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# Boolean games played in a triangle using bi-partite and tri-partite entanglement

A thesis presented in fulfilment of the requirements for the degree of Doctor of Philosophy
by


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Gràcies a totes i tots per acompanyar-me en este camí

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# Abstract

This dissertation analyses the Nash equilibrium points in a triangle network when the three nodes/players are playing pairwise boolean games using bi-partite and tri-partite entanglement. The players are given one bit as input and must output another bit; the boolean games are defined by choosing two boolean functions of two variables, one function for the input bits and another for the outputs. The players win jointly each of the games if the function of the inputs matches the function of the outputs. The players also share a 6-qubit quantum state, each owning two qubits, which will be used to play the games, i.e. to decide on their outputs given their inputs by measuring locally their two qubits. This 6-qubit state corresponds either to two GHZ-like quantum states (tri-partite entanglement) or three Bell-like quantum states (bi-partite entanglement). The aim is to compare the performance in terms of the (new) Nash equilibrium points of these two types of quantum resources in the described triangle-network situation for any choice of the two boolean functions defining the game. This research, that mixes quantum games, quantum networks, and quantum resources, presents an interesting and rather rich situation, with potential applications in quantum information, for example, in the quantum internet.

# Chapter 0

# Preliminaries

## Notation

The notation used in the present dissertation is the standard notation for quantum mechanics: the Dirac notation, also known as the **Bra-ket notation**.
**Quantum states** are represented by **complex-valued** $n$-dimensional **column vectors** in a **Hilbert space** $\mathcal{H}$, states denoted using a vertical bar and an angle bracket as $|v\rangle \in \mathbb{C}^n$. In the usual mathematical notation, a vector is represented using bold letters. Both formulations are equivalent. An example:

$$|v\rangle = \mathbf{v} = \begin{pmatrix} 1 \\ 3+i \\ \vdots \\ -8 \end{pmatrix}$$

This is referred to as the "**ket**" of such vector. The dual vector (or the "bra") is represented as $\langle v|$. The "**bra**" represents a complex-conjugated row vector. Using the same example as before, the "bra" is:

$$\langle v| = (\mathbf{v}^*)^T = (\mathbf{v}^T)^* = \begin{pmatrix} 1 & 3-i & \dots & -8 \end{pmatrix}$$

The Hilbert space $\mathcal{H}$ by definition has an **inner product**, which is denoted using the "bra" and the "ket" of two vectors: $\langle w\,|v\rangle \in \mathbb{C}$. For example:

$$\langle w|\, v\rangle = (|w\rangle, |v\rangle) = (\mathbf{w}^*, \mathbf{v}) = \begin{pmatrix} e^{i\pi/12} & 6174 & \dots & \sqrt{15}-9i \end{pmatrix} \begin{pmatrix} 1 \\ 3+i \\ \vdots \\ -8 \end{pmatrix}$$

$$= (1)e^{i\pi/12} + 6174(3+i) + \dots + (\sqrt{15}-9i)(-8)$$

It is easy to check that $\langle w|\, v\rangle$ and $\langle v|\, w\rangle$ are related $\langle v|\, w\rangle = \langle w|\, v\rangle^*$.
An **outter product** is also defined, denoted by $|w\rangle\langle v|$, which is a matrix. Keeping the same vectors as with the inner product, the outter product is:

$$|w\rangle\langle v| = \begin{pmatrix} e^{-i\pi/12} \\ 6174 \\ \vdots \\ \sqrt{15}+9i \end{pmatrix} \begin{pmatrix} 1 & 3-i & \dots & -8 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\pi/12} & e^{-i\pi/12}(3-i) & \dots & e^{-i\pi/12}(-8) \\ 6174 & 6174(3-i) & \dots & 6174(-8) \\ \vdots & \vdots & \dots & \vdots \\ \sqrt{15}+9i & (\sqrt{15}+9i)(3-i) & \dots & (\sqrt{15}+9i)(-8) \end{pmatrix}$$

Notice that $|w\rangle\langle v|$ and $|v\rangle\langle w|$ are completely different objects with no relation to each other, as opposed to the inner product. The resulting matrix from $|w\rangle\langle v|$ and $|v\rangle\langle w|$ will have in general different dimensions.

The standard basis, also called the **computational basis** in quantum mechanics, of a $n$-dimensional Hilbert space is an orthonormal basis that represents the standard basis of a vector space $\{|0\rangle, |1\rangle, |2\rangle, ..., |n-1\rangle\}$:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad ... \quad |n-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Any vector in the Hilbert space $|v\rangle \in \mathcal{H}$ can be written as a linear combination of the basis vectors; in this case, of the standard/computational basis:

$$|v\rangle = \sum_{k=0}^{n-1} \lambda_k |k\rangle$$

where $\lambda_k \in \mathbb{C}$.

# Postulates of Quantum Mechanics

This section will introduce very briefly the postulates of quantum mechanics. See, for instance, [1] for more details about the postulates. It is out of scope to discuss the interpretation and philosophical implications of the postulates. For that matter, see [2, 3].

### I. Description of the state

A quantum state is described by an $n$-dimensional complex vector in a Hilbert space $|\Psi\rangle \in \mathcal{H}$. The state can be written as:

$$|\Psi\rangle = \sum_k c_k |\psi_k\rangle \tag{1}$$

where: the index $k$ goes from 1 to the dimension of the Hilbert space; $\{|\psi_k\rangle\}$ is a basis of the Hilbert space; and $c_k$ are the **probability amplitudes**[1]. The probability of finding the state $|\Psi\rangle$ to be in $|\psi_k\rangle$ is then $|c_k|^2$. The **normalisation condition** of probabilities requires $\sum_k |c_k|^2 = 1$, which implies that the state vector $|\Psi\rangle$ has unit norm $\langle\Psi|\Psi\rangle = 1$.

[1] The other alternative representation of the state uses the wave-function $\Psi(x, y, z, t)$. The state vector and the wave-function representations are equivalent. For the present dissertation, only the state vector formulation will be used.

If the state is not pure, which means that it is a probabilistic mixture of pure states, it is said to be a **mixed state** and is represented by a density matrix $\rho$. In a similar fashion to pure states, the density matrix can be written as:

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \tag{2}$$

where $p_k$ is the probability of finding $\rho$ in state $|\psi_k\rangle$. Pure states correspond to $\rho = |\Psi\rangle\langle\Psi|$. The density matrix: 1) is Hermitian[2] $\rho^\dagger = \rho$; 2) has unit trace $Tr(\rho) = 1$; 3) is non-negative $\langle\phi| \rho |\phi\rangle \geqslant 0 \ \ \forall |\phi\rangle \in \mathcal{H}$, equivalently denoted as $\rho \geqslant 0$.

### II. System composition

If the quantum system is composed of $N$ individual sub-systems, then the resulting Hilbert space is the **tensor product** of the **Hilbert spaces** of each sub-system, $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$. Then, any state vector in that higher-dimensional space $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$ can also be written as a linear combination of the basis vectors in each space:

$$|\Psi\rangle = \sum_{k_1,k_2,...,k_N} c_{k_1 k_2 ... k_N} |\psi_{k_1} \phi_{k_2} ... \varphi_{k_N}\rangle \tag{3}$$

where: each summation index goes from 1 to the dimension of the corresponding Hilbert space; $|\psi_{k_1} \phi_{k_2} ... \varphi_{k_N}\rangle \equiv |\psi_{k_1}\rangle \otimes |\phi_{k_2}\rangle \otimes ... \otimes |\varphi_{k_N}\rangle$ are the corresponding basis vectors in each Hilbert space; and the coefficients are complex numbers $c_{k_1 k_2 ... k_N} \in \mathbb{C}$ such that $\sum_{k_1 k_2 ... k_N} |c_{k_1 k_2 ... k_N}|^2 = 1$.

Up to global phases, the **number of parameters** to describe such a state vector grows incredibly fast. If $n_k$ denotes the dimension of the $k$-th Hilbert space, the necessary real parameters to describe such a state is $2 n_1 n_2 n_3 ... n_N - 1$, where the factor 2 comes from the fact that each component of the vector is a complex number and the $-1$ comes from the normalisation condition of the coefficients.

### III. Observables and expectation values

**Observables** are quantities that can be measured in a physical system, e.g. position, momentum, energy, spin, and so on. Each observable is **represented** by a **linear Hermitian** (or self-adjoint) **operator** in the Hilbert space $\mathcal{O} \in \mathcal{H}$. The possible results of the measurement of an observable are the eigenvalues of its operator $\mathcal{O}$. The **expectation value** of the observable, denoted as $\langle\mathcal{O}\rangle$, given a pure (normalised) quantum state $|\Psi\rangle$ is computed as:

$$\langle\mathcal{O}\rangle = \langle\Psi| \mathcal{O} |\Psi\rangle \tag{4}$$

For a mixed state, represented by the density matrix $\rho$, the expectation value is:

$$\langle\mathcal{O}\rangle = Tr(\mathcal{O}\rho) \tag{5}$$

[2] In this dissertation, an Hermitian operator, which is also referred in mathematics as an Hermitian adjoint/conjugate, will be denoted using a dagger †.

### IV. Measurements

The process of measuring a physical system is represented by a set of measurement operators $\{M_m\}$, where $m$ labels the possible value of the measurements. If the state before the measurement is $|\Psi\rangle$, then the **probability of measuring** that state and **finding outcome** $m$ is:

$$Prob(m) = \langle\Psi| M_m^\dagger M_m |\Psi\rangle \tag{6}$$

For mixed states, the probability of outcome $m$ is:

$$Prob(m) = Tr(M_m^\dagger M_m \rho) \tag{7}$$

The measurement operators must satisfy the **completeness relation**: $\sum_m M_m^\dagger M_m = \mathbb{I}$, which implies that the sum of probabilities of all outcomes is 1, i.e. when measuring, an outcome will be obtained.

Immediately after the measurement with outcome $m$, the pre-measurement state $|\Psi\rangle$ goes to:

$$|\Psi\rangle \Longrightarrow \frac{M_m |\Psi\rangle}{\sqrt{\langle\Psi| M_m^\dagger M_m |\Psi\rangle}} \tag{8}$$

This situation is usually referred to as the *collapse of the wave-function* because it is an instantaneous and irreversible process. For mixed states:

$$\rho \Longrightarrow \frac{M_m \rho M_m^\dagger}{Tr(M_m^\dagger M_m \rho)} \tag{9}$$

The equation in (6) and the relation in (8) for pure states are usually referred to as the Born rule [4], proposed by Max Born in 1926.

In the literature, the general measurement process is described using only the formalism of **POVMs** (Positive Operator-Valued Measure) $\{E_m\}$ which are: positive semi-definite ($E_m \geqslant 0$) Hermitian operators ($E_m^\dagger = E_m$) that add up to the identity ($\sum_m E_m = \mathbb{I}$). Of special interest is a subset of the POVMs, called projective measurements[3], abbreviated as **PVM** (Projection-Valued Measure). A set of PVMs $\{P_m\}$ contains operators that are: Hermitian ($P_m^\dagger = P_m$); positive semi-definite ($P_m \geqslant 0$); idempotent ($P_m^2 = P_m$); pair-wise orthogonal ($P_m P_k = P_m \delta_{mk}$); and add up to the identity ($\sum_k P_m = \mathbb{I}$). The PVMs are a subset of POVMs when adding the idempotent property and the orthogonality relations.

The **POVM formalism** is **equivalent** to the **general measurement operators** $\{M_m\}$ described above by just defining $E_m = M_m^\dagger M_m$ to compute the probabilities in equations (6) and (7). In such case, the $M_m$ are said to be the Kraus operators[4] of $E_m$. The advantage of using the general formalism with $\{M_m\}$ is that each $M_m$ need not be Hermitian and the post-measurement states can be specified as in the relations in (8) and (9); in contrast to the POVMs $\{E_m\}$, which need to be Hermitian and the post-measurement state cannot be directly defined because there is not a unique decomposition of $\{E_m\}$, i.e. the Kraus operators of each $E_m$ are not unique.

[3]Some authors also refer to projective measurements as "von Neumann measurements".

[4]The name Kraus operators comes from Kraus' representation [5] that characterises and parametrises completely positive trace-preserving maps.

### V. Time evolution

The state $|\Psi(t)\rangle$ evolves in time according to the Schödinger equation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

where $H$ is the Hamiltonian operator of the system and $\hbar$ is the reduced Planck's constant $\hbar = h/2\pi$.

The time evolution of a mixed state represented by $\rho(t)$ follows the von Neumann equation:

$$i\hbar\frac{d}{dt}\rho(t) = [H, \rho]$$

where [ , ] denotes the commutator of operators $[H, \rho(t)] = H\rho(t) - \rho(t)H$. .

The reversible transformations, such as time evolution, in pure states are represented by unitary transformations $|\Psi\rangle \to \mathcal{U}|\Psi\rangle$ with $\mathcal{U} \in U(n)$.

### VI. Anti-symmetry of the wave function

This is an additional not-so-standard **postulate** that is mentioned mainly when using the **wave-function description**. It is the postulate of the anti-symmetry of the wave-function with respect to interchanging coordinates of fermions (including spin), which implies the Pauli exclusion principle. The Pauli exclusion principle states that two identical fermions cannot be at the same time in the same state or configuration. This postulate is not relevant to the present dissertation so it will not be elaborated any further.

NOTES

For higher-dimensional vector states, sometimes it will be written $|k_1k_2...k_m\rangle$ or $|k_1\rangle\otimes|k_2\rangle\otimes...\otimes|k_m\rangle$ indistinctly. It will depend on whether it is considered necessary to make the specification of each Hilbert space or not.

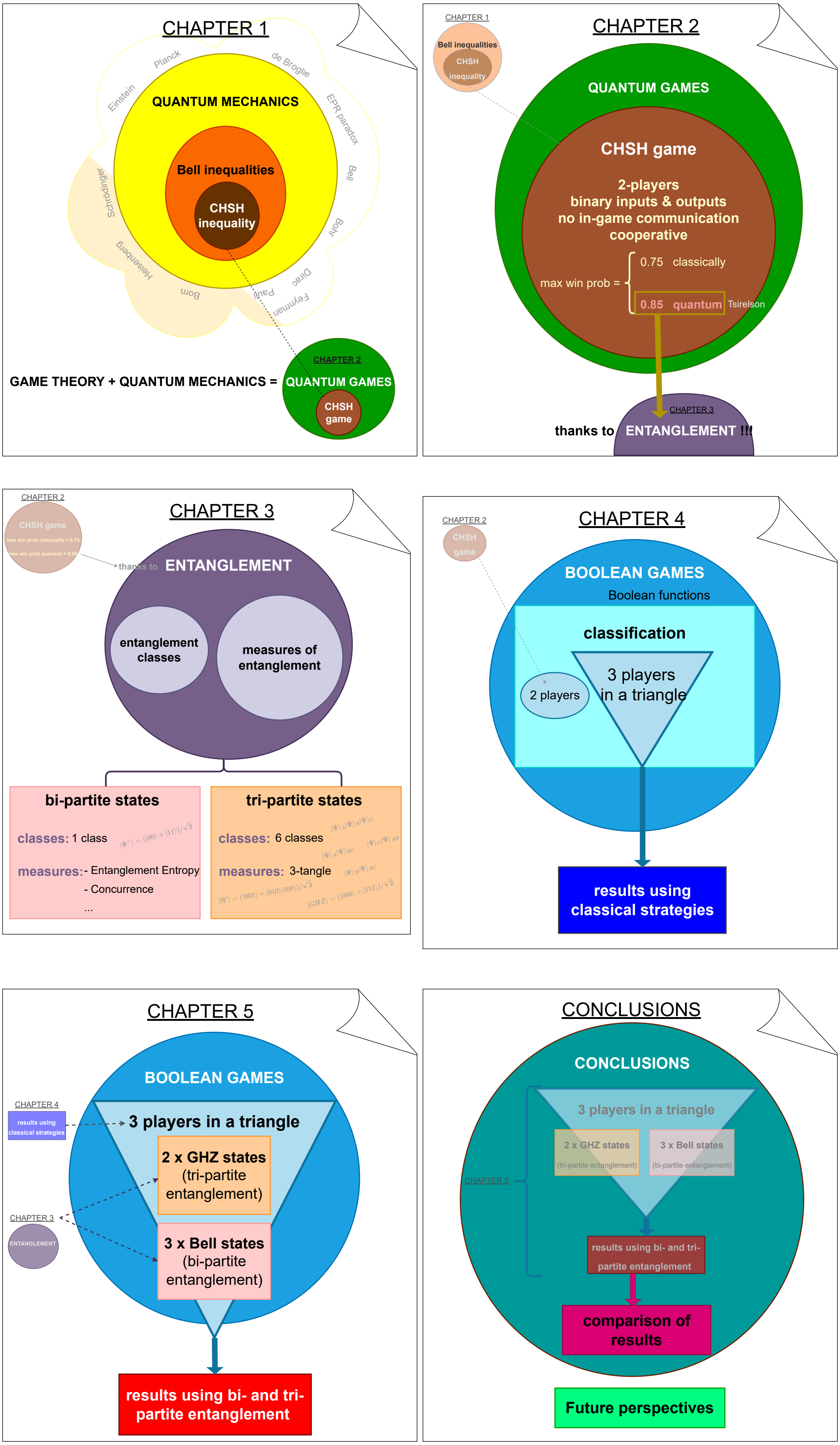
This dissertation only concerns with **bi-dimensional** quantum states, that is, **qubits** (quantum bits). That means that the Hilbert space of each component will always be two-dimensional.

## Structure of the dissertation

- Chapter 1: Quantum theory with its early development and its mix with game theory.
- Chapter 2: Description of the Clauser-Horne-Shimony-Holt (CHSH) game, one of the motivating pillars of this dissertation.
- Chapter 3: Definition of entanglement and its classification for bi-partite and tri-partite states.
- Chapter 4: Definition and classification of boolean games in a triangle network and the results using classical strategies.
- Chapter 5: Results for the boolean games in a triangle network when bi-partite and tri-partite quantum states are used[5].
- Chapter 6: Comparison of the results in chapter 5, the overall conclusions of this dissertation, and future perspectives.
- Bibliography.
- Appendices: Appendices with extra information and calculations.
- Drawing: an art drawing of the research in chapter 5.

The next six diagrams show a more-detailed description of each chapter and their connections to the previous chapters.

[5]Some of the results of this dissertation have been presented as a talk at All-Island Quantum Festival at University College Dublin (Ireland) in September 2022 and at Des journées quantiques à Aix-Marseille Université (France) in September 2023; and as a poster at Theory of Quantum Communication, Computing and Cryptography 2023 (Portugal) in July 2023.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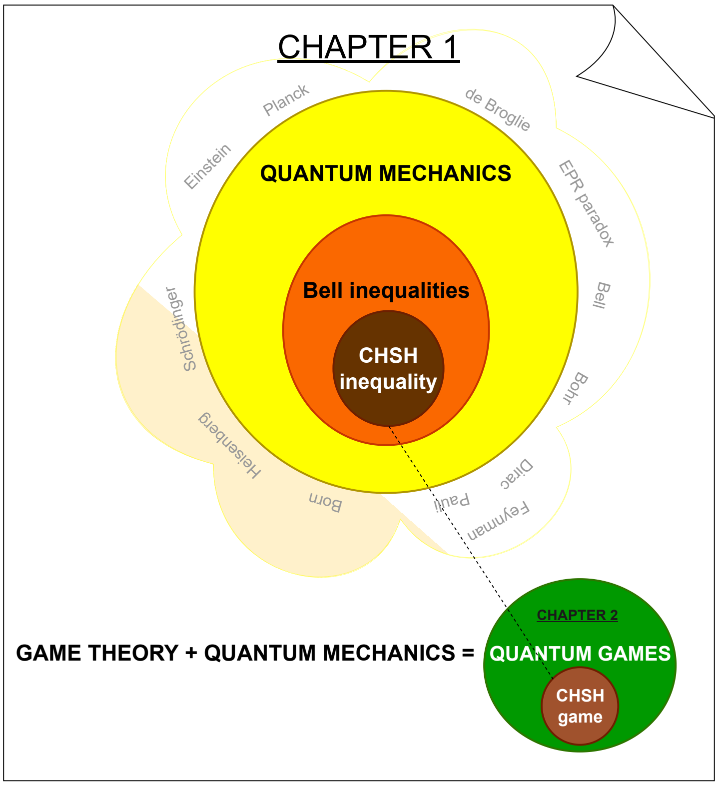
# Chapter 1

# Introduction




This introductory chapter explains briefly the early development of quantum physics and how quantum physics and game theory met.

## 1.1 Quantum Physics

Late in the $19^{\text{th}}$ century, with Maxwells's equations in 1865 successfully describing the electric and magnetic phenomena, it was believed that physics was finished; the laws of physics were well-established and predicted the experimental results so far. Nevertheless, in the same period, new experiments started to show results that could not be fully explained by the known physics at the time.

One of the problems that was troubling the physicists was finding an explanation to the spectrum of black-body radiation[1]. The available theory at the time – the

[1]The black-body radiation experiment consists of an black body – a perfect opaque object,

Rayleigh-Jeans law, in 1900 – worked well for explaining the spectrum at low frequencies, but it also predicted a divergence at higher frequencies, known as the ultra-violet catastrophe, divergence that was not seen experimentally. In 1901, Max Planck proposed the quantification of the energy of radiation, which is nowadays known as photons. With his assumption, Planck's theory was able to explain perfectly the spectrum of the black-body radiation. In his theory, he introduced the famous fundamental constant $h = 6.626 \times 10^{-34}\ J \cdot s$ (Planck's constant), which relates the energy of a photon and its frequency. Planck's quantification was the first step towards the birth of the quantum theory.

Another experiment with no satisfactory explanation at the time was the photo-electric effect[2]. The crucial figure solving the mystery was an unknown physicist at the time named Albert Einstein. In his paper in 1905 [3], he followed Planck's steps, proposing the quantification of light to explain the experimental results. Such was the success that he was awarded a Nobel Prize shortly after, in 1921, for his theory of the photo-electric effect.

In the following years many other experiments and physicists contributed to the full development of modern physics, and particularly, of quantum mechanics. The timeline of the the birth of quantum physics is:

| | | |
|---|---|---|
| 1901 | M. Planck | Black-body radiation |
| 1905 | A. Einstein | Photo-electric effect |
| 1913 | N. Bohr | Atomic model with quantified orbits |
| 1922 | A. Compton | Photon scattering with electrons |
| 1924 | W. Pauli | Exclusion principle |
| 1925 | L. de Broglie | Wave-particle duality |
| 1926 | E. Schrödinger | Wave equation |
| 1927 | W. Heisenberg | Uncertainty principle |
| 1927 | C. Davisson and L. Germer | Experiment on wave properties of electrons |
| 1927 | M. Born | Interpretation of the wave function |

For a more detailed description of the benchmark events during that period, see the chapter on the historical review of quantum mechanics in [6].

Despite the young theory of quantum mechanics predicting the experimental results, not everyone was completely satisfied with its physical interpretation (see in particular the section about the postulates of quantum mechanics in chapter 0); especially the probabilistic interpretation of the measurement results – one of the

that only emits and absorbs radiation – placed into an oven at a fixed temperature. When thermal equilibrium is reached, the emitted radiation of the black body is measured.

[2]The experiment of the photo-electric effect consists of a metal plate that is irradiated with light. The electrons in the plate are then excited by the radiation. The goal was to explain the relation between the irradiating light and the resulting excited electrons. This interaction between light and particles could not be properly explained using only the wave theory of light.

[3]This year Einstein published four ground-breaking papers: one explaining the photo-electric effect, another one about Brownian motion, one introducing special relativity, and one showing the mass-energy equivalence. 1905 is usually referred to as the *annus mirabilis* (miracle year) for Einstein's contribution to the development of modern physics.

pillars of the so-called Copenhagen interpretation[4] – discomforted many physicists back then. One of the distressed physicists with the quantum theory, which he helped to develop, was Einstein himself. He pronounced his famous quote "*Gott würfelt nicht*" (*God does not play dice*). Einstein accepted indeed the success of the quantum theory in explaining the experiments, but he and other physicists thought that the theory was not complete; there was still something missing. There were some hidden variables that if taken into account, the predictions of the measurement would be deterministic and not probabilistic.

In classical physics, the use of probabilities is associated to a lack of knowledge about the system. For instance, when tossing a coin, it is common to say that there is a 50% chance that the result is heads and a 50% chance that it is tails. However, if all the variables associated to the tossing of the coin were known (e.g. the force applied, the initial velocity of the coin at all points, the friction with the air, and so on), then it would be possible to predict with certainty the result of the toss. That was Einstein's argument regarding the probabilistic predictions of quantum mechanics. To him, the quantum theory was only considering probabilities of outcomes because the other (hidden) variables were being ignored, and if taken into account, the result of the measurement could be completely determined. Einstein, Podolsky, and Rosen wrote a famous paper in 1935, known as the EPR paradox[5] [7], questioning the completeness of quantum mechanics with a *Gedankenexperiment* (*thought experiment*). In the EPR thought-experiment, the result of a measurement on one particle would instantaneously affect the other particle, regardless of the distance between them[6]. In their view, information was propagating faster than the speed of light, thus violating causality, but that is not the case; the quantum theory does not violate causality.

The discussion on the matter continued for a few years until a groundbreaking paper in 1964 by a high-energy physicist from Northern Ireland named John Bell [8] gave the possibility of testing whether the predictions of quantum mechanics were intrinsically different than the predictions of a theory with hidden variables. Bell's model considered a generic local hidden-variable model[7] and studied the type of statistics that such a model would give. He proposed a linear combination of the statistics given by that local hidden-variables model and found that such a combination had an upper bound. According to Bell, any local theory of hidden variables in accordance with his axioms could never exceed that upper bound.

[4]The Copenhagen interpretation refers to the interpretation of quantum mechanics given primarily by N. Bohr and W. Heisenberg back in the 1950s. It consists of a set of principles about the meaning of quantities and other aspects of the quantum theory.

[5]To date, it is Einstein's most cited paper; even more than his papers on special and general relativity.

[6]Einstein sent a letter to Born in 1947 mentioning his famous *"spukhafte Fernwirkung"* (*spooky actions at a distance*) in relation to that interaction between two separated particles. Nowadays, Einstein's quote is popular when talking about entanglement in quantum mechanics.

[7]The term "local" for the hidden-variables model was crucial in the discussion. His model rested on the notion of local realism, that means 1) particles' attributes have a definite value despite measuring them or not (realism); and 2) the correlations between distant events need to be explained locally, for instance, the correlations were created at the common source of the experiment (locality). The locality condition might be also defined in the literature as "information cannot travel faster than the speed of light", which is also considered in the above definition of locality, but it might be a bit misleading.

Nevertheless, considering that same combination of probabilities, but computed from the quantum theory, that bound could be breached. With his research, Bell provided a way of testing *experimentally* the predictions of quantum mechanics against a local hidden-variable theory.

In honor to Bell, the combinations of probabilities that distinguish between the prediction of the quantum theory and those of a local hidden-variable theory are called **Bell inequalities**. If the inequality is violated, that process cannot be described using a local theory of hidden variables and is considered to be quantum. The quantum theory was labelled then as a non-local theory since in certain settings, the correlations found between spatially-separated (or non-communicating) parties could not come from a local theory. The set of formulations or results that aim at distinguishing between the quantum and the hidden-variable theories are collected under the name of Bell's theorem.

Shortly after Bell's groundbreaking paper, in 1969, four researchers, John Clauser, Michael Horne, Abner Shimony, and Richard Holt, published a paper proposing an inequality that gave that gap between the predictions of the local hidden-variables theory and the quantum theory; only that this time[8], it was possible to test it experimentally using photons and the technology available at the time. That inequality is known as the **CHSH** (Clauser-Horne-Shimony-Holt) **inequality**, which is a type of Bell inequality. Chapter 2 will elaborate more on the CHSH inequality and its implications. The subsequent years were prolific on the experimental side of testing the predictions of quantum mechanics using Bell inequalities, or Bell tests/experiments. These experiments – even conducted nowadays with improved settings – witnessed the violation of the inequalities and agreed with the predictions of quantum mechanics, thus rejecting the hidden-variable model[9].

Since then, the widely-established quantum theory has led to an incredible amount of applications that have the potential to revolutionise future technology. Applications that range from the popular field of quantum computing (proposed by Richard Feynman in 1981, see [10]) to quantum information and quantum cryptography. All these relatively-new fields use the framework of quantum mechanics to perform tasks in a more efficient way than with the classical framework. The huge impact of these fields will be more noticeable in the next few years. In fact, the Nobel Prize in Physics 2022 already recognised the importance of the people who were crucial in confirming the quantum theory at the onset of it[10]: Alain Aspect, John Clauser, and Anton Zeilinger were the prize winners "*for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science*".

[8]Bell's first inequality relied heavily on the assumption of a perfectly anti-correlated state, and faultless detectors and analysers, which is impossible to achieve experimentally.

[9]The Standford Encyclopedia of Philosophy has excellent and detailed entries about quantum mechanics and its early development. See in [9] the entries about Bell's theorem and the EPR paradox.

[10]John Bell was nominated for a Nobel Prize but he died of a stroke in 1990. Had he lived longer, he would have probably been awarded one.

## 1.2 Game Theory and Quantum Physics

Game theory studies the strategic interactions between parties (or players) that want to achieve some goal, which typically is having the best performance in a given game. Game theory itself is a field relatively new in mathematics. The first formal treatment was given by John von Neumann and Oskar Morgenstern in the book *The Theory of Games and Economic Behaviour* [11] in 1944. The next crucial figure in game theory was John Nash[11], a mathematician who published a set of articles in the 1950s [12, 13] that set the foundations of game theory with his definition and existence of equilibrium: the **Nash equilibrium**. A Nash equilibrium is a configuration of strategies in which no player wants to unilaterally deviate from it. This concept typically applies to non-cooperative games, i.e. games in which the players do not necessarily need to cooperate with the others. Nash's definition of equilibrium is of great importance because, as he showed in [13], all games have at least one Nash equilibrium[12]. In the following years, game theory flourished and has found applications in many fields; for instance, in theoretical economics, in today's networks, in political science, in the military, in biology, and in many other diverse fields. That is why John Nash's contribution was recognised in the Nobel Prize in Economic Sciences in 1994 – shared with game theorists John Harsanyi and Reinhard Selten – *"for their pioneering analysis of equilibria in the theory of non-cooperative games"*.

During the same period as Nash's crucial papers, in 1951, Melvin Dresher and Merrill Flood proposed the well-known game called the **Prisoner's Dilemma**, which was formalised by Albert Tucker [15]. The Prisoner's Dilemma has become one of the most famous examples of the application of game theory. The game involves two thieves getting caught and then brought into two separate interrogation rooms to avoid communication. The police officer tells each prisoner: "if you both confess, you both get 1 year in prison; if you both deny the robbery, you both get 5 years, and if you confess and the other denies it, you get 10 years and the other is set free (gets 0 years[13])". If the players act rationally on their own interest, the solution for this game is when they both deny the robbery, thus both getting 5 years in prison. From an external point of view, it seems that the best strategy for both would be confession, and getting 1 year; however, that solution is not "stable" since any of the players would prefer to deviate unilaterally to get a better deal (1 year against freedom). The solution of denying and getting 5 years is the one in which no player would want to change from it, thus it is the Nash equilibrium of this game. That is why it is coined as a game that poses a dilemma because the solution to this game brings forward the question of individual rationality against

[11]The biography of John Nash, with title *A Beautiful Mind*, was written by the journalist Sylvia Nasar in 1998. A subsequent cinematographic adaptation with the same title was made in 2001 by director Ron Howard, starring Russel Crow as John Nash.

[12]Nash used Kakutani's fixed-point theorem [14] to prove the existence of at least one equilibrium for all games when using mixed strategies, which are just a probabilistic mixture of (pure) strategies.

[13]These possible outcomes of a two-person game can be formulated as a matrix, known as the payoff matrix. Each row and column has the possible strategies for the players and the entries give the correspondent payoff for that combination of strategies. In the Prisoner's Dilemma, the number of sentence years (or payoff) varies across different versions, but the essence of the game is the same.

collective rationality[14].

In 1999, Jens Eisert, Martin Wilkens, and Maciej Lewenstein (EWL, for short) analysed the **Prisoner's Dilemma** in the context of the players having access to **quantum strategies** [16]. Simultaneously, David Meyer also proposed the term quantum strategies in the context of studying quantum algorithms[15] [17].

In general, all games can be analysed using mixed strategies, that is, a probabilistic mixture of (pure) strategies. Then, using the payoff matrix, the players' payoff is calculated as an average, i.e. probability of choosing that strategy times the payoff that such strategy gives. Essentially that is what EWL did with the Prisoner's Dilemma; but instead of computing the probabilities in the usual way, they considered that such probabilities came from a quantum system. More specifically, in their paper, they considered that there was a 2-qubit quantum state and each player would perform a local (unitary) operation on that state to choose between denying or confessing by measuring their qubit (measuring the qubit in state $|0\rangle$ meant confessing and in state $|1\rangle$ denying)[16]. Then, the four possible probabilities – of both confessing, of both denying, and of one confessing and the other denying – were computed. Finally, the average payoff for each player was obtained by multiplying those probabilities by the corresponding payoff in the payoff matrix.

EWL analysed the resulting average payoffs for the prisoners and found that the new solution with quantum strategies was Pareto optimal, i.e. by deviating from that set of strategies it is not possible to improve one player's payoff without decreasing the other player's payoff. In the classical version of the Prisoner's Dilemma, the Pareto optimal solution corresponds to both players confessing (and getting 1 year), but it is not an equilibrium. EWL's result implied that the dilemma disappeared when the players used quantum strategies since the new (quantum) equilibrium was the best solution for the players individually and collectively.

After EWL's paper, there was a heated debate about the validity of their analysis from the game-theoretic point of view. For instance, in [18], the authors argued that such quantum state acted as some sort of advice, which made the comparison with the classical unadvised set-up unfair, and also that the same (quantum) equilibrium could be reproduced by adding an extra strategy to the original classical game, thus

[14] Individual against collective rationality is also illustrated in the game The Tragedy of the Commons, proposed by William Forster Lloyd in 1833. In that game, the players acting on their own self-interest when consuming a common resource leads to the exhausting of that resource, contrary to everyone's benefit.

[15] He considered the Penny Flip game, which exhibits similarities with oracle problems in quantum algorithms. The Penny Flip game consists of two players blindly choosing to flip or not a penny over a sequence of turns. Meyer considered that one of the players had access to quantum strategies (i.e. setting the penny in a superposition of heads and tails) and the other did not. In that case, there was a strategy in which the quantum player would always win.

[16] This idea is a simplification of the original model. In their formulation, the state started in a separable state $|C\rangle \otimes |C\rangle$ (the vector $|C\rangle = (1\ 0)^T$ is identified as confessing, and $|D\rangle = (0\ 1)^T$ as denying) then an entangling gate $\hat{J}$ would act on both qubits. The entangling gate had a parameter $\gamma$ that controlled the amount of entanglement introduced. After that, the players would perform their strategies by picking local unitary operations. Finally a dis-entangling gate $\hat{J}^\dagger$ would act and the final state would be measured.

making the use of quantum strategies redundant. Another issue found is that the disappearance of the dilemma using quantum resources only happened when the set of strategies, i.e. the local unitary operations, were restricted. In contrast, if the full set of local unitaries were considered, there was no Nash equilibrium at all [19]. All of these arguments about the EWL's version of the Prisoner's Dilemma were fair and enriched the conversation about the meaning of a quantum game. That is why other quantisation models of games were also proposed, for instance, see [20]. In the following years, a bunch of other popular games were adapted to the quantum scenario. The study of games using quantum resources was born, leading to the field of quantum game theory.

The new framework of **quantum games** has also extended to other areas beyond game theory itself. For instance, quantum games have applications in quantum computing, complexity theory, quantum cryptography, and quantum foundations. The framework of games allows one to formulate (or re-formulate) existing or new problems in a different way, giving new tools for the analysis and its possible applications. One of the most well-known games in the area of quantum foundations is the CHSH game, named after the Clauser-Horne-Shimony-Holt (CHSH) inequality (see the end of section 1.1). This game, which will be fully explained in the next chapter, will be one of the basis of the present dissertation. For an extensive review of quantum games and their applications, see [21].

In fact, not only are quantum games a theoretical tool to analyse research questions, but also the quantumness in games has actually reached the general public. There is a long list of computer games and games for smartphones that use the attractiveness of playing a game to illustrate some concepts in quantum physics. One example is a quantum version of the popular Tic-Tac-Toe (or noughts and crosses or Xs and Os) called Quantum Tic-Tac-Toe, which was proposed in 2006 by Allan Goff [22] to introduce the concept of superposition[17]. There are some games that are actually helping research, for instance, the game Decodoku[18] proposes puzzles related to quantum error correction in the field of quantum computing. On another application, the quantum can also be used directly in conventional games; for instance with "quantum blurr" [23] for terrain generation. All these examples show the broadness of the topic of quantum games, either for research or just for fun.

## Summary of the chapter

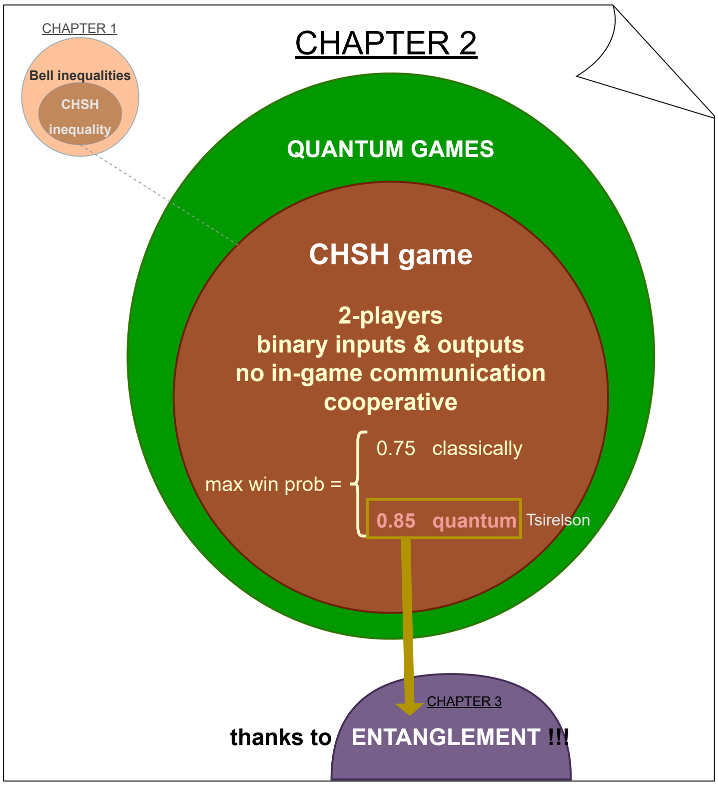
This chapter has briefly introduced the need and development of the quantum theory and all the obstacles and objections to it back in the 20$^{\text{th}}$ century. John Bell was crucial to settling the argument with his Bell inequalities. One of the first Bell-type inequalities to be tested experimentally was the CHSH inequality, confirming the validity of quantum mechanics. The CHSH inequality can be illustrated as a game – in the next chapter, chapter 2. The mix between the language of games and the quantum started back in the early 2000s, establishing the study of quantum games to be a sub-field in itself with many corners and applications still to be explored.

[17]The author of the present dissertation programmed a version of the Quantum Tic-Tac-Toe based on [22] that can be used in a workshop scenario for children and/or teenagers. It is available in different languages at `https://vickynititi.itch.io/quantum-tic-tac-toe` (accessed Jan 2024).

[18]Available at `https://citizensciencegames.com/games/decodoku/` (accessed Jan 2024).

# Chapter 2

# The CHSH game




This chapter explains the Clauser-Horne-Shimony-Holt (CHSH) game. First, the CHSH inequality [24], which is type of Bell inequality (see chapter 1), is explained. Then, the CHSH game, based on the inequality, is introduced and analysed in detail. This is the most popular quantum game, which served as inspiration for the research conducted in this dissertation.

## 2.1 The CHSH Inequality

During the period of John Bell and his (Bell) inequalities to test the completeness of quantum mechanics, John Clauser, Michael Horne, Abner Shimony, and Richard Holt proposed in 1969 their own test: the Clauser-Horne-Shimony-Holt (CHSH) inequality. It was one of the first inequalities that was actually possible to implement experimentally[1]. John Clauser himself and Stuart Freedman were able to conduct the first experiment [25] of the CHSH inequality in 1972, finding an experimental value that exceeded the bound of the inequality and confirming the prediction of quantum mechanics, thus discarding the local hidden-variables theory. However, in that first experiment there were loopholes[2] that (it could be argued) that could still save the hidden-variables theories. Later on, in 1982, Alain Aspect *et al.* [26, 27] performed several experiments[3] that addressed some of the loopholes. The technological improvements led finally in 2015 to loophole-free experiments of Bell inequalities [28–30].

Returning to the original CHSH inequality, the setup is as follows:

- A pair of (entangled) photons is distributed between two different and separated detectors.
- The detectors measure the polarisation of the photon, giving two possible outcomes: either horizontal polarisation, which is assigned a $+1$ value, or vertical polarisation, which is assigned a value of $-1$.
- The first detector is aligned with a certain orientation and measures the polarisation of one of the photons; the result of such measurement is labelled as **a**. Then, another orientation of that same detector is chosen to measure the polarisation of the photon again, labelled as **a'**. Similarly for the second detector with two orientations for measuring the polarisation **b** and **b'** of the other photon. Regardless of the orientation of the detectors, the possible results of the measurements are either $+1$ (horizontal polarisation) or $-1$ (vertical polarisation), i.e. $\mathbf{a}, \mathbf{a'}, \mathbf{b}, \mathbf{b'} \in \{+1, -1\}$.
- The correlations between the measurements of both detectors are studied.

The CHSH inequality is this particular combination of correlations between the measurements of both detectors:

$$\text{CHSH ineq} = \langle \mathbf{ab} \rangle + \langle \mathbf{ab'} \rangle + \langle \mathbf{a'b} \rangle - \langle \mathbf{a'b'} \rangle \tag{2.1}$$

where $\langle \mathbf{ab} \rangle$ denotes the expectation value of observable **ab**, since the measurement event in the detector is of probabilistic nature. The interesting feature of the CHSH inequality in equation (2.1) is that the maximal value predicted by the local-hidden variables model differs from the maximum coming from quantum mechanics. The maximum value coming from the hidden-variable model (or classical scenario) is 2 and the quantum is $2\sqrt{2} \approx 2.828$, as will be shown next.

[1]As mentioned in the previous chapter, Bell's first inequality required extreme experimental control and precision, making them unsuitable for real experiments.

[2]There are many types of loopholes when performing a Bell experiment. In this case, it was the locality (also known as communication) loophole, which asserts that the events are not space-like separated, allowing the possibility of communication or signalling between the two events.

[3]Alain Aspect was one of the awardees of the Nobel Prize in Physics 2022 for his experiments.

According to the classical view and interpretation of the experiment, two individual particles are sent to two separated detectors that measure either +1 or -1. The measurements of the particles at each detector are spatially-separated events, which means that no information can be transmitted from one to the other[4]. This setting implies that the measurements of each detector must be fully independent or that they were correlated via some underlying local hidden variables that dictated the outcomes for both particles. The classical bounds for the CHSH inequality are more easily illustrated by just considering the independence of the measurement outcomes, which leads to the following relation:

$$\mathbf{ab} + \mathbf{a'b} + \mathbf{ab'} - \mathbf{a'b'} = \mathbf{a}(\mathbf{b} + \mathbf{b'}) + \mathbf{a'}(\mathbf{b} - \mathbf{b'}) = \pm 2 \tag{2.2}$$

where it was used that all the variables take values either +1 or $-1$; which implies that any of the two terms with **b**s is 2 and the other term is 0. Then, the absolute value of the combination in equation (2.2) is 2. The classical bound of the CHSH inequality in equation (2.1) follows from the result in equation (2.2) and the triangle inequality:

$$\text{CHSH ineq} = |\langle \mathbf{ab} \rangle + \langle \mathbf{ab'} \rangle + \langle \mathbf{a'b} \rangle - \langle \mathbf{a'b'} \rangle| \leqslant \langle |\mathbf{ab} + \mathbf{a'b} + \mathbf{ab'} - \mathbf{a'b'}| \rangle = 2 \tag{2.3}$$

As mentioned above, this is a simple proof of the classical bound of the CHSH inequality using the independence of measurement outcomes and some basic mathematics. That same bound can also be obtained using the Bell-like local hidden-variable approach by writing the correlation functions as integrals over the hidden-variable space. See the original paper [24] of the CHSH inequality for that more-formal derivation.

The proof of the quantum bound of the CHSH inequality, which is $2\sqrt{2}$, will be made in the next section 2.2 when introducing the CHSH game.

## 2.2 The CHSH game

As it was already mentioned, the CHSH game is based on the CHSH inequality [24] from equation (2.1). The usual description of the game goes as follows:

- There are two players, typically called Alice and Bob.
- When the game starts, the players are separated and no in-game communication is allowed.
- Then, each player receives a binary input and must output a binary bit. For Alice, her input is $x \in \{0, 1\}$ and her output is $a \in \{0, 1\}$. Bob's input is $y \in \{0, 1\}$ and his output is $b \in \{0, 1\}$.
- The players only have information about their input and output and know nothing about the other player's input or outputs[5]. This situation is usually presented with a referee giving them the outputs and receiving the output bits.

[4] Assuming Einstein's special relativity principle, i.e. information cannot travel faster than the speed of light.

[5] This situation is known in the game-theoretic literature as "games with incomplete/imperfect information", or equivalently "Bayesian games" because Bayes' theorem is used to update the players' beliefs in the course of the game.

- The players (jointly) win if the next condition is satisfied:

$$x \wedge y = a \oplus b \tag{2.4}$$

where $\wedge$ represents the boolean AND operator and $\oplus$ the boolean XOR operator[6]. Following the standard game-theoretic tools, this winning condition can also be captured using a payoff matrix, which is divided into two situations depending on the value of $x \wedge y$:

| | | $x \wedge y = 0$ | | $x \wedge y = 1$ | |
|---|---|---|---|---|---|
| | | Bob | | Bob | |
| | | 0 | 1 | 0 | 1 |
| Alice | 0 | (1, 1) | (0, 0) | (0, 0) | (1, 1) |
| | 1 | (0, 0) | (1, 1) | (1, 1) | (0, 0) |

(2.5)

where Alice's strategies $a = 0$ and $a = 1$ are represented as rows and Bob's strategies $b = 0$ and $b = 1$ as columns. For a given combination of strategies (for certain inputs) in the payoff matrix in (2.5), the first number in each parenthesis refers to Alice's payoff while the second refers to Bob's payoff. In this case, when they win, they both receive a payoff of 1; and when they lose, they get nothing (zero). For this dissertation the payoff matrix formulation is not particularly useful, it is only mentioned here for completeness and for illustrating the connection of the winning condition in equation (2.4) with the payoff matrix formulation widely used in game theory.

- Figure 2.1 illustrates the set-up of the game.

### 2.2.1 The classical bounds

In the classical setting of the game, Alice and Bob may use any classical resources, but when the game starts and they are given their inputs, they cannot use any resource to communicate. This imposition means that either their outputs are completely uncorrelated/independent or might be correlated by some before-game communication and/or some in-game event that does not allow for communication[7]. This situation is analogous to the classical view of the experiment for the CHSH inequality in section 2.1, explained just before equation (2.2). Like before with the classical bounds for the CHSH *inequality*, the bounds of the CHSH *game* are also more easily illustrated by directly analysing the winning condition in equation (2.4) and assuming that both outputs are independent/uncorrelated. That winning condition implies that for inputs $x = y = 1$ the players win if $a = 0$ and $b = 1$, or if $a = 1$ and $b = 0$. In the other three possible combinations of inputs, the players win if $a = b$. Since it is a cooperative game, that means, both players win or lose jointly, the important quantity is the winning probability. Assuming a uniform distribution

[6] In this case, the winning condition was written using boolean operators, but that same winning condition can be written as: $x \cdot y = a + b \pmod 2$, where $\cdot$ is the usual multiplication and the outputs are added modulo 2.

[7] For instance, Alice and Bob could meet before the game and agree on always outputting 0 regardless of their inputs (before-game communication), or they could agree on outputting 0 only if it is raining when they perform the measurements (a common event that cannot be used for communication). The important part here is that no communication of any kind must occur when the game starts.

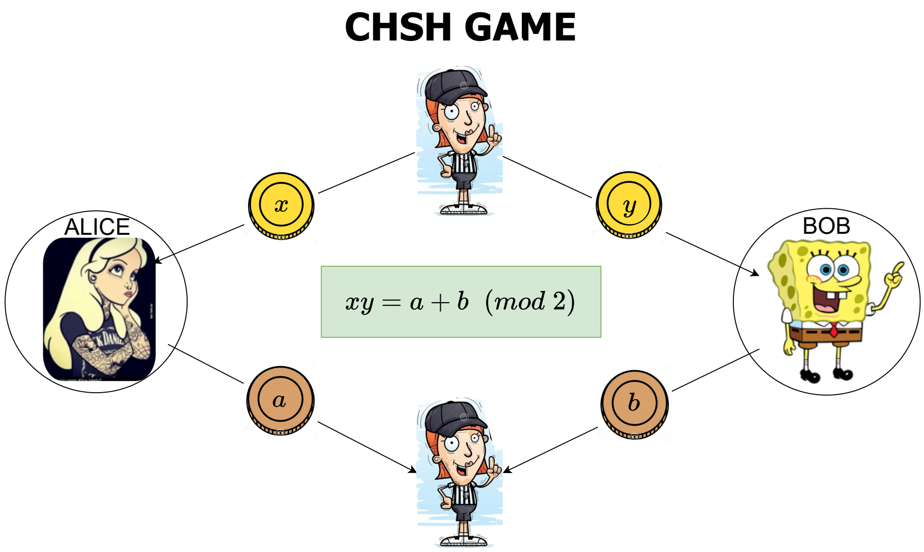



Fig. 2.1: *CHSH game with the referee and two players Alice and Bob. The players are separated and cannot communicate once they receive their binary input* $x, y \in \{0, 1\}$*, represented by a golden coin. They must output a bit* $a, b \in \{0, 1\}$*, represented by a bronze coin. The referee then compares the inputs and outputs, and the players win if the winning condition* $x \cdot y = a + b \pmod 2$ *is satisfied, which is equivalent to condition in equation (2.4) using boolean operators.*

of inputs, it is not hard to show that the maximum and minimum value of the winning probability classically is $3/4 = 0.75$ and $1/4 = 0.25$. To show that, let $a_0$ and $a_1$ denote Alice's output when she receives input $x = 0$ and $x = 1$, respectively. Similarly for Bob with $b_0$ and $b_1$ when $y = 0$ and $y = 1$. Then, the four equations coming from the winning condition in equation (2.4) are:

$$x = 0;\ y = 0 \ \rightarrow\ 0 = a_0 + b_0 \tag{2.6}$$
$$x = 0;\ y = 1 \ \rightarrow\ 0 = a_0 + b_1 \tag{2.7}$$
$$x = 1;\ y = 0 \ \rightarrow\ 0 = a_1 + b_0 \tag{2.8}$$
$$x = 1;\ y = 1 \ \rightarrow\ 1 = a_1 + b_1 \tag{2.9}$$

where it was assumed that the players' outputs are independent of each other. Adding all the four equations gives a new equation:

$$2\,(a_0 + a_1 + b_0 + b_1) = 1 \tag{2.10}$$

which cannot be satisfied since $a_0, a_1, b_0, b_1 \in \{0, 1\}$. That means that it is not possible to fulfill all four equations at the same time. However, it is possible to fulfill three of them. For instance, with both players outputting 0 regardless of their input (i.e. $a_0 = a_1 = b_0 = b_1 = 0$) then the first three equations are satisfied. If all the combinations of inputs are equally probably, then 3 out of 4 times the players will win, giving the maximum winning probability of $3/4 = 0.75$.

A similar argument follows for the minimum winning probability. It is not possible to fulfill all four losing equations, i.e. the resulting equations from flipping the left-hand side (exchanging 0 for 1 and vice-versa) in the winning equations in (2.6)-(2.9); only three out of four times the players can lose the game.

Assuming equally probable inputs, that leads to the minimum winning probability of $1/4 = 0.25$.

Using very simple arguments, it has just been shown that the winning probabilities of the CHSH game when using classical strategies lie in the range between $1/4 = 0.25$ and $3/4 = 0.75$.

### 2.2.2 The quantum scenario

In the quantum version of the CHSH game, the players share initially a 2-qubit quantum state, with one qubit for each player, and depending on their inputs they will perform a (local) measurement of their qubit and use the result of that measurement as their output.

The probabilistic nature of quantum mechanics requires the use of probabilities of events happening. In the present case, $Prob(a, b|x, y)$ denotes the (conditional) probability of outputs $a$ and $b$ given inputs $x$ and $y$. From the winning condition in equation (2.4) and assuming equally probable inputs, the winning probability for the CHSH game is:

$$\begin{aligned} Prob(\text{win}) = \frac{1}{4}\,[&Prob(0,0|0,0) + Prob(1,1|0,0) + Prob(0,0|0,1) + Prob(1,1|0,1) \\ &+Prob(0,0|1,0) + Prob(1,1|1,0) + Prob(0,1|1,1) + Prob(1,0|1,1)] \end{aligned} \tag{2.11}$$

Equation (2.11) is the standard form of writing the winning probability given the winning conditions of the CHSH game. Nevertheless, there is another way of writing it that makes the analysis of the quantum bounds much easier. The next subsection will derive the quantum bound in a formal way using the alternative form of the winning probability. For now, it is sufficient to consider the one in equation (2.11).

In the quantum version, the players share a 2-qubit state. The shared state will be a maximally entangled state[8]:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{2.12}$$

where the first qubit belongs to Alice and the second to Bob. This particular state is usually referred to as a Bell state or an EPR pair – from the Einstein-Podolsky-Rosen paradox, introduced in chapter 1. It is one of the simplest 2-qubit states that is maximally entangled. Along with three other states, they define the so-called Bell basis $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$, where $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. All of the states in the Bell basis are maximally entangled.

After receiving their input, both players will perform a projective measurement[9] on their qubit in a given basis. Figure 2.2 illustrates the quantum situation for the

[8] The notion of a "maximally entangled" state for two qubits will be explained in chapter 3.

[9] A projective measurement is a special case of a more general measurement operator, in which the measurement process is represented by a projector. See the measurement postulate in the postulates of quantum mechanics in chapter 0.

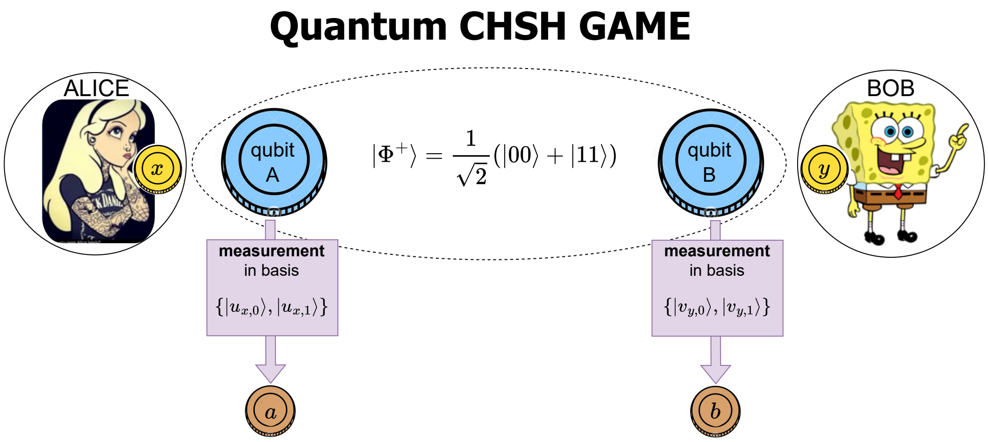



Fig. 2.2: *Situation of the CHSH game with players each having one qubit. The system of two qubits is described by* $|\Phi^+\rangle$. *After receiving their inputs* $x, y$, *the players perform a measurement on their qubit in a chosen basis. Their output bit will be the result of that measurement* $a, b$.

CHSH game. Alice's choice of basis will be a rotation of a certain angle along the X-axis:

$$|u_{x,0}\rangle = \cos(\alpha_x)\,|0\rangle - \sin(\alpha_x)\,|1\rangle \tag{2.13}$$
$$|u_{x,1}\rangle = \sin(\alpha_x)\,|0\rangle + \cos(\alpha_x)\,|1\rangle \tag{2.14}$$

while for Bob:

$$|v_{y,0}\rangle = \cos(\beta_y)\,|0\rangle - \sin(\beta_y)\,|1\rangle \tag{2.15}$$
$$|v_{y,1}\rangle = \sin(\beta_y)\,|0\rangle + \cos(\beta_y)\,|1\rangle \tag{2.16}$$

The set $\{|u_{x,0}\rangle, |u_{x,1}\rangle\}$ is Alice's new basis[10] that depends on the input $x$. Given $x$, if Alice measures her qubit to be in state $|u_{x,0}\rangle$ her output will be $a = 0$, whereas if her qubit is in state $|u_{x,1}\rangle$ she will output $a = 1$. Similar argument for Bob with the basis $\{|v_{y,0}\rangle, |v_{y,1}\rangle\}$ depending on the input $y$. In the quantum scenario, the players' strategies are in their choice of the basis vectors in which they will measure their qubit, which, in this case, in the choice of the angles $\alpha_x$ and $\beta_y$. In principle, the strategic space now is infinite[11] because they can choose any two vectors that form a basis.

Since the players will be measuring in another basis, it is convenient to re-express the initial quantum state in equation (2.12) in the new basis:

[10] The choice of the basis might seem very restrictive, and indeed it is, but it is a simple basis that depends on only one parameter and is good enough to achieve the maximum and minimum bounds in the quantum scenario. Other more-general choices of basis would be equally valid, but they might depend on more parameters and/or not reach the quantum bounds.

[11] In the classical version, only pure (or deterministic) strategies were considered, e.g. choose either 0 or 1. Nonetheless, a probabilistic mixture of strategies, known as mixed strategies, could also be considered, e.g. choose 0 with probability $p$ and choose 1 with probability $1 - p$. In that case, the strategic space in the classical scenario is also infinite. However, since the mixed strategies are a convex combination of pure strategies, in this case, it is sufficient to consider pure strategies to find the range of the winning probability. This range for the CHSH game is 0.25 to 0.75.

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\cos(\alpha_x-\beta_y)\,|u_{x,0}v_{y,0}\rangle - \sin(\alpha_x-\beta_y)\,|u_{x,0}v_{y,1}\rangle \right.\\
&\qquad \left. + \sin(\alpha_x-\beta_y)\,|u_{x,1}v_{y,0}\rangle + \cos(\alpha_x-\beta_y)\,|u_{x,1}v_{y,1}\rangle\right] \qquad (2.17)
\end{aligned}
$$

where two trigonometric identities were used.

The postulates of quantum mechanics say that the probability of measuring a (pure) state represented by $|\Psi\rangle$ to be in state $|\psi\rangle$ is $Prob(|\psi\rangle) = |\langle\psi\,|\Psi\rangle|^2$. In the present case with the game, the probability of the players outputting $a, b$ conditioned to the inputs $x, y$ is equal to the probability of the players measuring the shared state $|\Phi^+\rangle$ to be in state $|u_{x,a}v_{y,b}\rangle$:

$$
\begin{aligned}
Prob(a,b|x,y) &= \left|\langle u_{x,a}v_{y,b}\,|\Phi^+\rangle\right|^2 \\
&= \langle\Phi^+\,|u_{x,a}v_{y,b}\rangle\langle u_{x,a}v_{y,b}|\,\Phi^+\rangle \\
&= \langle\Phi^+|\left(|u_{x,a}\rangle\langle u_{x,a}|\otimes|v_{y,b}\rangle\langle v_{y,b}|\right)|\Phi^+\rangle \\
&= \langle\Phi^+|\,\Pi_{x,a}\otimes\Pi_{y,b}\,|\Phi^+\rangle \qquad (2.18)
\end{aligned}
$$

where, in the last line, it was defined the projective measurement operators for Alice $\Pi_{x,a} = |u_{x,a}\rangle\langle u_{x,a}|$ and $\Pi_{y,b} = |v_{y,b}\rangle\langle v_{y,b}|$ for Bob. The tensor product $\otimes$ was written explicitly to show that each projector acts on their own Hilber space. Note that the definition of the conditional probability in equation (2.18) is independent of the chosen basis; the only elements needed are the projective measurement operators and the chosen state (in a given dimension).

For the particular choice of the maximally entangled state in equation (2.12) and the new basis defined in equations (2.13)-(2.16), the probabilities are easily computed from equation (2.17) as just the coefficients squared. These four conditional probabilities are:

$$Prob(0,0|x,y) = Prob(1,1|x,y) = \frac{1}{2}\cos^2(\alpha_x-\beta_y) \qquad (2.19)$$

$$Prob(0,1|x,y) = Prob(1,0|x,y) = \frac{1}{2}\sin^2(\alpha_x-\beta_y) \qquad (2.20)$$

Finally, substituting those values of the probabilities into the winning probability of the CHSH game in equation (2.11) gives:

$$Prob(\text{win}) = \frac{1}{4}\left[\cos^2(\alpha_0-\beta_0)+\cos^2(\alpha_0-\beta_1)+\cos^2(\alpha_1-\beta_0)+\sin^2(\alpha_1-\beta_1)\right] \qquad (2.21)$$

The computation of the extreme values of equation (2.21) is shown in appendix A.1. The canonical values of the angles that maximise the winning probability are: $\alpha_0 = 0$; $\alpha_1 = \pi/4$; $\beta_0 = \pi/8$; $\beta_1 = -\pi/8$. Then, the maximum winning probability is:

$$\max\, Prob(\text{win}) = \frac{1}{4}\left[3\cos^2\left(\frac{\pi}{8}\right) + \sin^2\left(\frac{3\pi}{8}\right)\right]$$

$$= \cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4} \approx 0.853 \tag{2.22}$$

where it was used that $\sin^2(3\pi/8) = \cos^2(\pi/8)$.

As for some minimising angles, one choice can be $\alpha_0 = 0$; $\alpha_1 = \pi/4$; $\beta_0 = -3\pi/8$; $\beta_1 = 3\pi/8$, then the minimum probability is:

$$\min Prob(\text{win}) = \frac{1}{4}\left[2\cos^2\left(\frac{3\pi}{8}\right) + \cos^2\left(\frac{5\pi}{8}\right) + \sin^2\left(\frac{-\pi}{8}\right)\right]$$
$$= \sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4} \approx 0.146 \tag{2.23}$$

where it was used that $\cos^2(3\pi/8) = \cos^2(5\pi/8) = \sin^2(\pi/8)$.

As it was just shown, using a maximally entangled state and some projective measurements, the range of the winning probability for the CHSH game using quantum resources is larger than the classical range by around 20% !:

$$0.146 \approx \frac{2-\sqrt{2}}{4} \leqslant \text{Quantum } Prob(\text{win}) \leqslant \frac{2+\sqrt{2}}{4} \approx 0.853 \tag{2.24}$$
$$0.25 = \frac{1}{4} \leqslant \text{Classical } Prob(\text{win}) \leqslant \frac{3}{4} = 0.75 \tag{2.25}$$

This separation between the classical and the quantum value is referred to in the literature as non-locality, since, as Bell showed with his inequalities, no local hidden-variables theory could reproduce those correlations. Nonetheless, quantum mechanics, which is non-local, predicts indeed those correlations.

## The quantum bound

In this part, it will be shown formally that the maximum and minimum value of the winning probability for the CHSH game using the maximally entangled state $|\Phi^+\rangle$ in equation (2.12) and the chosen basis for the projective measurements in equations (2.13)-(2.16) achieve the optimum values and it is not possible to do any better (or worse) by using other states and/or measurements.

To show the optimality of the obtained bounds, the first step is re-writing the winning probability of the CHSH game in a way that resembles the original CHSH inequality. The second step is using Tsirelson's proof [31–33] from the 1980s for the bounds of the CHSH inequality.

For the step of re-writing the winning probability in equation (2.11), consider the trivial identity of all probabilities summing to 1 regardless of the inputs:

$$Prob(0,0|x,y) + Prob(0,1|x,y) + Prob(1,0|x,y) + Prob(1,1|x,y) = 1 \tag{2.26}$$

This completeness relation[12] and the winning probability of the CHSH game can be written in a shorter form by defining the coordination and anti-coordination probabilities. The coordination probability is defined as the probability of the players

[12] Completeness relation is used more in the context of operators that sum to the identity. In this case, it just means that the event actually happens, therefore, the probabilities of the different possible outcomes sum to 1.

coordinating their outputs, i.e. $Prob(coord|x,y) = Prob(0,0|x,y) + Prob(1,1|x,y)$. In a similar way, the anti-coordination probability is defined for when the players anti-coordinate their outputs $Prob(anti|x,y) = Prob(0,1|x,y) + Prob(1,0|x,y)$. Then, the completeness relation in equation (2.26) is just $Prob(coord|x,y) + Prob(anti|x,y) = 1$. The winning probability of the CHSH game in equation (2.11) becomes:

$$\begin{aligned} Prob(\text{win}) =& \frac{1}{4}\left[Prob(coord|0,0) + Prob(coord|0,1)\right. \\ & \left. + Prob(coord|1,0) + Prob(anti|1,1)\right] \end{aligned} \tag{2.27}$$

Moreover, the coordination and anti-coordination probabilities can be written in a more symmetric way using the completeness relation:

$$Prob(coord|x,y) = \frac{1}{2}\left[Prob(coord|x,y) + 1 - Prob(anti|x,y)\right] \tag{2.28}$$

$$Prob(anti|x,y) = \frac{1}{2}\left[Prob(anti|x,y) + 1 - Prob(coord|x,y)\right] \tag{2.29}$$

Then, substituting equations (2.28) and (2.29) into equation (2.27), the winning probability only depends on the difference between the coordination and anti-coordination probabilities:

$$\begin{aligned} Prob(\text{win}) = \frac{1}{2} + \frac{1}{8}&\left[Prob(coord|0,0) - Prob(anti|0,0)\right. \\ & + Prob(coord|0,1) - Prob(anti|0,1) \\ & + Prob(coord|1,0) - Prob(anti|1,0) \\ & \left. - (Prob(coord|1,1) - Prob(anti|1,1))\right] \end{aligned} \tag{2.30}$$

Remember that the conditional probability can be written as: $Prob(a,b|x,y) = \langle\Psi| \Pi_{x,a} \otimes \Pi_{y,b} |\Psi\rangle$ (see equation (2.18) ), where $|\Psi\rangle$ is a given quantum state and $\Pi_{x,a} = |u_{x,a}\rangle\langle u_{x,a}|$ and $\Pi_{y,b} = |v_{y,b}\rangle\langle v_{y,b}|$ are the measurements projector for each player in the new basis $\{|u_{x,a}\rangle \otimes |v_{y,b}\rangle\}_{a,b\in\{0,1\}}$ given the inputs $x$ and $y$. Even though the same notation was used in the previous section for the new measurement basis, $\{|u_{x,a}\rangle\}_{a\in\{0,1\}}$ and $\{|v_{y,b}\rangle\}_{b\in\{0,1\}}$ are assumed here to be a generic basis, i.e. not restricted to the choices in equations (2.13)-(2.16).

Using that the (conditional) probabilities come from the quantum measurement process, which is represented by the projector operators applied to the generic state $|\Psi\rangle$, then the difference between the coordination and anti-coordination probabilities can be written as:

$$\begin{aligned} Prob(coord|x,y) - Prob(anti|x,y) &= \langle\Psi| \Pi_{x,0} \otimes \Pi_{y,0} |\Psi\rangle + \langle\Psi| \Pi_{x,1} \otimes \Pi_{y,1} |\Psi\rangle \\ & - \langle\Psi| \Pi_{x,0} \otimes \Pi_{y,1} |\Psi\rangle - \langle\Psi| \Pi_{x,1} \otimes \Pi_{y,0} |\Psi\rangle \\ &= \langle\Psi| (\Pi_{x,0} - \Pi_{x,1}) \otimes (\Pi_{y,0} - \Pi_{y,1}) |\Psi\rangle \end{aligned} \tag{2.31}$$

This equation implies that the important operator is the difference between the two projective measurement operators. It is convenient then to define $A_x = \Pi_{x,0} - \Pi_{x,1}$ and $B_y = \Pi_{y,0} - \Pi_{y,1}$, which allows to re-write equation (2.31) in a more compact way:

$$Prob(coord|x,y) - Prob(anti|x,y) = \langle\Psi| A_x \otimes B_y |\Psi\rangle \tag{2.32}$$

Before characterising $A_x$ and $B_y$, it is worth recalling the properties of the projective measurement operators. Focusing on the projector $\Pi_{x,a}$, it fulfills these conditions:

- Projector: $\Pi_{x,a}^2 = (|u_{x,a}\rangle\langle u_{x,a}|)(|u_{x,a}\rangle\langle u_{x,a}|) = |u_{x,a}\rangle\langle u_{x,a}| = \Pi_{x,a}$
- Positive-definite: $\langle\Psi| \Pi_{x,a} |\Psi\rangle = \langle\Psi| u_{x,a}\rangle\langle u_{x,a} |\Psi\rangle \geqslant 0 \quad \forall\ |\Psi\rangle \in \mathcal{H}$.
- Hermitian: $\Pi_{x,a}^\dagger = \Pi_{x,a}$.
- Completeness relation: $\Pi_{x,0} + \Pi_{x,1} = |u_{x,0}\rangle\langle u_{x,0}| + |u_{x,1}\rangle\langle u_{x,1}| = \mathbb{I}$ since $\{|u_{x,0}\rangle, |u_{x,1}\rangle\}$ is a basis of the corresponding Hilbert space.

Similar relations hold for Bob's projectors $\Pi_{y,b}$. The projector operators are observables, that is, quantities that can be measured directly in the system, e.g. energy, position, momentum. Observables are represented in quantum mechanics by Hermitian operators. The possible values of an observable are the eigenvalues of the operator. The eigenvalues of a projector operator are either 0 or 1, which is why $\Pi_{x,a}$ and $\Pi_{y,b}$ are the chosen quantum operators to decide on the binary outputs for Alice and Bob $a, b \in \{0, 1\}$.

In contrast, the operators $A_x$ and $B_y$ have eigenvalues $+1$ and $-1$. Firstly, they are both Hermitian $(A_x)^\dagger = A_x$ and $(B_y)^\dagger = B_y$, so they are observables. To show that the eigenvalues are $+1$ and $-1$, it is enough to square the operators: $(A_x)^2 = (\Pi_{x,0}-\Pi_{x,1})^2 = (\Pi_{x,0})^2+(\Pi_{x,1})^2-\Pi_{x,0}\Pi_{x,1}-\Pi_{x,1}\Pi_{x,0} = \mathbb{I}$, where it was used the projector property $(\Pi_{x,a})^2 = \Pi_{x,a}$ and the completeness relation $\Pi_{x,0}+\Pi_{x,1} = \mathbb{I}$, which means the anti-commutator[13] is 0, i.e. $\Pi_{x,0}\Pi_{x,1} + \Pi_{x,1}\Pi_{x,0} = 0$. Therefore, since $(A_x)^2 = \mathbb{I}$, the eigenvalues of $A_x$ are $+1$ and $-1$. The same holds for Bob with $B_y$.

Returning to the winning probability of the CHSH game, equation (2.30) can be written in terms of the newly-defined operators $A_x$ and $B_y$ in a very compact manner:

$$Prob(\text{win}) = \frac{1}{2} + \frac{1}{8}\langle\Psi| (A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1) |\Psi\rangle \tag{2.33}$$

The combination of the operators $A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$ is the part related directly to the CHSH inequality from section 2.1. In that section, it was shown that, classically, for variables $\mathbf{a}, \mathbf{a'}, \mathbf{b}, \mathbf{b'} \in \{+1, -1\}$, it follows $|\langle\mathbf{ab}\rangle+\langle\mathbf{ab'}\rangle+\langle\mathbf{a'b}\rangle - \langle\mathbf{a'b'}\rangle| \leqslant 2$. However, as it will be explained using a similar technique, the quantum bound of that combination of operators is $2\sqrt{2}$.

First, consider the operator:

$$\mathcal{O} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 \tag{2.34}$$

The square of $\mathcal{O}$ is:

$$\mathcal{O}^2 = \left[A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1)\right]^2$$

[13]The anti-commutator of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, denoted as $\{\mathcal{O}_1, \mathcal{O}_2\}$ (not to be confused with the set notation) is $\{\mathcal{O}_1, \mathcal{O}_2\} = \mathcal{O}_1\mathcal{O}_2 + \mathcal{O}_2\mathcal{O}_1$.

$$
\begin{aligned}
&= (A_0)^2 \otimes (B_0 + B_1)^2 + (A_1)^2 \otimes (B_0 - B_1)^2 \\
&\quad + A_0 A_1 \otimes (B_0 + B_1)(B_0 - B_1) + A_1 A_0 \otimes (B_0 - B_1)(B_0 + B_1) \\
&= \mathbb{I}\big((B_0 + B_1)^2 + (B_0 - B_1)^2\big) + (A_0 A_1 - A_1 A_0) \otimes (B_1 B_0 - B_0 B_1) \\
&= 4\mathbb{I} \otimes \mathbb{I} + [A_0, A_1] \otimes [B_1, B_0] \qquad (2.35)
\end{aligned}
$$

where it was used that $(A_x)^2 = (B_y)^2 = \mathbb{I}$. The symbol $[\ ,\ ]$ denotes the commutator[14] of operators. The terms with the commutator $[A_0, A_1]$ and $[B_1, B_0]$ are both bounded by $2\mathbb{I}$ since $||[A_0, A_1]|| \leqslant 2||A_0||\ ||A_1|| \leqslant 2$ because $||A_x|| \leqslant 1$. Therefore, that means that $\mathcal{O}^2 \leqslant 8\mathbb{I}$, then taking the square root (which is operator-monotone), gives the bounds of equation (2.34) to be: $-2\sqrt{2}\mathbb{I} \leqslant \mathcal{O} \leqslant 2\sqrt{2}\mathbb{I}$. In the classical case, all combinations of operators $A_x$ and $B_y$ commute, thus giving the classical bound of 2. The key difference is then the non-commutativity of the quantum operators. This simple proof considering the square of $\mathcal{O}$ to find the bounds is a known alternative proof of what Tsirelson did in [31].

In the proof, notice that there was no mention whatsoever of the choice of quantum state $|\Psi\rangle$ or the choice of the measurement operators $\Pi_{x,a}$ and $\Pi_{y,b}$; the proof relies solely on the properties of quantum operators. Finally, these bounds on operators imply the already-mentioned bounds for the winning probability of the CHSH game in (2.33):

$$\max\ Prob(\text{win}) = \frac{1}{2} + \frac{1}{8}(2\sqrt{2}) = \frac{2+\sqrt{2}}{4} \approx 0.853 \qquad (2.36)$$

$$\min\ Prob(\text{win}) = \frac{1}{2} - \frac{1}{8}(2\sqrt{2}) = \frac{2-\sqrt{2}}{4} \approx 0.146 \qquad (2.37)$$

bounds, which, as shown in the previous section, can be attained using a maximally entangled state and some simple projection operators.

It is worth making a small comment on the role of the quantum state and the no-communication condition of the game. The correlations in the quantum state are created locally (by the source that emits the two photons, for example). These correlations are kept even though the individual parts are separated[15]. The players make use of the locally-created correlations but they the cannot extract or transmit any information to the other player by measuring their own part of the state. This means that they cannot use the quantum state for communication, thus fulfilling the no-communication condition of the game[16].

[14]The commutator of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, denoted as $[\mathcal{O}_1, \mathcal{O}_2]$ is defined as $[\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1\mathcal{O}_2 - \mathcal{O}_2\mathcal{O}_1$.

[15]Assuming no decoherent process has occurred along the way. The quantum systems are never fully isolated, which means that they interact with the environment. That interaction leads to the system to lose its "quantumness". This lost of quantum coherence is known as decoherence.

[16]The Einstein-Podolsky-Rosen paradox [7] in 1935 focused on that "communication issue", which led to a causality issue in their view (see section 1.1 in chapter 1). However, causality is not violated because the correlations are created locally and there is no possibility of transmitting information between the individual parts of the quantum state by measuring.

## 2.3 Further information on the CHSH game and quantum games

Besides its original connection to the CHSH inequality, the CHSH game is one of the simplest games that belongs to the so-called XOR games. XOR games – firstly formally introduced in [34] – are games whose winning condition involves the boolean function XOR (exclusive OR). Equivalently, only the parity of the players' output bit matters when deciding the winning condition. The XOR games are very important because it has been shown that there is a separation between the classical and the quantum bound. What is more, the quantum bound for the XOR games can be obtained by semi-definite programming[17]. For instance, in [35], using semi-definite programming techniques, the author computed the classical and the quantum bound for a generalised CHSH inequality with two parties, each having $n$ measurement settings, with each measurement having a binary outcome.

When a certain game defined with the no in-game communication condition (or, equivalently, the locality condition for the players and all the resources used) shows a separation between the classical bound and the quantum bound it is said to be a **non-local game**[18]. If the non-local game can be won perfectly (with probability 1) using quantum resources, but not classically, it is usually referred to by some authors as a pseudo-telepathy game[19] [36]. The XOR games are just a subset of non-local games, and the CHSH game is one example of an XOR non-local game, but there are others, for instance the Mermin-Peres magic square game[20] [37–39], the Odd Cycle game[21] [34, 40, 41], which is also an XOR game, and many other games.

The study of **non-local games** has a vast range of **applications**. In foundations of quantum mechanics, any Bell inequality can be mapped to a non-local game, but the converse map is not unique [42, 43], which implies that non-local games is a broader topic than Bell inequalities. On the more practical side, the non-local correlations arising from Bell inequalities, and therefore, also from non-local games, were firstly connected by Artur Ekert in 1991 to quantum cryptography [44]. In that cryptographic context, the crucial concept of self-testing appeared. Self-testing aims at certifying that a process has a non-classical (or quantum) nature without focusing

[17]The claims of the separation between the classical and quantum bounds and the efficient computation of the quantum bounds using semi-definite programming are deeply connected to Tsirelson's work mentioned previously. He showed that the quantum bound can be computed using inner products of unitary vectors. Again, see [31] for that connection.

[18]The denomination comes from the Bell inequalities, which would not be violated in a generic (Bell-like) local theory. In contrast, quantum mechanics, predicts their violation, and thus quantum mechanics was labelled as a non-local theory.

[19]The term *pseudo-telepathy* is justified in that winning the game gives the impression of a true telepathy between the players, even though there was no communication (that is why "pseudo"). Sometimes in the literature the terms *non-local games* and *pseudo-telepathy games* are used interchangeably, but they should not be confused.

[20]This is a very popular game in the quantum literature with two non-communicating players playing in a $3 \times 3$ grid. They need to fill in the spaces following certain rules. Classically this game has a winning probability of $8/9 \approx 0.89$. Using quantum operators as their strategies, the players can win this game with probability 1.

[21]The Odd Cycle game consists of two players trying to color a cycle with an odd number of vertices $n$. Graph theory assures that it is not possible to do the coloring perfectly, but with quantum resources, the players can do better than using any classical strategy.

on how the process works, but only on initial and resulting statistics of it. In the case of the CHSH game, if the winning probability is found in the range between $3/4 = 0.75$ and $(2+\sqrt{2})/4 \approx 0.853$ or between the range $(2-\sqrt{2})/4 \approx 0.146$ and $1/4 = 0.25$, it can be certified that the process is purely quantum without knowing the inner workings of it. Self-testing is closely related to the development of device-independent protocols, which are protocols that do not depend on the specific details of the system. These protocols are of high importance in quantum information and cryptography. Some examples of the applications of self-testing are: generation of device-independent randomness; device-independent quantum cryptography (for instance, with quantum key distribution, QKD); entanglement detection; delegated quantum computing; and many other applications. See [45] for a complete review of self-testing and its applications.

Returning to quantum games, in 2021, one of the most important results lately in computer science – the much celebrated $MIP^* = RE$ in [46] – used non-local games for the proof[22]. Quantum games that use quantum phenomena other than non-locality also have applications in quantum cryptography (see for example [52]) and in quantum and classical information (see, for instance, [53, 54]).

## Summary of the chapter

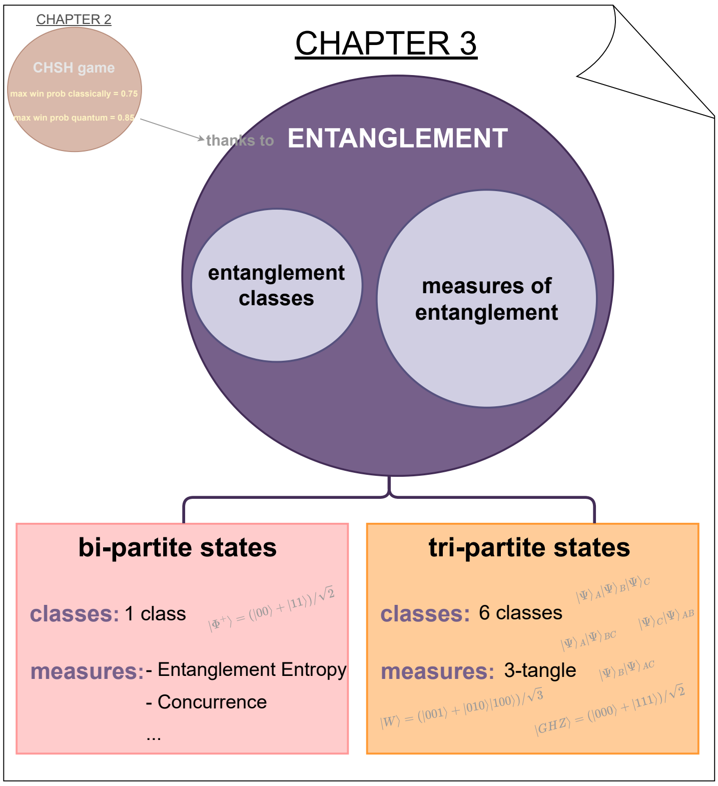
This chapter has introduced the CHSH inequality, and the corresponding CHSH game. It was shown that if the players use quantum resources the winning probability can increase (and decrease) by about 10% from the classical value, in which no "quantumness" is involved. The games that distinguish between the classical and the quantum value are called non-local games. Non-local games in particular and quantum games in general have many applications in quantum information, quantum cryptography, quantum foundations, and quantum computing.

The next chapter concerns with explaining more clearly the term *maximally entangled state*, used in the quantum version of the CHSH game. To do so, the next chapter introduces in a more-formal manner the concept of entanglement. With the basics of the CHSH game from this chapter and the concept of entanglement in the next chapter, the foundations for the conducted research explained in chapters 4 and 5 will have been laid.

[22] That research showed that the two classes $MIP^*$ (Multi-prover Interactive Proof with entangled provers) and $RE$ (Recursive Enumerable languages) were equivalent. That result has striking implications in directly solving a long-standing open problem in quantum mechanics (Tsirelson's problem [47]) and in operator algebras (Connes' Embedding Problem CEP [48, 49]), which were shown to be equivalent [50, 51].

# Chapter 3

# Entanglement in bi-partite and tri-partite states




## 3.1 Introduction

Entanglement is one of the most interesting effects or consequences of quantum mechanics. Entanglement has been seen to be responsible for better performance in many tasks, or even in new tasks with no classical analogue[1]. One example is the result for the CHSH game explained in the previous chapter. In that scenario, the

[1]For instance, entanglement is important in the speed up in quantum computing [55], in the teleportation of a quantum state [56], in dense coding [57], in quantum cryptography [44, 58], and other tasks.

use of a maximally entangled state[2] can provide the players with a higher winning probability than using any classical resource. That is why many efforts have been put into studying entanglement; especially into how to classify it, how to quantify it, and how to witness it[3].

The classification, quantification and witnessing of entanglement in a given (pure or mixed) state is a sub-field *per se*. The complexity of classifying and defining a measure of entanglement grows when considering not-so-large multi-partite systems. A multi-partite system is a system that can be divided into $N$ subsystems, also known as $N$-partite system. Then, $N = 2$ means that a system can be divided into two subsystems (bi-partite system); $N = 3$ for three subsystems (tri-partite system); and so on. The theory for bi-partite pure states is well-established; for tri-partite pure states is also quite well understood. In both cases, there are a (finite) number of equivalence classes to classify the type of entanglement[4]. However for four parties or more, the number of those equivalence classes goes to infinity, thus making that approach unsuitable. That is why, in the case of $N \geqslant 4$ the classification of entanglement is defined to depend on the task itself. If even classifying the type of entanglement is hard for more than four parties, it is not surprising that, in such case, there is no universal measure of entanglement. For a contained review of multi-partite entanglement see [59]. Reference [60] has a good overview of the measures of entanglement for the bi-partite case; and [61] gives the full flavour of quantum entanglement.

In the present chapter, only the bi-partite case will be explained in more detail; the tri-partite case will also be mentioned briefly – both cases considering only pure states. The reason for focusing only on bi-partite and tri-partite pure states is justified in the use of both types of systems in the research of the present dissertation explained in chapter 5.

## 3.2 Definition of entanglement

Entanglement is defined for what it is not. Consider a $N$-partite pure quantum state described by $|\Psi\rangle$. The state is said to be **entangled if it cannot be written as a tensor product of its individual components**. If it can, the state is said to be (fully) separable and no entanglement is present. Mathematically, a separable state is:

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_N\rangle \tag{3.1}$$

Again, if the state cannot be written as in equation (3.1), it is said to be entangled. For a mixed state, described by the density matrix $\rho$, a state is entangled if $\rho \neq \rho_1 \otimes \rho_2 \otimes ... \otimes \rho_N$. However, for mixed states, there are even subtleties with that

[2]It has still not been explained either: 1) what entanglement is; or 2) what maximum entanglement is or how it is computed. This chapter will address these two questions.

[3]For that task there are the so-called entanglement witnesses, that aim at identifying entanglement in a quantum state without having full knowledge of the state. For instance, these entanglement witnesses can be directly operators or even inequalities, such as Bell inequalities.

[4]These equivalence classes are defined as the set of pure states that are equivalent under stochastic local operations and classical communication (SLOCC), that is, if they can be converted into each other using local operations and classical communications with a non-zero probability of success.

definition, since an unentangled mixed state might be not written as a product, but a product state is always separable (unentangled). For instance, the mixed state $\rho = 1/2\left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1|\right)$ is fully separable (unentangled) but it is not a tensor product of density matrices. That simple example demonstrates the complexity[5] of studying entanglement in mixed states. It is out of the scope of the present dissertation going into the theory of entanglement with mixed states; only pure states will be considered.

In the bi-partite case ($N = 2$), the answer to the question "is it entangled?" has a binary answer: either is (fully) separable or not. However, for more than two parties $N \geqslant 3$, the question and the answer are not so simple. The state can be separable for a number of parties but entangled for a subset of them, which is known as partial separability. That is why it is so hard to analyse and quantify entanglement in general for the $N$-party case.

## 3.3 Measurement of entanglement

The measurement and quantification of the entanglement present in a given (pure or mixed) state is a sub-field itself. As was already mentioned, the bi-partite and tri-partite cases are quite well understood; for more than three parties, the theory is not so uniformly agreed upon. The quantification in such cases depends on the considered tasks or processes.

In general, an entanglement measure is a real non-negative function that: 1) assigns a value of 0 for separable (unentangled) states; and 2) does not increase under local operations and classical communication (LOCC) when applied to the state, property known as the monotonicity condition. In the next subsections, only two or three entanglement measures will be given for the bi-partite and tri-partite case.

### 3.3.1 Bi-partite pure states

One popular measure of entanglement for a bi-partite state is the **concurrence** C. The concurrence for a pure state $|\Psi\rangle$ is defined as:

$$C(|\Psi\rangle) = |\langle\Psi| \sigma_y \otimes \sigma_y |\Psi\rangle| \tag{3.2}$$

where $\sigma_y$ is one of the Pauli matrices, also known as the flip matrix:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.3}$$

If the state is not pure, which means it is described by a density matrix[6] $\rho$, the concurrence takes a more complex form: $C(\rho) = \max(0, \sqrt{\eta_1} - \sqrt{\eta_2} - \sqrt{\eta_3} - \sqrt{\eta_4})$,

[5] In the computational complexity sense too! Even in the bi-partite case, checking if a mixed state is separable is NP-Hard [62]. In complexity theory, an NP-Hard problem is a problem that is at least as hard as any NP-Problem. An NP-Problem is classified into the NP (non-deterministic polynomial time) class if it can be solved in polynomial time with a non-deterministic Turing machine.

[6] A pure state $|\Psi\rangle$ can also described with the density matrix $\rho = |\Psi\rangle\langle\Psi|$.

where $\eta_j$ are the eigenvalues in descending order of the matrix $\rho\tilde{\rho}$, and, $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, and $\rho^*$ denotes only conjugation. The concurrence is normalised $0 \leqslant C \leqslant 1$. A concurrence value of 1 implies a maximally entangled state, and a concurrence of 0 implies a separable (unentangled) state. Another related quantity to the concurrence is the **tangle** $\tau$. It is just the square of the concurrence: $\tau(\rho) = C^2(\rho)$. The use of the tangle is more justified in the tri-partite scenario, explained in the next subsection.

Another widely-used measure is the so-called von Neumann entropy[7] $S$. Given the state represented by the density matrix $\rho$, its von Neumann entropy is:

$$S(\rho) = -Tr\,(\rho \log \rho) = -\sum \nu_j \log \nu_j \tag{3.4}$$

where $\nu_j$ are the eigenvalues of $\rho$ and the base of the logarithm is arbitrary, but it is usually chosen with normalisation purposes. Written using the eigenvalues, $S(\rho)$ is just the Shannon entropy [63], a standard quantity in information theory. The concurrence and the von Neumann entropy are related: $S(\rho) = h(1/2 + 1/2\sqrt{1 - C(\rho)^2})$, where $h(x)$ is the binary Shannon entropy function $h(x) = -x \log_2(x) - (1-x)\log_2(1-x)$.

The von Neumann entropy vanishes for pure states, i.e. $\rho = |\Psi\rangle\langle\Psi|$, because in that case, one eigenvalue of $\rho$ is 1 and the others are 0. That is why the von Neumann entropy is used as a definition of the **entanglement entropy** in the bi-partite case. The bi-partite entanglement entropy is computed as the von Neumann entropy of either of the reduced density matrices of the system. Referring to the two parties as $A$ and $B$, the reduced density matrices are $\rho_A = Tr_B(\rho)$ and $\rho_B = Tr_A(\rho)$, where $Tr_B$ and $Tr_A$ is the partial trace over party B and A, respectively. The entanglement entropy $S_{EE}$ is defined as:

$$S_{EE} = S(\rho_A) = -Tr\,(\rho_A \log \rho_A) = -\sum \mu_j \log \mu_j \tag{3.5}$$

where $\mu_j$ are the eigenvalues of $\rho_A$. Using the Schmidt decomposition[8] of $\rho$ it can be shown that $S(\rho_A) = S(\rho_B)$, thus, any of the two reduced density matrices can be used to compute $S_{EE}$.

These definitions are not the only **measures of entanglement**. For instance, there is the geometric measure of entanglement [64], the Schmidt measure [65], the distillable entanglement [66], the entanglement cost [67], the entanglement of formation [68], the relative entropy of entanglement [69], the localisable entanglement [70], the squashed entanglement [71], and many other measures. This long list of

[7]This quantity is named after John von Neumann, an Hungarian-American mathematician and chemical engineer. He and Oskar Morgenstern helped to formally develop game theory (see section 1.2 in chapter 1, page 5). Von Neumann made major contributions to many others fields besides quantum mechanics, such as set theory, group theory, and operator theory, among others. During World War II, he worked on the Manhattan Project, which aimed at developing the first atomic bomb.

[8]The Schmidt decomposition states that any vector in the product of Hilbert spaces of dimensions $n$ and $m$ (with $n \geqslant m$), respectively, i.e. $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, can be written as a linear combination of orthonormal sets of vectors in each Hilbert space. That means, $|\psi\rangle = \sum_{k=1}^{m} \lambda_k |u_k\rangle \otimes |v_k\rangle$, where $\{|u_1\rangle, ...., |u_m\rangle\} \subset \mathcal{H}_1$ and $\{|v_1\rangle, ...., |v_m\rangle\} \subset \mathcal{H}_2$ are orthonormal in each Hilbert space. It is worth noticing that the index $k$ goes up to $m$, the lower dimension of the two spaces. The coefficients $\lambda_k$ are non-negative real numbers and unique up to re-ordering.

possible measures of entanglement demonstrates how rich the topic is and that the quantification of entanglement might depend directly on the considered task or process. In the case of bi-partite pure states, some of these measures reduce to the just-introduced entanglement entropy.

The next example of computing the entanglement entropy and the concurrence for a pure bi-partite state aims at justifying the claim of calling the state in equation (2.12) in chapter 2 (on page 13) *a maximally entangled state.*

**Example**

This short example will compare the measure of entanglement given by the concurrence and by the entanglement entropy, i.e. von Neumann entropy of the reduced density matrices. The considered state is a 2-qubit pure state $|\Psi\rangle = \sqrt{1-|\lambda|^2}\,|00\rangle + \lambda\,|11\rangle$, with $\lambda$ being a parameter[9] whose norm is between 0 and 1, i.e. $0 \leqslant |\lambda| \leqslant 1$. This state can be referred as a Bell-like state because for $\lambda = \pm 1/\sqrt{2}$, recovers two of the states in the Bell basis $|\Phi^{\pm}\rangle$. Then, the concurrence and the entanglement entropy of this state are:

$$C = 2|\lambda|\sqrt{1-|\lambda|^2} \tag{3.6}$$

$$S_{EE} = -|\lambda|^2 \log_2|\lambda|^2 - (1-|\lambda|^2)\log_2(1-|\lambda|^2) \tag{3.7}$$

where, for normalisation, the logarithm was taken in base 2. Figure 3.1 shows both measurements of entanglement in the Bell-like state as a function of the parameter $|\lambda|$. As can be seen from the plot, both the concurrence $C$ and the entanglement entropy $S_{EE}$ peak for $|\lambda| = 1/\sqrt{2}$, which, up to relative phases, corresponds to the state used in the CHSH game to achieve the maximum winning probability (see equation (2.12) on page 13): $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, thus justifying referring to it as a *maximally entangled state*. Moreover, the Bell state (or EPR pair) $|\Phi^+\rangle$ is the basic unit of bi-partite entanglement, since any other pure state can asymptotically be transformed by local operations and classical communication (LOCC) into it and vice-versa [72]. In fact, the conversion rate from a generic pure bi-partite state $|\psi\rangle$ into the Bell state $|\Phi^+\rangle$ is equal[10] to the entanglement entropy of the state $S_{EE}(|\psi\rangle)$ [73], thus justifying the use of the entanglement entropy as a meaningful measure of entanglement in the bi-partite case.

Entanglement quantification is important and is related to the CHSH game in the following manner: the achievement of a maximal winning probability in the game depends on the amount of entanglement present in the initial state. For a completely separable state, the maximum winning probability reduces to the classical value of $3/4 = 0.75$. Therefore, if, after playing the game, the winning probability is above 0.75, it is assured that there was some entanglement in the state. This simple

[9]The state is normalised, but it has been omitted the possible relative phase in $|00\rangle$. In general, the state would be written as $|\Psi\rangle = e^{i\psi}\sqrt{1-|\lambda|^2}\,|00\rangle + \lambda\,|11\rangle$, however in this case the phase $e^{i\psi}$ does not play an important role, so it is omitted.

[10]If there are $N$ copies of any bi-partite state $|\psi\rangle$, the conversion rate $r$ is defined as the number of Bell states (or EPR pairs) $|\Phi^+\rangle$ asymptotically needed to prepare the state by LOCC and vice-versa. Mathematically, $|\psi\rangle^{\otimes N} \leftrightarrow |\Phi^+\rangle^{\otimes \lfloor rN \rfloor}$, with $\lfloor\ \rfloor$ being the floor function and, as $N \to \infty$, the approximation error goes to 0.

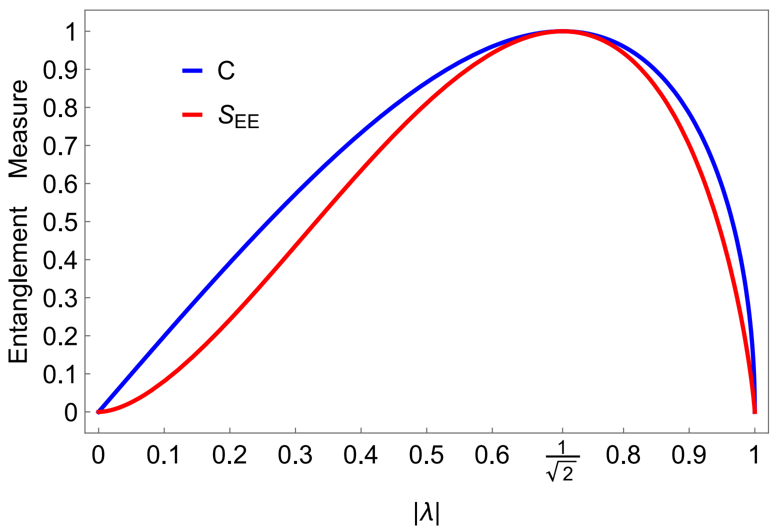



Fig. 3.1: *Values of the concurrence C (blue line) in equation (3.6) and the entanglement entropy $S_{EE}$ (red line) in equation (3.7) for the Bell-like state* $|\Psi\rangle = \sqrt{1-|\lambda|^2}\,|00\rangle + \lambda\,|11\rangle$ *as a function of* $|\lambda|$.

scenario can be used directly as an entanglement witness[11]. In that area of studying the resulting statistics of a process, such as the winning probability of the CHSH game, the concepts of rigidity [76] and robustness are defined. A process is said to be rigid if by just looking at the final (bounded) correlations, it can be inferred (up to certain transformations) the state used and the measurements performed[12]. A protocol or a process is said to be robust if it tolerates noise well; noise here meaning a deviation from the ideal case. The CHSH game has been proven to be both robust and rigid [77, 78].

### 3.3.2 Tri-partite pure states

The theory of tri-partite pure states classifies them into 6 equivalence classes[13]. Labelling the parties as A, B, C, the equivalence classes are:

1) fully separable state $|\psi_A\rangle\,|\psi_B\rangle\,|\psi_C\rangle$.
2) partially separable with respect to A: $|\psi_A\rangle\,|\psi_{B,C}\rangle$ with $|\psi_{B,C}\rangle$ non-separable.
3) partially separable with respect to B: $|\psi_B\rangle\,|\psi_{A,C}\rangle$ with $|\psi_{A,C}\rangle$ non-separable.
4) partially separable with respect to C: $|\psi_C\rangle\,|\psi_{A,B}\rangle$ with $|\psi_{A,B}\rangle$ non-separable.
5) GHZ-class, which is genuinely tri-partite entangled, i.e. is not partially separable with respect to any of the three parties.

[11] In [74] it is shown that, for pure states, quantum entanglement implies violation of a Bell inequality. For mixed states, there are entangled mixed states that do not violate any Bell inequality [75], thus a Bell inequality in that case cannot be used to witness entanglement.

[12] The term rigidity is closely related to self-testing (explained briefly in section 2.3 of chapter 2) and many times both terms are used interchangeably. In the rigidity definition, there are many copies of the state used, whereas in the self-testing scenario, there is only one copy of the state.

[13] From previously, the equivalence classes are defined in terms of equivalence between states under stochastic local operations and classical communication (SLOCC).

6) W-class, which also is genuinely tri-partite entangled.

The first four classes are not surprising; the last two are the interesting ones. The term **GHZ-class** comes from the tri-partite GHZ state, named after Daniel Greenberger, Michael Horne, and Anton Zeilinger[14] (GHZ) [79]. The **W-class** comes from the W state, after a joint paper by Wolfgang Dür *et al.* [80]. These two tri-partite states are defined as:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \tag{3.8}$$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \tag{3.9}$$

The fact that they belong to two different equivalent classes implies that it is not possible to convert one to the other using only SLOCC[15]. Even though both states are genuinely tri-partite, the type of entanglement present in each is not equivalent [80]. The explanation of why the they are different is a little out of the scope, but it is worth mentioning a few words about them.

If, say party A, were to measure their qubit in $|GHZ\rangle$, then, the resulting bi-partite state is fully separable: either $|0\rangle \otimes |0\rangle$ or $|1\rangle \otimes |1\rangle$. On the contrary, if the measurement is on the $|W\rangle$, the resulting bi-partite state can still be entangled (or not): either $\left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right)/\sqrt{2}$ or $|0\rangle \otimes |0\rangle$. In this sense, the entanglement present in $|W\rangle$ is more robust when loosing/measuring a qubit in comparison to $|GHZ\rangle$. In the same context of studying the type of entanglement, the robustness [82] of entanglement is also defined.

For completeness, in this section, it will be mentioned briefly one of the most-widely used measures of entanglement in the tri-partite case, which is related to a rather curious result, referred to as the monogamy of entanglement[16].

The tangle (or 2-tangle) is an entanglement measure in the bi-partite case but, from it, the **3-tangle** arises, which is a **measure of genuine tri-partite entanglement**. The tangle is just the square of the concurrence $\tau(\rho) = C(\rho)^2$ (see (3.2) on page 24 and the subsequent definition with density matrices). For a tri-partite state, represented by $\rho_{ABC}$, the tangle $\tau_{AB}$ is computed using the reduced density matrix $\rho_{AB} = Tr_C(\rho_{AB})$. Focusing on party A, the tangle fulfills the next inequality [84]:

$$\tau_{AB} + \tau_{AC} \leqslant \tau_{A(BC)} \tag{3.10}$$

where $\tau_{AB}$ and $\tau_{AC}$ are the tangles for subsystem $AB$ and $AC$, and $\tau_{A(BC)}$ represents the tangle between subsystem A and subsystem BC, which, for pure states[17] is

[14] Along with Alain Aspect and John Clauser, Anton Zeilinger was one of the awardees of the Nobel Prize in Physics in 2022.

[15] Actually, it is not possible to convert one to the other exactly, but the W state can be approximated with an arbitrary precision by the GHZ state [81]. However, the GHZ cannot be approximated by the W state.

[16] In the quantum foundational aspect, there is a fine distinction between the monogamy of entanglement and the monogamy of correlations, see [83].

[17] For mixed states it is defined as a minimum of the average over all possible pure-state decomposition of the density matrix $\tau^{min}_{A(BC)}$. For more details, see the arXiV version of [84], which is more complete.

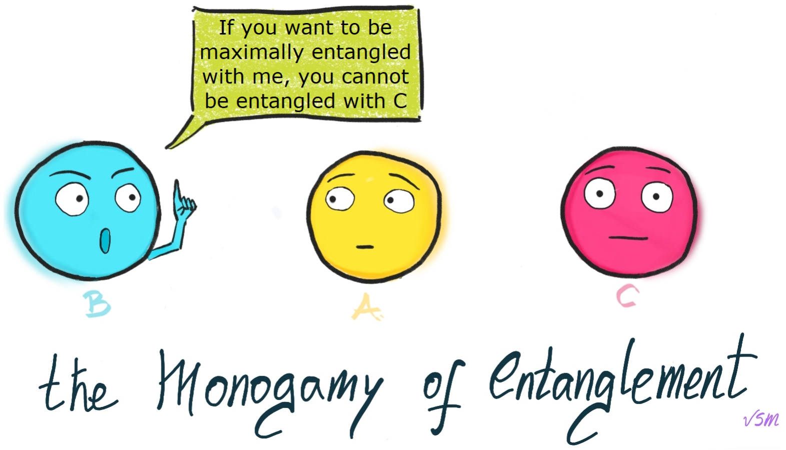



Fig. 3.2: *Drawing about what the monogamy of entanglement means with three qubits, labelled as parties A, B, and C.*

computed as $\tau_{A(BC)} = 4\det(\rho_A)$.

Equation (3.10) has crucial implications: party A sharing some entanglement with party BC constraints the amount of entanglement that A can share individually with B and C separately. This is know as the monogamy of entanglement (see the illustration in Figure 3.2). From equation (3.10), it is defined the 3-tangle $\tau_{ABC}$ as the difference between both sides:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC} \tag{3.11}$$

The 3-tangle $\tau_{ABC}$ in equation (3.11) is invariant under the permutation of parties because it is related to Cayley's hyperdeterminant [85] of the 3-qubit state, which is an invariant, thus making the 3-tangle a good measure of the genuine tri-partite entanglement present in a given 3-qubit state.

A quick calculation of the 3-tangle for the GHZ state and the W state gives $\tau_{ABC}(|GHZ\rangle) = 1$ and $\tau_{ABC}(|W\rangle) = 0$, reaffirming the statement that those two tri-partite states have a different type of entanglement. This calculation of the 3-tangle for both states can be checked in appendix A.2.

## Summary of the chapter

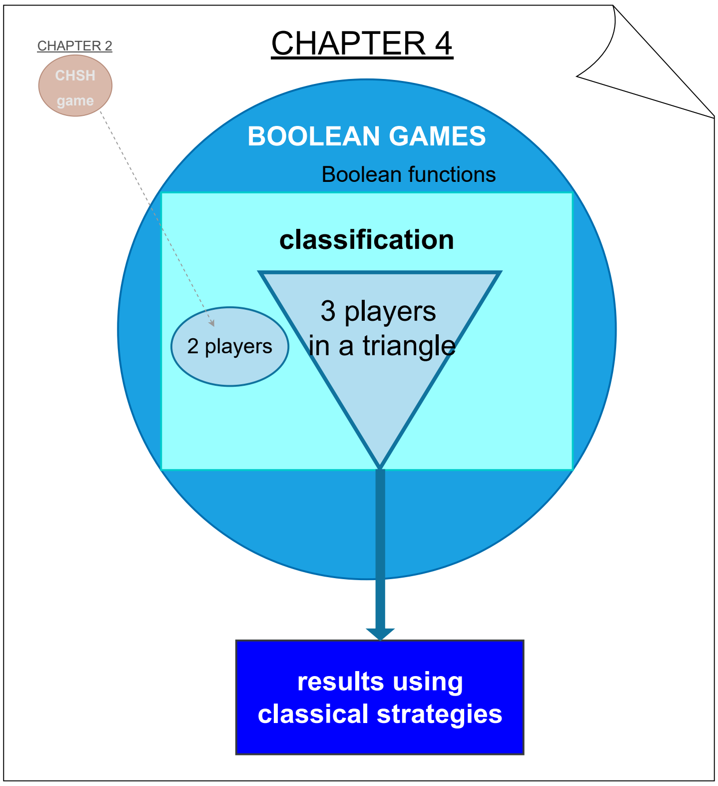
This chapter has defined entanglement and gave a small overview of the classification and quantification of entanglement for the bi-partite and tri-partite case. Some entanglement measures were briefly described, such as the concurrence, the entanglement entropy, the tangle and the 3-tangle. The classification of entanglement in the bi-partite scenario is only concerned with the distinction between separable and entangled state, giving 2 classes of entanglement[18]. The tri-partite case is richer; it has 6 classes of entanglement: one class is the fully separable case; from the second to the fourth class, they describe partially separable states with respect to one party; the last two classes are represented by the GHZ and the W states, which are states that are fully non-separable.

[18] This said, remember that all the bi-partite pure states can be reversibly transformed into the Bell state (or EPR pair) $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$

The next chapter, chapter 4, moves to introducing boolean games played with three players in a triangle configuration when the players use classical strategies. Chapter 5, however, analyses the same situation of boolean games in a triangle when the players use quantum resources, specifically, when they use bi-partite and tri-partite states, just presented in this chapter. Chapter 4 and 5 constitute the main contribution to new research of this dissertation.

# Chapter 4

# Boolean games with 3 players in a triangle




This chapter lays the foundations of the present dissertation. The research conducted focuses on analysing the situation of three players playing intertwined boolean games in a triangle-network configuration. The reason for choosing the triangle-network situation will be fully explained in chapter 5, since this triangle-network has been widely used as a simple form of a quantum network. More about that on the next chapter. Chapter 4 explains the layout for the games and gives the results in the classical case, whereas chapter 5 presents the results for this same situation when the players share quantum resources. Both chapter 4 and 5 constitute the author's new results from the research carried for this dissertation.

The boolean games considered are defined by **two boolean functions**: one function $I$ for the input bits and another function $O$ for the output bits. For short, the function **I** for the input bits will be referred to as the ***input function*** and **O**

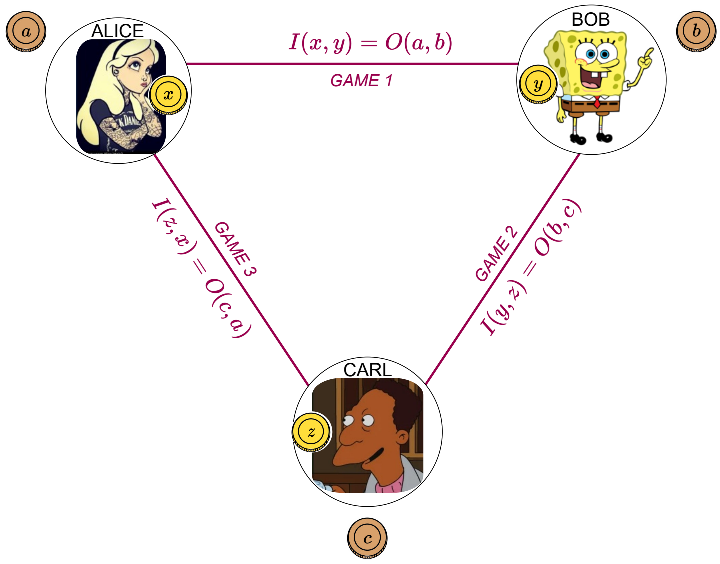



Fig. 4.1: *Boolean games played in a triangle network. Each player plays pairwise a game, which is defined by the choices of the boolean functions $I$ and $O$. The players' input bits $x, y, z \in \{0,1\}$ are represented by the yellow coins and the output bits $a, b, c \in \{0,1\}$ by the bronze coins. The players jointly win each game if $I(\ ,\ ) = O(\ ,\ )$.*

as the ***output function***. Subsection 4.1.1 defines and classifies all the boolean functions from which $I$ and $O$ are chosen.

The three players – Alice, Bob, and Carl – will **play pairwise** a two-player game with the other two players separately. At the start of the game, the players receive one bit and must output another bit. The inputs will be labelled as $x, y, z \in \{0,1\}$ and the outputs $a, b, c \in \{0,1\}$ for Alice, Bob, and Carl, respectively. During the game, as was the case with the CHSH game explained in section 2.2, the players are separated and no in-game communication is allowed, which implies that they have only access to partial information during the games. Figure 4.1 illustrates the situation. The winning condition for each game is:

$$\text{win GAME 1} \rightarrow I(x,y) = O(a,b) \tag{4.1}$$
$$\text{win GAME 2} \rightarrow I(y,z) = O(b,c) \tag{4.2}$$
$$\text{win GAME 3} \rightarrow I(z,x) = O(c,a) \tag{4.3}$$

Notice that the players have only one input and one output, which, both, will be used for the two games that they are playing. Also notice that the boolean functions $I$ and $O$ are the same for all three games. Each of the games is considered to be a cooperative game, that is, both players win or lose jointly. Since each player plays two games, the final **payoff** would be the **average** of both. For Alice:

$$\$_A = \frac{1}{2}\left[Prob(\text{win G1}) + Prob(\text{win G3})\right] \tag{4.4}$$

where $Prob(\text{win G1})$ and $Prob(\text{win G3})$ denote the probability of winning GAME 1 and GAME 3, respectively. Similarly for Bob and Carl.

Focusing for now on GAME 1, the $Prob(\text{win G1})$ will only depend on what Alice and Bob do in the game, that is, on the **conditional probability** of outputs

$a, b$ given inputs $x, y$, i.e. $Prob(a, b|x, y)$. The winning condition for GAME 1 in equation (4.1) allows to write the winning probability of this game in a compact way:

$$Prob(\text{win G1}) = \frac{1}{4} \sum_{x,y,a,b \in \{0,1\}} Prob(a, b|x, y)\delta_{I(x,y)O(a,b)} \tag{4.5}$$

where the 1/4 comes from assuming uniformly distributed inputs $x, y \in \{0, 1\}$ and $\delta_{I(x,y)O(a,b)}$ is the Kronecker delta, which selects the probabilities for which the winning condition in equation (4.1) is met. Note that $Prob(a, b|x, y)$ does not depend on Carl's input $z$ or output $c$; however, there is an initial probability distribution of inputs and outputs that does depend on $z$ and $c$, i.e. $Prob(a, b, c|x, y, z)$. To obtain $Prob(a, b|x, y)$ from $Prob(a, b, c|x, y, z)$ it is necessary to add all the possible results for $z$ and $c$ in $Prob(a, b, c|x, y, z)$, which gives a probability independent of Carl's inputs and outputs. Mathematically:

$$Prob(a, b|x, y) = \frac{1}{2} \sum_{z} \sum_{c} Prob(a, b, c|x, y, z) \tag{4.6}$$

where the 1/2 comes from the uniformly distributed inputs. This means, that given inputs $x, y \in \{0, 1\}$ there is a 50% chance (1/2) that $z$ is either 0 or 1. Then, the winning probability of GAME 1 in equation (4.5) can be written as:

$$Prob(\text{win G1}) = \frac{1}{8} \sum_{x,y,a,b} \sum_{z,c} Prob(a, b, c|x, y, z)\delta_{I(x,y)O(a,b)} \tag{4.7}$$

An analogous expression applies to $Prob(\text{win G2})$ and $Prob(\text{win G3})$. Focusing on Alice, her average payoff from both games is:

$$\begin{aligned} \$_A =& \frac{1}{16} \sum_{x,y,z} \sum_{a,b,c} \big[Prob(a, b, c|x, y, z)\delta_{I(x,y)O(a,b)} \\ &+ Prob(a, b, c|x, y, z)\delta_{I(z,x)O(c,a)}\big] \end{aligned} \tag{4.8}$$

A similar expression of the average payoffs follows for Bob and Carl. Basically, to compute the players' average payoff for this particular situation, the only two important ingredients are $Prob(a, b, c|x, y, z)$ and the $I$ and $O$.

A classification of the boolean games will follow in the next section 4.1 depending on the choice of $I$ and $O$, but first, the boolean functions are defined in section 4.1.1. Second, to warm up, all the two-player boolean games will be classified depending on the specific choice of $I$ and $O$ in section 4.1.2. Third, the boolean games in the proposed triangle-configuration are classified in section 4.1.3. Finally, the Nash equilibrium points using classical strategies of the classified games are explained in section 4.2.

## 4.1 Classification of the Boolean games in the triangle

Classifying the boolean games in the triangle configuration implies analysing all the possible different expressions of Alice's payoff in equation (4.8) depending on the choice of $I$ and $O$. To do that, it is helpful first to mention the classification of the two-player games, for which it is also useful to classify first all the boolean functions of two variables.

| set symbol | set elements | Meaning |
|---|---|---|
| $S_0$ | $f_1(x,y) = 0$<br>$f_2(x,y) = 1$ | constant functions |
| $S_1$ | $f_3(x,y) = x$<br>$f_4(x,y) = \overline{x}$<br>$f_5(x,y) = y$<br>$f_6(x,y) = \overline{y}$ | functions of one variable |
| $S_{2u}$ | $f_7(x,y) = x \cdot y$ (AND)<br>$f_8(x,y) = \overline{x \cdot y}$ (NAND)<br>$f_9(x,y) = x \cdot \overline{y}$ (NIMPLY)<br>$f_{10}(x,y) = \overline{x} + y$ (IMPLY)<br>$f_{11}(x,y) = \overline{x} \cdot y$<br>$f_{12}(x,y) = x + \overline{y}$<br>$f_{13}(x,y) = \overline{x + y}$ (NOR)<br>$f_{14}(x,y) = x + y$ (OR) | unbalanced functions of two variables |
| $S_{2b}$ | $f_{15}(x,y) = x \oplus y$ (XOR)<br>$f_{16}(x,y) = \overline{x \oplus y}$ (XNOR) | balanced functions of two variables |

Tab. 4.1: *Classification of the 16 boolean functions of two binary variables. Some of the functions are named after the most common and important binary gates, that implement the corresponding logical operation: AND, NAND (negated AND), NOR (negated OR), OR, XOR (exclusive OR), and XNOR (exclusive negated OR). Some other gates are less well-known such as the IMPLY and NIMPLY gates, that implement logical material implication and material non-implication.* $f_{11}$ *and* $f_{12}$ *implement the logical converse non-implication and converse implication, respectively.*

### 4.1.1 Boolean functions of two variables

There are **16 boolean functions of two binary variables** $x, y \in \{0, 1\}$. These functions can be **subdivided** into **four sets**:

- The set of functions that do not depend explicitly on any of the variables, i.e. constant functions, denoted by $S_0$.
- The set of functions that depend explicitly on only one variable, $S_1$.
- The set of unbalanced functions of two variables, $S_{2u}$.
- The set of balanced functions of two variables, $S_{2b}$.

For the functions that depend explicitly on two variables, it is considered to be **balanced** if for all the possible inputs, the function outputs the same number of 0s and 1s; mathematically, $\#\{(x,y) \mid f(x,y) = 0\} = \#\{(x,y) \mid f(x,y) = 1\} = 2$; it is **unbalanced** otherwise. **Table 4.1** contains the **definition** of the **16 boolean functions** along with the mentioned classification. The numbering of the functions is purposely chosen such that the even functions are just the negated version of the previous (odd) function, i.e. $f_{2k}(x,y) = \overline{f_{2k-1}(x,y)}$ for $k = 1, ..., 8$.

Furthermore, the unbalanced functions of two variables $S_{2u}$ can also be subdivided into two subsets: the ones unbalanced *towards the value 0* (i.e. with more 0s than 1s), denoted as $S_{2u0}$; and the ones unbalanced *towards the value 1*, $S_{2u1}$. That means, $S_{2u} = S_{2u0} \cup S_{2u1}$, with:

$$S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\} \tag{4.9}$$
$$S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\} \tag{4.10}$$

The presented numbering of the functions and the mentioned sets (and subsets) will be useful to classify the different types of boolean games.

As mentioned in the caption of Table 4.1, some of the boolean functions have their own names. For instance, $f_7$ is the AND function, $f_8$ is the NAND, and $f_{15}$ is the XOR function. Coming back to chapter 2, the winning of the CHSH game in equation (2.4) on page 11 was that the AND of the inputs should match the XOR of the outputs. With the present numbering of the boolean functions, the CHSH game is identified by having $f_7$ as input function and $f_{15}$ as output function, i.e. CHSH game $\rightarrow I = f_7(x, y)$ and $O = f_{15}(a, b)$ .

Before moving to classifying the boolean games in the triangle configuration with three players, it is useful to first **classify the boolean games with two players**. It also serves to understand the position of the CHSH game explained in chapter 2 among all the other possible two-player boolean games.

### 4.1.2 Two-player boolean games

The two players – now only Alice and Bob – receive each a binary input $x, y \in \{0, 1\}$ and binary output $a, b \in \{0, 1\}$. As in the explained triangle-scenario, no in-game communication is allowed. In this case, the boolean games considered are cooperative games, that is, both players win or lose jointly[1]. The winning condition of the game is defined using the input and output functions:

$$I(x, y) = O(a, b) \tag{4.11}$$

where $I, O$ are chosen from the set of boolean functions of two variables defined and classified in Table 4.1. The winning condition in equation (4.11) allows one to write the winning probability of any two-player (cooperative) boolean game in a compact way:

$$Prob(\text{win}) = \frac{1}{4} \sum_{x,y,a,b \in \{0,1\}} Prob(a, b|x, y) \delta_{I(x,y)\ O(a,b)} \tag{4.12}$$

where the 1/4 comes from assuming uniformly distributed inputs, $Prob(a, b|x, y)$ is the usual conditional probability of outputs $a, b$ given $x, y$, and $\delta$ is the Kronecker delta, that selects the probabilities for which the winning condition is met[2].

[1] In the triangle configuration, even though the players win or lose jointly each game, it is not correct to say they are in a cooperative scenario because, as will be explained later, this situation may be analysed from a selfish point of view, i.e. the players only caring about their own average payoff.

[2] Equations (4.11) and (4.12) are the same as equations (4.1) and (4.5), respectively, which, for the described triangle-configuration at the start of this chapter, define the winning condition and winning probability of GAME 1 – played only between Alice and Bob.

The interesting games are those whose $I, O$ depend explicitly on the two variables, that is, $I, O \in S_{2u} \cup S_{2b}$; otherwise, either it is a trivial game, or there is no equal power of decision for both players. All of this means that there are in principle **100 different games when $\mathbf{I, O \in S_{2u} \cup S_{2b}}$**, but, due to the symmetry properties of the boolean functions, not all games present an intrinsically different winning probability in equation (4.12). As will be shown next, there are **only 8 different types** of games.

### ◆ $\mathbf{O \in S_{2b} = \{f_{15}, f_{16}\}}$

The boolean functions XOR ($f_{15}$) and XNOR ($f_{16}$) have the common property of giving the same result if the two inputs bits are the same. That is, for any choice of $a \in \{0,1\}$, $f_{15}(a,a) = 0$ , $f_{15}(a,\overline{a}) = 1 = \overline{f_{15}(a,a)}$; and $f_{16}(a,a) = 1$ , $f_{16}(a,\overline{a}) = 0 = \overline{f_{16}(a,a)}$. Then, when $O$ is any of those, it is enough to only consider $O(a,a)$, since $O(\overline{a},\overline{a}) = O(a,a)$ and $O(a,\overline{a}) = O(\overline{a},a) = \overline{O(a,a)}$. To keep the notation tidier, to denote the negated bit of $O(a,a)$ , $O(a,\overline{a})$ will be used instead of $\overline{O(a,a)}$. Bearing that in mind, the winning probability in equation (4.12) can be expressed as:

$$
\begin{aligned}
Prob(\text{win}) &= \frac{1}{4}\sum_{x,y}\sum_{a}\left[Prob(a,a|x,y)\delta_{I(x,y)O(a,a)} + Prob(a,\overline{a}|x,y)\delta_{I(x,y)O(a,\overline{a})}\right] \\
&= \frac{1}{4}\sum_{x,y}\left[Prob(a,a|x,y) + Prob(\overline{a},\overline{a}|x,y)\right]\delta_{I(x,y)O(a,a)} \\
&+ \frac{1}{4}\sum_{x,y}\left[Prob(a,\overline{a}|x,y) + Prob(\overline{a},a|x,y)\right]\delta_{I(x,y)O(a,\overline{a})} \\
&= \frac{1}{4}\sum_{x,y}\delta_{I(x,y)O(a,\overline{a})} + \frac{1}{4}\sum_{x,y}Prob(coord|x,y)\left[\delta_{I(x,y)O(a,a)} - \delta_{I(x,y)O(a,\overline{a})}\right]
\end{aligned}
\tag{4.13}
$$

for any choice of $a \in \{0,1\}$; where $Prob(coord|x,y)$ is the probability of both players coordinating their outputs $Prob(coord|x,y) = Prob(a,a|x,y) + Prob(\overline{a},\overline{a}|x,y)$; and where it was used that $Prob(a,\overline{a}|x,y) + Prob(\overline{a},a|x,y) = 1 - Prob(a,a|x,y) + Prob(\overline{a},\overline{a}|x,y) = 1 - Prob(coord|x,y)$ for any $x,y$.

Depending on the choice of $I(x,y)$ the winning probability in equation (4.13) will look different, but, up to sign permutations, there are **only 4 truly distinct possibilities**:

- $O = f_{15}$ and $I \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; or $O = f_{16}$ and $I \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$; which, up to permutations of the negative sign, the winning probability looks:

$$
\begin{aligned}
Prob(\text{win}) = &\frac{1}{4} + \frac{1}{4}\left[Prob(coord|0,0) + Prob(coord|0,1)\right. \\
&\left. + Prob(coord|1,0) - Prob(coord|1,1)\right]
\end{aligned}
\tag{4.14}
$$

- $O = f_{15}$ and $I \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$; or $O = f_{16}$ and $I \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; which, again, up to permutations of the negative sign (in the brackets with the probabilities), the winning probability looks:

$$\begin{aligned} Prob(\text{win}) =& \frac{3}{4} - \frac{1}{4}\left[Prob(coord|0,0) + Prob(coord|0,1)\right. \\ & \left. + Prob(coord|1,0) - Prob(coord|1,1)\right] \end{aligned} \quad (4.15)$$

- $O = I = f_{15}$; or $O = I = f_{16}$, the winning probability is:

$$\begin{aligned} Prob(\text{win}) =& \frac{1}{2} + \frac{1}{4}\left[Prob(coord|0,0) - Prob(coord|0,1)\right. \\ & \left. - Prob(coord|1,0) + Prob(coord|1,1)\right] \end{aligned} \quad (4.16)$$

- $O = f_{15}$ and $I = f_{16}$; or $O = f_{16}$ and $I = f_{15}$, the winning probability is:

$$\begin{aligned} Prob(\text{win}) =& \frac{1}{2} - \frac{1}{4}\left[Prob(coord|0,0) - Prob(coord|0,1)\right. \\ & \left. - Prob(coord|1,0) + Prob(coord|1,1)\right] \end{aligned} \quad (4.17)$$

The winning probability of the CHSH game looks like the one in equation (4.14) – see equation (2.27) on page 17 of chapter 2. Notice that the winning probabilities in equation (4.14) and in equation (4.15) sum to 1, as well as in equations (4.16) and (4.17).

### ◆ $\mathbf{O \in S_{2u} = \{f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}\}}$

The set of unbalanced functions of two variables $S_{2u}$ is characterised by the fact that they output only one bit and the negated one three times. It was divided into two subsets: unbalanced towards 0 (outputting more 0s than 1s) $S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; and unbalanced towards 1, $S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$.

It will be useful to refer to $m, n$ as the input bits for which the minority bit is outputted $f_*(m,n)$. For example, the AND function $f_7(a,b)$ evaluates $f_7(0,0) = f_7(0,1) = f_7(1,0) = 0$ and $f_7(1,1) = 1$, then, the minority bit is 1 and the input for this minority bit is $m = 1$ and $n = 1$. Having this information of the inputs of the minority bit is enough to characterise the boolean function since for any function in $S_{2u}$, the following property holds: $f_*(m,\overline{n}) = f_*(\overline{m},n) = f_*(\overline{m},\overline{n}) = \overline{f_*(m,n)}$. Using a similar argument as previously when $O \in S_{2b}$, the winning probability can be written with respect to the input of the minority bit *of the output function* $O(m,n) = f_*(m,n)$ and $O(m,\overline{n}) = O(\overline{m},n) = O(\overline{m},\overline{n}) = \overline{O(m,n)}$:

$$\begin{aligned} Prob(\text{win}) &= \frac{1}{4}\sum_{x,y}\Big[Prob(m,n|x,y)\delta_{I(x,y)O(m,n)} + Prob(m,\overline{n}|x,y)\delta_{I(x,y)O(m,\overline{n})} \\ &\quad + Prob(\overline{m},n|x,y)\delta_{I(x,y)O(\overline{m},n)} + Prob(\overline{m},\overline{n}|x,y)\delta_{I(x,y)O(\overline{m},\overline{n})}\Big] \\ &= \frac{1}{4}\sum_{x,y} Prob(m,n|x,y)\delta_{I(x,y)O(m,n)} + \\ &\quad + \frac{1}{4}\sum_{x,y}\left[Prob(m,\overline{n}|x,y) + Prob(\overline{m},n|x,y) + Prob(\overline{m},\overline{n}|x,y)\right]\delta_{I(x,y)O(m,\overline{n})} \end{aligned}$$

$$= \frac{1}{4}\sum_{x,y}\delta_{I(x,y)O(m,\overline{n})} + \frac{1}{4}\sum_{x,y} Prob(m,n|x,y)\left[\delta_{I(x,y)O(m,n)} - \delta_{I(x,y)O(m,\overline{n})}\right] \tag{4.18}$$

where, in the last line, it was used that $Prob(m,\overline{n}|x,y) + Prob(\overline{m},n|x,y) + Prob(\overline{m},\overline{n}|x,y) = 1 - Prob(m,n|x,y)$. This equation is completely analogous to the one in equation (4.13) by just replacing $Prob(coord|x,y)$ with $Prob(m,n|x,y)$. Then, again, up to sign permutations, there will be **only 4 distinct situations**:

- $O \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ and $I \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$; or $O \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ and $I \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; which, up to permutations of the negative sign, the winning probability looks:

$$\begin{aligned} Prob(\text{win}) =& \frac{1}{4} + \frac{1}{4}\left[Prob(m,n|0,0) + Prob(m,n|0,1)\right. \\ & \left. + Prob(m,n|1,0) - Prob(m,n|1,1)\right] \end{aligned} \tag{4.19}$$

- $O, I \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; and $O, I \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$; up to permutations of the negative sign (in the brackets with the probabilities), the winning probability looks:

$$\begin{aligned} Prob(\text{win}) =& \frac{3}{4} - \frac{1}{4}\left[Prob(m,n|0,0) + Prob(m,n|0,1)\right. \\ & \left. + Prob(m,n|1,0) - Prob(m,n|1,1)\right] \end{aligned} \tag{4.20}$$

- $O \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ and $I = f_{16}$; or $O \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ and $I = f_{15}$; the winning probability looks:

$$\begin{aligned} Prob(\text{win}) =& \frac{1}{2} + \frac{1}{4}\left[Prob(m,n|0,0) - Prob(m,n|0,1)\right. \\ & \left. - Prob(m,n|1,0) + Prob(m,n|1,1)\right] \end{aligned} \tag{4.21}$$

- $O \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ and $I = f_{15}$; or $O \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ and $I = f_{16}$; the winning probability looks:

$$\begin{aligned} Prob(\text{win}) =& \frac{1}{2} - \frac{1}{4}\left[Prob(m,n|0,0) - Prob(m,n|0,1)\right. \\ & \left. - Prob(m,n|1,0) + Prob(m,n|1,1)\right] \end{aligned} \tag{4.22}$$

As before, the winning probabilities in equations (4.19)-(4.20) and in equations (4.21)-(4.22) sum to 1.

It was just shown the **8 different types** of winning probabilities for any (cooperative) boolean game with two players when $I, O \in S_{2u} \cup S_{2b}$. Now it is time to classify them depending on the maximum and minimum values of the winning probabilities using classical strategies.

#### 4.1.2.1 Classical strategies

When the players receive their input bit, they are not allowed to communicate, which implies that the probability of Alice outputting $a$ depends solely on her own variables, i.e. her input $x$, and not on what Bob does, and vice-versa[3]. In this case, they will not be using any source of (classical) advice that could correlate their outputs given their inputs. Moreover, Alice and Bob outputting their outputs given their inputs correspond to two independent events, meaning that the probabilities of those events factorise. These two conditions for **uncorrelated** and **independent events** are written in mathematical terms as $Prob(a, b|x, y) = Prob(a|x)Prob(b|y)$. If the players use **mixed strategies**, Alice outputs $a$ given $x$ with a certain probability $\mathbf{p_{a|x}}$, and the same for Bob, denoted as $\mathbf{q_{b|y}}$. For Alice, it is clear that $p_{0|x} + p_{1|x} = 1$ for any $x$, and pure strategies correspond to choosing the extreme values of $p_{0|x}$ and $p_{1|x}$. In this case with binary inputs and outputs, there are only four possible pure strategies, which are:

1) The *ZERO* strategy, $p_{0|x} = 1$, which means outputting 0 regardless of the input bit.

2) The *ONE* strategy, $p_{1|x} = 1$, which means outputting 1 regardless of the input bit.

3) The *IDENTITY* strategy, $p_{x|x} = 1$, which means outputting the input bit.

4) The *NOT* strategy, $p_{\overline{x}|x} = 1$, which means outputting the negated input bit.

To analyse the range of values of the winning probabilities for the different games when $O \in S_{2b}$ using classical strategies, the only thing to do is substituting $Prob(coord|x, y)$ in equations (4.14)-(4.17) by $p_{0|x}q_{0|y} + p_{1|x}q_{1|y}$ and numerically compute the **maximum and minimum values of the winning probabilities**. Similarly when $O \in S_{2u}$ in equations (4.19)-(4.22) replacing $Prob(m, n|x, y)$ by $p_{m|x}q_{n|y}$. Then the winning probabilities will be a convex combination of $p_{a|x}$ and $q_{b|y}$, which means that to obtain the range of values, it is enough to consider only the extreme values, i.e. the pure strategies explained earlier – the *ZERO*, *ONE*, *IDENTITY*, *NOT* strategies. This implies testing all the $4 \times 4 = 16$ possible combinations of pure strategies for both players and checking the maximum and minimum values of the corresponding winning probability. **Table 4.2** contains the **classification** of all the possible 100 **two-player boolean games** depending on the range of winning probabilities.

Remember that the CHSH game is identified by $I(x, y) = f_7(x, y)$ (AND function) and $O(a, b) = f_{15}(a, b)$ (XOR function), corresponding to the third row in Table 4.2, whose classical winning probability is bounded between 0.25 and 0.75 . As shown in section 2.2 of chapter 2, the quantum bounds of the CHSH are $(2 - \sqrt{2})/4 \approx 0.146$ and $(2 + \sqrt{2})/4 \approx 0.854$. From Table 4.2, the same classical bounds also correspond to the games when $I \in S_{2u} = S_{2u0} \cup S_{2u1}$ and $O \in S_{2b}$ (third and sixth row), which means that there are 16 CHSH-like games. The other games that cannot be won perfectly, i.e. $I \in S_{2u0}$ $O \in S_{2u1}$ (second row); $I \in S_{2u1}$ $O \in S_{2u0}$

[3]This no-communication condition is known in the literature as *non-signalling*. Using the standard language with probabilities, a non-signalling probability distribution with two parties fulfils two conditions: *i)* $\sum_b Prob(a, b|x, y) = \sum_b Prob(a, b|x, y')$ and *ii)* $\sum_a Prob(a, b|x, y) = \sum_a Prob(a, b|x', y)$; for all $x, y, x', y'$.

| $I \in$ | $O \in$ | no. games | min | max |
|---|---|---|---|---|
| $S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ | $S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ | 16 | 0.25 | 1 |
| | $S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ | 16 | 0 | 0.75 |
| | $S_{2b} = \{f_{15}, f_{16}\}$ | 8 | 0.25 | 0.75 |
| $S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ | $S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ | 16 | 0 | 0.75 |
| | $S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ | 16 | 0.25 | 1 |
| | $S_{2b} = \{f_{15}, f_{16}\}$ | 8 | 0.25 | 0.75 |
| $S_{2b} = \{f_{15}, f_{16}\}$ | $S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$ | 8 | 0.25 | 0.75 |
| | $S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$ | 8 | 0.25 | 0.75 |
| | $S_{2b} = \{f_{15}, f_{16}\}$ | 4 | 0 | 1 |

Tab. 4.2: *The first and second column specify from which set of boolean functions the $I$ and $O$ are chosen for all the (non-trivial) two-player boolean games. The third column enumerates the total number of games in that classification. The last two columns show the minimum and the maximum winning probability for that type of games. The definition of each $f_i$ is found in Table 4.1 on page 34.*

(fourth row); and $I \in S_{2b}$ $O \in S_{2u}$ (seventh and eight row), do not exhibit such a separation between the classical and the quantum bounds as the CHSH-like games do.

It has been proven that for two-party two-input two-output games without communication, the only games that show a separation when using classical resources (modelled by local hidden-variables LHV theories) versus using non-local resources correspond basically to CHSH games/inequalities. For instance, in [86], the authors characterise, among other things, the vertices from the local and the non-local polytope – which is the generalisation of three-dimensional polyhedra with flat faces, constructed in this case from all the possible sets of correlations satisfying the non-signalling condition. The authors show that the vertices of the non-local theories with two inputs and outputs are equivalent to CHSH games; it is also known that the only non-trivial facets of the LHV polytope are CHSH inequalities [87]. This separation between classical and non-local theories – quantum mechanics would be an example, but there are many other possible ones – is also related to the fact that in a Bell experiment with $n$ parties, LHV theories can only evaluate $n$-partite linear functions via correlators [88]. In the present two-party case, the correlator[4] would correspond to choosing the output function $O(a, b)$ to be XOR ($f_{15}$ in Table 4.1 on page 34), and the "functions to evaluate" would correspond to the input functions $I(x, y)$. That implies that only the 2-partite linear functions can be evaluated in a LHV theory, but since the AND function is not linear, it cannot be classically computed with correlators; situation that maps to a CHSH game with the AND function and the correlators as the XOR output function. This last example is a bit more obscure to understand in terms of the games defined here, but it provides with another interpretation/explanation of the difference between using local and non-local resources in terms of computationally expressiveness.

[4]A correlator is a correlating function; in this context, is a probability distribution over joint outcomes. In [88] the correlators considered are the sum of measurement outcomes $m_j$ modulo the total number of outcomes $d$.

### 4.1.3 Three-player boolean games in the triangle

Now that the two-player boolean games have been classified, it is the turn to **classify the boolean games in the triangle configuration**. This classification consists of analysing all the possible (truly) different expressions of Alice's payoff[5] in equation (4.8) on page 33 as a function of the choice of $I, O \in S_{2u} \cup S_{2b}$.

The previous classification of the winning probabilities in equations (4.14)-(4.22) is useful since Alice's average payoff for the two (identical) games played in the triangle configuration is easily obtained by just adding the winning probability of GAME 1 between Alice and Bob, and the same winning probability for GAME 3, between Carl and Alice. It is paramount then to label properly the probabilities from each game. To shorten the notation, for instance, $Prob_{ab}(1,0|1,1)$ should be read as $Prob(a = 1, b = 0|x = 1, y = 1)$, while $Prob_{ca}(1,0|1,1) \equiv Prob(c = 1, a = 0|z = 1, x = 1)$.

The next few equations only refer to Alice's average payoff depending on the choice of $I, O \in S_{2u} \cup S_{2b}$, since the expressions for Bob's and Carl's average payoffs are obtained by the cyclic permutations $(b, c, a)$ and $(y, z, x)$; and $(c, a, b)$ and $(z, x, y)$, respectively[6].

#### ◆ $\mathbf{O \in S_{2b} = \{f_{15}, f_{16}\}}$

This first classification is practically the same as the one in equations (4.14)-(4.17) for the two-player games.

- $O = f_{15}$ and $I \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; or $O = f_{16}$ and $I \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$; which, up to permutations of the negative signs with the same conditional probabilities[7], the average payoff looks like:

$$\begin{aligned}\$_A =& \frac{1}{4} + \frac{1}{8}\,[Prob_{ab}(coord|0,0) + Prob_{ca}(coord|0,0) + Prob_{ab}(coord|0,1) \\ &+ Prob_{ca}(coord|1,0) + Prob_{ab}(coord|1,0) + Prob_{ca}(coord|0,1) \\ &-Prob_{ab}(coord|1,1) - Prob_{ca}(coord|1,1)]\end{aligned} \tag{4.23}$$

- $O = f_{15}$ and $I \in S_{2u1} = \{f_8, f_{10}, f_{12}, f_{14}\}$; or $O = f_{16}$ and $I \in S_{2u0} = \{f_7, f_9, f_{11}, f_{13}\}$; which, again, up to permutations of the negative sign in the brackets with the same conditional probabilities (see footnote [7]), the average payoff looks like:

$$\begin{aligned}\$_A =& \frac{3}{4} - \frac{1}{8}\,[Prob_{ab}(coord|0,0) + Prob_{ca}(coord|0,0) + Prob_{ab}(coord|0,1) \\ &+ Prob_{ca}(coord|1,0) + Prob_{ab}(coord|1,0) + Prob_{ca}(coord|0,1) \\ &-Prob_{ab}(coord|1,1) - Prob_{ca}(coord|1,1)]\end{aligned} \tag{4.24}$$

[5] Now it is necessary to talk about payoffs and not only about the winning probabilities because each player might only care about what they obtain individually.

[6] For instance, the permutation $(b, c, a)$ and $(y, z, x)$ denote the mappings $(a, b, c) \to (b, c, a)$ and $(x, y, z) \to (y, z, x)$ .

[7] That means, that the two negative signs go in front of: $Prob_{ab}(coord|0,0)$ and $Prob_{ca}(coord|0,0)$; or $Prob_{ab}(coord|0,1)$ and $Prob_{ca}(coord|0,1)$; or $Prob_{ab}(coord|1,0)$ and $Prob_{ca}(coord|1,0)$; or $Prob_{ab}(coord|1,1)$ and $Prob_{ca}(coord|1,1)$.

- $O = I = f_{15}$; or $O = I = f_{16}$, the average payoff is (exactly):

$$\begin{aligned}\$_A =& \frac{1}{2} + \frac{1}{8}\left[Prob_{ab}(coord|0,0) + Prob_{ca}(coord|0,0) - Prob_{ab}(coord|0,1)\right.\\ &- Prob_{ca}(coord|1,0) - Prob_{ab}(coord|1,0) - Prob_{ca}(coord|0,1)\\ &\left.+Prob_{ab}(coord|1,1) + Prob_{ca}(coord|1,1)\right] \qquad (4.25)\end{aligned}$$

- $O = f_{15}$ and $I = f_{16}$; or $O = f_{16}$ and $I = f_{15}$, the average payoff is (exactly):

$$\begin{aligned}\$_A =& \frac{1}{2} - \frac{1}{8}\left[Prob_{ab}(coord|0,0) + Prob_{ca}(coord|0,0) - Prob_{ab}(coord|0,1)\right.\\ &- Prob_{ca}(coord|1,0) - Prob_{ab}(coord|1,0) - Prob_{ca}(coord|0,1)\\ &\left.+Prob_{ab}(coord|1,1) + Prob_{ca}(coord|1,1)\right] \qquad (4.26)\end{aligned}$$

To classify all the games, it is only necessary to analyse one example for each different type of payoff. That means, **choosing** specifically an **example** of the **two functions** $I, O$ **for each** of the **payoffs** in equations (4.23)-(4.26) since the results will also apply – up to possible permutations – to the other functions that share the same type of payoff. The **representative functions** chosen are: $I = f_7$ and $O = f_{15}$ (CHSH game) corresponding exactly, including the signs, to the payoff in equation (4.23); $I = f_8$ and $O = f_{15}$ corresponding exactly to the payoff in equation (4.24); $I = O = f_{15}$ corresponding to the payoff in equation (4.25); and $I = f_{15}$ and $O = f_{16}$ corresponding to the payoff in equation (4.26).

Furthermore, since the average payoff in equation (4.23) – corresponding to $I = f_7$ and $O = f_{15}$ – and the payoff in equation (4.24) – corresponding to $I = f_8$ and $O = f_{15}$ – sum to 1, it is enough to analyse only the payoff of $I = f_7$ and $O = f_{15}$. Similar argument applies to the payoffs in equations (4.25) and (4.26), leaving, for example $I = O = f_{15}$ as the only case to analyse. Then, in this case, **only 2 representative functions** are necessary.

### ◆ $\mathbf{O} \in \mathbf{S_{2u}} = \{\mathbf{f_7}, \mathbf{f_8}, \mathbf{f_9}, \mathbf{f_{10}}, \mathbf{f_{11}}, \mathbf{f_{12}}, \mathbf{f_{13}}, \mathbf{f_{14}}\}$

This situation presents some **subtleties** due to the mix of the two games. In the previous classification in equations (4.23)-(4.26), only the coordination probability $Prob(coord|\ \ )$ of Alice with Bob and Carl for the two games mattered . Now, it matters the **probability of certain outputs** $Prob(\mathbf{m}, \mathbf{n}|\ \ )$, where, remember, $m, n$ are the inputs for which the minority bit of $O$ is outputted – see the part of $O \in S_{2u}$ on page 37. In the present case of $O \in S_{2u}$, Alice's average payoff is of the form:

$$\begin{aligned}\$_A \sim \quad & + (-1)^{k_{00}}\left[Prob_{ab}(m, n|0, 0) + Prob_{ca}(m, n|0, 0)\right]\\ & + (-1)^{k_{01}}\left[Prob_{ab}(m, n|0, 1) + Prob_{ca}(m, n|0, 1)\right]\\ & + (-1)^{k_{10}}\left[Prob_{ab}(m, n|1, 0) + Prob_{ca}(m, n|1, 0)\right]\\ & + (-1)^{k_{11}}\left[Prob_{ab}(m, n|1, 1) + Prob_{ca}(m, n|1, 1)\right] \qquad (4.27)\end{aligned}$$

where $k_{ij}$ denotes the sign in front of the probability with inputs $i, j \in \{0, 1\}$, which depends on the specific choice of $O, I$; and the red colour highlights Alice's inputs and outputs for $x = 0$, and blue for $x = 1$. From the previous classification of the two-player games when $O \in S_{2u}$ in equations (4.19)-(4.22), either there is one negative sign in front of one of the probabilities, i.e. $(-,+,+,+)$ and its permutations[8] when $I, O \in S_{2u}$; or there are two negative signs of the form $(+,-,-,+)$ or $(-,+,+,-)$ when $I \in S_{2b}$ and $O \in S_{2u}$.

When $I, O \in S_{2u}$, there is a nuance between having, for example $(-,+,+,+)$ or $(+,-,+,+)$. Focusing only on Alice's payoff when $x = 0$, that is, the probabilities in equation (4.27) with red inputs and outputs, having $(-,+,+,+)$ implies two negative signs: $\$_A \sim -Prob_{ab}(m,n|0,0) - Prob_{ca}(m,n|0,0) + Prob_{ab}(m,n|0,1) + Prob_{ca}(m,n|1,0) + \dots$; whereas $(+,-,+,+)$ implies only one negative sign: $\$_A \sim +Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) - Prob_{ab}(m,n|0,1) + Prob_{ca}(m,n|1,0) + \dots$ . Clearly these two situations will yield to different results. Therefore, for $I, O \in S_{2u}$, there are four types of average payoff instead of only the two of the two-player classification in equations (4.19) and (4.20).

Here are all the **6 types of payoffs** when $O \in S_{2u}$:

- $I, O \in \{f_7, f_{13}\}$; or $I, O \in \{f_8, f_{14}\}$; $O \in \{f_9, f_{11}\}$ and $I \in \{f_7, f_{13}\}$; or $O \in \{f_{10}, f_{12}\}$ and $I \in \{f_8, f_{14}\}$; which up to moving the two negative signs of the probabilities with the same inputs[9] inside the brackets, the average payoff looks like:

$$\begin{aligned}\$_A =& \frac{3}{4} - \frac{1}{8}\left[Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) + Prob_{ab}(m,n|0,1)\right.\\ &+ Prob_{ca}(m,n|1,0) + Prob_{ab}(m,n|1,0) + Prob_{ca}(m,n|0,1)\\ &\left.- Prob_{ab}(m,n|1,1) - Prob_{ca}(m,n|1,1)\right]\end{aligned} \tag{4.28}$$

- $O \in \{f_7, f_{13}\}$ and $I \in \{f_8, f_{14}\}$; or $O \in \{f_8, f_{14}\}$ and $I \in \{f_7, f_{13}\}$; or $O \in \{f_{10}, f_{12}\}$ and $I \in \{f_7, f_{13}\}$; or $O \in \{f_9, f_{11}\}$ and $I \in \{f_8, f_{14}\}$; which, again, up to permutations of the two negative signs of the probabilities with the same inputs (see footnote [9]), the average payoff looks like:

$$\begin{aligned}\$_A =& \frac{1}{4} + \frac{1}{8}\left[Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) + Prob_{ab}(m,n|0,1)\right.\\ &+ Prob_{ca}(m,n|1,0) + Prob_{ab}(m,n|1,0) + Prob_{ca}(m,n|0,1)\\ &\left.- Prob_{ab}(m,n|1,1) - Prob_{ca}(m,n|1,1)\right]\end{aligned} \tag{4.29}$$

- $O \in \{f_7, f_{13}\}$ and $I \in \{f_9, f_{11}\}$; or $O \in \{f_8, f_{14}\}$ and $I \in \{f_{10}, f_{12}\}$; $I, O \in \{f_9, f_{11}\}$; or $I, O \in \{f_{10}, f_{12}\}$; which, up to permutations of the negative signs of the probabilities with different inputs[10] inside the brackets, the average payoff looks like:

$$\begin{aligned}\$_A =& \frac{3}{4} - \frac{1}{8}\left[Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) + Prob_{ab}(m,n|0,1)\right.\\ &- Prob_{ca}(m,n|1,0) - Prob_{ab}(m,n|1,0) + Prob_{ca}(m,n|0,1)\\ &\left.+ Prob_{ab}(m,n|1,1) + Prob_{ca}(m,n|1,1)\right]\end{aligned} \tag{4.30}$$

[8] The notation $(-,+,+,+)$ denotes a negative sign in front of the first two probabilities in (4.27), and positive for the rest, i.e. $k_{00} = 1$ and $k_{01} = k_{10} = k_{11} = 2$ .

[9] That means that the two negative signs go in front of the probabilities with $x = y = z$, that is, of $Prob_{ab}(m,n|0,0)$ and $Prob_{ca}(m,n|0,0)$; or in front of $Prob_{ab}(m,n|1,1)$ and $Prob_{ca}(m,n|1,1)$.

[10] That means that the negative signs go either in front of $Prob_{ab}(m,n|0,1)$ and $Prob_{ca}(m,n|0,1)$; or $Prob_{ab}(m,n|1,0)$ and $Prob_{ca}(m,n|1,0)$.

- $O \in \{f_8, f_{14}\}$ and $I \in \{f_9, f_{11}\}$; or $O \in \{f_7, f_{13}\}$ and $I \in \{f_{10}, f_{12}\}$; $O \in \{f_9, f_{11}\}$ and $I \in \{f_{10}, f_{12}\}$; or $O \in \{f_{10}, f_{12}\}$ and $I \in \{f_9, f_{11}\}$; which, again, up to permutations of the two negative signs of the probabilities with different inputs (see footnote [10]), the average payoff looks like:

$$\begin{aligned}\$_A =& \frac{1}{4} + \frac{1}{8}\left[Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) + Prob_{ab}(m,n|0,1)\right. \\ & - Prob_{ca}(m,n|1,0) - Prob_{ab}(m,n|1,0) + Prob_{ca}(m,n|0,1) \\ & \left. + Prob_{ab}(m,n|1,1) + Prob_{ca}(m,n|1,1)\right] \end{aligned} \tag{4.31}$$

- $O \in \{f_7, f_9, f_{11}, f_{13}\}$ and $I = f_{15}$; or $O \in \{f_8, f_{10}, f_{12}, f_{14}\}$ and $I = f_{16}$, the average payoff is exactly:

$$\begin{aligned}\$_A =& \frac{1}{2} - \frac{1}{8}\left[Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) - Prob_{ab}(m,n|0,1)\right. \\ & - Prob_{ca}(m,n|1,0) - Prob_{ab}(m,n|1,0) - Prob_{ca}(m,n|0,1) \\ & \left. + Prob_{ab}(m,n|1,1) + Prob_{ca}(m,n|1,1)\right] \end{aligned} \tag{4.32}$$

- $O \in \{f_8, f_{10}, f_{12}, f_{14}\}$ and $I = f_{15}$; or $O \in \{f_7, f_9, f_{11}, f_{13}\}$ and $I = f_{16}$, the average payoff is exactly:

$$\begin{aligned}\$_A =& \frac{1}{2} + \frac{1}{8}\left[Prob_{ab}(m,n|0,0) + Prob_{ca}(m,n|0,0) - Prob_{ab}(m,n|0,1)\right. \\ & - Prob_{ca}(m,n|1,0) - Prob_{ab}(m,n|1,0) - Prob_{ca}(m,n|0,1) \\ & \left. + Prob_{ab}(m,n|1,1) + Prob_{ca}(m,n|1,1)\right] \end{aligned} \tag{4.33}$$

As explained before, at the end of the previous analysis with $O \in S_{2b}$, it is only necessary to pick some representative functions for $I, O$ of each type of payoff. In the present case, it has been shown that there are 6 types of payoff. In fact, there are more than 6 because the payoffs are different when the output bits of each player are the same or not, that is, when $m = n$ and when $m \neq n$ (or $n = \overline{m}$). Focusing now on equation (4.28) and on the probabilities when $x = 0$, for instance, for $I = O = f_7$, that has $m = n = 1$, Alice's payoff goes, up to global signs, like $\sim Prob_{ab}(1,1|0,0) + Prob_{ca}(1,1|0,0) + Prob_{ab}(1,1|0,1) + Prob_{ca}(1,1|1,0) + \ldots$; meanwhile for $I = f_7$ , $O = f_9$, that has $m = 1$ and $n = 0$, Alice's payoff goes like $\sim Prob_{ab}(1,0|0,0) + Prob_{ca}(1,0|0,0) + Prob_{ab}(1,0|0,1) + Prob_{ca}(1,0|1,0) + \ldots$ . In the first case, Alice's output $a = 1$ serves for both games with Bob and Carl, whereas in the latter case, Alice has a "conflict" between her outputs, since for the game with Bob the important output is $a = 1$ and for Carl is $a = 0$. This reasoning explains why it is necessary to pick **12 representative functions** in total, 2 for each different payoff in equations (4.28)-(4.33).

The next section focuses on analysing all the different types of payoffs in this triangle-scenario by picking certain representatives functions $I, O$ when the players use classical strategies.

#### 4.1.3.1 Classical strategies

Analogously to what was explained in section 4.1.2.1 on page 39 for the classical strategies in the two-player games, the probabilities factorise and the **players'**

**strategies** will be denoted as $\mathbf{p_{a|x}}$, $\mathbf{q_{b|y}}$, and $\mathbf{r_{c|z}}$ for Alice, Bob, and Carl, respectively. Next, only Alice's payoff will be explicitly written for some chosen representative functions $I, O$ since Bob's and Carl's payoffs can be easily obtained with the permutations $(p_{a|x}\ , q_{b|y}\ , r_{c|z}) \rightarrow (q_{b|y}\ , r_{c|z}\ , p_{a|x})$ and $(p_{a|x}\ , q_{b|y}\ , r_{c|z}) \rightarrow (r_{c|z}\ , p_{a|x}\ , q_{b|y})$. As explained previously, some of the payoffs add up to 1, which reduces the number of payoffs to analyse. That is why, the **representative functions** of the chosen payoffs that will be **analysed** in detail will be highlighted **in bold**.

- $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_{15}}$ – CHSH game –, which corresponds to the payoff type in equation (4.23):

$$\begin{aligned}\$_A =& \frac{1}{4}\left[3 - (2p_{0|0} + q_{0|0} + r_{0|0}) + (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0})\right. \\ &\left. + (p_{0|0} - p_{0|1})(q_{0|1} + r_{0|1})\right]\end{aligned} \tag{4.34}$$

- $I = f_8$, $O = f_{15}$, which corresponds to the payoff type in equation (4.24):

$$\begin{aligned}\$_A =& \frac{1}{4}\left[1 + 2p_{0|0} + q_{0|0} + r_{0|0} - (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0})\right. \\ &\left. - (p_{0|0} - p_{0|1})(q_{0|1} + r_{0|1})\right]\end{aligned} \tag{4.35}$$

  The payoffs in equations (4.34) and (4.35) sum to 1, so only the case of $I = f_7$, $O = f_{15}$ will be fully analysed.

- $\boldsymbol{I = f_{15}}$, $\boldsymbol{O = f_{15}}$, which corresponds to the payoff type in equation (4.25):

$$\$_A = \frac{1}{4}\left[2 + (p_{0|0} - p_{0|1})(q_{0|0} + r_{0|0} - q_{0|1} - r_{0|1})\right] \tag{4.36}$$

- $I = f_{15}$, $O = f_{16}$, which corresponds to the payoff type in equation (4.26):

$$\$_A = \frac{1}{4}\left[2 - (p_{0|0} - p_{0|1})(q_{0|0} + r_{0|0} - q_{0|1} - r_{0|1})\right] \tag{4.37}$$

  The payoffs in equations (4.36) and (4.37) sum to 1, so only the case of $I = f_{15}$, $O = f_{15}$ will be fully analysed.

- $I = O = f_7$, which corresponds to the payoff type in equation (4.28) with $m = n = 1$:

$$\begin{aligned}\$_A =& \frac{1}{8}\left[2 + 2(2p_{0|0} + q_{0|0} + r_{0|0}) - (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0})\right. \\ &\left. - (p_{0|0} - p_{0|1})(q_{0|1} + r_{0|1})\right]\end{aligned} \tag{4.38}$$

- $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_8}$, which corresponds to the payoff type in equation (4.29) with $m = n = 1$:

$$\begin{aligned}\$_A =& \frac{1}{8}\left[6 - 2(2p_{0|0} + q_{0|0} + r_{0|0}) + (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0})\right. \\ &\left. + (p_{0|0} - p_{0|1})(q_{0|1} + r_{0|1})\right]\end{aligned} \tag{4.39}$$

  The payoffs in equations (4.38) and (4.39) sum to 1, so only the case of $I = f_7$, $O = f_8$ will be fully analysed.

- $I = f_7$, $O = f_9$, which corresponds to the payoff type in equation (4.28) with $m = 1$ and $n = 0$:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[6 - 2(p_{0|0} + q_{0|0}) + (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0}) \\ &+(p_{0|0} - p_{0|1})(q_{0|1} - r_{0|1})\big]\end{aligned} \tag{4.40}$$

- $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_{10}}$, which corresponds to the payoff type in equation (4.29) with $m = 1$ and $n = 0$:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[2 + 2(p_{0|0} + q_{0|0}) - (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0}) \\ &-(p_{0|0} - p_{0|1})(q_{0|1} + r_{0|1})\big]\end{aligned} \tag{4.41}$$

  The payoffs in equations (4.40) and (4.41) sum to 1, so only the case of $I = f_7$, $O = f_{10}$ will be fully analysed.

- $\boldsymbol{I = f_9}$, $\boldsymbol{O = f_7}$, which corresponds to the payoff type in equation (4.30) with $m = n = 1$:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[2 + 2(p_{0|0} + p_{0|1} + q_{0|1} + r_{0|0}) - (p_{0|0} + p_{0|1})(q_{0|1} + r_{0|0}) \\ &-(p_{0|0} - p_{0|1})(q_{0|0} - r_{0|1})\big]\end{aligned} \tag{4.42}$$

- $I = f_9$, $O = f_8$, which corresponds to the payoff type in equation (4.31) with $m = n = 1$:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[6 - 2(p_{0|0} + p_{0|1} + q_{0|1} + r_{0|0}) + (p_{0|0} + p_{0|1})(q_{0|1} + r_{0|0}) \\ &+(p_{0|0} - p_{0|1})(q_{0|0} - r_{0|1})\big]\end{aligned} \tag{4.43}$$

  The payoffs in equations (4.42) and (4.43) sum to 1, so only the case of $I = f_9$, $O = f_7$ will be fully analysed.

- $I = O = f_9$, which corresponds to the payoff type in equation (4.30) with $m = 1$ and $n = 0$:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[6 - 2(p_{0|1} + q_{0|1}) + (p_{0|0} + p_{0|1})(q_{0|1} + r_{0|0}) \\ &+(p_{0|0} - p_{0|1})(q_{0|0} - r_{0|1})\big]\end{aligned} \tag{4.44}$$

- $\boldsymbol{I = f_9}$, $\boldsymbol{O = f_{10}}$, which corresponds to the payoff type in equation (4.31) with $m = 1$ and $n = 0$:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[2 + 2(p_{0|1} + q_{0|1}) - (p_{0|0} + p_{0|1})(q_{0|1} + r_{0|0}) \\ &-(p_{0|0} - p_{0|1})(q_{0|0} - r_{0|1})\big]\end{aligned} \tag{4.45}$$

  The payoffs in equations (4.44) and (4.45) sum to 1, so only the case of $I = f_9$, $O = f_{10}$ will be fully analysed.

- $\boldsymbol{I = f_{15}}$, $\boldsymbol{O = f_7}$ ; $I = f_{15}$, $O = f_{10}$ , which correspond to the payoff types in equation (4.32) with $m = n = 1$, and in equation (4.33) with $m = 1$ $n = 0$, respectively:

$$\$_A = \frac{1}{8}\left[4 - (p_{0|0} - p_{0|1})(q_{0|0} + r_{0|0} - q_{0|1} - r_{0|1})\right] \tag{4.46}$$

- $I = f_{15}$, $O = f_8$ ; $I = f_{15}$, $O = f_9$, which correspond to the payoff types in equation (4.33) with $m = n = 1$, and in equation (4.32) with $m = 1$ $n = 0$, respectively:

$$\$_A = \frac{1}{8}\left[4 + (p_{0|0} - p_{0|1})(q_{0|0} + r_{0|0} - q_{0|1} - r_{0|1})\right] \tag{4.47}$$

  The payoffs in equations (4.46) and (4.47) sum to 1, so only the case of $I = f_{15}$, $O = f_7$ will be fully analysed.

In the present situation, the classification of the payoffs is not as straightforward as before with the two-player (cooperative) games that looked only at the maximum and minimum values of the winning probability – in Table 4.2 on page 40. Now, there are three players and their individual payoffs need to be taken into account. **Table 4.3** contains all the **possible payoffs** of Alice, Bob, and Carl, from the expressions in equations (4.34)-(4.47) depending on the choice of $I, O$ when the players use **pure strategies**, that is, when $p_{a|x}$ , $q_{b|xy}$ , $r_{c|z}$ are either 0 or 1, since the pure strategies give the bounds of such payoffs. The **7 representative functions** $I, O$ from the payoffs in equations (4.34), (4.36), (4.39), (4.41), (4.42), (4.45), (4.46) are also highlighted **in bold in the table**. If there is no highlighted function on that row, it means that the payoff sums to 1 with some other choices of functions. For example, on the second row the chosen functions would be $I = O = f_7$ but the payoff adds to 1 with that $I = f_7$ and $O = f_8$ on the sixth row. The payoffs in the table correspond to the **truly different payoffs up to permutations** of players, e.g. the payoff $(\$_A,\ \$_B,\ \$_C) = (0, 0.5, 0.5)$ on the first row represents also the payoffs $(\$_A,\ \$_B,\ \$_C) = (0.5, 0, 0.5)$ and $(\$_A,\ \$_B,\ \$_C) = (0.5, 0.5, 0)$ .

| $I$ , $O \in$ | $(\$_A, \$_B, \$_C)$ |
|---|---|
| $I = O = \boldsymbol{f_{15}}$ ; $I = O = f_{16}$ | $\{(0, 0.5, 0.5), (0.25, 0.25, 0.5), (0.5, 0.5, 0.5), (0.5, 0.75, 0.75), (1, 1, 1)\}$ |
| $I, O \in \{f_7, f_{13}\}$ ; $I, O \in \{f_8, f_{14}\}$ | $\{(0.25, 0.25, 0.25), (0.25, 0.375, 0.375), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75), (0.5, 0.75, 0.75), (0.625, 0.625, 0.75), (0.75, 0.75, 0.75), (0.75, 0.875, 0.875), (1, 1, 1)\}$ |
| $I, O \in \{f_9, f_{11}\}$; $I, O \in \{f_{10}, f_{12}\}$ | $\{(0.25, 0.5, 0.5), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75), (0.5, 0.75, 0.75), (0.625, 0.625, 0.75), (0.75, 0.75, 0.75), (0.75, 0.875, 0.875), (1, 1, 1)\}$ |
| $I = \boldsymbol{f_{15}}$ , $O = \boldsymbol{f_{16}}$ ; $I = f_{16}$ , $O = f_{15}$ | $\{(0, 0, 0), (0.25, 0.25, 0.5), (0.5, 0.5, 0.5), (0.5, 0.75, 0.75), (0.5, 0.5, 1)\}$ |
| $I \in \{\boldsymbol{f_9}, f_{11}\}$ , $O \in \{\boldsymbol{f_{10}}, f_{12}\}$ ; $I \in \{f_{10}, f_{12}\}$ , $O \in \{f_9, f_{11}\}$ | $\{(0, 0, 0), (0.125, 0.125, 0.25), (0.25, 0.25, 0.25), (0.25, 0.25, 0.5), (0.25, 0.375, 0.375), (0.25, 0.5, 0.5), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75)\}$ |
| $I \in \{\boldsymbol{f_7}, f_{13}\}$ , $O \in \{\boldsymbol{f_8}, f_{14}\}$ ; $I \in \{f_8, f_{14}\}$ , $O \in \{f_7, f_{13}\}$ | $\{(0, 0, 0), (0.125, 0.125, 0.25), (0.25, 0.25, 0.25), (0.25, 0.25, 0.5), (0.25, 0.375, 0.375), (0.25, 0.5, 0.5), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.625, 0.625, 0.75), (0.75, 0.75, 0.75)\}$ |
| $I \in S_{2u} = \{\boldsymbol{f_7}, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}\}$ , $O \in S_{2b} = \{\boldsymbol{f_{15}}, f_{16}\}$ | $\{(0.25, 0.25, 0.25), (0.25, 0.5, 0.5), (0.5, 0.5, 0.75), (0.75, 0.75, 0.75)\}$ |
| $I \in \{\boldsymbol{f_7}, f_{13}\}$ , $O \in \{\boldsymbol{f_{10}}, f_{12}\}$ ; $I \in \{f_8, f_{14}\}$ , $O \in \{f_9, f_{11}\}$ | $\{(0.125, 0.375, 0.5), (0.25, 0.25, 0.25), (0.25, 0.25, 0.5), (0.25, 0.375, 0.375), (0.25, 0.5, 0.5), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5)\}$ |
| $I \in \{f_7, f_{13}\}$ , $O \in \{f_9, f_{11}\}$ ; $I \in \{f_8, f_{14}\}$ , $O \in \{f_{10}, f_{12}\}$ | $\{(0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75), (0.5, 0.625, 0.875), (0.5, 0.75, 0.75), (0.625, 0.625, 0.75), (0.75, 0.75, 0.75)\}$ |
| $I \in \{\boldsymbol{f_9}, f_{11}\}$ , $O \in \{\boldsymbol{f_7}, f_{13}\}$ ; $I \in \{f_{10}, f_{12}\}$ , $O \in \{f_8, f_{14}\}$ | $\{(0.25, 0.25, 0.25), (0.25, 0.5, 0.5), (0.25, 0.625, 0.625), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75), (0.5, 0.75, 0.75), (0.625, 0.625, 0.75), (0.75, 0.75, 0.75), (0.75, 0.875, 0.875)\}$ |
| $I \in \{f_9, f_{11}\}$ , $O \in \{f_8, f_{14}\}$ ; $I \in \{f_{10}, f_{12}\}$ , $O \in \{f_7, f_{13}\}$ | $\{(0.125, 0.125, 0.25), (0.25, 0.25, 0.25), (0.25, 0.25, 0.5), (0.25, 0.375, 0.375), (0.25, 0.5, 0.5), (0.375, 0.375, 0.75), (0.375, 0.5, 0.625), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75), (0.75, 0.75, 0.75)\}$ |
| $I = \boldsymbol{f_{15}}$ , $O \in \{\boldsymbol{f_7}, f_{10}, f_{12}, f_{13}\}$ ; $I = f_{16}$ , $O \in \{f_8, f_9, f_{11}, f_{14}\}$ | $\{(0.25, 0.25, 0.25), (0.375, 0.375, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.75), (0.5, 0.625, 0.625)\}$ |
| $I = f_{15}$ , $O \in \{f_8, f_9, f_{11}, f_{14}\}$ ; $I = f_{16}$ , $O \in \{f_7, f_{10}, f_{12}, f_{13}\}$ | $\{(0.25, 0.5, 0.5), (0.375, 0.375, 0.5), (0.5, 0.5, 0.5), (0.5, 0.625, 0.625), (0.75, 0.75, 0.75)\}$ |

Tab. 4.3: *Table with the classification of the games in a triangle depending on the choice of $I$ and $O$ along with the possible combinations of payoffs for all three players (up to permutations) using classical pure strategies. The functions in bold denote the representative functions that will be analysed further. The definition of each $f_i$ is found in Table 4.1 on page 34.*

## 4.2 Nash Equilibrium for boolean games with 3 players in a triangle

Table 4.3 contains all the possible types of payoffs when players use (classical) pure strategies, but these are not the solutions of the game/s. As mentioned in section 1.2 in chapter 1 on page 5, one of the most important solution concepts for game theory is that of Nash equilibrium. A **Nash equilibrium is a configuration of strategies such that no player wants to unilaterally deviate from it given that the other players do not deviate**. In mathematical terms, a strategy set $s^* = \{s_1^*, s_2^*, ..., s_n^*\}$ is a Nash equilibrium if:

$$\$_i(s_i^*, s_{-i}^*) \geqslant \$_i(s_i^{'}, s_{-i}^*) \quad \forall i \tag{4.48}$$

where $\$_i$ and $s_i$ denote player's $i$ payoff and strategy, respectively; and $s_{-i}$ denotes the strategies of all the players except player $i$'s.

Then, for the present situation with three players using mixed strategies, a Nash equilibrium point is a set of strategies $s^* = \{p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*\}$ that satisfies:

$$\$_A(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \geqslant \$_A(p_{0|0}^{'}, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \tag{4.49}$$

$$\$_A(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \geqslant \$_A(p_{0|0}^*, p_{0|1}^{'}, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \tag{4.50}$$

$$\$_B(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \geqslant \$_B(p_{0|0}^*, p_{0|1}^*, q_{0|0}^{'}, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \tag{4.51}$$

$$\$_B(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \geqslant \$_B(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^{'}, r_{0|0}^*, r_{0|1}^*) \tag{4.52}$$

$$\$_C(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \geqslant \$_C(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^{'}, r_{0|1}^*) \tag{4.53}$$

$$\$_C(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^*) \geqslant \$_C(p_{0|0}^*, p_{0|1}^*, q_{0|0}^*, q_{0|1}^*, r_{0|0}^*, r_{0|1}^{'}) \tag{4.54}$$

where $\$_A$, $\$_B$, and $\$_C$ are Alice's, Bob's and Carl's payoffs and the strategies with a dash denote other strategies different from the Nash equilibrium ones.

Finding such equilibrium points means finding **optimal solutions** for all players. That means that Alice wants her strategy set $\{p_{0|0}, p_{0|1}\}$ to be the one that maximises (optimises) her payoff given that Bob and Carl's strategies $\{q_{0|0}, q_{0|1}, r_{0|0}, r_{0|1}\}$ are fixed. Similarly for Bob and Carl. Thus the problem of finding the **Nash equilibria** can be stated as a typical **optimisation problem**:

$$\begin{aligned} &\underset{p_{0|0},p_{0|1}}{\text{maximise}} && \$_A(p_{0|0}, p_{0|1}) \\ &\text{subject to} && 0 \leqslant p_{0|0} \leqslant 1, \;\; 0 \leqslant p_{0|1} \leqslant 1 \end{aligned} \tag{4.55}$$

$$\begin{aligned} &\underset{q_{0|0},q_{0|1}}{\text{maximise}} && \$_B(q_{0|0}, q_{0|1}) \\ &\text{subject to} && 0 \leqslant q_{0|0} \leqslant 1, \;\; 0 \leqslant q_{0|1} \leqslant 1 \end{aligned} \tag{4.56}$$

$$\begin{aligned} &\underset{r_{0|0},r_{0|1}}{\text{maximise}} && \$_C(r_{0|0}, r_{0|1}) \\ &\text{subject to} && 0 \leqslant r_{0|0} \leqslant 1, \;\; 0 \leqslant r_{0|1} \leqslant 1 \end{aligned} \tag{4.57}$$

where only the variables controlled by each player are explicitly written in the dependence of their own payoffs, since the other players' variables are supposed to be fixed. To be clearer, for instance, the expression for Alice's payoff depends on all variables $\$_A(p_{0|0}, p_{0|1}, q_{0|0}, q_{0|1}, r_{0|0}, r_{0|1})$, but she can control only $p_{0|0}, p_{0|1}$ , so for her optimisation problem $q_{0|0}, q_{0|1}, r_{0|0}, r_{0|1}$ act as just parameters; that is why Bob's and Carl's variables are omitted in Alice's optimisation problem in (4.55).

Finding the Nash equilibrium points for a certain game might be hard in general, since the difficulty is related to the type of objective functions (players' payoffs) to be optimised and the restrictions on the strategy set, e.g. linear, convex, non-convex, etc. In the cases considered here with the different payoffs for the different boolean games with 3 players in a triangle, finding the Nash equilibrium points is relatively easy to solve since it is a convex optimisation problem. Section B.1 in appendix B contains more about the basics of convex optimisation. In the same appendix, section B.2 contains the necessary generic equations to solve in order to find the Nash equilibrium points. Subsection B.2.1 has the concrete equations to solve to find the Nash equilibrium points for the 7 chosen representative functions from above. As for the other 7 possible choices of functions, which are not marked in bold on pages 45-47, the payoffs of these sum to 1 with one of the chosen representative functions. This means that the maximisation problem – to find the Nash equilibrium points – of these 7 non-chosen functions can be written as a minimisation problem of the chosen representative functions. To be more explicit, for example, Alice's payoff in the games defined by $I = f_{15}$, $O = f_7$; and $I = f_{15}$, $O = f_8$ in equations (4.46) and (4.47), respectively, sum to 1, i.e. $\$_A^{f_{15}-f_7} + \$_A^{f_{15}-f_8} = 1$, which means that the maximisation problem for Alice for the game $I = f_{15}$, $O = f_8$ can be written as a minimisation problem of the game $I = f_{15}$, $O = f_7$. In mathematical terms, $\max\left(\$_A^{f_{15}-f_8}\right) = \max\left(1 - \$_A^{f_{15}-f_7}\right) = \min\left(-1 + \$_A^{f_{15}-f_7}\right) = -1 + \min\left(\$_A^{f_{15}-f_7}\right)$. Analogous relations hold for Bob and Carl. Therefore the payoffs of the 7 chosen representative functions contain the necessary information to solve all the 14 different (boolean) games in this triangle-like configuration. For the sake of space, the explicit solutions will only be given for the 7 representative functions; the solutions of the other games can be directly obtained using the payoffs of the representative ones by changing the problem to a minimisation one.

One game might have many Nash equilibrium points that give different payoffs to the players. One way of deciding which of these equilibrium points would be the best is by adding all the players' payoffs $\$_A + \$_B + \$_C$ to see which one/s gives the highest value. Aggregating the players' payoffs at an equilibrium point is referred to as the **social welfare of that equilibrium point**[11].

**Tables 4.4-4.10** have all the **Nash equilibrium** points for these choices of the **7 representative functions** for the games. For simplicity, the tables only contain the truly different Nash equilibrium points up to permutation of players. This means that if a point on these tables is not invariant under player permutations, it should be kept in mind that it is also represented there. The solutions in bold denote the ones that give the highest social welfare, i.e. the ones that give the highest value of

[11]The term *social welfare* in the present dissertation will be used interchangeably with $\$_A + \$_B + \$_C$. However, in the context of economics and social sciences, the *social welfare function* is a map between the individual preference relations (an ordering/ranking of a set) of a group of players into a unique preference relation, with the aim of faithfully representing the preferences of the whole group. See the chapter on social choice in the book in [89] for further details.

$\$_A + \$_B + \$_C$. Coming back to the Nash equilibrium points, to prove, for instance, that the first solution $s^* = \{1, p_{0|1}, 1, 1, 1, 1\}$ in Table 4.4 for $I = f_7$, $O = f_{15}$ (CHSH game) is indeed a Nash equilibrium point, it is necessary to check that the conditions in equations (4.49)-(4.54) hold:

$$\$_A(1, p_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \geqslant \$_A(p'_{0|0}, p_{0|1}, 1, 1, 1, 1) = \frac{1 + 2p'_{0|0}}{4} \tag{4.58}$$

$$\$_A(1, p_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \geqslant \$_A(1, p'_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \tag{4.59}$$

$$\$_B(1, p_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \geqslant \$_B(1, p_{0|1}, q'_{0|0}, 1, 1, 1) = \frac{2 - p_{0|1} + q'_{0|0}(1 + p_{0|1})}{4} \tag{4.60}$$

$$\$_B(1, p_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \geqslant \$_B(1, p_{0|1}, 1, q'_{0|1}, 1, 1) = \frac{2 - p_{0|1} + q'_{0|1}(1 - p_{0|1})}{4} \tag{4.61}$$

$$\$_C(1, p_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \geqslant \$_C(1, p_{0|1}, 1, 1, r'_{0|0}, 1) = \frac{2 - p_{0|1} + r'_{0|0}(1 + p_{0|1})}{4} \tag{4.62}$$

$$\$_C(1, p_{0|1}, 1, 1, 1, 1) = \frac{3}{4} \geqslant \$_C(1, p_{0|1}, 1, 1, 1, r'_{0|1}) = \frac{2 - p_{0|1} + r'_{0|0}(1 - p_{0|1})}{4} \tag{4.63}$$

These equations are clearly satisfied since $0 \leqslant p'_{0|0}, q'_{0|0}, r'_{0|0}, p_{0|0} \leqslant 1$.

| games $I = f_7$ , $O = f_{15}$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\mathbf{\{1, 1, 1, 1, 1, r_{0\|1}\}}$ <br> $\mathbf{\{0, 0, 0, 0, 0, r_{0\|1}\}}$ | $\mathbf{\frac{3}{4}}$ | $\mathbf{\frac{3}{4}}$ | $\mathbf{\frac{3}{4}}$ |
| $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\{1, 0, 0, 1, 1 - r_{0\|1}, r_{0\|1}\}$ | $\frac{2+r_{0\|1}}{4}$ | $\frac{3-r_{0\|1}}{4}$ | $\frac{1}{2}$ |
| $\{0, 1, 1, 0, 1 - r_{0\|1}, r_{0\|1}\}$ | $\frac{3-r_{0\|1}}{4}$ | $\frac{2+r_{0\|1}}{4}$ | $\frac{1}{2}$ |

Tab. 4.4: *Nash Equilibrium solutions for the game when $I = f_7$, $O = f_{15}$ (CHSH game), game represented by Alice's average payoff in equation (4.34) on page 45. The bold solutions give the highest social welfare* $\$_A + \$_B + \$_C = 9/4 = 2.25$ .

| games $I = f_{15}$ , $O = f_{15}$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\mathbf{\{1, 0, 1, 0, 1, 0\}}$ <br> $\mathbf{\{0, 1, 0, 1, 0, 1\}}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $\{p_{0\|0}, p_{0\|0}, q_{0\|0}, q_{0\|0}, r_{0\|0}, r_{0\|0}\}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Tab. 4.5: *Nash Equilibrium solutions for the game when $I = O = f_{15}$, game represented by Alice's average payoff in equation (4.36) on page 45. The bold solutions give the highest social welfare* $\$_A + \$_B + \$_C = 3$ .

| games $I = f_7$ , $O = f_8$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\{\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{r_{0\|1}}\}$ | $\mathbf{\frac{3}{4}}$ | $\mathbf{\frac{3}{4}}$ | $\mathbf{\frac{3}{4}}$ |
| $\{1, 1, 1, 1, 1, 1\}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

Tab. 4.6: *Nash Equilibrium solutions for the game when* $I = f_7$*,* $O = f_8$*, game represented by Alice's average payoff in equation (4.39) on page 45. The bold solution gives the highest social welfare* $\$_A + \$_B + \$_C = 9/4 = 2.25$ *.*

| games $I = f_7$ , $O = f_{10}$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\{1, 1, 0, 0, r_{0\|0}, r_{0\|0}\}$ | $\frac{2-r_{0\|0}}{4}$ | $\frac{1+r_{0\|0}}{4}$ | $\frac{1}{2}$ |
| $\{\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{r_{0\|0}}, \mathbf{0}\}$ | $\mathbf{\frac{5-r_{0\|0}}{8}}$ | $\mathbf{\frac{3+r_{0\|0}}{8}}$ | $\mathbf{\frac{1}{2}}$ |
| $\{\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{r_{0\|0}}, \mathbf{1}\}$ | $\mathbf{\frac{4-r_{0\|0}}{8}}$ | $\mathbf{\frac{4+r_{0\|0}}{8}}$ | $\mathbf{\frac{1}{2}}$ |
| $\{0, 0, 1, 1, r_{0\|0}, r_{0\|0}\}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ |

Tab. 4.7: *Nash Equilibrium solutions for the game when* $I = f_7$*,* $O = f_{10}$*, game represented by Alice's average payoff in equation (4.41) on page 46. The bold solutions give the highest social welfare* $\$_A + \$_B + \$_C = 3/2 = 1.5$ *.*

| games $I = f_9$ , $O = f_7$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\{\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}\}$ | $\mathbf{\frac{7}{8}}$ | $\mathbf{\frac{7}{8}}$ | $\mathbf{\frac{3}{4}}$ |
| $\{1, 1, 1, 1, r_{0\|0}, r_{0\|0}\}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ |

Tab. 4.8: *Nash Equilibrium solutions for the game when* $I = f_9$*,* $O = f_7$*, game represented by Alice's average payoff in equation (4.42) on page 46. The bold solution gives the highest social welfare* $\$_A + \$_B + \$_C = 5/2 = 2.5$ *.*

| games $I = f_9$ , $O = f_{10}$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\mathbf{\{1, 1, 0, 0, 0, 1\}}$ | $\mathbf{\frac{1}{2}}$ | $\mathbf{\frac{1}{2}}$ | $\mathbf{\frac{3}{4}}$ |
| $\{0, 1, 0, 1, r_{0\|0}, 1 - r_{0\|0}\}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Tab. 4.9: *Nash Equilibrium solutions for the game when* $I = f_9$*,* $O = f_{10}$*, game represented by Alice's average payoff in equation (4.45) on page 46. The bold solution gives the highest social welfare* $\$_A + \$_B + \$_C = 7/4 = 1.75$ *.*

| games $I = f_{15}$ , $O = f_7$ | | | |
|---|---|---|---|
| $s^* = \{p^*_{0\|0}, p^*_{0\|1}, q^*_{0\|0}, q^*_{0\|1}, r^*_{0\|0}, r^*_{0\|1}\}$ | $\$_A$ | $\$_B$ | $\$_C$ |
| $\mathbf{\{1, 0, 0, 1, r_{0\|0}, r_{0\|1}\}}$ | $\mathbf{\frac{5-r_{0\|0}+r_{0\|1}}{8}}$ | $\mathbf{\frac{5+r_{0\|0}-r_{0\|1}}{8}}$ | $\mathbf{\frac{1}{2}}$ |
| $\mathbf{\{0, 1, 1, 0, r_{0\|0}, r_{0\|1}\}}$ | $\mathbf{\frac{5+r_{0\|0}-r_{0\|1}}{8}}$ | $\mathbf{\frac{5-r_{0\|0}+r_{0\|1}}{8}}$ | $\mathbf{\frac{1}{2}}$ |
| $\{p_{0\|0}, p_{0\|0}, q_{0\|0}, q_{0\|0}, r_{0\|0}, r_{0\|0}\}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

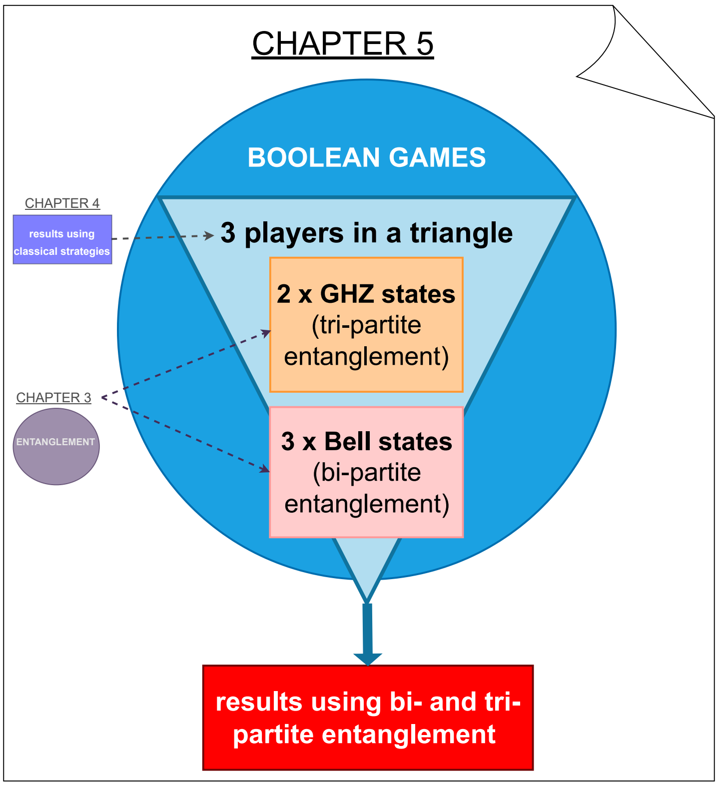
Tab. 4.10: *Nash Equilibrium solutions for the game when* $I = f_{15}$*,* $O = f_7$*, game represented by Alice's average payoff in equation (4.46) on page 47. The bold solutions give the highest social welfare* $\$_A + \$_B + \$_C = 7/4 = 1.75$ *.*

## Summary of the chapter

This chapter, chapter 4, has focused on classifying and analysing all the different boolean games played on the explained triangle configuration when the players use mixed strategies. Chapter 5 focuses on analysing this same situation when the players make use of quantum resources and how that changes the Nash equilibrium points in comparison to the classical case.

# Chapter 5

# Bi- and tri-partite entanglement for boolean games with 3 players in a triangle




This chapter focuses on the results for games played in a triangle when the players use quantum resources, specifically, when the players are provided with two tri-partite quantum states or three bi-partite states.

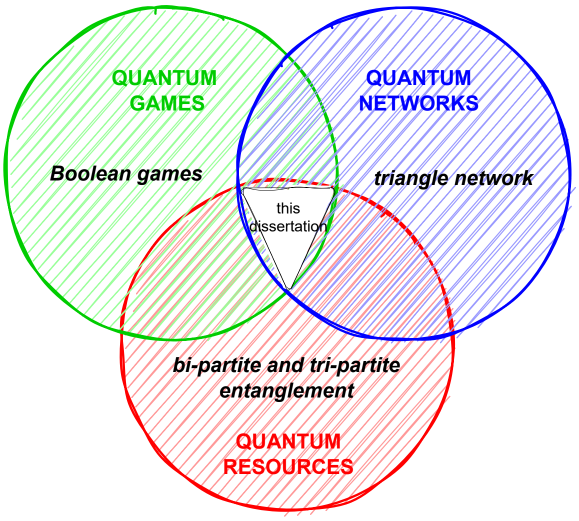
As mentioned at the beginning of chapter 4, the **triangle network** has been widely explored as a simple example of a quantum network, where quantum resources – typically, quantum states and local measurements – are distributed among the players/nodes[1]. The triangle network might seem very simple, but it exhibits many interesting features, which is why it has been extensively studied recently, in terms of non-locality [91–93], but also in terms of entanglement [94–

[1]For a comprehensive review on real-world quantum networks, see [90].

96]. Investigating how quantum resources perform when distributed in a network is crucial for the development of the quantum internet – see [97] for a current review.

Typically the resources distributed in the triangle network feature bi-partite states – see, for example [91, 92, 95]. The reason behind this is that the current technology is capable of producing and distributing such quantum states to spatially-separated parties (using photons) with relative ease. Experimentally producing and distributing tri-partite states is much more challenging, even though there has been some positive results lately [98]. This positive result shows that quantum technology is advancing towards the realisation and physical implementation of quantum networks.

In terms of quantum games and the triangle network, to the best of the author's knowledge, no comparison between bi-partite and tri-partite scenarios with the same number of qubits has been investigated. On another note, the use of quantum resources might also prove to be a better option for certain games in which there are conflicting interests [99, 100], in contrast to the CHSH game, which is a cooperative game. Those results also motivated the author to conduct the research presented in this dissertation by potentially including a conflict of interest between the different games played by each player in that triangle-like fashion. The next diagram shows the **author's inspiration** and motivation from different fields:




Having motivated the topic of focus of this dissertation, the **research questions** for the boolean games played in a triangle using bi-partite and tri-partite entanglement are:

1) *What are the (new) Nash equilibrium points and how do they compare to the classical ones using mixed strategies*[2]*?*

2) *Do the players prefer to use tri-partite or bi-partite (quantum) resources?*

[2]Remember that the (classical) results in the previous chapter considered only mixed strategies and not strategies coming from some (classical) piece of advice, so the comparison between the quantum results in this chapter and the classical ones from chapter 4 need to be taken with some caution. That being said, as will be seen next in the results, for certain quantum state/s the (quantum) Nash equilibrium points correspond exactly to the equilibrium points using mixed strategies.

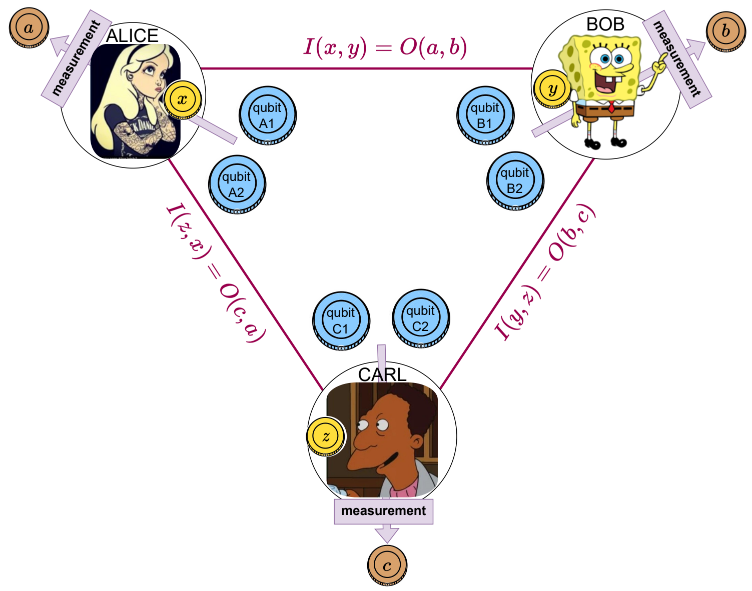



Fig. 5.1: *Quantum resources added to the boolean games in a triangle network. Each player owns two qubits, represented by the blue coins in the middle.*

3) *Which Nash equilibrium solutions give the highest social welfare for the tri-partite and bi-partite case?*

## 5.1 Quantum resources for games in a triangle network

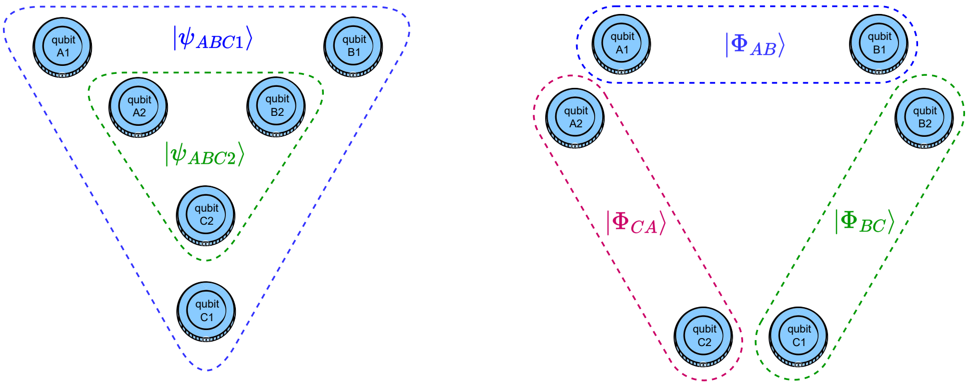
As with the CHSH game explained in chapter 2, in addition to the input and output coins, the players will also have access to qubits; in this case, to *two qubits each*, as represented in Figure 5.1. This means that the players will be using a **6-qubit quantum state** $|\Psi_{ABC}\rangle$ to decide on their output bits $a, b, c \in \{0, 1\}$. These outputs are obtained after the players perform a local measurement on their own two qubits. As in the CHSH game, the use of a quantum state acts as a piece of advice[3].

The 6-qubit quantum state $|\Psi_{ABC}\rangle$ considered for the games will be produced by two different types of sources:

- Two tri-partite sources, i.e. $|\Psi_{ABC}\rangle = |\psi_{ABC1}\rangle \otimes |\psi_{ABC2}\rangle$, as illustrated in Figure 5.2(a).
- Three bi-partite sources, i.e. $|\Psi_{ABC}\rangle = |\phi_{AB}\rangle \otimes |\phi_{BC}\rangle \otimes |\phi_{CA}\rangle$, as illustrated in Figure 5.2(b).

In general, the states produced by each source might be different, but the present dissertation considers that the **sources** are **identical**, that is, they produce the same state. The state considered to be produced by the tri-partite source will be a **GHZ-like state** – introduced in section 3.3.2 of chapter 3 –, and for the bi-partite source

[3]Since the players are using some (quantum) advice to play the game/s, it is a slightly different situation to playing without advice, which was the situation analysed in the previous chapter using mixed classical strategies. More on that later.




(a) *Two tri-partite sources.* (b) *Three bi-partite sources.*

Fig. 5.2: *Two ways of producing the 6-qubit state* $|\Psi_{ABC}\rangle$ *that will be used for the games. The dashed lines denote that those qubits were produced by a common source.*

a **Bell-like state** – introduced in section 2.2.2 of chapter 2. In mathematical terms, the two quantum states to be considered are:

$$|\Psi_{ABC}\rangle = \left(\sqrt{1-\lambda_{111}^2}\,|000\rangle + \lambda_{111}\,|111\rangle\right)^{\otimes 2} \tag{5.1}$$

$$|\Psi_{ABC}\rangle = \left(\sqrt{1-\lambda_{11}^2}\,|00\rangle + \lambda_{11}\,|11\rangle\right)^{\otimes 3} \tag{5.2}$$

where $\otimes$ denotes the tensor product; and $0 \leqslant \lambda_{111} \leqslant 1$ and $0 \leqslant \lambda_{11} \leqslant 1$. The two **entanglement parameters** $\lambda_{111}$ and $\lambda_{11}$ control the amount of entanglement in each state. The (pure) GHZ state and the (pure) Bell state correspond to choosing $\lambda_{111} = 1/\sqrt{2}$ and $\lambda_{11} = 1/\sqrt{2}$, respectively.

As mentioned before, the players will perform a **local measurement** on their own two qubits to decide on their output bit. Since they only have two options for their outputs, they will use two local projection operators that add to the identity. Focusing on Alice, given input $x$, she has a set of two projection-valued measure[4] $\{\Pi_{x,a}\}_{x,a\in\{0,1\}}$. When Alice receives $x = 0$ the projectors are:

$$\Pi_{0,0} = |a_0\rangle\langle a_0| \tag{5.3}$$

$$\Pi_{0,1} = \mathbb{I} - \Pi_{0,0} \tag{5.4}$$

and the two-qubit state for the first projector is:

$$|a_0\rangle = \sqrt{1-a_{11}^2}\,|00\rangle + a_{11}\,|11\rangle \tag{5.5}$$

with $0 \leqslant a_{11} \leqslant 1$. The physical meaning of the these equations is summarised as follows: Alice measures her two qubits and if the qubits are in the state $|a_0\rangle$ in equation (5.5) then she outputs $a = 0$, otherwise she outputs $a = 1$. A similar argument follows for input $x = 1$, but now, the projection operators are different:

$$\Pi_{1,0} = |\tilde{a}_0\rangle\langle \tilde{a}_0| \tag{5.6}$$

[4]Also called PVMs, see the postulates of quantum mechanics in chapter 0.

$$\Pi_{1,1} = \mathbb{I} - \Pi_{1,0} \tag{5.7}$$

with:

$$|\tilde{a}_0\rangle = \sqrt{1 - \tilde{a}_{11}^2}\,|00\rangle + \tilde{a}_{11}\,|11\rangle \tag{5.8}$$

with $0 \leqslant \tilde{a}_{11} \leqslant 1$.

The real **parameters** $a_{11}, \tilde{a}_{11}$ represent Alice's (**quantum**) **strategy** for each input. The situation is analogous for Bob with measurement operators $\{\Pi_{y,b}\}$ and parameters $b_{11}, \tilde{b}_{11}$; and for Carl with $\{\Pi_{z,c}\}$ and $c_{11}, \tilde{c}_{11}$. The choice of $\Pi_{0,0}$ and $\Pi_{1,0}$ in equations (5.3) and (5.6) seems limiting and it is indeed. Firstly, the quantum states $|a_0\rangle$ and $|\tilde{a}_0\rangle$ in equations (5.5) and (5.8) that are projected onto are restricted; a general 2-qubit quantum state to project onto would be $|\chi\rangle = \chi_{00}\,|00\rangle + \chi_{01}\,|01\rangle + \chi_{10}\,|10\rangle + \chi_{11}\,|11\rangle$ with $|\chi_{00}|^2 + |\chi_{01}|^2 + |\chi_{10}|^2 + |\chi_{11}|^2 = 1$. Even if such a generic state $|\chi\rangle$ was considered, the set of measurements operators $\Pi_{x,0} = |\chi_x\rangle\langle\chi_x|$ and $\Pi_{x,1} = \mathbb{I} - \Pi_{x,0}$ would still be a projection-valued measure (PVM). Using PVMs is still restricting since the most general situation would be using a set of positive operator-valued measure (POVM) – positive semi-definite Hermitian operators that add up to the identity. Being that general in both cases would increase significantly the total number of parameters – in this case, strategies – to manage because having $k$ parameters to define $\Pi_{x,0}$ translates into dealing with payoffs that would depend on $6k$ parameters (three players and each with two inputs). Simplicity was then one of the reasons behind choosing the measurement operators $\Pi_{x,0}$ to depend, as a first step, on only one parameter, as defined in equations (5.3)-(5.8).

As a simplifying statement, the quantum resources are just an elaborate way to compute the **conditional probability** as:

$$Prob(a, b, c|x, y, z) = \langle\Psi_{ABC}|\,\Pi_{x,a} \otimes \Pi_{y,b} \otimes \Pi_{z,c}\,|\Psi_{ABC}\rangle \tag{5.9}$$

where the state $|\Psi_{ABC}\rangle$ will be either the GHZ-like state in equation (5.1) or the Bell-like state in equation (5.2); and the projectors are defined in equations (5.3)-(5.4) and (5.6)-(5.7) with their corresponding projection states in equations (5.5) and (5.8). Then, this **conditional probability** will be a **function** of: the players' **strategies** $a_{11}, \tilde{a}_{11}, b_{11}, \tilde{b}_{11}, c_{11}, \tilde{c}_{11}$; and the **entanglement parameter** of the chosen state ($\lambda_{111}$ or $\lambda_{11}$). The expression for this conditional probability in equation (5.9) will be used to compute the two-player conditional probabilities $Prob(a, b|x, y)$ , $Prob(b, c|y, z)$ , and $Prob(a, c|x, z)$ , which in turn, these two-player conditional probabilities are used to obtain the winning probability of each game, as shown in equation (4.7) in chapter 4 on page 33. Finally, the players' average payoffs of both games $\$_A$, $\$_B$, and $\$_C$ can be computed from the winning probabilities of each game.

To **summarise** this new (quantum) situation:

0) The players own two qubits of the 6-qubit state $|\Psi_{ABC}\rangle$, which is either a GHZ-like – in equation (5.1) – or a Bell-like state – in equation (5.2).

1) The players receive their binary inputs $x, y, z$ ; drawn from an equal probability distribution.

2a) For Alice: if $x = 0$ she measures her two qubits over the state $|a_0\rangle$, and over the state $|\tilde{a}_0\rangle$ for $x = 1$, measurement process that gives the probability of her outputting $a = 0$ for that given input. Otherwise, she outputs $a = 1$.

2b) The players' strategies are their choice of the states to project over, e.g. for Alice $|a_0\rangle$ in equation (5.5) and $|\tilde{a}_0\rangle$ in equation (5.8), which depend on the parameters $a_{11}$ and $\tilde{a}_{11}$, taking values between 0 and 1.

3a) This process of measurement for all players gives $Prob(a, b, c|x, y, z)$ in equation (5.9), which is then used to obtain the two-player probabilities for each game $Prob(a, b|x, y)$ , $Prob(b, c|y, z)$ , and $Prob(a, c|x, z)$.

3b) These two-player probabilities are used to compute the winning probability for each game with the concrete winning conditions determined by the input and output functions $I$ and $O$.

4a) The players' payoff is the average between the winning probabilities of the two games played. For instance, for Alice, who plays GAME 1 and GAME 3, it is $\$_A = 1/2\,[Prob(\text{win G1}) + Prob(\text{win G3})]$.

4b) The payoffs will depend explicitly on the players' strategy choices, i.e. $a_{11}, \tilde{a}_{11}, b_{11}, \tilde{b}_{11}, c_{11}, \tilde{c}_{11}$ but also on the entanglement parameter of the quantum state $|\Psi_{ABC}\rangle$: either $\lambda_{111}$ for the sources producing identical GHZ-like states, or $\lambda_{11}$ for the Bell-like states. All of these parameters are real numbers between 0 and 1.

Once the players' payoffs $\$_A$, $\$_B$, and $\$_C$ are obtained for this quantum situation, the Nash equilibrium points can be found by optimising the payoffs over the strategies $a_{11}, \tilde{a}_{11}, b_{11}, \tilde{b}_{11}, c_{11}, \tilde{c}_{11}$. It is important to mention the fact that the players' are using a source of advice for their outputs; they are using a "common device" that correlates their answers/outputs. In such situation, one must talk about **correlated equilibrium**. This concept arises when in a game the players receive some information/advice about which pure strategy to choose[5], hence their chosen strategies might be correlated. The advice provider might be a third party/mediator/observer or even a common random event. The vector of recommended strategies for all players is chosen according to a probability distribution over the set of available pure strategies. After receiving the recommended strategy, the players are also free to choose whether to follow the advised strategy or not. In the present case with the use of the quantum state, it is being implied that the players strictly follow the advice[6], i.e. the result of their local measurement. Then, the equilibrium points can be found by the definition of Nash equilibrium – see equation (4.48) on page 49 in chapter 4.

The next section will compute the **new Nash equilibrium** points for the games when the players use the quantum resources stated previously. Technically, these equilibrium points are correlated equilibria because the players use a quantum state

[5]Each player only receives information about their own strategy, not about the others.

[6]Were it not the case, one would need to seek the formal definition of correlated equilibrium, that involves the probability distribution of strategies, the payoffs, and a function of the advised strategy for each player (the players might decide not to follow the advice or partially follow it). For more information about that, see the book in [89].

that correlates their outputs (and follow the "advice"), but the present author asks for some leniency to abuse the language and refer to them also as Nash equilibrium points, computed from a quantum probability distribution. Finding the Nash equilibrium points in this new situation will provide the answers to the research questions stated on page 55.

## 5.2 Nash Equilibrium using bi-partite and tri-partite states in a triangle

The first necessary ingredient is the players' payoffs. The next equations describe Alice's payoff when the players use the GHZ-like state – in equation (5.1) – and the Bell-like state – in equation (5.2) – for the chosen representative functions for the games. The payoffs for Bob and Carl are obtained by doing the appropriate permutation of players. As mentioned before, the payoffs will depend on the players' strategies but also on the entanglement parameter: $\lambda_{111}$ for the GHZ-like state, and $\lambda_{11}$ for the Bell-like state. However, the idea is the same, finding the Nash equilibrium points[7] by maximising the players' payoffs. To obtain the expressions for the players' payoffs and the corresponding Nash equilibrium points, the author of this dissertation has availed of the software *Wolfram Mathematica* [101] since it is very powerful for symbolic/analytical computations.

Starting with the CHSH game in this triangle-network situation:

- $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$ – CHSH game[8] –, Alice's payoff when the players share a **GHZ-like** state is

$$\text{(GHZ) } \$_A = \frac{1}{4}\Big[3 + \Big(-\left(2a_{11}^2 + b_{11}^2 + c_{11}^2\right) + (a_{11}^2 + \tilde{a}_{11}^2)(b_{11}^2 + c_{11}^2) \\ +(a_{11}^2 - \tilde{a}_{11}^2)(\tilde{b}_{11}^2 + \tilde{c}_{11}^2)\ \Big)\left(1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right)\Big] \tag{5.10}$$

A sanity check shows that for $\lambda_{111} = 0$ or $\lambda_{111} = 1$, i.e. no entanglement of the state, Alice's payoff[9] in equation (5.10) reduces to the payoff using classical mixed strategies in equation (4.34) on page 45 by identifying $\{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\} \equiv \{p_{0|0}, p_{0|1}, q_{0|0}, q_{0|1}, r_{0|0}, r_{0|1}\}$. This correspondence between the classical mixed strategies and the quantum strategies when there is no entanglement in the quantum state was another reason to keep the simple (and restricting) measurement operators defined in equations (5.3)-(5.8), since the classical results could be used as a benchmark.

[7]IMPORTANT NOTE: the next pages might incur into abusing the game-theoretic language, for which the author apologises in advance. The equilibrium points are the strategies, not the payoffs associated; however, in certain cases, there might be a finite/infinite amount of (correlated) Nash equilibrium points that give the same equilibrium payoffs. In such case, it will be referred/encompassed as being only an equilibrium point, but it should be read between the lines that it refers to a family/set of points giving that payoff.

[8]Continuing with the chosen notation from chapter 4, the input/output functions of the game discussed will still be shown in bold as a reminder that these are the chosen representative functions of the games.

[9]For a better and quicker distinction between the payoffs of GHZ- and the Bell-like state, the equations with Alice's payoffs will be marked with the text "(GHZ)" or "(Bell)".

Since the payoffs only depend on the square of the strategic terms, e.g. $a_{11}^2$, it is convenient to perform a change of variables to simplify the problem of finding the Nash equilibrium solutions, for instance, with the map: $p_a \equiv a_{11}^2$, $p_{\tilde{a}} \equiv \tilde{a}_{11}^2$, $q_b \equiv b_{11}^2$, $q_{\tilde{b}} \equiv \tilde{b}_{11}^2$, $r_c \equiv c_{11}^2$, $r_{\tilde{c}} \equiv \tilde{c}_{11}^2$. The equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – explicitly, in equations (B.63)-(B.67) – in terms of these “new” variables $p_a, p_{\tilde{a}}, q_b, q_{\tilde{b}}, r_c, r_{\tilde{c}}$ [10]. For this particular case of the CHSH game, the **results** for the Nash equilibria are shown in **Table 5.1**, note, in terms of the original strategic terms squared $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$.

These equilibrium points are identical to the equilibrium points in the classical case in chapter 4 in Table 4.4 on page 51; the only difference is that, in this case, the payoffs for these equilibrium points depend explicitly on the entanglement parameter $\lambda_{111}$. Again, by setting $\lambda_{111} = 0$ or $\lambda_{111} = 1$, the payoffs using (classical) mixed strategies are recovered. It should not be completely surprising that the Nash equilibrium points are the same as the ones in the classical case because, in this case, the entanglement parameter $\lambda_{111}$ acts just as a (strictly positive) multiplying factor of the term that contains the strategies in Alice's payoff in equation (5.10), thus not affecting the optimisation problem to find the Nash equilibrium points. Even though the quantum equilibrium points are exactly the same as the classical ones – after properly identifying the square of $a_{11}, \tilde{a}_{11}, b_{11}, \tilde{b}_{11}, c_{11}, \tilde{c}_{11}$ with mixed strategies –, they imply very different things. For instance, the third equilibrium point $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$ in Table 5.1 – and also in Table 4.4 – would correspond classically to just tossing a coin, that is, all players would output 0 half of the times, i.e. $p_{0|0} = p_{0|1} = q_{0|0} = q_{0|1} = r_{0|0} = r_{0|1} = 1/2$. Meanwhile, that same strategy in the quantum $a_{11}^2 = \tilde{a}_{11}^2 = b_{11}^2 = \tilde{b}_{11}^2 = c_{11}^2 = \tilde{c}_{11}^2 = 1/2$ implies that all the players would be making a fully-entangled measurement onto a Bell state (or EPR pair), i.e. $\Pi_{x|0} = \Pi_{y|0} = \Pi_{z|0} = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. These two scenarios are fundamentally different, but both can be identified with the same strategy. Another important issue to keep in mind is the presence of *entangled measurements* when any of the quantum strategies are different than 0 and different than 1, since then the projecting state is indeed entangled.

[10] From now on, all of the explicit equations to solve for each game and each state are found in subsection B.2.2 in appendix B.

| GHZ-like state with $I = f_7$ , $O = f_{15}$ | |
|---|---|
| $s^* = \{a^2_{11}, \tilde{a}^2_{11}, b^2_{11}, \tilde{b}^2_{11}, c^2_{11}, \tilde{c}^2_{11}\}$ | Payoffs |
| $\{1, 1, 1, 1, 1, \tilde{c}^2_{11}\}$<br>$\{0, 0, 0, 0, 0, \tilde{c}^2_{11}\}$ | $\$_A = \$_B = \$_C = \frac{3}{4}$ ● (blue) |
| $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}\left[1 + \lambda^2_{111}\left(1 - \lambda^2_{111}\right)\right]$ ● (red) |
| $\{1, 0, 0, 1, 1 - \tilde{c}^2_{11}, \tilde{c}^2_{11}\}$ | $\$_A = \frac{1}{4}\left[3 - \left(1 - \tilde{c}^2_{11}\right)\left(1 - 2\lambda^2_{111}\left(1 - \lambda^2_{111}\right)\right)\right]$ * (green)<br>$\$_B = \frac{1}{4}\left[3 - \tilde{c}^2_{11}\left(1 - 2\lambda^2_{111}(1 - \lambda^2_{111})\right)\right]$ * (green)<br>$\$_C = \frac{1}{2}\left[1 + \lambda^2_{111}\left(1 - \lambda^2_{111}\right)\right]$ ● (red) |
| $\{0, 1, 1, 0, 1 - \tilde{c}^2_{11}, \tilde{c}^2_{11}\}$ | $\$_A = \frac{1}{4}\left[3 - \tilde{c}^2_{11}\left(1 - 2\lambda^2_{111}(1 - \lambda^2_{111})\right)\right]$<br>$\$_B = \frac{1}{4}\left[3 - \left(1 - \tilde{c}^2_{11}\right)\left(1 - 2\lambda^2_{111}\left(1 - \lambda^2_{111}\right)\right)\right]$<br>$\$_C = \frac{1}{2}\left[1 + \lambda^2_{111}\left(1 - \lambda^2_{111}\right)\right]$ |

Tab. 5.1: *Nash Equilibria for the game defined by $I = f_7$, $O = f_{15}$ (CHSH game) using the GHZ-like state. The colour of the circles helps to identify the payoffs plotted in Figures 5.3 and Figure 5.4. The non-marked (last) solution gives the same payoff as the previous one by just permuting Alice and Bob.*

Figure 5.3 plots the payoffs as a function of the entanglement parameter $\lambda_{111}$ for the first two equilibrium points, identified in Table 5.1 by the **blue** and **red** circles[11]. To be clearer, for instance, the blue line in Figure 5.3 corresponds to the equilibrium points that give a constant payoff $\$_A = \$_B = \$_C = 3/4$, which is the first solution[12] of Table 5.1 marked with a blue circle. The red line corresponds to the payoff marked with a red circle on the same table. The constant payoff (blue line) is the same regardless of $\lambda_{111}$; however, the red line shows that, as the entanglement increases, the players can get a higher payoff than $1/2 = 0.5$ up until $5/8 = 0.625$ for the traditional/pure GHZ state ($\lambda_{111} = 1/\sqrt{2}$). For this red line/circle solution, even though the strategy is exactly the same as the classical one, the resulting payoff might show that it was performed on a quantum state. Therefore, for these two solutions, the presence of entanglement does not decrease the players' payoffs and can even increase on average those payoffs.

[11]To improve "distinguishability" in the analysis of the many possible solutions, the word of the colour of the solution marked with a coloured shape in the corresponding table will be shown in bold.

[12]This is where the game-theoretic language abuse will happen. For that particular row solution, there are an infinite amount of equilibrium points, one for each value of $\tilde{c}_{11}$; nonetheless, all of the them give the same payoff, so to keep things short, they will be encompassed as one.

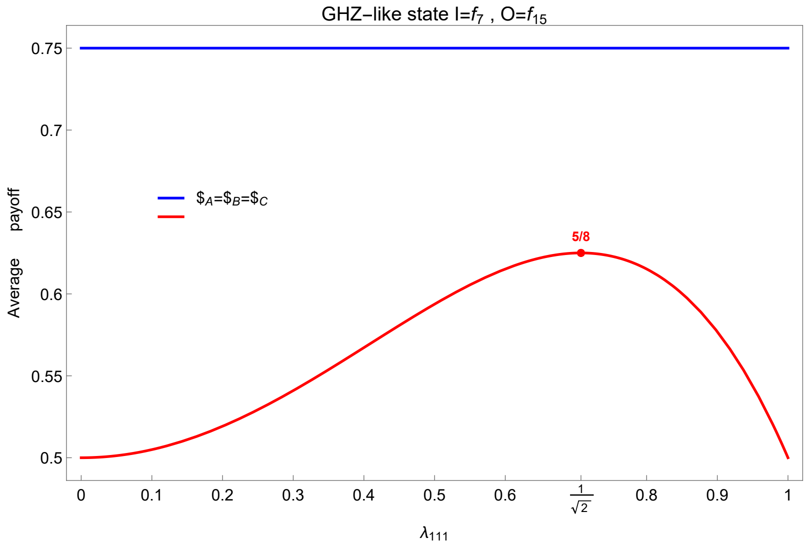
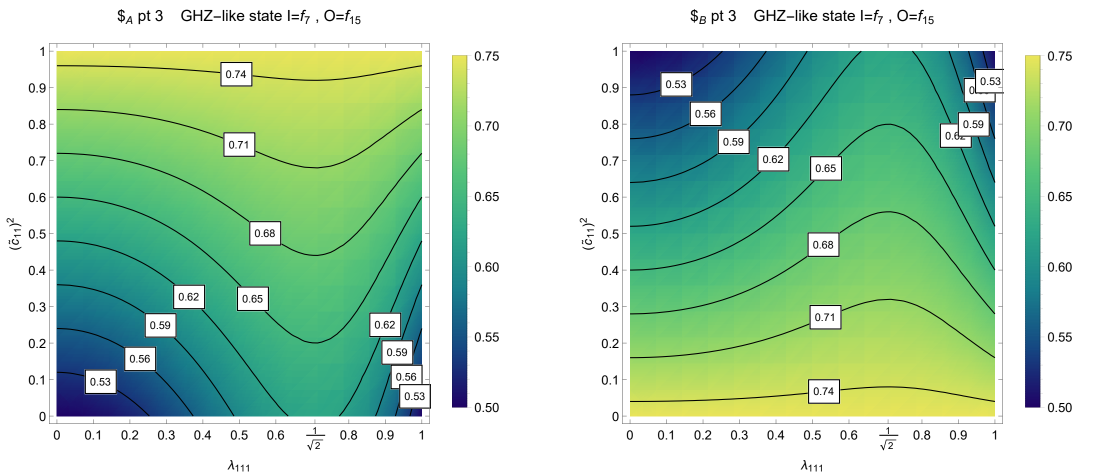



Fig. 5.3: *Plot that shows the players' average payoff for the Nash equilibrium points marked with a blue and red circle in Table 5.1 as a function of the entanglement parameter $\lambda_{111}$. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_7$, $O = f_{15}$ (CHSH game), using the GHZ-like state.*

Figure 5.4 shows a density plot of the payoff for Alice and Bob as a function of the square of Carl's strategy choice $\tilde{c}_{11}^2$ and the entanglement parameter $\lambda_{11}$ for the solutions marked in Table 5.1 with a **green** asterisk. The plots also contain some lines of constant payoff with the corresponding value. Carl's payoff is not plotted because it is the same as the payoff plotted with a red line in Figure 5.3. From Carl's perspective, he would prefer to have as much entanglement as possible to reach his maximum payoff $\$_C = 5/8 = 0.625$ at $\lambda_{111} = 1/\sqrt{2}$ (pure GHZ state). For Alice and Bob, being close to a maximally entangled state means less variation on their payoffs, that is, less influence by Carl's choice of $\tilde{c}_{11}$ – see the region around $\lambda_{111} = 1/\sqrt{2}$ for both plots in Figure 5.4, predominantly yellow and green, with no lower values in blue. The last non-marked equilibrium point on Table 5.1 gives, effectively, the same payoffs as the one with the green asterisk when permuting Alice and Bob.




(a) *Alice's payoff* $\$_A$ (b) *Bob's payoff* $\$_B$

Fig. 5.4: *Density plot showing Alice's and Bob's payoffs as a function of* $\tilde{c}_{11}^2$ *and the entanglement parameter* $\lambda_{11}$*. These payoffs correspond to the Nash equilibrium solutions marked with a green asterisk in Table 5.1. These results are for the CHSH game, identified with* $I = f_7$*,* $O = f_{15}$*, and using the GHZ-like state.*

The two density plots for Alice and Bob in Figure 5.4 can be combined into a unique density plot in which Bob's payoff is plotted *as a function of Alice's payoff* $\$_A$ and $\lambda_{111}$. This is done as follows: adding the two expressions for Alice's and Bob's payoff, marked with green asterisks in Table 5.1, Bob's payoff is:

$$\$_B = -\$_A + \frac{1}{4}\left[5 + 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right] \tag{5.11}$$

where $0 \leqslant \lambda_{111} \leqslant 1$. Alice's payoff is, nevertheless, restricted $\left[1 + \lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right]/4 \leqslant \$_A \leqslant 3/4$. This restriction comes from the range of possible values of Alice's payoff $\$_A(\lambda_{111}, \tilde{c}_{11})$ when Carl's strategy is $\tilde{c}_{11} = 0$ and when $\tilde{c}_{11} = \pm 1$. Then, the associated density plot of equation (5.11) is shown in Figure 5.5. The white region corresponds to $\$_A \leqslant \left[1 + \lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right]/4$, which is out of the possible values of $\$_A$.

Figure 5.5 shows more clearly the impact of entanglement in Alice's and Bob's payoff than the two plots in Figure 5.4. When there is no entanglement, both payoffs can vary more – see range of colours, from yellow (0.75) to blue (0.5), around $\lambda_{111} = 0,\ 1$ . In contrast, around the region $\lambda_{111} = 1/\sqrt{2}$, there are no low values in blue.

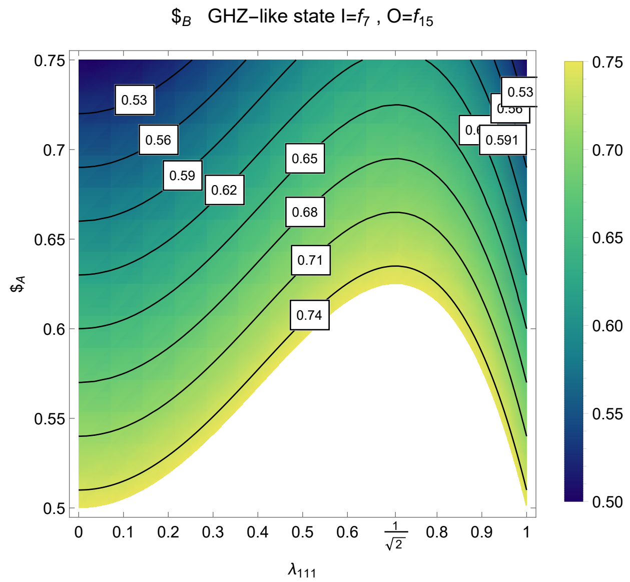



Fig. 5.5: *Density plot showing Bob's payoffs as a function of* $\$_A$ *and* $\lambda_{11}$. *These payoffs correspond to the Nash equilibrium solutions marked with a green asterisk in Table 5.1. These results are for the CHSH game, identified with* $I = f_7$, $O = f_{15}$, *and using the GHZ-like state.*

These are the Nash equilibrium solutions, and, as mentioned in section 4.2 in chapter 4, one way of assessing which solutions are better is computing the **social welfare** ($\$_A + \$_B + \$_C$) of each solution. In the classical case with mixed strategies, the social welfare of the solutions were constant and with values $9/4 = 2.25$ , $3/2 = 1.5$ , and $7/4 = 1.75$ – see Table 4.4 on page 51 that has the equilibrium points and add the payoffs. In this (quantum) case, the social welfare for each solution will also be a function of the entanglement parameter, but still, the extreme values of $\lambda_{111}$ – no entanglement – recover the classical value with mixed strategies. The specific expressions for the social welfare for all the equilibrium points in Table 5.1 are found in Table B.1 in appendix B. Figure 5.6 plots the social welfare in Table B.1 as a function of $\lambda_{111}$, with the colour of the line in the plot matching that of the circle in the same table (and also the colours in the main table, Table 5.1). The social welfare of the equilibrium points for $\lambda_{111} = 0$ and $\lambda_{111} = 1$ is the same as classically. Clearly from the plot, the preferred solution is the blue line because it has the highest collective payoff, also individual, and it does not depend on the amount of entanglement in the GHZ-like state; this solution corresponds exactly to the best classical solution – see again Table 4.4.

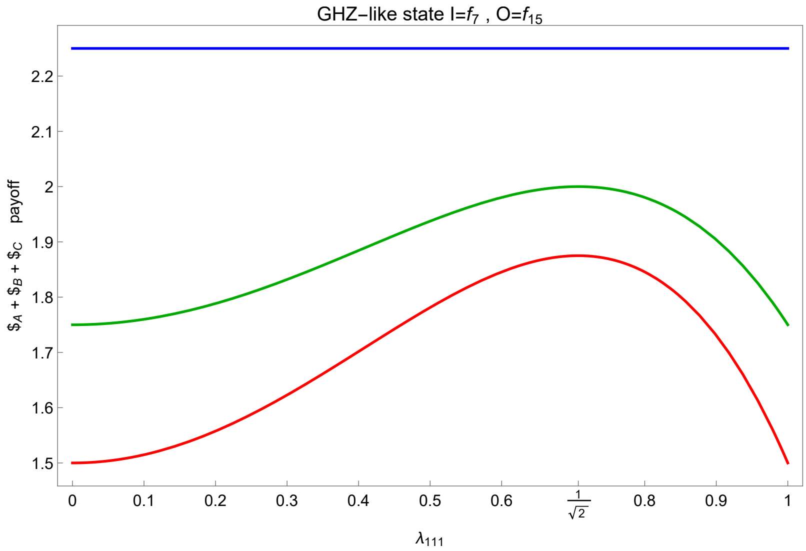



Fig. 5.6: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{111}$ using a GHZ-like state for $I = f_7$, $O = f_{15}$ (CHSH game). The Nash equilibrium solutions with the individual payoffs are found in Table 5.1, while the social welfare of them is in Table B.1. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ Continuing with the CHSH game, defined by $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$, Alice's payoff when the players share a **Bell-like** state is:

$$\begin{aligned}(\text{Bell})\ \$_A =& \frac{1}{4}\Big[3 + \Big(-\left(2a_{11}^2 + b_{11}^2 + c_{11}^2\right) + (a_{11}^2 + \tilde{a}_{11}^2)(b_{11}^2 + c_{11}^2) \\ &+ (a_{11}^2 - \tilde{a}_{11}^2)(\tilde{b}_{11}^2 + \tilde{c}_{11}^2)\Big)\left(1 - 3\lambda_{11}^2(1-\lambda_{11}^2)\right) - 4\lambda_{11}^2\left(1-\lambda_{11}^2\right)^2 \\ &+ \left(2a_{11}^2 + b_{11}^2 + c_{11}^2\right)\lambda_{11}^2\left(1 - 3\lambda_{11}^2 + 2\lambda_{11}^4\right)\Big] \end{aligned} \tag{5.12}$$

This payoff looks more interesting than the one using the GHZ-like state in equation (5.10) since the entanglement parameter of the Bell state $\lambda_{11}$ does not appear only as a multiplying factor, but it is mixed with the different strategic terms. That will give rise to different Nash equilibrium points. Nevertheless, again, setting $\lambda_{11} = 0$ or $\lambda_{11} = 1$ recovers the classical payoff using mixed strategies in equation (4.34).

As before, the payoffs depend only on the square of the strategic terms, so a change of variables is performed to find the Nash Equilibrium; the concrete equations to solve are shown in appendix in subsection B.2.2 in appendix B – equations (B.69)-(B.73). **Table 5.2** contains the **solutions** for this Bell-like state, with some of the solutions depending explicitly on functions of $\lambda_{11}$, labelled as $t_{B1}(\lambda_{11})$, $u_{B1}(\lambda_{11})$, and $v_{B1}(\lambda_{11})$. To shorten the notation in the table, the dependence on $\lambda_{11}$ of these functions has been omitted.

| Bell-like state with $I = f_7$ , $O = f_{15}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $0 \leqslant \lambda_{11} \leqslant 1$ | $\{1, 1, 1, 1, 1, \tilde{c}_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{3}{4} - \lambda_{11}^4 (1 - \lambda_{11}^2)$ |
| | $\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{3}{4} - \lambda_{11}^2 (1 - \lambda_{11}^2)^2$ |
| $0 \leqslant \lambda_{11} \leqslant 1$ | $\left\{\frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}\right\}$ | $\$_A = \$_B = \$_C = \frac{2 - 5\lambda_{11}^2 + \lambda_{11}^4 + 12\lambda_{11}^6 - 16\lambda_{11}^8 + 12\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4\left[1 - 3\lambda_{11}^2 (1 - \lambda_{11}^2)\right]}$ |
| $\lambda_{11} = 0, 1$ ; $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1, 0, 0, 1, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{2 + \tilde{c}_{11}^2}{4}$ ; $\$_A = \frac{9 + \tilde{c}_{11}^2}{16}$<br>$\$_B = \frac{3 - \tilde{c}_{11}^2}{4}$ ; $\$_B = \frac{10 - \tilde{c}_{11}^2}{16}$<br>$\$_C = \frac{1}{2}$ ; $\$_C = \frac{9}{16}$ |
| | $\{0, 1, 1, 0, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{3 - \tilde{c}_{11}^2}{4}$ ; $\$_A = \frac{10 - \tilde{c}_{11}^2}{16}$<br>$\$_B = \frac{2 + \tilde{c}_{11}^2}{4}$ ; $\$_B = \frac{9 + \tilde{c}_{11}^2}{16}$<br>$\$_C = \frac{1}{2}$ ; $\$_C = \frac{9}{16}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$, $\lambda_{11} = 1$ | $\{v_{B1}, 0, v_{B1}, 0, 0, 1\}$ | $\$_A = \$_B = \frac{1 - 2\lambda_{11}^2 - 3\lambda_{11}^4 + 16\lambda_{11}^6 - 22\lambda_{11}^8 + 15\lambda_{11}^{10} - 4\lambda_{11}^{12}}{2\left[1 - 3\lambda_{11}^2 (1 - \lambda_{11}^2)\right]}$ |

| | | |
|---|---|---|
| | | $\$_C = \dfrac{3 - 11\lambda_{11}^2 + 13\lambda_{11}^4 + 12\lambda_{11}^6 - 46\lambda_{11}^8 + 48\lambda_{11}^{10} - 16\lambda_{11}^{12}}{4\left[1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right]}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}},\ \lambda_{11} = 1$ | $\{0, u_{B1}, 0, u_{B1}, u_{B1}, 0\}$ | $\$_A = \$_B = \dfrac{2 - 5\lambda_{11}^2 + \lambda_{11}^4 + 12\lambda_{11}^6 - 16\lambda_{11}^8 + 12\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4\left[1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right]}$ <br> $\$_C = \dfrac{3}{4} - \lambda_{11}^2\left(1 - \lambda_{11}^2\right)^2$ |
| $\lambda_{11} = 0,\ \frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{2t_{B1}, 1, 2t_{B1}, 1, 1, 0)\}$ | $\$_A = \$_B = \dfrac{1 - 3\lambda_{11}^2 + 3\lambda_{11}^4 + 2\lambda_{11}^6 - 7\lambda_{11}^8 + 9\lambda_{11}^{10} - 4\lambda_{11}^{12}}{2\left[1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right]}$ <br> $\$_C = \dfrac{3 - 11\lambda_{11}^2 + 13\lambda_{11}^4 + 12\lambda_{11}^6 + 46\lambda_{11}^8 + 48\lambda_{11}^{10} - 16\lambda_{11}^{12}}{4\left[1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right]}$ |
| $\lambda_{11} = 0,\ \frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1, t_{B1}, 1, t_{B1}, t_{B1}, 1\}$ | $\$_A = \$_B = \dfrac{2 - 5\lambda_{11}^2 + \lambda_{11}^4 + 12\lambda_{11}^6 - 16\lambda_{11}^8 + 12\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4\left[1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right]}$ <br> $\$_C = \dfrac{3}{4} - \lambda_{11}^4\left(1 - \lambda_{11}^2\right)$ |

Tab. 5.2: *Nash Equilibria for the game defined by $I = f_7$, $O = f_{15}$ (CHSH game) using the Bell-like state. In this case, some of the solutions only exist in a certain interval of $\lambda_{11}$ or for certain values. The specific expressions for $t_{B1}(\lambda_{11})$, $u_{B1}(\lambda_{11})$, and $v_{B1}(\lambda_{11})$ are found in equations (5.13)-(5.15). The colour of the squares helps to identify the payoffs plotted in Figures 5.8, 5.9, and 5.11. The fifth solution not marked is equivalent to the previous one by permuting the players.*

Table 5.2 shows that there are in principle 8 different types of payoffs coming from the Nash equilibrium points, but notice that some of the solutions are only valid in a certain interval of the parameter $\lambda_{11}$, or even only for certain values. The reason will be explained next. Analogously to the previous case with the GHZ-like state, in this case, the coloured squares[13] help to identify the payoffs plotted in Figure 5.8, Figure 5.9, and Figure 5.11. The points marked with a blue, cyan, and green square are the same points as the classical and the GHZ-like case, but there are **new solutions** that explicitly **depend** on some **functions** of the **entanglement parameter**, labelled as $t_{B1}(\lambda_{11})$, $u_{B1}(\lambda_{11})$, and $v_{B1}(\lambda_{11})$. These functions[14] are defined as:

$$t_{B1}(\lambda_{11}) = \frac{-\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)} \tag{5.13}$$

$$u_{B1}(\lambda_{11}) = t_{B1}(\lambda_{11})+1 = \frac{1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)} \tag{5.14}$$

$$v_{B1}(\lambda_{11}) = 2t_{B1}(\lambda_{11})+1 = \frac{1-\lambda_{11}^2\left(5-9\lambda_{11}^2+4\lambda_{11}^4\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)} \tag{5.15}$$

Figure 5.7 shows the values of $t_{B1}(\lambda_{11})$, $u_{B1}(\lambda_{11})$, and $v_{B1}(\lambda_{11})$ as a function of $\lambda_{11}$.

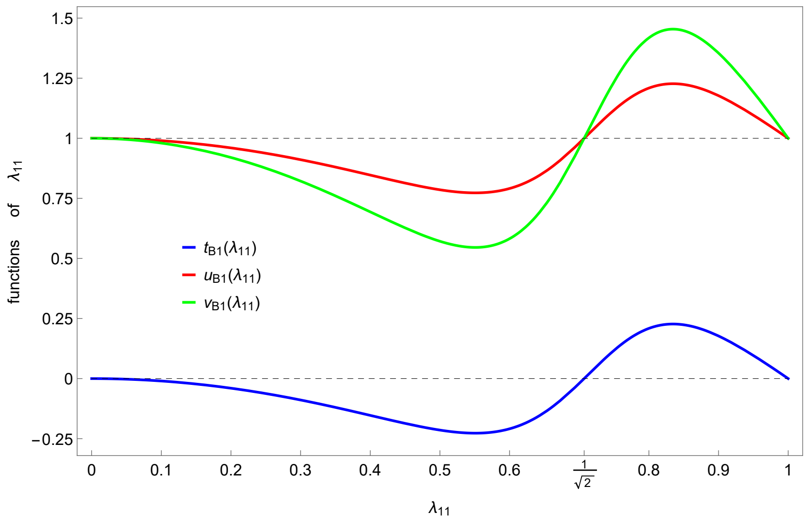



Fig. 5.7: *Plot of the three distinct functions of* $\lambda_{11}$ *that appear in the Nash equilibrium points for the Bell-like state shown in Table 5.2 for the CHSH game, identified by* $I = f_7$, $O = f_{15}$. *The specific expressions are found in equations (5.13)-(5.15).*

The **interval restriction** of $\lambda_{11}$ for certain solutions comes from imposing the optimisation conditions – specifically, the Karush-Kuhn-Tucker (KKT) conditions, explained in section B.1 of appendix B about convex optimisation – *and* imposing that the strategic terms $a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2$ must be between 0 and 1 because they represent probabilities. For instance, the third solution in Table 5.2, marked with a red square, corresponds to $a_{11}^2 = \tilde{a}_{11}^2 = b_{11}^2 = \tilde{b}_{11}^2 = c_{11}^2 = \tilde{c}_{11}^2 = u_{B1}(\lambda_{11})/2$,

[13]The payoffs of solutions for the Bell-like state will always be marked with a coloured *square*, while the ones for the GHZ-like state with a coloured *circle*.

[14]The subscript "B" in $t_{B1}$ stands for "Bell" and the number 1 denotes the first set of these functions, since, for some of the next games with the Bell-like state, other functions of $\lambda_{11}$ appear.

and by looking at the red line in Figure 5.7 representing $u_{B1}(\lambda_{11})$, it is easy to see that, in fact, $0 \leqslant u_{B1}(\lambda_{11})/2 \leqslant 1$ for $0 \leqslant \lambda_{11} \leqslant 1$. That is why that solution, which also fulfills the optimisation conditions, is allowed for any value of $\lambda_{11}$ between 0 and 1. Nonetheless, that is not the case for the payoff identified with a purple square whose solution is $a_{11}^2 = b_{11}^2 = v_{B1}(\lambda_{11})$ , $\tilde{a}_{11}^2 = \tilde{b}_{11}^2 = c_{11}^2 = 0$ , $\tilde{c}_{11}^2 = 1$ since, from looking at the green line in Figure 5.7 representing $v_{B1}(\lambda_{11})$, $0 \leqslant v_{B1}(\lambda_{11}) \leqslant 1$ only happens for $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$. That is why, that Nash equilibrium solution is only valid in that interval; again, also, because in that interval the optimisation conditions are fulfilled as well. As an illustrative example, the specific equations and restrictions to be satisfied for this particular solution are shown in equations (B.74)-(B.85) in appendix B.

It is worth mentioning a few words about the situation of the equilibrium points represented by the green square because, in that case, for those solutions to exist, there needs to be no entanglement at all ($\lambda_{11} = 0$ or $\lambda_{11} = 1$) or the Bell state needs to be maximally entangled ($\lambda_{11} = 1/\sqrt{2}$). The reason for such a restrictive case is related to the optimisation conditions that need to be satisfied to find the optimal points for all three payoffs. In particular, for such points, these optimisation conditions can only be satisfied[15] when $\lambda_{11} = 0$, or $\lambda_{11} = 1$, or $\lambda_{11} = 1/\sqrt{2}$.

The next paragraphs analyse in more detail the plots for the Nash equilibrium points using the Bell-like state.

Figure 5.8 shows the first three different payoffs for some of the Nash equilibrium points. For the first two payoffs shown with **blue** and **cyan** lines, that correspond to the payoffs marked with a blue and a cyan square in Table 5.2, it can be said that, as the entanglement increases, the players' payoff decreases to reach a minimum of $\$_A = \$_B = \$_C = 65/108 \approx 0.602$. That minimum for the cyan line is reached at $\lambda_{11} = 1/\sqrt{3} \approx 0.577$ , whereas for the blue line is reached at $\lambda_{11} = \sqrt{2/3} \approx 0.816$ . Both solutions give the same payoff $\$_A = \$_B = \$_C = 5/8 = 0.625$ for a maximally-entangled Bell state ($\lambda_{11} = 1/\sqrt{2}$). In contrast, the payoff represented by the red square increases as the amount of entanglement increases, to reach a maximum of $\$_A = \$_B = \$_C = 9/16 = 0.5625$ for $\lambda_{11} = 1/\sqrt{2}$.

It is interesting to analyse further this solution of the **red** line/square and compare it with the corresponding classical solution, which is the same as for the GHZ-like state. The classical solution is $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$ – see Table 4.4 –, which gives $\$_A = \$_B = \$_C = 1/2 = 0.5$, while the one for the Bell-like state is $s^* = \{u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2\}$, giving the corresponding payoff in Table 5.2 marked with the red square, and plotted with the red line in Figure 5.8. The equality $u_{B1}(\lambda_{11})/2 = 1/2$ is solved only for $\lambda_{11} = 0,\ 1,\ 1/\sqrt{2}$ – see also the red line in Figure 5.7 –, which would correspond to the classical strategy. In principle that means that having no entanglement at all or having a maximally-entangled state would correspond to the classical strategy $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$; however, from the red line in Figure 5.8, choosing $\lambda_{11} = 1/\sqrt{2}$ over $\lambda_{11} = 0,\ 1,$ gives a notably higher payoff (compare $1/2 = 0.5$ to $9/16 = 0.5625$), even though all of these correspond to the same

[15] See equations (B.86)-(B.92) for this particular case in appendix B.

strategy. This result might remind the reader of the use of non-locality to detect the possible "quantumness" of a device by just looking at the input and output statistics (self-testing) – see section 2.3 in chapter 2. In this sense, one could detect whether the strategy $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$ has been generated without entanglement ($\lambda_{11} = 0,\ 1,$ giving $\$_A = \$_B = \$_C = 1/2 = 0.5$) or with a maximally-entangled Bell state ($\lambda_{11} = 1/\sqrt{2}$ giving $\$_A = \$_B = \$_C = 9/16 = 0.5625$) by just looking at the players' payoffs. Of course, this statement needs to be taken with a pinch of salt because the players might use a classical correlating device giving that same strategy and maybe with the same payoff as the pure Bell state. It would be worth investigating if such classical strategies exist or not. If they do not exist, then one could use this game set-up as a self-testing scenario. As stated, this is something that needs to be researched into, falling into the future perspectives of this dissertation.

It is worth **comparing** this situation of the **Bell-like** state to the results for the **GHZ-like** state, shown in Table 5.1 and Figure 5.3 on pages 62 and 63. For starters, the payoff for the GHZ-like state marked with blue circle is constant $\$_A = \$_B = \$_C = 3/4$, whereas for the Bell-like state, the payoffs depend explicitly on $\lambda_{11}$, and are unfolded into two different ones: the one marked with a blue square and a cyan square. As for the payoffs marked with a red circle for the GHZ-like state and with a red square for the Bell-like state, both payoffs increase as the entanglement parameter increases, until reaching a maximum at $\lambda_{111} = \lambda_{11} = 1/\sqrt{2}$. The maximum for the GHZ-like case is higher $\$_A = \$_B = \$_C = 5/8 = 0.625$ than the one of the Bell-like state $\$_A = \$_B = \$_C = 9/16 = 0.5625$; but both are higher than the classical constant of $1/2 = 0.5$ using mixed non-correlated strategies.

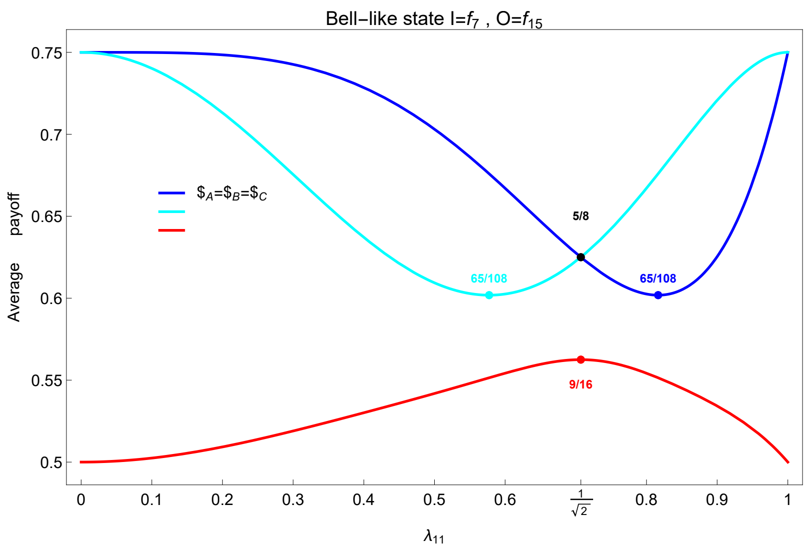



Fig. 5.8: *Plot that shows the players' average payoff for the Nash equilibrium points marked with blue, cyan, and red squares in Table 5.2 as a function of the entanglement parameter* $\lambda_{11}$*. The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_7$*,* $O = f_{15}$ *(CHSH game), using the Bell-like state.*

Figure 5.9 shows the players' payoffs for the solutions identified with the **green** square in Table 5.2 as a function of Carl's strategy $\tilde{c}_{11}$. The dark green lines correspond to the case without entanglement ($\lambda_{11} = 0$ or $\lambda_{11} = 1$) and the light-green lines to a maximally entangled Bell state ($\lambda_{11} = 1/\sqrt{2}$). Alice's payoff is denoted with a continuous line, Bob's with a dashed line, and Carl's with a dot-dashed line. For these solutions, Carl's payoff remains constant ($1/2 = 0.5$ or $9/16 = 0.5625$) but with his choice of $\tilde{c}_{11}$ he can increase Alice (Bob's) payoff at the expense of decreasing Bob's (Alice's). From Carl's perspective, he would prefer to have a maximally entangled state $\lambda_{11} = 1/\sqrt{2}$ over no entanglement at all – compare $1/2 = 0.5$ with $9/16 = 0.5625$. Alice would prefer the situation with $\lambda_{11} = 1/\sqrt{2}$ only when $0 \leqslant \tilde{c}_{11} < 1/\sqrt{3}$, whereas Bob would only prefer it when $\sqrt{2/3} < \tilde{c}_{11} \leqslant 1$.

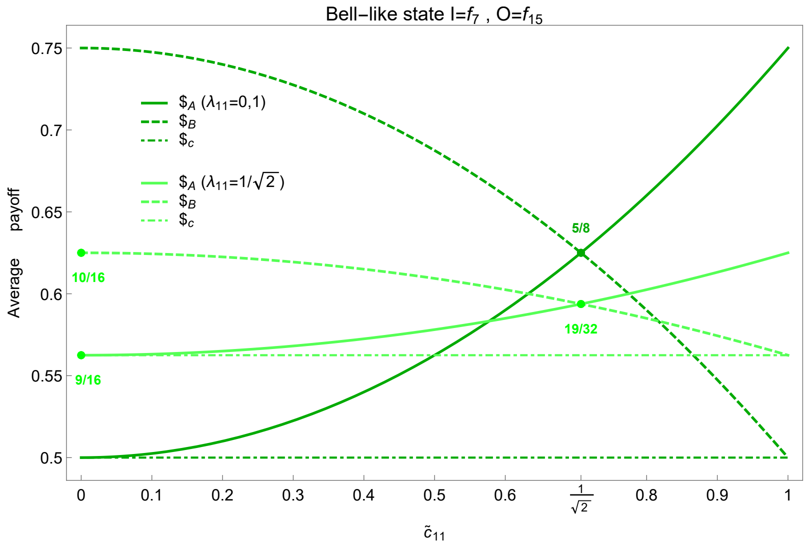



Fig. 5.9: *Plot that shows the players' average payoffs for the equilibrium point marked with a green square in Table 5.2 as a function of Carl's strategy $\tilde{c}_{11}$. The darker green lines are for $\lambda_{11} = 0, 1$ , and the light green for $\lambda_{11} = 1/\sqrt{2}$. These results are for the CHSH game, identified with $I = f_7$ and $O = f_{15}$, for the Bell-like state.*

Even though there are certain range of values of Carl's strategy $\tilde{c}_{11}$ for which Alice and Bob would prefer a maximally entangled state over no entanglement at all – explained before –, having a maximally entangled Bell state reduces pairwise the inequality between the players' payoffs. Figure 5.10 plots the payoff differences $\$_A - \$_B$ and $\$_A - \$_C$ as a function of *the square* of Carl's strategy $\tilde{c}^2_{11}$. The difference $\$_A - \$_B$ is shown with a continuous line and $\$_A - \$_C$ with a dotted line. The light gray lines correspond to not having entanglement ($\lambda_{11} = 0$ or $\lambda_{11} = 1$) and the dark gray / black to a maximally entangled Bell state $\lambda_{11} = 1/\sqrt{2}$. For any choice of $\tilde{c}^2_{11}$ the pairwise payoff difference is reduced by $1/4$ when $\lambda_{11} = 1/\sqrt{2}$ compared to when $\lambda_{11} = 0$ or $\lambda_{11} = 1$ .

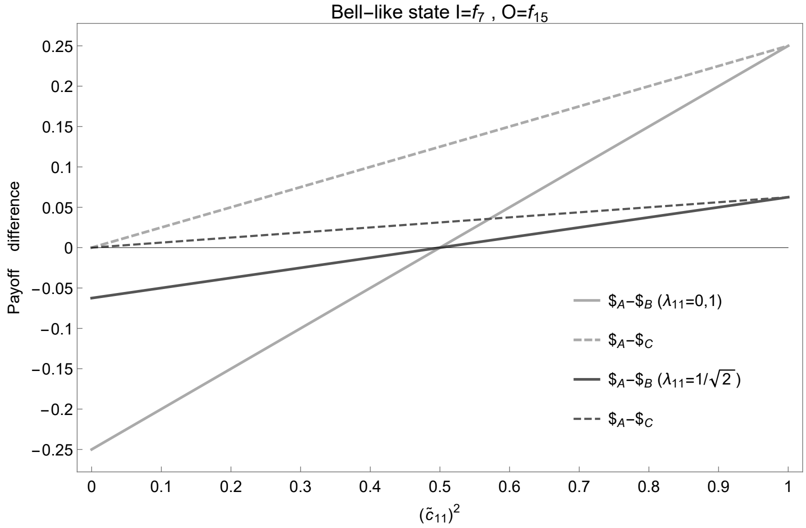



Fig. 5.10: *Payoff difference* $\$_A - \$_B$ *and* $\$_A - \$_C$ *for the payoffs in the solutions from Figure 5.9. Note that the horizontal axis is the* square *of Carl's strategy* $\tilde{c}_{11}^2$*, choice made so that the differences correspond to lines.*

Figure 5.11 shows the remaining different payoffs for the Bell-like state, which are exclusive to this state, so they cannot be compared with the GHZ-like state. These solutions give asymmetric payoffs, with one player receiving more – in the plotted case, Carl's payoff $\$_C$, in dotted lines – than the other two – Alice's and Bob's $\$_A = \$_B$ in solid lines. Note also that these solutions are only valid in a certain interval of $\lambda_{11}$. The solution giving the payoffs in the **purple** line, from the purple square in Table 5.2, show that as the entanglement increases, Alice and Bob get a better payoff than $1/2 = 0.5$, up until reaching $\$_A = \$_B \approx 0.568$ at $\lambda_{11} \approx 0.638$, whereas Carl gets worse payoff as the entanglement increases to reach a minimum of $\$_C \approx 0.6212$ at $\lambda_{11} \approx 0.609$. The solutions giving the payoffs in the **brown** line, from the brown square, give the same payoffs for Alice and Bob as the solution from the red square/line, while Carl gets the same payoff as the solution from the cyan square/line – see Figure 5.8. To summarise again, Alice and Bob increase their payoff from $1/2 = 0.5$ as the entanglement increases up to $\$_A = \$_B = 9/16 = 0.5625$ at $\lambda_{11} = 1/\sqrt{2}$; whereas for Carl it decreases until it reaches a minimum of $\$_C = 65/108 \approx 0.602$ at $\lambda_{11} = 1/\sqrt{3} \approx 0.577$. The solutions giving the payoffs in the **orange** line, is just the mirror image of the one from the purple square. In this case, as the entanglement starts to decrease $(1/\sqrt{2} \leqslant \lambda_{11} \leqslant 1)$, Carl's payoff decreases slightly to a minimum of $\$_C \approx 0.6212$ at $\lambda_{11} \approx 0.793$, and then increases again. For Alice and Bob, the payoff increases slightly to $\$_A = \$_B \approx 0.568$ at $\lambda_{11} \approx 0.770$. Finally, the solution giving the payoffs in the **magenta** line, from the magenta square, mirrors the payoffs in the brown line. This solution gives the same payoff for Alice and Bob as the red square/line, and the same for Carl as the blue square/line, both in Figure 5.8, with Carl's minimum $\$_C = 65/108 \approx 0.602$ located at $\lambda_{11} = \sqrt{2/3} \approx 0.816$ .

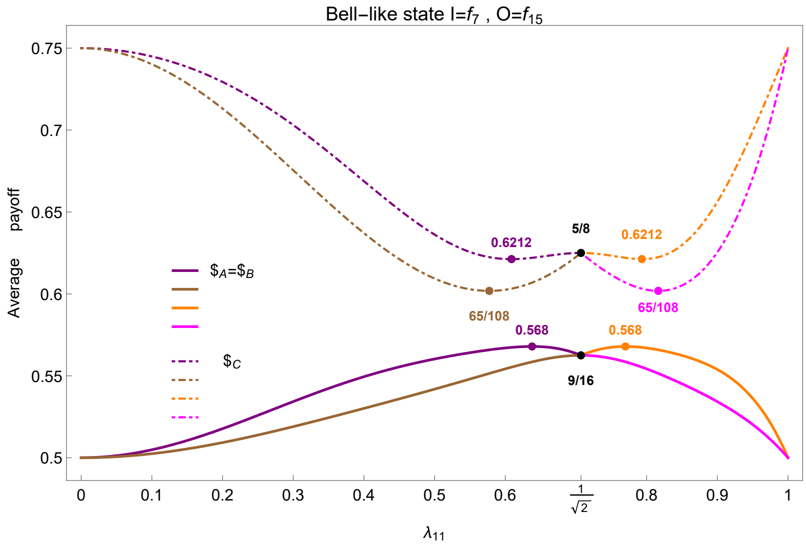



Fig. 5.11: *Plot that shows the players' average payoff for the Nash equilibrium points marked with purple, brown, orange, and magenta squares in Table 5.2 as a function of the entanglement parameter* $\lambda_{11}$. *The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_7$, $O = f_{15}$ *(CHSH game), using the Bell-like state.*

It is illustrative to have **all** of the **solutions** together in **one plot**, in Figure 5.12, except the solutions identified with the green square because they exist only for certain values of $\lambda_{11}$ and depend on $\tilde{c}_{11}$. The payoff from the Nash equilibrium solution from the red line coincides with the brown and magenta solid lines, that is why it is not visible. The cyan line also coincides exactly with the dot-dashed brown line when $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$, while the blue line coincides with the dot-dashed magenta line when $1/\sqrt{2} \leqslant \lambda_{11} \leqslant 1$.

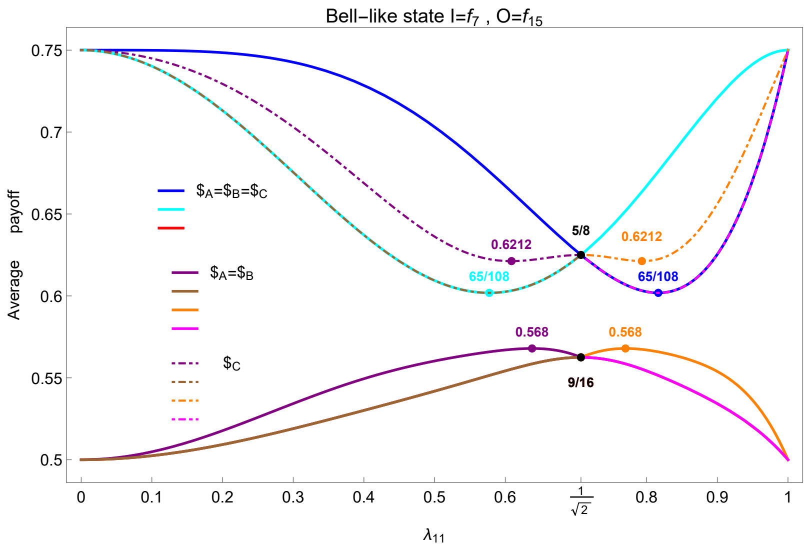



Fig. 5.12: *All payoffs from the seven different Nash equilibrium solutions showing the players' average payoff as a function of the entanglement parameter* $\lambda_{11}$ *in a Bell-like state for the CHSH game, identified by* $I = f_7$, $O = f_{15}$. *This plot is the combination of Figures 5.8 and 5.11. The payoffs are found in Table 5.2.*

As in the previous case with the GHZ-like state, the **social welfare** ($\$_A$+$\$_B$+$\$_C$)

of each Nash equilibrium solution for the Bell-like state can be computed, which depends on $\lambda_{11}$. The specific expressions for the social welfare for all the equilibrium points in Table 5.1 are found in Table B.1 in appendix B. Figure 5.6 plots the social welfare of the Nash equilibrium solutions as a function of $\lambda_{11}$, with the colour of the line in the plot matching that of the square in Table B.1 (also in the main table, Table 5.1). In this case, the preferred solution with the highest collective payoff would be the blue line for $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$, and the cyan line for $1/\sqrt{2} \leqslant \lambda_{11} \leqslant 1$ , while the red line gives the lower social welfare. The black points correspond to the solutions with constant social welfare $\$_A + \$_B + \$_C = 7/4$ for that particular value of $\lambda_{11}$, which are not marked in Table B.1. The **purple**, **brown**, **orange**, and **magenta** lines show an **interesting** behaviour, which is shown more clearly in another plot, in Figure 5.14.

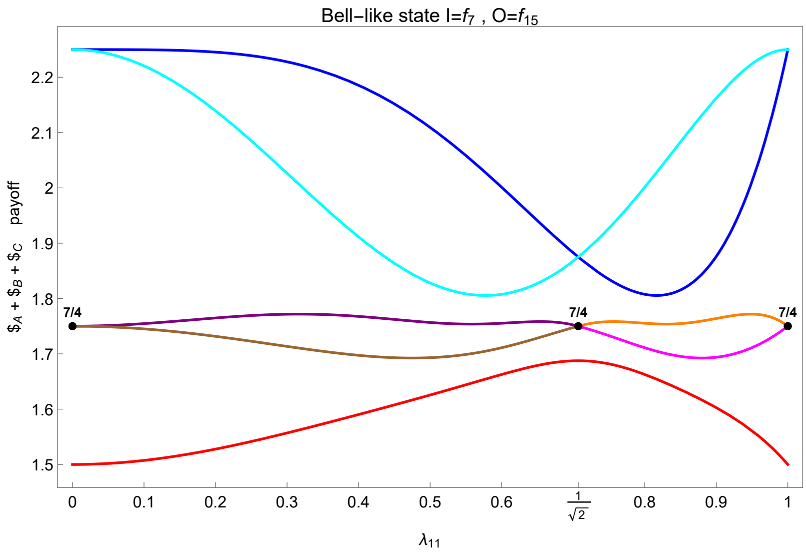



Fig. 5.13: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{11}$ using a Bell-like state for $I = f_7$, $O = f_{15}$ (CHSH game). The Nash equilibrium solutions with the individual payoffs are found in Table 5.2, while the social welfare of them is in Table B.2. The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the latter table. The interesting behaviour of the purple, brown, orange, and magenta lines is shown in Figure 5.14.*

Starting with the **brown** line in Figure 5.14, the maximum is $\$_A + \$_B + \$_C = 7/4 = 1.75$ at $\lambda_{11} = 0,\ 1/\sqrt{2}$ ; while the minimum is $\$_A + \$_B + \$_C \approx 1.692$ at $\lambda_{11} \approx 0.474$ . Looking now at the individual payoffs in Figure 5.12, at the minimum of the social welfare, the individual payoffs are $\$_A = \$_B \approx 0.539$ and $\$_C \approx 0.615$ , which is closer to the minimum of Alice and Bob than of Carl's. The maximum of the social welfare occurs at $\lambda_{11} = 0,\ 1/\sqrt{2}$ , which, for $\lambda_{11} = 0$, Alice's and Bob's payoff is maximum, while Carl's is minimum; whereas for $\lambda_{11} = 1/\sqrt{2}$, Alice's and Bob's payoff is not a maximum or minimum but Carl's is a maximum. Therefore for this solution, having a pure Bell state gives the same total payoff but distributed among the players in a "fairer" manner. The **magenta** solution is completely analogous to the brown one, with a shift of the minimum to $\lambda_{11} \approx 0.880$ . The **purple** solution presents a very interesting behaviour. It has three local minima and two maxima. Two minima are $7/4 = 1.75$ at $\lambda_{11} = 0,\ 1/\sqrt{2}$ , while the other minimum is slightly above $\$_A + \$_B + \$_C \approx 1.754$ at $\lambda_{11} \approx 0.558$ . The two maxima are: $\$_A + \$_B + \$_C \approx 1.772$ at $\lambda_{11} \approx 0.317$ ; and $\$_A + \$_B + \$_C \approx 1.758$ at $\lambda_{11} \approx 0.654$ . Again, looking at the *individual payoffs* for this purple solution at the minimum points:

- $\lambda_{11} = 0 \rightarrow \$_A = \$_B = 1/2 = 0.5$ ; $\$_C = 3/4 = 0.75$
- $\lambda_{11} = 1/\sqrt{2} \rightarrow \$_A = \$_B = 9/16 = 0.5625$ ; $\$_C = 5/8 = 0.625$
- $\lambda_{11} \approx 0.558 \rightarrow \$_A = \$_B \approx 0.545$ ; $\$_C \approx 0.624$

and at the maximum points:

- $\lambda_{11} \approx 0.317 \rightarrow \$_A = \$_B \approx 0.537$ ; $\$_C \approx 0.698$
- $\lambda_{11} \approx 0.654 \rightarrow \$_A = \$_B \approx 0.568$ ; $\$_C \approx 0.623$

The analysis is similar for the **orange** line by just shifting the positions of the minima and maxima. The purple and orange solution imply that if the players decided to share their payoffs evenly, in this case, having "a little" entanglement – not maximal, though – would be beneficial for them, with an absolute maximum reached for $\lambda_{11} \approx 0.317$ and $\lambda_{11} \approx 0.948$ .

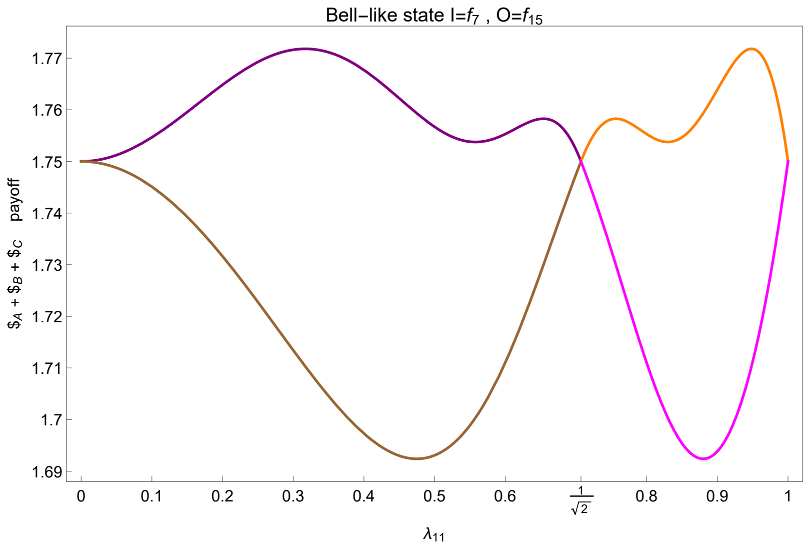



Fig. 5.14: *Sum of payoffs $\$_A + \$_B + \$_C$ from* some *of the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{11}$ using a Bell-like state for $I = f_7$, $O = f_{15}$ (CHSH game). These solutions correspond to the solutions marked with purple, brown, orange, and magenta squares in Table B.2.*

This concludes the analysis for the CHSH game using the GHZ- and Bell-like states. Now the same analysis will be performed for the rest of the representative functions of the games for both states.

➤ $\boldsymbol{I = f_{15}}$, $\boldsymbol{O = f_{15}}$, Alice's payoff when the players share a GHZ- *and* a Bell-like state:

$$\text{(GHZ) } \$_A = \frac{1}{4}\left[2 + (a_{11}^2 - \tilde{a}_{11}^2)(b_{11}^2 - \tilde{b}_{11}^2 + c_{11}^2 - \tilde{c}_{11}^2)\left(1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right)\right] \tag{5.16}$$

$$\text{(Bell) } \$_A = \frac{1}{4}\left[2 + (a_{11}^2 - \tilde{a}_{11}^2)(b_{11}^2 - \tilde{b}_{11}^2 + c_{11}^2 - \tilde{c}_{11}^2)\left(1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right)\right] \tag{5.17}$$

These two payoffs for the GHZ- and the Bell-like states are the same as the classical payoff using mixed strategies with an extra multiplying factor on the strategic terms – see equation (4.36) on page 45 in chapter 4. That means that the Nash equilibrium points will be the same as in the classical case; the only difference will be in the presence of entanglement, i.e. $\lambda_{111}$ or $\lambda_{11}$, in the payoffs that those solutions give, exactly as in the case with the CHSH game using the GHZ-like state. These Nash **equilibrium points** with the corresponding payoffs for the GHZ- and the Bell-like states are found in **Table 5.3**.

| GHZ-like state with $I = f_{15}$ , $O = f_{15}$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\{1, 0, 1, 0, 1, 0\}$<br>$\{0, 1, 0, 1, 0, 1\}$ | $\$_A = \$_B = \$_C = 1 - \lambda_{111}^2 + \lambda_{111}^4$ ● |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |

| Bell-like state with $I = f_{15}$ , $O = f_{15}$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\{1, 0, 1, 0, 1, 0\}$<br>$\{0, 1, 0, 1, 0, 1\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}[2 - 3\lambda_{11}^2 + 3\lambda_{11}^4]$ ■ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |

Tab. 5.3: *Nash Equilibria for the game defined by $I = f_{15}$, $O = f_{15}$ using the GHZ- and Bell-like states. The colour of the circle and square helps to identify the payoffs plotted in Figure 5.15.*

Figure 5.15 shows the values of the payoffs of all three players for the GHZ- and Bell-like states marked with a **blue** circle and a **red** square in Table 5.3 as a function of the entanglement parameter. From the plot, as the entanglement increases, the payoff decreases, to reach a minimum at the pure GHZ and Bell states, i.e. when $\lambda_{111} = \lambda_{11} = 1/\sqrt{2}$ . The minimum for the GHZ-like state curve is $3/4 = 0.75$, whereas for the Bell-like state is $5/8 = 0.625$. For the present game, the players would prefer to use the GHZ-like state, since the payoff is always strictly higher than (or equal to) that obtained using the Bell-like state.

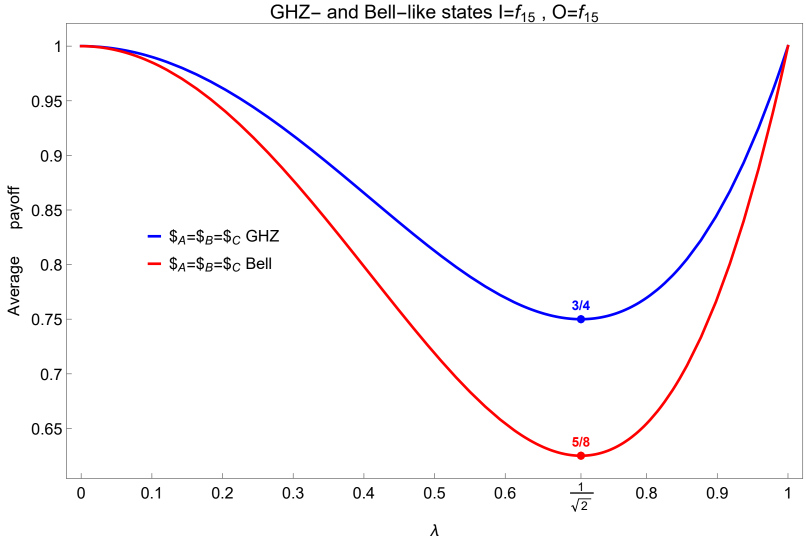



Fig. 5.15: *Plot that shows the players' average payoff for the Nash equilibrium points marked with a blue circle and a red square in Table 5.3 as a function of the corresponding entanglement parameter ($\lambda_{111}$ or $\lambda_{11}$). The colour of the lines matches the coloured circle and square, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_7$, $O = f_{15}$, using the GHZ-like state (blue line/circle) Bell-like state (red line/square).*

➤ $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_8}$, Alice's payoff when the players share a GHZ-like state is:

$$\begin{aligned}(\text{GHZ})\ \$_A =&\frac{1}{8}\Big[2 + 4\lambda_{111}^2(2-\lambda_{111}^2) - 2\lambda_{111}^4\left(2a_{11}^2 + b_{11}^2 + c_{11}^2\right)\\ &+ \left(1 - 2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(b_{11}^2 + c_{11}^2\right)\right.\\ &+ \left.\left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + \tilde{c}_{11}^2\right)\right)\Big]\end{aligned} \tag{5.18}$$

In this case, only when $\lambda_{111} = 1$ this payoff recovers the classical payoff using mixed strategies in equation (4.39) on page 45. $\lambda_{111} = 0$ does not recover the classical scenario because the players are using a correlating "agent", i.e. the quantum state, and choosing this correlating agent to be $|\Psi_{ABC}\rangle = |000000\rangle$ (for $\lambda_{111} = 0$) or $|\Psi_{ABC}\rangle = |111111\rangle$ (for $\lambda_{111} = 1$) leads to different results[16]. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.96)-(B.100).

**Table 5.4** contains the **solutions** with the corresponding interval of validity and associated payoffs. Analogously to what happened with the Bell-like state for the CHSH game ($I = f_7$, $O = f_{15}$), there are some **solutions** that **depend** on two functions of the **entanglement** parameter: $t_G(\lambda_{111})$, and $u_G(\lambda_{111})$. These functions[17] are:

$$t_G(\lambda_{111}) = \frac{\lambda_{111}^4}{1 - 2\lambda_{111}^2\left(1-\lambda_{111}^2\right)} \tag{5.19}$$

[16]In contrast, for the CHSH game $I = f_7$, $O = f_{15}$, choosing $\lambda_{111} = 0$ or $\lambda_{111} = 1$ did indeed recover the classical case. That is because it was parity of the outputs what mattered, i.e. equal outputs or not, so using $|\Psi_{ABC}\rangle = |000000\rangle$ or $|\Psi_{ABC}\rangle = |111111\rangle$ did not affect that.

[17]The subscript "G" in $t_G(\lambda_{111})$ denotes "GHZ" and there is no further numbering because in some of the upcoming solutions for different games using the GHZ-like state, these exact same functions appear again.

$$u_G(\lambda_{111}) = \frac{-1 + 2\lambda_{111}^2}{1 - 2\lambda_{111}^2\,(1 - \lambda_{111}^2)} \tag{5.20}$$

Figure 5.16 plots these functions $t_G(\lambda_{111})$ and $u_G(\lambda_{111})$. As can be seen, $t_G(\lambda_{111})$ lies between 0 and 1 for the whole interval, while $u_G(\lambda_{111})$ only for $1/\sqrt{2} \leqslant \lambda_{111} \leqslant 1$.

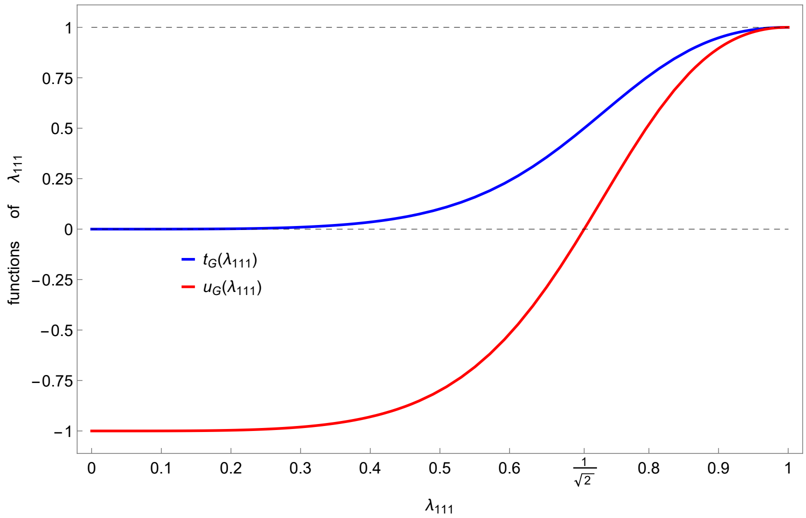



Fig. 5.16: *Plot of the two distinct functions of $\lambda_{111}$ that appear in the Nash equilibrium points for the GHZ-like state shown in Table 5.4 when $I = f_7$, $O = f_8$. The specific expressions are found in equations (5.19)-(5.20).*

| GHZ-like state with $I = f_7$ , $O = f_8$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{111} = 0$ , $\tilde{\#}_{11}^2 = 0$<br>$0 < \lambda_{111} \leqslant \sqrt{\frac{\sqrt{3}-1}{2}} \approx 0.605$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 4t_G$<br>$\sqrt{\frac{\sqrt{3}-1}{2}} < \lambda_{111} \leqslant 1$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$ | $\{0, \tilde{a}_{11}^2, 0, 0, 0, 0\}$<br>$\{0, 0, 0, \tilde{b}_{11}^2, 0, 0\}$<br>$\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + 2\lambda_{111}^2\left(2 - \lambda_{111}^2\right)\right]$ |
| $0 \leqslant \lambda_{111} \leqslant \sqrt{\frac{3-\sqrt{3}}{2}} \approx 0.796$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$<br>$\sqrt{\frac{3-\sqrt{3}}{2}} < \lambda_{111} < 1$ , $-1 + 2u_G \leqslant \tilde{\#}_{11}^2 \leqslant 1$<br>$\lambda_{111} = 1$ , $\tilde{\#}_{11}^2 = 1$ | $\{1, \tilde{a}_{11}^2, 1, 1, 1, 1\}$<br>$\{1, 1, 1, \tilde{b}_{11}^2, 1, 1\}$<br>$\{1, 1, 1, 1, 1, \tilde{c}_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[3 - 2\lambda_{111}^4\right]$ |
| $0 \leqslant \lambda_{111} \leqslant 1$ | $\{t_G, t_G, t_G, t_G, t_G, t_G\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + 2\lambda_{111}^2\left(2 - \lambda_{111}^2\right)\right] - \frac{\lambda_{111}^8}{2\left[1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right]}$ |
| $\lambda_{111} = \frac{1}{\sqrt{2}}$ | $\{1, 0, 0, 1, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$<br>$\{0, 1, 1, 0, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{10 - \tilde{c}_{11}^2}{16}$; $\$_B = \frac{9 + \tilde{c}_{11}^2}{16}$; $\$_C = \frac{9}{16}$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{0, 2t_G, 0, 2t_G, 2t_G, 0\}$ | $\$_A = \$_B = \frac{1}{4}\left[1 + 2\lambda_{111}^2\left(2 - \lambda_{111}^2\right)\right] - \frac{\lambda_{111}^8}{2\left[1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right]}$ |

| | | |
|---|---|---|
| | | $\$_C = \frac{1}{4}\left[1 + 2\lambda_{111}^2\left(2 - \lambda_{111}^2\right)\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{1, u_G, 1, u_G, u_G, 1\}$ | $\$_A = \$_B = \frac{1}{4}\left[1 + 2\lambda_{111}^2\left(2 - \lambda_{111}^2\right)\right] - \frac{\lambda_{111}^8}{2\left[1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right]}$ <br> $\$_C = \frac{1}{4}\left[3 - 2\lambda_{111}^4\right]$ |
| $\sqrt{\frac{\sqrt{3}-1}{2}} \approx 0.605 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{1 + 2u_G, 0, 1 + 2u_G, 0, 0, 1\}$ | $\$_A = \$_B = \frac{1}{4}\left[2\lambda_{111}^2 - 3\lambda_{111}^4 + \frac{1}{1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)}\right]$ <br> $\$_C = \frac{1}{2}\left[-1 + \lambda_{111}^2(1 - \lambda_{111}^2) + \frac{1}{1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant \sqrt{\frac{3-\sqrt{3}}{2}} \approx 0.796$ | $\{2u_G, 1, 2u_G, 1, 1, 0\}$ | $\$_A = \$_B = \frac{1}{4}\left[-1 + 4\lambda_{111}^2 - 3\lambda_{111}^4 + \frac{1}{1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)}\right]$ <br> $\$_C = \frac{1}{2}\left[-1 + \lambda_{111}^2(1 - \lambda_{111}^2) + \frac{1}{1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)}\right]$ |

Tab. 5.4: *Nash Equilibria for the game defined by $I = f_7$, $O = f_8$ using the GHZ-like state. The symbol $\tilde{\#}_{11}^2$ in the first two rows of solutions denotes either $\tilde{a}_{11}^2$, $\tilde{b}_{11}^2$ or $\tilde{c}_{11}^2$, depending on which solution is being considered. The specific expressions for $t_G$ and $u_G$ as functions of $\lambda_{111}$ are found in equations (5.19)-(5.20). The colour of the circles helps to identify the payoffs plotted in Figures 5.17, 5.18, and 5.19.*

From Table 5.4, the **interval restrictions** from the first two sets of solutions – payoff marked with blue and cyan circles[18] – come from the optimisation conditions[19]; and the symbol $\tilde{\#}^2_{11}$ in those denotes either $\tilde{a}^2_{11}$, $\tilde{b}^2_{11}$ or $\tilde{c}^2_{11}$, depending on which solution is being considered. For the rest of the solutions, the optimisation conditions and the requirement that the strategies must be between 0 and 1 sets the interval restrictions of the solution. The marked payoffs with a coloured circle are plotted in Figures 5.17 and 5.18. The non-marked solution is not plotted anywhere because it is the same solution as the one for the Bell-like state when $I = f_7$, $O = f_{15}$ (CHSH game), marked with a green square in Table 5.2 and plotted with a light-green line in Figure 5.9.

In comparison to the classical case, where there were only two different solutions – see Table 4.6 on page 52 –, there are many more using the GHZ-like state. The classical solutions correspond to the **blue** and **cyan** solutions when $\lambda_{111} = 1$ in Table 5.4. Figure 5.17 shows that the solutions giving the payoffs in the blue (cyan) line monotonically increases (decreases) from a value of 0.25 (0.75) to 0.75 (0.25), when moving from $\lambda_{111} = 0$ to $\lambda_{111} = 1$. This means that for each of these two sets of solutions, there is a clearly preferred initial correlating state: the solutions from the cyan line give a better payoff the closer to $\lambda_{111} = 0$, while for the blue line, it is better the closer to $\lambda_{111} = 1$. The **red** solution, however, starts at 0.25 when no entanglement is present and increases gradually until it reaches a maximum of $9/16 = 0.5625$ for the pure GHZ state ($\lambda_{111} = 1/\sqrt{2}$).

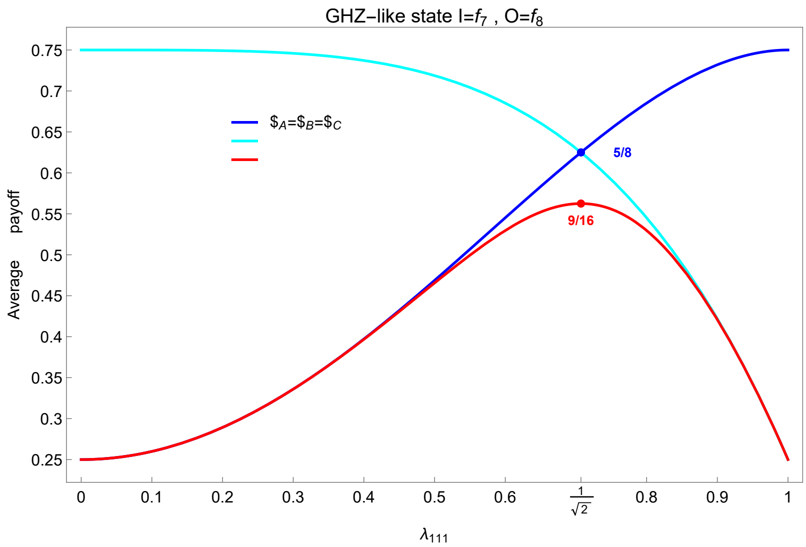



Fig. 5.17: *Plot that shows the players' average payoff for the Nash equilibrium points marked with blue, cyan, and red circles in Table 5.4 as a function of the entanglement parameter* $\lambda_{111}$*. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_7$*,* $O = f_8$*, using the GHZ-like state.*

[18]Technically, these solutions are defined in the whole interval $0 \leqslant \lambda_{111} \leqslant 1$; only that the specific values of $\tilde{a}^2_{11}$, $\tilde{b}^2_{11}$ or $\tilde{c}^2_{11}$ might be restricted, thus affecting the total number of solutions. In any case, the corresponding payoffs do not depend on the specific values of $\tilde{a}^2_{11}$, $\tilde{b}^2_{11}$ or $\tilde{c}^2_{11}$, only on $\lambda_{111}$.

[19]As illustrated in the example for the Bell-like state for the CHSH game in equations (B.74)-(B.85) in appendix B.

The solutions plotted in Figure 5.18 give different payoffs to Carl – dash-doted lines on the plot – in comparison to Alice and Bob – solid line. For the **purple** and **brown** solutions, the players' payoff increases up to a maximum of $9/16 = 0.5625$ for Alice and Bob, and a maximum of $5/8 = 0.625$ for Carl, at $\lambda_{111} = 1/\sqrt{2}$. In contrast, for the **magenta** and **orange** lines, the maximum for Alice and Bob occurs at a different point than for Carl: while Carl's maximum of 5/8 happens at $\lambda_{111} = 1/\sqrt{2}$, for Alice and Bob the maximum is $\$_A = \$_B \approx 0.5682$ at $\lambda_{111} \approx 0.67375$ and $\lambda_{111} \approx 0.7389$ . This fact creates a mismatch between the preferences of one player against the other two.

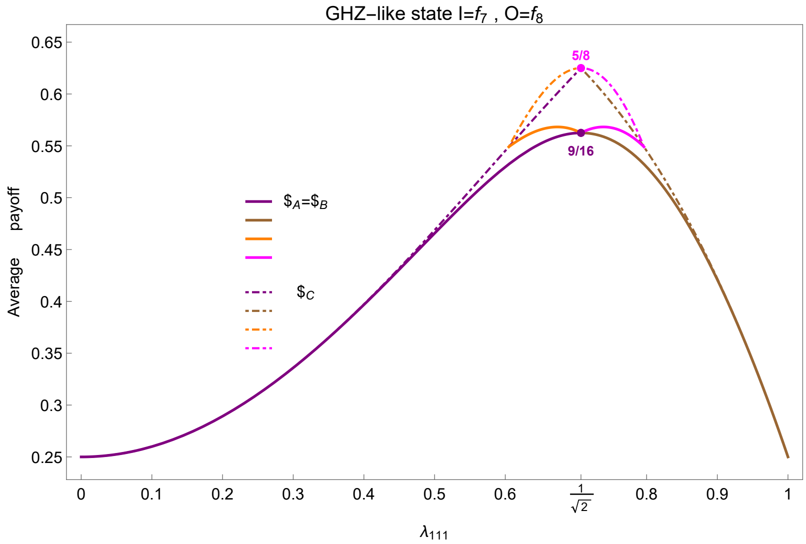



Fig. 5.18: *Plot that shows the players' average payoff for the Nash equilibrium points marked with purple, brown, orange, and magenta circles in Table 5.4 as a function of the entanglement parameter $\lambda_{111}$. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_7$, $O = f_8$, using the GHZ-like state.*

Figure 5.19 plots **all** the solutions together in one **unique plot**, which is an amalgamation of Figure 5.17 and Figure 5.18. The red solid line is exactly the same as the solid purple one in the first interval and the brown one in the second interval. The dot-dashed lines (Carl's payoff) for the purple and brown solutions coincide with the cyan and blue lines, respectively, on each interval. The orange and magenta solutions reduce to the cyan and blue solutions at $\lambda_{111} = \sqrt{(3-\sqrt{3})/2}$ and at $\lambda_{111} = \sqrt{(\sqrt{3}-1)/2}$, respectively. Not only these strategies give the same payoff at those points, as seen from the plot, but they also correspond to the same strategy[20].

[20]To be clearer, the orange strategy is $s* = \{2u_G, 1, 2u_G, 1, 1, 0\}$ and at $\lambda_{111} = \sqrt{(3-\sqrt{3})/2}$ becomes $s* = \{1, 1, 1, 1, 1, 0\}$, which is the same as the cyan strategy at the same value of $\lambda_{111}$.

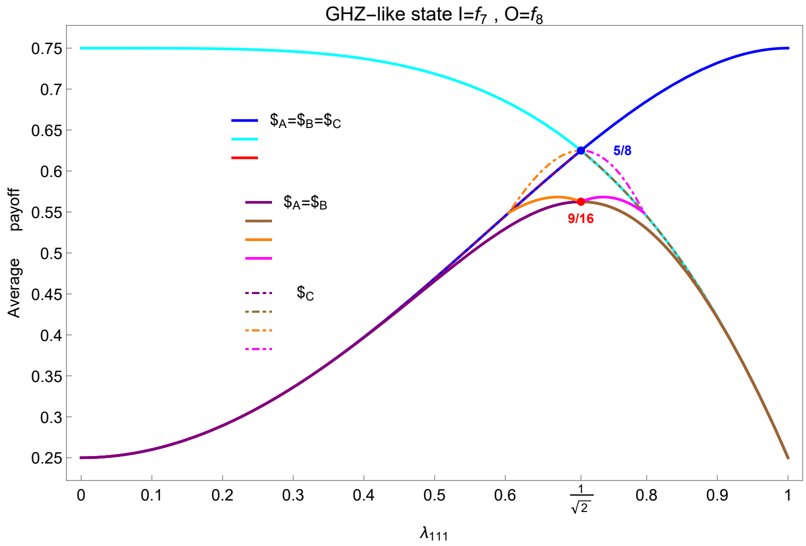



Fig. 5.19: *All payoffs from the different Nash equilibrium solutions showing the players' average payoff as a function of the entanglement parameter $\lambda_{111}$ in a GHZ-like state when $I = f_7$, $O = f_8$. This plot is the combination of Figures 5.17 and 5.18. The red solid line is hidden behind the purple and brown solid lines. The payoffs are found in Table 5.4.*

As before, the **social welfare** ($\$_A + \$_B + \$_C$) of each Nash equilibrium solution can be computed. The specific expressions for the social welfare for all the equilibrium points in Table 5.4 are found in Table B.4 in appendix B. Figure 5.20 plots the social welfare of all the solutions as a function of $\lambda_{111}$, with the colour of the line in the plot matching that of the circle in the latter table. The minimum and maximum values are, as in the classical case, $3/4 = 0.75$ and $9/4 = 2.25$, respectively, both achieved here only when there is no entanglement. In this case, the preferred solution with the highest collective payoff would be the cyan line for $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$ and the blue line for $1/\sqrt{2} \leqslant \lambda_{11} \leqslant 1$; while the worst social welfare is given by the red line. The black point corresponds to a solution giving a constant social welfare for the pure GHZ state. The maximum of the magenta and orange solutions is $\$_A + \$_B + \$_C \approx 1.756$ at $\lambda_{111} \approx 0.689$ and at $\lambda_{111} \approx 0.725$, respectively.

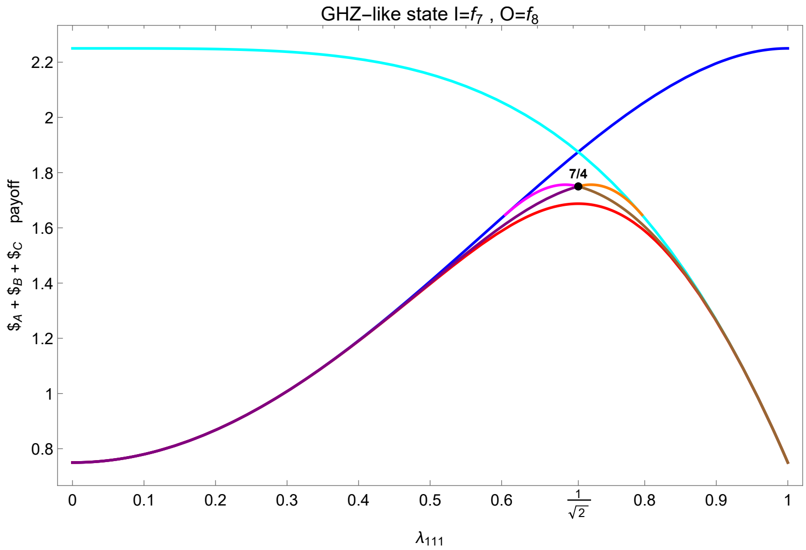



Fig. 5.20: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{111}$ using a GHZ-like state for $I = f_7$, $O = f_8$. The Nash equilibrium solutions with the individual payoffs are found in Table 5.4, while the social welfare of them is in Table B.4. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_8}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}(\text{Bell})\ \$_A =& \frac{1}{8}\left[2 + 4\lambda_{11}^2(1+\lambda_{11}^2(1-\lambda_{11}^2)) + 2\lambda_{11}^2(1-3\lambda_{11}^2+\lambda_{11}^4)\left(2a_{11}^2+b_{11}^2+c_{11}^2\right)\right.\\ &+\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\left(\left(a_{11}^2+\tilde{a}_{11}^2\right)\left(b_{11}^2+c_{11}^2\right)\right.\\ &\left.\left.+\left(a_{11}^2-\tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2+\tilde{c}_{11}^2\right)\right)\right] \end{aligned} \tag{5.21}$$

As with the GHZ-like state, $\lambda_{11} = 1$ recovers the classical payoff in equation (4.41), while $\lambda_{11} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.102)-(B.106).

**Table 5.5** contains the **solutions** with the corresponding interval of validity and associated payoffs. As with the GHZ-like state, the interval restrictions for the solutions whose payoff is marked with blue and cyan squares comes from imposing the optimisation conditions. For the rest of the solutions, the optimisation conditions and the requirement that the strategies must be between 0 and 1 sets the interval restrictions of the solution[21]. Some **solutions** depend on certain **functions** of $\boldsymbol{\lambda_{11}}$, and this time, these function are slightly different than $t_{B1}(\lambda_{11})$, $v_{B1}(\lambda_{11})$, and $v_{B1}(\lambda_{11})$, in equations (5.13)-(5.15) for the Bell-like state when $I = f_7$, $O = f_{15}$ (CHSH game). These **new functions**, labelled as $t_{B2}$, $u_{B2}$, and $v_{B2}$, look:

$$t_{B2}(\lambda_{11}) = \frac{-\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)} \tag{5.22}$$

$$u_{B2}(\lambda_{11}) = 2t_{B2}(\lambda_{11}) - 1 = \frac{-1+\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)} \tag{5.23}$$

[21] As an illustrative example, the optimisation equations of the strategy whose payoff is marked with a red square are worked out in equations (B.107)-(B.109) in appendix B.

$$v_{B2}(\lambda_{11}) = 4t_{B2}(\lambda_{11}) - 1 = \frac{-1 - \lambda_{11}^2 + 9\lambda_{11}^4 - 4\lambda_{11}^6}{1 - 3\lambda_{11}^2\,(1 - \lambda_{11}^2)} \tag{5.24}$$

Figure 5.21 plots $t_{B2}(\lambda_{11})$, $u_{B2}(\lambda_{11})$, and $v_{B2}(\lambda_{11})$. As can be seen from the plot, besides $\lambda_{11} = 0, 1$ ; $0 \leqslant t_{B2}(\lambda_{11}) \leqslant 1$ only for $(\sqrt{5}-1)/2 \leqslant \lambda_{11} \leqslant \sqrt{(\sqrt{5}-1)/2}$ [22]. For $0 \leqslant u_{B2}(\lambda_{11}) \leqslant 1$, then $1/\sqrt{2} \leqslant \lambda_{11} \leqslant \sqrt{(\sqrt{5}-1)/2}$. Finally, for $0 \leqslant v_{B2}(\lambda_{11}) \leqslant 1$, then $0.6695 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$.

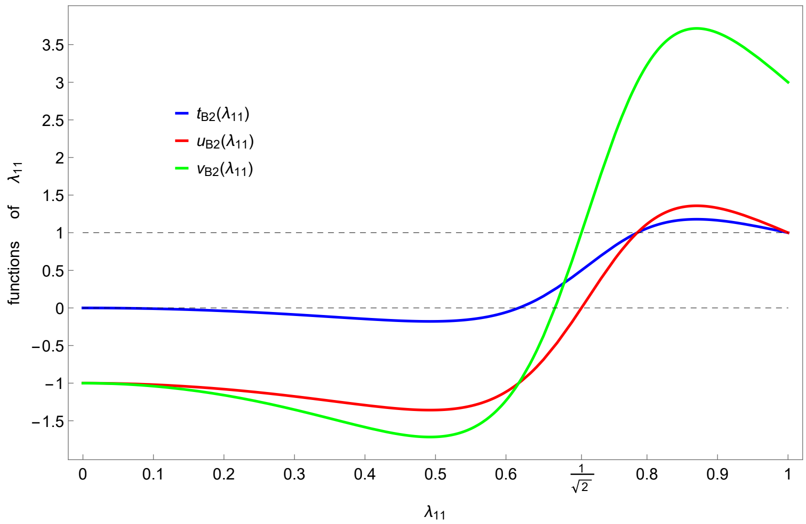



Fig. 5.21: *Plot of the three distinct functions of $\lambda_{11}$ that appear in the Nash equilibrium points for the Bell-like state shown in Table 5.5 when $I = f_7$, $O = f_8$. The specific expressions are found in equations (5.22)-(5.24).*

[22] As a mathematical curiosity, $(\sqrt{5}-1)/2 = \varphi - 1 = 1/\varphi$, where $\varphi = 1.6180\ldots$ is the golden ratio. Then $\sqrt{(\sqrt{5}-1)/2} = 1/\sqrt{\varphi}$.

| Bell-like state with $I = f_7$ , $O = f_8$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{11} = 0$ or $\lambda_{11} = \frac{\sqrt{5}-1}{2} \approx 0.618$, $\tilde{\#}_{11}^2 = 0$<br>$\frac{\sqrt{5}-1}{2} < \lambda_{11} \leqslant 0.6695$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 4t_{B2}$<br>$0.6695 < \lambda_{11} \leqslant 1$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$ | $\{0, \tilde{a}_{11}^2, 0, 0, 0, 0\}$<br>$\{0, 0, 0, \tilde{b}_{11}^2, 0, 0\}$<br>$\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right]$ |
| $0 \leqslant \lambda_{111} \leqslant 0.743$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$<br>$0.743 < \lambda_{111} < \sqrt{\frac{\sqrt{5}-1}{2}}$ , $-2 + v_{B2} \leqslant \tilde{\#}_{11}^2 \leqslant 1$<br>$\lambda_{11} = 1$ or $\lambda_{11} = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.786$ , $\tilde{\#}_{11}^2 = 1$ | $\{1, \tilde{a}_{11}^2, 1, 1, 1, 1\}$<br>$\{1, 1, 1, \tilde{b}_{11}^2, 1, 1\}$<br>$\{1, 1, 1, 1, 1, \tilde{c}_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[3 - 2\lambda_{11}^4(2 - \lambda_{11}^2)\right]$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right] - \frac{\lambda_{11}^4\left(1 - 3\lambda_{11}^2 + \lambda_{11}^4\right)^2}{2\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ |
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1, 0, 0, 1, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$<br>$\{0, 1, 1, 0, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{18 - \tilde{c}_{11}^2}{32}$ ; $\$_B = \frac{17 + \tilde{c}_{11}^2}{32}$ ; $\$_C = \frac{17}{32}$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{0, 2t_{B2}, 0, 2t_{B2}, 2t_{B2}, 0\}$ | $\$_A = \$_B = \frac{1}{4}\left[1 + 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right] - \frac{\lambda_{11}^4\left(1 - 3\lambda_{11}^2 + \lambda_{11}^4\right)^2}{2\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ |

| | | |
|---|---|---|
| | | $\$_C = \frac{1}{4}\left[1 + 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{1, u_{B2}, 1, u_{B2}, u_{B2}, 1\}$ | $\$_A = \$_B = \frac{1}{4}\left[1 + 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right] - \frac{\lambda_{11}^4\left(1 - 3\lambda_{11}^2 + \lambda_{11}^4\right)^2}{2\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ <br> $\$_C = \frac{1}{4}\left[3 - 2\lambda_{11}^4(2 - \lambda_{11}^2)\right]$ |
| $0.6695 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{v_{B2}, 0, v_{B2}, 0, 0, 1\}$ | $\$_A = \$_B = \frac{\left(1 - \lambda_{11}^2\right)\left(1 - \lambda_{11}^2 + 9\lambda_{11}^6 - 11\lambda_{11}^8 + 4\lambda_{11}^{10}\right)}{4\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ <br> $\$_C = \frac{-\lambda_{11}^2\left(1 - 12\lambda_{11}^2 + 14\lambda_{11}^4 + 13\lambda_{11}^6 - 24\lambda_{11}^8 + 8\lambda_{11}^{10}\right)}{4\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 0.743$ | $\{-1 + v_{B2}, 1, -1 + v_{B2}, 1, 1, 0\}$ | $\$_A = \$_B = \frac{\lambda_{11}^2\left(2 - 2\lambda_{11}^2 + \lambda_{11}^4 - 5\lambda_{11}^6 + 9\lambda_{11}^8 - 4\lambda_{11}^{10}\right)}{4\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ <br> $\$_C = \frac{-\lambda_{11}^2\left(1 - 12\lambda_{11}^2 + 14\lambda_{11}^4 + 13\lambda_{11}^6 - 24\lambda_{11}^8 + 8\lambda_{11}^{10}\right)}{4\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ |

Tab. 5.5: *Nash Equilibria for the game defined by $I = f_7$, $O = f_8$ using the Bell-like state. The symbol $\tilde{\#}_{11}^2$ in the first two rows of solutions denotes either $\tilde{a}_{11}^2$, $\tilde{b}_{11}^2$ or $\tilde{c}_{11}^2$, depending on which solution is being considered. The specific expressions for $t_{B2}$, $u_{B2}$, and $v_{B2}$ are found in equations (5.22)-(5.24). The colour of the squares helps to identify the payoffs plotted in Figures 5.22, 5.24, 5.23, and 5.25.*

The results of Table 5.5 for the Bell-like state and the results in Table 5.4 for the GHZ-like look quite similar: they have the same number of points – coloured-marked in a similar fashion –, but with a different dependence on the entanglement parameter and interval restrictions. In any case, the analysis of the plots will be rather similar.

Figure 5.22 plots the solutions whose payoffs are marked with **blue**, **cyan**, and **red** squares in Table 5.5 as a function of $\lambda_{11}$. The analysis is very similar to the first three solutions using the GHZ-like state in Figure 5.17, but in this case, the solutions are not defined for the whole interval of $\lambda_{11}$. The cyan (blue) solution decreases (increases) monotonically to a minimum (maximum) at the end of its defined interval, while the red one increases as the entanglement increases to reach the maximum of $17/32 = 0.53125$ at $\lambda_{11} = 1/\sqrt{2}$. The minimum value of these solutions at $\lambda_{11} = (\sqrt{5}-1)/2$ and $\lambda_{11} = \sqrt{(\sqrt{5}-1)/2}$ is $\$_A = \$_B = \$_C = \sqrt{5}-7/4 \approx 0.486$

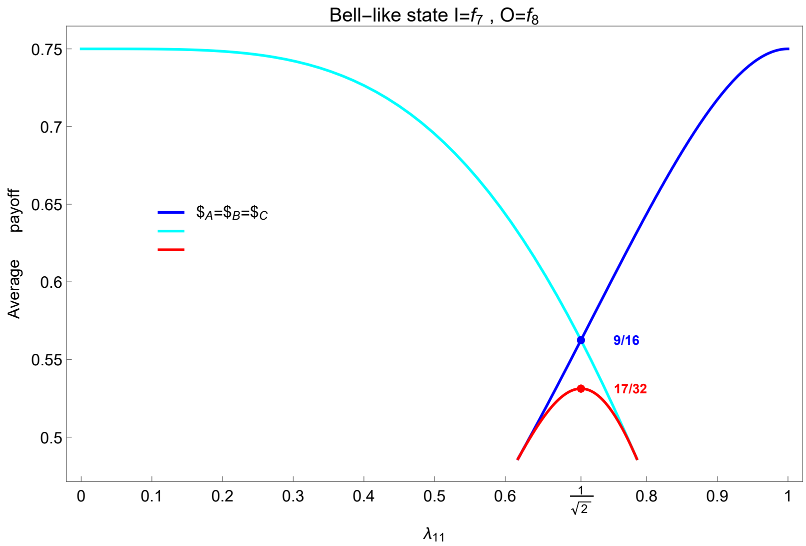
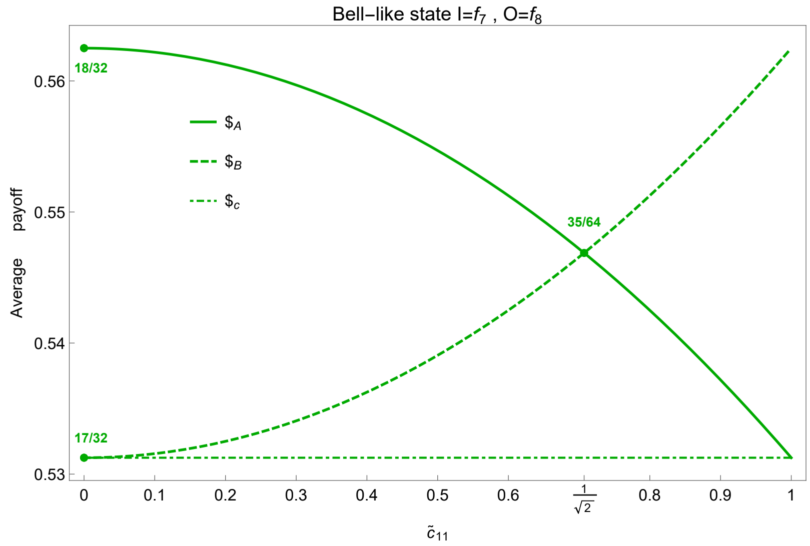
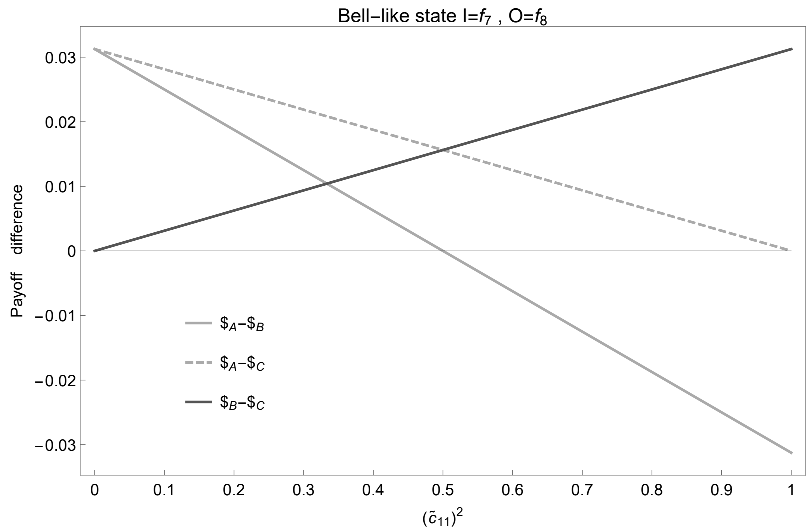



Fig. 5.22: *Plot that shows the players' average payoff for the Nash equilibrium points marked with blue, cyan, and red squares in Table 5.5 as a function of the entanglement parameter* $\lambda_{11}$*. The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_7$*,* $O = f_8$*, using the Bell-like state.*

Figure 5.23(a) plots the solution with payoffs marked with a **green** square in Table 5.5 as a function of Carl's strategy $\tilde{c}_{11}$. Carl's payoff, the dash-dotted green line, is constant 17/32, while Alice's and Bob's, the solid and dashed green lines, respectively, depend directly on $\tilde{c}_{11}$. As Carl's choice of $\tilde{c}_{11}$ increases, Alice's payoff decreases, while Bob's increases. Figure 5.23(b) plots the differences $\$_A - \$_B$, $\$_A - \$_C$, and $\$_B - \$_C$ with solid, dashed, and dot-dashed gray lines, respectively, as a function of $\tilde{c}_{11}^2$. In that case, Alice and Bob always get a higher or equal payoff than Carl (positive slope for both lines). For $\tilde{c}_{11}^2 = 1/2$, Alice and Bob get the same payoff, and for $\tilde{c}_{11}^2 = 1/3$ the difference between Alice's and Bob's payoff is the same as between Bob's and Carl's.




(a)




(b)

Fig. 5.23: (a) *Players' average payoffs for the equilibrium point marked with a green square in Table 5.2 as a function of Carl's strategy* $\tilde{c}_{11}$. *These results are for* $I = f_7$ *and* $O = f_8$, *for the Bell-like state.* (b) *Payoff difference* $\$_A - \$_B$, $\$_A - \$_C$, *and* $\$_B - \$_C$ *for the payoffs plotted in* (a). *Note that the horizontal axis is the* square *of Carl's strategy* $\tilde{c}_{11}^2$, *choice made so that the differences correspond to lines.*

The solutions plotted in Figure 5.24 give different payoffs to Carl – dash-dotted lines on the plot – in comparison to Alice and Bob – solid line. For the **purple** and **brown** solutions, the players' payoff increases up to a maximum of $17/32 = 0.53125$ for Alice and Bob, and a maximum of $9/16 = 0.5625$ for Carl, at $\lambda_{11} = 1/\sqrt{2}$. In contrast, for the **magenta** and **orange** lines, the maximum for Alice and Bob occurs at a different point than for Carl: while Carl's maximum of $9/16$ happens at $\lambda_{11} = 1/\sqrt{2}$, for Alice and Bob the maximum is $\$_A = \$_B \approx 0.5348$ at $\lambda_{11} \approx 0.6906$ and $\lambda_{11} \approx 0.7232$ [23]. Again, these two solutions create a mismatch between the preferences of one player against the other two.

[23] These solutions are close to the ones for the GHZ-like state, where the maximum for Alice and Bob was $\$_A = \$_B \approx 0.5682$ at $\lambda_{111} \approx 0.67375$ and at $\lambda_{111} \approx 0.7389$.

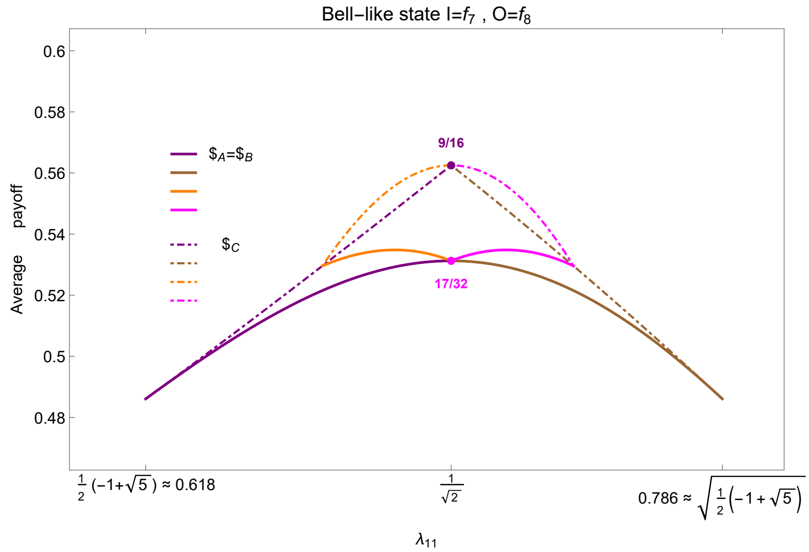



Fig. 5.24: *Plot that shows the players' average payoff for the Nash equilibrium points marked with purple, brown, orange, and magenta squares in Table 5.5 as a function of the entanglement parameter $\lambda_{11}$. The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. Note the short interval validity of those solutions. These results are for the choice of functions $I = f_7$, $O = f_8$, using the GHZ-like state.*

Figure 5.25 plots **all the solutions** that depend on $\lambda_{11}$ in a unique plot – not the green-square solution, since it does not depend on $\lambda_{11}$ –, which is an amalgamation of Figure 5.22 and Figure 5.24. In the exact same way as in the GHZ-like state, the red solid line is the same as the solid purple one in the first interval and the brown one in the second interval. The dot-dashed lines (Carl's payoff) for the purple and brown solutions coincide with the cyan and blue lines, respectively, on each interval.

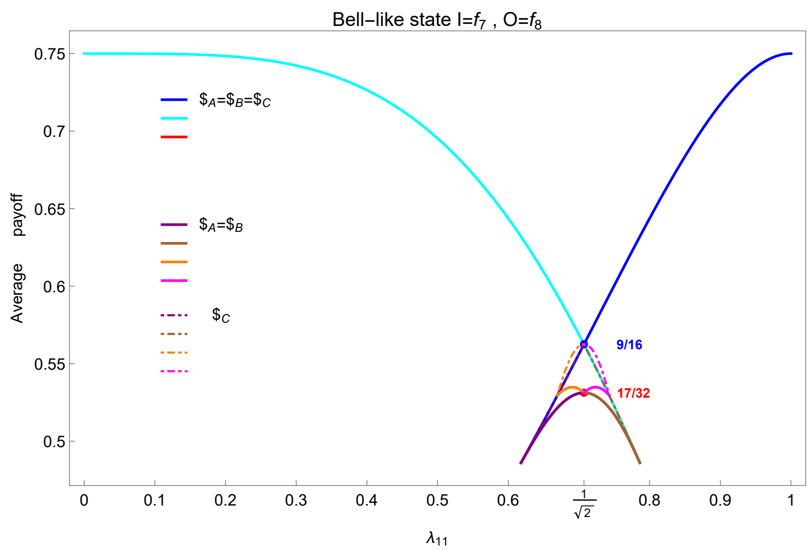



Fig. 5.25: *Payoffs from the different Nash equilibrium solutions showing the players' average payoff as a function of the entanglement parameter $\lambda_{11}$ in a Bell-like state when $I = f_7$, $O = f_8$. This plot is the combination of Figures 5.22 and 5.24. The payoffs are found in Table 5.5.*

The **social welfare** ($\$_A + \$_B + \$_C$) of each Nash equilibrium point in Table 5.5 is computed and shown in Table B.5 in appendix B. Figure 5.26 shows the social welfare for all solutions as a function of $\lambda_{11}$. Exactly as with the GHZ-like state, the preferred solutions are the cyan and blue ones at each interval, while the red one is the worst one. The maximum for the orange and magenta lines is $\$_A+\$_B+\$_C \approx 1.629$ at $\lambda_{11} \approx 0.699$ and at $\lambda_{11} \approx 0.715$, respectively.

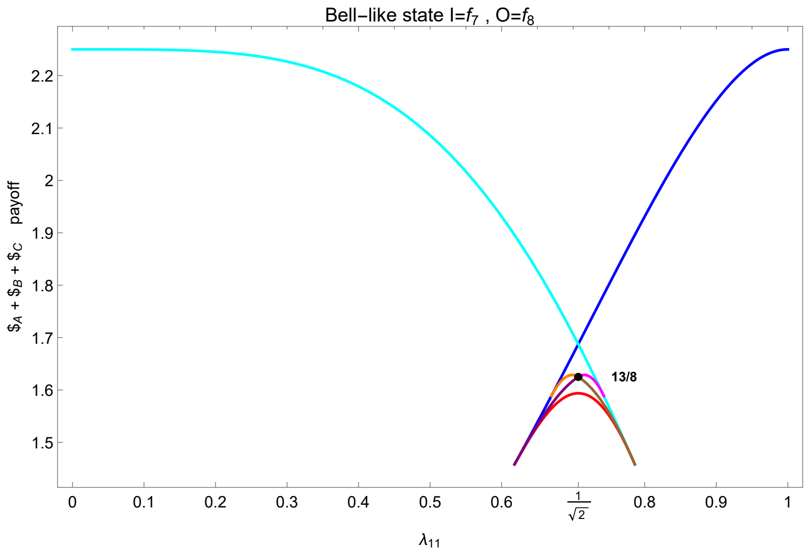



Fig. 5.26: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{11}$ using a Bell-like state for $I = f_7$, $O = f_8$. The Nash equilibrium solutions with the individual payoffs are found in Table 5.5, while the social welfare of them is in Table B.5. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_{10}}$, Alice's payoff when the players share a GHZ-like state:

$$\begin{aligned}(\text{GHZ})\ \$_A =& \frac{1}{8}\left[2 + 2(a_{11}^2 + c_{11}^2)(1-\lambda_{111}^2)^2 + 2\lambda_{111}^4\left(a_{11}^2 + b_{11}^2\right)\right.\\ &- \left(1 - 2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(b_{11}^2 + c_{11}^2\right)\right.\\ &\left.\left.+ \left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + \tilde{c}_{11}^2\right)\right)\right]\end{aligned} \tag{5.25}$$

In this case, only $\lambda_{111} = 1$ recovers the classical payoff in equation (4.41), while $\lambda_{111} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.111)-(B.115).

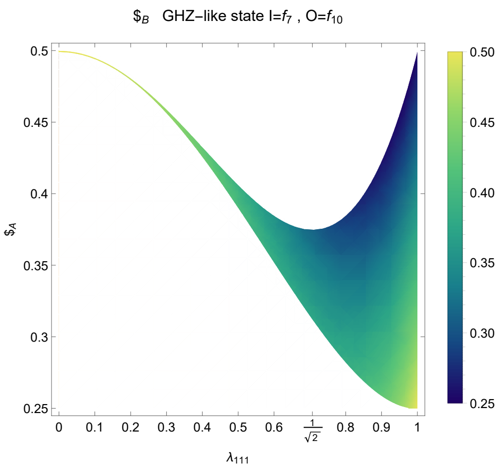
The **solutions** are found in **Table 5.6**. Interestingly enough, despite the payoff using the GHZ-like state in equation (5.25) being different to the **classical** one, the **points** that **optimise** both payoffs are the **same**[24]. That means, that the Nash equilibrium points are identical – compare Table 5.6 and Table 4.7 –, the only difference is the appearance of $\lambda_{111}$ in the corresponding payoffs.

[24]This result arises because the corresponding Karush–Kuhn–Tucker (KKT) conditions using the GHZ-like state are the same as the ones using mixed strategies up to a (positive) factor that depends on $\lambda_{111}$. See equations (B.111)-(B.115) in appendix B.

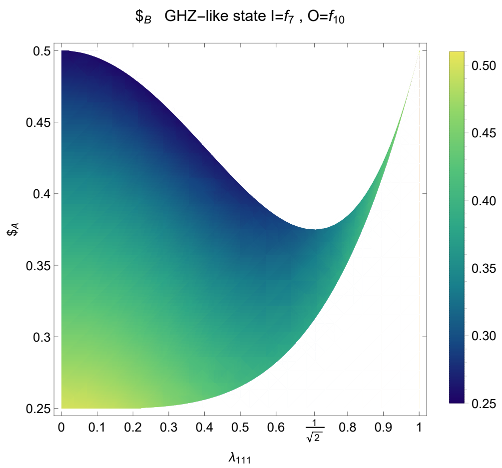
| GHZ-like state with $I = f_7$ , $O = f_{10}$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\{1, 1, 0, 0, c_{11}^2, c_{11}^2\}$ | $\$_A = \frac{1}{4}\left[2 - 2\lambda_{111}^2 + (2 - c_{11}^2)\,\lambda_{111}^4\right]$ * <br> $\$_B = \frac{1}{4}\left[2 - 2\lambda_{111}^2 + (1 + c_{11}^2)\,\lambda_{111}^4\right]$ * <br> $\$_C = \frac{1}{4}\left[1 + \lambda_{111}^4\right]$ ● |
| $\{0, 0, 1, 1, c_{11}^2, c_{11}^2\}$ | $\$_A = \frac{1}{4}\left[1 + \lambda_{111}^4 + c_{11}^2\,(1 - \lambda_{111}^2)^2\right]$ * <br> $\$_B = \frac{1}{4}\left[2\,(1 - \lambda_{111}^2 + \lambda_{111}^4) - c_{11}^2\,(1 - 2\lambda_{111}^2 + \lambda_{111}^4)\right]$ * <br> $\$_C = \frac{1}{4}\left[2 - 2\lambda_{111}^2 + \lambda_{111}^4\right]$ ● |
| $\{1, 0, 1, 0, c_{11}^2, 0\}$ | $\$_A = \frac{1}{8}\left[3 - 2\lambda_{111}^2 + 4\lambda_{111}^4 + c_{11}^2\,(1 - 2\lambda_{111}^2)\right]$ * <br> $\$_B = \frac{1}{8}\left[5 - 6\lambda_{111}^2 + 4\lambda_{111}^4 - c_{11}^2\,(1 - 2\lambda_{111}^2)\right]$ * <br> $\$_C = \frac{1}{2}\left[1 - \lambda_{111}^2\,(1 - \lambda_{111}^2)\right]$ ● |
| $\{0, 1, 0, 1, c_{11}^2, 1\}$ | $\$_A = \frac{1}{8}\left[4\,(1 - \lambda_{111}^2 + \lambda_{111}^4) + c_{11}^2\,(1 - 2\lambda_{111}^2)\right]$ * <br> $\$_B = \frac{1}{8}\left[4\,(1 - \lambda_{111}^2 + \lambda_{111}^4) - c_{11}^2\,(1 - 2\lambda_{111}^2)\right]$ * <br> $\$_C = \frac{1}{2}\left[1 - \lambda_{111}^2\,(1 - \lambda_{111}^2)\right]$ ● |
| $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$ | $\$_A = \$_B = \$_C = \frac{1}{8}\left[3 - 2\lambda_{111}^2\,(1 - \lambda_{111}^2)\right]$ ● |

Tab. 5.6: *Nash Equilibria for the game defined by $I = f_7$, $O = f_{10}$ using the GHZ-like state. The colour of the circles identifies the payoffs plotted in Figure 5.28. The solutions with asterisks are plotted in Figure 5.27.*

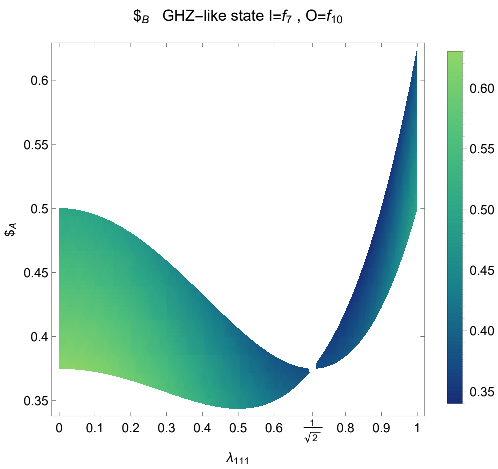
The payoffs marked with an **asterisk** in Table 5.6 depend on Carl's strategy $c_{11}^2$ and $\lambda_{111}$, while the ones marked with a circle only on $\lambda_{111}$. For each of the solutions marked with an asterisk, Bob's payoff $\$_B$ is plotted in a **density plot** as a function of Alice's payoff $\$_A$ and $\lambda_{111}$ in Figure 5.27, plots that also include the possible values of $\$_A$, as was done in the GHZ-like state for $I = f_7$, $O = f_{15}$ – see equation (5.11) and Figure 5.5.



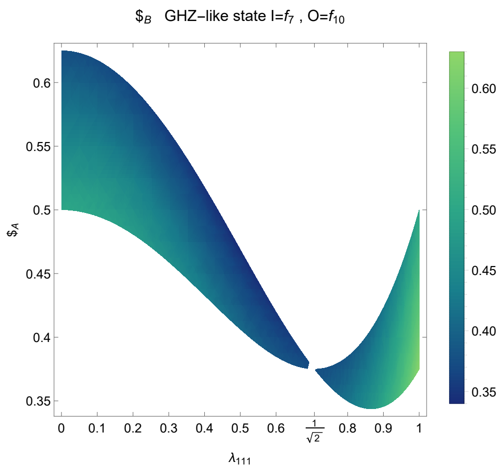

(a) *Bob's payoff* $\$_B$ *




(b) *Bob's payoff* $\$_B$ *




(c) *Bob's payoff* $\$_B$ *




(d) *Bob's payoff* $\$_B$ *

Fig. 5.27: *Density plots showing Bob's payoffs as a function of* $\$_A$ *and the entanglement parameter* $\lambda_{111}$*. These payoffs correspond to the Nash equilibrium solutions marked with purple, brown, orange, and magenta asterisks in Table 5.6. These results are when* $I = f_7$*,* $O = f_{10}$*, and using the GHZ-like state.*

A detailed analysis of each plot in Figure 5.27 is as follows:

- The players' payoffs in Figure 5.27(a) (**purple** asterisk) vary from $1/4 = 0.25$ to $1/2 = 0.5$ , and there is a clear distinction between $\lambda_{111} = 0$ and $\lambda_{111} = 1$. Around $\lambda_{111} = 0$, the payoff for both players is approximately the same $1/2 = 0.5$ (fine yellow "line" in the plot), while around $\lambda_{111} = 1$ the payoff for each player can vary widely at the expense of the other player's payoff (wide vertical region with all range of colours/payoff) – corresponding to the classical case. For $\lambda_{111} = 1/\sqrt{2}$, the players' payoff varies from $5/16 = 0.3125$ to $3/8 = 0.375$ .
- The analysis of Figure 5.27(b) (**brown** asterisk) is very similar to the one of Figure 5.27(a), but in this case, the players receive the same (maximum) payoff of $1/2 = 0.5$ at $\lambda_{111} = 1$, and the region around $\lambda_{111} = 0$ recovers the classical case with the high variability in payoffs. As before, at $\lambda_{111} = 1/\sqrt{2}$, the players' payoff varies from $5/16 = 0.3125$ to $3/8 = 0.375$ . The new feature is that this region with higher variability is much wider in terms of $\lambda_{111}$;

compare, for example, the region $0 \leqslant \lambda_{111} \leqslant 0.1$ in Figure 5.27(b) with the region $0.9 \leqslant \lambda_{111} \leqslant 1$ in Figure 5.27(a), the latter being much more reduced in terms of payoffs than the first one. That means that varying $\lambda_{111}$ slightly around those two "more-classical" regions, the solution marked with a purple asterisk will be impacted much more than the one with the brown asterisk.

- Figure 5.27(c) (**orange** asterisk) is different from the previous two. The players' payoffs in this case are higher since they vary from $11/32 = 0.34375$ to $5/8 = 0.625$ . For $\lambda_{111} = 0$, Alice's payoff varies from $3/8 = 0.375$ to $1/2 = 0.5$, while Bob's from $1/2 = 0.5$ to $5/8 = 0.625$ ; whereas for $\lambda_{111} = 1$, it is the other way around. At $\lambda_{111} = 1/\sqrt{2}$, Alice and Bob receive the same payoff $3/8 = 0.375$ , which is why it is not very visible in the density plot, because it corresponds to a unique point and the plotting software – *Wolfram Mathematica* – excluded it. It is interesting to see that the absolute minimum for Alice's payoff occurs exactly at $\lambda_{111} = 1/2$ (and $c_{11} = 0$), giving her $11/32 = 0.34375$, while Bob's minimum payoff is the same and occurs at $\lambda_{111} = \sqrt{3}/2 \approx 0.866$ (and $c_{11} = 0$). The highest payoff values for Alice are located around $\lambda_{111} = 1$, while for Bob, they are located around $\lambda_{111} = 0$

- The analysis for Figure 5.27(d) (**magenta** asterisk) is very similar to the one for Figure 5.27(c); it is just swapping some of the values. Again, the players' payoff ranges from $11/32 = 0.34375$ to $5/8 = 0.625$ . Analogously to the previous plot, in the region around $\lambda_{111} = 0$, Alice's payoff is between $1/2 = 0.5$ and $5/8 = 0.625$, while Bob's is between $3/8 = 0.375$ and $1/2 = 0.5$ ; whereas in the region around $\lambda_{111} = 1$ it is the other way around. Again, at $\lambda_{111} = 1/\sqrt{2}$ corresponds to a single point where both players get $3/8 = 0.375$ . The minimum for Alice of $11/32 = 0.34375$ occurs at $\lambda_{111} = \sqrt{3}/2 \approx 0.866$ (and $c_{11} = 1$), while is the same minimum for Bob at $\lambda_{111} = 1/2$ (and $c_{11} = 1$).

Moving now to the payoffs that only depend on $\lambda_{111}$, marked with a **circle** in Table 5.6, these are plotted in Figure 5.28. The **purple**, **brown**, and **orange** dot-dashed lines correspond to Carl's payoff, while the solid **red** line is the only solution that gives equal payoff to all players. The purple and brown dot-dashed lines increase and decrease monotonically, respectively. In any case, from Carl's perspective, he would prefer the orange solution – corresponding to the third and fourth solution in Table 5.6 – because it gives him a strictly higher payoff than the other solutions. Both for the red and orange solutions, they give worse payoff as the entanglement increases.

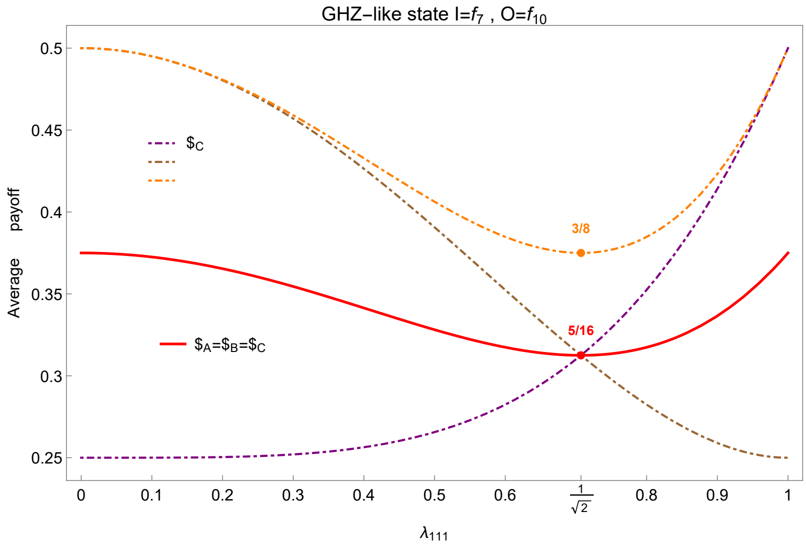



Fig. 5.28: *Plot that shows the players' average payoff for the Nash equilibrium points marked with purple, brown, orange, and red circles in Table 5.6 as a function of the entanglement parameter $\lambda_{111}$. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_7$, $O = f_{10}$, using the GHZ-like state.*

The expressions for the **social welfare** ($\$_A + \$_B + \$_C$) of each equilibrium solution in Table 5.6 are found in Table B.6 in appendix B. Figure 5.29 shows the social welfare as a function of $\lambda_{111}$. The social welfare of the payoffs coloured in purple and brown is the same, and also for the ones coloured in orange and magenta. That is why in the plot there are only three different lines. In this case, the social welfare worsens as the entanglement increases, therefore the players would prefer to have a fully non-entangled state.

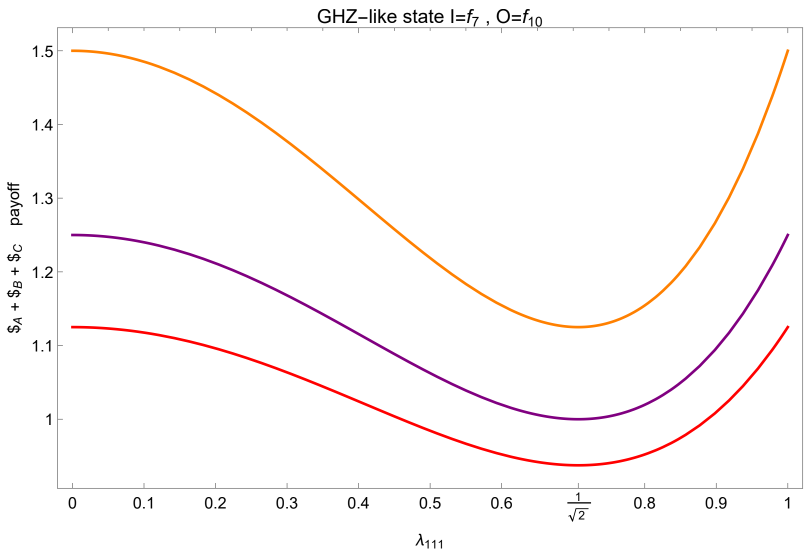



Fig. 5.29: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{111}$ using a GHZ-like state for $I = f_7$, $O = f_{10}$. The Nash equilibrium solutions with the individual payoffs are found in Table 5.6, while the social welfare of them is in Table B.6. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ $\boldsymbol{I = f_7}$, $\boldsymbol{O = f_{10}}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}\text{(Bell) } \$_A =& \frac{1}{8}\Big[2 + 4\lambda_{11}^2(1-\lambda_{11}^2)^2 + 2(a_{11}^2 + c_{11}^2)(1-\lambda_{11}^2)^3 \\ &- 2\lambda_{11}^2(1 - 3\lambda_{11}^2 + \lambda_{11}^4)\left(a_{11}^2 + b_{11}^2\right) \\ &- \left(1 - 3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\left(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(b_{11}^2 + c_{11}^2\right)\right. \\ &+ \left.\left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + \tilde{c}_{11}^2\right)\right)\Big]\end{aligned} \tag{5.26}$$

As before, $\lambda_{11} = 1$ recovers the classical payoff in equation (4.41), while $\lambda_{11} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.117)-(B.121).

The **results** are found in **Table 5.7**. This time there are quite a few more solutions and are different. The first four solutions, **strategies** marked with a coloured **star**, are the same as the ones for the GHZ-like state (and the classical) *except* that there are interval restrictions on $\lambda_{11}$ *and* Carl's strategy $c_{11}$, coming from the optimisation conditions. These restrictions change the solutions and their payoffs shown in that table. To avoid having a rather long table, these solutions marked with a coloured star are further detailed in another table, Table 5.8.

Looking now only at Table 5.7, some of the **solutions** explicitly depend on **functions** of the **entanglement** parameter: $t_{B1}(\lambda_{11})$, $u_{B1}(\lambda_{11})$, and $v_{B1}(\lambda_{11})$, which are exactly the **same** functions that appeared for the Bell-like state when $I = f_7$, $O = f_{15}$ (CHSH game). Their expressions are found in equations (5.13)-(5.15). Again, the interval restrictions of the solutions come from imposing the optimisation conditions and requiring that the strategies lie between 0 and 1. The marked payoffs with coloured squares will be shown in a plot next, while the two non-marked solutions (eighth and ninth solution) correspond, up to player permutations, to some solutions in the table detailing the star-marked solutions, Table 5.8.

| (1/2) Bell-like state with $I = f_7$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{11} = 0, 1, \frac{1}{\sqrt{2}},\ 0 \leqslant c_{11}^2 \leqslant 1$<br>$0 < \lambda_{11} < \frac{1}{\sqrt{2}},\ c_{11}^2 = 0$<br>$\frac{1}{\sqrt{2}} < \lambda_{11} < 1,\ c_{11}^2 = 1$ | $\{1, 1, 0, 0, c_{11}^2, c_{11}^2\}$ ★ | $\$_A = \frac{1}{4}\left[2\left(1-\lambda_{11}^2+\lambda_{11}^4\right) - c_{11}^2\lambda_{11}^6\right]$<br>$\$_B = \frac{1}{4}\left[2-\lambda_{11}^2-\lambda_{11}^4+\lambda_{11}^6 - c_{11}^2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right]$<br>$\$_C = \frac{1}{4}\left[1+\lambda_{11}^2-\lambda_{11}^4+\lambda_{11}^6 - c_{11}^2\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right]$ |
| | $\{0, 0, 1, 1, c_{11}^2, c_{11}^2\}$ ★ | $\$_A = \frac{1}{4}\left[1+\lambda_{11}^2-\lambda_{11}^4+\lambda_{11}^6 + c_{11}^2\left(1-\lambda_{11}^2\right)^3\right]$<br>$\$_B = \frac{1}{4}\left[2\left(1-\lambda_{11}^2+\lambda_{11}^4\right) - c_{11}^2\left(1-2\lambda_{11}^2+\lambda_{11}^6\right)\right]$<br>$\$_C = \frac{1}{4}\left[2-\lambda_{11}^2-\lambda_{11}^4+\lambda_{11}^6 - c_{11}^2\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right]$ |
| | $\{1, 0, 1, 0, c_{11}^2, 0\}$ ★ | $\$_A = \frac{1}{8}\left[3-3\lambda_{11}^2+7\lambda_{11}^4-2\lambda_{11}^6 + c_{11}^2\left(1-3\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6\right)\right]$<br>$\$_B = \frac{1}{8}\left[5-7\lambda_{11}^2+7\lambda_{11}^4-2\lambda_{11}^6 + c_{11}^2\left(-1+\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6\right)\right]$<br>$\$_C = \frac{1}{4}\left[2\left(1-\lambda_{11}^2+\lambda_{11}^4\right) - c_{11}^2\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right]$ |
| | $\{0, 1, 0, 1, c_{11}^2, 1\}$ ★ | $\$_A = \frac{1}{8}\left[4-2\lambda_{11}^2-2\lambda_{11}^4+4\lambda_{11}^6 + c_{11}^2\left(1-3\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6\right)\right]$<br>$\$_B = \frac{1}{8}\left[4-2\lambda_{11}^2-2\lambda_{11}^4+4\lambda_{11}^6 + c_{11}^2\left(-1+\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6\right)\right]$<br>$\$_C = \frac{1}{4}\left[2-\lambda_{11}^2-\lambda_{11}^4+2\lambda_{11}^6 - c_{11}^2\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{u_{B1}, 0, u_{B1}, 0, u_{B1}, 0\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[2\left(1-\lambda_{11}^2+\lambda_{11}^4\right) - \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ ■ |

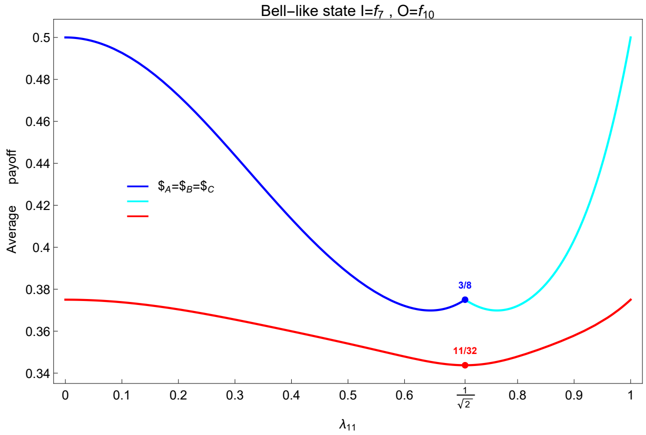
| | | |
|---|---|---|
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{t_{B1}, 1, t_{B1}, 1, t_{B1}, 1\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[2\left(1 - 2\lambda_{11}^4 + 2\lambda_{11}^6\right) - \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |
| $0 \leqslant \lambda_{11} \leqslant 1$ | $\{\frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[2\left(1 - \lambda_{11}^2 + \lambda_{11}^4\right) - \frac{\left(1-2\lambda_{11}^2+2\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{2\left[1-3\lambda_{11}^2(1-\lambda_{11}^2)\right]}\right]$ |
| $\lambda_{11} = 0, 1, \frac{1}{\sqrt{2}}, \tilde{c}_{11}^2 = 1$<br><br>$0 < \lambda_{11} < \frac{1}{\sqrt{2}}, v_{B1} \leqslant \tilde{c}_{11}^2 \leqslant 1$ | $\{0, 0, 0, 0, 1, \tilde{c}_{11}^2\}$ | $\$_A = \frac{1}{4}\left[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6\right]$<br>$\$_B = \frac{1}{4}\left[1 + \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6\right]$<br>$\$_C = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ |
| $\lambda_{11} = 0, 1, \frac{1}{\sqrt{2}}, \tilde{c}_{11}^2 = 0$<br><br>$0 < \lambda_{11} < \frac{1}{\sqrt{2}}, 0 \leqslant \tilde{c}_{11}^2 \leqslant 2u_{B1}$ | $\{1, 1, 1, 1, 0, \tilde{c}_{11}^2\}$ | $\$_A = \frac{1}{4}\left[1 + 2\lambda_{11}^4 - \lambda_{11}^6\right]$<br>$\$_B = \frac{1}{4}\left[2 - 2\lambda_{11}^2 + 2\lambda_{11}^4 - \lambda_{11}^6\right]$<br>$\$_C = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{u_{B1}, u_{B1}, u_{B1}, u_{B1}, 0, 0\}$ | $\$_A = \frac{1}{4}\left[2\left(1 - \lambda_{11}^2 + \lambda_{11}^4\right) - \frac{\left(1-2\lambda_{11}^2+\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$<br>$\$_B = \frac{1}{4}\left[2\left(1 - \lambda_{11}^2 + \lambda_{11}^4\right) - \frac{\lambda_{11}^6\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$<br>$\$_C = \frac{1}{4}\left[2\left(1 - \lambda_{11}^2 + \lambda_{11}^4\right) - \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{t_{B1}, t_{B1}, t_{B1}, t_{B1}, 1, 1\}$ | $\$_A = \frac{1}{4}\left[3 - 4\lambda_{11}^2 + 2\lambda_{11}^4 - \frac{\left(1-2\lambda_{11}^2+\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |

| | | |
|---|---|---|
| | | $\$_B = \frac{1}{4}\left[1 + 2\lambda_{11}^4 - \frac{\lambda_{11}^6\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_C = \frac{1}{4}\left[2\left(1 - 2\lambda_{11}^4 + 2\lambda_{11}^6\right) - \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{v_{B1}, 0, v_{B1}, 0, 1, 0\}$ | $\$_A = \frac{1}{4}\left[3 - 4\lambda_{11}^2 + 2\lambda_{11}^4 - \frac{\left(1-\lambda_{11}^2-3\lambda_{11}^4+2\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_B = \frac{1}{4}\left[1 + 2\lambda_{11}^4 + \frac{\left(1-3\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_C = \left[1 + 2\lambda_{11}^4 + \frac{\left(-1+2\lambda_{11}^2\right)\left(-1+4\lambda_{11}^2-9\lambda_{11}^4+14\lambda_{11}^6-13\lambda_{11}^8+4\lambda_{11}^{10}\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{2t_{B1}, 1, 2t_{B1}, 1, 0, 1\}$ | $\$_A = \frac{1}{4}\left[3 - 2\lambda_{11}^2 - 4\lambda_{11}^4 + 4\lambda_{11}^6 - \frac{\left(1-\lambda_{11}^2-3\lambda_{11}^4+2\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_B = \frac{1}{4}\left[1 + 2\lambda_{11}^2 - 4\lambda_{11}^4 + 4\lambda_{11}^6 + \frac{\left(1-3\lambda_{11}^2+3\lambda_{11}^4-2\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_C = \left[1 + 2\lambda_{11}^2 - 4\lambda_{11}^4 + 4\lambda_{11}^6 + \frac{\left(-1+2\lambda_{11}^2\right)\left(-1+4\lambda_{11}^2-9\lambda_{11}^4+14\lambda_{11}^6-13\lambda_{11}^8+4\lambda_{11}^{10}\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |

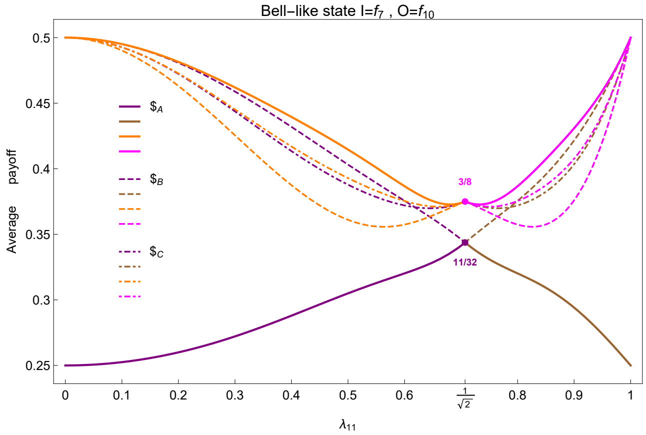
Tab. 5.7: *Nash Equilibria for the game defined by $I = f_7$, $O = f_{10}$ using the Bell-like state. The strategies marked with a star are detailed on the next Table 5.8 because of the interval restrictions. The specific expressions for $t_{B1}$, $u_{B1}$, and $v_{B1}$ as functions of $\lambda_{11}$ are found in equations (5.13)-(5.15) on page 69. The colour of the squares helps to identify the payoffs plotted in Figure 5.30. The non-marked solutions are the same (up to player permutations) as some in Table 5.8.*

Figure 5.30 contains two plots of all the marked solutions with coloured **squares** in Table 5.7. As always, the colour of the lines in the plots matches that of the square identifying the payoff in the same table. Figure 5.30(a) shows the solutions marked with blue, cyan, and red squares, which all give the same payoff to all players. The **blue** and **cyan** lines show that, without entanglement, the players get $1/2 = 0.5$, and as the entanglement increases, the payoff decreases. Interestingly, the minimum is reached closer to $\lambda_{11} = 1/\sqrt{2}$, but not at that exact point. This minimum is $\$_A = \$_B = \$_C \approx 0.36986$ at $\lambda_{11} \approx 0.646$ and at $\lambda_{11} \approx 0.764$. After this minimum, the payoff starts to increase again until reaching $3/8 = 0.375$ for the pure Bell state. For the **red** line, without entanglement, the players receive $3/8 = 0.375$, and as the entanglement increases, the payoff decreases to reach a minimum of $11/32 = 0.34375$ for the pure Bell state. This red solution $s^* = \{u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2\}$ is exactly the same as the solution in the CHSH game ($I = f_7$, $O = f_{15}$) for the Bell-like state – see red square solution in Table 5.2. In this case, it also exhibits a similar interesting behaviour – see the analysis on page 70 –, because, again, $\lambda_{11} = 0,\ 1,\ 1/\sqrt{2}$ recovers the classical strategy of $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$, but the payoff distinguishes between $\lambda_{11} = 0,\ 1$ , which gives $3/8 = 0.375$, and $\lambda_{11} = 1/\sqrt{2}$, which gives in this case a little less $11/32 = 0.34375$ . As mentioned in the analysis on page 70, it would be interesting to investigate whether this distinction between producing this exact same strategy with a maximally-entangled Bell state and a fully separable state (or classically), is a fundamental consequence of quantum mechanics, which cannot be reproduced using classical means, or not.

Figure 5.30(b) shows the solutions marked with purple, brown, orange, and magenta squares; these give different payoffs for all three players. For the **purple** and **brown** lines, Alice monotonically increases her payoff (solid line) as the entanglement increases, while Bob's (dashed line) decreases, both reaching $11/32 = 0.34375$ for the pure Bell state. Carl's payoff (dot-dashed line) for the purple and brown lines follows the same pattern as the blue and cyan lines in Figure 5.30(a): it decreases until it reaches the minimum of $\$_C \approx 0.36986$ at $\lambda_{11} \approx 0.646$ and at $\lambda_{11} \approx 0.764$. In this case, the pure Bell state gives Carl a payoff of $3/8 = 0.375$ – more than for Alice and for Bob. For the **orange** and **magenta** lines, the players' payoff decreases until it reaches a minimum and then increases again. Now, the minimum of all three players occurs at different points: for Alice, the minimum is $\$_A \approx 0.3727$ at $\lambda_{11} \approx 0.6809$ and at $\lambda_{11} \approx 0.7323$; for Bob, the minimum is $\$_B \approx 0.3557$ at $\lambda_{11} \approx 0.5637$ and at $\lambda_{11} \approx 0.826$; for Carl, the minimum is $\$_C \approx 0.3711$ at $\lambda_{11} \approx 0.66001$ and at $\lambda_{11} \approx 0.7512$. For a pure Bell state, all three players get $3/8 = 0.375$ .




(a)




(b)

Fig. 5.30: *Plots that show the players' average payoff for the Nash equilibrium points marked with* (a) *blue, cyan, and red squares in Table 5.7; and* (b) *purple, brown, orange, and magenta squares in Table 5.7 as a function of the entanglement parameter* $\lambda_{11}$. *The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_7$, $O = f_{10}$, *using the Bell-like state.*

As mentioned previously, the next table, **Table 5.8** details the **solutions** marked with a **star** in Table 5.7. That means that the first three solutions in Table 5.8, marked with a purple star, correspond to the solution marked with the same star in Table 5.7. The solutions in Table 5.8 are more classical in the sense that they do not depend explicitly on $\lambda_{11}$, only the corresponding payoffs do. To distinguish these solutions a bit more from the long list of solutions, the corresponding payoffs are marked with coloured diamonds and triangles, instead of squares. The marked solutions with the diamonds in Table 5.8 are plotted as a function of $\lambda_{11}$ in Figure 5.31, with the colour of the line matching that of the diamond; while the marked solutions with the triangles are plotted as a function of Carl's strategy $c_{11}$ in Figure 5.32, again, with the colour of the lines matching that of the triangle. The non-marked solutions correspond to a player permutation of the other ones, or give constant payoff for all three players.

| (2/2) Bell-like state with $I = f_7$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{11} = 0$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$<br>$\lambda_{11} = 1$ | $\{1, 1, 0, 0, c_{11}^2, c_{11}^2\}$ ★ | $\$_A = \$_B = \frac{1}{2}$ ; $\$_C = \frac{1}{4}$<br>$\$_A = \frac{12 - c_{11}^2}{32}$ ; $\$_B = \frac{11 + c_{11}^2}{32}$ ; $\$_C = \frac{11}{32}$ ▼<br>$\$_A = \frac{2 - c_{11}^2}{4}$ ; $\$_B = \frac{1 + c_{11}^2}{4}$ ; $\$_C = \frac{1}{2}$ ▼ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{1, 1, 0, 0, 0, 0\}$ ★ | $\$_A = \frac{1}{2}[1 - \lambda_{11}^2 + \lambda_{11}^4]$<br>$\$_B = \frac{1}{4}[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6]$ ♦<br>$\$_C = \frac{1}{4}[1 + \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6]$ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{1, 1, 0, 0, 1, 1\}$ ★ | $\$_A = \frac{1}{4}[2 - 2\lambda_{11}^2 + 2\lambda_{11}^4 - \lambda_{11}^6]$<br>$\$_B = \frac{1}{2}[1 - \lambda_{11}^2 + \lambda_{11}^4]$ ♦<br>$\$_C = \frac{1}{4}[1 + 2\lambda_{11}^4 - \lambda_{11}^6]$ |
| $\lambda_{11} = 0$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$<br>$\lambda_{11} = 1$ | $\{0, 0, 1, 1, c_{11}^2, c_{11}^2\}$ ★ | $\$_A = \frac{1 + c_{11}^2}{4}$ ; $\$_B = \frac{2 - c_{11}^2}{4}$ ; $\$_C = \frac{1}{2}$<br>$\$_A = \frac{11 + c_{11}^2}{32}$ ; $\$_B = \frac{12 - c_{11}^2}{32}$ ; $\$_C = \frac{11}{32}$<br>$\$_A = \$_B = \frac{1}{2}$ ; $\$_C = \frac{1}{4}$ |

| | | |
|---|---|---|
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{0, 0, 1, 1, 0, 0\}$ ★ | $\$_A = \frac{1}{4}[1 + \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6]$<br>$\$_B = \frac{1}{2}[1 - \lambda_{11}^2 + \lambda_{11}^4]$<br>$\$_C = \frac{1}{4}[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6]$ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{0, 0, 1, 1, 1, 1\}$ ★ | $\$_A = \frac{1}{2}[1 - \lambda_{11}^2 + \lambda_{11}^4]$<br>$\$_B = \frac{1}{4}[1 + 2\lambda_{11}^4 - \lambda_{11}^6]$<br>$\$_C = \frac{1}{4}[2 - 2\lambda_{11}^2 + 2\lambda_{11}^4 - \lambda_{11}^6]$ |
| $\lambda_{11} = 0$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$<br>$\lambda_{11} = 1$ | $\{1, 0, 1, 0, c_{11}^2, 0\}$ ★ | $\$_A = \frac{3 + c_{11}^2}{8}$ ; $\$_B = \frac{5 - c_{11}^2}{8}$ ; $\$_C = \frac{1}{2}$<br>$\$_A = \$_B = \$_C = \frac{3}{8}$<br>$\$_A = \frac{5 - c_{11}^2}{8}$ ; $\$_B = \frac{3 + c_{11}^2}{8}$ ; $\$_C = \frac{1}{2}$ ▼ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{1, 0, 1, 0, 0, 0\}$ ★ | $\$_A = \frac{1}{8}[3 - 3\lambda_{11}^2 + 7\lambda_{11}^4 - 2\lambda_{11}^6]$<br>$\$_B = \frac{1}{8}[5 - 7\lambda_{11}^2 + 7\lambda_{11}^4 - 2\lambda_{11}^6]$ ♦<br>$\$_C = \frac{1}{2}[1 - \lambda_{11}^2 + \lambda_{11}^4]$ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{1, 0, 1, 0, 1, 0\}$ ★ | $\$_A = \$_B = \$_C = \frac{1}{4}[2 - 3\lambda_{11}^2 + 5\lambda_{11}^4 - 2\lambda_{11}^6]$ ♦ |

| | | |
|---|---|---|
| $\lambda_{11} = 0$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$<br>$\lambda_{11} = 1$ | $\{0, 1, 0, 1, c_{11}^2, 1\}$ ★ | $\$_A = \frac{4 + c_{11}^2}{8}$ ; $\$_B = \frac{4 - c_{11}^2}{8}$ ; $\$_C = \frac{1}{2}$<br>$\$_A = \$_B = \$_C = \frac{3}{8}$<br>$\$_A = \frac{4 - c_{11}^2}{8}$ ; $\$_B = \frac{4 + c_{11}^2}{8}$ ; $\$_C = \frac{1}{2}$ ▼ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{0, 1, 0, 1, 0, 1\}$ ★ | $\$_A = \$_B = \$_C = \frac{1}{4}\left(1 + \lambda_{11}^2\right)\left(2 - 3\lambda_{11}^2 + 2\lambda_{11}^4\right)$ ♦ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{0, 1, 0, 1, 1, 1\}$ ★ | $\$_A = \frac{1}{8}\left[5 - 5\lambda_{11}^2 + \lambda_{11}^4 + 2\lambda_{11}^6\right]$<br>$\$_B = \frac{1}{8}\left[3 - \lambda_{11}^2 + \lambda_{11}^4 + 2\lambda_{11}^6\right]$ ♦<br>$\$_C = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ |

Tab. 5.8: *Detailed Nash equilibrium solutions of the cases marked with a star on Table 5.7, for the game defined by* $I = f_7$, $O = f_{10}$ *using the Bell-like state. The colour of the diamonds and the triangles help to identify the payoffs plotted in Figure 5.31 and Figure 5.32, respectively. The solutions not marked with any diamond are equivalent to other ones with the purple diamonds by permuting the players.*

For the solutions marked with a **diamond**, the **red** and **blue** lines in Figure 5.31 give the same payoff for all three players, which, as the entanglement increases, the payoff decreases to reach the minimum of $3/8 = 0.375$ for the pure Bell state. As is immediately seen from Table 5.8 and the plot, in the first interval[25], Alice's payoff for the purple solution (solid line) coincides with Carl's for the brown solution (dot-dashed line); in the second interval, Bob's payoff for the magenta line (dashed line) coincides with Carl's payoff for the orange line (dot-dashed). For the **purple** lines, Alice's and Bob's payoffs (solid and dashed lines) decrease as the entanglement increases, while Carl's (dot-dashed) increases. Similarly, for the **magenta** lines, which mirror the purple solutions in the second interval, Bob's and Carl's decreases as the entanglement increases, and Alice's increases. For the **brown** and **orange** lines, the maxima for Bob $\$_B = 5/8 = 0.625$ and for Carl $\$_C = 1/2 = 0.5$ are located at $\lambda_{11} = 0\ , 1$, and as the entanglement increases, their payoff decreases. However, for Alice, the maximum of her payoff is $\$_A = 3/8 = 0.375$ at $\lambda_{11} = 0\ , 1/\sqrt{2}\ , 1$, while the minimum is $\$_A = \left(316 - 31\sqrt{31}\right)/432 \approx 0.332$ at $\lambda_{11} = \sqrt{(7-\sqrt{31})/6} \approx 0.489$ and at $\lambda_{11} = \sqrt{(\sqrt{31}-1)/6} \approx 0.873$ .

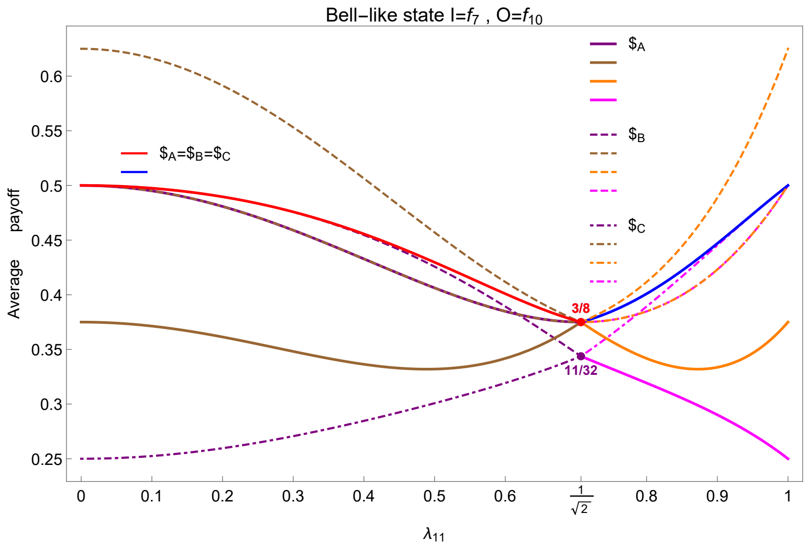
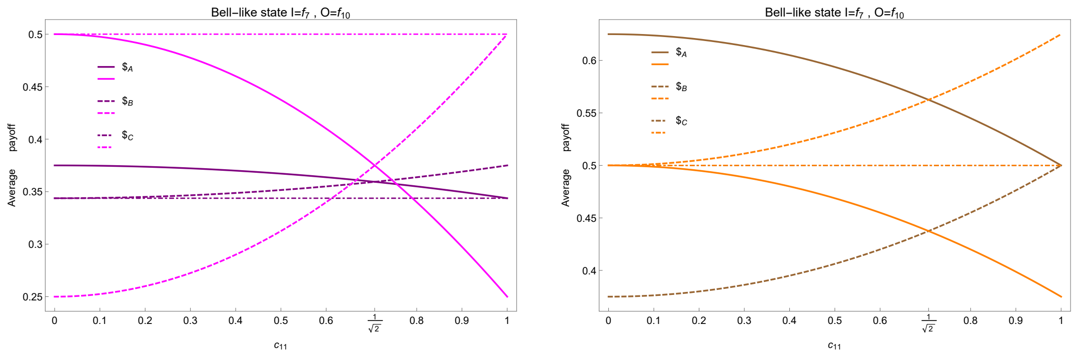



Fig. 5.31: *Plot that shows the players' average payoff for the Nash equilibrium points marked with purple, magenta, brown, blue, red, and orange* diamonds *in Table 5.8 as a function of the entanglement parameter $\lambda_{11}$. The colour of the lines matches the coloured diamonds, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_7$, $O = f_{10}$, using the Bell-like state.*

Figure 5.32 shows the solutions marked with a **triangle** in Table 5.8 as a function of Carl's strategy $c_{11}$. The behaviour of all the same lines with different colours is very similar: Carl's payoff (dot-dashed line) is constant; Alice's payoff (solid line) decreases monotonically as $c_{11}$ increases; while for Bob's (dashed line), it increases monotonically. The **magenta**, **orange**, and **brown** lines correspond to solutions with $\lambda_{11} = 1$, whereas the **purple** line corresponds to $\lambda_{11} = 1/\sqrt{2}$, which is why the difference between the players' payoffs is narrower[26].

[25]The first interval comprises $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$, while the second is $1/\sqrt{2} \leqslant \lambda_{11} \leqslant 1$.

[26]As happened with the Bell-like state for $I = f_7$, $O = f_{15}$ (CHSH game), see Figure 5.9 and Figure 5.10 on page 72.




(a) (b)

Fig. 5.32: *Plot that shows the players' average payoff for the Nash equilibrium points marked with* (a) *purple and magenta; and* (b) *brown, and orange* triangles *in Table 5.8 as a function of Carl's strategy* $c_{11}$*. The colour of the lines matches the coloured triangles, which mark/identify the payoffs, in the same table. The brown dot-dashed line coincides with the orange dot-dashed line. These results are for the choice of functions* $I = f_7$*,* $O = f_{10}$*, using the Bell-like state.*

The expressions for the **social welfare** ($\$_A + \$_B + \$_C$) of each equilibrium point in Table 5.7 and Table 5.8 are found in Table B.7 and Table B.8 in appendix B, respectively. Figure 5.33 plots the social welfare for all the solutions as a function of $\lambda_{11}$. The lines of the solutions marked with a **square** in the social welfare table, Table B.7, which are also marked with the same square in Table 5.7, are shown in **full colour**; while the ones marked with a **diamond** in Table B.8, also marked with the same diamond in Table 5.8, are shown with a **fainter line**. Finally, the points correspond to solutions with constant social welfare at that particular $\lambda_{11}$ shown non-marked in Table B.8, but marked with a triangle in Table 5.8. It is remarkable the small difference between the vivid-orange and the vivid-blue line, and between the vivid-magenta and the vivid-cyan line. The vivid-orange and vivid-magenta corresponded to solutions where each player would get a different payoff – see Figure 5.30(b) – while the vivid-blue and vivid-cyan to solutions with the same payoff for all – see Figure 5.30(a). For these four solutions, for the social welfare, the solutions that give different payoffs are slightly better than the ones giving the same to all. Focusing now on the overall picture, the vivid-colour solutions, corresponding to the squares and that are "more quantum", perform worse than the fainter lines, which correspond to the diamonds and are more classical. The solutions giving the highest social welfare correspond to the red and blue faint lines – see the same coloured lines in Figure 5.31 – while the lowest one is given by the vivid-red line – see the same red line in Figure 5.30(a).

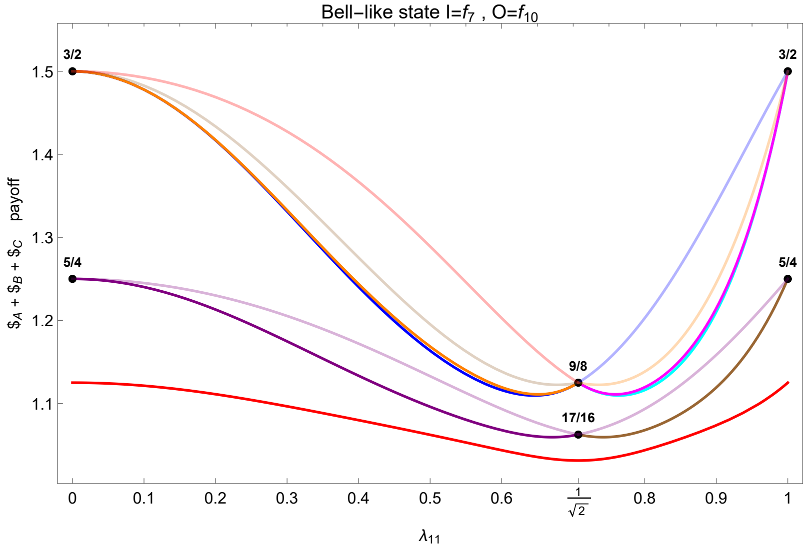



Fig. 5.33: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{11}$ using a Bell-like state for $I = f_7$, $O = f_{10}$. The Nash equilibrium solutions with the individual payoffs are found in Table 5.7 and Table 5.8; while the social welfare of them is found in Table B.7 and Table B.8. The colour of the lines matches the coloured squares and diamonds, which mark/identify the payoffs, in both the latter tables. The fainter lines (purple, brown, blue, red, and orange) correspond to the payoffs in Table B.8 with the diamonds, while the solid lines, to the payoffs in Table B.7 with the squares.*

➤ $\boldsymbol{I = f_9}$, $\boldsymbol{O = f_7}$, Alice's payoff when the players share a GHZ-like state:

$$\begin{aligned}\text{(GHZ) } \$_A =& \frac{1}{8}\Big[6 - 4\lambda_{111}^2(2-\lambda_{111}^2) + 2\lambda_{111}^4\left(a_{11}^2 + \tilde{a}_{11}^2 + \tilde{b}_{11}^2 + c_{11}^2\right) \\ &- \left(1 - 2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + c_{11}^2\right)\right. \\ &\left. + \left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(b_{11}^2 - \tilde{c}_{11}^2\right)\right)\Big] \qquad (5.27)\end{aligned}$$

In this case, only $\lambda_{111} = 1$ recovers the classical payoff in equation (4.42), while $\lambda_{111} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.123)-(B.127).

The **solutions** are found in **Table 5.9**. As always, the restrictions of the solutions come from imposing the optimisation conditions and requiring that the strategies must be between 0 and 1. For this GHZ-like, the **same functions of $\boldsymbol{\lambda_{111}}$** for the strategies appear again: $t_G(\lambda_{111})$ and $u_G(\lambda_{111})$, defined in equations (5.19)-(5.20) and plotted in Figure 5.16.

| GHZ-like state with $I = f_9$ , $O = f_7$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{111} = 0$ | $\{0,0,0,0,0,0\}$<br>$\{0,0,0,0,1,1\}$ | $\$_A = \$_B = \$_C = \frac{3}{4}$ |
| $\lambda_{111} = \frac{1}{\sqrt{2}}$ | $\{1,1,1,0,0,0\}$<br>$\{1,1,0,0,0,1\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |
| $\lambda_{111} = 1$ | $\{1,1,1,1,0,0\}$<br>$\{1,1,1,1,1,1\}$ | $\$_A = \$_B = \$_C = \frac{3}{4}$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{0,1,1,0,0,0\}$ | $\$_A = \$_B = \frac{7}{8} - \frac{1}{4}\left[5\lambda_{111}^2 - 4\lambda_{111}^4\right]$<br>$\$_C = \frac{3}{4} - \lambda_{111}^2 + \lambda_{111}^4$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{1,0,0,1,1,1\}$ | $\$_A = \$_B = \frac{5}{8} - \frac{1}{4}\left[3\lambda_{111}^2 - 4\lambda_{111}^4\right]$<br>$\$_C = \frac{3}{4} - \lambda_{111}^2 + \lambda_{111}^4$ |
| $0 \leqslant \lambda_{111} \leqslant 1$ | $\{t_G, t_G, t_G, t_G, t_G, t_G\}$ | $\$_A = \$_B = \$_C = \frac{1}{8}\left[7 - 6\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right] - \frac{1}{8\left[1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right]}$ |

| | | |
|---|---|---|
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{2t_G, 0, 0, 2t_G, 1, 1\}$ | $\$_A = \$_B = \frac{1}{8}\left[7 - 6\lambda_{111}^2 + 8\lambda_{111}^4\right] - \frac{1}{8\left[1 - 2\lambda_{111}^2(1-\lambda_{111}^2)\right]}$ <br> $\$_C = \frac{3}{4} - \lambda_{111}^2 + \lambda_{111}^4$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{u_G, 1, 1, u_G, 0, 0\}$ | $\$_A = \$_B = \frac{1}{8}\left[9 - 10\lambda_{111}^2 + 8\lambda_{111}^4\right] - \frac{1}{8\left[1 - 2\lambda_{111}^2(1-\lambda_{111}^2)\right]}$ <br> $\$_C = \frac{3}{4} - \lambda_{111}^2 + \lambda_{111}^4$ |

Tab. 5.9: *Nash Equilibria for the game defined by $I = f_9$, $O = f_7$ using the GHZ-like state. The specific expressions for $t_G$ and $u_G$ are found in equations (5.19)-(5.20) on page 78. The colour of the circles helps to identify the payoffs plotted in Figure 5.34.*

Figure 5.34 shows all the equilibrium points in Table 5.9 as a function of $\lambda_{111}$. As always, the colour of the lines in the plot matches the colour of the squares in the same table. There are only two classical solutions – see Table 4.8 to see the strategies – one that gives Alice and Bob $\$_A = \$_B = 7/8 = 0.875$ and Carl $\$_C = 3/4 = 0.75$, and one that gives $\$_A = \$_B = \$_C = 3/4 = 0.75$ to all three. These classical solutions correspond, respectively, to the solution marked with a cyan circle, and the third constant solution (for $\lambda_{111} = 1$) in Table 5.9. The red and purple solutions reduce to the mentioned constant solution (third solution) and the cyan line/square for $\lambda_{111} = 1$ and $\lambda_{111} = 1/\sqrt{2}$, respectively. This is true not only in the sense of giving the same payoff, but of corresponding to exactly the same strategy.

In this case, besides the first three solutions with constant payoff in Table 5.9, there is only one solution that gives the same payoff to all players (**red** line), and as the entanglement increases, the payoff decreases until it reaches $7/16 = 0.4375$. The other solutions give the same payoff for Alice and Bob, and different to Carl. Carl's payoff (dot-dashed lines) for the solutions identified with **blue**, **cyan**, **purple**, and **brown** lines/squares is the same – see the table to see that the payoff is the same –, which forms a continuous line without any jumps[27]. The blue and cyan solutions give $\$_A = \$_B = 7/8 = 0.875$ and $\$_C = 3/4 = 0.75$ at $\lambda_{111} = 0\ ,1$ . The payoffs gradually decrease as the entanglement increases, until they reach an equalising minimum of $\$_A = \$_B = \$_C = 1/2 = 0.5$ for the pure GHZ state. For the purple and brown solutions, the minimum for Alice and Bob is $\$_A = \$_B \approx 0.492$ at $\lambda_{111} \approx 0.661$ and at $\lambda_{111} \approx 0.751$ ; while for Carl, the minimum is $\$_C = 1/2 = 0.5$ at $\lambda_{111} = 1/\sqrt{2}$.

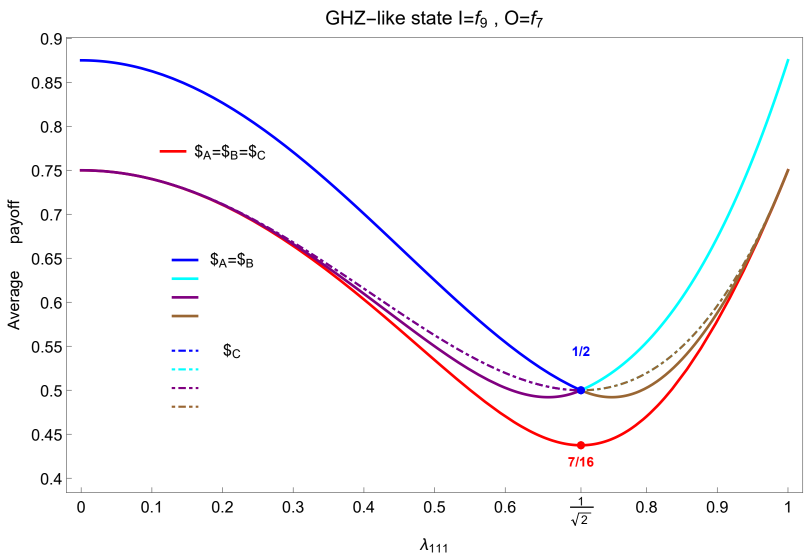



Fig. 5.34: *Plot that shows the players' average payoff for the Nash equilibrium points marked with blue, cyan, red, purple, and brown circles in Table 5.9 as a function of the entanglement parameter $\lambda_{111}$. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. The dash-dotted blue and dash-dotted purple lines coincide, and also the dash-dotted cyan and brown lines. These results are for the choice of functions $I = f_9$, $O = f_7$, using the GHZ-like state.*

[27]Technically, it cannot be said that it is a continuous line because that payoff is being given by different solutions defined locally at a certain interval. So it is not technically correct to talk about continuity and "differentiability" when matching the payoffs of different solutions.

The expressions for the **social welfare** ($\$_A + \$_B + \$_C$) for all the equilibrium points in Table 5.9 are found in Table B.9 in appendix B. Figure 5.35 shows the social welfare as a function of $\lambda_{111}$. The black points correspond to solutions giving a constant payoff for that particular value of $\lambda_{111}$. The plot shows that the highest social welfare comes from the blue and cyan solutions depending on the interval, while the lowest comes from the red solution. The minimum value for the purple and brown lines is $\$_A + \$_B + \$_C \approx 1.487$ at $\lambda_{111} \approx 0.670$ and at $\lambda_{111} \approx 0.742$ . For all of the solutions, the presence of entanglement in the GHZ-like state leads to a decrease of the payoffs.

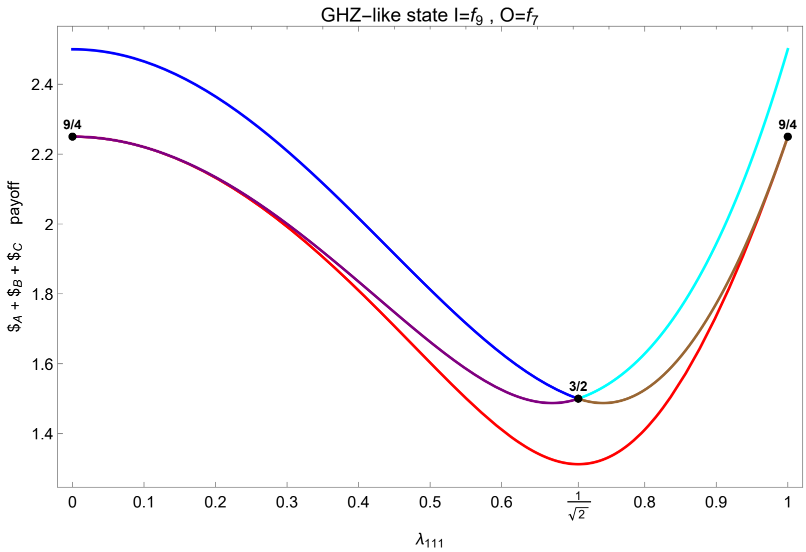



Fig. 5.35: *Sum of payoffs $\$_A + \$_B + \$_C$ from all the Nash equilibrium solutions as a function of the entanglement parameter $\lambda_{111}$ using a GHZ-like state for $I = f_9$, $O = f_7$. The Nash equilibrium solutions with the individual payoffs are found in Table 5.9, while the social welfare of them is in Table B.9. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ $\boldsymbol{I} = \boldsymbol{f_9}$, $\boldsymbol{O} = \boldsymbol{f_7}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}(\text{Bell})\ \$_A =& \frac{1}{8}\left[6 - 4\lambda_{11}^2(1 + \lambda_{11}^2(1 - \lambda_{11}^2))\right. \\ & - 2\lambda_{11}^2(1 - 3\lambda_{11}^2 + \lambda_{11}^4)\left(a_{11}^2 + \tilde{a}_{11}^2 + \tilde{b}_{11}^2 + c_{11}^2\right) \\ & - \left(1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right)\left(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + c_{11}^2\right)\right. \\ & \left.\left. + \left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(b_{11}^2 - \tilde{c}_{11}^2\right)\right)\right] \qquad (5.28)\end{aligned}$$

As before, $\lambda_{111} = 1$ recovers the classical payoff in equation (4.42), while $\lambda_{111} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.129)-(B.133).

The **solutions** are found in **Table 5.10**. In this case, there are some remarkable **differences** when considering the constant solutions for the Bell- and the GHZ-like state. The first constant solution, valid only for $\lambda_{11} = 1/\sqrt{2}$ is identical to that for the GHZ-like state in Table 5.9, and so are the solutions $s^* = \{0, 1, 1, 0, 0, 0\}$ and $s^* = \{1, 0, 0, 1, 1, 1\}$, both restricted to the same interval using either the

GHZ- or the Bell-like state. However, the solutions $s^* = \{0,0,0,0,0,0\}$ and $s^* = \{1,1,1,1,1,1\}$ are valid in an interval using the Bell-like state, while they were only valid, respectively, at $\lambda_{111} = 0$ and at $\lambda_{111} = 1$ for the GHZ-like state. In addition, the solutions $s^* = \{0,0,0,0,1,1\}$ and $s^* = \{1,1,1,1,0,0\}$ that appeared for the GHZ-like state are not a solution anymore using the Bell-like state.

The rest of the solutions depend on **functions** of the **entanglement** parameter: $t_{B2}(\lambda_{11})$, $u_{B2}(\lambda_{11})$, and $v_{B3}(\lambda_{11})$. The functions $t_{B2}(\lambda_{11})$ and $u_{B2}(\lambda_{11})$ are the **same** that appeared for the Bell-like state when $I = f_7$, $O = f_8$ in equations (5.22)-(5.23), while $v_{B3}(\lambda_{11})$ is a **new** function, exclusive to this game, that looks like:

$$v_{B3}(\lambda_{11}) = 3 - 2t_{B2}(\lambda_{11}) = \frac{3 - 7\lambda_{11}^2 + 3\lambda_{11}^4 + 2\lambda_{11}^6}{1 - 3\lambda_{11}^2\,(1 - \lambda_{11}^2)} \tag{5.29}$$

The interval for which $0 \leqslant v_{B3}(\lambda_{11}) \leqslant 1$ is $\sqrt{(\sqrt{5}-1)/2} \leqslant \lambda_{11} \leqslant 1$ . As a last illustrative example to see the restrictions imposed by the optimisation conditions and the requirement of the strategies being between 0 and 1, the equations for the solutions marked with a magenta square are shown in equations (B.134)-(B.138) in appendix B.

| Bell-like state with $I = f_9$ , $O = f_7$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1,1,1,0,0,0\}$<br>$\{1,1,0,0,0,1\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{\sqrt{5}-1}{2}$ | $\{0,0,0,0,0,0\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[3 - 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right]$ ■ |
| $\sqrt{\frac{\sqrt{5}-1}{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1,1,1,1,1,1\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + 4\lambda_{11}^4 - 2\lambda_{11}^6\right]$ ■ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + 4\lambda_{11}^4 - 2\lambda_{11}^6\right] + \frac{\left(1 - 2\lambda_{11}^2 + \lambda_{11}^6\right)^2}{2\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ ■ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{0,1,1,0,0,0\}$ | $\$_A = \$_B = \frac{1}{8}\left[7 - 9\lambda_{11}^2 + 5\lambda_{11}^4 + 2\lambda_{11}^6\right]$ ■<br>$\$_C = \frac{3}{4} - \lambda_{11}^2 + \lambda_{11}^4$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1,0,0,1,1,1\}$ | $\$_A = \$_B = \frac{1}{8}\left[5 - 7\lambda_{11}^2 + 11\lambda_{11}^4 - 2\lambda_{11}^6\right]$ ■ |

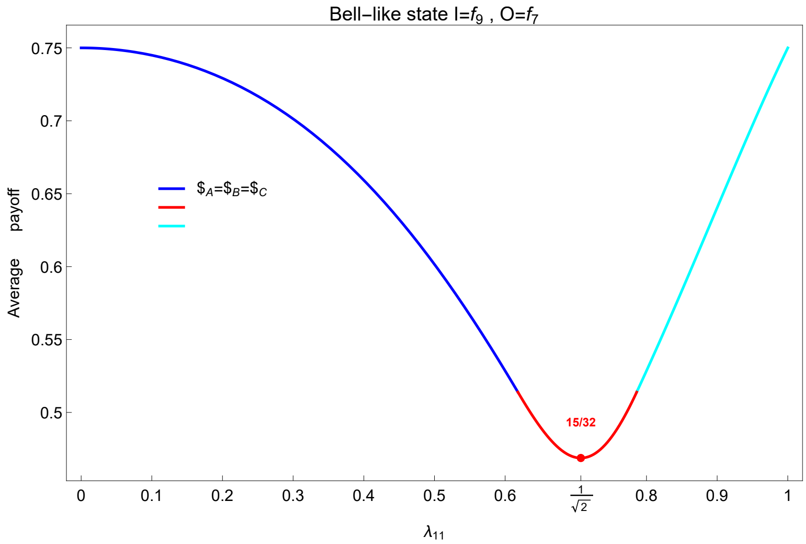
| | | |
|---|---|---|
| | | $\$_C = \frac{3}{4} - \lambda_{11}^2 + \lambda_{11}^4$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{\sqrt{5}-1}{2}$ | $\{0, -2t_{B2}, -2t_{B2}, 0, 0, 0\}$ | $\$_A = \$_B = \frac{1}{4}\left[3 - 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right]$ <br> $\$_C = \frac{1}{4}\left[1 + 4\lambda_{11}^4 - 2\lambda_{11}^6\right] + \frac{\left(1 - 2\lambda_{11}^2 + \lambda_{11}^6\right)^2 - 3\lambda_{11}^4\left(1 - 3\lambda_{11}^2 + \lambda_{11}^4\right)^2}{2\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{2t_{B2}, 0, 0, 2t_{B2}, 1, 1\}$ | $\$_A = \$_B = \frac{1}{4}\left[\lambda_{11}^2 + 7\lambda_{11}^4 - 4\lambda_{11}^6\right] - \frac{\left(1 - \lambda_{11}^2\right)\left(-1 + \lambda_{11}^2 + \lambda_{11}^4\right)\left(3 - 7\lambda_{11}^2 + 3\lambda_{11}^4 + 2\lambda_{11}^6\right)}{4\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ <br> $\$_C = \frac{3}{4} - \lambda_{11}^2 + \lambda_{11}^4$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{u_{B2}, 1, 1, u_{B2}, 0, 0\}$ | $\$_A = \$_B = \frac{1}{4}\left[1 + 4\lambda_{11}^4 - 2\lambda_{11}^6\right] - \frac{\left(1 - \lambda_{11}^2\right)\left(-1 + \lambda_{11}^2 + \lambda_{11}^4\right)\left(3 - 7\lambda_{11}^2 + 3\lambda_{11}^4 + 2\lambda_{11}^6\right)}{4\left[1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right]}$ <br> $\$_C = \frac{3}{4} - \lambda_{11}^2 + \lambda_{11}^4$ |
| $\sqrt{\frac{\sqrt{5}-1}{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1, v_{B3}, v_{B3}, 1, 1, 1\}$ | $\$_A = \$_B = \frac{1}{4}\left[1 + 4\lambda_{11}^4 - 2\lambda_{11}^6\right]$ <br> $\$_C = \frac{1}{4}\left[1 + 4\lambda_{11}^4 - 2\lambda_{11}^6\right] - \frac{\left(1 - 2\lambda_{11}^2 + \lambda_{11}^6\right)^2}{1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)}$ |

Tab. 5.10: *Nash Equilibria for the game defined by $I = f_9$, $O = f_7$ using the Bell-like state. The specific expressions for $t_{B2}$ and $u_{B2}$ are found in equations (5.22)-(5.23) on page 85, while the new $v_{B3}$ is found in equation (5.29). The colour of the squares helps to identify the payoffs plotted in Figure 5.36.*

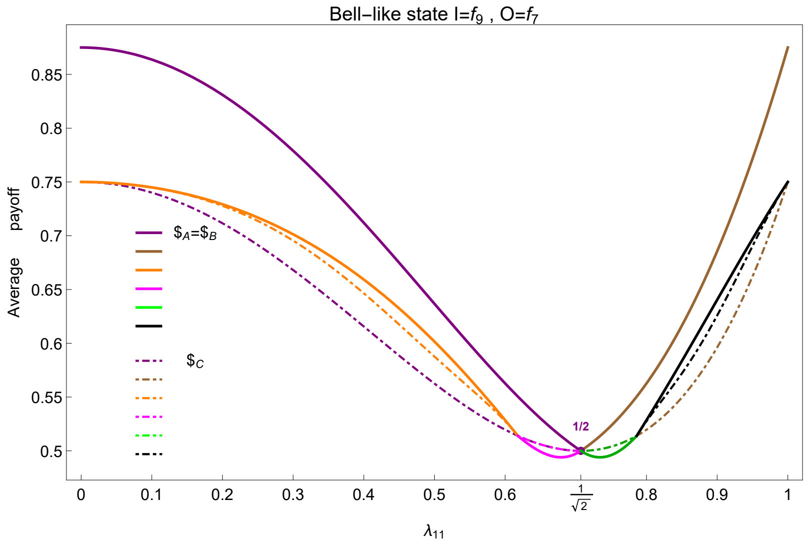
Figure 5.36 plots the solutions of Table 5.10 as a function of $\lambda_{11}$.

Figure 5.36(a) contains the solutions marked with **blue**, **cyan**, and **red** squares/lines, which give the same payoff to all the players. For all three solutions, as the entanglement increases, the payoff decreases. The blue and red lines are joined at $\lambda_{11} = (\sqrt{5}-1)/2 \approx 0.618$, while the cyan and red are joined at $\lambda_{11} = \sqrt{(\sqrt{5}-1)/2} \approx 0.786$; the payoff at these joined points is $\$_A = \$_B = \$_C = 11/4 - \sqrt{5} \approx 0.514$ . The red solution at the joining points with the blue and cyan line corresponds exactly with the blue and cyan solutions, i.e. identical strategy.

The solutions in Figure 5.36(b) give the same payoff to Alice and Bob (solid lines), but different to Carl (dot-dashed lines). The payoffs of the **purple** and **brown** solutions for all players monotonically decrease as the entanglement increases. Similarly for the **orange** and **black** solutions, which are defined between $0 \leqslant \lambda_{11} \leqslant (\sqrt{5}-1)/2$ and $\sqrt{(\sqrt{5}-1)/2} \leqslant \lambda_{11} \leqslant 1$, respectively. The players' payoffs where the orange and magenta, and green and black lines meet ( $\lambda_{11} = (\sqrt{5}-1)/2$ and $\lambda_{11} = \sqrt{(\sqrt{5}-1)/2}$ ) is $\$_A = \$_B = \$_C = 11/4 - \sqrt{5} \approx 0.514$. The most interesting solutions are the **magenta** and **green** solutions because Carl's payoff is strictly higher than Alice's and Bob's, as opposed to the other solutions in purple, brown, orange, and black lines, in which Alice and Bob always receive more than Carl. For the magenta and green lines, the minimum for Alice and Bob is $\$_A = \$_B \approx 0.494$, located at $\lambda_{11} \approx 0.679$ and at $\lambda_{11} \approx 0.734$ , respectively.




(a)




(b)

Fig. 5.36: *Plots that show the players' average payoff for the Nash equilibrium points marked with* (a) *blue, cyan, and red; and* (b) *purple, brown, orange, magenta, green, and black squares in Table 5.10 as a function of the entanglement parameter* $\lambda_{11}$. *The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_9$, $O = f_7$, *using the Bell-like state.*

The expressions for the **social welfare** ($\$_A + \$_B + \$_C$) for the equilibrium points in Table 5.10 are found in Table B.10 in appendix B. Figure 5.37 plots the social welfare as a function of $\lambda_{11}$, with the colour of the line matching that of the square in the latter table. For the social welfare, the best solutions would be the purple and brown, depending on the interval considered. The lowest social welfare is given by the orange, red, and black solutions in their respective intervals. It is worth mentioning the slight difference between the blue and orange lines, and between the cyan and black lines. As opposed to what happened with the social welfare for the Bell-like for $I = f_7$, $O = f_{10}$ – see the analysis on page 107 and see Figure 5.33 – this time the solutions that give the same to all three players (blue and cyan lines) are slightly better than the ones giving different payoffs to the players (orange and black lines). The minimum of the magenta and green lines is $\$_A + \$_B + \$_C \approx 1.489$ at $\lambda_{11} \approx 0.682$ and at $\lambda_{11} \approx 0.731$ .

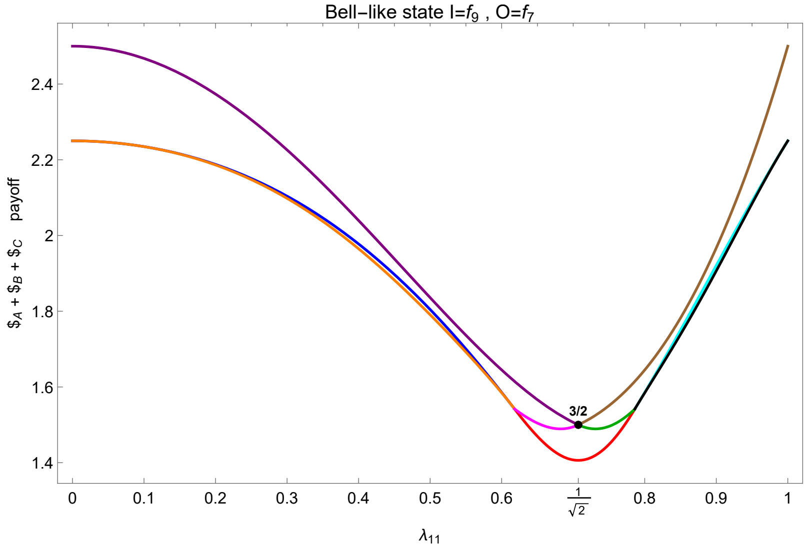



Fig. 5.37: *Sum of payoffs* $\$_A + \$_B + \$_C$ *from all the Nash equilibrium solutions as a function of the entanglement parameter* $\lambda_{11}$ *using a Bell-like state for* $I = f_9$, $O = f_7$. *The Nash equilibrium solutions with the individual payoffs are found in Table 5.10, while the social welfare of them is in Table B.10. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ $\boldsymbol{I = f_9}$, $\boldsymbol{O = f_{10}}$, Alice's payoff when the players share a GHZ-like state:

$$\text{(GHZ) } \$_A = \frac{1}{8}\Big[2 + 2(a_{11}^2 + c_{11}^2)\left(1 - \lambda_{111}^2\right)^2 + 2\lambda_{111}^4\left(\tilde{a}_{11}^2 + \tilde{b}_{11}^2\right) - \left(1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right)\left(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + c_{11}^2\right) + \left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(b_{11}^2 - \tilde{c}_{11}^2\right)\right)\Big] \tag{5.30}$$

Setting $\lambda_{111} = 1$ recovers the classical payoff in equation (4.45), while $\lambda_{111} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.140)-(B.144).

The **solutions** are found in **Table 5.11**. As always, the interval restrictions of the solutions come from imposing the optimisation conditions and requiring that the strategy is between 0 and 1. As in some of the previous results for the GHZ-like state, the **functions** of $\lambda_{111}$ for the strategies are $t_G(\lambda_{111})$ and $u_G(\lambda_{111})$, whose expressions are in equations (5.19)-(5.20), but in this case, there is also a **new** function: $v_G(\lambda_{111})$, that is given by:

$$v_G(\lambda_{111}) = t_G - u_G = \frac{\left(1 - \lambda_{111}^2\right)^2}{1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)} \tag{5.31}$$

Just like $t_G(\lambda_{111})$, this function as a strategy is not restricted to a certain interval, i.e. $0 \leqslant v_G(\lambda_{111}) \leqslant 1$ for $0 \leqslant \lambda_{111} \leqslant 1$, whereas $u_G(\lambda_{111})$ was restricted to $1/\sqrt{2} \leqslant \lambda_{111} \leqslant 1$.

| GHZ-like state with $I = f_9$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{111} = 0$ | $\{1,0,1,0,1,0\}$<br>$\{1,0,1,0,0,1\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |
| $\lambda_{111} = 1$ | $\{0,1,0,1,0,1\}$<br>$\{0,1,0,1,1,0\}$ | |
| $\lambda_{111} = \frac{1}{\sqrt{2}}$ | $\{0,0,0,1,1,0\}$<br>$\{1,1,1,0,0,1\}$ | $\$_A = \$_B = \$_C = \frac{3}{8}$ |
| $0 \leqslant \lambda_{111} \leqslant 1$ | $\{v_G, t_G, v_G, t_G, v_G, t_G\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}\left[1 - \lambda_{111}^2 + \lambda_{111}^4 - \frac{\lambda_{111}^4 (1-\lambda_{111}^2)^2}{1 - 2\lambda_{111}^2(1-\lambda_{111}^2)}\right]$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{0,0,1,1,1,0\}$ | $\$_A = \$_B = \frac{1}{2}[1 - \lambda_{111}^2 + \lambda_{111}^4]$<br>$\$_C = \frac{1}{4}[3 - 4\lambda_{111}^2 + 2\lambda_{111}^4]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{1,1,0,0,0,1\}$ | $\$_A = \$_B = \frac{1}{2}[1 - \lambda_{111}^2 + \lambda_{111}^4]$ |

| | | |
|---|---|---|
| | | $\$_C = \frac{1}{4}\left[1 + 2\lambda_{111}^4\right]$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{1, 2t_G, -u_G, 0, 0, 1\}$ | $\$_A = \$_B = \frac{1}{2}\left[1 - \lambda_{111}^2 + \lambda_{111}^4\right]$ <br> $\$_C = \frac{1}{4}\left[3 + 2\lambda_{111}^4 - \frac{1}{1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{0, u_G, 2v_G, 1, 1, 0\}$ | $\$_A = \$_B = \frac{1}{2}\left[1 - \lambda_{111}^2 + \lambda_{111}^4\right]$ <br> $\$_C = \frac{1}{4}\left[5 - 4\lambda_{111}^2 + 2\lambda_{111}^4 - \frac{1}{1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)}\right]$ |

Tab. 5.11: *Nash Equilibria for the game defined by $I = f_9$, $O = f_{10}$ using the GHZ-like state. The specific expressions for $t_G$ and $u_G$ are found in equations (5.19)-(5.20) on page 78, while the new $v_G$ is found in equation (5.31). The colour of the circles helps to identify the payoffs plotted in Figure 5.38.*

Figure 5.38 shows all the solutions marked with the corresponding coloured circle/line in Table 5.11. There are only two classical solutions – see Table 4.9 on page 53 – the first classical solution corresponds to the second set in Table 5.11 (only valid for $\lambda_{111} = 1$); and the second classical is the sixth in the same table (payoff marked with a brown circle), only that this time, that particular solution is restricted to $1/\sqrt{2} \leqslant \lambda_{111} \leqslant 1$. Now it comes the analysis of the solutions in this figure. The **red** solutions give the same payoff to all three players, and as the entanglement increases, the payoff decreases to reach a minimum of $5/16 = 0.3125$ for the pure GHZ state. From the same table and the plot itself, it can be seen that the orange and purple solid lines ($\$_A = \$_B$) coincide, and also the brown and magenta solid lines. The **purple** and **brown** solutions follow the same pattern: Carl's payoff (dot-dashed lines) is strictly higher than Alice's and Bob's, and as the entanglement increases, their payoffs decrease until they reach the minimum of $3/8 = 0.375$ . The **orange** and **magenta** solutions, however, give Alice and Bob a higher payoff than to Carl; Carl's minimum for these two solutions is $\$_C \approx 0.349$, located at $\lambda_{11} \approx 0.627$ and at $\lambda_{11} \approx 0.779$ .

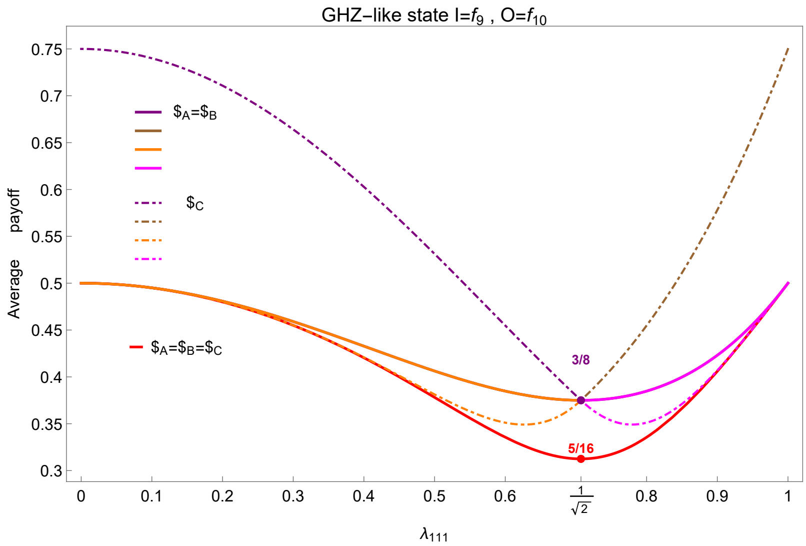



Fig. 5.38: *Plot that shows the players' average payoff for the Nash equilibrium points marked with red, purple, brown, orange, and magenta circles in Table 5.11 as a function of the entanglement parameter $\lambda_{111}$. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_9$, $O = f_{10}$, using the GHZ-like state.*

The expressions for the **social welfare** ($\$_A + \$_B + \$_C$) for the equilibrium points in Table 5.11 are found in Table B.11 in appendix B. Figure 5.39 plots the social welfare as a function of $\lambda_{111}$, with the colour of the line matching that of the square in the latter table. The black points correspond to giving a constant payoff for that particular value of $\lambda_{111}$. The social welfare has the same behaviour as the one for the GHZ-like for $I = f_9$, $O = f_7$ – see Figure 5.35. The purple and brown solutions are the best in their interval of validity, while the worst is the red line. The minimum value for the orange and magenta lines is $\$_A + \$_B + \$_C \approx 1.107$ at $\lambda_{111} \approx 0.653$ and at $\lambda_{111} \approx 0.757$ . For all the solutions, the presence of entanglement leads to a decrease of payoffs.

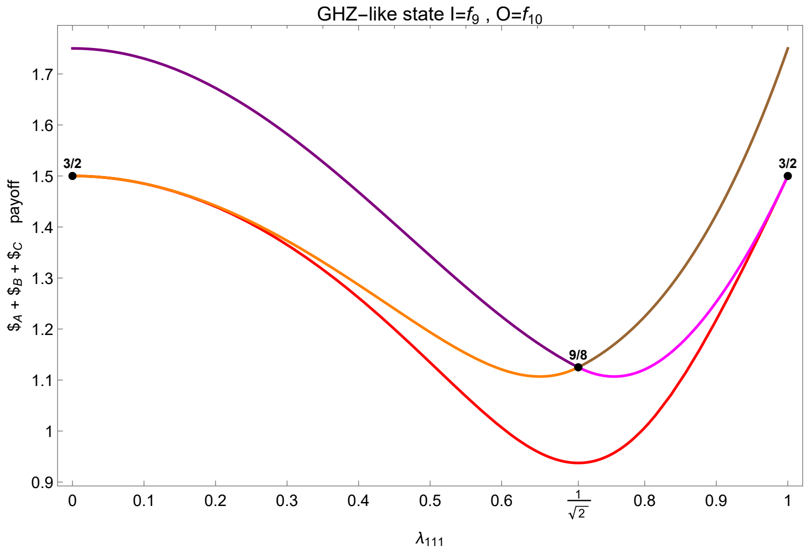



Fig. 5.39: *Sum of payoffs* $\$_A + \$_B + \$_C$ *from all the Nash equilibrium solutions as a function of the entanglement parameter* $\lambda_{111}$ *using a GHZ-like state for* $I = f_9$*,* $O = f_{10}$*. The Nash equilibrium solutions with the individual payoffs are found in Table 5.11, while the social welfare of them is in Table B.11. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the latter table.*

➤ $\boldsymbol{I = f_9}$, $\boldsymbol{O = f_{10}}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}\text{(Bell) } \$_A =& \frac{1}{8}\Big[2 + 4\lambda_{11}^2(1-\lambda_{11}^2)^2 + 2(a_{11}^2 + c_{11}^2)(1-\lambda_{11}^2)^3 \\ &- 2\lambda_{11}^2(1 - 3\lambda_{11}^2 + \lambda_{11}^4)(\tilde{a}_{11}^2 + \tilde{b}_{11}^2) \\ &- \left(1 - 3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\Big(\left(a_{11}^2 + \tilde{a}_{11}^2\right)\left(\tilde{b}_{11}^2 + c_{11}^2\right) \\ &+ \left(a_{11}^2 - \tilde{a}_{11}^2\right)\left(b_{11}^2 - \tilde{c}_{11}^2\right)\Big)\Big]\end{aligned} \tag{5.32}$$

As before, setting $\lambda_{111} = 1$ recovers the classical payoff in equation (4.45), while $\lambda_{111} = 0$ does not. The explicit equations to solve for the Nash equilibria are found in subsection B.2.2 in appendix B – equations (B.146)-(B.150).

The **solutions** are found in **Table 5.11**. In this case, there are quite a few **more** solutions than with the GHZ-like state, and **than in any other game**, though some of them are tightly restricted in a short interval. As with the GHZ-like state, the classical ones correspond to the second set/row of solutions in Table 5.11 (with $\lambda_{11} = 1$) and the one whose payoff is marked with a black square. Remarkably, besides solutions appearing at the usual points of $\lambda_{11} = 0,\ 1,\ 1/\sqrt{2}$, there are **new constant solutions** at $\lambda_{11} = (\sqrt{5}-1)/2 \approx 0.618$ and at $\lambda_{11} = \sqrt{(\sqrt{5}-1)/2} \approx 0.786$ [28]. The other solutions, whose payoff is marked with a coloured square, depend on some **functions** of $\lambda_{11}$. This time, the only function of $\lambda_{11}$ that is common to the ones from previous games is $t_{B2}(\lambda_{11})$ – in equation (5.22) – while **three new** expressions appear: $u_{B3}(\lambda_{11})$, $v_{B4}(\lambda_{11})$, and $w_{B1}(\lambda_{11})$. These are defined as:

[28] These numbers appeared before for the Bell-like state when $I = f_7$, $O = f_8$ – see footnote 22 on page 86. They are related to the golden ratio: $(\sqrt{5}-1)/2 = \varphi - 1$ and $\sqrt{(\sqrt{5}-1)/2} = 1/\sqrt{\varphi}$ .

$$u_{B3}(\lambda_{11}) = 2\left[t_{B1}(\lambda_{11}) - t_{B2}(\lambda_{11})\right] - 1 = \frac{1 - 3\lambda_{11}^2 + 3\lambda_{11}^4 - 2\lambda_{11}^6}{1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)} \tag{5.33}$$

$$v_{B4}(\lambda_{11}) = t_{B1}(\lambda_{11}) - t_{B2}(\lambda_{11}) + 1 = \frac{\left(1 - \lambda_{11}^2\right)^3}{1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)} \tag{5.34}$$

$$w_{B1}(\lambda_{11}) = 2t_{B2}(\lambda_{11}) - 2 = \frac{-2\left(1 - 2\lambda_{11}^2 + \lambda_{11}^6\right)}{1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)} \tag{5.35}$$

where the expression for $t_{B1}(\lambda_{11})$ is in equation (5.13). Some of the solutions in Table 5.12 might seem a bit strange because they are linear combinations of $u_{B3}(\lambda_{11})$, $v_{B4}(\lambda_{11})$, and $w_{B1}(\lambda_{11})$. Having those linear combinations could have been avoided by adding more definitions of functions of $\lambda_{11}$, but to keep the definitions to a minimum, the author decided not to. The bottom line is that **all** of the (**strategy**) **functions** of $\boldsymbol{\lambda_{11}}$ for the Bell-like state in **all games** are some **linear combinations** of $\mathbf{t_{B1}}(\boldsymbol{\lambda_{11}})$ and $\mathbf{t_{B2}}(\boldsymbol{\lambda_{11}})$. Again, the interval restrictions of the functions come from the optimisation conditions and requiring that the strategies must be between 0 and 1.

| Bell-like state with $I = f_9$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\lambda_{11} = 0$ | $\{1,0,1,0,1,0\}$ <br> $\{1,0,1,0,0,1\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |
| $\lambda_{11} = 1$ | $\{0,1,0,1,0,1\}$ <br> $\{0,1,0,1,1,0\}$ | |
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{0,0,0,1,1,0\}$ <br> $\{1,1,1,0,0,1\}$ | $\$_A = \$_B = \$_C = \frac{3}{8}$ |
| $\lambda_{11} = \frac{\sqrt{5}-1}{2}$ | $\left\{\frac{1+\sqrt{5}}{4}, 0, \frac{1+\sqrt{5}}{4}, 0, \frac{1+\sqrt{5}}{4}, 0\right\}$ | $\$_A = \$_B = \$_C = \frac{1}{16}\left[35 - 13\sqrt{5}\right] \approx 0.371$ |
| $\lambda_{11} = \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\left\{\frac{3-\sqrt{5}}{4}, 1, \frac{3-\sqrt{5}}{4}, 1, \frac{3-\sqrt{5}}{4}, 1\right\}$ | |
| $0 \leqslant \lambda_{11} \leqslant \frac{\sqrt{5}-1}{2}$ | $\{v_{B4}, 0, v_{B4}, 0, v_{B4}, 0\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6 - \frac{\lambda_{11}^6\left(1-\lambda_{11}^2\right)^3}{1 - 3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{v_{B4}, t_{B2}, v_{B4}, t_{B2}, v_{B4}, t_{B2}\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[4 - 4\lambda_{11}^2 - 4\lambda_{11}^4 + 3\lambda_{11}^6 + \frac{\left(-1+2\lambda_{11}^2\right)\left(2-5\lambda_{11}^2+\lambda_{11}^4+11\lambda_{11}^6-7\lambda_{11}^8+\lambda_{11}^{10}\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ |

| | | |
|---|---|---|
| $\sqrt{\frac{\sqrt{5}-1}{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{v_{B4}, 1, v_{B4}, 1, v_{B4}, 1\}$ | $\$_A = \$_B = \$_C = \frac{1}{4}\left[1 + \frac{\lambda_{11}^4\left(2-\lambda_{11}^2\left(2-\lambda_{11}^2\right)^3\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{0, 0, 1, 1, 1, 0\}$ | $\$_A = \$_B = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ <br> $\$_C = \frac{1}{4}\left[3 - 4\lambda_{11}^2 + 2\lambda_{11}^4\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1, 1, 0, 0, 0, 1\}$ | $\$_A = \$_B = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ <br> $\$_C = \frac{1}{4}\left[1 + 2\lambda_{111}^4\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{3}}$ | $\{u_{B3}, 0, 1, 0, u_{B3}, 0\}$ | $\$_A = \frac{1}{4}\left[5 - 5\lambda_{11}^2 - 7\lambda_{11}^4 + 5\lambda_{11}^6 + \frac{-3+13\lambda_{11}^2-15\lambda_{11}^4+2\lambda_{11}^6\left(-6+18\lambda_{11}^2-9\lambda_{11}^4+\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ <br> $\$_B = \frac{1}{4}\left[5 - 5\lambda_{11}^2 - 7\lambda_{11}^4 + 6\lambda_{11}^6 + \frac{-3+13\lambda_{11}^2-15\lambda_{11}^4+2\lambda_{11}^6\left(-6+18\lambda_{11}^2-9\lambda_{11}^4+\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ <br> $\$_C = \frac{1}{4}\left[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6\right]$ |
| | | |

| | | |
|---|---|---|
| $\frac{1}{\sqrt{3}} \leqslant \lambda_{11} \leqslant \sqrt{3-\sqrt{7}}$ | $\{w_{B1}+3, 0, 0, u_{B3}, 1, 0\}$ | $\$_A = \frac{1}{4}\left[4 - 4\lambda_{11}^2 - 4\lambda_{11}^4 + 3\lambda_{11}^6 + \frac{(-1+2\lambda_{11}^2)(1+\lambda_{11}^2-\lambda_{11}^4)(2-6\lambda_{11}^2+6\lambda_{11}^4-\lambda_{11}^6)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_B = \frac{1}{4}\left[4 - 5\lambda_{11}^2 - \lambda_{11}^4 + 2\lambda_{11}^6 + \frac{(-1+2\lambda_{11}^2)(1+\lambda_{11}^2-\lambda_{11}^4)(2-6\lambda_{11}^2+6\lambda_{11}^4-\lambda_{11}^6)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_C = \frac{1}{4}\left[2 - 4\lambda_{11}^4 + \lambda_{11}^6\right]$ |
| $\sqrt{3-\sqrt{7}} \leqslant \lambda_{11} \leqslant \sqrt{\frac{3-\sqrt{3}}{3}}$ | $\{v_{B4}, 0, 0, 2+w_{B1}+v_{B4},$ <br> $1, w_{B1}-v_{B4}+3\}$ | $\$_A = \frac{1}{4}\left[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6 + \frac{\lambda_{11}^2(1-3\lambda_{11}^2+\lambda_{11}^4)(-1+5\lambda_{11}^2-9\lambda_{11}^4+3\lambda_{11}^6)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_B = \frac{1}{4}\left[2 - \lambda_{11}^2 - \lambda_{11}^4 + \lambda_{11}^6 + \frac{\lambda_{11}^4(2-13\lambda_{11}^2+24\lambda_{11}^4-12\lambda_{11}^6+2\lambda_{11}^8)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ <br> $\$_C = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ |
| $\sqrt{\frac{3-\sqrt{3}}{3}} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{1, 2(v_{B4}-u_{B3}), -w_{B1}-1, 0, 0, 1)\}$ | $\$_A = \$_B = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ <br> $\$_C = \frac{1}{4}\left[-2 + \lambda_{11}^4 + 2\lambda_{11}^6 - \lambda_{11}^8 + \frac{4+\lambda_{11}^2(1-\lambda_{11}^2)(-13+\lambda_{11}^4-2\lambda_{11}^6+\lambda_{11}^8)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{\sqrt{3}}}$ | $\{0, -u_{B3}, -w_{B1}, 1, 1, 0\}$ | $\$_A = \$_B = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ |

| | | |
|---|---|---|
| | | $\$_C = \frac{1}{4}\left[-4\lambda_{11}^2 + \lambda_{11}^4 + 2\lambda_{11}^6 - \lambda_{11}^8 + \frac{4+\lambda_{11}^2\left(1-\lambda_{11}^2\right)\left(-13+\lambda_{11}^4-2\lambda_{11}^6+\lambda_{11}^8\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ |
| $\frac{1}{\sqrt{\sqrt{3}}} \leqslant \lambda_{11} \leqslant \sqrt{\sqrt{7}-2}$ | $\{v_{B4}, 1, 1, w_{B1}+v_{B4}, 0, w_{B1}-v_{B4}+1\}$ | $\$_A = \frac{1}{4}\left[3 - \lambda_{11}^2 - 7\lambda_{11}^4 + 5\lambda_{11}^6 + \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\left(-1+5\lambda_{11}^2-9\lambda_{11}^4+3\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ <br> $\$_B = \frac{1}{4}\left[4 - 4\lambda_{11}^2 - 4\lambda_{11}^4 + 3\lambda_{11}^6 + \frac{\lambda_{11}^4\left(2-13\lambda_{11}^2+24\lambda_{11}^4-12\lambda_{11}^6+2\lambda_{11}^8\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ <br> $\$_C = \frac{1}{2}\left[1 - \lambda_{11}^2 + \lambda_{11}^4\right]$ |
| $\sqrt{\sqrt{7}-2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{2}{3}}$ | $\{w_{B1}, 1, 1, 2v_{B4}, 0, 1\}$ | $\$_A = \frac{1}{4}\left[2 - 4\lambda_{11}^4 + 3\lambda_{11}^6 + \frac{\left(-1+2\lambda_{11}^2\right)\left(1+\lambda_{11}^2-\lambda_{11}^4\right)\left(2-6\lambda_{11}^2+6\lambda_{11}^4-\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ <br> $\$_B = \frac{1}{4}\left[3 - 2\lambda_{11}^2 - 4\lambda_{11}^4 + 4\lambda_{11}^6 + \frac{\left(-1+2\lambda_{11}^2\right)\left(1+\lambda_{11}^2-\lambda_{11}^4\right)\left(2-6\lambda_{11}^2+6\lambda_{11}^4-\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ <br> $\$_C = \frac{1}{4}\left[-1 + 5\lambda_{11}^2 - \lambda_{11}^4 - \lambda_{11}^6\right]$ |
| $\sqrt{\frac{2}{3}} \leqslant \lambda_{11} \leqslant 1$ | $\{2v_{B4}, 1, 0, 1, 2v_{B4}, 1\}$ | $\$_A = \frac{1}{4}\left[4 - 4\lambda_{11}^2 - 4\lambda_{11}^4 + 3\lambda_{11}^6 + \frac{-3+13\lambda_{11}^2-15\lambda_{11}^4+2\lambda_{11}^6\left(-6+18\lambda_{11}^2-9\lambda_{11}^4+\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ |

| | | |
|---|---|---|
| | | $\$_B = \frac{1}{4}\left[5 - 7\lambda_{11}^2 - \lambda_{11}^4 + 2\lambda_{11}^6 + \frac{-3+13\lambda_{11}^2-15\lambda_{11}^4+2\lambda_{11}^6\left(-6+18\lambda_{11}^2-9\lambda_{11}^4+\lambda_{11}^6\right)}{1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)}\right]$ ■ <br> $\$_C = \frac{1}{4}\left[1 + 2\lambda_{11}^4 - \lambda_{11}^6\right]$ |

Tab. 5.12: *Nash Equilibria for the game defined by* $I = f_9$, $O = f_{10}$ *using the Bell-like state. The specific expression of* $t_{B2}$ *as a function of* $\lambda_{11}$ *is found in equation (5.22) on page 85, while the expressions for the new* $u_{B3}$, $v_{B3}$, *and* $w_{B1}$ *are found in equations (5.33)-(5.35). The colour of the squares helps to identify the payoffs plotted in Figures 5.40, 5.41, and 5.42.*

Figure 5.40 plots the solutions from Table 5.12 marked with blue, cyan, red, green, and black squares. The **blue**, **cyan**, and **red** solutions give the same payoff to all three players, while the **green** and **black** solutions distinguish between Alice and Bob (solid line), and Carl (dot-dashed line). From the table and the plot, the blue solution ends, and matches the start of the cyan one, at $\lambda_{11} = (\sqrt{5}-1)/2$, while the red solution starts at $\lambda_{11} = \sqrt{(\sqrt{5}-1)/2}$, also matching the cyan one. The behaviour of all lines is similar: the payoff decreases as the entanglement increases. The crossing point between the blue and solid green line, and between the red and the solid black line, gives a payoff of 0.388 and is located at $\lambda_{11} \approx 0.582$ and at $\lambda_{11} \approx 0.812$ , respectively. That means that in the interval $0.582 \leqslant \lambda_{11} \leqslant 0.812$, the green and black solutions are preferable to all players over the existing solution for that particular $\lambda_{11}$ (either blue, cyan, or red solutions).

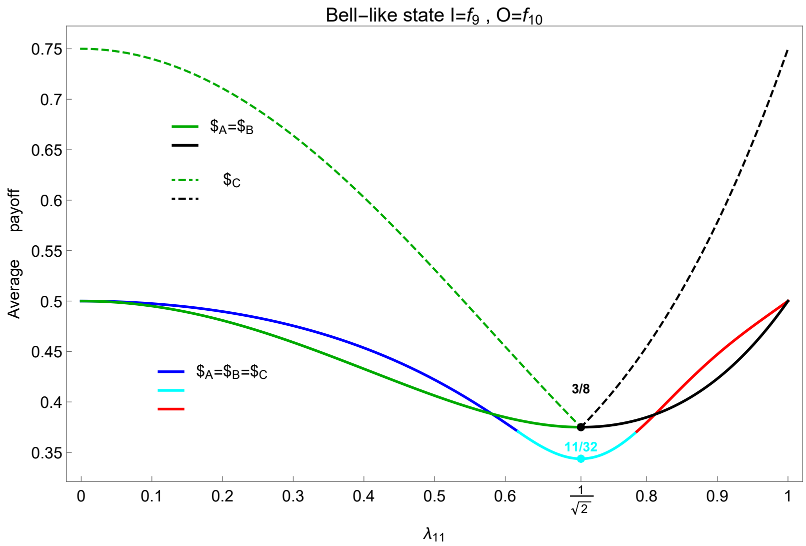
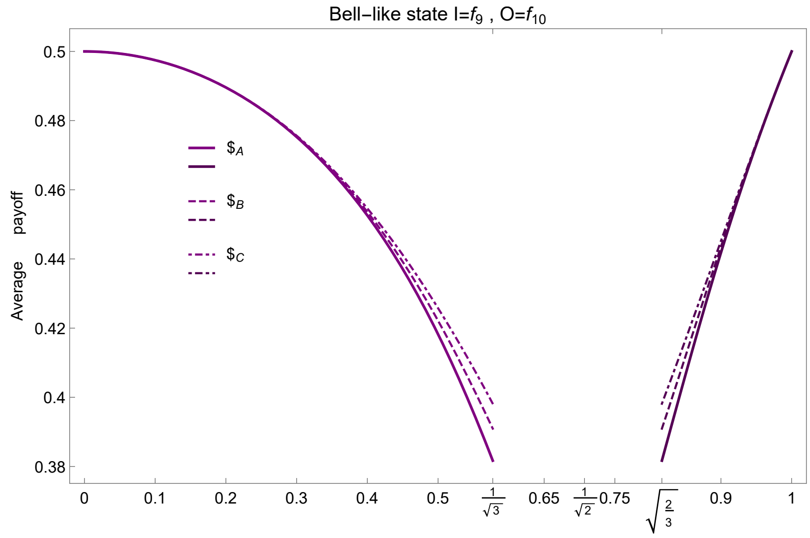


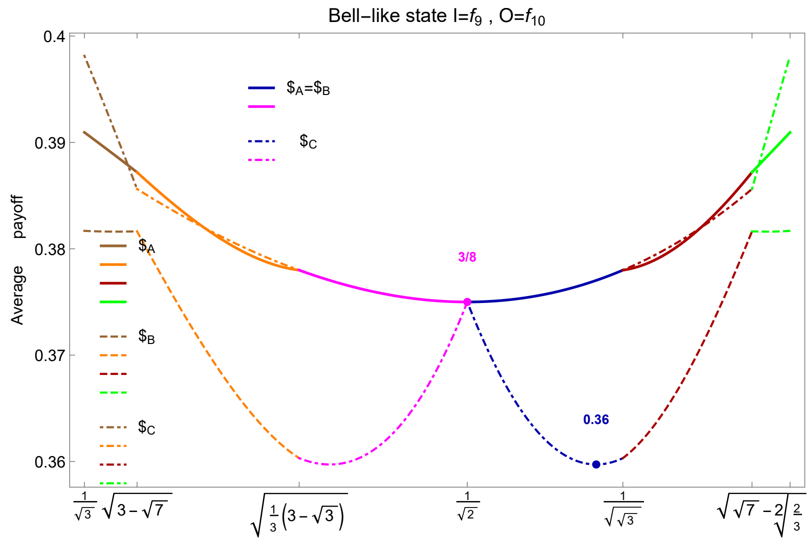

Fig. 5.40: *Plots that shows the players' average payoff for the Nash equilibrium points marked with blue, cyan, red, green, and black squares in Table 5.12 as a function of the entanglement parameter $\lambda_{11}$. The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions $I = f_9$, $O = f_{10}$, using the Bell-like state.*

The rest of the solutions from Table 5.12 are shown in Figure 5.41. Figure 5.41(a) plots the solutions marked with a **purple** square – the sixth marked solution – and a **dark-purple** square – the last solution – in the table, which are valid in the interval $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{3}$ and $\sqrt{2/3} \leqslant \lambda_{11} \leqslant 1$, respectively. These two solutions give different payoffs to all three players, with Carl (dot-dashed line) receiving the most and Alice the least (solid line); and as the entanglement increases, their payoffs decrease.

Figure 5.41(b), however, is more interesting because it shows that there are **six different solutions** with different behaviours in the very short interval $1/\sqrt{3} \leqslant \lambda_{11} \leqslant \sqrt{2/3}$. In any case, the solutions in dark-blue, dark-red, and light-green lines are a mirrored version of the solutions in brown, orange, and magenta, respectively, meaning that their analysis is the same, which is why it will be omitted. Starting with the **brown** solution – showing the same behaviour as the **light-green** – the interval of validity is the shortest: from $0.577 \approx 1/\sqrt{3} \leqslant \lambda_{11} \leqslant \sqrt{3-\sqrt{7}} \approx 0.595$. In that tiny interval, Bob's payoff (dashed line) is practically constant, while Alice's varies slightly (solid line), and Carl's (dot-dashed line) changes the most. While for most of the interval, Carl's payoff is the highest, at $\lambda_{11} \approx 0.592$, then Carl's and Alice's payoff are equal, and subsequently, Alice gets the highest payoff of all three. For the **orange** solution – similar to the **dark-red** solution – valid in $0.595 \approx \sqrt{3-\sqrt{7}} \leqslant \lambda_{11} \leqslant \sqrt{(3-\sqrt{3})/3} \approx 0.650$, Bob's payoff is the lowest and it decreases the most as the entanglement increases, while Alice's and Carl's do not vary much. In this case, Alice's payoff starts off to be higher than Carl's, and at $\lambda_{11} \approx 0.618$ the tendency changes: Carl's payoff is higher than Alice's, until the end of the interval at $\lambda_{11} = \sqrt{(3-\sqrt{3})/3} \approx 0.650$, where they both become equal $\$_A = \$_C = (4-\sqrt{3})/6 \approx 0.378$ . For the **magenta** solution – similar to the **dark-blue** solution – both Alice and Bob have the same payoff, while Carl's payoff is strictly less. For Alice and Bob, their payoff does not vary much in that interval, but it decreases slightly as $\lambda_{11}$ decreases. For Carl, his payoff decreases until it reaches a minimum of $\$_C \approx 0.36$ at $\lambda_{11} \approx 0.660$, and then starts to increase again until it reaches the equalising maximum for all three players of $\$_A = \$_B = \$_C = 3/8 = 0.375$ for the pure Bell state. As mentioned, the analysis of the dark-blue, dark-red, and light-green solutions corresponds to the analysis for the brown, orange, and magenta solutions, but with the values of $\lambda_{11}$ shifted.




(a)




(b)

Fig. 5.41: *Plots that show the players' average payoff for the Nash equilibrium points marked with* (a) *purple, and dark purple squares in Table 5.12; and* (b) *brown, orange, magenta, dark blue, dark red, and light green squares in Table 5.12, as a function of the entanglement parameter* $\lambda_{11}$. *The colour of the lines matches the coloured squares, which mark/identify the payoffs, in the same table. These results are for the choice of functions* $I = f_9$, $O = f_{10}$, *using the Bell-like state.*

To gain some perspective of the **solutions** in the two plots in Figure 5.41, these are plotted **all together** in Figure 5.42.

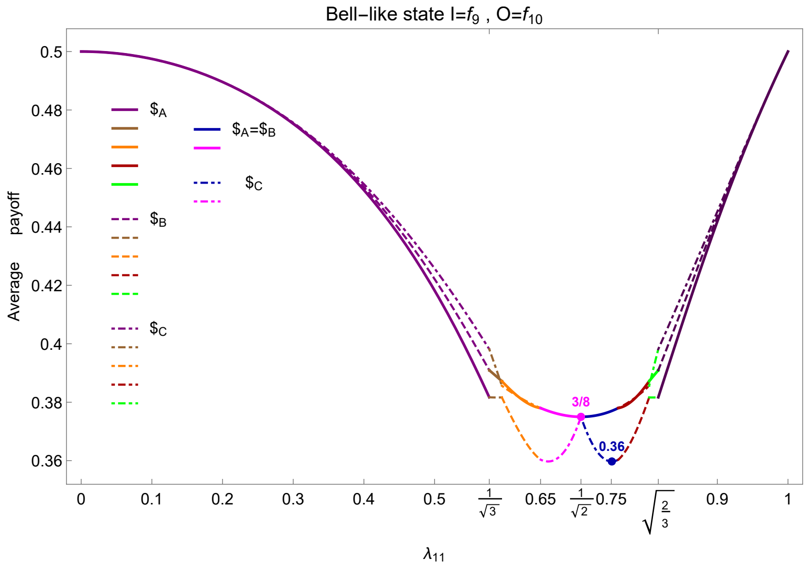



Fig. 5.42: *Payoffs from the different Nash equilibrium solutions showing the players' average payoff as a function of the entanglement parameter $\lambda_{11}$ in a Bell-like state when $I = f_9$, $O = f_{10}$. This plot is the combination of the two plots in Figure 5.41. The payoffs are found in Table 5.12.*

The expressions for the **social welfare** ($\$_A + \$_B + \$_C$) for the equilibrium points in Table 5.12 are found in Table B.12 in appendix B. Figure 5.43 plots the social welfare as a function of $\lambda_{11}$, with the colour of the line matching that of the square in the latter table. The blue and purple lines in the interval $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{3}$, and the red and dark-purple lines in the interval $\sqrt{2/3} \leqslant \lambda_{11} \leqslant 1$ seem to fully overlap, but they do not. The blue and red lines are slightly above the purple and dark-purple lines, but the difference between them is so small that it cannot be appreciated (difference ranges from 0 to $1/972 \approx 0.00102$). The minimum values of the magenta and dark-blue line is $\$_A + \$_B + \$_C \approx 1.113$ at $\lambda_{11} \approx 0.671$ and at $\lambda_{11} \approx 0.741$ . From the players' perspective, the best solutions would be the green and the black lines in their respective intervals. The lowest payoffs are given by the purple, blue, cyan, red, and dark-purple lines, in that order, each at their non-overlapping intervals; while the brown, orange, magenta, dark-blue, dark-red, and light-green solutions give an intermediate social welfare in their respective intervals of validity.

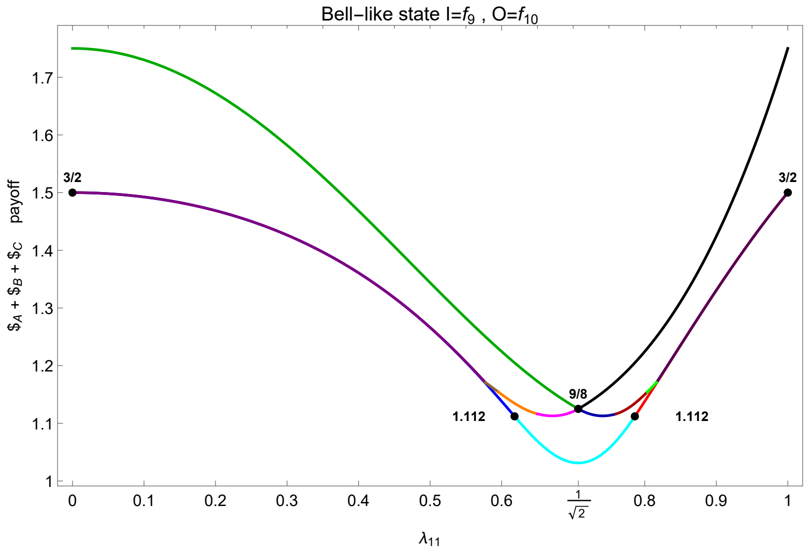



Fig. 5.43: *Sum of payoffs* $\$_A + \$_B + \$_C$ *from all the Nash equilibrium solutions as a function of the entanglement parameter* $\lambda_{11}$ *using a Bell-like state for* $I = f_9$, $O = f_{10}$. *The Nash equilibrium solutions with the individual payoffs are found in Table 5.12, while the social welfare of them is in Table B.12. The colour of the lines matches the coloured circles, which mark/identify the payoffs, in the same table. The black points correspond to the solutions with constant payoffs in the same table.*

➤ $\boldsymbol{I = f_{15}}$, $\boldsymbol{O = f_{7}}$, Alice's payoff when the players share a GHZ- *and* the Bell-like state:

$$\text{(GHZ) } \$_A = \frac{1}{8}\left[4 - (a_{11}^2 - \tilde{a}_{11}^2)(b_{11}^2 - \tilde{b}_{11}^2 + c_{11}^2 - \tilde{c}_{11}^2)\left(1 - 2\lambda_{111}^2(1 - \lambda_{111}^2)\right)\right] \tag{5.36}$$

$$\text{(Bell) } \$_A = \frac{1}{8}\left[4 - (a_{11}^2 - \tilde{a}_{11}^2)(b_{11}^2 - \tilde{b}_{11}^2 + c_{11}^2 - \tilde{c}_{11}^2)\left(1 - 3\lambda_{11}^2(1 - \lambda_{11}^2)\right)\right] \tag{5.37}$$

As was also the case with $I = O = f_{15}$, the payoffs are the **same** as the **classical** one using mixed strategies with a multiplying factor. That means, that the Nash equilibrium points will be the same as classically. These points with the corresponding payoffs are found in **Table 5.13**.

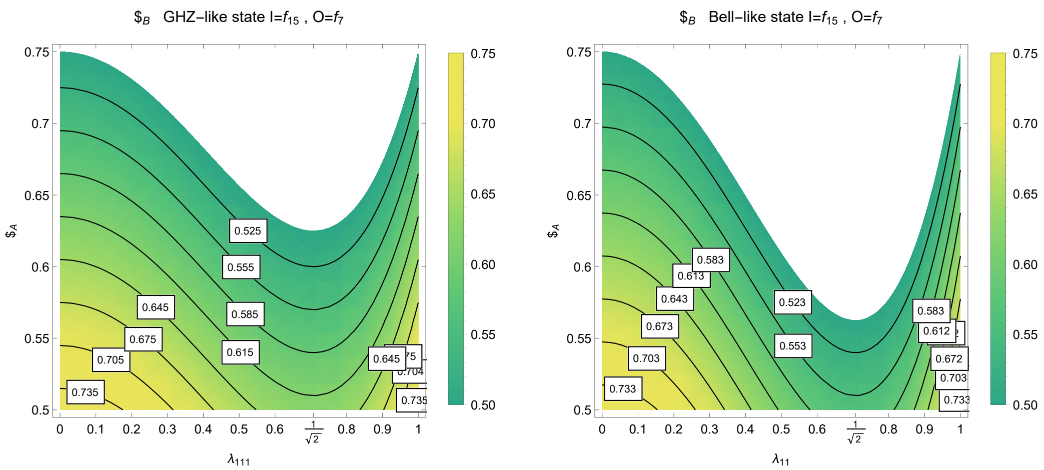
| GHZ-like state with $I = f_{15}$ , $O = f_7$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\{1, 0, 0, 1, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{1}{8}\left[4 + \left(1 - c_{11}^2 + \tilde{c}_{11}^2\right)\left(1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right)\right]$ * <br> $\$_B = \frac{1}{8}\left[4 + \left(1 + c_{11}^2 - \tilde{c}_{11}^2\right)\left(1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right)\right]$ * <br> $\$_C = \frac{1}{2}$ |
| $\{0, 1, 1, 0, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{1}{8}\left[4 + \left(1 + c_{11}^2 - \tilde{c}_{11}^2\right)\left(1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right)\right]$ <br> $\$_B = \frac{1}{8}\left[4 + \left(1 - c_{11}^2 + \tilde{c}_{11}^2\right)\left(1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right)\right]$ <br> $\$_C = \frac{1}{2}$ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |

| Bell-like state with $I = f_{15}$ , $O = f_7$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | Payoffs |
| $\{1, 0, 0, 1, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{1}{8}\left[4 + \left(1 - c_{11}^2 + \tilde{c}_{11}^2\right)\left(1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right)\right]$ * <br> $\$_B = \frac{1}{8}\left[4 + \left(1 + c_{11}^2 - \tilde{c}_{11}^2\right)\left(1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right)\right]$ * <br> $\$_C = \frac{1}{2}$ |
| $\{0, 1, 1, 0, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A = \frac{1}{8}\left[4 + \left(1 + c_{11}^2 - \tilde{c}_{11}^2\right)\left(1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right)\right]$ <br> $\$_B = \frac{1}{8}\left[4 + \left(1 - c_{11}^2 + \tilde{c}_{11}^2\right)\left(1 - 3\lambda_{11}^2\left(1 - \lambda_{11}^2\right)\right)\right]$ <br> $\$_C = \frac{1}{2}$ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\$_A = \$_B = \$_C = \frac{1}{2}$ |

Tab. 5.13: *Nash equilibrium points for the game defined by $I = f_{15}$, $O = f_7$ using GHZ- and Bell-like states. The blue and red asterisk help to identify the payoffs plotted in Figure 5.44. The non-marked points correspond to the other ones by permuting the players.*

The results show that there are three different solutions. The first solution – marked with asterisks – gives a constant payoff for Carl, and Alice's and Bob's payoffs depend on Carl's choice of $c_{11}$ and $\tilde{c}_{11}$. The second one – without asterisks – is just a permutation of the first one. The third one gives a constant payoff to all three players.

The first solution, marked with **asterisks**, gives payoffs for Alice and Bob that depend on the difference between Carl's strategy parameters $c_{11}^2 - \tilde{c}_{11}^2$ and the entanglement parameter ($\lambda_{111}$ or $\lambda_{11}$). As was done in some of the previous games, it is more illustrative to plot Bob's payoff as a function of $\$_A$ and the corresponding entanglement parameter. Figure 5.44 contains the two plots for the GHZ- and the Bell-like states. From the plots, Alice's and Bob's payoff varies from $1/2 = 0.5$ (green regions) to $3/4 = 0.75$ (yellow regions). The more remarkable feature of the plot is the decrease of both Alice's and Bob's payoff as the entanglement increases, but the decrease is more pronounced in the Bell-like state than in the GHZ-like state – see the deeper white region in Figure 5.44(b) in comparison to Figure 5.44(a). That means that, for this game, the entanglement from the GHZ-like state provides the players with better payoffs than the Bell-like state. The presence of entanglement, at any case, helps to reduce the difference between the players payoffs when compared to the non-entangled case, in which Alice's (Bob's) payoff $\$_A$ ($\$_B$) can vary from the minimum to the maximum at the expense of Bob's (Alice's) payoff ($\$_A$).




(a) $\$_B$ * *using the GHZ-like state.* (b) $\$_B$ * *using the Bell-like state.*

Fig. 5.44: *Density plot showing Bob's payoff* $\$_B$ *as a function of* $\$_A$ *and the corresponding entanglement parameter. These payoffs correspond to the Nash equilibrium solutions marked with a coloured asterisk in Table 5.13. These results are when* $I = f_{15}$, $O = f_7$, *and using the GHZ- and Bell-like state.*

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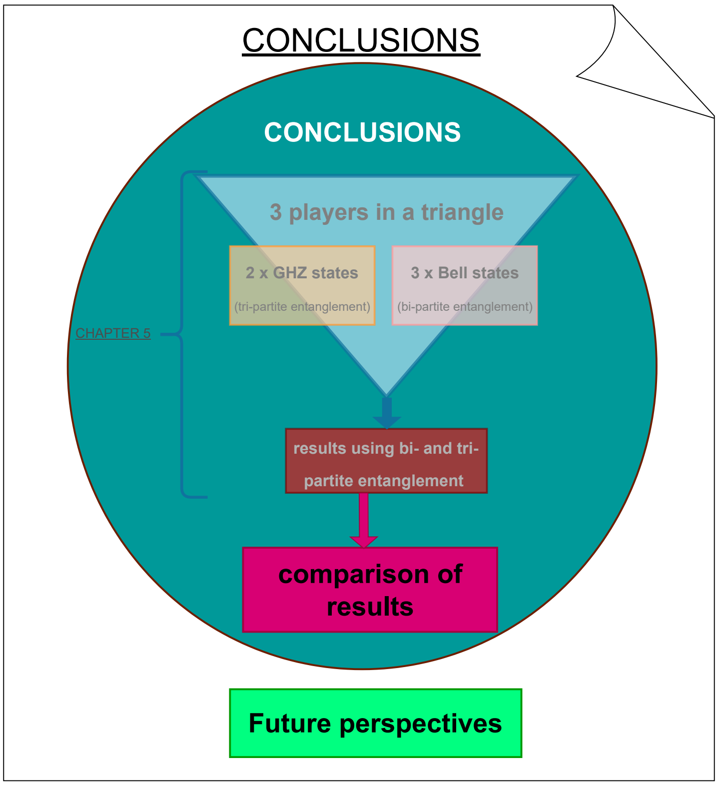
This was the last case to analyse, which completes the analysis of the 7 different boolean games, represented by the chosen input and output functions, played on the triangle using two GHZ-like states and three Bell-like states. As mentioned in chapter 4 on page 50, the solutions for the other 7 games represented by the other (non-chosen) functions can be obtained directly with the same payoffs analysed in this chapter by minimising them instead of maximising them.

## Summary of the chapter

This chapter has presented the results for all the different boolean games played in a triangle when the players use quantum resources in the form of bi-partite (Bell-like states) and tri-partite (GHZ-like states) entanglement. With this work, the research questions at the start of this chapter, on page 55, can be answered. The answers to the research questions, along with some of the conclusions to be drawn from this research (and its limitations), are found in the next and last chapter, chapter 6, in section 6.2 .

# Chapter 6

# Conclusions



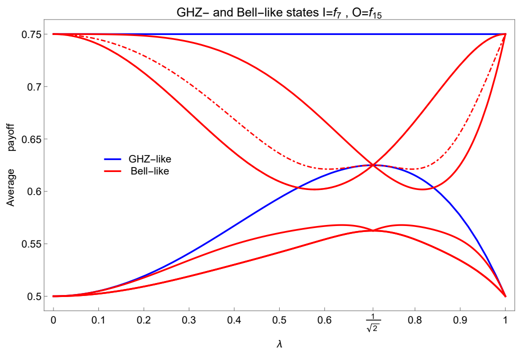

The previous chapter, chapter 5 contained all the results for the representative functions of the games when using the GHZ- and Bell-like states, which was the main research focus of this dissertation. This chapter gives a short summary of the highlights for each game and each state, and compares the overall performance of these two states. Finally, this chapter also focuses on the conclusions from this dissertation, as well as on the future perspectives and interesting avenues of research.

## 6.1 Comparison of results

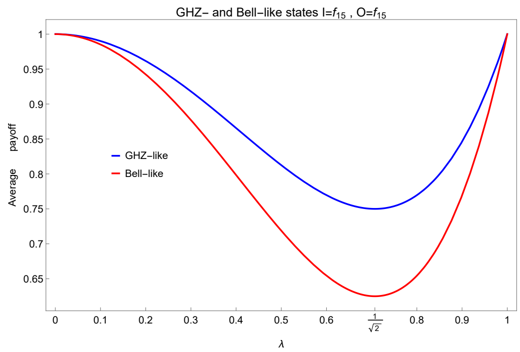
After having all of the results, now it is the time to compare the overall performance of each state for each of the games.

**Figure 6.1** contains **all seven plots comparing all** the **Nash equilibrium solutions** that depended *only* on the corresponding entanglement parameter ($\lambda_{111}$ or $\lambda_{11}$, denoted generically as $\lambda$ in the plots) **for all** the **representative functions**

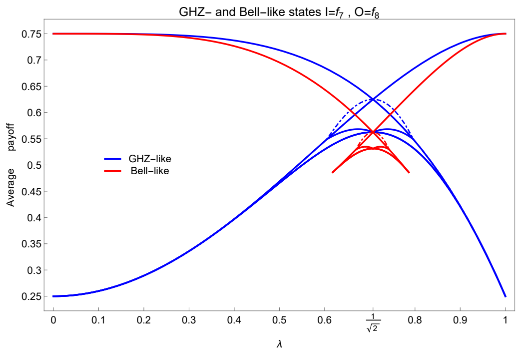
using the **GHZ**- and the **Bell**-like states. That means that some of the solutions are not plotted there because they did not depend on the entanglement parameter, or depended only on Carl's strategy ($c_{11}$ or $\tilde{c}_{11}$). Each sub-figure in Figure 6.1 collects into a unique plot the corresponding plots for the GHZ- and Bell-like states, showing all the **lines** in one unique colour: **blue** for the solutions using the **GHZ-like** state, and **red** for the **Bell-like** state. Similarly, **Figure 6.2** has all seven plots comparing the **social welfare** ($\$_A + \$_B + \$_C$) of all the Nash equilibrium solutions, also as a function of the entanglement parameter, for the GHZ-like state (blue lines) and the Bell-like state (red lines). In the case of the social welfare, the solutions that depended only on Carl's strategy ($c_{11}$ or $\tilde{c}_{11}$) are indeed included as points at their given $\lambda$, and the solutions that depended on both Carl's strategy and $\lambda$ give a social welfare that is a function of only $\lambda$, which means that these solutions are also included in Figure 6.2, whereas they were not included in Figure 6.1.



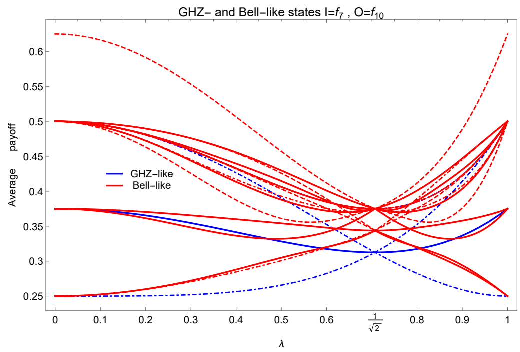

(a) $I = f_7$, $O = f_{15}$



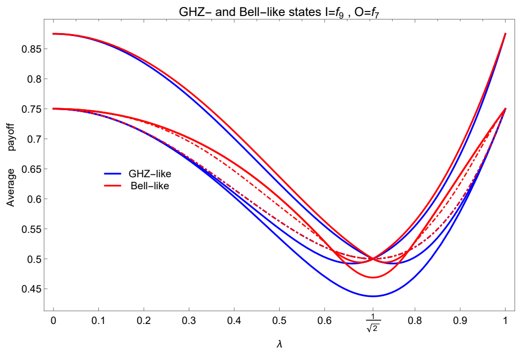

(b) $I = f_{15}$, $O = f_{15}$



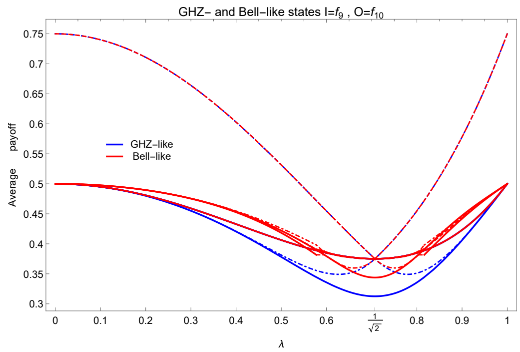

(c) $I = f_7$, $O = f_8$



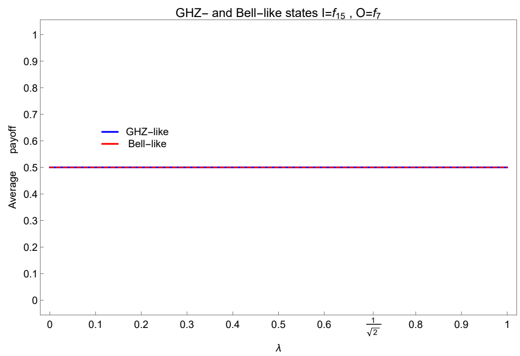

(d) $I = f_7$, $O = f_{10}$




(e) $I = f_9$, $O = f_7$

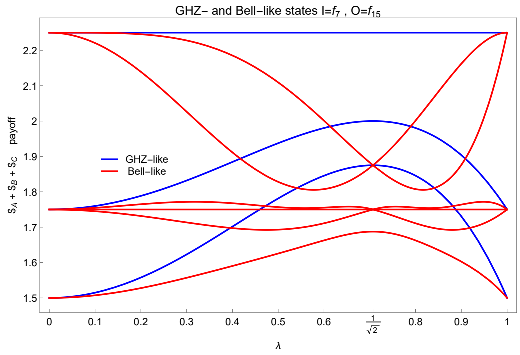



(f) $I = f_9$, $O = f_{10}$

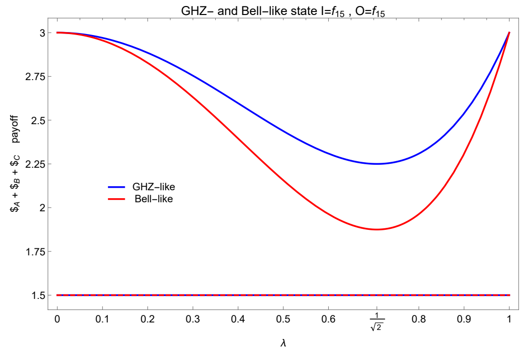



(g) $I = f_{15}$, $O = f_7$

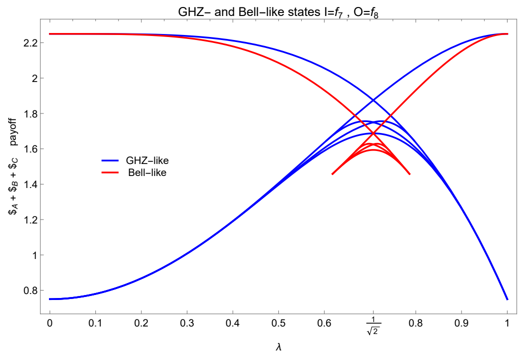
Fig. 6.1: *Comparison between the Nash equilibrium solutions that depended* only *on the corresponding entanglement parameter $\lambda$ for all the representative functions using the GHZ- and the Bell-like states. The sub-caption identifies the functions of the games $I$ and $O$ of that plot. These plots amalgamate the corresponding figures for both states and show all the lines in one unique colour: blue for the GHZ-like state, and red for the Bell-like state.*



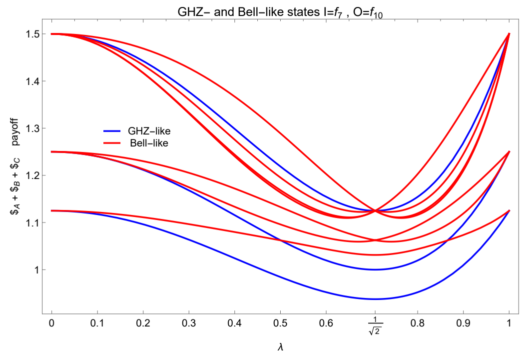

(a) $I = f_7$, $O = f_{15}$



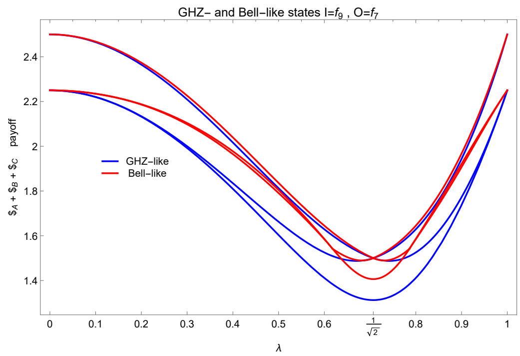

(b) $I = f_{15}$, $O = f_{15}$



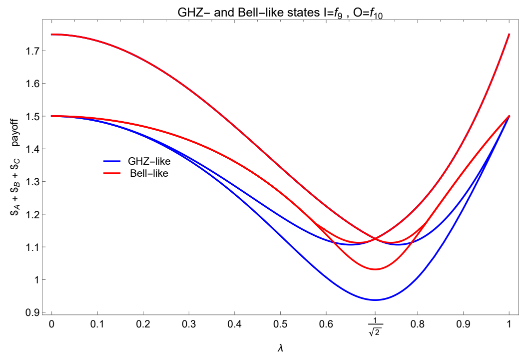

(c) $I = f_7$, $O = f_8$



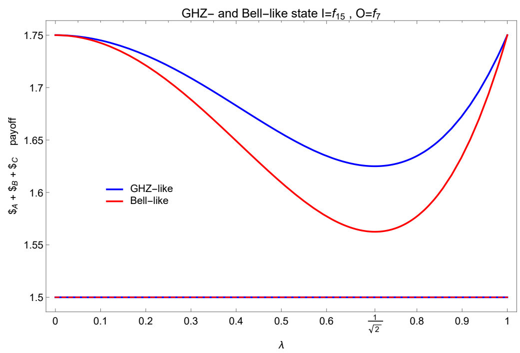

(d) $I = f_7$, $O = f_{10}$




(e) $I = f_9$, $O = f_7$




(f) $I = f_9$, $O = f_{10}$




(g) $I = f_{15}$, $O = f_7$

Fig. 6.2: *Comparison of the social welfare, i.e.* $\$_A + \$_B + \$_C$*, of all the Nash equilibrium solutions for all the representative functions using the GHZ- and the Bell-likes state, as a function of the corresponding entanglement parameter* $\lambda$*. These plots amalgamate the corresponding figures of the social welfare for both states and show all the lines in one unique colour: blue for the GHZ-like state, and red for the Bell-like state.*

Just for reference, the next list indicates for each of the sub-figures in Figure 6.1 and in Figure 6.2 which of the previous figures in chapter 5 these correspond to:

NASH EQUILIBRIUM POINTS (Figure 6.1)

- Figure 6.1(a) ($I = f_7$, $O = f_{15}$, CHSH game) is the combination of Figure 5.3 (GHZ-like) with all lines in blue, and Figure 5.12 (Bell-like) with all lines in red.
- Figure 6.1(b) ($I = f_{15}$, $O = f_{15}$) is the same as Figure 5.15.
- Figure 6.1(c) ($I = f_7$, $O = f_8$) is the combination of Figure 5.19 (GHZ-like) with all lines in blue, and Figure 5.25 (Bell-like) with all lines in red.
- Figure 6.1(d) ($I = f_7$, $O = f_{10}$) is the combination of Figure 5.28 (GHZ-like) with all lines in blue, and all the plots in Figure 5.30 and Figure 5.31 (Bell-like) with all lines in red.
- Figure 6.1(e) ($I = f_9$, $O = f_7$) is the combination of Figure 5.34 (GHZ-like) with all lines in blue, and the two plots in Figure 5.36 (Bell-like) with all lines in red.
- Figure 6.1(f) ($I = f_9$, $O = f_{10}$) is the combination of Figure 5.38 (GHZ-like) with all lines in blue, and Figure 5.40 and Figure 5.42 (Bell-like) with all lines in red.
- Figure 6.1(g) ($I = f_{15}$, $O = f_7$) is a horizontal line corresponding to the solution with constant payoff $\$_A = \$_B = \$_C = 1/2$ for the GHZ- and Bell-like state in Table 5.13.

SOCIAL WELFARE (Figure 6.2)

- Figure 6.2(a) ($I = f_7$, $O = f_{15}$, CHSH game) is the combination of Figure 5.6 with all lines in blue and Figure 5.13 with all lines in red.
- Figure 6.2(b) ($I = f_{15}$, $O = f_{15}$) is basically Figure 5.15 when the payoff is multiplied by 3, since for these solutions, the players' payoffs are identical.
- Figure 6.2(c) ($I = f_7$, $O = f_8$) is the combination of Figure 5.20 with all lines in blue and Figure 5.26 with all lines in red.
- Figure 6.2(d) ($I = f_7$, $O = f_{10}$) is the combination of Figure 5.29 with all lines in blue and Figure 5.33 with all lines in red.
- Figure 6.2(e) ($I = f_9$, $O = f_7$) is the combination of Figure 5.35 with all lines in blue and Figure 5.37 with all lines in red.
- Figure 6.2(f) ($I = f_9$, $O = f_{10}$) is the combination of Figure 5.39 with all lines in blue and Figure 5.43 with all lines in red.
- Figure 6.2(g) ($I = f_{15}$, $O = f_7$) is a new figure after adding the players' payoffs $\$_A+\$_B+\$_C$ from Table 5.13, which results in a function only of $\lambda_{111}$ (GHZ-like, blue line) and of $\lambda_{11}$ (Bell-like, red line) and a constant payoff of 3/2.

## 6.2 Conclusions

Looking at the results from chapter 5 and at the comparison between both states for all games, shown in Figure 6.1 and 6.2, a few **general statements** can be made:

- The game itself determines which of the two quantum states performs better, i.e. has Nash equilibrium points that give higher payoffs.
- The considered (restricted) states and strategies do not surpass the classical bounds of the solutions, i.e. no non-locality was found. This might be related to the choice here of the quantum states and strategies, since for certain values of some parameters in the quantum setting, the situation is no different than using classical mixed strategies; maybe using more-general quantum resources would lead to finding some non-local results. This is only speculation because, perhaps, the specific combination of probabilities, which come from the boolean games defined in the triangle, do not gather the conditions to exhibit non-locality. Clearly more research is necessary in this direction.
- The Bell-like state offers in general a richer situation with many new equilibrium points appearing for all games, in contrast to the GHZ-like state, which for some of the games gives exactly the same equilibrium points as classically using mixed strategies[1].
- Only for the CHSH game and for the game defined by $I = f_7$ (AND) $O = f_8$ (NAND), there are *certain Nash equilibrium points* that give higher payoffs as the entanglement increases compared to when there is less entanglement or none at all[2]. Some of these better-with-more-entanglement equilibrium points exist for the whole range of the entanglement parameter ($\lambda_{111}$ or $\lambda_{11}$) and some others only for a certain range. In some other games, the presence of entanglement might help to reduce the differences between the players' payoffs, but sometimes it does not.

Continuing with the tradition in this dissertation of having lots of tables, a very **short summary** of the results for each game for both states is found in **Table 6.1**, where the state whose solutions give, in general, higher payoffs is highlighted in green.

The research conducted in this dissertation provides the **answers** to the **research questions** on page 55 in chapter 5:

[1]Keep in mind the fundamental difference in the correspondence between the quantum and the classical mixed strategies; the quantum strategies imply performing (maybe entangled) measurements on a shared quantum state, while the mixed strategies could be implemented with just tossing a coin – biased coin when necessary.

[2]Only these two games of the 7 analysed here have some equilibrium points offering a higher-than-classical payoff thanks to entanglement. Nevertheless, remember that there were other 7 games to be analysed defined by the non-chosen representative functions – not marked in bold on pages 45-47 –, whose payoffs added to 1 with the payoffs of the representative functions, i.e. the games fully analysed here. That meant that the Nash equilibrium points for these 7 non-chosen representative games can be obtained directly from the payoffs of the 7 games fully analysed here by just minimising instead of maximising – see the long paragraph on page 50 for the justification. In the results for the 7 remaining games, not analysed here, there could be some equilibrium points showing the same payoff-higher-than-classical behaviour.

1) *What are the (new) Nash equilibrium points and how do they compare to the classical ones using mixed strategies?* See all the results in section 5.2 in chapter 5.

2) *Do the players prefer to use tri-partite or bi-partite (quantum) resources?* It depends on the game. See the green-highlighted states in the summary table, Table 6.1.

3) *Which Nash equilibrium solutions give the highest social welfare for the tri-partite and bi-partite case?* See the comparison in Figure 6.2 for each game, and the green-highlighted states in Table 6.1.

| Game | Summary |
|---|---|
| $I = f_7$, $O = f_{15}$<br>AND - XOR<br>(CHSH game) | **GHZ**: same equilibrium points as classically. **Bell**: richer situation in terms of many new equilibrium points, but give lower payoffs than the GHZ. |
| $I = f_{15}$, $O = f_{15}$<br>XOR - XOR | Same equilibrium points as classically for **GHZ** and **Bell**. The entanglement of the GHZ-like state is better because it decreases less the payoffs than Bell-like. |
| $I = f_7$, $O = f_8$<br>AND - NAND | Same number of points for the **GHZ** and **Bell**, but the entanglement in the GHZ gives higher payoffs. |
| $I = f_7$, $O = f_{10}$<br>AND - IMPLY | **GHZ:** same equilibrium points as classically even though the (quantum) payoff function is very different. **Bell**: very rich situation with many equilibrium points, and they give higher payoffs than the GHZ. |
| $I = f_9$, $O = f_7$<br>NIMPLY - NAND | Same number of total points for the **GHZ** and **Bell**, and different than classical, but the entanglement in the Bell gives slightly higher payoffs. |
| $I = f_9$, $O = f_{10}$<br>NIMPLY - IMPLY | **GHZ**: new equilibrium points different than classical. **Bell**: richest situation with the highest number of new equilibrium points of all games, and they give higher payoffs than the GHZ. |
| $I = f_{15}$, $O = f_7$<br>XOR - AND | Same equilibrium points as classically for **GHZ** and **Bell**. The entanglement of the GHZ-like state is better because it decreases less the payoffs than the Bell-like state. |

Tab. 6.1: *Summary of the results for each game, represented by their input $I = f_*$ and output functions $O = f_*$ – identified by their gate names, and whose boolean expressions are found in Table 4.1 in chapter 4 – and for each quantum state: the GHZ-like and the Bell-like state. The quantum state highlighted in green would be the "winner" when comparing the performance of both states.*

Even though the research conducted was interesting, it has two main sources of **limitations**:

1) **Difference between the classical and quantum setting**. As mentioned previously, the results in chapter 4 for the classical part stem from the players using mixed strategies (probabilistic mixture of pure strategies). This means that the players are not using any source of advice in the classical setting; whereas in the quantum results from chapter 5, the players are indeed using a quantum state to inform their output bits. Naturally, this difference in the setting leads to different qualitative behaviour, which explains why the quantum setting gives, in some of the games, more – and different – Nash equilibrium points than classically. Even though for certain non-entangled states the quantum results "reduce" to the classical ones, a direct comparison without warning is not correct. A proper comparison would require obtaining first the classical results when the players use some classical source of advice, and compare them with the ones using a quantum source of advice, i.e. some quantum state and measurements.

2) **Restrictive measurement settings**. The measurement operators $\{\Pi_{x,a}\}$, $\{\Pi_{y,b}\}$, $\{\Pi_{z,c}\}$ used in this research are a small subset of projection-valued measures (PVMs). The next step would be using a generic set of PVMs, and ultimately climb up to full generality by considering a generic set of positive operator-valued measures (POVMs). Expanding the allowed set of measurements would lead perhaps to different results.

The **main conclusion** that can be drawn from this dissertation is that **the proposed research**, even with its many assumptions/restrictions/limitations, **offers an interesting and rich situation worth exploring, since comparing two different types of quantum resources performing tasks** (or playing games) **in a quantum-network setting is crucial in the present and future of quantum information.**

## 6.3 Future perspectives

In the **results** of this dissertation one very **interesting question** arose that was not answered due to the lack of time, but that would be worth exploring next. The question appeared for the Bell-like state in the CHSH game ($I = f_7$, $O = f_{15}$) and also in the game defined by $I = f_7$, $O = f_{10}$. In both games, one of the Nash equilibrium solutions was $s^* = \{u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2\}$ [3], which reduced to the the classical solution $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$ at $\lambda_{11} = 0,\ 1,\ 1/\sqrt{2}$. However, the corresponding players' payoffs were different for $\lambda_{11} = 0,\ 1$ and for $\lambda_{11} = 1/\sqrt{2}$. As mentioned in the analysis on page 70 and on page 101, that might open the possibility of (**maybe**) being able to **distinguish** if the **strategy** $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$ was **generated** with a **quantum device** using a pure Bell state or with a classical/unentangled device. The next task would be to investigate if there exist classical correlated strategies that reduce to $s^* = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2\}$ *and* give the same payoff

[3]See equation (5.14) on page 69 for the specific expression of $u_{B1}(\lambda_{11})$, which was also plotted in Figure 5.7.

as $s^* = \{u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2, u_{B1}/2\}$ for $\lambda_{11} = 1/\sqrt{2}$. If they *do not* exist, then this result would be encompassed into the vast and useful field of self-testing.

It would be natural to gap the limitations of this research – described above – by, firstly, properly studying the games classically when the players use indeed some classical source of advice, and not only mixed strategies. The second step would be studying the games in the quantum setting when the players use a generic set of measurements (POVMs). Lastly and more importantly, it would be very useful to have a more-proper characterisation of the payoffs – in equation (4.8) on page 33 – only in terms of conditional probabilities, so that these combinations of probabilities can be analysed, leading maybe to general statements; as is typically done for local and non-local correlations in general local hidden-variables theories and non-signalling theories, e.g. [86, 88].

The previous two paragraphs mentioned one immediate avenue to be explored and addressing the limitations of this research, but since this dissertation **mixed** the **sub-fields** of quantum games, quantum networks, and quantum resources – see the diagram of the author's inspiration on page 55 – it gives **infinite possibilities** to **extend this work**. Just to name a few in each area:

- *Quantum resources*: 1) Do the same analysis but considering a more-generic quantum strategy with more parameters than the ones considered – in equations (5.5) and (5.8) on page 57 – to see if there exist better strategies. 2) Do the same analysis of the Nash equilibrium points for all the games with a W-like state – see equation (3.9) on page 28 – , and maybe add some asymmetry in the used state for the players. 3) Investigate if there are states (in any dimension) that are optimal, and if such states achieve a higher payoff than the classical bounds (non-locality).

- *Quantum games*: 1) Choose a different input/output boolean function for each game instead of only two functions $I, O$ for all games as presented here, i.e. $I_1, O_1$ (game between Alice and Bob); $I_2, O_2$ (game between Bob and Carl); $I_3, O_3$ (game between Carl and Alice). 2) Use input/output function of the 3 input/output variables at the same time, e.g. $I(x, y, z)$ and $O(a, b, c)$, and from these define pairwise – to maintain the triangle network structure – some winning conditions for each game.

- *Quantum networks*: 1) Consider the situation of a square network with an extra player. 2) Have one player playing more games than the others, e.g. remove the game between Bob and Carl, or have a central player playing 2 or 3 games.

The focus of this dissertation and of these possible extensions is purely theoretical, just for the sake of knowledge, so one important task is **finding practical applications** of the results in this dissertation, most likely to be found within quantum information and maybe quantum cryptography. In any case, regardless of the possible applicability or not of the research conducted, **this dissertation serves as a significant step for the author in this research direction**.

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# Appendix A

## A.1 Optimisation of the angles for the CHSH game

The winning probability $Prob(\text{win})$, in the next calculation shortened as $P_{\text{win}}$, of the CHSH game in equation (2.21) in chapter 2 is:

$$P_{\text{win}} = \frac{1}{4}\left[\cos^2(\alpha_0 - \beta_0) + \cos^2(\alpha_0 - \beta_1) + \cos^2(\alpha_1 - \beta_0) + \sin^2(\alpha_1 - \beta_1)\right] \quad \text{(A.1)}$$

To optimise this winning probability, the partial derivatives are needed:

$$\frac{\partial P_{\text{win}}}{\partial \alpha_0} = -\frac{1}{2}\cos(\beta_0 - \beta_1)\sin(2\alpha_0 - \beta_0 - \beta_1) \quad \text{(A.2)}$$

$$\frac{\partial P_{\text{win}}}{\partial \alpha_1} = \frac{1}{2}\sin(\beta_0 - \beta_1)\cos(2\alpha_1 - \beta_0 - \beta_1) \quad \text{(A.3)}$$

$$\frac{\partial P_{\text{win}}}{\partial \beta_0} = -\frac{1}{2}\cos(\alpha_0 - \alpha_1)\sin(2\beta_0 - \alpha_0 - \alpha_1) \quad \text{(A.4)}$$

$$\frac{\partial P_{\text{win}}}{\partial \beta_1} = \frac{1}{2}\sin(\alpha_0 - \alpha_1)\cos(2\beta_1 - \alpha_0 - \alpha_1) \quad \text{(A.5)}$$

where some trigonometric identities were used to write the partial derivatives as products. The partial derivatives must vanish[1], which implies the following relations:

$$\frac{\partial P_{\text{win}}}{\partial \alpha_0} = 0 \begin{cases} \nearrow \; \beta_0 - \beta_1 = \frac{\pi}{2}(2n_1 + 1) & \text{(A.6)} \\ \searrow \; 2\alpha_0 - \beta_0 - \beta_1 = \pi n_2 & \text{(A.7)} \end{cases}$$

$$\frac{\partial P_{\text{win}}}{\partial \alpha_1} = 0 \begin{cases} \nearrow \; \beta_0 - \beta_1 = \pi k_1 & \text{(A.8)} \\ \searrow \; 2\alpha_1 - \beta_0 - \beta_1 = \frac{\pi}{2}(2k_2 + 1) & \text{(A.9)} \end{cases}$$

[1] To shorten the notation for the next calculation, the name of the variables at the critical points will be the same as the variables themselves. For instance, $\frac{\partial P_{\text{win}}}{\partial \alpha_0} = 0$ needs to be read as the partial derivative with respect to variable $\alpha_0$ evaluated at the critical angles $cr = \{\alpha_0^{cr}, \alpha_1^{cr}, \beta_0^{cr}, \beta_1^{cr}\}$, usually denoted by $\left.\frac{\partial P_{\text{win}}}{\partial \alpha_0}\right|_{cr} = 0$.

$$\frac{\partial P_{\text{win}}}{\partial \beta_0} = 0 \begin{cases} \nearrow \; \alpha_0 - \alpha_1 = \frac{\pi}{2}(2m_1 + 1) & \text{(A.10)} \\ \searrow \; 2\beta_0 - \alpha_0 - \alpha_1 = \pi m_2 & \text{(A.11)} \end{cases}$$

$$\frac{\partial P_{\text{win}}}{\partial \beta_1} = 0 \begin{cases} \nearrow \; \alpha_0 - \alpha_1 = \pi l_1 & \text{(A.12)} \\ \searrow \; 2\beta_1 - \alpha_0 - \alpha_1 = \frac{\pi}{2}(2l_2 + 1) & \text{(A.13)} \end{cases}$$

where $n_1, n_2, k_1, k_2, m_1, m_2, l_1, l_2 \in \mathbb{Z}$.

The next step is finding all the possible sets of solutions such that all four derivatives vanish simultaneously. Not all combinations are possible, for example, if equation (A.6) holds, then equation (A.8) cannot hold, so equation (A.9) must hold. The specific steps to obtain all the (compatible) solutions are skipped because it is a long tedious calculation. In terms of the free angle $\alpha_0$, the solutions $\{\theta\} \equiv \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$ that make all derivatives in equations (A.2)–(A.5) vanish are:

$$\{\theta^{(1)}\} = \{\alpha_0,\ \alpha_0 - \frac{\pi}{2}(2m_1 + 1),\ \alpha_0 + \frac{\pi}{2}(2n_1 + l_2 - m_1 + 1),\ \alpha_2 + \frac{\pi}{2}(l_2 - m_1)\} \tag{A.14}$$

$$\{\theta^{(2)}\} = \{\alpha_0,\ \alpha_0 - l_1\pi,\ \alpha_0 + \frac{\pi}{2}(m_2 - l_1),\ \alpha_0 - \frac{\pi}{2}(3l_1 + 2k_2 + m_2 + 1)\} \tag{A.15}$$

$$\{\theta^{(3)}\} = \{\alpha_0,\ \alpha_0 - \frac{\pi}{2}(2m_1 + 1),\ \alpha_0 + \frac{\pi}{2}(2k_1 + l_2 - m_1),\ \alpha_0 + \frac{\pi}{2}(l_2 - m_1)\} \tag{A.16}$$

$$\{\theta^{(4)}\} = \{\alpha_0,\ \alpha_0 - \pi(n_2 + m_2 - k_1),\ \alpha_0 + \frac{\pi}{2}(k_1 - n_2),\ \alpha_0 - \frac{\pi}{2}(k_1 - n_2)\} \tag{A.17}$$

$$\{\theta^{(5)}\} = \{\alpha_0,\ \alpha_0 + \frac{\pi}{4}(2k_2 - 2n_2 + 1),\ \alpha_0 + \frac{\pi}{8}(2k_2 - 2n_2 + 4m_2 + 1),\ \alpha_0 - \frac{\pi}{8}(2k_2 + 6n_2 + 4m_2 + 1)\} \tag{A.18}$$

Substituting the solutions in equations (A.14)-(A.18) into the winning probability in equation (A.1) and using some trigonometric identities, the extreme values are:

$$P_{\text{win}}(\{\theta^{(1)}\}) = \frac{1}{4}\left[2 + (-1)^{l_2}\right] \qquad \text{with } m_1 = n_1 = 0 \tag{A.19}$$

$$P_{\text{win}}(\{\theta^{(2)}\}) = \frac{1}{4}\left[2 + (-1)^{m_2}\right] \qquad \text{with } l_1 = k_2 = 0 \tag{A.20}$$

$$P_{\text{win}}(\{\theta^{(3)}\}) = \frac{1}{4}\left[2 + (-1)^{l_2}\right] \qquad \text{with } m_1 = k_1 = 0 \tag{A.21}$$

$$P_{\text{win}}(\{\theta^{(4)}\}) = \frac{1}{4}\left[2 + (-1)^{n_2}\right] \qquad \text{with } m_2 = k_1 = 0 \tag{A.22}$$

$$P_{\text{win}}(\{\theta^{(5)}\}) = \frac{1}{4}\left[2 + (-1)^{m_2}\sqrt{2}\right] \qquad \text{with } k_2 = n_2 = 0 \tag{A.23}$$

where, to simplify the expressions, some of the free parameters $n_1, n_2, k_1, k_2, m_1, m_2, l_1, l_2 \in \mathbb{Z}$ were set to 0.

Clearly, the global maximum and minimum of the winning probability is achieved with $\{\theta^{(5)}\}$ in equation (A.23). The maximum is $(2 + \sqrt{2})/4 \approx 0.853$ and the minimum is $(2 - \sqrt{2})/4 \approx 0.146$. The canonical choice of angles for the maximum corresponds to choosing $m_2 = k_2 = n_2 = 0$, which translates into angles $\{\theta^{(5)}_{max}\} = \{\alpha_0,\ \alpha_0 + \frac{\pi}{4},\ \alpha_0 + \frac{\pi}{8},\ \alpha_0 - \frac{\pi}{8}\}$; for the minimum, the choice $m_2 = -1$ and $k_2 = n_2 = 0$ gives some minimising angles $\{\theta^{(5)}_{min}\} = \{\alpha_0,\ \alpha_0 + \frac{\pi}{4},\ \alpha_0 - \frac{3\pi}{8},\ \alpha_0 + \frac{3\pi}{8}\}$.

## A.2 3-tangle of the GHZ and W states

This section contains a detailed computation of the **3-tangle** $\tau_{ABC}$ of a **GHZ-like** state and a **W-like** state. Remember that the tangle between parties A and B is computed from the density matrix $\rho_{AB}$:

$$\tau_{AB} = \max(0, \sqrt{\eta_1} - \sqrt{\eta_2} - \sqrt{\eta_3} - \sqrt{\eta_4})^2 \tag{A.24}$$

with $\eta_j$ being the eigenvalues in descending order of the matrix $\rho_{AB}\tilde{\rho}_{AB}$, where $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$; $\sigma_y$ is the Pauli matrix, and $\rho^*_{AB}$ denotes only conjugation of $\rho_{AB}$. The 3-tangle is then computed as:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC} \tag{A.25}$$

where, for pure states, $\tau_{A(BC)} = 4\det(\rho_A)$. With all these definitions, the only ingredients needed to compute the $\tau_{ABC}$ of a tri-partite pure state are the reduced density matrices $\rho_{AB}$, $\rho_{AC}$, and $\rho_A$ [2].

### A.2.1 GHZ-like state

The chosen GHZ-like state is $|GHZ\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$ [3]. The first step is computing the density matrix $\rho_{ABC} = |GHZ\rangle\langle GHZ|$, which takes a very simple form:

$$\rho_{ABC} = |\alpha|^2\,|000\rangle\langle 000| + |\beta|^2\,|111\rangle\langle 111| + \alpha\beta^*\,|000\rangle\langle 111| + \beta\alpha^*\,|111\rangle\langle 000| \tag{A.26}$$

The **reduced density matrices** take a simple form too:

$$\rho_{AB} = Tr_C(\rho_{ABC}) = |\alpha|^2\,|00\rangle\langle 00| + |\beta|^2\,|11\rangle\langle 11| = \begin{pmatrix} |\alpha|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix} \tag{A.27}$$

[2] Remember that $\tau_{ABC}$ does not depend on the chosen party. The 3-tangle is an invariant and could be computed using also the reduced density matrices $\rho_{BA}$, $\rho_{BC}$, and $\rho_B$; or $\rho_{CA}$, $\rho_{CB}$, and $\rho_C$.

[3] It has been chosen to write the state using two parameters $\alpha$ and $\beta$ and not with $\alpha$ and $\sqrt{1-|\alpha|^2}$ to make evident that $\beta$ can be purely imaginary too, since it is usually assumed that $\sqrt{1-|\alpha|^2}$ is a real number.

$$\rho_{AC} = Tr_B(\rho_{ABC}) = |\alpha|^2 \left|00\right\rangle\left\langle 00\right| + |\beta|^2 \left|11\right\rangle\left\langle 11\right| = \begin{pmatrix} |\alpha|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix} \tag{A.28}$$

$$\rho_A = Tr_{AB}(\rho_{ABC}) = |\alpha|^2 \left|0\right\rangle\left\langle 0\right| + |\beta|^2 \left|1\right\rangle\left\langle 1\right| = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{A.29}$$

From $\rho_A$ in equation (A.29), the tangle between A and BC can directly be computed: $\tau_{A(BC)} = 4\det(\rho_A) = 4|\alpha|^2|\beta|^2 = 4|\alpha|^2(1-|\alpha|^2)$, where the normalisation condition has been used $|\beta|^2 = 1-|\alpha|^2$.

The next step to compute $\tau_{AB}$ needs the flipped density matrix $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$. The reduced density matrix is diagonal $\rho_{AB} = diag[|\alpha|^2, 0, 0, |\beta|^2]$, and the effect of the flipping operators $\sigma_y \otimes \sigma_y$ over a diagonal matrix with diagonal entries $a_{11}, a_{22}, a_{33}, a_{44}$ is easily explained: it flips the order of the diagonal. That is, the new diagonal is $a_{44}, a_{33}, a_{22}, a_{11}$. Taking this into account, then $\tilde{\rho}_{AB} = diag[|\beta|^2, 0, 0, |\alpha|^2]$. Finally, the $\rho_{AB}\tilde{\rho}_{AB}$ can be easily computed, whose result is:

$$\rho_{AB}\tilde{\rho}_{AB} = \begin{pmatrix} |\alpha|^2|\beta|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\alpha|^2|\beta|^2 \end{pmatrix} \tag{A.30}$$

The **eigenvalues** of $\rho_{AB}\tilde{\rho}_{AB}$ are clearly $\eta_1 = \eta_2 = |\alpha|^2|\beta|^2$; $\eta_3 = \eta_4 = 0$, which, by the definition of the tangle in equation (A.24), implies $\tau_{AB} = 0$. The **tangle** between **subsystem AB vanishes**, which means there is no entanglement; they share a separable state. This can be seen directly from the density matrix in equation (A.27): $\rho_{AB} = |\alpha|^2 \left|00\right\rangle\left\langle 00\right| + |\beta|^2 \left|11\right\rangle\left\langle 11\right| = |\alpha|^2 \left|\phi_1\right\rangle\left\langle \phi_1\right| + |\beta|^2 \left|\phi_2\right\rangle\left\langle \phi_2\right|$, where $\left|\phi_1\right\rangle = \left|0\right\rangle_A \otimes \left|0\right\rangle_B$ and $\left|\phi_2\right\rangle = \left|1\right\rangle_A \otimes \left|1\right\rangle_B$. This result implies the following: upon measurement of subsystem AB, state $\left|\phi_1\right\rangle = \left|0\right\rangle_A \otimes \left|0\right\rangle_B$ will be measured with probability $|\alpha|^2$, and state $\left|\phi_2\right\rangle = \left|1\right\rangle_A \otimes \left|1\right\rangle_B$ will be measured with probability $|\beta|^2 = 1-|\alpha|^2$. Clearly, state $\left|\phi_1\right\rangle$ is separable and so is state $\left|\phi_2\right\rangle$, even though the outcomes for A and B will be correlated. This means that if A measures state $\left|0\right\rangle$ ($\left|1\right\rangle$), B will measure $\left|0\right\rangle$ ($\left|1\right\rangle$) too with probability one. In this sense, the (classical) correlation is there but the subsystem AB is not entangled.

The subsystem AC has the same results as subsystem AB because both reduced density matrices in equations (A.27) and (A.28) are identical: $\tau_{AC} = \tau_{AB} = 0$. The **3-tangle** of the **GHZ-like** state is:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{BC} = 4|\alpha|^2(1 - |\alpha|^2) \tag{A.31}$$

It is interesting to note that $\tau_{\mathbf{ABC}}$ and $\tau_{A(BC)}$, which are the same for this GHZ-like state, $\tau_{ABC} = \tau_{A(BC)} = 4|\alpha|^2(1-|\alpha|^2)$ is **identical** to the **tangle** of a (bi-partite) **Bell-like** state. That is, the tangle of the 2-qubit state $|\psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$ is also $\tau_{AB} = 4|\alpha|^2(1-|\alpha|^2)$. This means that the GHZ-like state has the same tri-partite entanglement (measured with $\tau_{ABC}$) as the bi-partite entanglement (measured with $\tau_{AB}$) in a Bell-like state.

The maximum value of the $\tau_{ABC}$ for the GHZ-like state is 1; achieved when $|\alpha| = 1/\sqrt{2}$, which, up to relative phases, corresponds to the standard GHZ state $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$.

### A.2.2 W-like state

The chosen W-like state is: $|W\rangle = \alpha\,|100\rangle + \beta\,|010\rangle + \gamma\,|001\rangle$, and by normalisation, $|\alpha|^2+|\beta|^2+|\gamma|^2 = 1$. As before, the first step is computing the density matrix $\rho_{ABC} = |W\rangle\langle W|$, which is:

$$\begin{aligned}\rho_{ABC} =&|\alpha|^2\,|100\rangle\langle 100| + \alpha\beta^*\,|100\rangle\langle 010| + \alpha\gamma^*\,|100\rangle\langle 001|\\ &+ \beta\alpha^*\,|010\rangle\langle 100| + |\beta|^2\,|010\rangle\langle 010| + \beta\gamma^*\,|010\rangle\langle 001|\\ &+ \alpha^*\gamma\,|001\rangle\langle 100| + \beta^*\gamma\,|001\rangle\langle 010| + |\gamma|^2\,|001\rangle\langle 001|\end{aligned} \tag{A.32}$$

The **reduced density matrices** are:

$$\begin{aligned}\rho_{AB} =&|\alpha|^2\,|10\rangle\langle 10| + \alpha\beta^*\,|10\rangle\langle 01| + \beta\alpha^*\,|01\rangle\langle 10| + |\beta|^2\,|01\rangle\langle 01| + |\gamma|^2\,|00\rangle\langle 00|\\ =&\begin{pmatrix}|\gamma|^2 & 0 & 0 & 0\\ 0 & |\beta|^2 & \alpha^*\beta & 0\\ 0 & \beta^*\alpha & |\alpha|^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\end{aligned} \tag{A.33}$$

$$\begin{aligned}\rho_{AC} =&|\alpha|^2\,|10\rangle\langle 10| + \alpha\gamma^*\,|10\rangle\langle 01| + \gamma\alpha^*\,|01\rangle\langle 10| + |\beta|^2\,|00\rangle\langle 00| + |\gamma|^2\,|01\rangle\langle 01|\\ =&\begin{pmatrix}|\beta|^2 & 0 & 0 & 0\\ 0 & |\gamma|^2 & \alpha\gamma^* & 0\\ 0 & \gamma\alpha^* & |\alpha|^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\end{aligned} \tag{A.34}$$

$$\rho_A = |\alpha|^2 \, |1\rangle\langle 1| + (|\beta|^2 + |\gamma|^2) \, |0\rangle\langle 0| = \begin{pmatrix} |\beta|^2 + |\gamma|^2 & 0 \\ 0 & |\alpha|^2 \end{pmatrix} \tag{A.35}$$

The tangle between A and BC is $\tau_{A(BC)} = 4\det(\rho_A) = 4|\alpha|^2(|\beta|^2 + |\gamma|^2)$.

The flipped density matrix $\tilde{\rho}_{AB}$ is:

$$\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\alpha|^2 & \alpha^*\beta & 0 \\ 0 & \beta^*\alpha & |\beta|^2 & 0 \\ 0 & 0 & 0 & |\gamma|^2 \end{pmatrix} \tag{A.36}$$

The product $\rho_{AB}\tilde{\rho}_{AB}$ is:

$$\rho_{AB}\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2|\alpha|^2|\beta|^2 & 2|\beta|^2\alpha^*\beta & 0 \\ 0 & 2|\alpha|^2\alpha\beta^* & 2|\alpha|^2|\beta|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.37}$$

The **eigenvalues** of $\rho_{AB}\tilde{\rho}_{AB}$ are: $\eta_1 = 4|\alpha|^2|\beta|^2$ and $\eta_2 = \eta_3 = \eta_4 = 0$. Then, the tangle between subsystem A and B is $\tau_{AB} = 4|\alpha|^2|\beta|^2$. Since the reduced density matrix $\rho_{AC}$ in equation (A.34) can be obtained from $\rho_{AB}$ in equation (A.33) by interchanging $\beta$ and $\gamma$, the result for $\tau_{AC}$ is straightforward from $\tau_{AB}$ with that swap: $\tau_{AC} = 4|\alpha|^2|\gamma|^2$.

With these results for the tangles, the **3-tangle** of the **W-like** state is **0** regardless of the parameters $\alpha$, $\beta$, $\gamma$:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{BC} = 4|\alpha|^2(|\beta|^2 + |\gamma|^2) - 4|\alpha|^2|\beta|^2 - 4|\alpha|^2|\gamma|^2 = 0 \tag{A.38}$$

# Appendix B

## B.1 Convex optimisation

This section has been extracted from the book about convex optimisation in [102] and it follows a similar notation. An objective (real) function $f(\mathbf{x})$ is to be minimised subject to some inequality constraints defined by $g_i(\mathbf{x}) \leqslant 0$, and some equality constraints defined by $h_j(\mathbf{x}) = 0$. In the standard form, this optimisation problem is usually written as:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \\ & g_i(\mathbf{x}) \leqslant 0, \quad i = 1, ..., m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, ..., p \end{aligned} \tag{B.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $f : \mathcal{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. The optimal value of this problem will be denoted as $p^\star$. If instead of minimising, the objective is maximising a function, then it is enough to change the sign of the function to turn it into a minimisation problem, i.e. $\max(f(\mathbf{x})) = \min(-f(\mathbf{x}))$

The problem defined in (B.1) is said to be a **convex** optimisation **problem** when $f(\mathbf{x})$ and $g_i(\mathbf{x})$ are **convex functions** and $h_j(\mathbf{x})$ are **linear**. Many of the convex optimisation problems can be solved using well-established algorithms that run in polynomial time.

The idea behind optimising a function with constraints is to transform it into optimising a weighted function that incorporates the constraints. This weighted function is the *Lagrangian* of the problem stated in (B.1). The Lagrangian is defined as:

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x}) \tag{B.2}$$

where $\lambda_i$ and $\nu_j$ are known as the Lagrange multipliers of the inequality and equality constraints $g_i(\mathbf{x}) \leqslant 0$ and $h_j(\mathbf{x}) = 0$, respectively. The minimum value of the Lagrangian over $\mathbf{x}$, for $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in \mathbb{R}^p$ is called the *dual function* $g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \inf_{x \in \mathcal{D}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})$ [1]. In the case of $\lambda_i \geqslant 0$ and for any value of $\nu_j$, the dual function $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ provides a lower bound of the optimal value of the primal problem in (B.1),

[1]The dual function $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ should not be confused with the functions of the inequality constraints $g_i(\mathbf{x})$

i.e. $g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leqslant p^\star$ for $\lambda_i \geqslant 0$. This lower bound can also be optimised and written as an optimisation problem in standard form:

$$\begin{aligned} &\underset{\boldsymbol{\lambda},\boldsymbol{\nu}}{\text{maximise}} \; g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \\ &\text{subject to } \lambda_i \geqslant 0 \end{aligned} \tag{B.3}$$

This optimisation problem in (B.3) is referred to as the *dual problem* of the *primal problem* in (B.1). Even if the primal problem is not convex, the dual problem is indeed convex. The reason is that the function to maximise $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$ is concave since it is the pointwise infimum of a family of affine functions of $(\boldsymbol{\lambda}, \boldsymbol{\nu})$, and the constraints are convex, therefore the optimisation (dual) problem in (B.3) can be solved using standard techniques. The optimum value of the dual problem in (B.3), denoted as $d^\star$, is the best lower bound possible from the Lagrange dual function of the original – possibly non-convex – (primal) problem in (B.1), i.e. $d^\star \leqslant p^\star$. This relation is called weak duality. **Strong duality** is when $d^\star = p^\star$, that is, the **optimal** solution of the **dual** problem gives exactly the **optimal** solution of the **primal** problem. Strong duality is also phrased in the literature as a problem with zero duality gap. For convex problems, that is, when $f(\mathbf{x})$ and $g_i(\mathbf{x})$ are convex functions and $h_j(\mathbf{x})$ linear, strong duality usually holds – but not always.

There are cases and conditions under which strong duality can be proven; such conditions are called constraint qualifications. One of the simplest of these constraints comes from Slater's theorem. Slater's theorem states that strong duality holds if the problem is convex and the next condition (**Slater's condition**) holds: if there exists a relative interior $\mathbf{x} \in$ *relint* $\mathcal{D}$ such that $g_i(\mathbf{x}) < 0$ , $i = 1, ..., m$ , where the equality constraints are $A\mathbf{x} = b$, previously denoted generically with $h_j(\mathbf{x})$. Slater's condition says that the feasible region must have an interior point, that is, a point for which the equality constraints $A\mathbf{x} = b$ are satisfied and the inequality constraints *strictly* hold.

When strong duality holds and the $f(\mathbf{x}), g_i(\mathbf{x}), h_j(\mathbf{x})$ are differentiable, it can be proven that the optimal value of the primal and dual problem, achieved by the pairs $\mathbf{x}^\star$, $(\lambda^\star, \nu^\star)$, must fulfill certain conditions. These are known as the *Karush-Kuhn-Tucker conditions* (**KKT conditions**), which are:

$$\begin{aligned} &\text{1) Primal feasibility:} && g_i(\mathbf{x}^\star) \leqslant 0 && i = 1, ..., m \\ & && h_j(\mathbf{x}^\star) = 0 && j = 1, ..., p \\ &\text{2) Dual feasibility:} && \lambda_i^\star \geqslant 0 && i = 1, ..., m \\ &\text{3) Complementary slackness:} && \lambda_i^\star g_i(\mathbf{x}^\star) = 0 && i = 1, ..., m \\ &\text{4) Stationarity:} && \nabla f(\mathbf{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla g_i(\mathbf{x}^\star) + \sum_{j=1}^{p} \nu_j^\star \nabla h_j(\mathbf{x}^\star) = 0 \end{aligned} \tag{B.4}$$

where the last equation is just the gradient of the Lagrangian in equation (B.2) $\nabla L = 0$.

The KKT conditions generalise the method of Lagrange multipliers used to optimise a function with only equality constraints. If the primal problem is convex, then any point satisfying the KKT conditions in the equations in (B.4) is optimal for the primal and dual problem, with zero duality gap (strong duality). In other words, for a convex problem, the KKT conditions are sufficient to find the optimal solution.

## B.2 Nash equilibrium and convex optimisation

Remember that the condition for a **Nash Equilibrium** to happen is that no player wants to unilaterally deviate from it given that the other players do not deviate. Mathematically, a strategy profile $s^* = \{s_1^*, s_2^*, ..., s_n^*\}$ is a Nash equilibrium if: $\$_i(s_i^*, s_{-i}^*) \geqslant \$_i(s_i', s_{-i}^*)\ \forall i$, where $s_i$ denotes player $i$'s strategy and $s_{-i}$ the strategies of all the players except player $i$'s.

Then, in this case with three players, Alice, Bob and Carl must maximise their own payoffs as a function of their own strategies, denoted as $\$_A(p_{0|0}, p_{0|1})$, $\$_B(q_{0|0}, q_{0|1})$, and $\$_C(r_{0|0}, r_{0|1})$, respectively. For **Alice**, this **maximisation problem** can be written in standard form:

$$\begin{aligned} \underset{p_{0|0}, p_{0|1}}{\text{maximise}} \quad & \$_A(p_{0|0}, p_{0|1}) \\ \text{subject to} \quad & -p_{0|0} \leqslant 0, \\ & -p_{0|1} \leqslant 0, \\ & -1 + p_{0|0} \leqslant 0, \\ & -1 + p_{0|1} \leqslant 0 \end{aligned} \tag{B.5}$$

This problem is convex[2] and fulfills Slater's condition[3], which implies that strong duality holds. This fact means that this problem can be solved using the techniques explained in the previous section B.1. To do that, the **Lagrangian** of the primal problem stated in (B.5) is:

$$L_A(p_{0|0}, p_{0|1}) = -\$_A - \lambda_1 p_{0|0} - \lambda_2 p_{0|1} + \lambda_3(-1 + p_{0|0}) + \lambda_4(-1 + p_{0|1}) \tag{B.6}$$

where the minus sign in front of $\$_A$ comes from the fact that it is a maximisation problem, instead of the usual minimisation. The objective is to maximise $L_A$ with respect to variables $p_{0|0}$ and $p_{0|1}$.

The corresponding **KKT conditions** encompassed in (B.4), which, in this case, are sufficient to find optimality, are:

$$1) \quad -p_{0|0} \leqslant 0, \quad -p_{0|1} \leqslant 0, \quad -1 + p_{0|0} \leqslant 0, \quad -1 + p_{0|1} \leqslant 0 \tag{B.7}$$

[2]The function to optimise $\$_A(p_{0|0}, p_{0|1})$ and the inequality constraints $g_1(p_{0|0}) = -p_{0|0} \leqslant 0$, $g_2(p_{0|1}) = -p_{0|1} \leqslant 0$, $g_3(p_{0|0}) = -1 + p_{0|0} \leqslant 0$, $g_4(p_{0|1}) = -1 + p_{0|1} \leqslant 0$ are both convex functions, that is, they all fulfill the convexity condition: for any pair of points $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ and any $0 \leqslant \mu \leqslant 1$ $\rightarrow f(\mu \mathbf{x} + (1-\mu)\mathbf{y}) \leqslant \mu f(\mathbf{x}) + (1-\mu) f(\mathbf{y})$. In fact, they are all affine functions, which means they are both convex and concave.

[3]In this case, there are no equality constraints which implies that the feasible region has an interior point, i.e. a point for which the inequality constraints strictly hold $g_i(\mathbf{x}) < 0$.

$$2)\quad \tilde{\lambda}_1 \geqslant 0,\ \ \tilde{\lambda}_2 \geqslant 0,\ \ \tilde{\lambda}_3 \geqslant 0,\ \ \tilde{\lambda}_4 \geqslant 0, \tag{B.8}$$

$$3)\quad p_{0|0}\tilde{\lambda}_1 = 0,\ \ p_{0|1}\tilde{\lambda}_2 = 0,\ \ (1-p_{0|0})\tilde{\lambda}_3 = 0,\ \ (1-p_{0|1})\tilde{\lambda}_4 = 0 \tag{B.9}$$

$$4)\quad -\frac{\partial \$_A}{\partial p_{0|0}} - \tilde{\lambda}_1 + \tilde{\lambda}_3 = 0 \tag{B.10}$$

$$-\frac{\partial \$_A}{\partial p_{0|1}} - \tilde{\lambda}_2 + \tilde{\lambda}_4 = 0 \tag{B.11}$$

By combining equations (B.9), (B.10) and (B.11), it is simple to reduce the system of equations by eliminating the Lagrange multipliers[4] $\tilde{\lambda}_i$ so that the equations depend only on $p_{0|0}$ and $p_{0|1}$:

$$1)\quad -p_{0|0} \leqslant 0,\ \ -p_{0|1} \leqslant 0,\ \ -1+p_{0|0} \leqslant 0,\ \ -1+p_{0|1} \leqslant 0 \tag{B.12}$$

$$2)\quad \tilde{\lambda}_1 = -(1-p_{0|0})\frac{\partial \$_A}{\partial p_{0|0}} \geqslant 0 \tag{B.13}$$

$$\tilde{\lambda}_2 = -(1-p_{0|1})\frac{\partial \$_A}{\partial p_{0|1}} \geqslant 0 \tag{B.14}$$

$$\tilde{\lambda}_3 = p_{0|0}\frac{\partial \$_A}{\partial p_{0|0}} \geqslant 0 \tag{B.15}$$

$$\tilde{\lambda}_4 = p_{0|1}\frac{\partial \$_A}{\partial p_{0|1}} \geqslant 0 \tag{B.16}$$

The Nash equilibrium solutions $s^* = \{p^*_{0|0}, p^*_{0|1}, q^*_{0|0}, q^*_{0|1}, r^*_{0|0}, r^*_{0|1}\}$ are obtained from the solutions to equations (B.12)-(B.16) and the analogous equations for Bob and Carl.

### B.2.1 Nash Equilibria for boolean games in a triangle using classical strategies

This section gives the KKT conditions in equations (B.12)-(B.16) for Alice's payoff for all the representative functions when the players are using mixed (**classical**) strategies.

- $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$, – CHSH game –, Alice's payoff in equation (4.34) on page 45 is:

$$\begin{aligned}\$_A(p_{0|0}, p_{0|1}) =& \frac{1}{4}\big[3 - (2p_{0|0} + q_{0|0} + r_{0|0}) + (p_{0|0} + p_{0|1})(q_{0|0} + r_{0|0}) \\ &+ (p_{0|0} - p_{0|1})(q_{0|1} + r_{0|1})\big]\end{aligned} \tag{B.17}$$

[4]These Lagrange multipliers have been denoted with a tilde $\tilde{\lambda}_i$ instead of $\lambda_i^\star$ because, in general for non-convex problems, the KKT conditions in (B.4) might not be sufficient, though they are necessary conditions. In this case, the problem is convex and strong duality holds, hence these $\tilde{\lambda}_i$ provide the optimal solution.

In this particular case, after computing the partial derivatives of $\$_A$, the KKT conditions are:

$$1) \quad -p_{0|0} \leqslant 0, \quad -p_{0|1} \leqslant 0, \quad -1+p_{0|0} \leqslant 0, \quad -1+p_{0|1} \leqslant 0 \tag{B.18}$$

$$2) \quad \tilde{\lambda}_1 = -(1-p_{0|0})\frac{1}{4}\left[-2+q_{0|0}+q_{0|1}+r_{0|0}+r_{0|1}\right] \geqslant 0 \tag{B.19}$$

$$\tilde{\lambda}_2 = -(1-p_{0|1})\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.20}$$

$$\tilde{\lambda}_3 = p_{0|0}\frac{1}{4}\left[-2+q_{0|0}+q_{0|1}+r_{0|0}+r_{0|1}\right] \geqslant 0 \tag{B.21}$$

$$\tilde{\lambda}_4 = p_{0|1}\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.22}$$

Analogous equations apply for Bob and Carl by performing an appropriate permutation of players. The solutions that fulfill these equations (B.18)-(B.22) for all players are found in Table 4.4 on page 51.

- $\boldsymbol{I} = \boldsymbol{f_{15}}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$, Alice's payoff in equation (4.36) on page 45 is:

$$\$_A = \frac{1}{4}\left[2+(p_{0|0}-p_{0|1})(q_{0|0}+r_{0|0}-q_{0|1}-r_{0|1})\right] \tag{B.23}$$

The KKT conditions are:

$$1) \quad -p_{0|0} \leqslant 0, \quad -p_{0|1} \leqslant 0, \quad -1+p_{0|0} \leqslant 0, \quad -1+p_{0|1} \leqslant 0 \tag{B.24}$$

$$2) \quad \tilde{\lambda}_1 = -(1-p_{0|0})\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.25}$$

$$\tilde{\lambda}_2 = (1-p_{0|1})\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.26}$$

$$\tilde{\lambda}_3 = p_{0|0}\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.27}$$

$$\tilde{\lambda}_4 = -p_{0|1}\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.28}$$

The results are found in Table 4.5 on page 51.

- $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_8}$, Alice's payoff in equation (4.39) on page 45 is:

$$\$_A = \frac{1}{8}\left[6-2(2p_{0|0}+q_{0|0}+r_{0|0})+(p_{0|0}+p_{0|1})(q_{0|0}+r_{0|0}) \right.$$
$$\left. +(p_{0|0}-p_{0|1})(q_{0|1}+r_{0|1})\right] \tag{B.29}$$

The KKT conditions are:

$$1)\quad -p_{0|0} \leqslant 0,\quad -p_{0|1} \leqslant 0,\quad -1+p_{0|0} \leqslant 0,\quad -1+p_{0|1} \leqslant 0 \tag{B.30}$$

$$2)\quad \tilde{\lambda}_1 = -(1-p_{0|0})\frac{1}{4}\left[-4+q_{0|0}+q_{0|1}+r_{0|0}+r_{0|1}\right] \geqslant 0 \tag{B.31}$$

$$\tilde{\lambda}_2 = -(1-p_{0|1})\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.32}$$

$$\tilde{\lambda}_3 = p_{0|0}\frac{1}{4}\left[-4+q_{0|0}+q_{0|1}+r_{0|0}+r_{0|1}\right] \geqslant 0 \tag{B.33}$$

$$\tilde{\lambda}_4 = p_{0|1}\frac{1}{4}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.34}$$

The results are found in Table 4.6 on page 52.

- $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{10}}$, Alice's payoff in equation (4.41) on page 46 is:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[2+2(p_{0|0}+q_{0|0})-(p_{0|0}+p_{0|1})(q_{0|0}+r_{0|0})\\ &-(p_{0|0}-p_{0|1})(q_{0|1}+r_{0|1})\big]\end{aligned} \tag{B.35}$$

The KKT conditions are:

$$1)\quad -p_{0|0} \leqslant 0,\quad -p_{0|1} \leqslant 0,\quad -1+p_{0|0} \leqslant 0,\quad -1+p_{0|1} \leqslant 0 \tag{B.36}$$

$$2)\quad \tilde{\lambda}_1 = (1-p_{0|0})\frac{1}{8}\left[-2+q_{0|0}+q_{0|1}+r_{0|0}+r_{0|1}\right] \geqslant 0 \tag{B.37}$$

$$\tilde{\lambda}_2 = (1-p_{0|1})\frac{1}{8}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.38}$$

$$\tilde{\lambda}_3 = -p_{0|0}\frac{1}{8}\left[-2+q_{0|0}+q_{0|1}+r_{0|0}+r_{0|1}\right] \geqslant 0 \tag{B.39}$$

$$\tilde{\lambda}_4 = -p_{0|1}\frac{1}{8}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.40}$$

The results are found in Table 4.7 on page 52.

- $\boldsymbol{I} = \boldsymbol{f_9}$, $\boldsymbol{O} = \boldsymbol{f_7}$, Alice's payoff in equation (4.42) on page 46 is:

$$\begin{aligned}\$_A =& \frac{1}{8}\big[2+2(p_{0|0}+p_{0|1}+q_{0|1}+r_{0|0})-(p_{0|0}+p_{0|1})(q_{0|1}+r_{0|0})\\ &-(p_{0|0}-p_{0|1})(q_{0|0}-r_{0|1})\big]\end{aligned} \tag{B.41}$$

The KKT conditions are:

$$1)\quad -p_{0|0} \leqslant 0,\quad -p_{0|1} \leqslant 0,\quad -1+p_{0|0} \leqslant 0,\quad -1+p_{0|1} \leqslant 0 \tag{B.42}$$

$$2) \quad \tilde{\lambda}_1 = (1-p_{0|0})\frac{1}{8}\left[-2+q_{0|0}+q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.43}$$

$$\tilde{\lambda}_2 = -(1-p_{0|1})\frac{1}{8}\left[2+q_{0|0}-q_{0|1}-r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.44}$$

$$\tilde{\lambda}_3 = -p_{0|0}\frac{1}{8}\left[-2+q_{0|0}+q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.45}$$

$$\tilde{\lambda}_4 = p_{0|1}\frac{1}{8}\left[2+q_{0|0}-q_{0|1}-r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.46}$$

The results are found in Table 4.8 on page 52.

- $\boldsymbol{I} = \boldsymbol{f_9}$, $\boldsymbol{O} = \boldsymbol{f_{10}}$, Alice's payoff in equation (4.45) on page 46 is:

$$\begin{aligned}\$_A =& \frac{1}{8}\left[2+2(p_{0|1}+q_{0|1})-(p_{0|0}+p_{0|1})(q_{0|1}+r_{0|0})\right. \\ & \left. -(p_{0|0}-p_{0|1})(q_{0|0}-r_{0|1})\right]\end{aligned} \tag{B.47}$$

The KKT conditions are:

$$1) \quad -p_{0|0} \leqslant 0, \quad -p_{0|1} \leqslant 0, \quad -1+p_{0|0} \leqslant 0, \quad -1+p_{0|1} \leqslant 0 \tag{B.48}$$

$$2) \quad \tilde{\lambda}_1 = (1-p_{0|0})\frac{1}{8}\left[q_{0|0}+q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.49}$$

$$\tilde{\lambda}_2 = -(1-p_{0|1})\frac{1}{8}\left[2+q_{0|0}-q_{0|1}-r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.50}$$

$$\tilde{\lambda}_3 = -p_{0|0}\frac{1}{8}\left[q_{0|0}+q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.51}$$

$$\tilde{\lambda}_4 = p_{0|1}\frac{1}{8}\left[2+q_{0|0}-q_{0|1}-r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.52}$$

The results are found in Table 4.9 on page 53.

- $\boldsymbol{I} = \boldsymbol{f_{15}}$, $\boldsymbol{O} = \boldsymbol{f_7}$, Alice's payoff in equation (4.46) on page 47 is:

$$\$_A = \frac{1}{8}\left[4-(p_{0|0}-p_{0|1})(q_{0|0}+r_{0|0}-q_{0|1}-r_{0|1})\right] \tag{B.53}$$

The KKT conditions are:

$$1) \quad -p_{0|0} \leqslant 0, \quad -p_{0|1} \leqslant 0, \quad -1+p_{0|0} \leqslant 0, \quad -1+p_{0|1} \leqslant 0 \tag{B.54}$$

$$2) \quad \tilde{\lambda}_1 = (1-p_{0|0})\frac{1}{8}\left[q_{0|0}-q_{0|1}+r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.55}$$

$$\tilde{\lambda}_2 = -(1-p_{0|1})\frac{1}{8}\left[q_{0|0}-q_{0|1}-r_{0|0}-r_{0|1}\right] \geqslant 0 \tag{B.56}$$

$$\tilde{\lambda}_3 = -p_{0|0}\frac{1}{8}\left[q_{0|0} - q_{0|1} + r_{0|0} - r_{0|1}\right] \geqslant 0 \tag{B.57}$$

$$\tilde{\lambda}_4 = p_{0|1}\frac{1}{8}\left[q_{0|0} - q_{0|1} + r_{0|0} - r_{0|1}\right] \geqslant 0 \tag{B.58}$$

The results are found in Table 4.10 on page 53

### B.2.2 Nash Equilibria for boolean games in a triangle using quantum states

To optimise the payoffs to find the Nash equilibrium points for the **GHZ**- and **Bell**-like states, it is convenient to perform a **change of variables** to **linearise** the problem because the payoffs depend only on the square of the players' strategies, i.e. on $a_{11}^2$, $\tilde{a}_{11}^2$, $b_{11}^2$, $\tilde{b}_{11}^2$, $c_{11}^2$, $\tilde{c}_{11}^2$. The new variables are defined as:

$$p \equiv a_{11}^2 \quad ; \quad \tilde{p} \equiv \tilde{a}_{11}^2 \tag{B.59}$$
$$q \equiv b_{11}^2 \quad ; \quad \tilde{q} \equiv \tilde{b}_{11}^2 \tag{B.60}$$
$$r \equiv c_{11}^2 \quad ; \quad \tilde{r} \equiv \tilde{c}_{11}^2 \tag{B.61}$$

The **payoffs** can then be **rewritten** in terms of $\{p, \tilde{p}, q, \tilde{q}, r, \tilde{r}\}$ and the corresponding KKT conditions to find the equilibrium points. Since the KKT conditions will depend on $\lambda_{111}$ for the GHZ-like state and on $\lambda_{11}$ for the Bell-like state, and to avoid any confusion with the notation, the **Lagrange multipliers** for this quantum case will be denoted as $\tilde{\mu}_{\mathbf{i}}$, instead of using $\tilde{\lambda}_i$.

➤ $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$ – CHSH game –, Alice's payoff when the players share a GHZ-like state is found in equation (5.10) on page 60. Here, the same **payoff** is written in terms of the **newly defined variables**:

$$\begin{aligned}(\text{GHZ})\ \$_A =& \frac{1}{4}\big[3 + \big(-(2p+q+r) + (p+\tilde{p})(q+r) \\ &+ (p-\tilde{p})(\tilde{q}+\tilde{r})\big)\left(1 - 2\lambda_{111}^2(1-\lambda_{111}^2)\right)\big]\end{aligned} \tag{B.62}$$

The KKT conditions for Alice's payoffs with the new variables are:

$$1)\quad -p \leqslant 0,\ \ -\tilde{p} \leqslant 0,\ \ -1+p \leqslant 0,\ \ -1+\tilde{p} \leqslant 0 \tag{B.63}$$

$$2)\quad \tilde{\mu}_1 = \frac{-1}{4}(1-p)\left[-2+q+\tilde{q}+r+\tilde{r}\right]\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right] \geqslant 0 \tag{B.64}$$

$$\tilde{\mu}_2 = \frac{-1}{4}(1-\tilde{p})\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right] \geqslant 0 \tag{B.65}$$

$$\tilde{\mu}_3 = \frac{1}{4}p\left[-2+q+\tilde{q}+r+\tilde{r}\right]\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right] \geqslant 0 \tag{B.66}$$

$$\tilde{\mu}_4 = \frac{1}{4}\tilde{p}\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right] \geqslant 0 \tag{B.67}$$

These KKT conditions are the same as the corresponding conditions in the classical case with mixed strategies in equations (B.18)-(B.22) with the extra multiplying factor $1-2\lambda_{111}^2(1-\lambda_{111}^2)$, which is clearly always positive. That is why, in this case, the Nash equilibrium points using the GHZ-like state are the same as in the classical case; the only change now is that the specific payoffs might depend on $\lambda_{111}$. The results are found in Table 5.1 on page 62.

➤ $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$, Alice's payoff when the players share a Bell-like state is:

$$\begin{aligned}
\text{(Bell) } \$_A =& \frac{1}{4}\left[3 + (-(2p+q+r) + (p+\tilde{p})(q+r)\right. \\
& +(p-\tilde{p})(\tilde{q}+\tilde{r}))\left(1-3\lambda_{11}^2(1-\lambda_{11}^2)\right) - 4\lambda_{11}^2\left(1-\lambda_{11}^2\right)^2 \\
& \left. + (2p+q+r)\,\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right]
\end{aligned} \tag{B.68}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1) \qquad -p \leqslant 0, \quad -\tilde{p} \leqslant 0, \quad -1+p \leqslant 0, \quad -1+\tilde{p} \leqslant 0 \tag{B.69}$$

$$\begin{aligned}
2) \quad \tilde{\mu}_1 =& \frac{-1}{4}(1-p)\left[(-2+q+\tilde{q}+r+\tilde{r}) \quad \left(1-3\lambda_{11}^2(1-\lambda_{11}^2)\right)\right. \\
& \left. +2\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right] \geqslant 0
\end{aligned} \tag{B.70}$$

$$\tilde{\mu}_2 = \frac{-1}{4}(1-\tilde{p})\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-3\lambda_{11}^2(1-\lambda_{11}^2)\right] \geqslant 0 \tag{B.71}$$

$$\begin{aligned}
\tilde{\mu}_3 =& \frac{1}{4}p\left[(-2+q+\tilde{q}+r+\tilde{r})\left(1-3\lambda_{11}^2(1-\lambda_{11}^2)\right)\right. \\
& \left. +2\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\right] \geqslant 0
\end{aligned} \tag{B.72}$$

$$\tilde{\mu}_4 = \frac{1}{4}\tilde{p}\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-3\lambda_{11}^2(1-\lambda_{11}^2)\right] \geqslant 0 \tag{B.73}$$

The results are found in Table 5.2 on page 68. **Two** particular **solutions** will be **checked** to **illustrate** some of the **interval restrictions** that appear in some of the solutions. To do that, the KKT conditions for Bob and Carl are needed, and they are easily obtained from permuting the players in the KKT conditions for Alice.

As a **first example**, considering the strategy $s = \{v_{B1}, 0, v_{B1}, 0, 0, 1\}$ in Table 5.2, with $v_{B1}(\lambda_{11})$ defined in equation (5.15), the corresponding KKT conditions for Alice, Bob, and Carl are:

$$\tilde{\mu}_1 = 0 \ \geqslant 0 \tag{B.74}$$

$$\tilde{\mu}_2 = \frac{1}{2}\left(\lambda_{11}^2 - 3\lambda_{11}^4 + 2\lambda_{11}^6\right) \ \geqslant 0 \tag{B.75}$$

$$\tilde{\mu}_3 = 0 \ \geqslant 0 \tag{B.76}$$

$$\tilde{\mu}_4 = 0 \ \geqslant 0 \tag{B.77}$$

$$\tilde{\mu}_5 = 0 \geqslant 0 \tag{B.78}$$

$$\tilde{\mu}_6 = \frac{1}{2}\left(\lambda_{11}^2 - 3\lambda_{11}^4 + 2\lambda_{11}^6\right) \geqslant 0 \tag{B.79}$$

$$\tilde{\mu}_7 = 0 \geqslant 0 \tag{B.80}$$

$$\tilde{\mu}_8 = 0 \geqslant 0 \tag{B.81}$$

$$\tilde{\mu}_9 = \frac{1}{2}\left(\lambda_{11}^2 - 3\lambda_{11}^4 + 2\lambda_{11}^6\right) \geqslant 0 \tag{B.82}$$

$$\tilde{\mu}_{10} = 0 \geqslant 0 \tag{B.83}$$

$$\tilde{\mu}_{11} = 0 \geqslant 0 \tag{B.84}$$

$$\tilde{\mu}_{12} = \frac{1}{2}\left(1 - 5\lambda_{11}^2 + 9\lambda_{11}^4 - 4\lambda_{11}^6\right) \geqslant 0 \tag{B.85}$$

where Alice's Lagrange multipliers are $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4$; Bob's are $\tilde{\mu}_5, \tilde{\mu}_6, \tilde{\mu}_7, \tilde{\mu}_8$; and Carl's $\tilde{\mu}_9, \tilde{\mu}_{10}, \tilde{\mu}_{11}, \tilde{\mu}_{12}$. The strategy must also be between 0 and 1, which, in this case, it requires that $0 \leqslant v_{B1}(\lambda_{11}) \leqslant 1$ – see Figure 5.7 on page 69. All of these equations and restrictions are only satisfied for $0 \leqslant \lambda_{11} \leqslant 1/\sqrt{2}$ or $\lambda_{11} = 1$, so that is why this solution is restricted to that particular interval in Table 5.2. Such is the case with many other solutions in other games.

As a **second example**, considering the solution $s = \{1, 0, 0, 1, 1 - \tilde{r}, \tilde{r}\}$ in Table 5.2, using the optimisation variables $\tilde{r} \equiv \tilde{c}_{11}^2$, the corresponding KKT conditions for Alice, Bob, and Carl are:

$$\tilde{\mu}_1 = \tilde{\mu}_4 = \tilde{\mu}_6 = \tilde{\mu}_7 = \tilde{\mu}_{10} = \tilde{\mu}_{12} = 0 \geqslant 0 \tag{B.86}$$

$$\tilde{\mu}_2 = \frac{1}{2}\tilde{r}\left(1 - 3\lambda_{11}^2 + 3\lambda_{11}^4\right) \geqslant 0 \tag{B.87}$$

$$\tilde{\mu}_3 = \frac{1}{2}\left(\lambda_{11}^2 - 3\lambda_{11}^4 + 2\lambda_{11}^6\right) \geqslant 0 \tag{B.88}$$

$$\tilde{\mu}_5 = \frac{-1}{2}\lambda_{11}^2\left(1 - 3\lambda_{11}^2 + 2\lambda_{11}^4\right) \geqslant 0 \tag{B.89}$$

$$\tilde{\mu}_8 = \frac{1}{2}(1 - \tilde{r})\left(1 - 3\lambda_{11}^2 + 3\lambda_{11}^4\right) \geqslant 0 \tag{B.90}$$

$$\tilde{\mu}_9 = \frac{-1}{2}\tilde{r}\left(\lambda_{11}^2 - 3\lambda_{11}^4 + 2\lambda_{11}^6\right) \geqslant 0 \tag{B.91}$$

$$\tilde{\mu}_{11} = \frac{1}{2}(1 - \tilde{r})\left(\lambda_{11}^2 - 3\lambda_{11}^4 + 2\lambda_{11}^6\right) \geqslant 0 \tag{B.92}$$

The only possible solution to all of these equations is when $\lambda_{11} = 0$ or $\lambda_{11} = 1$ or $\lambda_{11} = 1/\sqrt{2}$, for $0 \leqslant \tilde{r} \leqslant 1$. That is why in Table 5.2 that particular solution is restricted to these values of $\lambda_{11}$.

➤ $\boldsymbol{I} = \boldsymbol{f_{15}}$, $\boldsymbol{O} = \boldsymbol{f_{15}}$, Alice's payoff when the players share a GHZ-like and a Bell-like state:

$$\text{(GHZ) } \$_A = \frac{1}{4}\left[2 + (p-\tilde{p})(q-\tilde{q}+r-\tilde{r})\left(1-2\lambda_{111}^2(1-\lambda_{111}^2)\right)\right] \quad \text{(B.93)}$$

$$\text{(Bell) } \$_A = \frac{1}{4}\left[2 + (p-\tilde{p})(q-\tilde{q}+r-\tilde{r})\left(1-3\lambda_{11}^2(1-\lambda_{11}^2)\right)\right] \quad \text{(B.94)}$$

As it happened with the GHZ-like state in the CHSH game, the entanglement of the states, i.e. $\lambda_{111}$ and $\lambda_{11}$, appear on these payoffs as a multiplying factor on the strategic terms, thus leaving the Nash equilibrium points unchanged. The results are found in Tables 5.3 on page 77.

➤ $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_8}$, Alice's payoff when the players share a GHZ-like state is:

$$\begin{aligned}\text{(GHZ) } \$_A = &\frac{1}{8}\left[2 + 4\lambda_{111}^2(2-\lambda_{111}^2) - 2\lambda_{111}^4\left(2p+q+r\right)\right.\\ &+\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left(\left(p+\tilde{p}\right)\left(q+r\right)\right.\\ &\left.\left.+\ \left(p-\tilde{p}\right)\left(\tilde{q}+\tilde{r}\right)\right)\right]\end{aligned} \quad \text{(B.95)}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1) \quad -p \leqslant 0, \;\; -\tilde{p} \leqslant 0, \;\; -1+p \leqslant 0, \;\; -1+\tilde{p} \leqslant 0 \quad \text{(B.96)}$$

$$2) \quad \tilde{\mu}_1 = \frac{-1}{8}(1-p)\left[-4\lambda_{111}^4 + (q+\tilde{q}+r+\tilde{r})\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right] \geqslant 0 \quad \text{(B.97)}$$

$$\tilde{\mu}_2 = \frac{-1}{8}(1-\tilde{p})\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right] \geqslant 0 \quad \text{(B.98)}$$

$$\tilde{\mu}_3 = \frac{1}{8}p\left[-4\lambda_{111}^4 + (q+\tilde{q}+r+\tilde{r})\left(1-2\lambda_{111}^2\left((1-\lambda_{111}^2\right)\right)\right] \geqslant 0 \quad \text{(B.99)}$$

$$\tilde{\mu}_4 = \frac{1}{8}\tilde{p}\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right] \geqslant 0 \quad \text{(B.100)}$$

The results are found in Table 5.4 on page 81.

➤ $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_8}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}\text{(Bell) } \$_A = &\frac{1}{8}\left[2 + 4\lambda_{11}^2(1+\lambda_{11}^2(1-\lambda_{11}^2)) + 2\lambda_{11}^2(1-3\lambda_{11}^2+\lambda_{11}^4)\left(2p+q+r\right)\right.\\ &+\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\left(\left(p+\tilde{p}\right)\left(q+r\right)\right.\\ &\left.\left.+\left(p-\tilde{p}\right)\left(\tilde{q}+\tilde{r}\right)\right)\right]\end{aligned} \quad \text{(B.101)}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1) \qquad -p \leqslant 0, \quad -\tilde{p} \leqslant 0, \quad -1+p \leqslant 0, \quad -1+\tilde{p} \leqslant 0 \tag{B.102}$$

$$\begin{aligned} 2) \quad \tilde{\mu}_1 =& \frac{-1}{8}(1-p)\left[(q+\tilde{q}+r+\tilde{r})\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right. \\ & \left.+4\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right] \geqslant 0 \end{aligned} \tag{B.103}$$

$$\tilde{\mu}_2 = \frac{-1}{8}(1-\tilde{p})\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right] \geqslant 0 \tag{B.104}$$

$$\begin{aligned} \tilde{\mu}_3 =& \frac{1}{8}p\left[(q+\tilde{q}+r+\tilde{r})\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right. \\ & \left.+4\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right] \geqslant 0 \end{aligned} \tag{B.105}$$

$$\tilde{\mu}_4 = \frac{1}{8}\tilde{p}\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right] \geqslant 0 \tag{B.106}$$

The results are found in Table 5.5 on page 88. As an illustrative **example**, considering the strategy $s = \{t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}\}$, with $t_{B1}(\lambda_{11})$ defined in equation (5.13), the corresponding KKT conditions for Alice, Bob, and Carl vanish:

$$\tilde{\mu}_1 = \tilde{\mu}_2 = \tilde{\mu}_3 = \tilde{\mu}_4 = 0 \ \geqslant 0 \tag{B.107}$$

$$\tilde{\mu}_5 = \tilde{\mu}_6 = \tilde{\mu}_7 = \tilde{\mu}_8 = 0 \ \geqslant 0 \tag{B.108}$$

$$\tilde{\mu}_9 = \tilde{\mu}_{10} = \tilde{\mu}_{11} = \tilde{\mu}_{12} = 0 \ \geqslant 0 \tag{B.109}$$

but there is the requirement that $0 \leqslant t_{B2}(\lambda_{11}) \leqslant 1$, which only happens for $(\sqrt{5}-1)/2 \leqslant \lambda_{11} \leqslant \sqrt{(\sqrt{5}-1)/2}$. That is why in in Table 5.5 that solution is restricted to that particular interval.

➤ $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{10}}$, Alice's payoff when the players share a GHZ-like state:

$$\begin{aligned} \text{(GHZ)}\ \$_A =& \frac{1}{8}\left[2+2(p+r)(1-\lambda_{111}^2)^2+2\lambda_{111}^4\left(p+q\right)\right. \\ & -\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left((p+\tilde{p})\left(q+r\right)\right. \\ & \left.\left.+\left(p-\tilde{p}\right)\left(\tilde{q}+\tilde{r}\right)\right)\right] \end{aligned} \tag{B.110}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1) \quad -p \leqslant 0, \quad -\tilde{p} \leqslant 0, \quad -1+p \leqslant 0, \quad -1+\tilde{p} \leqslant 0 \tag{B.111}$$

$$2) \quad \tilde{\mu}_1 = \frac{1}{8}(1-p)\left[-2+q+\tilde{q}+r+\tilde{r}\right]\left[1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right] \geqslant 0 \tag{B.112}$$

$$\tilde{\mu}_2 = \frac{1}{8}(1-\tilde{p})\left[q-\tilde{q}+r-\tilde{r}\right]\left[1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right] \geqslant 0 \tag{B.113}$$

$$\tilde{\mu}_3 = \frac{-1}{8} p \left[-2 + q + \tilde{q} + r + \tilde{r}\right] \left[1 - 2\lambda_{111}^2 \left(1 - \lambda_{111}^2\right)\right] \geqslant 0 \tag{B.114}$$

$$\tilde{\mu}_4 = \frac{-1}{8} \tilde{p} \left[q - \tilde{q} + r - \tilde{r}\right] \left[1 - 2\lambda_{111}^2 \left(1 - \lambda_{111}^2\right)\right] \geqslant 0 \tag{B.115}$$

These last four equations are the same KKT conditions as in the classical case in equations (B.36)-(B.40) with the multiplying factor $1 - 2\lambda_{111}^2 \left(1 - \lambda_{111}^2\right)$. Even though the payoff for the GHZ-like state in equation (B.110) is different from the classical case in equation (B.35), the partial derivatives only differ by that multiplying factor. This implies that the Nash Equilibrium points using the GHZ-like state will be the same as the classical ones using mixed strategies; the only difference is the presence of entanglement, i.e. the parameter $\lambda_{111}$, in the payoffs that those equilibrium points give. The results are found in Table 5.6 on page 93.

➤ $\boldsymbol{I} = \boldsymbol{f_7}$, $\boldsymbol{O} = \boldsymbol{f_{10}}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}
\text{(Bell) } \$_A =& \frac{1}{8} \left[2 + 4\lambda_{11}^2 (1 - \lambda_{11}^2)^2 + 2(p + r)(1 - \lambda_{11}^2)^3\right. \\
& - 2\lambda_{11}^2 (1 - 3\lambda_{11}^2 + \lambda_{11}^4) (p + q) \\
& - \left(1 - 3\lambda_{11}^2 \left(1 - \lambda_{11}^2\right)\right) ((p + \tilde{p}) (q + r) \\
& \left. + \ (p - \tilde{p}) (\tilde{q} + \tilde{r}))\right]
\end{aligned} \tag{B.116}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1) \qquad -p \leqslant 0, \quad -\tilde{p} \leqslant 0, \quad -1 + p \leqslant 0, \quad -1 + \tilde{p} \leqslant 0 \tag{B.117}$$

$$\begin{aligned}
2) \quad \tilde{\mu}_1 =& \frac{1}{8}(1 - p) \left[(q + \tilde{q} + r + \tilde{r}) \left(1 - 3\lambda_{11}^2 \left(1 - \lambda_{11}^2\right)\right)\right. \\
& \left. +2 \left(-1 + 4\lambda_{11}^2 - 6\lambda_{11}^4 + 2\lambda_{11}^6\right)\right] \geqslant 0
\end{aligned} \tag{B.118}$$

$$\tilde{\mu}_2 = \frac{1}{8}(1 - \tilde{p}) \left[q - \tilde{q} + r - \tilde{r}\right] \left[1 - 3\lambda_{11}^2 \left(1 - \lambda_{11}^2\right)\right] \geqslant 0 \tag{B.119}$$

$$\begin{aligned}
\tilde{\mu}_3 =& \frac{-1}{8} p \left[(q + \tilde{q} + r + \tilde{r}) \left(1 - 3\lambda_{11}^2 \left(1 - \lambda_{11}^2\right)\right)\right. \\
& \left. +2 \left(-1 + 4\lambda_{11}^2 - 6\lambda_{11}^4 + 2\lambda_{11}^6\right)\right] \geqslant 0
\end{aligned} \tag{B.120}$$

$$\tilde{\mu}_4 = \frac{-1}{8} \tilde{p} \left[q - \tilde{q} + r - \tilde{r}\right] \left[1 - 3\lambda_{11}^2 \left(1 - \lambda_{11}^2\right)\right] \geqslant 0 \tag{B.121}$$

The results are found in Table 5.7 on page 100.

➤ $\boldsymbol{I} = \boldsymbol{f_9}$, $\boldsymbol{O} = \boldsymbol{f_7}$, Alice's payoff when the players share a GHZ-like state:

$$\text{(GHZ) } \$_A = \frac{1}{8} \left[6 - 4\lambda_{111}^2 (2 - \lambda_{111}^2) + 2\lambda_{111}^4 (p + \tilde{p} + \tilde{q} + r)\right.$$

$$-\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left((p+\tilde{p})\left(\tilde{q}+r\right)\right.$$
$$\left.+\left(p-\tilde{p}\right)\left(q-\tilde{r}\right)\right)\Big] \quad \text{(B.122)}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1)\quad -p\leqslant 0,\quad -\tilde{p}\leqslant 0,\quad -1+p\leqslant 0,\quad -1+\tilde{p}\leqslant 0 \quad \text{(B.123)}$$

$$2)\quad \tilde{\mu}_1=\frac{-1}{8}(1-p)\left[2\lambda_{111}^4-(q+\tilde{q}+r-\tilde{r})\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \quad \text{(B.124)}$$

$$\tilde{\mu}_2=\frac{-1}{8}(1-\tilde{p})\left[2\lambda_{111}^4-(-q+\tilde{q}+r+\tilde{r})\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \quad \text{(B.125)}$$

$$\tilde{\mu}_3=\frac{1}{8}p\left[2\lambda_{111}^4-(q+\tilde{q}+r-\tilde{r})\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \quad \text{(B.126)}$$

$$\tilde{\mu}_4=\frac{1}{8}\tilde{p}\left[2\lambda_{111}^4-(-q+\tilde{q}+r+\tilde{r})\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \quad \text{(B.127)}$$

The results are found in Table 5.9 on page 110.

➤ $\boldsymbol{I}=\boldsymbol{f_9}$, $\boldsymbol{O}=\boldsymbol{f_7}$, Alice's payoff when the players share a Bell-like state:

$$\text{(Bell) }\$_A=\frac{1}{8}\Big[6-4\lambda_{11}^2(1+\lambda_{11}^2(1-\lambda_{11}^2))$$
$$-2\lambda_{11}^2(1-3\lambda_{11}^2+\lambda_{11}^4)\left(p+\tilde{p}+\tilde{q}+r\right)$$
$$-\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\left((p+\tilde{p})\left(\tilde{q}+r\right)\right.$$
$$\left.+\left(p-\tilde{p}\right)\left(q-\tilde{r}\right)\right)\Big] \quad \text{(B.128)}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1)\qquad -p\leqslant 0,\quad -\tilde{p}\leqslant 0,\quad -1+p\leqslant 0,\quad -1+\tilde{p}\leqslant 0 \quad \text{(B.129)}$$

$$2)\quad \tilde{\mu}_1=\frac{1}{8}(1-p)\left[(q+\tilde{q}+r-\tilde{r})\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right.$$
$$\left.+2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right]\geqslant 0 \quad \text{(B.130)}$$

$$\tilde{\mu}_2=\frac{1}{8}(1-\tilde{p})\left[(-q+\tilde{q}+r+\tilde{r})\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right.$$
$$\left.+2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right]\geqslant 0 \quad \text{(B.131)}$$

$$\tilde{\mu}_3=\frac{-1}{8}p\left[(q+\tilde{q}+r-\tilde{r})\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right.$$
$$\left.+2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right]\geqslant 0 \quad \text{(B.132)}$$

$$\tilde{\mu}_4=\frac{-1}{8}\tilde{p}\left[(-q+\tilde{q}+r+\tilde{r})\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right.$$

$$+2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\big]\geqslant 0 \tag{B.133}$$

The results are found in Table 5.10 on page 115. As an illustrative **example**, considering the strategy $s=\{2t_{B2},0,0,2t_{B2},1,1\}$, with $t_{B2}(\lambda_{11})$ defined in equation (5.22), from Table 5.10, the corresponding KKT conditions for Alice, Bob, and Carl are:

$$\tilde{\mu}_1=\tilde{\mu}_3=\tilde{\mu}_4=\tilde{\mu}_6=\tilde{\mu}_7=\tilde{\mu}_8=\tilde{\mu}_9=\tilde{\mu}_{10}=0\ \geqslant 0 \tag{B.134}$$

$$\tilde{\mu}_2=\frac{1}{4}\left(1-3\lambda_{11}^2+3\lambda_{11}^4\right)\ \geqslant 0 \tag{B.135}$$

$$\tilde{\mu}_5=\frac{1}{4}\left(1-3\lambda_{11}^2+3\lambda_{11}^4\right)\ \geqslant 0 \tag{B.136}$$

$$\tilde{\mu}_{11}=\frac{-1}{4}\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\ \geqslant 0 \tag{B.137}$$

$$\tilde{\mu}_{12}=\frac{-1}{4}\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\ \geqslant 0 \tag{B.138}$$

whose solution is the interval $(\sqrt{5}-1)/2\leqslant\lambda_{11}\leqslant 1$; however, requiring that the strategic term $0\leqslant 2t_{B2}(\lambda_{11})\leqslant 1$, that only happens when $(\sqrt{5}-1)/2\leqslant\lambda_{11}\leqslant 1/\sqrt{2}$. That is why that solution in Table 5.10 is restricted to the latter interval.

➤ $\boldsymbol{I}=\boldsymbol{f_9}$, $\boldsymbol{O}=\boldsymbol{f_{10}}$, Alice's payoff when the players share a GHZ-like state:

$$\begin{aligned}(\text{GHZ})\ \$_A=&\frac{1}{8}\Big[2+2(p+r)\left(1-\lambda_{111}^2\right)^2+2\lambda_{111}^4\left(\tilde{p}+\tilde{q}\right)\\&-\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\left((p+\tilde{p})\left(\tilde{q}+r\right)\right.\\&\left.+\left(p-\tilde{p}\right)\left(q-\tilde{r}\right)\right)\Big]\end{aligned} \tag{B.139}$$

The corresponding KKT conditions for Alice's payoffs are:

$$1)\quad -p\leqslant 0,\quad -\tilde{p}\leqslant 0,\quad -1+p\leqslant 0,\quad -1+\tilde{p}\leqslant 0 \tag{B.140}$$

$$2)\quad \tilde{\mu}_1=\frac{-1}{8}(1-p)\left[2\left(1-\lambda_{111}^2\right)^2-\left(q+\tilde{q}+r-\tilde{r}\right)\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \tag{B.141}$$

$$\tilde{\mu}_2=\frac{-1}{8}(1-\tilde{p})\left[2\lambda_{111}^4-\left(-q+\tilde{q}+r+\tilde{r}\right)\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \tag{B.142}$$

$$\tilde{\mu}_3=\frac{1}{8}p\left[2\left(1-\lambda_{111}^2\right)^2-\left(q+\tilde{q}+r-\tilde{r}\right)\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \tag{B.143}$$

$$\tilde{\mu}_4=\frac{1}{8}\tilde{p}\left[2\lambda_{111}^4-\left(-q+\tilde{q}+r+\tilde{r}\right)\left(1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right)\right]\geqslant 0 \tag{B.144}$$

The results are found in Table 5.11 on page 120.

➤ $\boldsymbol{I} = \boldsymbol{f_9}$, $\boldsymbol{O} = \boldsymbol{f_{10}}$, Alice's payoff when the players share a Bell-like state:

$$\begin{aligned}\text{(Bell) } \$_A =& \frac{1}{8}\left[2 + 4\lambda_{11}^2(1-\lambda_{11}^2)^2 + 2(p+r)(1-\lambda_{11}^2)^3\right. \\ & - 2\lambda_{11}^2(1-3\lambda_{11}^2+\lambda_{11}^4)(\tilde{p}+\tilde{q}) \\ & - \left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\left((p+\tilde{p})\left(\tilde{q}+r\right)\right. \\ & \left.\left.+\left(p-\tilde{p}\right)\left(q-\tilde{r}\right)\right)\right]\end{aligned} \tag{B.145}$$

The corresponding KKT conditions for Alice's payoffs are:

1) $$-p \leqslant 0, \quad -\tilde{p} \leqslant 0, \quad -1+p \leqslant 0, \quad -1+\tilde{p} \leqslant 0 \tag{B.146}$$

2) $$\tilde{\mu}_1 = \frac{1}{8}(1-p)\left[2\left(1-\lambda_{11}^2\right)^3 + \left(q+\tilde{q}+r-\tilde{r}\right)\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right] \geqslant 0 \tag{B.147}$$

$$\begin{aligned}\tilde{\mu}_2 =& \frac{1}{8}(1-\tilde{p})\left[\left(-q+\tilde{q}+r+\tilde{r}\right)\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right. \\ & \left.+2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right] \geqslant 0\end{aligned} \tag{B.148}$$

$$\tilde{\mu}_3 = \frac{-1}{8}p\left[2\left(1-\lambda_{11}^2\right)^3 + \left(q+\tilde{q}+r-\tilde{r}\right)\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right] \geqslant 0 \tag{B.149}$$

$$\begin{aligned}\tilde{\mu}_4 =& \frac{-1}{8}\tilde{p}\left[\left(-q+\tilde{q}+r+\tilde{r}\right)\left(1-3\lambda_{11}^2\left(1-\lambda_{11}^2\right)\right)\right. \\ & \left.+2\lambda_{11}^2\left(1-3\lambda_{11}^2+\lambda_{11}^4\right)\right] \geqslant 0\end{aligned} \tag{B.150}$$

The results are found in Table 5.12 on page 128.

➤ $\boldsymbol{I} = \boldsymbol{f_{15}}$, $\boldsymbol{O} = \boldsymbol{f_7}$, Alice's payoff when the players share a GHZ-like and a Bell-like state:

$$\text{(GHZ) } \$_A = \frac{1}{8}\left[4 - (p-\tilde{p})(q-\tilde{q}+r-\tilde{r})\left(1-2\lambda_{111}^2(1-\lambda_{111}^2)\right)\right] \tag{B.151}$$

$$\text{(Bell) } \$_A = \frac{1}{8}\left[4 - (p-\tilde{p})(q-\tilde{q}+r-\tilde{r})\left(1-3\lambda_{11}^2(1-\lambda_{11}^2)\right)\right] \tag{B.152}$$

As before, the equilibrium points remain the same as in the classical case because the entanglement is only present as a multiplying factor on the strategic terms. The results are found in Table 5.13 on page 134.

### B.2.3 Social welfare for the Nash equilibrium points using quantum states

The next few tables contain the **social welfare** ($\$_A + \$_B + \$_C$) of the **Nash equilibrium points** for **all games** using the **GHZ**- and the **Bell**-like state. The values of the social welfare have been obtained by adding the payoffs of the players of all the equilibrium points, points which are found in their respective tables in chapter 5.

| GHZ-like state with $I = f_7$ , $O = f_{15}$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\{1, 1, 1, 1, 1, \tilde{c}_{11}^2\}$<br>$\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | $\frac{9}{4}$ |
| $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$ | $\frac{3}{2}\left[1 + \lambda_{111}^2 \left(1 - \lambda_{111}^2\right)\right]$ |
| $\{1, 0, 0, 1, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$<br>$\{0, 1, 1, 0, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$ | $\frac{7}{4} + \lambda_{111}^2 - \lambda_{111}^4$ |

Tab. B.1: *Social welfare for the Nash Equilibrium points in Table 5.1 for the game defined by* $I = f_7$, $O = f_{15}$ *(CHSH game) using the GHZ-like state. The colour of the circles help to identify the payoffs plotted in Figure 5.6.*

| Bell-like state with $I = f_7$ , $O = f_{15}$ | | |
|---|---|---|
| $\lambda_{11}$ interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $0 \leqslant \lambda_{11} \leqslant 1$ | $\{1, 1, 1, 1, 1, \tilde{c}_{11}^2\}$ | $\frac{9}{4} - 3\lambda_{11}^4 \left(1 - \lambda_{11}^2\right)$ |
| | $\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | $\frac{9}{4} - 3\lambda_{11}^2 \left(1 - \lambda_{11}^2\right)^2$ |

| | | |
|---|---|---|
| | | |
| $0 \leqslant \lambda_{11} \leqslant 1$ | $\left\{\frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}\right\}$ | $\frac{3\,(2-5\lambda_{11}^2+\lambda_{11}^4+12\lambda_{11}^6-16\lambda_{11}^8+12\lambda_{11}^{10}-4\lambda_{11}^{12})}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |
| $\lambda_{11}=0,1\ ;\ \lambda_{11}=\frac{1}{\sqrt{2}}$ | $\{1,0,0,1,1-\tilde{c}_{11}^2,\tilde{c}_{11}^2\}$ <br> $\{0,1,1,0,1-\tilde{c}_{11}^2,\tilde{c}_{11}^2\}$ | $\frac{7}{4}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}},\ \lambda_{11}=1$ | $\{v_{B1},0,v_{B1},0,0,1\}$ | $\frac{7-19\lambda_{11}^2+\lambda_{11}^4+76\lambda_{11}^6-134\lambda_{11}^8+108\lambda_{11}^{10}-32\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}},\ \lambda_{11}=1$ | $\{0,u_{B1},0,u_{B1},u_{B1},0\}$ | $\frac{7-23\lambda_{11}^2+31\lambda_{11}^4-16\lambda_{11}^6+4\lambda_{11}^8+12\lambda_{11}^{10}-8\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |
| $\lambda_{11}=0,\ \frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{2t_{B1},1,2t_{B1},1,1,0)\}$ | $\frac{7-23\lambda_{11}^2+25\lambda_{11}^4+20\lambda_{11}^6-74\lambda_{11}^8+84\lambda_{11}^{10}-32\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |
| $\lambda_{11}=0,\ \frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1,t_{B1},1,t_{B1},t_{B1},1\}$ | $\frac{7-19\lambda_{11}^2+7\lambda_{11}^4+40\lambda_{11}^6-56\lambda_{11}^8+36\lambda_{11}^{10}-8\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |

Tab. B.2: *Social welfare for the Nash Equilibrium points in Table 5.2 for the game defined by $I = f_7$, $O = f_{15}$ (CHSH game) using the Bell-like state. The colour of the squares help to identify the payoffs plotted in Figure 5.13 and Figure 5.14. The non-marked payoff, which is constant, is marked as a point in the same figure.*

| GHZ-like state with $I = f_{15}$ , $O = f_{15}$ | |
|---|---|
| $s* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\{1, 0, 1, 0, 1, 0\}$<br>$\{0, 1, 0, 1, 0, 1\}$ | $3 - 3\lambda_{111}^2 + 3\lambda_{111}^4$ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\frac{3}{2}$ |

| Bell-like state with $I = f_{15}$ , $O = f_{15}$ | |
|---|---|
| $s* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\{1, 0, 1, 0, 1, 0\}$<br>$\{0, 1, 0, 1, 0, 1\}$ | $\frac{3}{2}\left[2 - 3\lambda_{11}^2 + 3\lambda_{11}^4\right]$ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\frac{3}{2}$ |

Tab. B.3: *Social welfare for the Nash Equilibrium points in Table 5.3 for the game defined by* $I = f_{15}$, $O = f_{15}$ *using the GHZ- and Bell-like states. The colour of the circle and square helps to identify the payoffs plotted in Figure 6.2(b).*

| GHZ-like state with $I = f_7$ , $O = f_8$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\lambda_{111} = 0$ , $\tilde{\#}_{11}^2 = 0$<br>$0 < \lambda_{111} \leqslant \sqrt{\frac{\sqrt{3}-1}{2}} \approx 0.605$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 4t_G$<br>$\sqrt{\frac{\sqrt{3}-1}{2}} < \lambda_{111} \leqslant 1$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$ | $\{0, \tilde{a}_{11}^2, 0, 0, 0, 0\}$<br>$\{0, 0, 0, \tilde{b}_{11}^2, 0, 0\}$<br>$\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | $\frac{3}{4}\left[1 + 2\lambda_{111}^2\left(2 - \lambda_{111}^2\right)\right]$ |
| $0 \leqslant \lambda_{111} \leqslant \sqrt{\frac{3-\sqrt{3}}{2}} \approx 0.796$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$<br>$\sqrt{\frac{3-\sqrt{3}}{2}} < \lambda_{111} < 1$ , $-1 + 2u_G \leqslant \tilde{\#}_{11}^2 \leqslant 1$<br>$\lambda_{111} = 1$ , $\tilde{\#}_{11}^2 = 1$ | $\{1, \tilde{a}_{11}^2, 1, 1, 1, 1\}$<br>$\{1, 1, 1, \tilde{b}_{11}^2, 1, 1\}$<br>$\{1, 1, 1, 1, 1, \tilde{c}_{11}^2\}$ | $\frac{3}{4}\left[3 - 2\lambda_{111}^4\right]$ |
| $0 \leqslant \lambda_{111} \leqslant 1$ | $\{t_G, t_G, t_G, t_G, t_G, t_G\}$ | $\frac{3}{8}\left[1 + 6\lambda_{111}^2 - 6\lambda_{111}^4 + \frac{1}{1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)}\right]$ |
| $\lambda_{111} = \frac{1}{\sqrt{2}}$ | $\{1, 0, 0, 1, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$<br>$\{0, 1, 1, 0, 1 - \tilde{c}_{11}^2, \tilde{c}_{11}^2\}$ | $\frac{7}{4}$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{0, 2t_G, 0, 2t_G, 2t_G, 0\}$ | $\frac{1}{4}\left[2 + 10\lambda_{111}^2 - 8\lambda_{111}^4 + \frac{1}{1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{1, u_G, 1, u_G, u_G, 1\}$ | $\frac{1}{4}\left[4 + 6\lambda_{111}^2 - 8\lambda_{111}^4 + \frac{1}{1 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)}\right]$ |

| | | |
|---|---|---|
| | | |
| $\sqrt{\frac{\sqrt{3}-1}{2}} \approx 0.605 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{1+2u_G, 0, 1+2u_G, 0, 0, 1\}$ | $\frac{-1}{2} + \frac{3}{2}\lambda_{111}^2 - 2\lambda_{111}^4 + \frac{1}{1-2\lambda_{111}^2(1-\lambda_{111}^2)}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant \sqrt{\frac{3-\sqrt{3}}{2}} \approx 0.796$ | $\{2u_G, 1, 2u_G, 1, 1, 0\}$ | $-1 + \frac{5}{2}\lambda_{111}^2 - 2\lambda_{111}^4 + \frac{1}{1-2\lambda_{111}^2(1-\lambda_{111}^2)}$ |

Tab. B.4: *Social welfare for the Nash Equilibrium points in Table 5.4 for the game defined by $I = f_7$, $O = f_8$ using the GHZ-like state. The colour of the circles help to identify the payoffs plotted in Figure 5.20. The non-marked payoff, which is constant, is marked as a point in the same figure.*

| Bell-like state with $I = f_7$ , $O = f_8$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\lambda_{11} = 0$ or $\lambda_{11} = \frac{\sqrt{5}-1}{2} \approx 0.618$, $\tilde{\#}_{11}^2 = 0$ | $\{0, \tilde{a}_{11}^2, 0, 0, 0, 0\}$ | |
| $\frac{\sqrt{5}-1}{2} < \lambda_{11} \leqslant 0.6695$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 4t_{B2}$ | $\{0, 0, 0, \tilde{b}_{11}^2, 0, 0\}$ | $\frac{3}{4}\left[1 + 2\lambda_{11}^2\left(1 + \lambda_{11}^2 - \lambda_{11}^4\right)\right]$ |
| $0.6695 < \lambda_{11} \leqslant 1$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$ | $\{0, 0, 0, 0, 0, \tilde{c}_{11}^2\}$ | |
| $0 \leqslant \lambda_{111} \leqslant 0.743$ , $0 \leqslant \tilde{\#}_{11}^2 \leqslant 1$ | $\{1, \tilde{a}_{11}^2, 1, 1, 1, 1\}$ | |
| $0.743 < \lambda_{111} < \sqrt{\frac{\sqrt{5}-1}{2}}$ , $-2 + v_{B2} \leqslant \tilde{\#}_{11}^2 \leqslant 1$ | $\{1, 1, 1, \tilde{b}_{11}^2, 1, 1\}$ | $\frac{3}{4}\left[3 - 2\lambda_{11}^4(2 - \lambda_{11}^2)\right]$ |

| | | |
|---|---|---|
| $\lambda_{11} = 1$ or $\lambda_{11} = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.786$ , $\tilde{\#}_{11}^{2} = 1$ | $\{1, 1, 1, 1, 1, \tilde{c}_{11}^{2}\}$ | |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}, t_{B2}\}$ | $\frac{3}{4}\left[1 + 2\lambda_{11}^{2}\left(1 + \lambda_{11}^{2} - \lambda_{11}^{4}\right)\right] - \frac{3\lambda_{11}^{4}\left(1 - 3\lambda_{11}^{2} + \lambda_{11}^{4}\right)^{2}}{2\left[1 - 3\lambda_{11}^{2}(1 - \lambda_{11}^{2})\right]}$ |
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1, 0, 0, 1, 1 - \tilde{c}_{11}^{2}, \tilde{c}_{11}^{2}\}$ <br> $\{0, 1, 1, 0, 1 - \tilde{c}_{11}^{2}, \tilde{c}_{11}^{2}\}$ | $\frac{13}{8}$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{0, 2t_{B2}, 0, 2t_{B2}, 2t_{B2}, 0\}$ | $\frac{3 - 3\lambda_{11}^{2} - 7\lambda_{11}^{4} + 18\lambda_{11}^{6} - 8\lambda_{11}^{8} + 6\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4 - 12\lambda_{11}^{2} + 12\lambda_{11}^{4}}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{1, u_{B2}, 1, u_{B2}, u_{B2}, 1\}$ | $\frac{5 - 11\lambda_{11}^{2} - \lambda_{11}^{4} + 34\lambda_{11}^{6} - 38\lambda_{11}^{8} + 18\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4 - 12\lambda_{11}^{2} + 12\lambda_{11}^{4}}$ |
| $0.6695 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{v_{B2}, 0, v_{B2}, 0, 0, 1\}$ | $\frac{2 - 5\lambda_{11}^{2} + 14\lambda_{11}^{4} + 4\lambda_{11}^{6} - 53\lambda_{11}^{8} + 54\lambda_{11}^{10} - 16\lambda_{11}^{12}}{4 - 12\lambda_{11}^{2} + 12\lambda_{11}^{4}}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 0.743$ | $\{-1 + v_{B2}, 1, -1 + v_{B2}, 1, 1, 0\}$ | $\frac{\lambda_{11}^{2}\left(3 + 8\lambda_{11}^{2} - 12\lambda_{11}^{4} - 23\lambda_{11}^{6} + 42\lambda_{11}^{8} - 16\lambda_{11}^{10}\right)}{4 - 12\lambda_{11}^{2} + 12\lambda_{11}^{4}}$ |

Tab. B.5: *Social welfare for the Nash Equilibrium points in Table 5.5 for the game defined by* $I = f_7$, $O = f_8$ *using the Bell-like state. The colour of the squares help to identify the payoffs plotted in Figure 5.26. The non-marked payoff, which is constant, is marked as a point in the same figure.*

| GHZ-like state with $I = f_7$ , $O = f_{10}$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\{1, 1, 0, 0, c_{11}^2, c_{11}^2\}$<br>$\{0, 0, 1, 1, c_{11}^2, c_{11}^2\}$ | $\frac{5}{4} - \lambda_{111}^2 + \lambda_{111}^4$ |
| $\{1, 0, 1, 0, c_{11}^2, 0\}$<br>$\{0, 1, 0, 1, c_{11}^2, 1\}$ | $\frac{3}{2}\left[1 - \lambda_{111}^2 + \lambda_{111}^4\right]$ |
| $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$ | $\frac{3}{8}\left[3 - 2\lambda_{111}^2\left(1 - \lambda_{111}^2\right)\right]$ |

Tab. B.6: *Social welfare for the Nash Equilibrium points in Table 5.6 for the game defined by $I = f_7$, $O = f_{10}$ using the GHZ-like state. The colour of the circles identify the payoffs plotted in Figure 5.29.*

| (1/2) Bell-like state with $I = f_7$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{u_{B1}, 0, u_{B1}, 0, u_{B1}, 0\}$ | $\frac{3}{4}\left[2\left(1 - \lambda_{11}^2 + \lambda_{11}^4\right) - \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{t_{B1}, 1, t_{B1}, 1, t_{B1}, 1\}$ | $\frac{3}{4}\left[2\left(1 - 2\lambda_{11}^4 + 2\lambda_{11}^6\right) - \frac{\lambda_{11}^2\left(1-3\lambda_{11}^2+2\lambda_{11}^4\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{1-3\lambda_{11}^2(1-\lambda_{11}^2)}\right]$ |

| | | |
|---|---|---|
| | | |
| $0 \leqslant \lambda_{11} \leqslant 1$ | $\{\frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}, \frac{u_{B1}}{2}\}$ | $\frac{3}{4}\left[2\left(1-\lambda_{11}^2+\lambda_{11}^4\right)-\frac{\left(1-2\lambda_{11}^2+2\lambda_{11}^6\right)\left(1-4\lambda_{11}^2+6\lambda_{11}^4-2\lambda_{11}^6\right)}{2\left[1-3\lambda_{11}^2(1-\lambda_{11}^2)\right]}\right]$ |
| $\lambda_{11} = 0, 1, \frac{1}{\sqrt{2}}, \tilde{c}_{11}^2 = 1$<br>$0 < \lambda_{11} < \frac{1}{\sqrt{2}}, v_{B1} \leqslant \tilde{c}_{11}^2 \leqslant 1$ | $\{0, 0, 0, 0, 1, \tilde{c}_{11}^2\}$ | $\frac{1}{4}\left[5 - 2\lambda_{11}^2 + 2\lambda_{11}^6\right]$ |
| $\lambda_{11} = 0, 1, \frac{1}{\sqrt{2}}, \tilde{c}_{11}^2 = 0$<br>$0 < \lambda_{11} < \frac{1}{\sqrt{2}}, 0 \leqslant \tilde{c}_{11}^2 \leqslant 2u_{B1}$ | $\{1, 1, 1, 1, 0, \tilde{c}_{11}^2\}$ | $\frac{1}{4}\left[5 - 4\lambda_{11}^2 + 6\lambda_{11}^4 - 2\lambda_{11}^6\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{u_{B1}, u_{B1}, u_{B1}, u_{B1}, 0, 0\}$ | $\frac{5 - 19\lambda_{11}^2 + 35\lambda_{11}^4 - 44\lambda_{11}^6 + 50\lambda_{11}^8 - 30\lambda_{11}^{10} + 8\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{t_{B1}, t_{B1}, t_{B1}, t_{B1}, 1, 1\}$ | $\frac{5 - 17\lambda_{11}^2 + 23\lambda_{11}^4 - 16\lambda_{11}^6 + 20\lambda_{11}^8 - 18\lambda_{11}^{10} + 8\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{v_{B1}, 0, v_{B1}, 0, 1, 0\}$ | $\frac{6 - 27\lambda_{11}^2 + 64\lambda_{11}^4 - 102\lambda_{11}^6 + 115\lambda_{11}^8 - 66\lambda_{11}^{10} + 16\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |

| | | |
|---|---|---|
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{2t_{B1}, 1, 2t_{B1}, 1, 0, 1\}$ | $\dfrac{6 - 21\lambda_{11}^2 + 28\lambda_{11}^4 - 18\lambda_{11}^6 + 25\lambda_{11}^8 - 30\lambda_{11}^{10} + 16\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ ■ |

Tab. B.7: *Social welfare for the Nash Equilibrium points in Table 5.7 – except the ones marked with a star – for the game defined by* $I = f_7$, $O = f_{10}$ *using the Bell-like state. The colour of the squares help to identify the payoffs plotted in Figure 5.33. The non-marked payoffs are the same as the marked with a purple and magenta diamond in the next table, Table B.8.*

| (2/2) Bell-like state with $I = f_7$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\lambda_{11} = 0,\ 1$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1, 1, 0, 0, c_{11}^2, c_{11}^2\}$ ★ | $\frac{5}{4}$<br>$\frac{17}{16}$ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{1, 1, 0, 0, 0, 0\}$ ★ | $\dfrac{1}{4}\left[5 - 2\lambda_{11}^2 + 2\lambda_{11}^6\right]$ ◆ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{1, 1, 0, 0, 1, 1\}$ ★ | $\dfrac{1}{4}\left[5 - 4\lambda_{11}^2 + 6\lambda_{11}^4 - 2\lambda_{11}^6\right]$ ◆ |
| $\lambda_{11} = 0,\ 1$ | $\{0, 0, 1, 1, c_{11}^2, c_{11}^2\}$ ★ | $\frac{5}{4}$ |

| | | |
|---|---|---|
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | | $\frac{17}{16}$ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{0,0,1,1,0,0\}$ ★ | $\frac{1}{4}\left[5 - 2\lambda_{11}^2 + 2\lambda_{11}^6\right]$ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{0,0,1,1,1,1\}$ ★ | $\frac{1}{4}\left[5 - 4\lambda_{11}^2 + 6\lambda_{11}^4 - 2\lambda_{11}^6\right]$ |
| $\lambda_{11} = 0,\ 1$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1,0,1,0,c_{11}^2,0\}$ ★ | $\frac{3}{2}$<br>$\frac{9}{8}$ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{1,0,1,0,0,0\}$ ★ | $\frac{1}{4}\left[6 - 7\lambda_{11}^2 + 9\lambda_{11}^4 - 2\lambda_{11}^6\right]$ ♦ |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{1,0,1,0,1,0\}$ ★ | $\frac{3}{4}\left[2 - 3\lambda_{11}^2 + 5\lambda_{11}^4 - 2\lambda_{11}^6\right]$ ♦ |
| $\lambda_{11} = 0,\ 1$<br>$\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{0,1,0,1,c_{11}^2,1\}$ ★ | $\frac{3}{2}$<br>$\frac{9}{8}$ |
| $0 < \lambda_{11} < \frac{1}{\sqrt{2}}$ | $\{0,1,0,1,0,1\}$ ★ | $\frac{3}{4}\left(1 + \lambda_{11}^2\right)\left(2 - 3\lambda_{11}^2 + 2\lambda_{11}^4\right)$ ♦ |

| | | |
|---|---|---|
| | | |
| $\frac{1}{\sqrt{2}} < \lambda_{11} < 1$ | $\{0,1,0,1,1,1\}$ ★ | $\frac{1}{4}\left[6 - 5\lambda_{11}^2 + 3\lambda_{11}^4 + 2\lambda_{11}^6\right]$ ◆ |

Tab. B.8: *Social welfare for the Nash equilibrium points in Table 5.8 for the game defined by $I = f_7$, $O = f_{10}$ using the Bell-like state. The colour of the diamonds help to identify the payoffs plotted in Figure 5.33, shown there as fainter lines. The non-marked constant payoffs are marked as points in the same figure; while the non-marked that are not constant are the same as the marked with a purple and magenta diamond.*

| GHZ-like state with $I = f_9$ , $O = f_7$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\lambda_{111} = 0$ | $\{0,0,0,0,0,0\}$<br>$\{0,0,0,0,1,1\}$ | $\frac{9}{4}$ |
| $\lambda_{111} = \frac{1}{\sqrt{2}}$ | $\{1,1,1,0,0,0\}$<br>$\{1,1,0,0,0,1\}$ | $\frac{3}{2}$ |
| $\lambda_{111} = 1$ | $\{1,1,1,1,0,0\}$<br>$\{1,1,1,1,1,1\}$ | $\frac{9}{4}$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{0,1,1,0,0,0\}$ | $\frac{1}{2}\left[5 - 7\lambda_{111}^2 - 6\lambda_{111}^4\right]$ ● |

| | | |
|---|---|---|
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{1,0,0,1,1,1\}$ | $2-\frac{5}{2}\lambda_{111}^2+3\lambda_{111}^4$ |
| $0 \leqslant \lambda_{111} \leqslant 1$ | $\{t_G,t_G,t_G,t_G,t_G,t_G\}$ | $\frac{3}{8}\left[7-6\lambda_{111}^2\left(1-\lambda_{111}^2\right)\right]-\frac{3}{8\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right]}$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{2t_G,0,0,2t_G,1,1\}$ | $\frac{1}{2}\left[5-5\lambda_{111}^2+6\lambda_{111}^4\right]-\frac{1}{4\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right]}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{u_G,1,1,u_G,0,0\}$ | $\frac{1}{2}\left[6-7\lambda_{111}^2+6\lambda_{111}^4\right]-\frac{1}{4\left[1-2\lambda_{111}^2(1-\lambda_{111}^2)\right]}$ |

Tab. B.9: *Social welfare for the Nash Equilibrium points in Table 5.9 for the game defined by $I = f_9$, $O = f_7$ using the GHZ-like state. The colour of the circles helps to identify the payoffs plotted in Figure 5.35. The non-marked payoffs, which are constant, are marked as points in the same figure.*

| Bell-like state with $I = f_9$ , $O = f_7$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | payoffs |
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{1,1,1,0,0,0\}$<br>$\{1,1,0,0,0,1\}$ | $\frac{3}{2}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{\sqrt{5}-1}{2}$ | $\{0,0,0,0,0,0\}$ | $\frac{3}{4}\left[3-2\lambda_{11}^2\left(1+\lambda_{11}^2-\lambda_{11}^4\right)\right]$ |

| | | |
|---|---|---|
| $\sqrt{\frac{\sqrt{5}-1}{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1,1,1,1,1,1\}$ | $\frac{3}{4}\left[1+4\lambda_{11}^4-2\lambda_{11}^6\right]$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{t_{B2},t_{B2},t_{B2},t_{B2},t_{B2},t_{B2}\}$ | $\frac{3}{4}\left[1+4\lambda_{11}^4-2\lambda_{11}^6\right]+\frac{3\left(1-2\lambda_{11}^2+\lambda_{11}^6\right)^2}{2\left[1-3\lambda_{11}^2(1-\lambda_{11}^2)\right]}$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{0,1,1,0,0,0\}$ | $\frac{1}{4}\left[10-13\lambda_{11}^2+9\lambda_{11}^4+2\lambda_{11}^6\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1,0,0,1,1,1\}$ | $\frac{1}{4}\left[8-11\lambda_{11}^2+15\lambda_{11}^4-2\lambda_{11}^6\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{\sqrt{5}-1}{2}$ | $\{0,-2t_{B2},-2t_{B2},0,0,0\}$ | $\frac{9-33\lambda_{11}^2+35\lambda_{11}^4+30\lambda_{11}^6-80\lambda_{11}^8+42\lambda_{11}^{10}-4\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{2t_{B2},0,0,2t_{B2},1,1\}$ | $\frac{9-37\lambda_{11}^2+67\lambda_{11}^4-70\lambda_{11}^6+56\lambda_{11}^8-18\lambda_{11}^{10}+4\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{u_{B2},1,1,u_{B2},0,0\}$ | $\frac{11-45\lambda_{11}^2+73\lambda_{11}^4-54\lambda_{11}^6+26\lambda_{11}^8-6\lambda_{11}^{10}+4\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ |

| | | |
|---|---|---|
| $\sqrt{\frac{\sqrt{5}-1}{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1, v_{B3}, v_{B3}, 1, 1, 1\}$ | $\frac{-1+7\lambda_{11}^2+5\lambda_{11}^4-50\lambda_{11}^6+70\lambda_{11}^8-18\lambda_{11}^{10}-4\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ ■ |

Tab. B.10: *Social welfare for the Nash Equilibrium points in Table 5.10 for the game defined by* $I = f_9$, $O = f_7$ *using the Bell-like state. The colour of the squares help to identify the payoffs plotted in Figure 5.37. The non-marked payoff, which is constant, is marked as a point in the same figure.*

| GHZ-like state with $I = f_9$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\lambda_{111} = 0$ | $\{1,0,1,0,1,0\}$<br>$\{1,0,1,0,0,1\}$ | $\frac{3}{2}$ |
| $\lambda_{111} = 1$ | $\{0,1,0,1,0,1\}$<br>$\{0,1,0,1,1,0\}$ | |
| $\lambda_{111} = \frac{1}{\sqrt{2}}$ | $\{0,0,0,1,1,0\}$<br>$\{1,1,1,0,0,1\}$ | $\frac{9}{8}$ |
| $0 \leqslant \lambda_{111} \leqslant 1$ | $\{v_G, t_G, v_G, t_G, v_G, t_G\}$ | $\frac{3\left(1-3\lambda_{111}^2+4\lambda_{111}^4-2\lambda_{111}^6+\lambda_{111}^8\right)}{2-4\lambda_{111}^2+4\lambda_{111}^4}$ ● |

| | | |
|---|---|---|
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{0,0,1,1,1,0\}$ | $\frac{1}{4}\left[7 - 8\lambda_{111}^2 + 6\lambda_{111}^4\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{1,1,0,0,0,1\}$ | $\frac{1}{4}\left[5 - 4\lambda_{111}^2 + 6\lambda_{111}^4\right]$ |
| $0 \leqslant \lambda_{111} \leqslant \frac{1}{\sqrt{2}}$ | $\{1,2t_G,-u_G,0,0,1\}$ | $\frac{1}{4}\left[7 - 4\lambda_{111}^2 + 6\lambda_{111}^4 - \frac{1}{1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)}\right]$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{111} \leqslant 1$ | $\{0,u_G,2v_G,1,1,0\}$ | $\frac{1}{4}\left[9 - 8\lambda_{111}^2 + 6\lambda_{111}^4 - \frac{1}{1-2\lambda_{111}^2\left(1-\lambda_{111}^2\right)}\right]$ |

Tab. B.11: *Social welfare for the Nash Equilibrium points in Table 5.11 for the game defined by $I = f_9$, $O = f_{10}$ using the GHZ-like state. The colour of the circles help to identify the payoffs plotted in Figure 5.39. The non-marked payoffs, which are constant, are marked as points in the same figure.*

| Bell-like state with $I = f_9$ , $O = f_{10}$ | | |
|---|---|---|
| Interval | $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\lambda_{11} = 0$ | $\{1,0,1,0,1,0\}$<br>$\{1,0,1,0,0,1\}$ | $\frac{3}{2}$ |

| | | |
|---|---|---|
| $\lambda_{11} = 1$ | $\{0, 1, 0, 1, 0, 1\}$<br>$\{0, 1, 0, 1, 1, 0\}$ | |
| $\lambda_{11} = \frac{1}{\sqrt{2}}$ | $\{0, 0, 0, 1, 1, 0\}$<br>$\{1, 1, 1, 0, 0, 1\}$ | $\frac{9}{8}$ |
| $\lambda_{11} = \frac{\sqrt{5}-1}{2}$ | $\left\{\frac{1+\sqrt{5}}{4}, 0, \frac{1+\sqrt{5}}{4}, 0, \frac{1+\sqrt{5}}{4}, 0\right\}$ | $\frac{3}{16}\left[35 - 13\sqrt{5}\right] \approx 1.112$ |
| $\lambda_{11} = \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\left\{\frac{3-\sqrt{5}}{4}, 1, \frac{3-\sqrt{5}}{4}, 1, \frac{3-\sqrt{5}}{4}, 1\right\}$ | |
| $0 \leqslant \lambda_{11} \leqslant \frac{\sqrt{5}-1}{2}$ | $\{v_{B4}, 0, v_{B4}, 0, v_{B4}, 0\}$ | $\frac{3\left(2 - 7\lambda_{11}^2 + 8\lambda_{11}^4 - 3\lambda_{11}^8 + \lambda_{11}^{12}\right)}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ (blue) |
| $\frac{\sqrt{5}-1}{2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{\sqrt{5}-1}{2}}$ | $\{v_{B4}, t_{B2}, v_{B4}, t_{B2}, v_{B4}, t_{B2}\}$ | $\frac{3\left(2 - 7\lambda_{11}^2 + 9\lambda_{11}^4 - 6\lambda_{11}^6 + 8\lambda_{11}^8 - 6\lambda_{11}^{10} + 2\lambda_{11}^{12}\right)}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ (cyan) |
| $\sqrt{\frac{\sqrt{5}-1}{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{v_{B4}, 1, v_{B4}, 1, v_{B4}, 1\}$ | $\frac{3\left[1 - 3\lambda_{11}^2 + \lambda_{11}^4\left(5 - \lambda_{11}^2\left(2 - \lambda_{11}^2\right)^3\right)\right]}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ (red) |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{0, 0, 1, 1, 1, 0\}$ | $\frac{1}{4}\left[7 - 8\lambda_{11}^2 + 6\lambda_{11}^4\right]$ (green) |

| | | |
|---|---|---|
| | | |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant 1$ | $\{1, 1, 0, 0, 0, 1\}$ | $\frac{1}{4}\left[5 - 4\lambda_{11}^2 + 6\lambda_{11}^4\right]$ |
| $0 \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{3}}$ | $\{u_{B3}, 0, 1, 0, u_{B3}, 0\}$ | $\frac{6 - 21\lambda_{11}^2 + 24\lambda_{11}^4 - 9\lambda_{11}^8 + 4\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $\frac{1}{\sqrt{3}} \leqslant \lambda_{11} \leqslant \sqrt{3 - \sqrt{7}}$ | $\{w_{B1} + 3, 0, 0, u_{B3}, 1, 0\}$ | $\frac{(-1 + \lambda_{11}^2)(-6 + 17\lambda_{11}^2 - 19\lambda_{11}^4 + 5\lambda_{11}^6 - 8\lambda_{11}^8 + 4\lambda_{11}^{10})}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $\sqrt{3 - \sqrt{7}} \leqslant \lambda_{11} \leqslant \sqrt{\frac{3-\sqrt{3}}{3}}$ | $\{v_{B4}, 0, 0, 2 + w_{B1} + v_{B4},$<br>$1, w_{B1} - v_{B4} + 3\}$ | $\frac{6 - 23\lambda_{11}^2 + 40\lambda_{11}^4 - 48\lambda_{11}^6 + 53\lambda_{11}^8 - 24\lambda_{11}^{10} + 5\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $\sqrt{\frac{3-\sqrt{3}}{3}} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{2}}$ | $\{1, 2(v_{B4} - u_{B3}), -w_{B1} - 1, 0, 0, 1)\}$ | $\frac{6 - 23\lambda_{11}^2 + 36\lambda_{11}^4 - 24\lambda_{11}^6 + 5\lambda_{11}^8 + 12\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $\frac{1}{\sqrt{2}} \leqslant \lambda_{11} \leqslant \frac{1}{\sqrt{\sqrt{3}}}$ | $\{0, -u_{B3}, -w_{B1}, 1, 1, 0\}$ | $\frac{8 - 33\lambda_{11}^2 + 54\lambda_{11}^4 - 36\lambda_{11}^6 + 5\lambda_{11}^8 + 12\lambda_{11}^{10} - 4\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |
| $\frac{1}{\sqrt{\sqrt{3}}} \leqslant \lambda_{11} \leqslant \sqrt{\sqrt{7} - 2}$ | $\{v_{B4}, 1, 1, w_{B1} + v_{B4}, 0, w_{B1} - v_{B4} + 1\}$ | $\frac{9 - 35\lambda_{11}^2 + 49\lambda_{11}^4 - 24\lambda_{11}^6 + 8\lambda_{11}^8 - 6\lambda_{11}^{10} + 5\lambda_{11}^{12}}{4 - 12\lambda_{11}^2 + 12\lambda_{11}^4}$ |

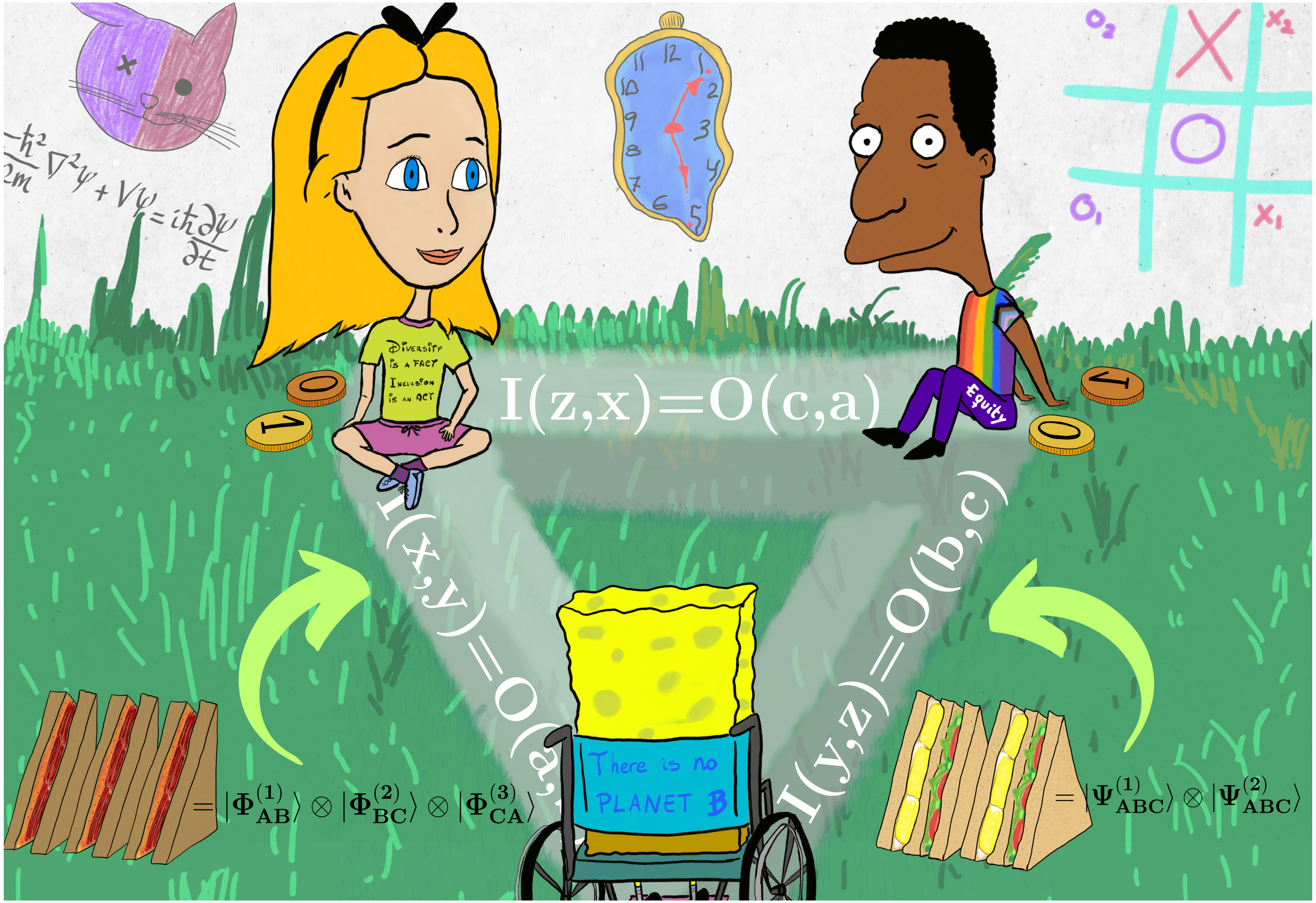
| | | |
|---|---|---|
| $\sqrt{\sqrt{7}-2} \leqslant \lambda_{11} \leqslant \sqrt{\frac{2}{3}}$ | $\{w_{B1}, 1, 1, 2v_{B4}, 0, 1\}$ | $\frac{\lambda_{11}^2\left(7-18\lambda_{11}^2+12\lambda_{11}^4+13\lambda_{11}^6-12\lambda_{11}^8+4\lambda_{11}^{10}\right)}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ ■ |
| $\sqrt{\frac{2}{3}} \leqslant \lambda_{11} \leqslant 1$ | $\{2v_{B4}, 1, 0, 1, 2v_{B4}, 1\}$ | $\frac{4-15\lambda_{11}^2+30\lambda_{11}^4-44\lambda_{11}^6+51\lambda_{11}^8-24\lambda_{11}^{10}+4\lambda_{11}^{12}}{4-12\lambda_{11}^2+12\lambda_{11}^4}$ ■ |

Tab. B.12: *Social welfare for the Nash Equilibrium points in Table 5.12 for the game defined by $I = f_9$, $O = f_{10}$ using the Bell-like state. The colour of the squares help to identify the payoffs plotted in Figure 5.43. The non-marked payoffs, which are constant, are marked as points in the same figure.*

| GHZ-like state with $I = f_{15}$ , $O = f_7$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\{1, 0, 0, 1, c_{11}^2, \tilde{c}_{11}^2\}$<br>$\{0, 1, 1, 0, c_{11}^2, \tilde{c}_{11}^2\}$ | $\frac{1}{4}\left[7 - 2\lambda_{111}^2 + 2\lambda_{111}^4\right]$ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\frac{3}{2}$ |

| Bell-like state with $I = f_{15}$ , $O = f_7$ | |
|---|---|
| $s^* = \{a_{11}^2, \tilde{a}_{11}^2, b_{11}^2, \tilde{b}_{11}^2, c_{11}^2, \tilde{c}_{11}^2\}$ | $\$_A + \$_B + \$_C$ |
| $\{1, 0, 0, 1, c_{11}^2, \tilde{c}_{11}^2\}$<br>$\{0, 1, 1, 0, c_{11}^2, \tilde{c}_{11}^2\}$ | $\frac{1}{4}\left[7 - 3\lambda_{11}^2 + 3\lambda_{11}^4\right]$ |
| $\{a_{11}^2, a_{11}^2, b_{11}^2, b_{11}^2, c_{11}^2, c_{11}^2\}$ | $\frac{3}{2}$ |

Tab. B.13: *Social welfare for the Nash equilibrium points in Table 5.13 for the game defined by* $I = f_{15}$, $O = f_7$ *using the GHZ-and Bell-like states. The colour of the circle and square helps to identify the payoffs plotted in Figure 6.2g.*




I created this drawing about this research for the Intercultural Project at the 10th Heidelberg Laureate Forum from 24th - 29th September 2023 in Heidelberg, Germany. The two types of sandwiches represent the bi-partite and tri-partite entanglement.