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Reappraising Fix-Point Formalizations of Rational Expectations Equilibria

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Abstract

The paper begins with a brief introduction to the origins of the tradition of formalizing the determination of a rational expectations equilibrium using topological fix points. It is then suggested that fidelity to the twin economic conceptual bases of rational expectations - self-reference and infinite regress - are not encapsulated in the topological formalism. To remedy this defect, a recursion theoretic formalism for modelling rational expectations equilibria is developed.

1 Introduction

In standard mathematical economics, *topological* fixed-point theorems have been used fruitfully to encapsulate and formalize *self-reference* (*rational expectations* and policy ineffectiveness), *infinite-regress* (*rational expectations*) and *self-reproduction* and *self-reconstruction* (growth), in economic dynamic contexts. This is in addition to, and quite apart from, their widespread use in proving existence of equilibria in a wide variety of economic and game theoretical contexts. The mathematical foundations of topology is, in general, based on set theory. Set theory, however, is only one of four branches of mathematical logic; the other three being, model theory, proof theory and recursion theory[1]. One can associate, roughly speaking, real analysis, non-standard analysis, constructive analysis and computable analysis with these four branches of mathematical logic. Economists, in choosing to formalize economic notions almost exclusively in terms of real analysis, may not always succeed in capturing the intended conceptual underpinnings of economic notions with the required fidelity. The claim in this paper is that the use of topological fix point theorems to formalize rational expectations does not capture the two fundamental behavioural notions that are crucial in its definition: *self-reference* and *infinite-regress*. I try, therefore, to reformalize the notion of rational expectations using a recursion theoretic formalism such that two fundamental theorems from this field can be invoked and utilized[2]. The idea of self-referential behaviour is, for example, formalized by considering the *action of a program* or an *algorithm* on its *own description*. Infinite regress is, of course, short-circuited, in the usual way, by a fix-point theorem.

Thus, I formalize the notion of *Rational Expectations Equilibria, REE*, recursion theoretically, eschewing all topological assumptions. The emphasis is on suggesting an alternative modelling strategy that can be mimicked for other concepts and areas of macroeconomics: *policy ineffectiveness, time inconsistency, growth, fluctuations* and other dynamic issues in macroeconomics.

A theoretical framework must mesh smoothly with - be consistent with - the empirical data generating process that could underpin it from methodological and epistemological points of view. I do not use these loaded words with grand aims in mind; I refer to the simple fact that a process that generates the macroeconomic data that is the basis on which the processes of scientific validations of any sort can be performed must do so in a way that is consistent with the way the theoretical model postulates the use of the data. I refer to this as a 'simple fact' in the elementary and intuitive sense that data that must be used by rational agents will have to respect their cognitive structures and the structures of the processing and measuring instruments with which they - and the macroeconomic system as a whole - will analyze and theorize with them. There is no point in postulating data generating mechanisms that are incompatible with the cognitive and processing and measuring structures of the analyzing

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[1] Some add the *higher arithmetic* (i.e., *number theory*) as an independent fifth branch of modern mathematical logic.

[2] One of which is also called a *fix point theorem*.

agents of the economy - at the individual and collective levels.

All recursion theoretic formalizations and results come, almost invariably, 'open ended' - meaning, even when uniqueness results are demonstrated there will be, embedded in the recesses of the procedures generating equilibria and other types of solutions, an indeterminacy. This is due to a generic result in computability theory called the *Halting Problem for Turing Machines*. It is a kind of generic *undecidability* result, a counterpart to the more formal, and more famous, Gödelian *undecidability* results. It is this fact, lurking as a backdrop to all the theorems in this essay, that makes it possible to claim that *Computable Macroeconomics* is not as determinate as Newclassical Macroeconomics. To be categorical about policy - positively or negatively - on the basis of mathematical models is a dangerous sport.

The essay is organized as follows. In the next section I outline the origins of the rational expectations problem as a (topological) fixed-point problem. In the third section, I suggest its reformulation in recursion theoretic terms. This reformulation makes it possible to re-interpret a rational expectations equilibrium as a recursion theoretic fixed-point problem in such a way that it is intrinsically computable, i.e., computable *ab initio*. Thus, there is no separation between a first step in which the existence of a rational expectations equilibrium is 'proved' and, then, an *ad hoc* mechanism devised to determine it - via uncomputable or *ad hoc* learning processes. Moreover, every recursion theoretic assumption, and their consequent formalisms I have employed or invoked, in this essay, is consistent with the known results and constraints on human cognitive structures and all known computing devices, artificial or natural, ideal or less-than-ideal.

In the concluding section I try to fashion a fabric, or at least its design, from the sketches of the threads outlined earlier, that depicts a possible research program on Computable Macroeconomics as an alternative to the Newclassical Recursive Macroeconomics.

2   Background

In a critical discussion of the use of the Brouwer fixed point theorem by Herbert Simon, [13], that presaged its decisive use in what became the definition of a rational expectations equilibrium, Karl Egil Aubert, a respected mathematician, suggested that economists - and political scientists - were rather cavalier about the domain of definition of economic variables and, hence, less than careful about the mathematics they invoked to derive economic propositions. I was left with the impression, after a careful reading of the discussion between Aubert and Simon ([1], [14], [2] and [15]), that the issue was not the use of a fixed point framework but its *nature, scope* and *underpinnings*. However, particularly in a rational expectations context, it is not only a question of the nature of the domain of definition but also the fact that there are self-referential and infinite-regress elements intrinsic to the problem. This makes *the appropriate choice of the fixed point theorem* within which to embed the question of a rational expectations equilibrium particularly sensitive to the kind of mathematics and

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logic that underpins it. In this section I trace the origins of the 'topologisation' of the mathematical problem of rational expectations equilibrium and discuss the possible infelicities inherent in such a formalization.

There are two crucial aspects to the notion of rational expectations equilibrium - henceforth, *REE* - ([12], pp.6-10): an individual optimization problem, subject to *perceived constraints,* and a system wide, autonomous, set of constraints imposing a consistency across the collection of the perceived constraints of the individuals. The latter would be, in a most general sense, the accounting constraint, generated autonomously, by the logic of the macroeconomic system. In a representative agent framework the determination of *REE*s entails the solution of a general fix point problem. Suppose the representative agent's *perceived law of motion* of the macroeconomic system (as a function of state variables and exogenous 'disturbances') as a whole is given by H^3. The system wide autonomous set of constraints, implied, partially at least, by the optimal decisions based on perceived constraints by the agents, on the other hand, imply an *actual law of motion* given by, say, H^0. The search for fixed-points of a mapping, T, linking the individually perceived macroeconomic law of motion, H, and the actual law of motion, H^0 is assumed to be given by a general functional relationship subject to the standard mathematical assumptions:

$$H^0 = T(H) \tag{1}$$

Thus, the fixed-points of H^* of T^4:

$$H^* = T(H^*) \tag{2}$$

determine *REE*s.

What is the justification for T? What kind of 'animal' is it? It is variously referred to as a 'reaction function', a 'best response function', a 'best response mapping', etc. But whatever it is called, eventually the necessary mathematical assumptions are imputed to it such that it is amenable to a topological interpretation whereby appeal can be made to the existence of a fix point for it as a mapping from a structured domain into itself. So far as I know, there is no optimizing economic theoretical justification for it.

There is also a methodological asymmetry in the determination of H and H^0, respectively. The former has a self-referential aspect to it; the latter an infinite regress element in it. Transforming, mechanically, (1) into (2) hides this fact and reducing it to a topological fixed-point problem does little methodological justice to the contents of the constituent elements of the problem. These elements are brought to the surface at the second, separate, step in which ostensible learning mechanisms are devised, in *ad hoc* ways, to determine, explicitly the uncomputable and non-constructive fixed-points. But is it really impossible to consider the twin problems in one fell swoop, so to speak?

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[3] Readers familiar with the literature will recognise that the notation H reflects the fact that, in the underlying optimisation problem, a *Hamiltonian* function has to be formed.

[4] In a space of functions.

This kind of tradition to the formalization and determination of *REE*s has almost by default forced the problem into a particular mathematical straitjacket. The mapping is given topological underpinnings, automatically endowing the underlying assumptions with real analytic content[5]. As a consequence of these default ideas the problem of determining any *REE* is dichotomized into two sub-problems: a first part where non-constructive and non-computable proofs of the existence of *REE*s are provided; and a subsequent, quite separate, second part where mechanisms - often given the sobriquet 'learning mechanisms' - are devised to show that such *REE*s can be determined by individual optimizing agents[6]. It is in this second part where standard economic theory endows agents with varieties of 'bounded rationality' postulates, without modifying the full rationality postulates of the underlying, original, individual optimization problem.

Now, how did this *topological fixed-point REE tradition* come into being? Not, as might conceivably be believed, as a result of Muth's justly celebrated original contribution,[11], but from the prior work of Herbert Simon on a problem of predicting the behaviour of rational agents in a political setting, [13] and an almost simultaneous economic application by Franco Modigliani and Emile Grunberg, [6]. Let me explain, albeit briefly, to the extent necessary in the context of this essay.[7]

Simon, in considering the general issue of the feasibility of public prediction in a social science context, formalized the problem for the particular case of investigating how 'the publication of an election prediction (particularly one based on poll data) might influence [individual] voting behaviour, and, hence - ... - falsify the prediction'. Simon, as he has done so often in so many problem situations, came up with the innovative suggestion that the self-referential and infinite-regress content of such a context may well be solved by framing it as a mathematical fixed-point problem:

> "Is there not involved here a vicious circle, whereby any attempt

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[5] In the strict technical sense, as suggested in the opeing section above, of the mathematics of *real analysis* as distinct from, say, *constructive, computable or non-standard analysis*.

[6] Perceptive readers may wonder whether there should not also be an optimization exercise over the set of feasible or perceived learning mechanisms? Carried to its logical conclusion, this would entail the determination of a set of *REE*s over the collection of learning mechanisms, *ad infinitum* (or *ad nauseum*, whichever one prefers).

[7] My aim is to show that the framing the *REE* problem as a *topological fixed-point problem* was not necessary. Moreover, by forcing the *REE* problem as a topological fixed-point problem it was necessary to dichotomize into the proof of existence part and a separate part to demonstrate the feasibility of constructing mechanisms to determine them. This is mainly - but not only - due to the utilization of non-constructive or uncomputable topological fixed-point theorems in the first, 'proof of *REE* existence', part. In this sense the *REE learning* research program is very similar to the earlier dichotomizing of the *general equilibrium* problem. In that earlier phase, a long tradition of using topological fixed-point theorem to prove the existence of a economic equilibria was separated from devising *constructive* or *computable* mechanisms to determine them. The later phase resulted in the highly successful *Computable General Equilibrium (CGE)* models. It remains a melancholy fact, however, that even after over forty years of sustained and impressive work on CGE models, they are neither constructive nor computable, contrary to assertions by proponents of the theory (cf. [20] for a rigorous demonstration of this claim).

to anticipate the reactions of the voters alters those reactions and hence invalidates the prediction?

In principle, the last question can be answered in the negative: there is no vicious circle.

....

We [can prove using a 'classical theorem of topology due to Brouwer (the 'fixed-point' theorem)] that it is always possible in principle to take account of reactions to a published prediction in such a way that the prediction will be confirmed by the event."

Simon, op.cit, [13], pp. 82-4; italics added.

Grunberg and Modigliani recognized, clearly and explicitly, both the self-referential nature of the problem of consistent individually rational predictions in the face of being placed in an economic environment where their predictions are reactions to, and react upon (*ad infinitum* – i.e., *infinite regress)*, the aggregate outcome, but also were acutely aware of the technical difficulties of infinite regression that was also inherent in such situations (cf., in particular, [6], p. 467 and p. 471). In their setting an individual producer faced the classic problem of expected price and quantity formation in a single market, subject to public prediction of the market clearing price. It was not dissimilar to the crude cobweb model, as was indeed recognized by them ([6], p.468, footnote 13). Interestingly, what eventually came to be called *rational expectations* by Muth was called a *warranted expectation*[8] by Grunberg and Modigliani (ibid, pp. 469-70). In any event, their claim that it was '*normally* possible' to prove the *existence* of 'at least one correct public prediction in the face of effective reaction by the agents' was substantiated by invoking Brouwer's Fixed Point Theorem (ibid, p. 472). To facilitate the application of the theorem, the constituent functions[9] and variables - in particular, the *reaction function* and the conditions on the domain of definition of prices - were assumed to satisfy the necessary real number and topological conditions (continuity, boundedness, etc).

Thus it was that the tradition, in the rational expectations literature of 'solving' the conundrums of self-reference and infinite-regress via *topological fixed-*

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[8]I am reminded that Phelps, in one of his early, influential, papers that introduced the concept of the natural rate of unemployment in its modern forms, first referred to it as a *warranted rate*. Eventually, of course, the Wicksellian term *natural rate*, introduced by Friedman, prevailed. Phelps and Grunberg-Modigliani were, presumably, influenced by Harrodian thoughts in choosing the eminently suitable word 'warranted' rather than 'natural' or 'rational', respectively. Personally, for aesthetic as well as reasons of economic content, I wish the Phelps and Grunberg-Modigliani suggestions had prevailed.

[9]The relation between a market price and its predicted value was termed the *reaction function*: "Relations of this form between the variable to be predicted and the prediction will be called *reaction functions*." ([6], p.471; italics in original).

As became the tradition in the whole rational expectations literature, the functional form for the reaction functions were chosen with a clear eye on the requirements for the application of an appropriate topological fixed point theorem. The self-reference and infinite-regress underpinnings were thought to have been adequately subsumed in the existence results that were guaranteed by the fixed point solution. That the twin conundrums were not subsumed but simply camouflaged was not to become evident till all the later activity on trying to devise learning processes for identifying *REE*s.

point theorems was etched in the collective memory of the profession. And so, four decades after the Simon and the Grunberg-Modigliani contributions, Sargent, in his influential *Arne Ryde Lectures* ([12]) was able to refer to the fixed-point approach to rational expectations, referring to equation (2), above:

> "A rational expectations equilibrium is a fixed point of the mapping T."
> [12], p.10.

Now, fifty years after that initial introduction of the topological fixed-point tradition by Simon and Grunberg-Modigliani, economists automatically and uncritically accept that this is the only way to solve the *REE* existence problem - and they are not to be blamed. After all, the same complacency dominates the fundamentals of general equilibrium theory, as if the equilibrium existence problem can only be framed as a fixed-point solution. Because of this complacency, the existence problem has forever been severed of all connections with the problem of determining - or finding or constructing or locating - the processes that may lead to the non-constructive and uncomputable equilibrium. The recursion theoretic fixed-point tradition not only preserves the unity of equilibrium existence demonstration with the processes that determine it; but it also retains, in the forefront, the self-referential and infinite-regress aspects of the problem of the interaction between individual and social prediction and individual and general equilibrium.

3 Recursion Theoretic Rational Expectations

3.1 Recursion Theoretic Formalisms

There is nothing sacrosanct about a topological interpretation of the operator T, the reaction or response function. It could equally well be interpreted recursion theoretically, which is what I shall do in the sequel[10]. I need some unfamiliar, but elementary, formal machinery – concepts, definitions, new or alternative connotations for familiar words, etc., – not normally available to the mathematical economist or, in particular, to the macroeconomist.

Definition 1 *An **operator** is a function:*

$$\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n \tag{3}$$

where $\mathcal{F}_k$ $(k \geqq 1)$ is the class of all partial (recursive) functions from $\mathbb{N}^k$ to $\mathbb{N}$.

Definition 2 *Φ is a **recursive operator** if there is a computable function ϕ such that $\forall f \in \mathcal{F}_m$ and $\mathbf{x} \in \mathbb{N}^k$, $y \in \mathbb{N}$:*

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[10] I have relied on the following four excellent texts for the formalisms and results of recursion theory that I am using in this part of the essay: [4], [5], [9] and [17].

$$\Phi\left(f\right)\left(\mathbf{x}\right) \simeq y \ iff \ \exists \ a \ finite \ \theta \sqsubseteq f \ such \ that \ \phi\left(\widetilde{\theta},\mathbf{x}\right) \simeq y$$

where[11] $\widetilde{\theta}$ is a standard *coding* of a *finite* function θ, which is extended by f.

Definition 3 *An operator* $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ *is* **continuous** *if, for any* $f \in \mathcal{F}_m$, *and* $\forall \mathbf{x}, y$:

$$\Phi\left(f\right)\left(\mathbf{x}\right) \simeq y \ iff \ \exists \ a \ finite \ \theta \sqsubseteq f \ \ such \ that \ \Phi\left(\theta\right)\left(\mathbf{x}\right) \simeq y$$

Definition 4 *An operator* $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ *is* **monotone** *if, whenever* $f, g \in \mathcal{F}_m$ *and* $f \sqsubseteq g$, *then* $\Phi\left(f\right) \sqsubseteq \Phi\left(g\right)$.

Theorem 5 *A recursive operator is continuous and monotone.*

Example 6 *Consider the following* **recursive program,** $Þ$ *,(also a recursive operator) over the integers:*
$Þ:\ F\left(x,y\right) \Longleftarrow if \ x = y \ then \ y+1, \ else \ F\left(x, F\left(x-1, y+1\right)\right)$
Now replace each occurrence of F *in* $Þ$ *by each of the following functions:*

$$f_1\left(x,y\right):\ if \ x = y \ then \ y+1, \ else \ x+1 \tag{4}$$

$$f_2\left(x,y\right):\ if \ x \geqq y \ then \ x+1, \ else \ y-1 \tag{5}$$

$$f_3\left(x,y\right):\ if \ \left(x \geqq y\right) \wedge \left(x-y \ even\right) \ then \ x+1, \ else \ undefined. \tag{6}$$

Then, on either side of $\Longleftarrow$ *in* $Þ$, *we get the* **identical** *partial functions:*

$$\forall i \left(1 \leqq i \leqq 3\right), \ f_i\left(x,y\right) \equiv if \ x = y \ then \ y=1, \ else \ f_i\left(x-1, y+1\right) \tag{7}$$

Such functions f_i *(* $\forall i \left(1 \leqq i \leqq 3\right)$ *) are referred to as* **fixed-points** *of the recursive program* $Þ$ *(recursive operator).*
Note that these are fixed-points of functionals.

Remark 7 *Note that* f_3, *in contrast to* f_1 *and* f_2, *has the following special property.* $\forall \langle x, y \rangle$ *of pairs of integers such that* $f_3\left(x,y\right)$ *is defined, both* f_1 *and* f_2 *are also defined and have the same value as does* f_3.

- f_3 *is, then, said to be* **less defined than or equal to** f_1 *and* f_2 *and this property is denoted by* $f_3 \sqsubseteq f_1$ *and* $f_3 \sqsubseteq f_2$.

- *In fact, in this particular example, it so happens that* f_3 **is less defined than or equal to** *all fixed points of* $Þ$.

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[11] If $f\left(\mathbf{x}\right)$ and $g\left(\mathbf{x}\right)$ are expressions involving the variables $\mathbf{x} = \left(x_1, x_2, ..., x_k\right)$, then:

$$f\left(\mathbf{x}\right) \simeq g\left(\mathbf{x}\right)$$

means: for any $\mathbf{x}$, $f\left(\mathbf{x}\right)$ and $g\left(\mathbf{x}\right)$ are either both defined or undefined, and if defined, they are equal.

- *In addition, f_3 is the **only** partial function with this property for $Þ$ and is, therefore called the **least fixed point of $Þ$**.*

We now have the minimal formal machinery needed to state one of the classic theorems of recursive function theory, known variously as the *first recursion theorem, Kleene's theorem* or, sometimes, as the *fixed point theorem for complete partial orders.*

Theorem 8 *Suppose that $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ is a recursive operator (or a recursive program $Þ$). Then there is a partial function f_ϕ that is the least fixed point of Φ :*
$$\Phi\left(f_\phi\right) = f_\phi;$$
$$\text{If } \Phi\left(g\right) = g, \text{ then } f_\phi \sqsubseteq g.$$

Remark 9 *If, in addition to being partial, f_ϕ is also total, then it is the **unique least fixed point**. Note also that a recursive operator is characterized by being continuous and monotone. There would have been some advantages in stating this famous theorem highlighting the domain of definition, i.e., complete partial orders, but the formal machinery becomes slightly unwieldy.*

Remark 10 *Although this way of stating the (first) recursion theorem almost highlights its non-constructive aspect – i.e., the theorem guarantees the **existence** of a fix-point without indicating a way of finding it – it is possible to use a slightly stronger form of the theorem to amend this 'defect' (cf. [10], p.59).*

3.2 Recursion Theoretic *REE*

Before stating formally, as a summarizing theorem, the main result (i.e., theorem 11, below) it is necessary to formalize the rational agent and the setting in which rationality is exercised in the expectational domain in recursion theoretic formalisms, too. This means, at a minimum, the rational agent as a *recursion theoretic agent*[12].

The topological fix-point theorems harnessed by a rational agent are, as mentioned previously, easily done in standard economic theory where the agents themselves are set-theoretically formalized. There is no dissonance between the formalism in which the rational agent is defined and the economic setting in which such an agent operates. The latter setting is also set theoretically defined.

The recursion theoretic formalism introduced in the previous sub-section presupposes that the rational agent is now recursion theoretically defined and so too the setting - i.e., the economy. Defining the rational agent recursion theoretically means defining the preferences characterizing the agent and the choice theoretic actions recursion theoretically. This means, firstly, defining the domain of choice for the agent number theoretically and, secondly, the choice

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[12] This should not cause any disquiet in expectational economics, at least not to those of us who have accepted the Lucasian case for viewing agents as 'signal processors' who use optimal filters in their rational decision processing activities (cf. [8], p.). Agents as 'signal processors' is only a special variant of being 'optimal computing units'.

of maximal (sub)sets over such a domain in a computably viable way. Such a redefinition and reformalization should mean equivalences between the rational choice of an agent over well defined preferences and the computing activities of an ideal computer, i.e., Turing Machine (or any of its own formal equivalences, by the *Church-Turing Thesis*). Since a complete formalism and the relevant equivalences are described, defined and, where necessary, rigorously proved in Chapter 3 of my *Ryde Lectures* ([18]), I shall simply assume the interested reader can be trusted to refer to it for any detailed clarification and substantiation.

It is now easy to verify that the domain over which the recursive operator and the partial functions are defined are weaker[13] than the conventional domains over which the economist works. Similarly, the continuity and monotonicity of the recursive operator is naturally satisfied by the standard assumptions in economic theory for the reaction or response function, T. Hence, we can apply the *first recursion theorem* to equation (2), interpreting T as a recursive operator and not as a topological mapping. Then, from theorem 8, we know that there is a partial function - i.e., a computable function - f_t that is the least fixed point of T.

Theorem 11 *Suppose that the reaction or response function, $T : H_m \longrightarrow H_n$ is a recursive operator (or a recursive program Γ). Then there is a computable function f_t that is a least fixed point of T :*
$T(f_t) = f_t;$
If $T(g) = g$, then $f_t \sqsubseteq g$

Remark 12 *Theorem 8 can be used directly to show that $\exists$ a (recursive) program that, under any input, outputs exactly itself. It is this program that acts as the relevant reaction or response **function** for an economy in REE. The existence of such a recursive program justifies the Newclassical methodological stand on the ubiquity of rational expectations equilibria. However, since theorem 8 is stated above in its non-constructive version, finding this particular recursive program requires a little effort. Hence, the need for learning processes to find this program, unless the theorem is utilized in its constructive version. Even with these caveats, the immediate advantage is that there is no need to deal with non-recursive reals or non-computable functions in the recursion theoretic formalism. In the tradition formalism the fix-point that is the REE is, except for flukes, a non-recursive real; constructing learning processes to determine non-recursive reals is either provably impossible or formally intractable (computationally complex).*

What are the further advantages of recasting the problem of solving for the *REE recursion theoretically* rather than retaining the traditional topological formalizations?

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[13]They are 'weaker' in a very special sense. A domain of definition that is number theoretically defined – i.e., over only the rational or the natural numbers – rather than over the whole of the real number system pose natural diophantine and combinatorial conundrums that cannot easily be resolved by the standard operators of optimization.

An advantage at the superficial level but nevertheless not unimportant in policy oriented economic theoretic contexts is the simple fact that, as even the name indicates, recursion encapsulates, explicitly, the idea of self-reference because functions are defined, naturally, in terms of themselves. Secondly the existence of a least fix point is a solution to the infinite-regress problem. Thus the two 'birds' are encapsulated in one fell swoop - and, that too, with a computable function.

Think of the formal discourse of economic analysis as being conducted in a programming language; call it $\Im$. We know that we choose the underlying terminology for economic formalisms with particular meanings in mind for the elemental units: preferences, endowments, technology, information, expectation and so on; call the generic element of the set ς. When we form a compound economic proposition out of the ς units, the meaning is natural and clear. We can, therefore, suppose that evaluating a compound expression in $\Im$ is immediate: given an expression in $\Im$, say $\lambda(\varsigma)$, the variables in λ, when given specific values α, are to be evaluated according to the *semantics* of $\Im$. To actually *evaluate* a compound expression, $\lambda(\varsigma)$, we write a *recursive program* in the language $\Im$, the language of economic theory.

But that leaves a key question unanswered: what is the computable function that is implicitly defined by the recursive program? The first recursion theorem answers this question with the answer: the least fixed-point. In this case, therefore, there is a direct application of the first recursion theorem to the semantics of the language $\Im$. The artificial separation between the syntax of economic analysis, when formalized, and its natural semantics can, therefore, be bridged *effectively*.

If the language of economic theory is best regarded as a very high level programming language, $\Im$, to understand a *theorem* in economics, in recursion theoretic terms, represent the *assumptions* - i.e., *axioms* and the *variables* - as *input data* and the *conclusions* as *output data*. State the theorem as an expression in the language $\Im$.Then try to convert the proof into a program in the language $\Im$, which will take in the inputs and produce the desired output. If one is unable to do this, it is probably because the proof relies essentially on some infusion of non-constructive or uncomputable elements. This step will identify any inadvertent infusion of non-algorithmic reasoning, which will have to be resolved - sooner or later, if computations are to be performed on the variables as input data. The computations are not necessarily numerical; they can also be symbolic.

In other words, if we take algorithms and data structures to be fundamental,then it is natural to define and understand functions in these terms. If a function does not correspond to an algorithm, what can it be? The topological definition of a function is not naturally algorithmic. Therefore, the expressions formed from the language of economic theory, in a topological formalization, are not necessarily implementable by a program, except by fluke, appeal to magic or by illegitimate, intractable and vague approximations. Hence the need to dichotomize every topological existence proof. In the case of *REE,* this is the root cause of the artificial importance granted to a separate problem of learning

*REE*s.

4 Concluding Notes

In recent years Sargent and his collaborators have developed what they call a *Recursive Macroeconomics* and before that there was the encyclopedic treatise by Lucas and Stokey (with Prescott) on *Recursive Methods in Economic Dynamics* ([7], [16]). *Recursive Macroeconomic Theory*, as Sargent et.al see it, is *recursive* in view of the three basic theoretical technologies that underpin the economic hypotheses: *sequential analysis*, *dynamic programming* and *optimal filtering*. To put it in terms of the pioneers whose theories underpin *Recursive Macroeconomic Theory*, the core of this approach harnesses the theoretical technologies of Abraham Wald's *sequential analysis*, Richard Bellman's *dynamic programming* and Rudolf Kalman's *filtering* frameworks. This means, the underlying economic hypotheses of *Recursive Macroeconomic Theory* will be framed and formalized in such a way as to be based on the mathematics of sequential analysis, dynamic programming and optimal filtering - whether or not economic reality demands it; whether or not economic behaviour warrants it; whether or not economic institutions justify it; and most basically, whether or not economic data conform to their requirements.

The word *recursive* is heavily loaded with connotations of dynamics, computation and numerical methods. But these connotations are also fraught with dangers. For example the methods of dynamic programming are provably complex in a precise computational sense; the equations that have to be solved to implement optimal filtering solutions are also provably intractable; ditto for sequential analysis.

The *recursion theoretic* framework for rational expectations equilibria that I have suggested in the main part of this essay is explicitly computational, algorithmically dynamic and meaningfully numerical. Moreover, the theorems that I have derived above, have an open-ended character about them. To put in blunt words, these theorems tell an implementable story about things that can be done; but they are silent about things that cannot be done[14]. But the stories are always about what can be done with well defined methods to do them - the algorithms. They are never about pseudo-recursive operators that are disconnect to, or independent of, computations and numerical methods.

The exercise presented in the third section of this paper is a prototype of a strategy to be applied to defining areas of macroeconomics: growth, fluctuations, policy, capital, monetary and unemployment theories. The general idea is to

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[14]I cannot resist recalling those famous 'last lines' of the early Wittgenstein:

"What we cannot speak about we must pass over in silence." ([21], §7).

The sense in which this famous aphorism comes to mind is that in the recursion theoretic approach one does not invoke magic, metaphysics or other formal or informal tricks to solve equations. A problem is always posed in a specific context of *effective* methods of solution. The formal mathematical approach in standard economic theory is replete with magical and metaphysical methods to 'solve', 'prove' or determine solutions, equilibria, etc.

strip the formal models in the respective fields of their topological underpinnings and replace them, systematically, with recursion theoretic elements in such a way that the open-endedness is enhanced and the numerical and computational contents made explicit and implementable. The specific way it was done in §3 was to concentrate on the use of the topological fixed-point theorem and replace it with a recursion theoretic fixed-point theorem. Similarly, in the case, of growth theory, say of the von Neumann variety, an analogous exercise can be carried out. This will lead to the use of the *second recursion theorem* rather than the one I have harnessed in this paper and growth will mean self-reconstruction and self-reproduction. In the case of fluctuations, the idea would be to replace all reliance on differential or difference equation modelling of economic dynamics and replace them with naturally recursion theoretic entities such as *cellular automata*[15]. The aim, ultimately, is to produce a corpus of theories of the central macroeconomic issues so that they can be collected under the alternative umbrella phrase: *Computable Macroeconomics.*

The question will be asked, quite legitimately, whether this line of attack aims also to maintain fidelity with microeconomic, rationality, postulates and, if so, in what way it will differ in the foundations from, say, *Recursive Macroeconomic Theory*. The canonical workhorse on which *Recursive Macroeconomic Theory* rides is the (*competitive*) *equilibrium model of a dynamic stochastic economy*. A rational agent in such an economic environment is, as mentioned above, essentially, a *signal processor*. Hence, *optimal filtering* plays a pivotal role in this approach to macroeconomic theory. The simple answer of a *Computable Macroeconomist* would be that the rational agent of microeconomics would be reinterpreted as a *Turing Machine* - a construction I have developed in great detail in, for example, [18], chapter 3. The analogous construction for the other side of the market is equally feasible, starting from re-interpreting the production function as a Turing Machine. This endows the production process with the natural dynamics that belonged to it in the hands of the classical economists and the early Austrians but was diluted by the latter-day Newclassicals. What of market structure - i.e., *economic institutions?* Here, too, following in the giant footsteps of Simon and Scarf, there is a path laid out whereby an algorithmic interpretation of institutions is formally natural.

That leaves only, almost, that sacrosanct disciplining rule of economic theory: optimization. Recursion theoretic problem formulations eschew optimizations and replace them with *decision problems*. Simply stated, one asks whether problems are *solvable (*or *not*) and if they are solvable, how hard is it to solve them and if they are not how must one change the problem formulation to make them solvable. *Decidability*, *solvability* and *computability* are the touch-

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[15] There is more to this suggestion than can be discussed here. It has to do with the connections between dynamical systems theory, numerical analysis and recursion theory, if digital computers are the vehicles for experimental and simulation exercises. If, on the other hand, one is prepared to work with special purpose analogue computers, then the connection between dynamical systems and recursion theory can be more direct and it may not be necessary to eschew the use of differential or difference equations in investigating and modelling economic dynamics. I have discussed these issues in [19].

stones of a modelling strategy in Computable Macroeconomics. This is to place macroeconomics squarely in the *satisficing* world.

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