preferences, a central question is how to represent the transitivity of a fuzzy binary relation. Arguably the most general way of doing this is to assume a form of transitivity called max-star transitivity. The star operator in this formulation is commonly taken to be a triangular norm. The familiar max-min transitivity condition is a member of this family, but there are infinitely many others. Restricting attention to fuzzy aggregation rules that satisfy counterparts of unanimity and independence of irrelevant alternatives, we characterise the set of max-star transitive relations that permit preference aggregation to be non-dictatorial. This set contains all and only those triangular norms that contain a zero divisor.

1 Introduction

A fuzzy set is the extension of a vague predicate, so if "small" is vague then the set of small objects is a fuzzy set. More precisely, let X denote the universal set and let W denote a subset of X in the classical sense, $W \subseteq X$. The set W is characterised by the function $f_W : X \to \{0, 1\}$ where $f_W(x) = 1$ if $x \in W$, and $f_W(x) = 0$ if $x \notin W$. Given $x \in X$, $f_W(x)$ is the degree to which

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x belongs to W. The generalisation to a fuzzy subset occurs by permitting this degree to take more than two values, typically by allowing any value in $[0, 1]$.[1] If G denotes a fuzzy subset of X and $f_G(x) = 1$ then x "clearly" belongs to G, and if $f_G(x) = 0$ then x "clearly does not" belong to G. In between there are various degrees of belonging.

A fuzzy (binary) relation F defined on a choice space X is characterised by a function $f_F : X \times X \to [0, 1]$. If this relation represents an individual's (weak) preferences, then $f_F(x, y)$ can be interpreted as the degree to which this individual is confident that "x is at least as good as y". This is not the only possible interpretation of $f_F(x, y)$; another is that it measures the intensity of an individual's belief, or how true they regard the proposition "x is at least as good as y".

For a fuzzy relation to count as a representation of preferences, it must satisfy certain criteria.[2] One is reflexivity, $f_F(x, x) = 1$ for all $x \in X$. Another is connectedness, $f_F(x, y) = 0$ implies $f_F(y, x) = 1$ for all $x, y \in X$. The most difficult condition to formulate is transitivity. There are many possible ways to model the transitivity of a fuzzy binary relation. Any condition is legitimate provided that it satisfies the weak constraint that, for all $x, y, z \in X$, $f_F(x, y) = 1$ and $f_F(y, z) = 1$ implies $f_F(x, z) = 1$. For example, this condition is met by the familiar max-min transitivity condition $f_F(x, z) \geq \min \{f_F(x, y), f_F(y, z)\}$, and also by the Lukasiewicz transitivity condition $f_F(x, z) \geq f_F(x, y) + f_F(y, z) - 1$.

Arguably the most general way of expressing the transitivity property is to assume a form of transitivity called max-star transitivity. If $\star$ is a binary operation on $[0, 1]$ then this condition says that $f_F(x, z) \geq f_F(x, y) \star f_F(y, z)$. The star operator in this formulation is commonly taken to be a triangular norm[3], i.e. a function T from $[0, 1]^2$ to $[0, 1]$ such that for all $x, y, z \in [0, 1]$ the following conditions are satisfied,

(i) $T(x, y) = T(y, x)$,
(ii) $T(x, T(y, z)) = T(T(x, y), z)$,
(iii) $T(x, y) \leq T(x, z)$ if $y \leq z$,

[1] In so-called "ordinal" versions of fuzzy set theory [0,1] is replaced by an abstract set on which a particular mathematical structure is defined. See Goguen (1967), Barrett, Pattanaik and Salles (1992), Basu, Deb and Pattanaik (1992) and Duddy, Perote-Peña and Piggins (2008).

[2] A sample of the literature on fuzzy preferences is Orlovsky (1978), Ovchinnikov (1981), Basu (1984), Billot (1995), Dutta, Panda, Pattanaik (1986), Dutta (1987), Jain (1990), Ponsard (1990), Dasgupta and Deb (1991, 1996, 2001), Ovchinnikov and Roubens (1991, 1992) and Banerjee (1993, 1994). The philosophical underpinnings of these ideas are discussed in Piggins and Salles (2007).

[3] Klement, Mesiar and Pap (2000) is a detailed account of triangular norms.

(iv) $T(x,1) = x$.

Throughout this paper we use the notation $x \star y$ and $T(x,y)$ interchangeably.

It is easy to see that max-min transitivity and Lukasiewicz transitivity are particular max-star transitive relations. There are infinitely many others. Of course, some valid transitivity conditions are not max-star transitive relations; $f_F(x,z) \geq \frac{1}{2}(f_F(x,y) + f_F(y,z))$ is an example.

Max-min transitivity possesses a technical property that is not shared by Lukasiewicz transitivity. It contains no zero divisor. A triangular norm T contains no zero divisor if and only if for all $x, y \in (0,1)$, $T(x,y) \neq 0$. The Lukasiewicz triangular norm $T_L(x,y) = 0$ when $x = \frac{1}{2}, y = \frac{1}{2}$ and so this norm contains a zero divisor. This condition is central to this paper.

2 Social choice

This paper is a contribution to the literature on social choice with fuzzy preferences. A comprehensive survey of the literature is Salles (1998).[4] The literature has been motivated by the idea that fuzziness can have a "smoothing" effect on preference aggregation and so perhaps the famous impossibility results of Arrow (1951) and others can be avoided.[5]

The literature typically focuses on criteria that imply that preference aggregation must be undemocratic in some sense. These results are akin to Arrow's impossibility theorem. In these papers the relevant conditions are sufficient conditions; if an aggregation rule satisfies them then it implies that there must be an undesirable concentration of power in society. Our approach is different, we identify both a necessary and sufficient condition for preference aggregation to be undemocratic in a particular sense. More specifically, we show that an aggregation rule satisfying certain criteria is dictatorial if and only if the triangular norm used in the formulation of the transitivity condition has no zero divisor. A consequence of this result is that

[4]Various results can be found in, among others, Barrett, Pattanaik and Salles (1986), Dutta (1987), Ovchinnikov (1991), Banerjee (1994), Billot (1995), Richardson (1998), Dasgupta and Deb (1999), Fono and Andjiga (2005), Perote-Peña and Piggins (2007, 2008a, 2008b), Duddy, Perote-Peña and Piggins (2008). See also Leclerc (1984, 1991) and Leclerc and Monjardet (1995).

[5]Our approach like others in the literature allows for social preferences to be vague even if the underlying profile of individual preferences is exact. This is what we mean by smoothing. A similar suggestion is made by Sen (1970). Note that an exact preference is a fuzzy preference $f_F(x,y)$ such that $f_F(x,y) \in \{0,1\}$ for all $x, y \in X$. A vague preference is a fuzzy preference $f_F(x,y)$ such that $f_F(x,y) \notin \{0,1\}$ for some $x, y \in X$.

max-min transitivity leads to dictatorship, whereas Lukasiewicz transitivity does not.

An equivalent way of putting the matter is this. Restricting attention to aggregation rules that satisfy counterparts of unanimity and independence of irrelevant alternatives, we characterise the set of max-star transitive relations that permit preference aggregation to be non-dictatorial. This set contains all and only those triangular norms that contain a zero divisor.

Preliminaries

X is a set of social alternatives with $\#X \geq 3$.
$N = \{1, ..., n\}$ with $n \geq 2$ is a finite set of individuals.
A fuzzy binary relation (FBR) over X is a function $f : X \times X \to [0, 1]$.
An exact binary relation over X is an FBR g such that $g(X \times X) \subseteq \{0, 1\}$.
S is the set of all FBRs over X.
H is the set of all $r \in S$ satisfying the conditions

(i) for all $x \in X$, $r(x, x) = 1$,

(ii) for all $x, y \in X$, $r(x, y) = 0$ implies $r(y, x) = 1$,

(iii) for all $x, y, z \in X$, $r(x, z) \geq r(x, y) \star r(y, z)$ where $\star$ is a triangular norm.

The FBRs in H will be interpreted as fuzzy weak preference relations.[6]
A fuzzy aggregation rule (FAR) is a function $\Phi : H^n \to H$. We write $r = \Phi(r_1, ..., r_n)$, $r' = \Phi(r'_1, ..., r'_n)$ and so on (where Φ is the FAR). We write $r(x, y)$ to denote the restriction of r to (x, y), and $r'(x, y)$ to denote the restriction of r' to (x, y) and so on.
Φ is *independent (I)* if and only if, for all $(r_1, ..., r_n), (r'_1, ..., r'_n) \in H^n$ and all $x, y \in X$,

$r_j(x, y) = r'_j(x, y)$ for all $j \in N$ implies $r(x, y) = r'(x, y)$.

Φ is *unanimous (U)* if and only if, for all $(r_1, \ldots, r_n) \in H^n$, all $x, y \in X$ and all $v \in [0, 1]$, $r_j(x, y) = v$ for all $j \in N$ implies $r(x, y) = v$.
Φ is *neutral* if and only if, for all $(r_1, ..., r_n), (r'_1, ..., r'_n) \in H^n$ and all $x, y, z, w \in X$,

$r_j(x, y) = r'_j(z, w)$ for all $j \in N$ implies $r(x, y) = r'(z, w)$.

[6]It is possible to factor out of a fuzzy weak preference relation a fuzzy strict preference relation, and a fuzzy indifference relation. There are several ways of doing this (Dasgupta and Deb, 2001). However, this issue does not arise in this paper. Our theorem requires the fuzzy weak preference relation only. Moreover, we adopt the philosophical position that indifference is not a vague concept. It is perhaps more natural to think of preferences as being vague when neither exact strict preference nor exact indifference exist, and in these cases no degree of preference or degree of indifference is defined. For this reason, we prefer to work with the fuzzy weak preference relation as a primitive.

Φ is *dictatorial* if and only if there exists an individual $i \in N$ such that for all $x, y \in X$, and for every $(r_1, \ldots, r_n) \in H^n$, $r_i(x, y) = r(x, y)$.

I is stronger than the condition commonly used in the literature, but it can be shown to follow from the requirement that a non-constant FAR cannot be manipulated.[7] The same is true for U, which is stronger than the requirement that the FAR is compensative.[8] Our dictatorship condition is strong too, but it is important to characterise when dictatorship in this strong sense arises. This is what we accomplish in this paper.

3 Theorem

Theorem. *If $\star$ has no zero divisor then any FAR satisfying I and U is dictatorial. Moreover, if $\star$ has a zero divisor then a non-dictatorial FAR exists that satisfies I and U.*

We first prove sufficiency. The following lemma holds for any triangular norm.[9]

Lemma 1. *Any FAR satisfying I and U is neutral under any triangular norm.*

Proof. Let Φ be an FAR. Case 1: If $(a, b) = (c, d)$ then the result follows immediately from the fact that Φ is I.

Case 2: $(a, b), (a, c) \in X \times X$. Take $(r_1, ..., r_n) \in H^n$ such that $r_j(b, c) = 1$ for all $j \in N$. U implies that $r(b, c) = 1$. Since r is max-star transitive, we have $r(a, c) \geq r(a, b)$. In addition, since $r_j(b, c) = 1$ for all $j \in N$ and individual preferences are max-star transitive, it follows that $r_j(a, c) \geq r_j(a, b)$ for all $j \in N$. Select a profile $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ such that $\overline{r}_j(b, c) = 1$ and $\overline{r}_j(c, b) = 1$ for all $j \in N$. From the argument above we know that $\overline{r}(a, c) \geq \overline{r}(a, b)$ and $\overline{r}_j(a, c) \geq \overline{r}_j(a, b)$ for all $j \in N$. However, an identical argument shows that $\overline{r}(a, b) \geq \overline{r}(a, c)$ and $\overline{r}_j(a, b) \geq \overline{r}_j(a, c)$ for all $j \in N$. Therefore, it must be the case that $\overline{r}(a, b) = \overline{r}(a, c)$ and $\overline{r}_j(a, b) = \overline{r}_j(a, c)$ for all $j \in N$. Since $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1, ..., r_n) \in H^n$ such that $r_j(b, c) = 1$ and $r_j(c, b) = 1$ for all $j \in N$. Let F^n denote the set of such profiles. Take any profile $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ such that $\widehat{r}_j(a, b) = \widehat{r}_j(a, c)$ for all $j \in N$. Then there exists a profile $(r'_1, ..., r'_n) \in F^n$ such that $\widehat{r}_j(a, b) = \widehat{r}_j(a, c) = r'_j(a, b) = r'_j(a, c)$ for all

[7] Perote-Peña and Piggins (2007, 2008a, 2008b), Duddy, Perote-Peña and Piggins (2008).

[8] García-Lapresta and Llamazares (2001).

[9] This lemma generalises Lemma 1 in Perote-Peña and Piggins (2007).

$j \in N$. I implies that $\widehat{r}(a,b) = \widehat{r}(a,c) = r'(a,b) = r'(a,c)$. Take any pair of distinct profiles $(r_1'', ..., r_n''), (r_1^*, ..., r_n^*) \in H^n$ such that $r_j''(a,b) = r_j^*(a,c)$ for all $j \in N$. Then there exists a profile $(r_1^{**}, ..., r_n^{**}) \in F^n$ such that $r_j''(a,b) = r_j^*(a,c) = r_j^{**}(a,b) = r_j^{**}(a,c)$ for all $j \in N$. I implies that $r''(a,b) = r^*(a,c) = r^{**}(a,b) = r^{**}(a,c)$.

Case 3: $(a,b),(c,b) \in X \times X$. Take $(r_1, ..., r_n) \in H^n$ such that $r_j(a,c) = 1$ for all $j \in N$. U implies that $r(a,c) = 1$. Since r is max-star transitive, we have $r(a,b) \geq r(c,b)$. In addition, since $r_j(a,c) = 1$ for all $j \in N$ and individual preferences are max-star transitive, it follows that $r_j(a,b) \geq r_j(c,b)$ for all $j \in N$. Select a profile $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ such that $\overline{r}_j(a,c) = 1$ and $\overline{r}_j(c,a) = 1$ for all $j \in N$. From the argument above we know that $\overline{r}(a,b) \geq \overline{r}(c,b)$ and $\overline{r}_j(a,b) \geq \overline{r}_j(c,b)$ for all $j \in N$. However, an identical argument shows that $\overline{r}(c,b) \geq \overline{r}(a,b)$ and $\overline{r}_j(c,b) \geq \overline{r}_j(a,b)$ for all $j \in N$. Therefore, it must be the case that $\overline{r}(a,b) = \overline{r}(c,b)$ and $\overline{r}_j(a,b) = \overline{r}_j(c,b)$ for all $j \in N$. Since $(\overline{r}_1, ..., \overline{r}_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1, ..., r_n) \in H^n$ such that $r_j(a,c) = 1$ and $r_j(c,a) = 1$ for all $j \in N$. Let G^n denote the set of such profiles. Take any profile $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ such that $\widehat{r}_j(a,b) = \widehat{r}_j(c,b)$ for all $j \in N$. Then there exists a profile $(r_1', ..., r_n') \in G^n$ such that $\widehat{r}_j(a,b) = \widehat{r}_j(c,b) = r_j'(a,b) = r_j'(c,b)$ for all $j \in N$. I implies that $\widehat{r}(a,b) = \widehat{r}(c,b) = r'(a,b) = r'(c,b)$. Take any pair of distinct profiles $(r_1'', ..., r_n''), (r_1^*, ..., r_n^*) \in H^n$ such that $r_j''(a,b) = r_j^*(c,b)$ for all $j \in N$. Then there exists a profile $(r_1^{**}, ..., r_n^{**}) \in G^n$ such that $r_j''(a,b) = r_j^*(c,b) = r_j^{**}(a,b) = r_j^{**}(c,b)$ for all $j \in N$. I implies that $r''(a,b) = r^*(c,b) = r^{**}(a,b) = r^{**}(c,b)$.

Case 4: $(a,b),(c,d) \in X \times X$ with a,b,c,d distinct. Take $(r_1, ..., r_n) \in H^n$ such that $r_j(b,d) = r_j(d,b) = r_j(a,c) = r_j(c,a) = 1$ for all $j \in N$. U implies that $r(d,b) = 1$. Since r is max-star transitive, we have $r(a,b) \geq r(a,d)$. However, an identical argument shows that $r(a,d) \geq r(a,b)$ and so $r(a,b) = r(a,d)$. In addition, since $r_j(d,b) = r_j(b,d) = 1$ for all $j \in N$ and individual preferences are max-star transitive, it follows that $r_j(a,b) = r_j(a,d)$ for all $j \in N$. We can repeat this argument to show that $r(a,d) = r(c,d)$ and $r_j(a,d) = r_j(c,d)$ for all $j \in N$. Since $(r_1, ..., r_n) \in H^n$ is arbitrary, this condition holds for all profiles $(r_1, ..., r_n) \in H^n$ such that $r_j(b,d) = r_j(d,b) = r_j(a,c) = r_j(c,a) = 1$ for all $j \in N$. Let J^n denote the set of such profiles. Take any profile $(\widehat{r}_1, ..., \widehat{r}_n) \in H^n$ such that $\widehat{r}_j(a,b) = \widehat{r}_j(c,d)$ for all $j \in N$. Then there exists a profile $(r_1', ..., r_n') \in J^n$ such that $\widehat{r}_j(a,b) = \widehat{r}_j(c,d) = r_j'(a,b) = r_j'(c,d)$ for all $j \in N$. I implies that $\widehat{r}(a,b) = \widehat{r}(c,d) = r'(a,b) = r'(c,d)$. Take any pair of distinct profiles $(r_1'', ..., r_n''), (r_1^*, ..., r_n^*) \in H^n$ such that $r_j''(a,b) = r_j^*(c,d)$ for all $j \in N$. Then there exists a profile $(r_1^{**}, ..., r_n^{**}) \in J^n$ such that $r_j''(a,b) = r_j^*(c,d) = r_j^{**}(a,b) = r_j^{**}(c,d)$ for all $j \in N$. I implies that $r''(a,b) = r^*(c,d) = r^{**}(a,b) = r^{**}(c,d)$.

Case 5: $(a,b),(b,a) \in X \times X$. Take any profile $(r_1,...,r_n) \in H^n$ such that $r_j(a,b) = r_j(a,c) = r_j(b,c) = r_j(b,a)$ for all $j \in N$. Cases (2) and (3) imply that $r(a,b) = r(a,c) = r(b,c) = r(b,a)$. Let W^n denote the set of such profiles. Take any profile $(\overline{r}_1,...,\overline{r}_n) \in H^n$ such that $\overline{r}_j(a,b) = \overline{r}_j(b,a)$ for all $j \in N$. Then there exists a profile $(r'_1,...,r'_n) \in W^n$ such that $\overline{r}_j(a,b) = \overline{r}_j(b,a) = r'_j(a,b) = r'_j(b,a)$ for all $j \in N$. I implies that $\overline{r}(a,b) = \overline{r}(b,a) = r'(a,b) = r'(b,a)$. Take any pair of distinct profiles $(r''_1,...,r''_n),(r^*_1,...,r^*_n) \in H^n$ such that $r''_j(a,b) = r^*_j(b,a)$ for all $j \in N$. Then there exists a profile $(r^{**}_1,...,r^{**}_n) \in W^n$ such that $r''_j(a,b) = r^*_j(b,a) = r^{**}_j(a,b) = r^{**}_j(b,a)$ for all $j \in N$. I implies that $r''(a,b) = r^*(b,a) = r^{**}(a,b) = r^{**}(b,a)$. □

Lemma 2. *If $\star$ has no zero divisor then any FAR satisfying I and U is dictatorial.*

Proof. By the previous lemma, Φ is neutral. Let $(r_1,...,r_n) \in H^n$ denote a profile such that $r_i(a,b) = 0$ for all $i \in N$. U implies that $r(a,b) = 0$. Let $(r'_1,...,r'_n) \in H^n$ denote a profile such that $r'_i(a,b) = 1$ for all $i \in N$. U implies that $r'(a,b) = 1$. Consider the following sequence of profiles:

$$\begin{aligned}
\mathbf{R}^{(0)} &= (r_1,...,r_n),\\
\mathbf{R}^{(1)} &= (r'_1,r_2,..,r_n),\\
\mathbf{R}^{(2)} &= (r'_1,r'_2,r_3,..,r_n),\\
&\dots\\
\mathbf{R}^{(n)} &= (r'_1,...,r'_n).
\end{aligned}$$

At some stage in this sequence, the social value of (a,b) rises from 0 to some number greater than 0. Without loss of generality, assume that this happens at $\mathbf{R}^{(2)}$ when individual 2 changes his or her preferences from $r(a,b)$ to $r'(a,b)$. We prove that this individual is a dictator.

First of all, consider a profile $(r'_1,r_2,r'_3,..,r'_n) \in H^n$. We claim that at this profile the social value of (a,b) is zero. To see this consider the profile $(r^*_1,...,r^*_n) \in H^n$. At this profile, every individual's (a,c) preference is the same as their (a,b) preference at $\mathbf{R}^{(1)}$. Everyone's (a,b) preference is the same as their (a,b) preference at $(r'_1,r_2,r'_3,..,r'_n)$. Finally, everyone's (b,c) preference is the same as their (a,b) preference at $\mathbf{R}^{(2)}$.

Max-star transitivity implies $r^*(a,c) \geq r^*(a,b) \star r^*(b,c)$. Since Φ is neutral, this means that $0 \geq T(r^*(a,b),\alpha)$ where $\alpha > 0$. If $\alpha = 1$ then $r^*(a,b) = 0$. If $\alpha < 1$ then because T contains no zero divisor, $r^*(a,b) = 0$. I implies that at $(r'_1,r_2,r'_3,..,r'_n) \in H^n$ the social value of (a,b) is zero, which is what we wanted to demonstrate.

Note, however, that at this profile connectedness implies that $r_2(b, a) = 1$ and also that the social value of (b, a) must be equal to 1. This is true irrespective of everyone else's (b, a) values. Neutrality therefore implies that for all $(r_1, ..., r_n) \in H^n$ and for all $(a, b) \in X \times X$, $r_2(a, b) = 1$ implies $r(a, b) = 1$.

The proof can now be completed as follows. Take a profile $(r_1, \ldots, r_n) \in H^n$ such that $r_2(c, b) = r_2(b, c) = 1$, and $r_i(a, c) = r_2(a, b)$ for all $i \in N$. The other individuals can assign any value they choose to (a, b). We know from the argument above that $r(c, b) = r(b, c) = 1$, and that U implies $r(a, c) = r_2(a, b)$. Since r is max-star transitive, we have $r(a, c) \star r(c, b) \leq r(a, b)$ and $r(a, b) \star r(b, c) \leq r(a, c)$. In other words, $r_2(a, b) \star 1 \leq r(a, b)$ and $r(a, b) \star 1 \leq r_2(a, b)$. Since $\star$ is a triangular norm it must be the case that $r(a, b) = r_2(a, b)$. Again, neutrality implies that for all $(r_1, ..., r_n) \in H^n$ and for all $(a, b) \in X \times X$, $r_2(a, b) = r(a, b)$. □

We now prove necessity. Before we do so, we note the following lemma.

Lemma 3. *If $\star$ is a triangular norm with a zero divisor, then there exists a zero divisor x such that $T(x, x) = 0$.*

Proof. Assume, by way of contradiction, that no such divisor exists. Therefore, there exists $x, y \in (0, 1)$ with $x \neq y$ such that $T(x, y) = 0$. Without loss of generality assume that $x > y$. But then $T(y, y) = 0$ from the requirement that every triangular norm satisfies property (iii). This is a contradiction. □

We now define the following sets.

Let $M(a, b) = \{x \in [0, 1]$ such that at $(r_1, \ldots, r_n) \in H^n$ there exists an $i \in N$ such that $r_i(a, b) = x$ and $r_i(a, b) \geq r_j(a, b)$ for all $j \in N\}$.

Let $m(a, b) = \{x \in [0, 1]$ such that at $(r_1, \ldots, r_n) \in H^n$ there exists an $i \in N$ such that $r_i(a, b) = x$ and $r_j(a, b) \geq r_i(a, b)$ for all $j \in N\}$.

Lemma 4. *If $\star$ has a zero divisor then a non-dictatorial FAR exists that satisfies I and U.*

Proof. Define the function $\Phi : H^n \to H$ as follows. For all $a, b \in X$ and all $(r_1, \ldots, r_n) \in H^n$, let $r(a, b)$ be equal to the median value of the three numbers $M(a, b), x$ and $m(a, b)$ where x is a zero divisor with the property $T(x, x) = 0$. This function satisfies I and U and is non-dictatorial. All we have to prove is that the function takes values in H. The function clearly satisfies reflexivity and connectedness, we just need to prove that it satisfies max-star transitivity.

Assume, by way of contradiction, that the function does not satisfy max-star transitivity. Then there exists $(r_1, \ldots, r_n) \in H^n$ and $a, b, c \in X$ such that $r(a,b) \star r(b,c) > r(a,c)$. First of all, let us rule out the possibility that $r(a,b) \leq x$ and $r(b,c) \leq x$. We know that $T(x,x) = 0$ and so, given that every triangular norm satisfies property (iii), if it is the case that $r(a,b) \leq x$ and $r(b,c) \leq x$ then we would have $T(r(a,b), r(b,c)) = 0$ which contradicts the assumption that $r(a,b) \star r(b,c) > r(a,c)$.

Secondly, we can rule out the possibility that $r(a,b) > x$ and $r(b,c) > x$. Suppose it is the case that $r(a,b) > x$ and $r(b,c) > x$. Recalling the definition of Φ, $r(a,b) > x$ and $r(b,c) > x$ implies that $r(a,b) = m(a,b)$ and $r(b,c) = m(b,c)$. Let $j \in N$ be an individual such that $r_j(a,c) = m(a,c)$. Since individual j's preferences are max-star transitive, we know that $r_j(a,b) \star r_j(b,c) \leq r_j(a,c)$. From the definition of $m(.)$ it must be the case that $r_j(a,b) \geq m(a,b)$ and $r_j(b,c) \geq m(b,c)$, which implies $r_j(a,b) \geq r(a,b)$ and $r_j(b,c) \geq r(b,c)$. Given that T satisfies property (iii), it must be the case that $r(a,b) \star r(b,c)$ is less than or equal to $r_j(a,b) \star r_j(b,c)$. Hence $r(a,b) \star r(b,c) \leq r_j(a,c)$. We have $r_j(a,b) = m(a,c)$ and we know that $m(a,c) \leq r(a,c) \leq M(a,c)$, and so we have $r(a,b) \star r(b,c) \leq r(a,c)$. This contradicts our assumption that $r(a,b) \star r(b,c) > r(a,c)$.

Only two possibilities remain. Either (i) $r(a,b) > x$ and $r(b,c) \leq x$, or (ii) $r(a,b) \leq x$ and $r(b,c) > x$. Assume, without loss of generality, that (i) is true. Given that T satisfies property (iii), we know that $r(a,b) \star x$ is greater than or equal to $r(a,b) \star r(b,c)$. Therefore, given our earlier assumption that $r(a,b) \star r(b,c) > r(a,c)$, it must be the case that $r(a,b) \star x > r(a,c)$. We know by property (iv) that $1 \star x = x$. Given that T satisfies property (iii) and $r(a,b) \leq 1$, we have $r(a,b) \star x \leq x$. Since we have $r(a,b) \star x > r(a,c)$ and $r(a,b) \star x \leq x$, it must be true that $x > r(a,c)$. Returning again to the definition of Φ, note that $x > r(a,c)$ implies $r(a,c) = M(a,c)$. Note too that $r(a,b) > x$ implies $r(a,b) = m(a,b)$. We know then that for all $i \in N$, $r_i(a,b) \geq r(a,b)$ and $r_i(a,c) \leq r(a,c)$. So there must exist an individual $k \in N$ such that $r_k(a,b) \geq r(a,b)$, $r_k(a,c) \leq r(a,c)$ and $r_k(b,c) = M(b,c)$. Since individual k's preferences are max-star transitive, we know that $r_k(a,b) \star r_k(b,c) \leq r_k(a,c)$. We know that $m(b,c) \leq r(b,c) \leq M(b,c)$, and so we have $r(b,c) \leq M(b,c) = r_k(b,c)$. Given that T satisfies property (iii) and $r(a,b) \leq r_k(a,b)$ and $r(b,c) \leq r_k(b,c)$, it must be true that $r(a,b) \star r(b,c)$ is less than or equal to $r_k(a,b) \star r_k(b,c)$. Hence $r(a,b) \star r(b,c) \leq r_k(a,c)$ and so, since $r_k(a,c) \leq r(a,c)$, we have $r(a,b) \star r(b,c) \leq r(a,c)$. However, this contradicts our assumption that $r(a,b) \star r(b,c) > r(a,c)$. $\square$

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